The Frobenius Endomorphism and Multiplicities

by

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To my parents

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CHAPTER I

Introduction

Commutative algebraists and algebraic geometers have long used the Frobenius or p-th power map to study rings and schemes in positive characteristic. The algebraic analogue of a smooth variety is a regular ring, such as a polynomial ring over a field. The failure of rings to be regular (i.e., the singular points on a variety) can be detected using the Frobenius map in characteristic $p > 0$. This leads to the definitions of F-regular, F-rational and F-pure singularities [33], [29]. Quite surprisingly, these singularities have a mysterious correspondence to certain singularities in characteristic 0, whose definitions usually require resolution of singularities. For instance, it is known that a variety over $\mathbb{C}$ has rational singularities if and only if its mod p reductions are F-rational for almost all primes $p > 0$ [63], [18]. Moreover, it is conjectured that a similar correspondence holds for log canonical and F-pure singularities [19], [50].

Local cohomology captures several algebraic and geometric properties of a commutative ring. It has close relations with sheaf cohomology and singular cohomology in algebraic geometry and algebraic topology. For example, the elements in the first local cohomology module supported at a certain ideal give the obstruction to extending sections off the subvariety defined by the ideal to the whole variety. In other words, it measures the difficulty in extending holomorphic functions defined on an open sub-manifold to the whole manifold. In positive characteristic, the Frobenius endomorphism of the ring naturally induces Frobenius actions on all the local cohomology modules, which leads to the definition of F-injective singularities [13].

The first goal in this thesis is to understand these "F-singularities," in particular the Frobenius structure of local cohomology modules of F-pure and F-injective rings. One of my main interests is to understand when a local ring $(R, \mathfrak{m})$ has the property that there are only finitely many F-stable [1] submodules of each local cohomology module $H^i_{\mathfrak{m}}(R)$ (we refer to Chapter III for detailed definitions). Rings with this property are called *FH-finite* and have been studied in [58] and [12], where the following was proved:

Theorem I.1 (Sharp [58], Enescu-Hochster [12])**.** *Let $(R, \mathfrak{m})$ be an F-pure Gorenstein local ring of dimension d. Then R is FH-finite, i.e., there are only finitely many F-stable submodules of $H^d_{\mathfrak{m}}(R)$.*

It was also proved in [12] that Stanley-Reisner rings are FH-finite based on a detailed analysis of the structure of the local cohomology modules of these rings. Moreover, Enescu and Hochster asked whether the F-pure property itself is enough for FH-finiteness (see Discussion 4.4 in [12] and Conjecture 1.2 in [11]). We provide a positive answer to this question. We emphasize that our result does not need any extra condition on the ring, and it works for every local cohomology module supported at the maximal ideal (i.e., not only the top one).

Theorem I.2. *Let $(R, \mathfrak{m})$ be an F-pure local ring. Then R is FH-finite, i.e., there are only finitely many F-stable submodules of $H^i_{\mathfrak{m}}(R)$ for every i.*

[1] F-stable submodules are originally introduced by Smith in [62] and [63], this terminology is also used in [12], but in [11] and [43], they are called F-compatible submodules.

The results of Sharp [58] and Enescu-Hochster [12] have close connections with recent work of Blickle and Böckle [6], Kumar and Mehta [35] as well as Schwede and Tucker [56], [57]. For example, Blickle and Böckle [6] recovered and generalized Theorem I.1 in the dual setting, in the language of *Cartier modules*. Kumar and Mehta [35] globalize this result to show that there are finitely many Frobenius compatibly split subvarieties (for a fixed splitting). This is also proved independently by Schwede [56] in a generalized setting (e.g., for pairs). Moreover, Schwede and Tucker [57] give an explicit upper bound on the number of F-ideals, i.e., ideals that can be annihilators of F-stable submodules of $H^d_{\mathfrak{m}}(R)$. So in the Gorenstein case Schwede and Tucker's results give an upper bound on the number of F-stable submodules of $H^d_{\mathfrak{m}}(R)$ by duality.

However, in the non-Gorenstein case, studying F-stable submodules of $H^d_{\mathfrak{m}}(R)$ can be difficult. As Matlis duality does not take $H^d_{\mathfrak{m}}(R)$ to R, the results in [6], [56] or [57] won't provide us much information about the finiteness of the number of F-stable submodules of $H^d_{\mathfrak{m}}(R)$. So our result as well as its proof give new insight in this area. In fact, our result has lots of applications. Recently, Horiuchi, Miller and Shimomoto [34] applied our Theorem I.2 to prove that F-purity deforms to F-injectivity, an outstanding case of the still open conjecture that F-injectivity deforms. We will discuss these recent applications in detail in Chapter III.

Among the techniques for studying singularities and local cohomology in characteristic $p > 0$, the theory of Lyubeznik's F-module is a very powerful tool. For example, using this technique, Lyubeznik has shown that all local cohomology modules of a regular ring of characteristic $p > 0$ have only finitely many associated primes [41]. In [28], Hochster proved that, for any regular ring R of characteristic $p > 0$, the category of Lyubeznik's F-modules has enough injectives, i.e., every F-module can be embedded in an injective F-module. It is therefore quite natural to ask what is the global dimension of this category. Our main result in Chapter IV is the following:

Theorem I.3. *Let R be a regular ring which is essentially of finite type over an F-finite regular local ring. Then the category of Lyubeznik's F-modules has finite global dimension $d+1$ where $d = \dim R$.*

The proof of the above theorem utilizes lots of ideas in Cartier module theory of Blickle and Böckle [6] as well as part of the early constructions in Emerton and Kisin's work [9].

We also proved that for any regular local ring $(R, \mathfrak{m})$ of dimension at least one, the injective hull of the residue field $E(R/\mathfrak{m})$, with its standard F_R-module structure, is *not* injective in the category of F-finite F_R-modules. Recently in [42], Lyubeznik, Singh and Walther have obtained a surprising result compared to our result.

Example I.4. Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$ and dimension $d \geq 1$, and let $E = E(R/\mathfrak{m}) \cong H^d_{\mathfrak{m}}(R)$ be the injective hull of the residue field. Then $\mathrm{Ext}^1_{F_R}(R, E) \neq 0$. Moreover, when $R/\mathfrak{m}$ is an infinite field, $\mathrm{Ext}^1_{F_R}(R, E)$ is also infinite. In particular, E is not injective in the category of F_R-modules.

Theorem I.5 (*cf.* Corollary 2.10 in [42])**.** *Let $(R, \mathfrak{m})$ be a standard graded polynomial ring of dimension n over a separably closed field ($\mathfrak{m}$ stands for the homogeneous maximal ideal). Then $H^d_{\mathfrak{m}}(R)$, with its standard graded F_R-module structure, is injective in the category of graded F-finite F_R-modules.*

The Hilbert-Samuel multiplicity e_R of a local ring R is a classical invariant that measures the singularity of R. In general, e_R is always a positive integer, and the larger the e_R, the worse the singularity of R. It is well known that under mild conditions, $e_R = 1$ if and only if R is a regular local ring. Quite surprisingly, how the Hilbert-Samuel multiplicity behaves under flat local extensions is not understood. Christer Lech conjectured around 1960 [39],[40] that $e_R \leq e_S$ for every flat local extension $R \to S$. Because it is natural to expect that if $R \to S$ is a flat local extension, then R cannot have a worse singularity than S. Hence, from this point of view, Lech's conjecture seems quite natural. However, after over fifty years, very little is known on Lech's conjecture! In Chapter V, we will attack this conjecture using Cohen-factorization coupled with the Frobenius endomorphism. We give some positive results in low dimension, for example we show that Lech's conjecture is true if R is Gorenstein of dimension ≤ 3 in equal characteristic $p > 0$. We also relate Lech's conjecture to some natural questions on modules of finite length and finite projective dimension and extend many previous results to the generalized setting.

CHAPTER II

Preliminaries and notations

In this chapter we will collect the basic definitions and theorems in commutative algebra that we will use throughout this thesis, although we will sometime repeat these definitions and notations in context.

We will use $(R, \mathfrak{m})$ or $(R, \mathfrak{m}, K)$ to denote a Noetherian local ring R with unique maximal ideal $\mathfrak{m}$. When we use the second notion we also specify that the residue field of R is K. We will always use d to denote the dimension of the ring $(R, \mathfrak{m})$. Sometimes we will also use $(R, \mathfrak{m})$ to denote a Noetherian graded ring with unique homogeneous maximal ideal $\mathfrak{m}$. This will be clear in context.

In Chapter III and Chapter IV we will mostly work over rings of characteristic $p > 0$. In this case there is a natural Frobenius endomorphism $F\colon R \to R$, as well as its iterates $F^e\colon R \to R$. Since we often need to distinguish the source and target ring, we will use $R^{(e)}$ to denote the target ring of the e-th Frobenius map $F^e\colon R \to R^{(e)}$. Thus, $R^{(e)}$ is R viewed as an R-algebra with structural homomorphism F^e. When M is an R-module and $x \in M$ is an element, we use $M^{(e)}$ to denote the corresponding module over $R^{(e)}$ and $x^{(e)}$ to denote the corresponding element in $M^{(e)}$. We shall let $F^e_R(-)$ denote the Frobenius functor of Peskine-Szpiro from R-modules to R-modules (we will omit the subscript R when R is clear from the context). In detail, $F^e_R(M)$ is given by base change to $R^{(e)}$ and then identifying $R^{(e)}$ with R. We say R is *F-finite* if $R^{(1)}$ is finitely generated as an R-module. By Kunz's result [36], we know that $R^{(e)}$ is faithfully flat as an R-module when R is regular. So for an F-finite regular ring, $R^{(1)}$ (and hence $R^{(e)}$ for every e) is finite and projective as an R-module.

We use $R\{F\}$ to denote the Frobenius skew polynomial ring, which is the non-commutative ring generated over R by the symbols $1, F, F^2, \dots$ by requiring that $Fr = r^pF$ for $r \in R$. Note that $R\{F\}$ is always free as a left R-module. When R is regular and F-finite, $R\{F\}$ is projective as a right R-module (because $R^{(1)}$ is projective in this case).

We say that an R-module M is an *$R\{F\}$-module* if M is a left module over the ring $R\{F\}$. This is the same as saying that there is a Frobenius action $F\colon M \to M$ such that for all $u \in M$, $F(ru) = r^pu$, and also the same as saying that there is an R-linear map: $F_R(M) \to M$. We say an R-module N is an *F-stable* submodule of an $R\{F\}$-module M if N is an $R\{F\}$-submodule of M. We say an $R\{F\}$-module M is *F-nilpotent* if some power of the Frobenius action on M kills the whole module M, i.e., $F^e\colon M \to M$ is zero for some e.

We say an R-module M is a *right $R\{F\}$-module* if it is a right module over the ring $R\{F\}$, or equivalently, there exists a morphism $\phi\colon M \to M$ such that for all $r \in R$ and $x \in M$, $\phi(r^px) = r\phi(x)$ (the right action of F can be identified with ϕ). This morphism can be also viewed as an R-linear map $\phi\colon M^{(1)} \to M$. We note that a right $R\{F\}$-module is the same as a *Cartier module* defined in [6] (we will recall this in Chapter III).

Using the Frobenius endomorphism one can define the so called "F-singularities." These include F-regular, F-rational, F-pure and F-injective singularities. Since in this thesis we will mainly work with the latter two, we only give the definition for

F-pure and F-injective rings. We first recall that a map of R-modules $N \to N'$ is *pure* if for every R-module M the map $N \otimes_R M \to N' \otimes_R M$ is injective. This implies that $N \to N'$ is injective, and is weaker than the condition that $0 \to N \to N'$ be split. R is called *F-pure* (respectively, *F-split*) if the Frobenius endomorphism F: $R \to R$ is pure (respectively, split). Evidently, an F-split ring is F-pure and an F-pure ring is reduced. When R is either F-finite or complete, F-pure and F-split are equivalent [33].

The Frobenius endomorphism on R induces a natural Frobenius action on each localization of R. So it induces a natural action on the Čech complex $C^\bullet(x_1, \dots, x_n, R) = 0 \to R \to \oplus R_{x_i} \to \cdots \to R_{x_1 \cdots x_n} \to 0$ of R, and hence also on the cohomology of the Čech complex. In particular, it induces a natural action on each local cohomology module $H^i_{\mathfrak{m}}(R)$. We say a local ring is *F-injective* if F acts injectively on all of the local cohomology modules of R with support in $\mathfrak{m}$. This holds if R is F-pure [12].

The Hilbert-Samuel multiplicity of an R-module with respect to an $\mathfrak{m}$-primary ideal I is defined as

$$e(I, M) = d! \cdot \lim_{t \to \infty} \frac{l_R(M/I^t M)}{t^d}.$$

When R has characteristic $p > 0$, one also defines the Hilbert-Kunz multiplicity [49] to be

$$e_{HK}(I, M) = \lim_{e \to \infty} \frac{l_R(M/I^{[p^e]} M)}{p^{de}},$$

where $I^{[p^e]}$ is the ideal generated by all x^{p^e} for $x \in I$ (in the context we use q to denote p^e). We use $e_R(M)$, $e_{HK}(M)$ (resp. e_R, $e_{HK}(R)$) to denote the Hilbert-Samuel multiplicity and Hilbert-Kunz multiplicity of the module M (resp. the ring R) with respect to the maximal ideal $\mathfrak{m}$.

