Rota's conjecture and positivity of algebraic cycles in permutohedral varieties

by

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CHAPTER I

Introduction

Attempting to solve the four color problem, George Birkhoff introduced a polynomial associated to a graph G which coherently encodes the answers to the analogous q color problem for all natural numbers q [Bir12]. This polynomial $\chi_G(q)$, called the *chromatic polynomial*, is determined by the property that

$$\chi_G(q) = (\text{number of proper colorings of } G \text{ using } q \text{ colors}), \qquad q \geq 1.$$

Any other numerical invariant of a simple graph which can be recursively computed by deletion and contraction of edges is a specialization of the chromatic polynomial.

A sequence of real numbers $a_0, a_1, \ldots, a_r$ is said to be *log-concave* if

$$a_i^2 \geq a_{i-1}a_{i+1} \quad \text{for all } i.$$

In previous work, we proved that the coefficients of the chromatic polynomial form a log-concave sequence for any graph, thus resolving a conjecture of Ronald Read [Huh12]. An important step in the proof was to construct a complex algebraic variety associated to a graph and ask a more general question on the characteristic class of the algebraic variety. It turned out that the property of the characteristic class responsible for log-concavity is that it is *realizable*, meaning that the homology class is the class of a subvariety (an irreducible algebraic subset).

Theorem 1. *If ξ is an element in the homology group*

$$\xi = \sum_j x_j \left[\mathbb{P}^{d-j} \times \mathbb{P}^j\right] \in H_{2d}(X;\mathbb{Z}), \quad X = \mathbb{P}^{n-m} \times \mathbb{P}^m,$$

then some positive integer multiple of ξ is the class of a subvariety if and only if the x_j form a nonzero log-concave sequence of nonnegative integers with no internal zeros.

In general, for any compact complex variety X, one may define the space of realizable homology classes as a closed subset of $H_{2d}(X;\mathbb{R})$ which consists of limits of homology classes of subvarieties up to a constant multiple (Definition 57). This subset, showing distribution of primes in the homology of X, plays a key role in the solution to the graph theory problem. A motivating observation for further investigation is that, even for very simple varieties such as the product of two projective spaces, the orderly structure of the space of realizable homology classes becomes visible only after taking positive multiples of homology classes. For example, there is no subvariety of $\mathbb{P}^5 \times \mathbb{P}^5$ which has the homology class

$$1[\mathbb{P}^5\times\mathbb{P}^0]+2[\mathbb{P}^4\times\mathbb{P}^1]+3[\mathbb{P}^3\times\mathbb{P}^2]+4[\mathbb{P}^2\times\mathbb{P}^3]+2[\mathbb{P}^1\times\mathbb{P}^4]+1[\mathbb{P}^0\times\mathbb{P}^5] \in H_{10}(\mathbb{P}^5\times\mathbb{P}^5;\mathbb{Z}),$$

although $(1, 2, 3, 4, 2, 1)$ is a log-concave sequence with no internal zeros.

Read's conjecture was later extended by Gian-Carlo Rota and Dominic Welsh to combinatorial geometries, also called matroids, whose defining axioms are modeled on the relation of linear independence in a vector space (Conjecture 21). The aim of this thesis is to show that the above mentioned algebro-geometric proof does not work in this more general setting for one and only one very interesting reason: not every matroid is realizable as a configuration of vectors in a vector space.

Since the discovery of a finite projective plane which is not coordinatizable over any field, mathematicians have been interested in this tension between the axioms of combinatorial geometry and algebraic geometry. After numerous unsuccessful quests for the “missing axiom” which guarantees realizability, logicians found that one cannot add finitely many new axioms to matroid theory to resolve the tension [MNW14, Vam78]. On the other hand, computer experiments revealed that numerical invariants of small matroids behave as if they were realizable, confirming the log-concavity conjecture in particular for all matroids within the range of our computational capabilities.

We ask whether every matroid is realizable over every field in some generalized sense (Question 60). Here a matroid on $\{0, 1, \ldots, n\}$ is viewed as an integral homology class in the toric variety X_{A_n} constructed from the n-dimensional permutohedron, a polytope which reflects the structure of the root system A_n. This homology class is nef and effective for any matroid (Corollary 34), and the usual realizability of a matroid translates to the statement that the corresponding homology class is the class of a subvariety (Theorem 46).

The anticanonical divisor of the toric variety X_{A_n} is a sum of two nef and big divisors. The associated linear systems define a map

$$\pi_1 \times \pi_2 : X_{A_n} \longrightarrow \mathbb{P}^n \times \mathbb{P}^n.$$

We show that the chromatic (characteristic) polynomials appear through the push-forward of the matroid homology classes:

$$H_{2d}(X_{A_n}; \mathbb{Z}) \longrightarrow H_{2d}(\mathbb{P}^n \times \mathbb{P}^n; \mathbb{Z}), \quad \text{(matroid)} \longmapsto \text{(characteristic polynomial)},$$

see Theorem 54. Under this framework, Theorem 1 together with numerical evidence for the log-concavity conjecture suggest an intriguing possibility that any matroid homology class is a limit of realizable homology classes up to a constant multiple. If true, this will not only prove the log-concavity conjecture but also explain the subtle discrepancy between combinatorial geometry and algebraic geometry.

In Chapter II, we introduce the permutohedral variety X_{A_n} and recall the basics of matroid theory. The matroid homology classes are defined in Chapter III. It is shown there that a matroid homology class is effective, and generates an extremal ray of the nef cone of X_{A_n}. The main result of Chapter IV is that the anticanonical push-forward of a matroid is the reduced characteristic polynomial of the matroid. This purely combinatorial computation is used in the last section to prove the log-concavity conjecture for matroids that are realizable over some field.

CHAPTER II

Matroids and the permutohedron

2.1 The permutohedral variety

Let n be a nonnegative integer and let E be the set $\{0, 1, \ldots, n\}$.

Definition 2. The n-dimensional *permutohedron* is the convex hull

$$\Xi_n = \operatorname{conv}\Big\{(x_0, \ldots, x_n) \mid x_0, x_1, \ldots, x_n \text{ is a permutation of } 0, 1, \ldots, n\Big\} \subseteq \mathbb{R}^{n+1}.$$

The symmetric group on E acts on the permutohedron Ξ_n by permuting coordinates, and hence each one of the above $(n+1)!$ points is a vertex of Ξ_n.

The n-dimensional permutohedron is contained in the hyperplane

$$x_0 + x_1 + \cdots + x_n = \frac{n(n+1)}{2}.$$

The permutohedron has one facet for each nonempty proper subset S of E, denoted Ξ_S, is the convex hull of those vertices whose coordinates in positions in S are smaller than any coordinate in positions not in S. For example, if S is a set with one element i, then the corresponding facet is

$$\Xi_{\{i\}} = \operatorname{conv}\Big\{(x_0, \ldots, x_n) \mid \ (x_0, \ldots, x_n) \text{ is a vertex of } \Xi_n \text{ with } x_i = 0\Big\} \subseteq \mathbb{R}^{n+1}.$$

Similarly, if S is the entire set minus one element $E \setminus \{i\}$, then the corresponding facet is

$$\Xi_{E\setminus\{i\}} = \operatorname{conv}\Big\{(x_0, \ldots, x_n) \mid \ (x_0, \ldots, x_n) \text{ is a vertex of } \Xi_n \text{ with } x_i = n\Big\} \subseteq \mathbb{R}^{n+1}.$$

These facets can be identified with the permutohedron of one smaller dimension. In general, a facet of a permutohedron can be identified with the product of two permutohedrons of smaller dimensions:

$$\Xi_S \simeq \Xi_{|S|-1} \times \Xi_{|E\setminus S|-1}.$$

More generally, the codimension d faces of the permutohedron Ξ_n bijectively correspond to the ordered partitions of E into $d+1$ parts. Explicitly, the codimension d face corresponding to a flag of nonempty proper subsets $\big(S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_d\big)$ is the convex hull of those vertices whose coordinates in positions in $S_j \setminus S_{j-1}$ are smaller than any coordinate in positions in $S_{j+1} \setminus S_j$ for all j.

The normal fan of the n-dimensional permutohedron is a complete fan in an n-dimensional quotient of the vector space $\mathbb{R}^{n+1}$:

$$|\Delta_{A_n}| := \mathbb{R}^{n+1}/\operatorname{span}(1, 1, \ldots, 1).$$

The quotient space $|\Delta_{A_n}|$ is generated by the vectors $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n$, where $\mathbf{u}_i$ is the primitive ray generator in the normal fan corresponding to the facet $\Xi_{\{i\}}$. In coordinates,

$$\mathbf{u}_0 = (1, 0, \ldots, 0),\ \mathbf{u}_1 = (0, 1, \ldots, 0),\ \ldots\ \mathbf{u}_n = (0, 0, \ldots, 1) \quad \bmod (1, 1, \ldots, 1).$$

Notation. For a subset S of E, we define

$$\mathbf{u}_S := \sum_{i \in S} \mathbf{u}_i.$$

If S is a nonempty and proper subset of E, then $\mathbf{u}_S$ generates a ray in the normal fan corresponding to the facet Ξ_S.

Definition 3. The n-dimensional *permutohedral fan* is the complete fan Δ_{A_n} whose d-dimensional cones are of the form

$$\sigma_{\mathcal{S}} = \operatorname{cone}(\mathbf{u}_{S_1}, \mathbf{u}_{S_2}, \ldots, \mathbf{u}_{S_d}), \qquad \mathcal{S} = \Big(S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_d\Big),$$

where $\mathcal{S}$ is a flag of nonempty proper subsets of E. We call $\sigma_{\mathcal{S}}$ the cone determined by the flag $\mathcal{S}$.

The permutohedral fan Δ_{A_n} is the normal fan of the permutohedron Ξ_n, and can be identified with the fan of Weyl chambers of the root system A_n.

The geometry of the permutohedral fan is governed by the combinatorics of the Boolean lattice of all subsets of E. Let $\mathcal{T} = \big(T_1 \subsetneq T_2 \subsetneq \cdots \subsetneq T_{d-1}\big)$ be a flag of nonempty proper subsets of E. We say that a subset S of E is *strictly compatible* with $\mathcal{T}$ if $S \subsetneq T_j$ or $T_j \subsetneq S$ for each j. Then the d-dimensional cones in Δ_{A_n} containing the cone determined by $\mathcal{T}$ bijectively correspond to the nonempty proper subsets of E that are strictly compatible with $\mathcal{T}$.

The symmetric group on E acts on the Boolean lattice of subsets of E, and hence on the permutohedral fan Δ_{A_n}. In addition, the permutohedral fan has an automorphism of order 2, sometimes called the *Cremona symmetry*:

$$\operatorname{Crem} : |\Delta_{A_n}| \longrightarrow |\Delta_{A_n}|, \qquad x \longmapsto -x.$$

This automorphism associates to a flag the flag that consists of complements:

$$\left(S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_d\right) \longmapsto \left(\hat{S}_d \subsetneq \cdots \subsetneq \hat{S}_2 \subsetneq \hat{S}_1\right), \qquad \hat{S}_j = E \setminus S_j.$$

Let k be a field. The normal fan of the permutohedron defines a smooth projective toric variety over k. This variety is the main character of the thesis.

Definition 4. The n-dimensional *permutohedral variety* X_{A_n} is the toric variety of the permutohedral fan Δ_{A_n} with respect to the lattice $\mathbb{Z}^{n+1}/\text{span}(1, \ldots, 1)$.

When the field k is relevant to a statement, we will say that X_{A_n} is the permutohedral variety over k. Otherwise, we do not explicitly mention the field k. Our basic reference for toric varieties is [Ful93].

Notation.

(i) If S is a nonempty proper subset of E, we write D_S for the torus-invariant prime divisor of X_{A_n} corresponding to the ray generated by $\mathbf{u}_S$.

(ii) If $\mathcal{S}$ is a flag of nonempty proper subsets of E, we write $V(\mathcal{S})$ for the torus orbit closure in X_{A_n} corresponding to the cone determined by $\mathcal{S}$.

The codimension of $V(\mathcal{S})$ in X_{A_n} is equal to the length d of the flag

$$\mathcal{S} = \left(S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_d\right).$$

The torus orbit closure $V(\mathcal{S})$ is a transversal intersection of smooth hypersurfaces

$$V(\mathcal{S}) = D_{S_1} \cap D_{S_2} \cap \cdots \cap D_{S_d}.$$

A fundamental geometric fact is that X_{A_n} can be obtained by blowing up all the torus-invariant linear subspaces of the projective space $\mathbb{P}^n$. In fact, there are two essentially different ways of identifying X_{A_n} with the blown up projective space.

Consider the composition of blowups

$$X_{n-1} \longrightarrow X_{n-2} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \;=\; \mathbb{P}^n,$$

where $X_{d+1} \longrightarrow X_d$ is the blowup of the strict transform of the union of all the torus-invariant d-dimensional linear subspaces of $\mathbb{P}^n$. We identify the rays of the fan of $\mathbb{P}^n$ with the vectors

$$\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n,$$

and index the homogeneous coordinates of the projective space by the set E:

$$z_0, z_1, \ldots, z_n.$$

This gives one identification between X_{A_n} and X_{n-1}. We denote the above composition of blowups by $\pi_1 : X_{A_n} \longrightarrow \mathbb{P}^n$.

Definition 5. The map $\pi_2 : X_{A_n} \longrightarrow \mathbb{P}^n$ is the composition of the Cremona symmetry and $\pi_1 : X_{A_n} \longrightarrow \mathbb{P}^n$.

We have the commutative diagram

$$\begin{array}{ccc} & X_{A_n} & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ \mathbb{P}^n & \underset{\text{Crem}}{\dashrightarrow} & \mathbb{P}^n, \end{array}$$

where Crem is the standard Cremona transformation

$$\text{Crem} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n, \qquad (z_0 : z_1 : \cdots : z_n) \longmapsto (z_0^{-1} : z_1^{-1} : \cdots : z_n^{-1}).$$

All three maps in the diagram are torus-equivariant, and they restrict to isomorphisms between the n-dimensional tori.