We say an ideal I is a minimal reduction of $\mathfrak{m}$ if I is generated by a system of parameters and the integral closure of I is $\mathfrak{m}$ (this is slightly different from the usual definition, but is easily seen to be equivalent). A minimal reduction of $\mathfrak{m}$ always exists if $K = R/\mathfrak{m}$ is an infinite field. The only thing we will use about minimal reductions is that $e(I, R) = e_R$. When R is a Cohen-Macaulay ring and I is an ideal generated by a system of parameters, we always have $e(I, R) = l_R(R/I)$, in particular, when I is a minimal reduction of $\mathfrak{m}$, $e_R = l_R(R/I)$.

We use "MCM" to denote "maximal Cohen-Macaulay module" over the local ring R, i.e., a finitely generated R-module M such that $\operatorname{depth}_{\mathfrak{m}} M = \dim R$. We use $\nu_R(\cdot)$ to denote the minimal number of generators of a module over R ($\nu(\cdot)$ when R is clear from the context). We use $\operatorname{edim} R$ to mean the embedding dimension of R, i.e., $\operatorname{edim} R = \dim_K \mathfrak{m}/\mathfrak{m}^2$. The associated graded ring of R with respect to $\mathfrak{m}$ will be denoted by $gr_{\mathfrak{m}}R$. A module M over R is said to have *finite flat dimension* (resp. *finite projective dimension*) if there is a finite resolution of M by flat (resp. projective) R-modules. We use the notation $fd_R M < \infty$ (resp. $pd_R M < \infty$).

We will use $E_R(K)$ or simply E_R to denote the injective hull of the residue field $K = R/\mathfrak{m}$ of R. We define $M^\vee = \operatorname{Hom}_R(M, E_R)$ to be the Matlis dual of an R-module M. We will use ω_R to mean the canonical module of an local ring R, that is, $\omega_R^\vee = H^d_{\mathfrak{m}}(R)$. Canonical modules exist under very mild conditions, for example when R is a homomorphic image of a Gorenstein ring (e.g., when R is complete). In Chapter III and IV we need to understand some of the theory of canonical modules for non-Cohen-Macaulay rings and also the definition for possibly non-local rings. We will explain in detail when we use these notions.

Finally we recall the definition and basic properties of excellent rings. We say a homomorphism $R \to S$ of Noetherian rings is *geometrically regular* if it is flat and all the fibers $\kappa_P (= R_P/PR_P) \to \kappa_P \otimes S$ are geometrically regular (i.e., $\kappa'_P \otimes S$ is regular for every algebraic field extension κ' of κ). An *excellent* ring is a universally catenary

Noetherian ring such that in every finitely generated R-algebra S, the singular locus $\{P \in \operatorname{Spec} S\colon\ S_P \text{ is not regular}\}$ is Zariski closed, and for every local ring A of R, the map $A \to \widehat{A}$ is geometrically regular. In this thesis we will use the definition of excellent rings as well as two important facts about excellent rings: that every complete local or F-finite ring is excellent, and that every algebra essentially of finite type over an excellent ring is still excellent. We refer to [46] and [37] for details about excellent rings.

CHAPTER III

Frobenius structure on local cohomology

One of our interests in studying the Frobenius structure on local cohomology modules is to understand when a local ring $(R, \mathfrak{m})$ of equal characteristic $p > 0$ has the property that there are only finitely many F-stable submodules for each $H^i_{\mathfrak{m}}(R)$, $1 \leq i \leq \dim R$. Rings with this property are called *FH-finite* and have been studied in [12] and [58]. Our first goal in this chapter is to show that for an F-pure local ring $(R, \mathfrak{m})$, all local cohomology modules $H^i_{\mathfrak{m}}(R)$ have only finitely many F-stable submodules. This answers positively the open question raised by Enescu and Hochster in [12]. We will also discuss recent applications of this result. Most results in this Chapter have appeared in my papers [43] and [45].

3.1 FH-finite, FH-finite length and anti-nilpotency

Definition III.1 (*cf.* Definition 2.5 in [12])**.** A local ring $(R, \mathfrak{m})$ of dimension d is called *FH-finite* if for all $0 \leq i \leq d$, there are only finitely many F-stable submodules of $H^i_{\mathfrak{m}}(R)$. We say $(R, \mathfrak{m})$ has *FH-finite length* if for each $0 \leq i \leq d$, $H^i_{\mathfrak{m}}(R)$ has finite length in the category of $R\{F\}$-modules.

It was proved in [12] that an F-pure Gorenstein ring is FH-finite (see Theorem 3.7 in [12]). This also follows from results in [58]. It was then asked in [12] whether the F-pure property itself is enough for FH-finiteness (see Discussion 4.4 in [12]). In order to attack this question, Enescu and Hochster introduced the anti-nilpotency condition for $R\{F\}$-modules in [12], which turns out to be very useful. In fact, it is proved in [12] that the anti-nilpotency of $H^i_{\mathfrak{m}}(R)$ for all i is equivalent to the condition that all power series rings over R be FH-finite.

Definition III.2. Let $(R, \mathfrak{m})$ be a local ring and let W be an $R\{F\}$-module. We say W is *anti-nilpotent* if for every F-stable submodule $V \subseteq W$, F acts injectively on W/V.

Theorem III.3 (*cf.* Theorem 4.15 in [12])**.** *Let $(R, \mathfrak{m})$ be a local ring and let $x_1, \ldots, x_n$ be formal power series indeterminates over R. Let $R_0 = R$ and $R_n = R[[x_1, \ldots, x_n]]$. Then the following conditions on R are equivalent:*

1. *All local cohomology modules $H^i_{\mathfrak{m}}(R)$ are anti-nilpotent.*
2. *The ring R_n is FH-finite for every n.*
3. *$R_1 \cong R[[x]]$ has FH-finite length.*

When R satisfies these equivalent conditions, we call it stably FH-finite.

We will also need some results in [6] about Cartier modules. We recall some definitions in [6]. The definitions and results in [6] work for schemes and sheaves, but we will only give the corresponding definitions for local rings for simplicity (we will not use the results on schemes and sheaves).

Definition III.4. A *Cartier module* over R is an R-module equipped with a p^{-1} linear map $C_M \colon M \to M$, that is, an additive map satisfying $C(r^p x) = rC(x)$ for every $r \in R$ and $x \in M$. A Cartier module (M, C) is called *nilpotent* if $C^e(M) = 0$ for some e.

Remark III.5. 1. A Cartier module is precisely a right module over the ring $R\{F\}$ (see [60] for corresponding properties of right $R\{F\}$-modules). In chapter IV, we will study right $R\{F\}$-modules in detail when R is regular.

2. If (M, C) is a Cartier module, then $C_P\colon M_P \to M_P$ defined by

$$C_P(\frac{x}{r}) = \frac{C(r^{p-1}x)}{r}$$

for every $x \in M$ and $r \in R - P$ gives M_P a Cartier module structure over R_P.

Next we recall the notion of *Frobenius closure*: for any ideal $I \subseteq R$, $I^F = \{x \in R | x^{p^e} \in I^{[p^e]}$ for some $e\}$. If R is F-pure, then every ideal is Frobenius closed. We will see that under mild conditions on the ring, the converse also holds [32].

We also need the notion of *approximately Gorenstein ring* introduced in [26]: $(R, \mathfrak{m})$ is approximately Gorenstein if there exists a decreasing sequence of $\mathfrak{m}$-primary ideals $\{I_t\}$ such that every R/I_t is a Gorenstein ring and the $\{I_t\}$ are cofinal with the powers of $\mathfrak{m}$. That is, for every $N > 0$, $I_t \subseteq \mathfrak{m}^N$ for all $t \gg 1$. We will call such a sequence of ideals an *approximating sequence of ideals*. Note that for an $\mathfrak{m}$-primary ideal I, R/I is Gorenstein if and only if I is an irreducible ideal, i.e., it is not the intersection of two strictly larger ideals. Every reduced excellent local ring is approximately Gorenstein [26]. The following lemma is well-known. We give a proof because we cannot find a good reference.

Lemma III.6. *Let $(R, \mathfrak{m})$ be an approximately Gorenstein ring (e.g., R is reduced and excellent). The following are equivalent:*

1. *R is F-pure.*

2. *Every ideal is Frobenius closed.*

3. *There exists an approximating sequence of ideals $\{I_t\}$ such that $I_t^F = I_t$.*

Proof. The only nontrivial direction is (3) $\Rightarrow$ (1). We want to show $R \to R^{(1)}$ is pure when $I_t^F = I_t$. It suffices to show that $E_R \hookrightarrow R^{(1)} \otimes_R E_R$ is injective where E_R denotes the injective hull of the residue field of R. But it is easy to check that $E_R = \varinjlim_t \dfrac{R}{I_t}$. Hence $E_R \hookrightarrow R^{(1)} \otimes_R E_R$ is injective if $\dfrac{R}{I_t} \hookrightarrow \dfrac{R^{(1)}}{I_t R^{(1)}}$ is injective for all t. But this is true because $I_t^F = I_t$. $\square$

We end this section with a simple lemma which will reduce most problems to the F-split case (recall that for complete local rings, F-pure is equivalent to F-split).

Lemma III.7 (*cf.* Lemma 2.7(a) in [12])**.** *Let $(R, \mathfrak{m})$ be a local ring. Then R has FH-finite length (resp. is FH-finite or stably FH-finite) if and only if $\widehat{R}$ has FH-finite length (resp. is FH-finite or stably FH-finite).*

3.2 F-pure implies stably FH-finite

In order to prove the main result, we begin with some simple Lemmas III.8, III.9, III.10 and a Proposition III.11 which are characteristic free. In fact, in all these lemmas we only need to assume I is a finitely generated ideal in a (possibly non-Noetherian) ring R so that the Čech complex characterization of local cohomology can be applied (the proof will be the same). However, we only state these results when R is Noetherian.

Lemma III.8. *Let R be a Noetherian ring, I be an ideal of R and M be any R-module. We have a natural map:*

$$M \otimes_R H_I^i(R) \xrightarrow{\phi} H_I^i(M)$$

Moreover, when $M = S$ is an R-algebra, ϕ is S-linear.

Proof. Given maps of R-modules $L_1 \xrightarrow{\alpha} L_2 \xrightarrow{\beta} L_3$ and $M \otimes_R L_1 \xrightarrow{id\otimes\alpha} M \otimes_R L_2 \xrightarrow{id\otimes\beta}$

$M \otimes_R L_3$ such that $\beta \circ \alpha = 0$, there is a natural map:

$$M \otimes_R \frac{\ker \beta}{\operatorname{im} \alpha} \to \frac{\ker(id \otimes \beta)}{\operatorname{im}(id \otimes \alpha)}$$

sending $m \otimes \overline{z}$ to $\overline{m \otimes z}$. Now the result follows immediately by the Čech complex characterization of local cohomology. $\square$

Lemma III.9. *Let R be a Noetherian ring, S be an R-algebra, and I be an ideal of R. We have a commutative diagram:*

$$\begin{tikzcd} & S \otimes_R H^i_I(R) \arrow[d, "\phi"] \\ H^i_I(R) \arrow[ur, "j_2"] \arrow[r, "j_1"] & H^i_{IS}(S) \end{tikzcd}$$

where j_1, j_2 are the natural maps induced by $R \to S$. In particular, j_2 sends z to $1 \otimes z$.

Proof. This is straightforward to check. $\square$

Lemma III.10. *Let R be a Noetherian ring and S be an R-algebra such that the inclusion $\iota\colon R \hookrightarrow S$ splits. Let γ be the splitting $S \to R$. Then we have a commutative diagram:*

$$\begin{tikzcd} & S \otimes_R H^i_I(R) \arrow[dl, "q_2"'] \arrow[d, "\phi"] \\ H^i_I(R) & H^i_{IS}(S) \arrow[l, "q_1"] \end{tikzcd}$$

where q_1, q_2 are induced by γ, in particular q_2 sends $s \otimes z$ to $\gamma(s)z$.