Like π_1, the induced map π_2 is the blowup of all the torus-invariant linear subspaces of the target projective space. The rays in the fan of the image of π_2 are generated by the vectors

$$\mathbf{u}_{\hat{0}}, \mathbf{u}_{\hat{1}}, \ldots, \mathbf{u}_{\hat{n}},$$

where $\hat{i}$ is the complement of $\{i\}$ in E. The ray generated by $\mathbf{u}_{\hat{i}}$ correspond to the facet $\Xi_{\hat{i}}$ of the permutohedron. The homogeneous coordinates of this projective space will be written

$$z_{\hat{0}}, z_{\hat{1}}, \ldots, z_{\hat{n}}.$$

If S is a nonempty proper subset of E with $|S| \geq 2$, then D_S is the exceptional divisor of π_1 corresponding to the codimension $|S|$ linear subspace

$$\bigcap_{j \in S} \big\{ z_j = 0 \big\} \subseteq \mathbb{P}^n.$$

If S is a nonempty subset of E with $|E \setminus S| \geq 2$, then D_S is the exceptional divisor of π_2 corresponding to the dimension $|S|$ linear subspace

$$\bigcap_{j \notin S} \big\{ z_{\hat{j}} = 0 \big\} \subseteq \mathbb{P}^n.$$

The Cremona symmetry of the permutohedral fan

$$\text{Crem} : |\Delta_{A_n}| \longrightarrow |\Delta_{A_n}|, \qquad x \longmapsto -x$$

changes the role of π_1 and π_2.

The anticanonical linear system of X_{A_n} has a simple description in terms of π_1 and π_2. Choose an element i of E, and consider the corresponding hyperplanes in the two projective spaces:

$$H_i := \{z_i = 0\} \subseteq \mathbb{P}^n, \qquad H_{\hat{i}} := \{z_{\hat{i}} = 0\} \subseteq \mathbb{P}^n.$$

The pullbacks of the hyperplanes in the permutohedral variety are

$$\pi_1^{-1}(H_i) = \sum_{i \in S} D_S \quad \text{and} \quad \pi_2^{-1}(H_{\hat{i}}) = \sum_{i \notin S} D_S.$$

Since any subset of E either contains i or does not contain i, the sum of the two divisors is the union of all torus-invariant prime divisors in X_{A_n}. In other words, the sum is the torus-invariant anticanonical divisor of the permutohedral variety:

$$-K_{X_{A_n}} = \pi_1^{-1}(H_i) + \pi_2^{-1}(H_{\hat{i}}).$$

The decomposition of the anticanonical linear system gives the map

$$\pi_1 \times \pi_2 : X_{A_n} \longrightarrow \mathbb{P}^n \times \mathbb{P}^n,$$

whose image is the closure of the graph of the Cremona transformation. This gives another proof of the result of Batyrev and Blume that $-K_{X_{A_n}}$ is nef and big [BB11].

Proposition 6. *The anticanonical divisor of X_{A_n} is nef and big.*

The true anticanonical map of the linear system $|-K_{X_{A_n}}|$ fits into the commutative diagram

$$\begin{array}{ccc} X_{A_n} & \xrightarrow{\pi_1 \times \pi_2} & \mathbb{P}^n \times \mathbb{P}^n \\ \Big\downarrow{\scriptstyle -K} & & \Big\downarrow{\scriptstyle \mathfrak{s}} \\ \mathbb{P}^{n^2+n} & \xrightarrow{L} & \mathbb{P}^{n^2+2n}. \end{array}$$

Here $-K$ is the anticanonical map, L is a linear embedding of codimension n, and $\mathfrak{s}$

is the Segre embedding.

Remark 7. The permutohedral variety X_{A_n} can be viewed as the torus orbit closure of a general point in the flag variety $\mathrm{Fl}(\mathbb{C}^{n+1})$, see [Kly85, Kly95]. Under this identification, π_1 and π_2 are projections onto the Grassmannians

$$\begin{array}{ccccc} & & X_{A_n} & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ \mathbb{P}^n \simeq \mathrm{Gr}(1, \mathbb{C}^{n+1}) & & & & \mathrm{Gr}(n, \mathbb{C}^{n+1}) \simeq \mathbb{P}^n. \end{array}$$

Recall that torus-invariant divisors on X_{A_n} may be viewed as piecewise linear functions on Δ_{A_n}. For later use, we give names to the piecewise linear functions of the divisors $\pi_1^{-1}(H_i)$ and $\pi_2^{-1}(H_{\hat{i}})$.

Definition 8. Let i be an element of E. We define $\alpha = \alpha(i)$ to be the piecewise linear function on Δ_{A_n} determined by the values

$$\alpha(\mathbf{u}_S) = \begin{cases} -1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S, \end{cases}$$

and define $\beta = \beta(i)$ to be the piecewise linear function on Δ_{A_n} determined by the values

$$\beta(\mathbf{u}_S) = \begin{cases} 0 & \text{if } i \in S, \\ -1 & \text{if } i \notin S. \end{cases}$$

The dependence of α and β on i will often be invisible from their notation. Different choices of i will give rationally equivalent divisors, and piecewise linear functions which are equal to each other modulo linear functions. The functions α and β pullback to each other under the Cremona symmetry of Δ_{A_n}.

2.2 The Bergman fan of a matroid

A matroid is a combinatorial structure that captures the notion of linear independence in vector spaces. We will show in Section 3.2 that a matroid on E defines an extremal nef homology class in the permutohedral variety X_{A_n}. In principle, a question on matroids on E can be translated to a question on the geometry of the permutohedral variety X_{A_n}. Two general references on matroid theory are [Oxl11] and [Wel76].

As before, n is a nonnegative integer and $E = \{0, 1, \ldots, n\}$.

Definition 9. A *matroid* M on E is a collection of subsets of E, called *independent* sets, which satisfies the following properties.

(i) The empty subset of E is an independent set.

(ii) Every subset of an independent set is an independent set.

(iii) If I_1 and I_2 are independent sets and I_1 has more elements than I_2, then there is an element in I_1 which, when added to I_2, gives a larger independent set than I_2.

A *loop* of M is a singleton subset of E which is not independent. We will often assume that a matroid has no loops.

Example 10. Let G be a graph and E be the set of edges. Define a subset of E to be independent if it does not contain any cycle. This defines a matroid M on E, called the *graphic matroid* of G.

Example 11. Let k be a field, and let E be a finite subset of a vector space over k. Define a subset of E to be independent if it is linearly independent. This defines a matroid M on E which is *realizable* over k. The matroid M has no loops if and only if all the vectors are nonzero.

A matroid M assigns a nonnegative integer, called *rank*, to each subset S of E:

$$\text{rank}_M(S) := (\text{the cardinality of any maximal independent subset of } S)$$

The rank of the entire set E is called the rank of M. It is the common cardinality of any one of the *bases* of M.

Notation. A matroid has rank 0 if and only if every element of E is a loop. If the rank of a matroid M is positive, we write

$$\text{rank}_M(E) = r + 1.$$

The use of the symbols n, r, M, E will be consistent throughout the thesis.

If M is a realizable matroid given by a set of vectors E, then a flat of M is an intersection of E with a subspace of the ambient vector space. In general, a flat of a matroid is defined as follows.

Definition 12. A *flat* of M is a subset F of E with the following property:

The addition of any element not in F to F increases the rank.

In short, a flat of M is a subset of E which is maximal for its rank. The empty subset is a flat of M if and only if M has no loops.

We say that a flat F_1 *covers* another flat F_2 if F_1 properly contains F_2 and there is no other flat between F_1 and F_2. We will often use the following property of the collection of flats of a matroid:

Proposition 13. *If F is a flat of M, then the flats of M that cover F partition the elements of $E \setminus F$.*

In other words, each element in $E \setminus F$ is contained in exactly one flat that cover F. In fact, together with the statements that E is a flat and the intersection of any two flats is a flat, Proposition 13 can be used to define the term 'matroid on E' [Oxl11, Wel76].

Definition 14. The *lattice of flats* of a matroid M is the poset $\mathscr{L}_M$ of all flats of M, ordered by inclusion.

The lattice of flats has a unique minimal element, the set of all loops, and a unique maximal element, the entire set E. It is graded by the rank function, and every maximal chain in $\mathscr{L}_M \setminus \{\min \mathscr{L}_M, \max \mathscr{L}_M\}$ has the same number of flats r.

Assumption. In the remainder of this chapter, we assume that M is a loopless matroid on E with rank $r+1$.

Definition 15. The *Bergman fan* of M, denoted Δ_M, is the fan in $|\Delta_{A_n}|$ consisting of cones corresponding to flags of nonempty proper flats of M. In other words, the Bergman fan of M is a collection of cones of the form

$$\sigma_{\mathcal{F}} = \text{cone}(\mathbf{u}_{F_1}, \mathbf{u}_{F_2}, \dots, \mathbf{u}_{F_d}),$$

where $\mathcal{F}$ is a flag of nonempty proper flats

$$\mathcal{F} = \Big(F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_d\Big).$$

When $r = 0$, by definition, Δ_M is the 0-dimensional fan at the origin.

The Bergman fan Δ_M is an r-dimensional subfan of the permutohedral fan Δ_{A_n}. Ardila and Klivans introduced this fan in [AK06] and called it the fine subdivision of the Bergman fan of the matroid. If every subset of E is independent, then $r = n$ and the permutohedral fan is the Bergman fan of M. In other words, the permutohedral fan is the Bergman fan of the uniform matroid of full rank.

We next prove a fundamental property of the Bergman fan Δ_M that it satisfies the *balancing condition*. Geometrically, the condition says that, for every $(r-1)$-

dimensional cone τ, the sum of the ray generators of the cones in Δ_M that contain τ is contained in τ. Combinatorially, the condition is a translation of the flat partition property for matroids (Proposition 13).

Proposition 16. *Let $F_1, \ldots, F_m$ be the nonempty proper flats of M that are strictly compatible with a flag of nonempty proper flats*

$$\mathcal{G} = \Big(G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_{r-1}\Big).$$

If we set $G_0 = \varnothing$ and $G_r = E$, then there is exactly one index l such that each F_j covers G_{l-1} and is covered by G_l, and

$$\sum_{j=1}^{m} \mathbf{u}_{F_j} = \mathbf{u}_{G_l} + (m-1)\mathbf{u}_{G_{l-1}}.$$

When M has loops, the same formula holds if we replace G_0 by the set of all loops. For geometric and combinatorial reasons, we choose not to define the Bergman fan for matroids with loops.

Proof. Since any maximal flag of nonempty proper flats of M has length r, there is exactly one index l such that each F_j covers G_{l-1} and is covered by G_l.

To justify the displayed equality between the two vectors, we construct a new matroid N on G_l from the old matroid M on E by deleting all elements not in G_l. By definition, a subset I of G_l is independent for N if and only if it is independent for M. Using the fact that G_l is a flat of M, one can check that a subset of G_l is a flat for N if and only if it is a flat for M. Applying Proposition 13 to the flat G_{l-1} of the new matroid N, we see that every element of $G_l \setminus G_{l-1}$ is contained in exactly one of the F_j. In other words,

$$\coprod_{j=1}^{m} \big(F_j \setminus G_{l-1}\big) = G_l \setminus G_{l-1}.$$

This implies what we want. $\square$

2.3 The Möbius function of a matroid

The Möbius function of the lattice of flats will play a fundamental role in the intersection theory of matroids in the permutohedral variety. We continue to assume that M is a matroid which has no loops.

Definition 17. Let $\mathscr{L}$ be a finite poset. The *Möbius function* of $\mathscr{L}$ is the function

$$\mu_{\mathscr{L}} : \mathscr{L} \times \mathscr{L} \longrightarrow \mathbb{Z}$$

determined by the following properties:

(i) If $x \not\leq y$, then $\mu_{\mathscr{L}}(x, y) = 0$.

(ii) If $x = y$, then $\mu_{\mathscr{L}}(x, y) = 1$.

(iii) If $x < y$, then

$$\mu_{\mathscr{L}}(x, y) = -\sum_{x \leq z < y} \mu_{\mathscr{L}}(x, z).$$

When $\mathscr{L}$ is the lattice of flats of M, we write μ_M for the Möbius function $\mu_{\mathscr{L}}$. The Möbius function of the lattice of flats of a matroid has several special properties that the Möbius function of posets in general do not have. For example, Rota's theorem says that, if F_1 is a flat contained in a flat F_2, then

$$(-1)^{\mathrm{rank}_M(F_2) - \mathrm{rank}_M(F_1)} \mu_M(F_1, F_2) > 0.$$

Another basic result on matroids is the following theorem of Weisner. For proofs, see [Rot64, Zas87].

Theorem 18. *Let F be a flat of M, and let i be an element of F. If $F_1, F_2, \ldots, F_m$ are the flats of M covered by F which do not contain i, then*

$$\mu_M(\varnothing, F) = -\sum_{j=1}^{m} \mu_M(\varnothing, F_j).$$

When M has loops, one should replace $\varnothing$ by the set of loops and choose i among elements of F which is not a loop. We will frequently use Weisner's theorem later in intersection theoretic computations in X_{A_n}. In a sense, Weisner's theorem plays a role which is Cremona symmetric to the role played by the flat partition property for matroids (Proposition 13).

Definition 19. The *characteristic polynomial* of M is the polynomial

$$\chi_M(q) = \sum_{F \in \mathscr{L}_M} \mu_M(\varnothing, F)\; q^{\mathrm{rank}_M(E) - \mathrm{rank}_M(F)}.$$

By definition of the Möbius function,

$$\chi_M(1) = \sum_{F \in \mathscr{L}_M} \mu_M(\varnothing, F) = 0.$$

We define the *reduced characteristic polynomial* of M by

$$\overline{\chi}_M(q) := \chi_M(q)/(q-1).$$

By Rota's theorem, the coefficients of the characteristic polynomial of a matroid alternate in sign. The same is true for the coefficients of the reduced characteristic polynomial.

Proposition 20. *Let i be an element of E and let l be a nonnegative integer. If $F_1, F_2, \ldots, F_m$ are the flats of rank l which do not contain i, then the coefficient of*

q^{r-l} in the reduced characteristic polynomial is

$$\sum_{j=1}^{m} \mu_M(\varnothing, F_j).$$

Proof. Let $G_1, G_2, \ldots, G_s$ be the flats of rank $l+1$ which contain i. By the flat partition property (Proposition 13), each F_j is covered by exactly one G_k, the "span" of F_j and the element i. Applying Weisner's theorem to each G_k and the element i, we have

$$\sum_{k=1}^{s} \mu_M(\varnothing, G_k) = -\sum_{j=1}^{m} \mu_M(\varnothing, F_j).$$

In other words, the sum of the values of the Möbius function over all rank l flats not containing i and all rank $l+1$ flats containing i is zero. Since the coefficient of q^{r-l+1} of $\chi_M(q)$ is the sum of the values of the Möbius function over all rank l flats, the above identity gives the desired expression for the coefficient of the reduced characteristic polynomial $\chi_M(q)/(q-1)$. □

We now introduce a series of conjectures on the characteristic polynomial of a matroid which motivated this work. A sequence of real numbers $a_0, a_1, \ldots, a_r$ is said to be *log-concave* if

$$a_i^2 \geq a_{i-1}a_{i+1} \quad \text{for all } i.$$

If the sequence has no internal zeros, then the log-concavity implies the *unimodality*:

$$a_0 \leq a_1 \leq \cdots \leq a_j \geq \cdots \geq a_{r-1} \geq a_r \quad \text{for some } j.$$

The log-concavity conjecture of Rota, Welsh, and Heron states that the coefficients of the characteristic polynomial of a matroid form a log-concave sequence [Rot71, Wel71, Her72]. A detailed treatment of the conjecture and various partial results can be found in [Aig87]. See also [Kun95, Sta95, Sta00, Oxl11]. For a fascinating collection of log-concavity conjectures coming from various areas of mathematics, see [Sta89] and [Bre94].

Conjecture 21. *The coefficients of the characteristic polynomial of a matroid form a log-concave sequence.*

In Section 4.3, we prove that the coefficients of the reduced characteristic polynomial form a log-concave sequence for matroids which are realizable over some field. It follows that Conjecture 21 holds for matroids which are realizable over some field. Since graphic matroids are realizable over every field, this confirms the following conjecture of Read [Rea68], for unimodality, and Hoggar [Hog74], for log-concavity.

Conjecture 22. *The coefficients of the chromatic polynomial of a graph form a log-concave sequence.*

A closely related is the following conjecture of Mason [Mas72].

Conjecture 23. *The number of independent subsets of size i of a matroid form a log-concave sequence in i.*

In Section 4.3, we give a proof of Conjecture 23 for matroids which are realizable over some field. Following Lenz [Len12], we deduce Conjecture 23 from Conjecture 21.

CHAPTER III

Algebraic cycles in the permutohedral variety

3.1 Intersection theory of the permutohedral variety

Let X be an n-dimensional smooth toric variety defined from a complete fan Δ_X. An element of the Chow cohomology group $A^l(X)$ gives a homomorphism of Chow groups from $A_l(X)$ to $\mathbb{Z}$. The resulting homomorphism of abelian groups is the Kronecker duality homomorphism

$$A^l(X) \longrightarrow \mathrm{Hom}\big(A_l(X), \mathbb{Z}\big).$$

The Kronecker duality homomorphism for X is, in fact, an isomorphism [FS97]. Since $A_l(X)$ is generated by the classes of l-dimensional torus orbit closures, the isomorphism identifies Chow cohomology classes with certain integer valued functions on the set of d-dimensional cones in Δ_X, where $d = n - l$.

Notation. If σ is a d-dimensional cone containing a $(d-1)$-dimensional cone τ in the fan of X, then there is exactly one ray in σ not in τ. The primitive generator of this ray will be denoted $\mathbf{u}_{\sigma/\tau}$.

Definition 24. A d-dimensional *Minkowski weight* on Δ_X is a function Δ from the set of d-dimensional cones to the integers which satisfies the *balancing condition*: For every $(d-1)$-dimensional cone τ,

$$\sum_{\tau \subset \sigma} \Delta(\sigma) \mathbf{u}_{\sigma/\tau} \text{ is contained in the lattice generated by } \tau,$$

where the sum is over all d-dimensional cones σ containing τ.

One may think a d-dimensional Minkowski weight as a d-dimensional subfan of Δ_X with integer weights on its d-dimensional cones. The balancing condition imposed on d-dimensional Minkowski weights on Δ_X is a translation of the rational equivalence relations between l-dimensional torus orbit closures in X [FS97].

Theorem 25. *The Chow cohomology group $A^l(X)$ is isomorphic to the group of d-dimensional Minkowski weights on Δ_X:*

$$A^l(X) \simeq Hom\big(A_l(X), \mathbb{Z}\big) \simeq \big(\textit{the group of d-dimensional Minkowski weights}\big).$$

These groups are also isomorphic to the d-dimensional homology group of X through the 'degree' map

$$A_d(X) \longrightarrow \operatorname{Hom}\big(A_l(X), \mathbb{Z}\big), \qquad \xi \longmapsto \Big(\eta \longmapsto \deg(\xi \cdot \eta)\Big).$$

Its inverse is the composition of the isomorphism $\operatorname{Hom}\big(A_l(X), \mathbb{Z}\big) \simeq A^l(X)$ and the Poincaré duality isomorphism

$$A^l(X) \longrightarrow A_d(X), \qquad \Delta \longmapsto \Delta \cap [X].$$

We say that Δ and $\Delta \cap [X]$ are *Poincaré dual* to each other.

Theorem 25, when applied to the permutohedral variety, says that a cohomology class of X_{A_n} is a function from the set of flags in E which satisfies the balancing condition. The balancing condition for a d-dimensional Minkowski weight Δ on the permutohedral fan can be translated as follows: For every flag of nonempty proper subsets $\mathcal{S} = \big(S_1 \subsetneq \cdots \subsetneq S_{d-1}\big)$, if $T_1, \ldots, T_m$ are the nonempty proper subsets of E that are strictly compatible with $\mathcal{S}$, then

$$\sum_{j=1}^{m} \Delta(\sigma_j)\mathbf{u}_{T_j} \quad \text{is contained in the lattice generated by } \mathbf{u}_{S_1}, \ldots, \mathbf{u}_{S_{d-1}},$$

where σ_j is the cone generated by $\mathbf{u}_{T_j}$ and $\mathbf{u}_{S_1}, \ldots, \mathbf{u}_{S_{d-1}}$.

Let M be a loopless matroid of rank $r+1$. Proposition 16 shows that the balancing condition is satisfied by the indicator function of the Bergman fan of M. To be more precise, we have the following.

Proposition 26. *The Bergman fan Δ_M defines an r-dimensional Minkowski weight on the permutohedral fan Δ_{A_n}, denoted by the same symbol Δ_M, such that*

$$\Delta_M(\sigma_{\mathcal{S}}) = \begin{cases} 1 & \text{if } \mathcal{S} \text{ is a maximal flag of nonempty proper flats of } M, \\ 0 & \text{if otherwise.} \end{cases}$$

When $r = 0$, by definition, $\Delta_M = 1$.

The cup product of a divisor and a cohomology class of a smooth complete toric variety defines a product of a piecewise linear function and a Minkowski weight. If φ is a piecewise linear function and Δ is a d-dimensional Minkowski weight, then $\varphi \cup \Delta$ is a $(d-1)$-dimensional Minkowski weight. We will often use the following explicit formula for the cup product [AR10].

Theorem 27. *Let φ be a piecewise linear function and let Δ be a d-dimensional Minkowski weight on Δ_X. If τ is a $(d-1)$-dimensional cone in Δ_X, then*

$$\Big(\varphi \cup \Delta\Big)(\tau) = \varphi\left(\sum_{\tau \subset \sigma} \Delta(\sigma)\mathbf{u}_{\sigma/\tau}\right) - \sum_{\tau \subset \sigma} \varphi\Big(\Delta(\sigma)\mathbf{u}_{\sigma/\tau}\Big),$$

where the sums are over all d-dimensional cones σ containing τ.

In particular, if Δ is nonnegative and φ is linear on the cone generated by the cones containing τ, then

$$\Big(\varphi \cup \Delta\Big)(\tau) = 0.$$

Similarly, if Δ is nonnegative and φ is concave on the cone generated by the cones containing τ, then

$$\Big(\varphi \cup \Delta\Big)(\tau) \geq 0.$$

Corollary 28. *If φ is concave and Δ is nonnegative, then $\varphi \cup \Delta$ is nonnegative.*

Theorem 27 can be used to compute the cup product of the piecewise linear function α and the Bergman fan Δ_M. Proposition 30 below shows that the result of the cup product is the Bergman fan of another matroid. Recall that $\alpha = \alpha(i)$ is the piecewise linear function on the permutohedral fan Δ_{A_n} determined by its values

$$\alpha(\mathbf{u}_S) = \begin{cases} -1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

For any element i of E and any nonempty proper subset G of E, the function α is linear when restricted to the cone

$$\text{cone}(\mathbf{u}_j \mid j \in G) = \bigcup_{\mathcal{F}} \sigma_{\mathcal{F}},$$

where the union is over all flag of nonempty proper subsets contained in G.

Definition 29. When $r \geq 1$, we define the *truncation* of M to be the matroid $\overline{M}$ on the same set E defined by the following condition:

A subset I is independent for $\overline{M}$ if and only if I is independent for M and $|I| \leq r$.

We do not define the truncation for rank 1 matroids.

A proper subset F of E is a flat for $\overline{M}$ if and only if F is a flat for M and $\text{rank}_M(F) < r$. In other words, $\mathscr{L}_{\overline{M}} \setminus \{E\}$ is obtained from $\mathscr{L}_M \setminus \{E\}$ by deleting all flats of rank r.

Proposition 30. *If $r \geq 1$ and $\overline{M}$ is the truncation of M, then*

$$\alpha \cup \Delta_M = \Delta_{\overline{M}}.$$

A repeated application of Proposition 30 gives the equality between the 0-dimensional Minkowski weights

$$\Big(\underbrace{\alpha \cup \cdots \cup \alpha}_{r}\Big) \cup \Delta_M = 1.$$

Proof. Let τ be an $(r-1)$-dimensional cone in Δ_{A_n} determined by a flag of nonempty proper subsets

$$\mathcal{G} = \Big(G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_{r-1}\Big).$$

When $r = 1$, write $\mathcal{G}$ for the empty flag and set G_{r-1} for the empty subset.

We compute the value of the Minkowski weight $\alpha \cup \Delta_M$ at τ using the formula in Theorem 27. If one of the G_k is not a flat of M, then all the $\Delta_M(\sigma)$ in the formula are zero, and hence

$$\Big(\alpha \cup \Delta_M\Big)(\tau) = 0.$$

Suppose that all the G_k are flats of M. We need to prove the following statements.

(i) If $\text{rank}_M(G_{r-1}) = r$, then $\Big(\alpha \cup \Delta_M\Big)(\tau) = 0$.

(ii) If $\text{rank}_M(G_{r-1}) = r - 1$, then $\Big(\alpha \cup \Delta_M\Big)(\tau) = 1$.

In both cases, the r-dimensional cones σ containing τ with nonzero $\Delta_M(\sigma)$ bijectively correspond to the flats $F_1, \ldots, F_m$ of M that are strictly compatible with $\mathcal{G}$. In the first case, all the F_j should be contained in G_{r-1}. Since α is linear on the cone generated by rays corresponding to nonempty proper subsets in G_{r-1}, Theorem 27 implies that

$$\Big(\alpha \cup \Delta_M\Big)(\tau) = 0.$$

In the second case, all the F_j should cover G_{r-1} and should be covered by E. By the matroid balancing condition of Proposition 16, we have

$$\Big(\alpha \cup \Delta_M\Big)(\tau) = \alpha\Big(\sum_{j=1}^{m} \mathbf{u}_{F_j}\Big) - \sum_{j=1}^{m} \alpha(\mathbf{u}_{F_j}) = (m-1)\alpha(\mathbf{u}_{G_{r-1}}) - \sum_{j=1}^{m} \alpha(\mathbf{u}_{F_j}).$$

Let i be the element defining α. If G_{r-1} contains i, then all the F_j contain i. If G_{r-1} does not contain i, then (by the flat partition property) exactly one of the F_j contains i. It follows that the right-hand side of the above equation is 1. □

This remark below is for readers familiar with the language of matroid theory [Oxl11]. We will not use the result of this remark in the remainder of this thesis.

Remark 31. Let $\mathscr{M}$ be a modular cut of M, and let $M \longrightarrow M(\mathscr{M})$ be the corresponding elementary quotient map. The argument used in the proof of Proposition 30 shows that, in fact,

$$(\alpha - \alpha_{\mathscr{M}}) \cup \Delta_M = \Delta_{M(\mathscr{M})},$$

where $\alpha_{\mathscr{M}}$ is the piecewise linear function on Δ_{A_n} defined by

$$\alpha_{\mathscr{M}}(\mathbf{u}_S) = \begin{cases} -1 & \text{if } S \in \mathscr{M}, \\ 0 & \text{if } S \notin \mathscr{M}. \end{cases}$$

When combined with the Higgs factorization theorem, this implies that any matroid cohomology class is a product of divisor classes. To be more precise, let U be the uniform matroid of full rank on E, and consider the Higgs factorization of the canonical quotient map $U \longrightarrow M$. If $\mathscr{M}_1, \ldots, \mathscr{M}_{n-r}$ is the corresponding sequence of modular cuts, then the above generalization of Proposition 30 implies that

$$\Delta_M = (\alpha - \alpha_{\mathscr{M}_1}) \cup (\alpha - \alpha_{\mathscr{M}_2}) \cup \cdots \cup (\alpha - \alpha_{\mathscr{M}_{n-r}}).$$

The classes of $\alpha - \alpha_{\mathscr{M}_j}$ are not effective in general.