Proof. We may identify S with $R \oplus W$ and $R \hookrightarrow S$ with $R \hookrightarrow R \oplus W$ which sends r to $(r, 0)$, and $S \to R$ with $R \oplus W \to R$ which sends (r, w) to r (we may take W to be the R-submodule of S generated by $s - \iota \circ \gamma(s)$). Under this identification, we have:

$$S \otimes_R H^i_I(R) = H^i_I(R) \oplus W \otimes_R H^i_I(R)$$

$$H^i_{IS}(S) = H^i_I(R) \oplus H^i_I(W)$$

and q_1, q_2 are just the projections onto the first factors. Now the conclusion is clear because by Lemma III.8, ϕ: $S \otimes_R H^i_I(R) \to H^i_{IS}(S)$ is the identity on $H^i_I(R)$ and sends $W \otimes_R H^i_I(R)$ to $H^i_I(W)$. $\square$

Proposition III.11. *Let R be a Noetherian ring and S be an R-algebra such that $R \hookrightarrow S$ splits. Let y be an element in $H^i_I(R)$ and N be a submodule of $H^i_I(R)$. If the image of y is in the S-span of the image of N in $H^i_{IS}(S)$, then $y \in N$.*

Proof. We know there are two commutative diagrams as in Lemma III.9 and III.10 (note that here j_1 and j_2 are inclusions since $R \hookrightarrow S$ splits). We use γ to denote the splitting $S \to R$. The condition says that $j_1(y) = \sum s_k \cdot j_1(n_k)$ for some $s_k \in S$ and $n_k \in N$. Applying q_1 we get:

$$\begin{aligned} y &= q_1 \circ j_1(y) \\ &= \sum q_1(s_k \cdot j_1(n_k)) \\ &= \sum q_1(s_k \cdot \phi \circ j_2(n_k)) \\ &= \sum q_1 \circ \phi(s_k \cdot j_2(n_k)) \\ &= \sum q_2(s_k \otimes n_k) \\ &= \sum \gamma(s_k) \cdot n_k \in N \end{aligned}$$

where the first equality is by definition of q_1, the third equality is by Lemma III.9, the fourth equality is because ϕ is S-linear, the fifth equality is by Lemma III.10 and the definition of j_2 and the last equality is by the definition of q_2. This finishes the proof. $\square$

Now we return to the situation in which we are interested. We assume $(R, \mathfrak{m})$ is a Noetherian local ring of equal characteristic $p > 0$. We first prove an immediate corollary of Proposition III.11, which explains how FH-finite and stably FH-finite properties behave under split maps.

Corollary III.12. *Suppose $(R,\mathfrak{m}) \hookrightarrow (S,\mathfrak{n})$ is split and $\mathfrak{m}S$ is primary to $\mathfrak{n}$. Then if S is FH-finite (respectively, stably FH-finite), so is R.*

Proof. First notice that, when $R \hookrightarrow S$ is split, so is $R[[x_1,\ldots,x_n]] \hookrightarrow S[[x_1,\ldots,x_n]]$. So it suffices to prove the statement for FH-finite. Since $\mathfrak{m}S$ is primary to $\mathfrak{n}$, for every i, we have a natural commutative diagram:

$$\begin{array}{ccc} H^i_{\mathfrak{m}}(R) & \longrightarrow & H^i_{\mathfrak{n}}(S) \\ \big\downarrow F & & \big\downarrow F \\ H^i_{\mathfrak{m}}(R) & \longrightarrow & H^i_{\mathfrak{n}}(S) \end{array}$$

where the horizontal maps are induced by the inclusion $R \hookrightarrow S$, and the vertical maps are the Frobenius action. It is straightforward to check that if N is an F-stable submodule of $H^i_{\mathfrak{m}}(R)$, then the S-span of N is also an F-stable submodule of $H^i_{\mathfrak{n}}(S)$.

If N_1 and N_2 are two different F-stable submodules of $H^i_{\mathfrak{m}}(R)$, then their S-spans in $H^i_{\mathfrak{n}}(S)$ must be different by Proposition III.11. But since S is FH-finite, each $H^i_{\mathfrak{n}}(S)$ only has finitely many F-stable submodules. Hence so is $H^i_{\mathfrak{m}}(R)$. This finishes the proof. $\square$

Now we start proving our main result. First we prove a lemma:

Lemma III.13. *Let W be an $R\{F\}$-module. Then W is anti-nilpotent if and only if for every $y \in W$, $y \in \operatorname{span}_R\langle F(y), F^2(y), F^3(y), \ldots\ldots\rangle$.*

Proof. Suppose W is anti-nilpotent. For each $y \in W$,

$$V := \operatorname{span}_R\langle F(y), F^2(y), F^3(y), \ldots\ldots\rangle$$

is an F-stable submodule of W. Hence, F acts injectively on W/V by anti-nilpotency of W. But clearly $F(\overline{y}) = 0$ in W/V, so $\overline{y} = 0$, so $y \in V$.

For the other direction, suppose there exists some F-stable submodule $V \subseteq W$ such that F does not act injectively on W/V. We can pick some $y \notin V$ such that $F(y) \in V$. Since V is an F-stable submodule and $F(y) \in V$,

$$\mathrm{span}_R\langle F(y), F^2(y), F^3(y), \ldots\ldots \rangle \subseteq V.$$

So

$$y \in \mathrm{span}_R\langle F(y), F^2(y), F^3(y), \ldots\ldots \rangle \subseteq V$$

which is a contradiction. □

Theorem III.14. *Let $(R, \mathfrak{m})$ be a local ring which is F-split. Then $H^i_{\mathfrak{m}}(R)$ is anti-nilpotent for every i.*

Proof. By Lemma III.13, it suffices to show for every $y \in H^i_{\mathfrak{m}}(R)$, we have

$$y \in \mathrm{span}_R\langle F(y), F^2(y), F^3(y), \ldots\ldots \rangle.$$

Let $N_j = \mathrm{span}_R\langle F^j(y), F^{j+1}(y) \ldots\ldots \rangle$, consider the descending chain:

$$N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots\cdots \supseteq N_j \supseteq \cdots\cdots$$

Since $H^i_m(R)$ is Artinian, this chain stabilizes, so there exists a smallest e such that $N_e = N_{e+1}$. If $e = 0$ we are done. Otherwise we have $F^{e-1}(y) \notin N_e$. Since R is F-split, we apply Proposition III.11 to the Frobenius map $R \xrightarrow{r \to r^p} R = S$ (and $I = \mathfrak{m}$). In order to make things clear we use S to denote the target R, but we keep in mind that $S = R$.

From Proposition III.11 we know that the image of $F^{e-1}(y)$ is not contained in the S-span of the image of N_e under the map $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}S}(S) \cong H^i_{\mathfrak{m}}(R)$. But this map is exactly the Frobenius map on $H^i_{\mathfrak{m}}(R)$, so the image of $F^{e-1}(y)$ is $F^e(y)$, and after identifying S with R, the S-span of the image of N_e is the R-span of $F^{e+1}(y), F^{e+2}(y), F^{e+3}(y), \ldots\ldots$ which is N_{e+1}. So $F^e(y) \notin N_{e+1}$, which contradicts our choice of e. $\square$

Theorem III.15. *Let $(R, \mathfrak{m})$ be an F-pure local ring. Then R and all power series rings over R are FH-finite (i.e., R is stably FH-finite).*

Proof. We first show that every $\mathfrak{m}$-primary ideal in $\widehat{R}$ is Frobenius closed. Since there is a one-one correspondence between $\mathfrak{m}$-primary ideals in R and $\widehat{R}$, it suffices to prove that $I\widehat{R}$ is Frobenius closed for every $\mathfrak{m}$-primary $I \subseteq R$. Suppose there exists x such that $x^q \in (I\widehat{R})^{[q]} = I^{[q]}\widehat{R}$ but $x \notin I\widehat{R}$. Pick $y \in R$ with $y \equiv x \bmod I$, so $y^q \equiv x^q \bmod I^{[q]}$. We still have $y \notin I\widehat{R}$. But $y^q \in I^{[q]}\widehat{R} \cap R = I^{[q]}$, so $y \in I^F = I \subseteq I\widehat{R}$ which is a contradiction.

Next we observe that every $\mathfrak{m}$-primary ideal in $\widehat{R}$ is Frobenius closed easily implies $\widehat{R}$ is reduced. So we know that $\widehat{R}$ is excellent and reduced, hence it is approximately Gorenstein [26]. By Lemma III.6, $\widehat{R}$ is F-pure and hence F-split (the point is that we don't need to assume R is excellent in this argument).

Because $\widehat{R}$ is F-split, we can apply Theorem III.14, Theorem III.3 and Lemma III.7, and we get that R is stably FH-finite. $\square$

3.3 F-pure on the punctured spectrum implies FH-finite length

In this section, we will finally prove that for excellent local rings, F-pure on the punctured spectrum implies FH-finite length. We first show that for a complete and F-finite local ring $(R, \mathfrak{m})$, the condition that R_P be stably FH-finite for all $P \in \operatorname{Spec} R - \{\mathfrak{m}\}$ is equivalent to the condition that R have FH-finite length. Then we make use of the Γ-construction introduced in [30] to prove the general case. We also prove that the properties such as having FH-finite length, being FH-finite, and being stably FH-finite localize.

First we recall the following important theorem of Lyubeznik:

Theorem III.16 (*cf.* Theorem 4.7 in [41] or Theorem 4.7 in [12])**.** *Let W be an $R\{F\}$-module which is Artinian as an R-module. Then W has a finite filtration*

$$0 = L_0 \subseteq N_0 \subseteq L_1 \subseteq N_1 \subseteq \cdots \subseteq L_s \subseteq N_s = W \tag{3.1}$$

by F-stable submodules of W such that every N_j/L_j is F-nilpotent, while every L_j/N_{j-1} is simple in the category of $R\{F\}$-modules, with a nonzero Frobenius action. The integer s and the isomorphism classes of the modules L_j/N_{j-1} are invariants of W.

The following proposition in [12] characterizes being anti-nilpotent and having finite length in the category of $R\{F\}$-modules in terms of Lyubeznik's filtration:

Proposition III.17 (*cf.* Proposition 4.8 in [12])**.** *Let the notations and hypothesis be as in Theorem III.16. Then:*

1. *W has finite length as an $R\{F\}$-module if and only if each of the factors N_j/L_j has finite length in the category of R-modules.*

2. *W is anti-nilpotent if and only if in some (equivalently, every) filtration, the nilpotent factors $N_j/L_j = 0$ for every j.*

Remark III.18. It is worth pointing out that an Artinian R-module W is Noetherian over $R\{F\}$ if and only if in some (equivalently, every) filtration as in Theorem III.16, each of the factors N_j/L_j is Noetherian as an R-module. So W is Noetherian over $R\{F\}$ if and only if it has finite length as an $R\{F\}$-module. Hence R has FH-finite length if and only if all local cohomology modules $H^i_{\mathfrak{m}}(R)$ are Noetherian $R\{F\}$-modules.

We also need the following important theorem in [6] which relates $R\{F\}$-modules and Cartier modules. This result was also proved independently by Sharp and Yoshino in [60] in the language of left and right $R\{F\}$-modules.

Theorem III.19 (*cf.* Proposition 5.2 in [6] and Corollary 1.21 in [60])**.** *Let $(R,\mathfrak{m})$ be complete, local and F-finite. Then Matlis duality induces an equivalence of categories between $R\{F\}$-modules which are Artinian as R-modules and Cartier modules which are Noetherian as R-modules. The equivalence preserves nilpotence.*

We will use $^\vee$ to denote the Matlis dual over R and $^{\vee_P}$ to denote the Matlis dual over PR_P. We begin by proving some lemmas.

Lemma III.20. *Let $(R,\mathfrak{m})$ be a complete local ring. We have*

$$(H^i_{\mathfrak{m}}(R)^\vee)_P^{\vee_P} \cong H^{i-\dim R/P}_{PR_P}(R_P).$$

Proof. Write $R = T/J$ and $P = Q/J$ for T a regular local ring of dimension n. By local duality, we have

$$(H^i_{\mathfrak{m}}(R)^\vee)_P \cong \mathrm{Ext}^{n-i}_T(R,T)_P \cong \mathrm{Ext}^{n-i}_{T_Q}(R_P,T_Q).$$

Now by local duality over R_P,

$$(H^i_{\mathfrak{m}}(R)^\vee)_P^{\vee_P} \cong H^{\dim T_Q-(n-i)}_{PR_P}(R_P) \cong H^{i-\dim R/P}_{PR_P}(R_P).$$

$\square$

Lemma III.21. *We have the following:*

1. *If M is a nilpotent Cartier module over R, then M_P is a nilpotent Cartier module over R_P*

2. *If (M, C) is a simple Cartier module over R with a nontrivial C-action, then (M_P, C_P) is a simple Cartier module over R_P, and if $M_P \neq 0$, then the C_P-action is also nontrivial.*

Proof. (1) is obvious, because if C^e kills M, then C_P^e kills M_P. Now we prove (2). Let N be a Cartier R_P submodule of M_P. Consider the contraction of N in M, call it N'. Then it is easy to check that N' is a Cartier R-submodule of M. So it is either 0 or M because M is simple. But if $N' = 0$ then $N = 0$ and if $N' = M$ then $N = M_P$ because N is an R_P-submodule of M_P. This proves M_P is simple as a Cartier module over R_P. To see the last assertion, notice that if M is a simple Cartier module with a nontrivial C-action, then $C\colon M \to M$ must be surjective: otherwise the image would be a proper Cartier submodule. Hence $C_P\colon M_P \to M_P$ is also surjective. But we assume $M_P \neq 0$, so C_P is a nontrivial action. □