3.2 Every matroid is nef, effective, and extremal.

We have seen that the Bergman fan of a matroid on E defines a cohomology class of the permutohedral variety X_{A_n}. The goal of this section is to show that its Poincaré dual is an effective homology class which generates an extremal ray of the nef cone of X_{A_n}.

Definition 32. Let X be an n-dimensional smooth complete variety over a field k, and $d = n - l$.

(i) A d-dimensional Chow homology class of X is *nef* if it intersects all l-dimensional effective cycles nonnegatively.

(ii) A d-dimensional Chow homology class of X is *effective* if it is the class of an d-dimensional effective cycle.

If X is a toric variety, then every effective cycle is rationally equivalent to a torus-invariant effective cycle [FMSS95]. Therefore, in this case, a d-dimensional Chow homology class ξ is nef if and only if

$$\xi \cdot \big[V(\sigma)\big] \geq 0$$

for every l-dimensional torus orbit closure $V(\sigma)$ of X. In other words, ξ is nef if and only if its Poincaré dual is a nonnegative function when viewed as a d-dimensional

Minkowski weight. For example, the Bergman fan of a matroid M on E defines a nef homology class $\Delta_M \cap [X_{A_n}]$ of the permutohedral variety X_{A_n}.

If X is a toric variety, then every nef class of X is effective. This is a special case of the result of Li on spherical varieties [Li13].

Theorem 33. *If X is a toric variety, then every nef class of X is effective.*

Proof. The main observation is that every effective cycle in a toric variety is rationally equivalent to a torus-invariant effective cycle. Applying this to the diagonal embedding

$$\iota : X \longrightarrow X \times X, \qquad x \longmapsto (x, x),$$

we have an expression

$$\big[\iota(X)\big] = \sum_{\sigma,\tau} m_{\sigma,\tau} \big[V(\sigma) \times V(\tau)\big] \in A_n(X \times X), \qquad m_{\sigma,\tau} \geq 0,$$

where the sum is over all cones σ, τ in the fan of X such that $\dim \sigma + \dim \tau = n$. The choice of the integers $m_{\sigma,\tau}$ is in general not unique, but the knowledge of such constants characterizes both the cap product and the cup product of (co)homology classes on X [FMSS95].

Let ξ be a d-dimensional nef class of X. We show that ξ is the class of a torus-invariant effective cycle. If Δ is the Poincaré dual of ξ, viewed as a d-dimensional Minkowski weight, then

$$\xi = \Delta \cap [X] = \sum_{\sigma,\tau} m_{\sigma,\tau}\ \Delta(\sigma)\ \big[V(\tau)\big],$$

where the sum is over all d-dimensional cones σ and l-dimensional cones τ. Since ξ is nef, for all σ, we have

$$\Delta(\sigma) = \deg\Big(\xi \cdot \big[V(\sigma)\big]\Big) \geq 0.$$

Therefore ξ is the class of a torus-invariant effective cycle in X. □

An application of Theorem 33 to the permutohedral variety X_{A_n} gives the following.

Corollary 34. *If M is a loopless matroid on E, then $\Delta_M \cap [X_{A_n}]$ is effective.*

We stress that the statement does not involve the field k which is used to define the permutohedral variety X_{A_n}. The proof of Theorem 33 shows that an explicit effective cycle with the matroid homology class $\Delta_M \cap [X_{A_n}]$ can be found by degenerating the diagonal of the permutohedral variety in $X_{A_n} \times X_{A_n}$.

Definition 35. Let $N_d(X)$ be the real vector space of d-dimensional algebraic cycles with real coefficients modulo numerical equivalence on a smooth complete variety X.

(i) The *nef cone* of X in dimension d, denoted $\text{Nef}_d(X)$, is the cone in $N_d(X)$ generated by d-dimensional nef classes.

(ii) The *pseudoeffective cone* of X in dimension d, denoted $\text{Peff}_d(X)$, is the closure in $N_d(X)$ of the cone generated by the d-dimensional effective classes.

The nef cone in dimension d and the pseudoeffective cone in dimension l are dual to each other under the intersection pairing

$$N_d(X) \times N_l(X) \longrightarrow \mathbb{R}.$$

If X is the toric variety of a complete fan Δ_X, then $N_d(X) \simeq A_d(X) \otimes \mathbb{R}$ can be identified with the set of real valued functions Δ from the set of d-dimensional cones in Δ_X which satisfy the balancing condition over $\mathbb{R}$: For every $(d-1)$-dimensional cone τ,

$$\sum_{\tau \subset \sigma} \Delta(\sigma) \mathbf{u}_{\sigma/\tau} \quad \text{is contained in the subspace generated by } \tau,$$

where the sum is over all d-dimensional cones σ containing τ.

In the toric case, any integral effective cycle is rationally equivalent to an effective torus-invariant cycle, and hence there is no need to take the closure when defining the pseudoeffective cone. Furthermore, the pseudoeffective cone and the nef cone of X are polyhedral cones. These polyhedral cones depend only on the fan Δ_X and not on the field k used to define X. Theorem 33 shows that one is contained in the other.

Theorem 36. *If X is a toric variety, then the nef cone of X in dimension d is contained in the pseudoeffective cone of X in dimension d, for every d.*

We remark that there is a 4-dimensional complex abelian variety whose nef cone in dimension 2 is not contained in the pseudoeffective cone in dimension 2 [DELV11].

We now show that a loopless matroid on E gives an *extremal* nef class of X_{A_n}. The main combinatorial ingredient is the following theorem of Björner [Bjo92]. Recall that the *order complex* of a finite poset $\mathscr{L}$ is a simplicial complex which has the underlying set of $\mathscr{L}$ as vertices and the finite chains of $\mathscr{L}$ as faces.

Theorem 37. *The order complex of the lattice of flats of a matroid is shellable.*

The shellability of the order complex of the lattice of flats $\mathscr{L}_M$ implies, among many other things, that the Bergman fan Δ_M is *connected in codimension* 1: If σ and $\tilde{\sigma}$ are r-dimensional cones in Δ_M, then there are r-dimensional cones $\sigma_0, \sigma_1, \ldots, \sigma_l$ and $(r-1)$-dimensional cones $\tau_1, \ldots, \tau_l$ in Δ_M such that

$$\sigma = \sigma_0 \supset \tau_1 \subset \sigma_1 \supset \tau_2 \subset \cdots \supset \tau_{l-1} \subset \sigma_{l-1} \supset \tau_l \subset \sigma_l = \tilde{\sigma}.$$

Theorem 38. *If M is a loopless matroid on E of rank $r+1$, then the class $\Delta_M \cap [X_{A_n}]$ generates an extremal ray of the nef cone of X_{A_n} in dimension r.*

Proof. The claim is that Δ_M cannot be written as a sum of two nonnegative real Minkowski weights in a nontrivial way. Suppose Δ is an r-dimensional Minkowski weight with the following property:

If $\mathcal{S} = \big(S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_r\big)$ is a flag of nonempty proper subsets of E and one of the S_j is not a flat of M, then $\Delta(\sigma_{\mathcal{S}}) = 0$.

In short, we suppose that Δ is a Minkowski weight whose support is contained in the support of Δ_M. Note that any nonnegative summand of Δ_M should have this property. We show that there is a constant c such that $\Delta = c\, \Delta_M$.

Let τ be an $(r-1)$-dimensional cone determined by a flag of nonempty proper flats $\mathcal{G} = \big(G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_{r-1}\big)$. If $F_1, \ldots, F_m$ are the flats which are strictly compatible with $\mathcal{G}$, then the balancing condition for Δ at τ says that

$$\sum_{j=1}^{m} \Delta(\sigma_j)\mathbf{u}_{F_j} \ \text{ is in the subspace generated by } \tau,$$

where σ_j is the r-dimensional cone generated by $\mathbf{u}_{F_j}$ and $\mathbf{u}_{G_1}, \ldots, \mathbf{u}_{G_{r-1}}$. Writing G_{l-1} for the flat in $\mathcal{G}$ which is covered by (any) one of the F_j, we have

$$\sum_{j=1}^{m} \Delta(\sigma_j)\mathbf{u}_{F_j \setminus G_{l-1}} = c_1 \mathbf{u}_{G_1} + c_2 \mathbf{u}_{G_2 \setminus G_1} + \cdots + c_{r-1}\mathbf{u}_{G_{r-1} \setminus G_{r-2}}$$

for some real numbers $c_1, c_2, \ldots, c_{r-1}$. One can solve this equation explicitly using the fact that $G_l \setminus G_{l-1}$ is a disjoint union of the nonempty sets $F_j \setminus G_{l-1}$:

(i) If $l \neq r$, then $\Delta(\sigma_1) = \cdots = \Delta(\sigma_m) = c_l$ and all the other c_k are zero.

(ii) If $l = r$, then $\Delta(\sigma_1) = \cdots = \Delta(\sigma_m) = c_1 = \cdots = c_{r-1}$.

In any case, we write c for the common value of $\Delta(\sigma_j)$ and repeat the above argument for all $(r-1)$-dimensional cones τ. Since Δ_M is connected in codimension 1, we have $\Delta = c\, \Delta_M$. □

Example 39. The permutohedral surface X_{A_2} is the blowup of the three torus invariant points of $\mathbb{P}^2$. Let $\pi_1 : X_{A_2} \longrightarrow \mathbb{P}^2$ be the blowup map, D_0, D_1, D_2 be the exceptional curves, and H be the pull-back of a general line. The nef cone of curves in X_{A_2} is a four-dimensional polyhedral cone with five rays. The rays are generated by the classes of

$$H, \quad H - D_0, \quad H - D_1, \quad H - D_2, \quad \text{and} \quad 2H - D_0 - D_1 - D_2.$$

The first four classes come from matroids on $E = \{0, 1, 2\}$. The matroid corresponding to H has five flats

$$\varnothing,\ \{0\},\ \{1\},\ \{2\},\ E,$$

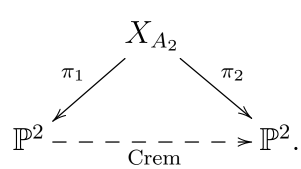
and the matroid corresponding to $H - D_i$ has four flats

$$\varnothing,\ \{i\},\ E \setminus \{i\},\ E.$$

The remaining class is the class of the strict transform under π_1 of a general conic passing through the three torus-invariant points of $\mathbb{P}^2$. It is Cremona symmetric to the class of H, and comes from the matroid on $\hat{E} = \{\hat{0}, \hat{1}, \hat{2}\}$, where $\hat{i} = E \setminus \{i\}$, whose flats are

$$\varnothing,\ \{\hat{0}\},\ \{\hat{1}\},\ \{\hat{2}\},\ \hat{E}.$$

It is the class of the pull-back of a general line through the map π_2 in the diagram



Example 40. The fan displacement rule of [FS97] shows that the product of two nef classes in a toric variety is a nef class. However, one should not expect that the product of two extremal nef classes in a toric variety is an extremal nef class. For

example, consider the diagram

$$\begin{array}{ccc} & X_{A_3} & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_2} \\ \mathbb{P}^3 & \underset{\text{Crem}}{\dashrightarrow} & \mathbb{P}^3. \end{array}$$

If H_1, H_2 are hyperplanes in $\mathbb{P}^3$, then the classes of their pullbacks $\pi_1^{-1}(H_1)$, $\pi_2^{-1}(H_2)$ are extremal nef classes in X_{A_3}. We note that the class of the product $\pi_1^{-1}(H_1) \cdot \pi_1^{-1}(H_2)$ is a sum of three different extremal nef curve classes in X_{A_3}. One may show this by directly computing the cup product of the piecewise linear functions α and β using Theorem 27. Alternatively, one may see this geometrically as follows. Let H_1 be the plane

$$z_0 + z_1 + z_2 + z_3 = 0,$$

and choose H_2 so that $\pi_2^{-1}(H_2)$ is the strict transform under π_1 of the cubic surface

$$z_0z_1z_2 + z_0z_1z_3 + z_0z_2z_3 + z_1z_2z_3 = 0.$$

Then the intersection in $\mathbb{P}^3$ is the union of three lines

$$\{z_0 + z_1 = z_2 + z_3 = 0\} \cup \{z_0 + z_2 = z_1 + z_3 = 0\} \cup \{z_0 + z_3 = z_1 + z_2 = 0\}.$$

The strict transform of any one of the three lines under π_1 generates an extremal ray of the nef cone of X_{A_3}. Their classes correspond to, respectively, to rank 2 matroids whose nonempty proper flats are

$$\big\{\{0,1\},\{2,3\}\big\} \quad \text{and} \quad \big\{\{0,3\},\{1,3\}\big\} \quad \text{and} \quad \big\{\{0,3\},\{1,2\}\big\}.$$

In this case, the intersection of the strict transforms is the strict transform of the intersection. It follows that the sum of the three nef curve classes in X_{A_3} is the product $\pi_1^{-1}(H_1) \cdot \pi_1^{-1}(H_2)$.

In general, finding all extremal rays of the nef cone seems to be a difficult combinatorial problem, even for relatively simple toric varieties. Here is a sample question: How many extremal rays are there in the nef cone of X_{A_n} in dimension two for some small values of n?

Let Δ be a two-dimensional nonnegative Minkowski weight on the fan of X. It is convenient to think the support of Δ as a geometric graph G_Δ, whose vertices are the primitive generators of the rays of the cones in the support of Δ. Two vertices of G_Δ are connected by an edge if and only if they generate a cone on which Δ is nonzero.

The main idea used in the proof of Theorem 38 gives a simple condition on G_Δ which guarantees that the class $\Delta \cap [X]$ generates an extremal ray of the nef cone of X. A few graphs G_Δ, including those of the Bergman fans of rank 3 matroids, satisfy this condition.