Our first main theorem in this section is the following:

Theorem III.22. *Let $(R, \mathfrak{m})$ be a complete and F-finite local ring. Then the following conditions are equivalent:*

1. *R_P is stably FH-finite for every $P \in \operatorname{Spec} R - \{\mathfrak{m}\}$.*

2. *R has FH-finite length.*

Proof. By Theorem III.16, for every $H^i_{\mathfrak{m}}(R)$, $0 \leq i \leq d$, we have a filtration

$$0 = L_0 \subseteq N_0 \subseteq L_1 \subseteq N_1 \subseteq \cdots \subseteq L_s \subseteq N_s = H^i_{\mathfrak{m}}(R) \tag{3.2}$$

of $R\{F\}$-modules such that every N_j/L_j is F-nilpotent while every L_j/N_{j-1} is simple in the category of $R\{F\}$-modules, with nontrivial F-action. Now we take the Matlis dual of the above filtration (3.2), we have

$$H^i_{\mathfrak{m}}(R)^\vee = N_s^\vee \twoheadrightarrow L_s^\vee \twoheadrightarrow \cdots \twoheadrightarrow N_0^\vee \twoheadrightarrow L_0^\vee = 0$$

such that each $\ker(L_j^\vee \twoheadrightarrow N_{j-1}^\vee)$ is Noetherian as an R-module and is simple as a Cartier module with nontrivial C-action, and each $\ker(N_j^\vee \twoheadrightarrow L_j^\vee)$ is Noetherian as an R-module and is nilpotent as a Cartier module by Lemma III.19. When we localize at $P \neq \mathfrak{m}$, we get

$$(H^i_{\mathfrak{m}}(R)^\vee)_P = (N_s^\vee)_P \twoheadrightarrow (L_s^\vee)_P \twoheadrightarrow \cdots \twoheadrightarrow (N_0^\vee)_P \twoheadrightarrow (L_0^\vee)_P = 0$$

with each $\ker((L_j^\vee)_P \twoheadrightarrow (N_{j-1}^\vee)_P)$ a simple Cartier module over R_P whose C_P action is nontrivial if it is nonzero, and each $\ker((N_j^\vee)_P \twoheadrightarrow (L_j^\vee)_P)$ a nilpotent Cartier module over R_P by Lemma III.21. Now, when we take the Matlis dual over R_P, we get a filtration of $R_P\{F\}$-modules

$$0 = L'_0 \subseteq N'_0 \subseteq L'_1 \subseteq \cdots \subseteq L'_s \subseteq N'_s = H^{i-\dim R/P}_{PR_P}(R_P) \tag{3.3}$$

where $L'_j = (L_j^\vee)_P^{\vee_P}$, $N'_j = (N_j^\vee)_P^{\vee_P}$, N'_j/L'_j is F-nilpotent and each L'_j/N'_{j-1} is either 0 or simple as an $R_P\{F\}$-module with nontrivial F-action by Lemma III.19 again. And we notice that

$$\begin{aligned}
& N'_j/L'_j = 0, \forall P \in \operatorname{Spec} R - \{\mathfrak{m}\} \\
\Leftrightarrow\ & (N_j^\vee)_P^{\vee_P}/(L_j^\vee)_P^{\vee_P} = 0, \forall P \in \operatorname{Spec} R - \{\mathfrak{m}\} \\
\Leftrightarrow\ & \ker((N_j^\vee)_P \twoheadrightarrow (L_j^\vee)_P) = 0, \forall P \in \operatorname{Spec} R - \{\mathfrak{m}\} \\
\Leftrightarrow\ & l_R(\ker(N_j^\vee \twoheadrightarrow L_j^\vee)) < \infty \\
\Leftrightarrow\ & l_R(N_j/L_j) < \infty
\end{aligned}$$

R_P is stably FH-finite for every $P \in \operatorname{Spec} R - \{\mathfrak{m}\}$ if and only if $H^{i-\dim R/P}_{PR_P}(R_P)$ is anti-nilpotent for every $0 \le i \le d$ and every $P \in \operatorname{Spec} R - \{\mathfrak{m}\}$. This is because when $0 \le i \le d$, $i - \dim R/P$ can take all values between 0 and $\dim R_P$, and if $i - \dim R/P$ is out of this range, then the local cohomology is 0 so it is automatically anti-nilpotent. By Proposition III.17, this is equivalent to the condition that for every $P \in \operatorname{Spec} R - \{\mathfrak{m}\}$, the corresponding N'_j/L'_j is 0. By the above chain of equivalence relations, this is equivalent to the condition that each N_j/L_j have finite length as an R-module. By Proposition III.17, this is equivalent to the condition that R have FH-finite length. $\square$

Corollary III.23. *Let $(R, \mathfrak{m})$ be a complete and F-finite local ring. If R_P is F-pure for every $P \neq \mathfrak{m}$, then R has FH-finite length.*

Proof. This is clear from Theorem III.15 and Theorem III.22. $\square$

Our next goal is to use the Γ-construction introduced by Hochster and Huneke in [30] to generalize Corollary III.23 to all excellent local rings. We first recall the Γ-construction.

Let K be a field of positive characteristic $p > 0$ with a p-base Λ. Let Γ be a fixed cofinite subset of Λ. For $e \in \mathbb{N}$ we denote by $K^{\Gamma,e}$ the purely inseparable field extension of K that is the result of adjoining p^e-th roots of all elements in Γ to K.

Now suppose that $(R, \mathfrak{m})$ is a complete local ring with $K \subseteq R$ a coefficient field. Let $x_1, \ldots, x_d$ be a system of parameters for R, so that R is module-finite over $A = K[[x_1, \ldots, x_d]] \subseteq R$. Let A^Γ denote

$$\bigcup_{e \in \mathbb{N}} K^{\Gamma,e}[[x_1, \ldots, x_d]],$$

which is a regular local ring that is faithfully flat and purely inseparable over A. The maximal ideal of A expands to that of A^Γ. Let R^Γ denote $A^\Gamma \otimes_A R$, which is module-finite over the regular ring A^Γ and is faithfully flat and purely inseparable over R. The maximal ideal of R expands to the maximal ideal of R^Γ. The residue field of R^Γ is $K^\Gamma = \bigcup_{e \in \mathbb{N}} K^{\Gamma,e}$. It is of great importance that R^Γ is F-finite. Moreover, we can preserve some good properties of R if we choose a sufficiently small cofinite subset Γ:

Lemma III.24 (*cf.* Lemma 6.13 in [30])**.** *Let R be a complete local ring. If P is a prime ideal of R then there exists a cofinite set $\Gamma_0 \subseteq \Lambda$ such that $Q = PR^{\Gamma}$ is a prime ideal in R^{Γ} for all $\Gamma \subseteq \Gamma_0$.*

Lemma III.25 (*cf.* Lemma 2.9 and Lemma 4.3 in [12])**.** *Let R be a complete local ring. Let W be an $R\{F\}$-module that is Artinian as an R-module such that the F-action is injective. Then for any sufficiently small choice of Γ cofinite in Λ, the action of F on $R^{\Gamma} \otimes_R W$ is also injective. Moreover, if R^{Γ} is FH-finite (resp. has FH-finite length), then so is R.*

Now we start proving our main theorems. We first show that F-purity is preserved under nice base change. This is certainly well-known to experts. We refer to [21] and [55] for some (even harder) results on base change problems. Since in most of these references the results are only stated for F-finite rings, we provide a proof which works for all excellent rings. In fact, the proof follows from essentially the same argument as in the proof of Theorem 7.24 in [30].

Proposition III.26. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a faithfully flat extension of excellent local rings such that the closed fibre $S/\mathfrak{m}S$ is Gorenstein and F-pure. If R is F-pure, then S is F-pure.*

Proof. Since R is excellent and reduced, R is approximately Gorenstein. Let $\{I_k\}$ be an approximating sequence of ideals in R. Let $x_1, \ldots, x_n$ be elements in S such that their image form a system of parameters in $S/\mathfrak{m}S$. Since $S/\mathfrak{m}S$ is Gorenstein, $(x_1^t, \ldots, x_n^t)$ is an approximating sequence of ideals in $S/\mathfrak{m}S$. Therefore $I_k + (x_1^t, \ldots, x_n^t)$ is an approximating sequence of ideals in S (see the proof of Theorem

7.24 in [30]).

To show that S is F-pure, it suffices to show every $I_k + (x_1^t, \ldots, x_n^t)$ is Frobenius closed, by Lemma III.6. Therefore we reduce to showing that if I is an irreducible ideal primary to $\mathfrak{m}$ in R and $x_1, \ldots, x_n$ are elements in S such that their image form a system of parameters in $S/\mathfrak{m}S$, then $(IS + (x_1, \ldots, x_n))^F = IS + (x_1, \ldots, x_n)$ in S.

Let v and w be socle representatives of R/I and $(S/\mathfrak{m}S)/(x_1, \ldots, x_n)(S/\mathfrak{m}S)$ respectively. It suffices to show $vw \notin (IS + (x_1, \ldots, x_n))^F$. Suppose we have

$$v^q w^q \in I^{[q]}S + (x_1^q, \ldots, x_n^q).$$

Taking images in $S/(x_1^q, \ldots, x_n^q)$, we have

$$\overline{v}^q \overline{w}^q \in \overline{I^{[q]}S},$$

therefore

$$\overline{w}^q \in (\overline{I^{[q]}S} : \overline{v}^q) = (I^{[q]} : v^q)(S/(x_1^q, \ldots, x_n^q)S)$$

where the second equality is because $S/(x_1^q, \ldots, x_n^q)S$ is faithfully flat over R ([46]). So we have

$$w^q \in (I^{[q]} : v^q)S + (x_1^q, \ldots, x_n^q).$$

Since R is F-pure, $(I^{[q]} : v^q) \in \mathfrak{m}$. So after taking images in $S/\mathfrak{m}S$, we get that

$$w^q \in (x_1^q, \ldots, x_n^q)(S/\mathfrak{m}S).$$

Hence $w \in (x_1^q, \ldots, x_n^q)^F$ in $S/\mathfrak{m}S$, which contradicts the condition that $S/\mathfrak{m}S$ be F-pure. $\square$

Next we want to observe that for an exact sequence of finitely generated R-modules

$$0 \to M \to N \to Q \to 0,$$

the non-split locus is the support of $\operatorname{Hom}_R(Q,Q)/\operatorname{im}(\operatorname{Hom}_R(Q,N))$. In particular the non-split locus is always closed. This easily implies the following:

Lemma III.27. *Let $(R,\mathfrak{m})$ be an F-finite local ring. Then the F-pure locus $\{P \in \operatorname{Spec} R | R_P$ is F-pure $\}$ is open in $\operatorname{Spec} R$.*

Proof. The F-pure locus is the complement of the non-split locus of the exact sequence $0 \to R \to R^{(1)} \to R^{(1)}/R \to 0$. Since R is F-finite, this is an exact sequence of finitely generated R-modules. Hence the non-split locus is closed, and so the F-pure locus is open. □

Now we show that F-purity on the punctured spectrum is preserved under the Γ-construction when we pick Γ sufficiently small and cofinite in Λ.

Proposition III.28. *Let $(R,\mathfrak{m})$ be a complete local ring such that R_P is F-pure on the punctured spectrum $\operatorname{Spec} R - \{\mathfrak{m}\}$. Then for any sufficiently small choice of Γ cofinite in Λ, R^Γ is F-pure on the punctured spectrum $\operatorname{Spec} R^\Gamma - \{\mathfrak{m}R^\Gamma\}$.*

Proof. Because R^Γ is purely inseparable over R, for all $P \in \operatorname{Spec} R$ there is a unique prime ideal $P^\Gamma \in \operatorname{Spec} R^\Gamma$ lying over P. In particular, $\operatorname{Spec} R^\Gamma \cong \operatorname{Spec} R$. Since R^Γ is F-finite, we know the F-pure locus of each R^Γ, call it X_Γ, is open in $\operatorname{Spec} R^\Gamma = \operatorname{Spec} R$ by Lemma III.27. Since open sets in $\operatorname{Spec} R$ satisfy ACC, we know that there exists Γ such that X_Γ is maximal. We will show that $X_\Gamma \supseteq \operatorname{Spec} R - \{\mathfrak{m}\}$. This will prove R^Γ is F-pure on the punctured spectrum $\operatorname{Spec} R^\Gamma - \{\mathfrak{m}R^\Gamma\}$.

Suppose there exists $Q \neq \mathfrak{m}$ such that $Q \notin X_\Gamma$. We may pick $\Gamma' \subseteq \Gamma$ sufficiently small and cofinite in Λ such that $QR^{\Gamma'}$ is prime (that is, $QR^{\Gamma'} = Q^{\Gamma'}$) by Lemma III.24. So $R_Q \to R^{\Gamma'}_{Q^{\Gamma'}}$ is faithfully flat whose closed fibre is a field. By Proposition III.26, $R^{\Gamma'}_{Q^{\Gamma'}}$ is F-pure. Since $\Gamma' \subseteq \Gamma$, $R^{\Gamma'} \to R^\Gamma$ is faithfully flat so $R^{\Gamma'}_{P^{\Gamma'}} \to R^\Gamma_{P^\Gamma}$ is faithfully flat for each $P \in \operatorname{Spec} R$. Now for $P \in X_\Gamma$, $R^\Gamma_{P^\Gamma}$ is F-pure, hence so is $R^{\Gamma'}_{P^{\Gamma'}}$. So $X_{\Gamma'} \supseteq X_\Gamma \cup \{Q\}$, which is a contradiction since we assume that X_Γ is maximal. $\square$

The next is the main theorem of this section.

Theorem III.29. *Let* $(R, \mathfrak{m})$ *be an excellent local ring such that* R_P *is* F*-pure for every* $P \in \operatorname{Spec} R - \{\mathfrak{m}\}$*. Then* R *has FH-finite length.*

Proof. We look at the chain of faithfully flat ring extensions:

$$R \to \widehat{R} \to \widehat{R}^\Gamma \to \widehat{\widehat{R}^\Gamma}.$$

Since R is excellent, for every $Q_0 \in \operatorname{Spec} \widehat{R} - \{\mathfrak{m}\widehat{R}\}$ lying over P in R, $R_P \to \widehat{R}_{Q_0}$ has geometrically regular fibres, so $\widehat{R}_{Q_0}$ is F-pure by Proposition III.26. Hence $\widehat{R}$ is F-pure on the punctured spectrum. So by Proposition III.28, we can pick Γ sufficiently small and cofinite in Λ such that $\widehat{R}^\Gamma$ is F-pure on the punctured spectrum.

For every $Q \in \operatorname{Spec} \widehat{\widehat{R}^\Gamma} - \{\mathfrak{m}\widehat{\widehat{R}^\Gamma}\}$, let $Q_1 \neq \mathfrak{m}\widehat{R}^\Gamma$ be the contraction of Q to $\widehat{R}^\Gamma$. Since $\widehat{R}^\Gamma_{Q_1}$ is F-finite, it is excellent ([37]). So the closed fibre of $\widehat{R}^\Gamma_{Q_1} \to \widehat{\widehat{R}^\Gamma}_Q$ is geometrically regular. So $\widehat{\widehat{R}^\Gamma}_Q$ is F-pure by Proposition III.26.

It follows that, for sufficiently small choice of Γ cofinite in Λ, $\widehat{\widehat{R}^\Gamma}_Q$ is F-pure for every $Q \in \operatorname{Spec} \widehat{\widehat{R}^\Gamma} - \{\mathfrak{m}\widehat{\widehat{R}^\Gamma}\}$. Now by Theorem III.15 applied to $\widehat{\widehat{R}^\Gamma}_Q$ and Theorem III.22 applied to $\widehat{\widehat{R}^\Gamma}$, we know $\widehat{\widehat{R}^\Gamma}$ has FH-finite length. Hence so does R by Lemma III.7 and Lemma III.25. $\square$

We end this section by showing that properties such as being FH-finite and having FH-finite length behave well under localizations. In the case that R is complete and F-finite, the results follows easily from Theorem III.22. Now we prove the general case. We start with a lemma.

Lemma III.30. *Let R be a complete local ring. Let W be a simple $R\{F\}$-module that is Artinian over R with nontrivial F-action. Then for any sufficiently small choice of Γ cofinite in Λ, $W \otimes_R R^\Gamma$ is a simple $R^\Gamma\{F\}$-module with nontrivial F-action.*

Proof. By Lemma III.25, we may pick Γ sufficiently small and cofinite in Λ such that F acts injectively on $W \otimes_R R^\Gamma$. I claim such a $W \otimes_R R^\Gamma$ must be a simple $R^\Gamma\{F\}$-module. If not, then by Theorem III.16, we have $0 \subsetneqq L \subsetneqq W \otimes_R R^\Gamma$ where L is simple with nontrivial Frobenius action. Now we pick $0 \neq x \in L$. Because R^Γ is purely inseparable over R, there exists e such that $0 \neq F^e(x) \in W$. Hence $L \cap W \neq 0$. But it is straightforward to check that $L \cap W$ is an $R\{F\}$-submodule of W. So $L \cap W = W$ since W is simple. Hence $L \supseteq W \otimes_R R^\Gamma$ which is a contradiction. $\square$

Proposition III.31. *Let $(R, \mathfrak{m})$ be a complete local ring. Then*

1. *If R has FH-finite length , then so does R^Γ for Γ sufficiently small and cofinite in Λ.*
2. *If R is stably FH-finite, then so is R^Γ for Γ sufficiently small and cofinite in Λ.*

Proof. By Theorem III.16, for every $0 \leq i \leq d = \dim R$, we have a filtration of $R\{F\}$-modules

$$0 = L_0 \subseteq N_0 \subseteq L_1 \subseteq N_1 \subseteq \cdots \subseteq L_s \subseteq N_s = H^i_{\mathfrak{m}}(R)$$

with each N_j/L_j F-nilpotent and L_j/L_{j-1} simple as an $R\{F\}$-module with nonzero F-action. By Lemma III.30, we can pick Γ sufficiently small and cofinite in Λ such that, for all i, all $L_j/L_{j-1} \otimes_R R^\Gamma$ are simple with nonzero F-action. Hence

$$0 = L_0^\Gamma \subseteq N_0^\Gamma \subseteq L_1^\Gamma \subseteq N_1^\Gamma \subseteq \cdots \subseteq L_s^\Gamma \subseteq N_s^\Gamma = H^i_{\mathfrak{m}}(R^\Gamma)$$

where $L_j^\Gamma = L_j \otimes_R R^\Gamma$ and $N_j^\Gamma = N_j \otimes_R R^\Gamma$ is a corresponding filtration of $H^i_{\mathfrak{m}}(R^\Gamma)$. Now both (1) and (2) are clear from Proposition III.17. $\square$

Theorem III.32. *Let $(R, \mathfrak{m})$ be a local ring that has FH-finite length (resp. is FH-finite or stably FH-finite). Then the same holds for R_P for every $P \in \operatorname{Spec} R$.*

Proof. It suffices to show that if $(R, \mathfrak{m})$ has FH-finite length, then R_P is stably FH-finite for every $P \neq \mathfrak{m}$. We first notice that $\widehat{R}$ has FH-finite length by Lemma III.7. We pick Γ sufficiently small and cofinite in Λ such that $\widehat{R}^{\Gamma}$ still has FH-finite length by Proposition III.31. Now we complete again, and we get that $B = \widehat{\widehat{R}^{\Gamma}}$ is an F-finite complete local ring that has FH-finite length by Lemma III.7, and the maximal ideal in B is $\mathfrak{m}B$. Notice that $R \to B$ is faithfully flat, hence for every $P \neq \mathfrak{m}$, we may pick $Q \in \operatorname{Spec} B - \{\mathfrak{m}B\}$ such that $R_P \to B_Q$ is faithfully flat (in particular, pure) and PB_Q is primary to QB_Q. So $\widehat{R_P} \to \widehat{B_Q}$ is split. By Theorem III.22 applied to B, B_Q is stably FH-finite. Hence so is $\widehat{B_Q}$ by Lemma III.7. Now we apply Corollary III.12, we see that $\widehat{R_P}$ is stably FH-finite. Hence so is R_P by Lemma III.7. □

3.4 Applications to the study of F-pure and F-injective singularities

The results proved in the previous sections have nice applications to the study of F-pure and F-injective singularities. In this section, we will discuss two such applications. The first one greatly generalizes a sufficient condition for F-purity by Enescu [10]. The second one shows that F-purity deforms to F-injectivity, an outstanding special case of the still open conjecture that F-injectivity deforms (this result was first proved in [34], but our approach here is much simpler).

One of the main results in [10] is that: if a Cohen-Macaulay F-injective ring $(R, \mathfrak{m})$ admits a canonical ideal $I \cong \omega_R$ such that R/I is F-pure, then R is F-pure. We want to study this condition carefully, and prove the same conclusion holds without assuming R is Cohen-Macaulay and F-injective: we only to assume R is equidimensional and S_2. The anti-nilpotency condition will be crucial in the proof.

We begin by summarizing some basic properties of canonical modules of non-Cohen-Macaulay local rings (for those we cannot find references, we give proofs). All these properties are characteristic free.

Proposition III.33 (*cf.* Remark 2.2(c) in [31])**.** *Let* $(R, \mathfrak{m})$ *be a homomorphic image of a Gorenstein local ring* $(S, \mathfrak{n})$*. Then* $\mathrm{Ext}_S^{\dim S - \dim R}(R, S) \cong \omega_R$*.*

Lemma III.34 (*cf.* Lemma 4.2 in [27])**.** *Let* $(R, \mathfrak{m})$ *be a local ring that admits a canonical module* ω_R*. Then every nonzerodivisor in* R *is a nonzerodivisor on* ω_R*.*

Proposition III.35 (*cf.* Corollary 4.3 in [1] or Remark 2.2(i) in [31])**.** *Let* $(R, \mathfrak{m})$ *be a local ring with canonical module* ω_R*. If* R *is equidimensional, then for every* $P \in \operatorname{Spec} R$*,* $(\omega_R)_P$ *is a canonical module for* R_P*.*

The following proposition is well-known to experts. But we include a proof as we cannot find a good reference.

Proposition III.36. *Let* $(R, \mathfrak{m})$ *be a local ring with canonical module* ω_R*. If* R *is equidimensional and unmixed, then the following are equivalent*

1. *There exists an ideal* $I \cong \omega_R$*.*
2. R *is generically Gorenstein (i.e.,* R_P *is Gorenstein for every minimal prime of* R*).*

When the equivalent conditions above hold, I *contains a nonzerodivisor of* R*.*

Proof. Since R is equidimensional, we know that $\omega_{R_P} \cong (\omega_R)_P$ for every prime ideal P of R by Proposition III.35. Let W be the multiplicative system of R consisting of all nonzerodivisors and let Λ be the set of minimal primes of R. Since R is equidimensional and unmixed, W is simply the complement of the union of the minimal primes of R.

(1) $\Rightarrow$ (2): If we have $I \cong \omega_R$, then for every $P \in \Lambda$, $\omega_{R_P} \cong (\omega_R)_P \cong IR_P \subseteq R_P$. But R_P is an Artinian local ring, $l(\omega_{R_P}) = l(R_P)$, so we must have $\omega_{R_P} \cong R_P$. Hence R_P is Gorenstein for every $P \in \Lambda$, that is, R is generically Gorenstein.

(2) $\Rightarrow$ (1): Since R is generically Gorenstein, we know that for $P \in \Lambda$, $\omega_{R_P} \cong R_P$. Now we have:

$$W^{-1}\omega_R \cong \prod_{P \in \Lambda} (\omega_R)_P \cong \prod_{P \in \Lambda} \omega_{R_P} \cong \prod_{P \in \Lambda} R_P \cong W^{-1}R.$$

Therefore we have an isomorphism $W^{-1}\omega_R \cong W^{-1}R$. The restriction of the isomorphism to ω_R then yields an injection $j\colon\ \omega_R \hookrightarrow W^{-1}R$ because elements in W are nonzero divisors on ω_R by Lemma III.34. The images of a finite set of generators of ω_R can be written as r_i/w_i. Let $w = \prod w_i$, we have $wj\colon\ \omega_R \hookrightarrow R$ is an injection. So ω_R is isomorphic to an ideal $I \subseteq R$.

Finally, when these equivalent conditions hold, we know that $W^{-1}I \cong \prod_{P \in \Lambda} R_P$ is free. So $W^{-1}I$ contains a nonzerodivisor. But whether I contains a nonzerodivisor is unaffected by localization at W. So I contains a nonzerodivisor. $\square$

Lemma III.37 (*cf.* Proposition 4.4 in [1] or Remark 2.2(f) in [31])**.** *Let $(R, \mathfrak{m})$ be a local ring with canonical module ω_R. Then ω_R is always S_2, and R is equidimensional and S_2 if and only if $R \to \operatorname{Hom}_R(\omega_R, \omega_R)$ is an isomorphism.*

The next proposition is also well-known to experts. Again we give the proof as we cannot find a good reference.

Proposition III.38. *Let $(R, \mathfrak{m})$ be an equidimensional and unmixed local ring that admits a canonical ideal $I \cong \omega_R$. Then I is a height one ideal and R/I is equidimensional and unmixed.*

Proof. By Proposition III.36, I contains a nonzerodivisor, so its height is at least one. Now we choose a height h associated prime P of I with $h \geq 2$. We localize at

P, PR_P becomes an associated prime of IR_P. In particular, R_P/IR_P has depth 0 so $H^0_{PR_P}(R_P/IR_P) \neq 0$.