Proposition 41. *If G_Δ is connected and the set of neighbors of any vertex is linearly independent, then $\Delta \cap [X]$ generates an extremal ray of the nef cone of X in dimension two.*

The condition on G_Δ is not necessary for $\Delta \cap [X]$ to be extremal.

Example 42. Consider the two-dimensional Minkowski weight Δ on Δ_{A_4} which has value 1 on the cones corresponding to flags of the form

$$\{i\} \subsetneq \{i, j, k\}, \qquad i \neq j \neq k,$$

and has value 0 on all other two-dimensional cones of Δ_{A_4}. A direct computation shows that $\Delta \cap [X_{A_4}]$ generates an extremal ray of the nef cone of X_{A_4}. The graph of Δ has 10 vertices at which the set of neighbors is linearly dependent.

We note that $\Delta \cap [X_{A_4}]$ is an intersection of two extremal nef divisor classes. In fact, there is a single irreducible family of surfaces in X_{A_4} whose members have the class $\Delta \cap [X_{A_4}]$. The family consists of strict transforms under π_1 of the cubic surfaces in $\mathbb{P}^4$ defined by the equations

$$c_0 z_0 + c_1 z_1 + c_2 z_2 + c_3 z_3 + c_4 z_4 = 0,$$

$$\begin{aligned} c_{234} z_2 z_3 z_4 + c_{c134} z_1 z_3 z_4 + c_{124} z_1 z_2 z_4 + c_{123} z_1 z_2 z_3 + c_{034} z_0 z_3 z_4 \\ + c_{024} z_0 z_2 z_4 + c_{023} z_0 z_2 z_3 + c_{014} z_0 z_1 z_4 + c_{013} z_0 z_1 z_3 + c_{012} z_0 z_1 z_2 = 0, \end{aligned}$$

where c_i and c_{ijk} are parameters. Each one of the above two equations defines a basepoint free linear system on X_{A_4} whose class generates an extremal ray of the nef cone of divisors.

Example 43. Consider the two-dimensional Minkowski weight Δ on Δ_{A_n} which has value 1 on the cones corresponding to flags of the form

$$\{i\} \subsetneq E \setminus \{j\}, \qquad i \neq j,$$

and has value 0 on all other two-dimensional cones of Δ_{A_n}. Then the graph G_Δ satisfies the condition of Proposition 41, and hence $\Delta \cap [X_{A_n}]$ generates an extremal ray of the nef cone of X_{A_n}. This homology class is invariant under the Cremona symmetry of X_{A_n} and the action of the symmetric group on E.

When $n = 3$, the graph is that of a three-dimensional cube. Since the codimension of Δ is 1, it is not difficult to describe families of surfaces in X_{A_3} whose members have the class $\Delta \cap [X_{A_3}]$. There is a single irreducible family, and it consists of strict transforms under π_1 of the quadric surfaces in $\mathbb{P}^3$ defined by

$$c_{01} z_0 z_1 + c_{02} z_0 z_2 + c_{03} z_0 z_3 + c_{12} z_1 z_2 + c_{13} z_1 z_3 + c_{23} z_2 z_3 = 0,$$

where c_{ij} are parameters. The equation defines a basepoint free linear system on X_{A_3} which is invariant under the Cremona symmetry of X_{A_3} and the action of the symmetric group on E.

When $n = 4$, the extremal nef class $\Delta \cap [X_{A_4}]$ has the interesting property that

(i) $\Delta \cap [X_{A_4}]$ is not a product of two nef (integral) divisor classes, and

(ii) $2\Delta \cap [X_{A_4}]$ is a product of two nef (integral) divisor classes.

To see that $\Delta \cap [X_{A_4}]$ is not an intersection of two nef divisor classes, one notes that any surface S in X_{A_4} which has the class $\Delta \cap [X_{A_4}]$ should map to a cubic surface in $\mathbb{P}^4$ under the maps π_1 and π_2. The cubic surfaces $\pi_1(S)$ and $\pi_2(S)$ are obtained by intersecting general members of torus-invariant linear systems on $\mathbb{P}^4$ outside their common base locus. If the common base locus has dimension less than 2, then the degrees of the linear systems are 1 and 3, and one can check directly that the class of S is not invariant either under the Cremona symmetry of X_{A_4} or under the action of the symmetric group on E. If the common base locus has dimension 2, then, since complete intersections are connected in codimension 1, $\pi_1(S)$ and $\pi_2(S)$ intersect some 2-dimensional torus orbits in $\mathbb{P}^4$ in curves. This contradicts that the Minkowski weight Δ has value zero on all flags involving two element subsets of E.

On the other hand, $2\Delta \cap [X_{A_4}]$ is an intersection of two nef divisor classes. The corresponding family comes from sextic surfaces in $\mathbb{P}^4$ defined by the equations

$$c_{01}z_0z_1 + c_{02}z_0z_2 + c_{03}z_0z_3 + c_{04}z_0z_4 + c_{12}z_1z_2 + c_{13}z_1z_3 + c_{14}z_1z_4 + c_{23}z_2z_3 + c_{24}z_2z_4 + c_{34}z_3z_4 = 0,$$

$$c_{234}z_2z_3z_4 + c_{134}z_1z_3z_4 + c_{124}z_1z_2z_4 + c_{123}z_1z_2z_3 + c_{034}z_0z_3z_4 + c_{024}z_0z_2z_4 + c_{023}z_0z_2z_3 + c_{014}z_0z_1z_4 + c_{013}z_0z_1z_3 + c_{012}z_0z_1z_2 = 0,$$

where c_{ij} and c_{ijk} are parameters. The family of strict transforms under π_1 is invariant under the Cremona symmetry of X_{A_4} and the action of the symmetric group on E. Each of its members has the class $2\Delta \cap [X_{A_4}]$. Each one of the above two equations defines a basepoint free linear system on X_{A_4} whose class generates an extremal ray of the nef cone of divisors.

It is more difficult to describe families of surfaces in X_{A_4} whose members have the homology class $\Delta \cap [X_{A_4}]$. In fact, there is a single irreducible family, and it is the family of strict transforms of cubic surfaces in $\mathbb{P}^4$ defined by the 2×2 minors of the matrix

$$\begin{bmatrix} c_{110}z_0 + c_{111}z_1 + c_{112}z_2 + c_{113}z_3 + c_{114}z_4 & c_{120}z_0 + c_{121}z_1 + c_{122}z_2 + c_{123}z_3 + c_{124}z_4 \\ c_{210}z_0 + c_{211}z_1 + c_{212}z_2 + c_{213}z_3 + c_{214}z_4 & c_{220}z_0 + c_{221}z_1 + c_{222}z_2 + c_{223}z_3 + c_{224}z_4 \\ c_{310}z_0 + c_{311}z_1 + c_{312}z_2 + c_{313}z_3 + c_{314}z_4 & c_{320}z_0 + c_{321}z_1 + c_{322}z_2 + c_{323}z_3 + c_{324}z_4 \end{bmatrix}$$

which is given by five sufficiently general rank 1 matrices

$$\begin{bmatrix} c_{110} & c_{120} \\ c_{210} & c_{220} \\ c_{310} & c_{320} \end{bmatrix}, \begin{bmatrix} c_{111} & c_{121} \\ c_{211} & c_{221} \\ c_{311} & c_{321} \end{bmatrix}, \begin{bmatrix} c_{112} & c_{122} \\ c_{212} & c_{222} \\ c_{312} & c_{322} \end{bmatrix}, \begin{bmatrix} c_{113} & c_{123} \\ c_{213} & c_{223} \\ c_{313} & c_{323} \end{bmatrix}, \begin{bmatrix} c_{114} & c_{124} \\ c_{214} & c_{224} \\ c_{314} & c_{324} \end{bmatrix}.$$

The family of strict transforms under π_1 is invariant under the Cremona symmetry of X_{A_4} and the action of the symmetric group on E.

In general, for $n \geq 3$, there is a single irreducible family whose members have the extremal nef class

$$\Delta \cap [X_{A_n}].$$

The family consists of strict transforms of rational scrolls in $\mathbb{P}^n$ which contain all the torus-invariant points and intersect no other torus orbits of codimension ≥ 2. The homology class is not an intersection of nef divisor classes. On the other hand,

$$(n-1)! \cdot \Delta \cap [X_{A_n}]$$

is an intersection of nef divisor classes. The corresponding family is the family of strict transforms of complete intersections in $\mathbb{P}^n$ defined by general linear combinations of square-free monomials in $z_0, z_1, \ldots, z_n$ with degrees $2, 3, \ldots, n-1$.

3.3 Realizing matroids in the permutohedral variety

This is the first section where the field k comes into play. The main result of this section says that a matroid M is realizable over k if and only if the corresponding homology class $\Delta_M \cap [X_{A_n}]$ in the permutohedral variety is the class of a subvariety over k. This sharply contrasts Corollary 34, which says that the class $\Delta_M \cap [X_{A_n}]$ is the class of an effective cycle over k for any matroid M and any field k.

Let X_{A_n} be the permutohedral variety over k, and let M be a loopless matroid on $E = \{0, 1, \ldots, n\}$. By a subvariety of X_{A_n}, we mean a geometrically reduced and geometrically irreducible closed subscheme of finite type over k. As usual, the rank of M is $r+1$.

Definition 44. A *realization* $\mathscr{R}$ of M over k is a collection of vectors $f_0, f_1, \ldots, f_n$ in an $(r+1)$-dimensional vector space V over k with the following property:

> A subset I of E is independent for M if and only if $\{f_i \mid i \in I\}$ is linearly independent in V.

Since M has no loops, all the f_j are nonzero. The *arrangement* associated to $\mathscr{R}$, denoted $\mathcal{A}_{\mathscr{R}}$, is the hyperplane arrangement

$$\mathcal{A}_{\mathscr{R}} := \{f_0 f_1 \cdots f_n = 0\} \subseteq \mathbb{P}(V^\vee),$$

where $\mathbb{P}(V^\vee)$ is the projective space of hyperplanes in V. We say that a linear subspace of $\mathbb{P}(V^\vee)$ is a *flat* of $\mathcal{A}_{\mathscr{R}}$ if it is an intersection of hyperplanes in $\mathcal{A}_{\mathscr{R}}$. There is an inclusion reversing bijection between the flats of M and the flats of $\mathcal{A}_{\mathscr{R}}$:

$$F \longmapsto \bigcap_{j \in F} \{f_j = 0\}.$$

The *embedding* associated to $\mathscr{R}$, denoted $L_{\mathscr{R}}$, is the map from the projectivized dual

$$L_{\mathscr{R}} : \mathbb{P}(V^\vee) \simeq \mathbb{P}^r \longrightarrow \mathbb{P}^n, \qquad L_{\mathscr{R}} = [f_0 : f_1 : \cdots : f_n].$$

Since $f_0, f_1, \ldots, f_n$ generate V, the linear map $L_{\mathscr{R}}$ is well-defined and is an embedding. Furthermore, since M has no loops, the generic point of $\mathbb{P}(V^\vee)$ maps to the open torus orbit of $\mathbb{P}^n$. If k is infinite, then a general point of $\mathbb{P}(V^\vee)$ maps to the open torus orbit of $\mathbb{P}^n$. Under the embedding $L_{\mathscr{R}}$, the union of the torus-invariant hyperplanes in $\mathbb{P}^n$ pullbacks to the arrangement $\mathcal{A}_{\mathscr{R}}$.

Definition 45. The variety of $\mathscr{R}$, denoted $Y_{\mathscr{R}}$, is the strict transform of the image of $L_{\mathscr{R}}$ under the composition of blowups $\pi_1 : X_{A_n} \longrightarrow \mathbb{P}^n$. By definition, there is a commutative diagram

$$\begin{array}{ccc} Y_{\mathscr{R}} & \xrightarrow{\iota_{\mathscr{R}}} & X_{A_n} \\ \downarrow{\scriptstyle \pi_{\mathscr{R}}} & & \downarrow{\scriptstyle \pi_1} \\ \mathbb{P}(V^\vee) & \xrightarrow[L_{\mathscr{R}}]{} & \mathbb{P}^n, \end{array}$$

where $\iota_{\mathscr{R}}$ is the inclusion and $\pi_{\mathscr{R}}$ is the induced blowup.

Recall that π_1 can be factored into

$$X_{A_n} = X_{n-1} \longrightarrow X_{n-2} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = \mathbb{P}^n,$$

where $X_{d+1} \longrightarrow X_d$ is the blowup of the strict transform of the union of all the torus-invariant d-dimensional linear subspaces of $\mathbb{P}^n$. If S is a proper subset of E with $|S| \geq 2$, then D_S is the exceptional divisor of π_1 corresponding to the codimension $|S|$ linear subspace

$$\bigcap_{i \in S} \{z_i = 0\} \subseteq \mathbb{P}^n.$$

If $S = \{i\}$, then D_S is the strict transform of the hyperplane $\{z_i = 0\}$. The union of all the D_S is a simple normal crossings divisor whose complement in X_{A_n} is the open torus orbit.

Similarly, $\pi_{\mathscr{R}}$ is the blowup of all the flats of the hyperplane arrangement $\mathcal{A}_{\mathscr{R}}$. It is the composition of maps

$$Y_{\mathscr{R}} \;=\; Y_{r-1} \longrightarrow Y_{r-2} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 \;=\; \mathbb{P}(V^\vee),$$

where $Y_{d+1} \longrightarrow Y_d$ is the blowup of the strict transform of the union of all the d-dimensional flats of $\mathcal{A}_{\mathscr{R}}$. Exceptional divisors of $\pi_{\mathscr{R}} : Y_{\mathscr{R}} \longrightarrow \mathbb{P}(V^\vee)$ are indexed by flats with rank at least 2.

Notation. If F is a flat of rank at least 2, then we write E_F for the exceptional divisor of $\pi_{\mathscr{R}}$ corresponding to the codimension $\mathrm{rank}_M(F)$ linear subspace

$$\bigcap_{j \in F} \{f_j = 0\} \subseteq \mathbb{P}(V^\vee).$$

When F is a flat of rank 1, we define E_F to be the strict transform of the hyperplane of $\mathbb{P}(V^\vee)$ corresponding to F.