However, by Proposition III.35, IR_P is a canonical ideal of R_P, which has dimension $h \geq 2$. Now the long exact sequence of local cohomology gives

$$\to H^0_{PR_P}(R_P) \to H^0_{PR_P}(R_P/IR_P) \to H^1_{PR_P}(IR_P) \to .$$

We have depth $R_P \geq 1$ (I contains a nonzerodivisor) and depth $IR_P \geq 2$ (the canonical module is always S_2 by Lemma III.37). Hence $H^0_{PR_P}(R_P) = H^1_{PR_P}(IR_P) = 0$. The above sequence thus implies $H^0_{PR_P}(R_P/IR_P) = 0$ which is a contradiction.

Hence we have shown that every associated prime of I has height one. Since R is equidimensional, this proves I has height one and R/I is equidimensional and unmixed. □

Proposition III.39 (*cf.* Page 531 of [27])**.** *Let $(R, \mathfrak{m})$ be a local ring of dimension d which admits a canonical module ω_R. Then for every finitely generated R-module M, $H^d_{\mathfrak{m}}(M) \cong \operatorname{Hom}_R(M, \omega_R)^\vee$.*

Remark III.40.

1. When $(R, \mathfrak{m})$ is catenary, R is S_2 implies R is equidimensional. Hence, if we assume R is excellent, then in the statement of Lemma III.37 and Proposition III.39, we don't need to assume R is equidimensional.

2. For example, when $(R, \mathfrak{m})$ is a complete local domain, then both canonical modules and canonical ideals exist. And the canonical ideal must have height one and contains a nonzerodivisor.

Next we recall the following result of Sharp in [59]:

Theorem III.41 (*cf.* Theorem 3.2 in [59])**.** *A local ring $(R, \mathfrak{m})$ is F-pure if and only if E_R has an injective Frobenius action compatible with its R-module structure.*

We will also need the following lemma:

Lemma III.42. *Let $(R,\mathfrak{m})$ be an equidimensional local ring of dimension d that admits a canonical module ω_R. Let I be a height one ideal of R that contains a nonzerodivisor. Then $H^d_{\mathfrak{m}}(I) \to H^d_{\mathfrak{m}}(R)$ induced by $I \hookrightarrow R$ is not injective.*

Proof. By Proposition III.39, to show $H^d_{\mathfrak{m}}(I) \to H^d_{\mathfrak{m}}(R)$ is not injective, it suffices to show

$$\operatorname{Hom}_R(R,\omega_R) \to \operatorname{Hom}_R(I,\omega_R) \tag{3.4}$$

is not surjective.

It suffices to show (3.4) is not surjective after we localize at a height one minimal prime P of I. Since I contains a nonzerodivisor and P is a height one minimal prime of I, it is straightforward to see that R_P is a one-dimensional Cohen-Macaulay ring with IR_P a PR_P-primary ideal. By Proposition III.35, $(\omega_R)_P$ is a canonical module of R_P. Hence to show $\operatorname{Hom}_R(R,\omega_R)_P \to \operatorname{Hom}_R(I,\omega_R)_P$ is not surjective, we can apply Proposition III.39 (taking Matlis dual over R_P) and we see it is enough to prove that

$$H^1_{PR_P}(IR_P) \to H^1_{PR_P}(R_P)$$

is not injective. But this is obvious because we know from the long exact sequence that the kernel is $H^0_{PR_P}(R_P/IR_P)$, which is nonzero because I is PR_P-primary. $\square$

The following result was first proved in [10] using pseudocanonical covers under the hypothesis that R be Cohen-Macaulay and F-injective. We want to drop these conditions and only assume R is equidimensional and S_2 (as in Remark III.40, when R is excellent, we only need to assume R is S_2). Our argument here is quite different. Here is our main result of this section:

Theorem III.43. *Let $(R, \mathfrak{m})$ be an equidimensional and S_2 local ring of dimension d which admits a canonical ideal $I \cong \omega_R$ such that R/I is F-pure. Then R is F-pure.*

Proof. First we note that I is a height one ideal by Proposition III.38. In particular we know that $\dim R/I < \dim R = d$. We have a short exact sequence:

$$0 \to I \to R \to R/I \to 0.$$

Moreover, if we endow I with an $R\{F\}$-module structure induced from R, then the above is also an exact sequence of $R\{F\}$-modules. Hence the tail of the long exact sequence of local cohomology gives an exact sequence of $R\{F\}$-modules, that is, a commutative diagram (we have 0 on the right because $\dim R/I < d$):

$$\begin{tikzcd}
H^{d-1}_{\mathfrak{m}}(R) \arrow[r, "\varphi_1"] \arrow[d, "F"] & H^{d-1}_{\mathfrak{m}}(R/I) \arrow[r, "\varphi_2"] \arrow[d, "F"] & H^{d}_{\mathfrak{m}}(I) \arrow[r, "\varphi_3"] \arrow[d, "F"] & H^{d}_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "F"] & 0 \\
H^{d-1}_{\mathfrak{m}}(R) \arrow[r, "\varphi_1"] & H^{d-1}_{\mathfrak{m}}(R/I) \arrow[r, "\varphi_2"] & H^{d}_{\mathfrak{m}}(I) \arrow[r, "\varphi_3"] & H^{d}_{\mathfrak{m}}(R) \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps denote the Frobenius actions on each module.

Since R is equidimensional and S_2, we know that

$$H^d_{\mathfrak{m}}(I) \cong H^d_{\mathfrak{m}}(\omega_R) \cong \operatorname{Hom}_R(\omega_R, \omega_R)^{\vee} \cong R^{\vee} \cong E_R$$

by Proposition III.39 and Lemma III.37. We want to show that, under the hypothesis, the Frobenius action on $H^d_{\mathfrak{m}}(I) \cong E_R$ is injective. Then we will be done by Theorem III.41.

Suppose the Frobenius action on $H^d_{\mathfrak{m}}(I)$ is not injective, then the nonzero socle element $x \in H^d_{\mathfrak{m}}(I) \cong E_R$ is in the kernel, i.e., $F(x) = 0$. From Proposition III.38 and Lemma III.42 we know that φ_3 is not injective. So we also have $\varphi_3(x) = 0$. Hence $x = \varphi_2(y)$ for some $y \in H^{d-1}_{\mathfrak{m}}(R/I)$. Because $0 = F(x) = F(\varphi_2(y)) = \varphi_2(F(y))$, we get that $F(y) \in \operatorname{im} \varphi_1$. Using the commutativity of the diagram, it is straightforward to check that $\operatorname{im}\varphi_1$ is an F-stable submodule of $H^d_{\mathfrak{m}}(R/I)$. Since R/I is F-pure, $H^{d-1}_{\mathfrak{m}}(R/I)$ is anti-nilpotent by Theorem III.15. Hence F acts injectively on $H^{d-1}_{\mathfrak{m}}(R/I)/\operatorname{im}\varphi_1$. But clearly $F(\overline{y}) = \overline{F(y)} = 0$ in $H^{d-1}_{\mathfrak{m}}(R/I)/\operatorname{im}\varphi_1$, so $\overline{y} = 0$. Therefore $y \in \operatorname{im}\varphi_1$. Hence $x = \varphi_2(y) = 0$ which is a contradiction because we assume x is a nonzero socle element. □

Remark III.44. If we assume that R is Cohen-Macaulay and F-injective in Theorem III.43, then the diagram used in the proof of Theorem III.43 reduces to the following:

$$\begin{tikzcd}
0 \arrow[r] & H^{d-1}_{\mathfrak{m}}(R/I) \arrow[r] \arrow[d, "F"] & H^d_{\mathfrak{m}}(I) \arrow[r] \arrow[d, "F"] & H^d_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "F"] & 0 \\
0 \arrow[r] & H^{d-1}_{\mathfrak{m}}(R/I) \arrow[r] & H^d_{\mathfrak{m}}(I) \arrow[r] & H^d_{\mathfrak{m}}(R) \arrow[r] & 0
\end{tikzcd}$$

Since R is F-injective, the Frobenius action on $H^d_{\mathfrak{m}}(R)$ is injective. So this diagram and the five lemma tell us immediately that the Frobenius action on $E_R \cong H^d_{\mathfrak{m}}(I)$ is injective if and only if the Frobenius action on $H^{d-1}_{\mathfrak{m}}(R/I)$ is injective, i.e., if and only if R/I is F-injective (or equivalently, F-pure since when R is Cohen-Macaulay, R/I is Gorenstein). This gives a quick proof of Enescu's original result.

It is quite natural to ask, when R is an F-pure Cohen-Macaulay ring and has a canonical module, can we always find $I \cong \omega_R$ such that R/I is F-pure? Note that by Proposition III.36, in this situation R has a canonical ideal $I \cong \omega_R$ because R is F-pure, hence reduced, in particular generically Gorenstein.

However the following example shows that this is not always true. So in view of Remark III.44, even when R is Cohen-Macaulay and F-pure, the injective Frobenius action on E_R may *not* be compatible with the natural Frobenius action $H^d_{\mathfrak{m}}(R)$ under the surjection $E_R \cong H^d_{\mathfrak{m}}(I) \twoheadrightarrow H^d_{\mathfrak{m}}(R)$, no matter how one picks $I \cong \omega_R$.

Example III.45 (*cf.* Example 2.8 in [15])**.** Let $R = K[[x_1, \ldots, x_n]]/(x_ix_j, i \neq j)$ where $n \geq 3$. Then R is a 1-dimensional complete F-pure non-Gorenstein Cohen-

Macaulay local ring. So R/I will be a 0-dimensional local ring (non-Gorenstein property ensures that I is not the unit ideal). If it is F-pure, it must be a field (since F-pure implies reduced). So R/I is F-pure if and only if $\omega_R \cong I \cong \mathfrak{m}$. But clearly $\omega_R \neq \mathfrak{m}$, because one can easily compute that the type of $\mathfrak{m}$ is n: $x_1 + \cdots + x_n$ is a regular element, and each x_i is in the socle of $\mathfrak{m}/(x_1 + \cdots + x_n)\mathfrak{m}$.

We also point out a connection between our main theorem and some theory in F-adjunction. In fact, results of Schwede in [56] imply that if $(R, \mathfrak{m})$ is an F-finite normal local ring with a canonical ideal $I \cong \omega_R$ which is principal in codimension 2 and R/I is normal and F-pure, then R is F-pure (take $X = \operatorname{Spec} R$, $\Delta = 0$ and $D = -K_R$ in Proposition 7.2 in [56]). The argument in [56] is geometrical and is in terms of Frobenius splitting. Our Theorem III.43 is a natural generalization (we don't require any F-finite, normal or principal in codimension 2 conditions) and we use the dualized argument, i.e., studying the Frobenius actions on local cohomology modules.

Our next goal is to apply the anti-nilpotency to prove F-purity deforms to F-injectivity. This is recently proved in [34] using slight different methods. Our treatment here is much simpler than the method in [34].

Theorem III.46. *Let $(R, \mathfrak{m})$ be a local ring and x be a nonzerodivisor of R. If all $H^i_{\mathfrak{m}}(R/xR)$ are anti-nilpotent, then R is F-injective. Hence, F-pure deforms to F-injective by Theorem III.15.*

Proof. We have short exact sequence of R-module with Frobenius action:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{\cdot x} & R & \longrightarrow & R/xR & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle x^{p-1}\cdot F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & & \\
0 & \longrightarrow & R & \xrightarrow{\cdot x} & R & \longrightarrow & R/xR & \longrightarrow & 0
\end{array}$$

which induces long exact sequence of local cohomology modules with Frobenius action:

$$\begin{tikzcd}
\arrow[r] & H^{i-1}_{\mathfrak{m}}(R/xR) \arrow[r, "\phi"] \arrow[d, "F"] & H^i_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] \arrow[d, "x^{p-1}\cdot F"] & H^i_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "F"] & {} \\
\arrow[r] & H^{i-1}_{\mathfrak{m}}(R/xR) \arrow[r, "\phi"] & H^i_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] & H^i_{\mathfrak{m}}(R) \arrow[r] & {}
\end{tikzcd}$$

It is enough to show $x^{p-1}\cdot F$ acts injectively on $H^i_{\mathfrak{m}}(R)$. Suppose not, we pick a nonzero $y \in \ker(x^{p-1}\cdot F)\bigcap \operatorname{socle} H^i_{\mathfrak{m}}(R)$. Then y maps to 0 in $H^i_{\mathfrak{m}}(R)$ under multiplication by x. Hence we know that $y=\phi(z)$ for some z in $H^{i-1}_{\mathfrak{m}}(R/xR)$. Moreover, since $y\in\ker(x^{p-1}\cdot F)$, by commutativity of the diagram we see that $F(z)\in\ker(\phi)$. But by assumption $H^{i-1}_{\mathfrak{m}}(R/xR)$ is anti-nilpotent and $\ker(\phi)$ is F-stable, we know that F acts injectively on $H^{i-1}_{\mathfrak{m}}(R/xR)/\ker(\phi)$. So $F(z)\in\ker(\phi)$ implies $z\in\ker(\phi)$. Hence $y=\phi(z)=0$. $\square$

Remark III.47. We don't really need anti-nilpotency in the above proof, we only need that F acts injectively on $H^i_{\mathfrak{m}}(R/xR)/\ker(\phi)=H^i_{\mathfrak{m}}(R/xR)/\operatorname{im}(H^i_{\mathfrak{m}}(R))$ for all i. I don't know whether this is always true when R/xR is F-injective. If this is true, then it will settle the long standing conjecture that F-injectivity deforms.