The union of all the E_F is a simple normal crossings divisor whose complement in $Y_{\mathscr{R}}$ is the intersection of $Y_{\mathscr{R}}$ with the open torus orbit of X_{A_n}. In the language of De Concini and Procesi [DP95], the variety $Y_{\mathscr{R}}$ is the wonderful compactification of the arrangement complement $\mathbb{P}(V^\vee) \setminus \mathcal{A}_{\mathscr{R}}$ corresponding to the maximal building set.

The statement below is a classical variant of the tropical statement of Katz and Payne [KP11].

Theorem 46. *Let X_{A_n} be the n-dimensional permutohedral variety over k.*

(i) If $\mathscr{R}$ is a realization of M over k, then

$$[Y_{\mathscr{R}}] = \Delta_M \cap [X_{A_n}] \in A_r(X_{A_n}).$$

(ii) If Y is an r-dimensional subvariety of X_{A_n} such that

$$[Y] = \Delta_M \cap [X_{A_n}] \in A_r(X_{A_n}),$$

then $Y = Y_{\mathscr{R}}$ for some realization $\mathscr{A}$ of M over k.

In particular, M is realizable over k if and only if $\Delta_M \cap [X_{A_n}]$ is the class of a subvariety over k.

Proof. For a nonempty proper subset F of E, the subvariety $Y_{\mathscr{R}}$ intersects the torus-invariant divisor D_F if and only if F is a flat of M. In this case,

$$Y_{\mathscr{R}} \cap D_F = E_F.$$

Let $\mathcal{F} = \big(F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r\big)$ be a flag of nonempty proper subsets of E. If one of the F_j is not a flat of M, then

$$Y_{\mathscr{R}} \cap V(\mathcal{F}) = Y_{\mathscr{R}} \cap D_{F_1} \cap \cdots \cap D_{F_r} = \varnothing.$$

If all the F_j are flats of M, then

$$Y_{\mathscr{R}} \cap V(\mathcal{F}) = Y_{\mathscr{R}} \cap D_{F_1} \cap \cdots \cap D_{F_r} = E_{F_1} \cap \cdots \cap E_{F_r},$$

and this intersection is a reduced point. Therefore, the class of $Y_{\mathscr{R}}$ in X_{A_n} is Poincaré dual to the Bergman fan of M. In other words,

$$[Y_{\mathscr{R}}] = \Delta_M \cap [X_{A_n}] \in A_r(X_{A_n}).$$

This proves the first assertion.

Conversely, suppose that Y is an r-dimensional subvariety of X_{A_n} defined over k such that

$$[Y] = \Delta_M \cap [X_{A_n}] \in A_r(X_{A_n}).$$

As an intermediate step, we prove that Y is not contained in any torus-invariant hypersurface of X_{A_n}.

Consider a torus-invariant prime divisor of X_{A_n}. It is of the form D_S for some nonempty proper subset S of E. We show that Y is not contained in D_S. Choose a rank 1 flat F_1 which is not comparable to S. This is possible because M has no loops and E is a disjoint union of the rank 1 flats of M. We extend F_1 to a maximal flag of proper flats

$$\mathcal{F} = \Big(F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r\Big).$$

By definition of the Bergman fan Δ_M, we have

$$D_{F_1} \cdot D_{F_2} \cdot \ldots \cdot D_{F_r} \cdot [Y] = 1.$$

On the other hand, if Y is contained in D_S, then the above intersection product can be computed by pulling back the divisors D_{F_j} under the inclusion $\iota : D_S \to X_{A_n}$. Since F_1 is not comparable to S, the pull-back of D_{F_1} to D_S is equivalent to zero. This leads to the contradiction that

$$\iota^* D_{F_1} \cdot \iota^* D_{F_2} \cdot \ldots \cdot \iota^* D_{F_r} \cdot [Y] = 0.$$

We now show that $Y = Y_{\mathscr{R}}$ for some realization $\mathscr{R}$ of M. Let i be an element of E, and let $H_i = \{z_i = 0\}$ be the corresponding hyperplane of $\mathbb{P}^n$. Proposition 30 shows that

$$\underbrace{\pi_1^{-1}(H_i)\cdot \ldots \cdot \pi_1^{-1}(H_i)}_{r} \cdot [Y] = \Big(\underbrace{\alpha \cup \cdots \cup \alpha}_{r} \cup \Delta_M\Big) \cap [X_{A_n}] = 1.$$

The projection formula tells us that the image $\pi_1(Y)$ is an r-dimensional subvariety of $\mathbb{P}^n$ which has degree 1. In other words, the image is an r-dimensional linear subspace

$$\pi_1(Y) = \mathbb{P}^r \subseteq \mathbb{P}^n.$$

Write the equations defining the above linear embedding by

$$L : \mathbb{P}^r \longrightarrow \mathbb{P}^n, \qquad f = [f_0 : f_1 : \cdots : f_n].$$

Since Y is not contained in any torus-invariant hypersurface of X_{A_n}, the image $\pi_1(Y)$ is not contained in any torus-invariant hyperplane of $\mathbb{P}^n$. Therefore all the linear forms f_j are nonzero. Let $\mathscr{R}$ be the set of vectors $\{f_0, f_1, \ldots, f_n\}$ in the $(r+1)$-dimensional vector space $H^0\big(\mathbb{P}^r, \mathcal{O}(1)\big)$. This defines a loopless matroid N on E which is realizable over k.

By definition of the strict transform, $Y = Y_{\mathscr{R}}$. Applying the first part of the theorem to $Y_{\mathscr{R}}$, we have

$$[Y] = [Y_{\mathscr{R}}] = \Delta_N \cap [X_{A_n}],$$

Since the set of flats of a matroid determines the matroid, $M = N$. This proves the second assertion. □

Example 47. We work with the permutohedral variety $X = X_{A_6}$ over the integers.

Consider the embedding

$$L : \mathbb{P}^2 \longrightarrow \mathbb{P}^6, \quad [x_0 : x_1 : x_2] \longmapsto [x_0 : x_1 : x_2 : x_0{+}x_1 : x_0{+}x_2 : x_1{+}x_2 : x_0{+}x_1{+}x_2].$$

The image of L is the intersection of the ten hyperplanes

$$\begin{aligned}
&H_1 = \{z_5 = z_1 + z_2\},\ H_2 = \{z_4 = z_0 + z_2\},\ H_3 = \{z_3 = z_0 + z_1\},\\
&H_4 = \{z_6 = z_2 + z_3\},\ H_5 = \{z_6 = z_1 + z_4\},\ H_6 = \{z_6 = z_0 + z_5\},\\
&H_7 = \{2z_0 = z_3 + z_4 - z_5\},\ H_8 = \{2z_1 = z_3 - z_4 + z_5\},\ H_9 = \{2z_2 = -z_3 + z_4 + z_5\},\\
&H_{10} = \{2z_6 = z_3 + z_4 + z_5\}.
\end{aligned}$$

Let $\tilde{H}_j$ be the strict transform of H_j under the blowup $\pi_1 : X \longrightarrow \mathbb{P}^6$. For each prime number p, we have the commutative diagram over $\mathbb{Z}/p$

$$\begin{array}{ccc}
Y_p & \longrightarrow & X_p \\
\big\downarrow & & \big\downarrow{\scriptstyle \pi_{1,p}} \\
\mathbb{P}^r_p & \xrightarrow{L_p} & \mathbb{P}^n_p,
\end{array}$$

where Y_p is the strict transform of the image of L_p under the blowup $\pi_{1,p}$. Write $\tilde{H}_{j,p}$ for the intersection of $\tilde{H}_j$ and X_p. For any prime number p, we have

$$\Big[\bigcap_{j=0}^{9} \tilde{H}_{j,p}\Big] = \Delta_M \cap [X_p],$$

where M is the rank 3 matroid on E whose rank 2 flats are

$$\{0,1,3\}, \quad \{0,2,4\}, \quad \{1,2,5\}, \quad \{0,5,6\}, \quad \{1,4,6\}, \quad \{2,3,6\}.$$

If $p \neq 2$, then

$$\bigcap_{j=1}^{10} \tilde{H}_{j,p} = Y_p.$$

If $p = 2$, then

$$\bigcap_{j=1}^{10} \tilde{H}_{j,p} = Y_p \cup S_p,$$

for some surface S_p in X_p. As a family of subschemes of X over $\mathrm{Spec}(\mathbb{Z})$, the latter is the limit of the former. When $p = 2$, we have

$$[Y_p] = \Delta_N \cap [X_p],$$

where N is the rank 3 matroid on E whose rank 2 flats are

$$\{0,1,3\}, \quad \{0,2,4\}, \quad \{1,2,5\}, \quad \{0,5,6\}, \quad \{1,4,6\}, \quad \{2,3,6\}, \quad \{3,4,5\}.$$

The matroid N is realized by the Fano plane, the configuration of the seven nonzero vectors in the three-dimensional vector space over the field with two elements. This matroid is not realizable over fields with characteristic not equal to 2.

CHAPTER IV

The anticanonical image of a matroid

4.1 Truncating the Bergman fan of a matroid

Let M be a loopless matroid on E, and let r_1, r_2, and d be integers which satisfy

$$1 \leq r_1 \leq r_2 \leq r \quad \text{and} \quad d = (r_2 - r_1) + 1.$$

We define a d-dimensional weighted fan $\Delta_{M[r_1,r_2]}$, called the truncated Bergman fan of M. The truncated Bergman fan defines a d-dimensional nef and effective homology class of X_{A_n}. When M is realizable, the combinatorics of the truncated Bergman fan is governed by the geometry of certain subvarieties of X_{A_n} which are far from being linear. Understanding the balancing condition for the truncated Bergman fan will be important for computing the anticanonical image of a matroid in Section 4.2.

As usual, when we use the words "flat", "cover", or "rank", we are referring to the matroid M and its lattice of flats.

Definition 48. The *truncated Bergman fan of type* (r_1, r_2), denoted $\Delta_{M[r_1,r_2]}$, is a function from the set of d-dimensional cones in Δ_{A_n} to the set of integers defined as follows. Let σ be the d-dimensional cone determined by a flag of nonempty proper subsets

$$\mathcal{F} = \Big(F_{r_1} \subsetneq F_{r_1+1} \subsetneq \cdots \subsetneq F_{r_2-1} \subsetneq F_{r_2}\Big).$$

(i) If each F_j is a flat of rank j, then the value of the truncated Bergman fan at σ is $|\mu_M(\varnothing, F_{r_1})|$.

(ii) Otherwise, the value of the truncated Bergman fan at σ is 0.

When $r_1 = 1$, the truncated Bergman fan is the Bergman fan of a repeated truncation of M. In general, the truncated Bergman fan has values other than $0, 1$, and is not the Bergman fan of any matroid.

Proposition 49. *The truncated Bergman fan $\Delta_{M[r_1,r_2]}$ is a d-dimensional Minkowski weight on Δ_{A_n}. In other words, the truncated Bergman fan satisfies the balancing condition.*

The homology class of the truncated Bergman fan of a matroid often, but not always, generates an extremal ray of the nef cone of X_{A_n}.

Proof. Let τ be a $(d-1)$-dimensional cone in the permutohedral fan generated by a flag of nonempty proper subsets

$$\mathcal{G} = \mathcal{G}(\tau) = \Big(G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_{d-1}\Big).$$

We check the balancing condition for the truncated Bergman fan at τ. For this we may assume that each G_j is a flat with rank at least r_1 and at most r_2, since otherwise the truncated Bergman fan has value 0 on each d-dimensional cone containing τ. Now there are two cases to be considered separately:

(i) $\text{rank}_M(G_1) = r_1 + 1$.

In this case, the balancing condition is another way of expressing Weisner's theorem. To see this, note that each d-dimensional cone σ containing τ which is in the support of $\Delta_{M[r_1,r_2]}$ corresponds to a unique flag of flats of the form

$$\mathcal{F} = \mathcal{F}(\sigma) = \Big(F \subsetneq G_1 \subsetneq \cdots \subsetneq G_{d-1}\Big).$$

If $F_1, \dots, F_m$ are the flats covered by G_1, then the vector to be balanced is

$$\sum_{\tau \subset \sigma} \Delta_{M[r_1,r_2]}(\sigma)\mathbf{u}_{\sigma/\tau} = \sum_{j=1}^{m} \big|\mu_M(\varnothing, F_j)\big|\mathbf{u}_{F_j}.$$

We need to show that this vector lies in the lattice generated by $\mathbf{u}_{G_1}, \dots, \mathbf{u}_{G_{d-1}}$. We show that the vector is a multiple of $\mathbf{u}_{G_1}$ by computing its coordinates before taking the quotient by $(1, \dots, 1)$. For any nonzero element i not in G_1, the i-th coordinate of the right-hand side is 0, and for any nonzero element i in G_1, the i-th coordinate of the right-hand side is

$$\sum_{i \in F_j} \big|\mu_M(\varnothing, F_j)\big|,$$

where the sum is over all F_j which contain i. By Weisner's theorem, this quantity is independent of i. Explicitly, for any element i of G_1,

$$\sum_{i \in F_j} \big|\mu_M(\varnothing, F_j)\big| = \sum_{j=1}^{m} \big|\mu_M(\varnothing, F_j)\big| - \big|\mu_M(\varnothing, G_1)\big|,$$

hence

$$\sum_{j=1}^{m} \big|\mu_M(\varnothing, F_j)\big|\mathbf{u}_{F_j} = \left[\sum_{j=1}^{m} \big|\mu_M(\varnothing, F_j)\big| - \big|\mu_M(\varnothing, G_1)\big|\right]\mathbf{u}_{G_1}.$$

This shows that the balancing condition for $\Delta_{M[r_1,r_2]}$ is satisfied at τ.

(ii) $\text{rank}_M(G_1) = r_1$.

In this case, if we set $G_d = E$, then there is a unique index $l \geq 2$ which satisfies

$$\text{rank}_M(G_l) \geq \text{rank}_M(G_{l-1}) + 2.$$

Each d-dimensional cone σ containing τ which is in the support of $\Delta_{M[r_1,r_2]}$ corresponds to a unique flag of flats of the form

$$\mathcal{F} = \mathcal{F}(\sigma) = \Big(G_1 \subsetneq \cdots \subsetneq G_{l-1} \subsetneq F \subsetneq G_l \subsetneq \cdots \subsetneq G_{d-1}\Big).$$

If $F_1, \ldots, F_m$ are the flats that cover G_{l-1} and are covered by G_l, then the vector to be balanced is

$$\sum_{\tau \subset \sigma} \Delta_{M[r_1,r_2]}(\sigma)\mathbf{u}_{\sigma/\tau} = \sum_{j=1}^{m} \big|\mu_M(\varnothing, G_1)\big|\mathbf{u}_{F_j}.$$

We need to show that this vector lies in the lattice generated by $\mathbf{u}_{G_1}, \ldots, \mathbf{u}_{G_{d-1}}$. By Proposition 16, we have

$$\sum_{j=1}^{m} \big|\mu_M(\varnothing, G_1)\big|\mathbf{u}_{F_j} = \big|\mu_M(\varnothing, G_1)\big|\Big(\mathbf{u}_{G_l} + (m-1)\mathbf{u}_{G_{l-1}}\Big).$$

This shows that the balancing condition for $\Delta_{M[r_1,r_2]}$ is satisfied at τ. □

For later use, we record here the main formula obtained in the proof of the balancing condition for the truncated Bergman fan of M. This formula, which is a version of Weisner's theorem, will play a role in the proof of Theorem 55.

Proposition 50. *Let $F_1, \ldots, F_m$ be the flats of a loopless matroid M that are covered by a flat G_1. For any element i of G_1, we have*

$$\sum_{j=1}^{m} \big|\mu_M(\varnothing, F_j)\big|\mathbf{u}_{F_j} = \left[\sum_{i \in F_j} \big|\mu_M(\varnothing, F_j)\big|\right]\mathbf{u}_{G_1}.$$

Equivalently,

$$\sum_{j=1}^{m} \big|\mu_M(\varnothing, F_j)\big|\mathbf{u}_{F_j} = \left[-\big|\mu_M(\varnothing, G_1)\big| + \sum_{j=1}^{m} \big|\mu_M(\varnothing, F_j)\big|\right]\mathbf{u}_{G_1}.$$

Example 51. Consider the quartic surface S in $\mathbb{P}^4$ defined by

$$z_0 + z_1 + z_2 + z_3 + z_4 = 0, \quad z_1z_2z_3z_4 + z_0z_2z_3z_4 + z_0z_1z_3z_4 + z_0z_1z_2z_4 + z_0z_1z_2z_3 = 0.$$

This quartic surface contains ten lines

$$z_{i_0} + z_{i_1} + z_{i_2} = z_{i_3} = z_{i_4} = 0$$

and ten double points

$$z_{i_0} = z_{i_1} = z_{i_2} = z_{i_3} + z_{i_4} = 0,$$

where $(i_0, i_1, i_2, i_3, i_4)$ is a permutation of $(0, 1, 2, 3, 4)$. Every line contains three of the ten points, and every point is contained in three of the ten lines. In fact, the incidence between the lines and the points is that of the Desargues configuration in a projective plane. We have a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\iota_{\tilde{S}}} & X_{A_4} \\ {\scriptstyle \pi_S}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ S & \xrightarrow[\iota_S]{} & \mathbb{P}^4, \end{array}$$

where ι_S, $\iota_{\tilde{S}}$ are inclusions and π_S is the blowup of the ten singular points. The smooth surface $\tilde{S}$ is invariant under the Cremona symmetry of X_{A_4} and the action of the symmetric group on E. It contains twenty smooth rational curves with self-intersection (-2), namely the strict transforms of the ten lines and the exceptional curves over the ten singular points. Any two of the twenty curves are either disjoint or intersect transversely at one point.

The homology class of $\tilde{S}$ in X_{A_4} is given by the truncated Bergman fan of type $(2, 3)$ of the rank 4 uniform matroid M:

$$[\tilde{S}] = \Delta_{M[2,3]} \cap [X_{A_4}].$$

The graph of the truncated Bergman fan satisfies the condition in Proposition 41, hence the homology class generates an extremal ray of the nef cone of X_{A_4} in dimension 2.

Example 52. Consider the quartic surface S in $\mathbb{P}^4$ defined by

$$z_1 + z_2 + z_3 + z_4 = 0, \quad z_1z_2z_3z_4 + z_0z_2z_3z_4 + z_0z_1z_3z_4 + z_0z_1z_2z_4 + z_0z_1z_2z_3 = 0.$$

This quartic surface contains four lines of the form

$$z_{i_1} + z_{i_2} + z_{i_3} = z_{i_4} = z_0 = 0,$$

where (i_1, i_2, i_3, i_4) is a permutation of $(1, 2, 3, 4)$, six lines of the form

$$z_{i_1} + z_{i_2} = z_{i_3} = z_{i_4} = 0,$$

where (i_1, i_2, i_3, i_4) is a permutation of $(1, 2, 3, 4)$, six double points

$$z_0 = z_{i_1} = z_{i_2} = z_{i_3} + z_{i_4} = 0,$$

where (i_1, i_2, i_3, i_4) is a permutation of $(1, 2, 3, 4)$, and one triple point

$$z_1 = z_2 = z_3 = z_4 = 0.$$

The incidence between the ten lines and the seven points is that of the rank 2 flats and rank 3 flats of the matroid M on $\{0, 1, 2, 3, 4\}$ which has one minimal dependent set $\{1, 2, 3, 4\}$. We have a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\iota_{\tilde{S}}} & X_{A_4} \\ {\scriptstyle \pi_S}\big\downarrow & & \big\downarrow{\scriptstyle \pi_1} \\ S & \xrightarrow[\iota_S]{} & \mathbb{P}^4, \end{array}$$

where ι_S, $\iota_{\tilde{S}}$ are inclusions and π_S is the blowup of the seven singular points. The strict transforms of the ten lines are smooth rational curves in $\tilde{S}$ disjoint from each other. Four of the ten curves, those corresponding to the lines containing the triple point, have self-intersection (-2). The remaining six have self-intersection (-1). The six exceptional curves over the double points of S are smooth rational curves in $\tilde{S}$ with self-intersection (-2), and the exceptional curve over the triple point is an elliptic curve with self-intersection (-3). A curve corresponding to a line meets a curve corresponding to a singular point if and only if the line contains the point. In this case, the two curves intersect transversely at one point.

The homology class of $\tilde{S}$ in X_{A_4} is given by truncated Bergman fan of type $(2, 3)$ of M:

$$[\tilde{S}] = \Delta_{M[2,3]} \cap [X_{A_4}].$$

Although the graph of the truncated Bergman fan does not satisfy the condition in Proposition 41, the homology class generates an extremal ray of the nef cone of X_{A_4} in dimension 2.

The previous examples generalize as follows. Let M be a rank 4 matroid on E which has a realization $\mathscr{R}$ over k. Consider the associated embedding

$$L_{\mathscr{R}} : \mathbb{P}(V^\vee) \longrightarrow \mathbb{P}^n, \qquad L_{\mathscr{R}} = (f_0 : f_1 : \cdots : f_n),$$

and the degree n surface S in $\mathbb{P}(V^\vee) \simeq \mathbb{P}^3$ defined by

$$\sum_{i=0}^{n} f_0 \cdots \hat{f}_i \cdots f_n = 0.$$

The surface S contains the 1-dimensional flats of the arrangement $\mathcal{A}_{\mathscr{R}}$ and the 0-dimensional flats of the arrangement $\mathcal{A}_{\mathscr{R}}$. The incidence between the points and lines is that of the rank 2 and rank 3 flats of M. We have a commutative diagram

$$\begin{array}{ccccc}
\tilde{S} & \xrightarrow{\iota_{\tilde{S}}} & Y_{\mathscr{R}} & \xrightarrow{\iota_{\mathscr{R}}} & X_{A_n} \\
\downarrow{\scriptstyle \pi_S} & & \downarrow{\scriptstyle \pi_{\mathscr{R}}} & & \downarrow{\scriptstyle \pi_1} \\
S & \xrightarrow[\iota_S]{} & \mathbb{P}(V^\vee) & \xrightarrow[L_{\mathscr{R}}]{} & \mathbb{P}^n,
\end{array}$$

where ι_S, $\iota_{\tilde{S}}$ are inclusions and π_S is the blowup of the singular points and lines. The surface $\tilde{S}$ is smooth and it contains an arrangement of (smooth but not necessarily connected) curves indexed by the rank 3 and rank 2 flats of M. The intersections of these curves in $\tilde{S}$ are controlled by the lattice of flats of M and its Möbius function:

(i) Two curves C_{F_1}, C_{F_2} indexed by flats F_1, F_2 intersect each other if and only if F_1 and F_2 are compatible. If $F_1 \subsetneq F_2$, then C_{F_1} and C_{F_2} intersect transversely at $\mu_M(\varnothing, F_1)$ points.

(ii) If F is a rank 2 flat and i is any element of F, then the self-intersection number of C_F in $\tilde{S}$ is

$$-\sum_{G} \mu_M(\varnothing, F),$$

where the sum is over all flats G that cover F and not containing i.

(iii) If F is a rank 3 flat and i is any element of F, then the self-intersection number of C_F in $\tilde{S}$ is

$$-\sum_{G} \mu_M(\varnothing, G),$$

where the sum is over all flats G that are covered by F and containing i.

The homology class of $\tilde{S}$ in X_{A_n} is Poincaré dual to the cohomology class defined by the type $(2,3)$ truncated Bergman fan of M.

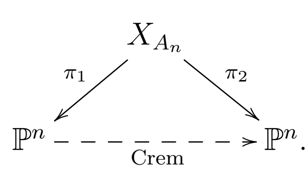
Example 53. The truncated Bergman fan of a matroid, in general, does not generate an extremal ray of the nef cone of the permutohedral variety. Here is a two-dimensional example. Let M be the rank 4 matroid on $E = \{0,1,2,3,4,5\}$ whose rank 3 flats are

$$\{0,1,2,3\}, \quad \{2,3,4,5\}, \quad \{0,1,4,5\},$$
$$\{0,2,4\}, \quad \{0,2,5\}, \quad \{0,3,4\}, \quad \{0,3,5\},$$
$$\{1,2,4\}, \quad \{1,2,5\}, \quad \{1,3,4\}, \quad \{1,3,5\}.$$

One can check that the rank 2 flats of M are all subsets of E with two elements, and the two-dimensional nef class $\Delta_{M[2,3]} \cap [X_{A_5}]$ can be written as a sum of two extremal nef classes in a nontrivial way. For an one-dimensional example, see Example 40.

4.2 The characteristic polynomial is the anticanonical image.

The anticanonical linear system of the permutohedral variety X_{A_n} is basepoint free and big. The product map $\pi_1 \times \pi_2$ may be viewed as the anticanonical map of X_{A_n}, where π_1, π_2 are the blowups in the commutative diagram



Write the reduced characteristic polynomial of a loopless matroid M as

$$\overline{\chi}_M(q) = \chi_M(q)/(q-1) = \sum_{l=0}^{r} (-1)^l \mu_M^l q^{r-l}.$$

We show that the reduced characteristic polynomial $\overline{\chi}_M(q)$ represents the push-forward of the homology class of M under the anticanonical mapping. This is the main result of [HK12], and we give a refined proof here. Conjectures 21 and 23 for realizable matroids follow from this computation.

Theorem 54. *Under the anticanonical map*

$$\pi_1 \times \pi_2 : X_{A_n} \longrightarrow \mathbb{P}^n \times \mathbb{P}^n,$$

the homology class of M push-forwards to its reduced characteristic polynomial $\overline{\chi}_M(q)$:

$$\Delta_M \cap [X_{A_n}] \longmapsto \sum_{l=0}^{r} \mu_M^l [\mathbb{P}^{r-l} \times \mathbb{P}^l].$$

Theorem 54 follows from a more general statement relating the Bergman fan Δ_M, the truncated Bergman fan $\Delta_{M[r_1,r_2]}$, and the piecewise linear functions α, β. Recall that $\alpha = \alpha(i)$ and $\beta = \beta(i)$ are piecewise linear functions on the fan Δ_{A_n} defined by the values

$$\alpha(\mathbf{u}_S) = \begin{cases} -1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S, \end{cases} \qquad \text{and} \qquad \beta(\mathbf{u}_S) = \begin{cases} 0 & \text{if } i \in S, \\ -1 & \text{if } i \notin S. \end{cases}$$

For any i and any nonempty proper subset G of E, the function α is linear when restricted to the cone

$$\text{cone}(\mathbf{u}_j \mid j \in G) = \bigcup_{\mathcal{F}} \sigma_{\mathcal{F}},$$

where the union is over all flags of nonempty proper subsets contained in G. The function β is linear when restricted to the cone

$$\text{cone}(\mathbf{u}_{\hat{j}} \mid j \notin G) = \bigcup_{\mathcal{F}} \sigma_{\mathcal{F}},$$

where the union is over all flags of nonempty proper subsets which contain G.

Proposition 55. *If r_1, r_2 are integers which satisfy $1 \leq r_1 < r_2 \leq r$, then*

(i) $\alpha \cup \Delta_{M[r_1,r_2]} = \Delta_{M[r_1,r_2-1]}$,

(ii) $\beta \cup \Delta_{M[r_1,r_2]} = \Delta_{M[r_1+1,r_2]}$.

Proof. We prove the second equality. The proof of the first equality is similar to that of Proposition 30, and will be omitted.