3.5 Some examples

Since stably FH-finite trivially implies F-injective, it is quite natural to ask whether FH-finite implies F-injective. The following example studied in [12] shows this does not hold in general.

Example III.48 (*cf.* Example 2.15 in [12])**.** Let $R=K[[x,y,z]]/(x^3+y^3+z^3)$ where K is a field of characteristic different from 3. This is a Gorenstein ring of dimension 2. And it can be checked that the only nontrivial F-stable submodule in $H^2_{\mathfrak{m}}(R)$ is its socle, a copy of K. Hence R is FH-finite. But it is known that R is F-pure (equivalently, F-injective since R is Gorenstein) if and only if the characteristic of K is congruent to 1 mod 3. Hence if the characteristic is congruent to 2 mod 3, we get an example of FH-finite ring which is not F-injective.

Another natural question to ask is whether the converse of Theorem III.15 is true. The next example will show this is also false in general. We recall a theorem in [63]:

Theorem III.49 (*cf.* Theorem 2.6 in [63]). *Let $(R, \mathfrak{m})$ be an excellent Cohen-Macaulay local ring of dimension d. Then R is F-rational if and only if $H^d_{\mathfrak{m}}(R)$ is a simple $R\{F\}$-module.*

Corollary III.50. *Let $(R, \mathfrak{m})$ be an excellent F-rational local ring of dimension d. Then R is stably FH-finite.*

Proof. This follows immediately from Theorem III.49 and Theorem III.3 because when $H^d_{\mathfrak{m}}(R)$ is a simple $R\{F\}$-module, it is obviously anti-nilpotent. $\square$

Example III.51 (*cf.* Example 7.15 in [32]). Let $R = K[t, xt^4, x^{-1}t^4, (x+1)^{-1}t^4]_{\mathfrak{m}} \subseteq K(x,t)$ where $\mathfrak{m} = (t, xt^4, x^{-1}t^4, (x+1)^{-1}t^4)$. Then R is F-rational but not F-pure. Hence by our Corollary III.50, R is a stably FH-finite Cohen-Macaulay ring that is not F-pure.

Also note that even when R is Cohen-Macaulay and F-injective, it is not always FH-finite, and does not even always have FH-finite length. We have the following example:

Example III.52 (*cf.* Example 2.16 in [12]). Let k be an infinite perfect field of characteristic $p > 2$, $K = k(u,v)$, where u and v are indeterminates, and let $L = K[y]/(y^{2p} + uy^p - v)$. Let $R = K + xL[[x]] \subseteq L[[x]]$. Then R is a complete F-injective Cohen-Macaulay domain of dimension 1 which is not FH-finite. Notice that by Theorem III.3, $R[[x]]$ is an F-injective Cohen-Macaulay domain of dimension 2 that does not have FH-finite length (this was not pointed out in [12]).

CHAPTER IV

Lyubeznik's F-modules

Results in this Chapter appeared in my paper [44]. Lyubeznik's F-modules in characteristic $p > 0$ are morally the counterpart of D-modules in characteristic 0. They have remarkable applications to the study of F-singularities and local cohomology modules in characteristic $p > 0$. Therefore it is quite interesting and natural to study their intrinsic properties. In [28], Hochster showed some properties of Lyubeznik's F-modules:

Theorem IV.1 (*cf.* Theorem 3.1 in [28])**.** *The category of F_R-modules over a Noetherian regular ring R of prime characteristic $p > 0$ has enough injectives, i.e., every F_R-module can be embedded in an injective F_R-module.*

Theorem IV.2 (*cf.* Theorem 5.1 and Corollary 5.2(b) in [28])**.** *Let R be a Noetherian regular ring of prime characteristic $p > 0$. Let M and N be F_R-finite F_R-modules. Then $\mathrm{Hom}_{F_R}(M, N)$ is a finite-dimensional vector space over $\mathbb{Z}/p\mathbb{Z}$ and, hence, is a finite set. Moreover, when R is local, every F_R-finite F_R-module has only finitely many F_R-submodules.*

The main purpose of this chapter is to get some further results based on Hochster's results. In connection with Theorem IV.1, we will prove the following (this can be viewed as an analogue of the corresponding statement for D-modules in characteristic

0):

Theorem IV.3. *Let R be an F-finite regular ring of characteristic $p > 0$ such that there exists a canonical module ω_R with $F^!\omega_R \cong \omega_R$ (this holds if R is essentially of finite type over an F-finite regular local ring). Then the category of F_R-modules has finite global dimension $d + 1$ where $d = \dim R$.*

Theorem IV.2 makes it quite natural to ask whether the higher Ext groups are also finite in this category (when M and N are F_R-finite F_R-modules). We show that in general this fails even for Ext^1:

Example IV.4. Let $(R, \mathfrak{m}, K)$ be a regular local ring of characteristic $p > 0$ and dimension $d \geq 1$, and let $E = E(R/\mathfrak{m})$ be the injective hull of the residue field. Then $\mathrm{Ext}^1_{F_R}(R, E) \neq 0$. Moreover, when K is infinite, $\mathrm{Ext}^1_{F_R}(R, E)$ is also infinite. In particular, E is not injective in the category of F_R-modules.

Throughout this chapter, R will always denote a not necessarily local Noetherian *regular* ring of characteristic $p > 0$ and dimension d.

4.1 F-modules and unit right $R\{F\}$-modules

We collect some definitions from [41]. These are the main objects that we shall study in this chapter.

Definition IV.5 (*cf.* Definition 1.1 in [41])**.** An F_R-*module* is an R-module M equipped with an R-linear isomorphism $\theta\colon M \to F(M)$ which we call the *structure morphism* of M. A homomorphism of F_R-modules is an R-module homomorphism $f\colon M \to M'$ such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta'} \\ F(M) & \xrightarrow{F(f)} & F(M') \end{array}$$

Definition IV.6 (*cf.* Definition 1.9 and Definition 2.1 in [41])**.** A generating morphism of an F_R-module M is an R-module homomorphism $\beta\colon M_0 \to F(M_0)$, where M_0 is some R-module, such that M is the limit of the inductive system in the top row of the commutative diagram

$$\begin{tikzcd}
M_0 \arrow[r, "\beta"] \arrow[d, "\beta"] & F(M_0) \arrow[r, "F(\beta)"] \arrow[d, "F(\beta)"] & F^2(M_0) \arrow[r, "F^2(\beta)"] \arrow[d, "F^2(\beta)"] & \cdots \\
F(M_0) \arrow[r, "F(\beta)"] & F^2(M_0) \arrow[r, "F^2(\beta)"] & F^3(M_0) \arrow[r, "F^3(\beta)"] & \cdots
\end{tikzcd}$$

and $\theta\colon M \to F(M)$, the structure isomorphism of M, is induced by the vertical arrows in this diagram. An F_R-module M is called *F_R-finite* if M has a generating morphism $\beta\colon M_0 \to F(M_0)$ with M_0 a finitely generated R-module.

Now we introduce the notion of *unit right $R\{F\}$-modules* which are an analogue of *unit left $R\{F\}$-modules* in [9]. This is a key concept in relating Lyubeznik's F_R-modules with right $R\{F\}$-modules. The idea can be also found in Section 5.2 in [6]. We first recall the functor $F^!(-)$ in the case that R is regular and F-finite: for any R-module M, $F^!(M)$ is the R-module obtained by first considering $\mathrm{Hom}_R(R^{(1)}, M)$ as an $R^{(1)}$-module and then identifying $R^{(1)}$ with R. Remember that giving an R-module M a right $R\{F\}$-module structure is equivalent to giving an R-linear map $M^{(1)} \to M$. But this is the same as giving an $R^{(1)}$-linear map $M^{(1)} \to \mathrm{Hom}_R(R^{(1)}, M)$. Hence after identifying $R^{(1)}$ with R, we find that giving M a right $R\{F\}$-module structure is equivalent to giving an R-linear map $\tau\colon M \to F^!M$. Moreover, it is straightforward to check that a homomorphism of right $R\{F\}$-modules is an R-module homomorphism $g\colon M \to M'$ such that the following diagram commutes

$$\begin{tikzcd}
M \arrow[r, "g"] \arrow[d, "\tau"] & M' \arrow[d, "\tau'"] \\
F^!M \arrow[r, "F^!(g)"] & F^!M'
\end{tikzcd}$$

Definition IV.7. A unit right $R\{F\}$-module is a right $R\{F\}$-module M such that the structure map $\tau\colon M \to F^! M$ is an isomorphism.

Remark IV.8. As in Definition IV.6, we introduce the notion of a generating morphism of unit right $R\{F\}$-modules. Let M_0 be a right $R\{F\}$-module with structure morphism $\tau_0\colon M_0 \to F^!(M_0)$. Let M be the limit of the inductive system in the top row of the commutative diagram

$$\begin{tikzcd}
M_0 \arrow[r, "\tau_0"] \arrow[d, "\tau_0"] & F^!(M_0) \arrow[r, "F^!(\tau_0)"] \arrow[d, "F^!(\tau_0)"] & (F^!)^2(M_0) \arrow[r, "(F^!)^2(\tau_0)"] \arrow[d, "(F^!)^2(\tau_0)"] & \cdots \\
F^!(M_0) \arrow[r, "F^!(\tau_0)"] & (F^!)^2(M_0) \arrow[r, "(F^!)^2(\tau_0)"] & (F^!)^3(M_0) \arrow[r, "(F^!)^3(\tau_0)"] & \cdots
\end{tikzcd}$$

Since R is F-finite, it is easy to see that $F^!(-)$ commutes with direct limit. Hence $\tau\colon M \to F^! M$ induced by the vertical arrows in the above diagram is an isomorphism. M is a unit right $R\{F\}$-module.

For an F-finite regular ring R, any rank 1 projective module is a canonical module ω_R of R (we refer to [20] for a detailed definition of canonical module and dualizing complex for possibly non-local rings). When R is local, $\omega_R = R$ is unique. It is easy to see that $F^!\omega_R$ is always a canonical module of R (see [20] for more general results). However, to the best of our knowledge, it is still unknown whether there always exists ω_R such that $F^!\omega_R \cong \omega_R$ for an F-finite regular ring R. Nonetheless, this is true if either R is essentially of finite type over an F-finite regular local ring or R is sufficiently affine. We refer to Proposition 2.20 and 2.21 in [6] as well as [20] for more details on this question.

The next theorem is well known. It follows from duality theory in [20]. In the context of the Frobenius morphism it is explained in Theorem 5.9 in [6]. Since we need to use this repeatedly throughout, we give a short proof for completeness.

Theorem IV.9. *Let R be an F-finite regular ring such that there exists a canonical module ω_R with $F^!\omega_R \cong \omega_R$. Then the category of unit right $R\{F\}$-modules is equivalent to the category of F_R-modules. Moreover, the equivalence is given by tensoring with ω_R^{-1}, and its inverse is given by tensoring with ω_R.*

Proof. We first note that, for any R-module M,

$$(\omega_R^{-1})^{(1)} \otimes_{R^{(1)}} \operatorname{Hom}_R(R^{(1)}, M) \cong (\omega_R^{-1})^{(1)} \otimes_{R^{(1)}} \operatorname{Hom}_R(R^{(1)}, \omega_R) \otimes_R (\omega_R^{-1} \otimes_R M).$$

Hence after identifying $R^{(1)}$ with R, the above equality becomes

$$\omega_R^{-1} \otimes_R F^!M \cong \omega_R^{-1} \otimes_R F^!\omega_R \otimes_R F(\omega_R^{-1} \otimes_R M) \cong F(\omega_R^{-1} \otimes_R M)$$

where the last equality is by our assumption $F^!\omega_R \cong \omega_R$. Now for any unit right $R\{F\}$-module M, we have an isomorphism $M \xrightarrow{\tau} F^!M$. Hence after tensoring with ω_R^{-1}, we get $\omega_R^{-1} \otimes_R M \xrightarrow{\mathrm{id} \otimes_R \tau} \omega_R^{-1} \otimes_R F^!M \cong F(\omega_R^{-1} \otimes_R M)$. This shows that $\omega_R^{-1} \otimes_R M$ is an F_R-module with structure morphism θ given by $\mathrm{id} \otimes_R \tau$. The converse can be proved similarly. $\square$

Throughout the rest of this chapter, we will use Ext^i_R, $\operatorname{Ext}^i_{R\{F\}}$, $\operatorname{Ext}^i_{uR\{F\}}$, and $\operatorname{Ext}^i_{F_R}$ (respectively, id_R, $\operatorname{id}_{R\{F\}}$, $\operatorname{id}_{uR\{F\}}$, id_{F_R}) to denote the i-th Ext group (respectively, the injective dimension) computed in the category of R-modules, right $R\{F\}$-modules, unit right $R\{F\}$-modules, and F_R-modules.