Let τ be a $(d-1)$-dimensional cone in Δ_{A_n} generated by a flag of nonempty proper subsets

$$\mathcal{G} = \Big(G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_{d-1}\Big).$$

The d-dimensional cones σ containing τ bijectively correspond to nonempty proper subsets of E that are strictly compatible with $\mathcal{G}$. We show that

$$\Big(\beta \cup \Delta_{M[r_1,r_2]}\Big)(\tau) = \Delta_{M[r_1+1,r_2]}(\tau).$$

If one of the G_j is not a flat with rank at least r_1 and at most r_2, then

$$\Big(\beta \cup \Delta_{M[r_1,r_2]}\Big)(\tau) = 0 \quad \text{and} \quad \Delta_{M[r_1+1,r_2]}(\tau) = 0.$$

Suppose that each G_j is a flat with rank at least r_1 and at most r_2. We need to prove the following statements.

(i) If $\text{rank}_M(G_1) = r_1$, then $\Big(\beta \cup \Delta_{M[r_1,r_2]}\Big)(\tau) = 0$.

(ii) If $\text{rank}_M(G_1) = r_1 + 1$, then $\Big(\beta \cup \Delta_{M[r_1,r_2]}\Big)(\tau) = \big|\mu_M(\varnothing, G_1)\big|$.

In the first case, the d-dimensional cones containing τ with nonzero value of $\Delta_{M[r_1,r_2]}$ correspond to some flats containing G_1. Since β is linear on the union of cones corresponding to flags of nonempty proper subsets which contain G_1, Theorem 27 implies that

$$\Big(\beta \cup \Delta_{M[r_1,r_2]}\Big)(\tau) = 0.$$

For the second case, let $F_1, \ldots, F_m$ be the flats that are covered by G_1. These flats bijectively correspond to the d-dimensional cones containing τ with nonzero value of $\Delta_{M[r_1,r_2]}$. By Theorem 27, we have

$$\Big(\beta \cup \Delta_{M[r_1,r_2]}\Big)(\tau) = \beta\left(\sum_{j=1}^{m} \big|\mu_M(\varnothing, F_j)\big| \mathbf{u}_{F_j} \right) - \sum_{j=1}^{m} \beta\left(\big|\mu_M(\varnothing, F_j)\big| \mathbf{u}_{F_j} \right),$$

and, by Proposition 50, the right-hand side is equal to

$$\left[-\big|\mu_M(\varnothing, G_1)\big| + \sum_{j=1}^{m} \big|\mu_M(\varnothing, F_j)\big| \right] \beta(\mathbf{u}_{G_1}) - \sum_{j=1}^{m} \big|\mu_M(\varnothing, F_j)\big| \beta(\mathbf{u}_{F_j}).$$

Let i be the element defining β. If G_1 does not contain i, then all the F_j do not contain i, and hence

$$\beta(\mathbf{u}_{G_1}) = \beta(\mathbf{u}_{F_1}) = \cdots = \beta(\mathbf{u}_{F_m}) = -1.$$

If G_1 contains i, then $\beta(\mathbf{u}_{G_1}) = 0$, and by Weisner's theorem applied to G_1 and i,

$$-\sum_{j=1}^{m} \big|\mu_M(\varnothing, F_j)\big| \beta(\mathbf{u}_{F_j}) = \big|\mu_M(\varnothing, G_1)\big|.$$

It follows that

$$\Big(\beta \cup \Delta_{M[r_1,r_2]}\Big)(\tau) = \big|\mu_M(\varnothing, G_1)\big|.$$

□

Since $\Delta_{M[1,r]}$ is Δ_M, a repeated application of Proposition 55 shows that, for $1 \le r_1 \le r_2 \le r$,

$$\Big(\underbrace{\beta \cup \cdots \cup \beta}_{r_1-1}\Big) \cup \Big(\underbrace{\alpha \cup \cdots \cup \alpha}_{r-r_2}\Big) \cup \Delta_M = \Delta_{M[r_1,r_2]}.$$

Corollary 56. *We have the equality between* 0*-dimensional Minkowski weights*

$$\Big(\underbrace{\beta \cup \cdots \cup \beta}_{l}\Big) \cup \Big(\underbrace{\alpha \cup \cdots \cup \alpha}_{r-l}\Big) \cup \Delta_M = \mu_M^l,$$

where μ_M^l *is the absolute value of the coefficient of* q^{r-l} *in the reduced characteristic polynomial* $\overline{\chi}_M(q)$*.*

Proof. By Proposition 55,

$$\beta \cup \Big(\underbrace{\beta \cup \cdots \cup \beta}_{l-1}\Big) \cup \Big(\underbrace{\alpha \cup \cdots \cup \alpha}_{r-l}\Big) \cup \Delta_M = \beta \cup \Delta_{M[l,l]}.$$

By Theorem 27 and the balancing condition for $\Delta_{M[l,l]}$ at the origin, we have

$$\beta \cup \Delta_{M[l,l]} = \sum_{j=1}^{m} \big|\mu_M(\varnothing, F_j)\big|,$$

where F_j are the flats of rank l not containing the element i defining β. Proposition 20 says that this quantity is equal to μ_M^l. □

Proof of Theorem 54. One computes the Poincaré dual of the push-forward using the projection formula and Corollary 56. □

4.3 Is every matroid realizable over every field?

Let X be an n-dimensional smooth complete variety over an algebraically closed field k. The group of numerical equivalence classes of d-dimensional cycles $N_d(X)$ is a finitely generated abelian group with several additional structures. In particular, it contains

(i) the set of prime classes, the classes of subvarieties,

(ii) the set of effective classes, the nonnegative linear combinations of prime classes,

(iii) the set of nef classes, the classes which intersect all codimension d primes non-negatively.

The semigroups (ii) and (iii) define cones in the finite-dimensional vector space $N_d(X)_{\mathbb{R}}$, the peusdoeffective cone and the nef cone of X. When X is a toric variety, the group $N_d(X)$ and its subsets (ii), (iii) are determined by the fan of X. On the other hand, in general, the subset (i) depends on the field k, as we have seen in Theorem 46 for permutohedral varieties.

Definition 57. A homology class $\xi \in N_d(X)_{\mathbb{R}}$ is said to be *prime* if some positive multiple of ξ is the class of a subvariety of X. Define

$$P_d(X) := \Big(\text{the closure of the set of prime classes in } N_d(X)_{\mathbb{R}}\Big).$$

The set $P_d(X)$ is a closed subset of the finite-dimensional vector space $N_d(X)_{\mathbb{R}}$ invariant under scaling by positive real numbers. It contains all extremal rays of the pseudoeffective cone of X in dimension d.

The theorem of Kleiman says that a nef divisor class is a limit of ample divisor classes [Kle66]. This shows that

$$\mathrm{Nef}_{n-1}(X) \subseteq P_{n-1}(X) \subseteq \mathrm{Peff}_{n-1}(X).$$

The theorem of Boucksom-Demailly-Paun-Peternell says that a nef curve class is a limit of movable curve classes [BDPP]. This shows that

$$\mathrm{Nef}_1(X) \subseteq P_1(X) \subseteq \mathrm{Peff}_1(X).$$

In general, the set $P_d(X)$ does not contain all the pseudoeffective nef classes of X. If X is a product of two projective spaces, then $P_d(X)$ is the set of log-concave sequences of nonnegative numbers with no internal zeros [Huh12].

Theorem 58. *If ξ is an element in the homology group*

$$\xi = \sum_j x_j \left[\mathbb{P}^{d-j} \times \mathbb{P}^j\right] \in A_d(X), \quad X = \mathbb{P}^{n-m} \times \mathbb{P}^m,$$

then some positive integer multiple of ξ is the class of a subvariety if and only if the x_j form a nonzero log-concave sequence of nonnegative integers with no internal zeros.

Therefore, in the vector space $N_d(X)_{\mathbb{R}}$ of numerical cycle classes in the product of two projective spaces, the elements of the subset $P_d(X)$ correspond to log-concave sequences of nonnegative real numbers with no internal zeros, while the elements of the cones $\mathrm{Nef}_d(X)$ and $\mathrm{Peff}_d(X)$ correspond to sequences of nonnegative real numbers. We remark that it is possible to obtain sharper statements in some cases. For example, if

$$\xi = x_0[\mathbb{P}^2 \times \mathbb{P}^0] + x_1[\mathbb{P}^1 \times \mathbb{P}^1] + x_2[\mathbb{P}^0 \times \mathbb{P}^2] \in A_2(\mathbb{P}^2 \times \mathbb{P}^2),$$

then ξ is the class of a subvariety if and only if x_0, x_1, x_2 are nonnegative and one of the following conditions is satisfied:

$$(x_1 > 0,\ x_1^2 \geq x_0 x_2) \quad \text{or} \quad (x_0 = 1, x_1 = 0, x_2 = 0) \quad \text{or} \quad (x_0 = 0, x_1 = 0, x_2 = 1).$$

We refer to [Huh13] for a proof.

Example 59. There is no five-dimensional subvariety of $\mathbb{P}^5 \times \mathbb{P}^5$ which has the homology class

$$\xi = 1[\mathbb{P}^5 \times \mathbb{P}^0] + 2[\mathbb{P}^4 \times \mathbb{P}^1] + 3[\mathbb{P}^3 \times \mathbb{P}^2] + 4[\mathbb{P}^2 \times \mathbb{P}^3] + 2[\mathbb{P}^1 \times \mathbb{P}^4] + 1[\mathbb{P}^0 \times \mathbb{P}^5],$$

although $(1, 2, 3, 4, 2, 1)$ is a log-concave sequence with no internal zeros. This follows from the classification of the quadro-quadric Cremona transformations of Pirio and Russo [PR12]. On the other hand, the proof of Theorem 58 shows that there is a five-dimensional subvariety of $\mathbb{P}^5 \times \mathbb{P}^5$ which has the homology class $48 \cdot \xi$.

Recall that the anticanonical push-forward of the homology class of a matroid M in X_{A_n} is the reduced characteristic polynomial $\overline{\chi}_M(q)$:

$$\pi_1 \times \pi_2 : X_{A_n} \longrightarrow \mathbb{P}^n \times \mathbb{P}^n, \qquad \Delta_M \cap [X_{A_n}] \longmapsto \overline{\chi}_M(q).$$

Therefore, the coefficients of the reduced characteristic polynomial $\overline{\chi}_M(q)$ form a log-concave sequence if and only if

$$(\pi_1 \times \pi_2)_* \, \Delta_M \cap [X_{A_n}] \in P_r(\mathbb{P}^n \times \mathbb{P}^n).$$

We ask whether the same inclusion holds in the permutohedral variety X_{A_n}.

Question 60. *For any matroid M and any algebraically closed field k, do we have*

$$\Delta_M \cap [X_{A_n}] \in P_r(X_{A_n})?$$

In view of Theorem 46, the question asks whether every matroid is realizable over every field, perhaps not as an integral homology class, but as a limit of homology class with real coefficients. Since $P_r(X_{A_n})$ maps to $P_r(\mathbb{P}^n \times \mathbb{P}^n)$ under the anticanonical push-forward, an affirmative answer to Question 60 implies the log-concavity conjectures of Section 2.3. We point out that the inclusion in Question 60 has purely combinatorial implications which are strictly stronger than the log-concavity. A computer verification of those implications for matroids with at most nine elements will be reported in a followup article.

Question 60 is related to movability of effective cycles in X_{A_n}. Let's say that the moving lemma holds for an r-dimensional effective cycle if it is equivalent to another effective cycle which intersects properly the union of codimension r torus orbits of X_{A_n}. Let $M_r(X_{A_n})$ be the closure of the cone generated by the r-dimensional effective cycles which satisfy the moving lemma. Clearly, we have

$$M_r(X_{A_n}) \subseteq \mathrm{Nef}_r(X_{A_n}).$$

If the other inclusion also holds, then, using the fact that a matroid homology class generates an extremal ray of the nef cone, one can show that Question 60 has an affirmative answer.

We close with proofs of the log-concavity conjectures for matroids that are realizable over some field.

Theorem 61. *Let M be a matroid.*

(i) If M is realizable over some field, then the coefficients of the reduced characteristic polynomial of M form a log-concave sequence.

(ii) If M is realizable over some field, then the coefficients of the characteristic polynomial of M form a log-concave sequence.

(iii) If M is realizable over some field, then the number of independent subsets of size i of M form a log-concave sequence in i.

(iv) For any graph G, the coefficients of the chromatic polynomial of G form a log-concave sequence.

The second item proves the conjecture of Rota, Welsh, and Heron for realizable matroids (Conjecture 21). The third item proves the conjecture of Mason for realizable matroids (Conjecture 23). The last item proves the conjecture of Read and Hoggar for all graphs (Conjecture 22).

Proof. Suppose M is realizable over k. By Theorem 46, $\Delta_M \cap [X_{A_n}]$ is the class of a subvariety of the permutohedral variety X_{A_n} over k. Since the push-forward maps prime classes to prime classes, Theorem 54 implies that

$$\overline{\chi}_M(q) \in P_r(\mathbb{P}^n \times \mathbb{P}^n).$$

It follows that the coefficients of the reduced characteristic polynomial of M form a log-concave sequence. Since the convolution of two log-concave sequences is a log-concave sequence, the coefficients of the characteristic polynomial of M also form a log-concave sequence.

To justify the third assertion, we consider the free dual extension of M. It is defined by taking the dual of M, placing a new element p in general position (taking the free extension), and again taking the dual. In symbols,

$$M \times p := (M^* + p)^*.$$

The free dual extension $M \times p$ has the following properties:

(i) If M is realizable over k, then $M \times p$ is realizable over a finite extension of k.

(ii) The number of independent subsets of size i of M is the coefficient of q^{r-i} of the reduced characteristic polynomial of M.

We refer to [Len12] and also [Bry77, Bry86] for these facts. It follows that the number of independent subsets of size i of M form a log-concave sequence in i.

For the last assertion, we recall that the chromatic polynomial of a graph is given by the characteristic polynomial of the associated graphic matroid [Wel76]. More precisely, we have

$$\chi_G(q) = q^{n_G} \cdot \chi_{M_G}(q),$$

where n_G is the number of connected components of G. Since graphic matroids are realizable over every field, the coefficients of the chromatic polynomial of G form a log-concave sequence. $\square$

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