We end this section by studying some examples of F_R-modules. The simplest example of an F_R-module is R equipped with structure isomorphism the identity map, that is, sending 1 in R to 1 in $F(R) \cong R$. Note that this corresponds to the unit right $R\{F\}$-module $\omega_R \cong F^!\omega_R$ under Theorem IV.9. Another important example is $E = E(R/\mathfrak{m})$, the injective hull of $R/\mathfrak{m}$ for a maximal ideal $\mathfrak{m}$ of R. We can give it a generating morphism $\beta\colon R/\mathfrak{m} \to F(R/\mathfrak{m})$ by sending $\overline{1}$ to $\overline{x_1^{p-1} \cdots x_d^{p-1}}$

(where $x_1, \ldots, x_d$ represents minimal generators of $\mathfrak{m}R_{\mathfrak{m}}$). We will call these structure isomorphisms of R and E the *standard* F_R-module structures on R and E. Note that in particular R and E with the standard F_R-module structures are F_R-finite F_R-modules. Now we provide a nontrivial example of an F_R-module:

Example IV.10. Let $R^\infty := \oplus_{i\in\mathbb{Z}} Rz_i$ denote the infinite direct sum of copies of R equipped with the F_R-module structure by setting

$$\theta : z_i \to z_{i+1}.$$

Then R^∞ is *not* F_R-finite. It is easy to see that we have a short exact sequence of F_R-modules:

$$0 \to R^\infty \xrightarrow{z_i \mapsto z_i - z_{i+1}} R^\infty \xrightarrow{z_i \mapsto 1} R \to 0$$

where the last R is equipped with the standard F_R-module structure.

We want to point out that the above sequence does not split in the category of F_R-modules. Suppose g: $R \to R^\infty$ is a splitting, say $g(1) = \{y_j\}_{j\in\mathbb{Z}} \neq 0$. Then a direct computation shows that $\theta(\{y_j\}) = \{y_j^p\}$, which is impossible by the definition of θ. Hence, by Yoneda's characterization of Ext groups, we know that $\mathrm{Ext}^1_{F_R}(R, R^\infty) \neq 0$.

4.2 The global dimension of F-modules

Our goal in this section is to prove Theorem IV.3. First we want to show that, when R is F-finite, the category of right $R\{F\}$-modules has finite global dimension $d+1$. We start with a lemma which is an analogue of Lemma 1.8.1 in [9].

Lemma IV.11. *Let R be a regular ring and let M be a right $R\{F\}$-module, so that there is an R-linear map ϕ: $M^{(1)} \to M$ (so for every i, we get an R-linear map ϕ^i: $M^{(i)} \to M$ by composing ϕ i times). Then we have an exact sequence of right*

$R\{F\}$-modules

$$0 \to M^{(1)} \otimes_R R\{F\} \xrightarrow{\alpha} M \otimes_R R\{F\} \xrightarrow{\beta} M \to 0$$

where for every $x^{(1)} \in M^{(1)}$,

$$\alpha(x^{(1)} \otimes F^i) = \phi(x^{(1)}) \otimes F^i - x \otimes F^{i+1}$$

and for every $y \in M$,

$$\beta(y \otimes F^i) = \phi^i(y^{(i)}).$$

Proof. It is clear that every element in $M^{(1)} \otimes_R R\{F\}$ (resp. $M \otimes_R R\{F\}$) can be written uniquely as a finite sum $\sum x_i^{(1)} \otimes F^i$ where $x_i^{(1)} \in M^{(1)}$ (resp. $x_i \in M$) because $R\{F\}$ is free as a left R-module (this verifies that our maps α and β are well-defined). It is straightforward to check that α, β are morphisms of right $R\{F\}$-modules and that $\beta \circ \alpha = 0$ and β is surjective (because $\beta(y \otimes 1) = \phi^0(y) = y$). So it suffices to show α is injective and $\ker(\beta) \subseteq \mathrm{im}(\alpha)$.

Suppose $\alpha(\sum x_i^{(1)} \otimes F^i) = 0$. By definition of α we get $\sum(\phi(x_i^{(1)}) - x_{i-1}) \otimes F^i = 0$. Hence by uniqueness we get $\phi(x_i^{(1)}) = x_{i-1}$ for all i. Hence $x_i = 0$ for all i (because it is a finite sum). This proves α is injective.

Now suppose $\beta(\sum_{i=0}^n y_i \otimes F^i) = 0$. We want to find x_i such that

$$\alpha(\sum_{i=0}^{n} x_i^{(1)} \otimes F^i) = \sum_{i=0}^{n} y_i \otimes F^i. \tag{4.1}$$

By definition of β we know that $\sum_{i=0}^n \phi^i(y_i^{(i)}) = 0$. Now one can check that

$$x_0 = -(y_1 + \phi(y_2^{(1)}) + \cdots + \phi^{n-1}(y_n^{(n-1)}))$$

$$x_2 = -(y_2 + \phi(y_3^{(1)}) + \cdots + \phi^{n-2}(y_n^{(n-2)}))$$

$$\ldots$$

$$x_{n-1} = -y_n$$

$$x_n = 0$$

is a solution of (4.1). This proves $\ker(\beta) \subseteq \operatorname{im}(\alpha)$. $\square$

In [9], a similar two-step resolution is proved for left $R\{F\}$-modules (see Lemma 1.8.1 in [9]). Using the two-step resolution it is proved in [9] that the category of left $R\{F\}$-modules has Tor-dimension at most $d+1$ (see Corollary 1.8.4 in [9]). We want to mimic the strategy and prove the corresponding results for right $R\{F\}$-modules. We can actually improve the result: we show that when R is F-finite, the category of right $R\{F\}$-modules has finite *global* dimension *exactly* $d+1$.

Theorem IV.12. *Let R be an F-finite regular ring of dimension d. Then the category of right $R\{F\}$-modules has finite global dimension $d+1$.*

Proof. We first note that for every right $R\{F\}$-module M with structure map $\tau\colon M \to F^! M$, a projective resolution of M *in the category of R-modules* can be given a structure of right $R\{F\}$-module such that it becomes an exact sequence of right $R\{F\}$-modules. This is because we can lift the natural map $\tau\colon M \to F^! M$ to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P_k \arrow[r] \arrow[d, dotted] & P_{k-1} \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, dotted] & P_0 \arrow[r] \arrow[d, dotted] & M \arrow[r] \arrow[d, "\tau"] & 0 \\
0 \arrow[r] & F^!(P_k) \arrow[r] & F^!(P_{k-1}) \arrow[r] & \cdots \arrow[r] & F^!(P_1) \arrow[r] & F^!(P_0) \arrow[r] & F^!(M) \arrow[r] & 0
\end{tikzcd}$$

because we can always lift a map from a complex of projective modules to an acyclic complex ($F^!(-)$ is an exact functor when R is F-finite).

By Lemma IV.11, we have an exact sequence of right $R\{F\}$-modules

$$0 \to M^{(1)} \otimes_R R\{F\} \xrightarrow{\alpha} M \otimes_R R\{F\} \xrightarrow{\beta} M \to 0. \tag{4.2}$$

Now, when we tensor the above (4.2) with the projective resolution of M over R, we have the following commutative diagram

(4.3)

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & P_k & \longrightarrow & P_{k-1} & \longrightarrow \cdots \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P_k \otimes_R R\{F\} & \longrightarrow & P_{k-1} \otimes_R R\{F\} & \longrightarrow \cdots \longrightarrow & P_1 \otimes_R R\{F\} & \longrightarrow & P_0 \otimes_R R\{F\} & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha_k}\uparrow & & {\scriptstyle\alpha_{k-1}}\uparrow & & {\scriptstyle\alpha_1}\uparrow & & {\scriptstyle\alpha_0}\uparrow & & \\
0 & \longrightarrow & P_k^{(1)} \otimes_R R\{F\} & \longrightarrow & P_{k-1}^{(1)} \otimes_R R\{F\} & \longrightarrow \cdots \longrightarrow & P_1^{(1)} \otimes_R R\{F\} & \longrightarrow & P_0^{(1)} \otimes_R R\{F\} & \longrightarrow & 0
\end{array}$$

The first line is a projective resolution of M over R. By the above discussion we can give each P_i a right $R\{F\}$-module structure such that it is an exact sequence of right $R\{F\}$-modules. The second line (resp., the third line) is obtained from the first line by tensoring with $R\{F\}$ (resp., applying $^{(1)}$ and then tensoring with $R\{F\}$). Each column is the map described in Lemma IV.11. In particular, all columns are exact sequences of right $R\{F\}$-modules.

Let $C_\bullet$ be the complex of the third line and $D_\bullet$ be the complex of the second line of (4.3). The homology of the mapping cone of $C_\bullet \to D_\bullet$ is the same as the homology of the quotient complex $D_\bullet/C_\bullet$, which is the first line in (4.3). Hence the mapping cone is acyclic. Since each P_i is projective as an R-module, we know that P_i is a direct summand of a free R-module G. So $P_i^{(1)}$ is a direct summand of $G^{(1)}$. Since R is F-finite, $G^{(1)}$ is projective as an R-module, and so $P_i^{(1)}$ is also projective as an R-module. Hence $P_i^{(1)} \otimes_R R\{F\}$ and $P_i \otimes_R R\{F\}$ are projective as right $R\{F\}$-modules for every i. Thus the mapping cone of $C_\bullet \to D_\bullet$ gives a right $R\{F\}$-projective resolution of M. We note that this resolution has length $k+1$. Since we can always take a projective resolution of M of length $k \leq d$, the right $R\{F\}$-projective resolution we obtained has length $\leq d+1$.

We have already seen that the global dimension of right $R\{F\}$-modules is $\leq d+1$. Now we let M and N be two right $R\{F\}$-modules with trivial right F-action (i.e., the structure maps of M and N are the zero maps). I claim that in this case we have

$$\mathrm{Ext}^j_{R\{F\}}(M,N) = \mathrm{Ext}^j_R(M,N) \oplus \mathrm{Ext}^{j-1}_R(M^{(1)},N). \tag{4.4}$$

To see this, we look at (4.3) applied to M with trivial right F-action. It is clear that in this case that each P_i in the first line of (4.3) also has trivial right $R\{F\}$-module structure. Hence as described in Lemma IV.11, we have

$$\alpha_j(x^{(1)} \otimes F^i) = -x \otimes F^{i+1} \tag{4.5}$$

for every $x^{(1)} \otimes F^i \in P_j^{(1)} \otimes_R R\{F\}$. The key observation is that, since N has trivial right F-action, when we apply $\mathrm{Hom}_{R\{F\}}(-,N)$ to

$$\alpha_j : P_j^{(1)} \otimes_R R\{F\} \to P_j \otimes_R R\{F\},$$

the dual map $\alpha_j^\vee$ is the zero map (one can check this by a direct computation using (4.5)). Hence when we apply $\mathrm{Hom}_{R\{F\}}(-,N)$ to the mapping cone of $C_\bullet \to D_\bullet$, the j-th cohomology is the same as the direct sum of the j-th cohomology of $\mathrm{Hom}_{R\{F\}}(C_\bullet[-1],N)$ and the j-th cohomology of $\mathrm{Hom}_{R\{F\}}(D_\bullet,N)$. That is,

$$\mathrm{Ext}^j_{R\{F\}}(M,N) = H^j(\mathrm{Hom}_{R\{F\}}(D_\bullet,N)) \oplus H^j(\mathrm{Hom}_{R\{F\}}(C_\bullet[-1],N)). \tag{4.6}$$

But for every right $R\{F\}$-module N, $\mathrm{Hom}_{R\{F\}}(- \otimes_R R\{F\}, N) \cong \mathrm{Hom}_R(-,N)$. Therefore applying $\mathrm{Hom}_{R\{F\}}(-,N)$ to $D_\bullet$ and $C_\bullet$ are the same as applying $\mathrm{Hom}_R(-,N)$ to

$$0 \to P_k \to P_{k-1} \to \cdots \to P_0 \to 0$$

and

$$0 \to P_k^{(1)} \to P_{k-1}^{(1)} \to \cdots \to P_0^{(1)} \to 0,$$

which are R-projective resolutions of M and $M^{(1)}$ respectively. Hence we know that

(4.7)

$$H^j(\mathrm{Hom}_{R\{F\}}(D_\bullet,N)) \oplus H^j(\mathrm{Hom}_{R\{F\}}(C_\bullet[-1],N)) = \mathrm{Ext}^j_R(M,N) \oplus \mathrm{Ext}^{j-1}_R(M^{(1)},N).$$

Now (4.4) follows from (4.6) and (4.7).

In particular, we can take two R-modules M and N such that $\mathrm{Ext}^d_R(M^{(1)}, N) \neq 0$ (for example, take $N = R$ and $M = R/(x_1, \ldots, x_d)$ where $x_1, \ldots, x_d$ is a regular sequence in R). Applying (4.4) to $j = d+1$ gives

$$\mathrm{Ext}^{d+1}_{R\{F\}}(M, N) = \mathrm{Ext}^d_R(M^{(1)}, N) \neq 0.$$

Hence the global dimension of right $R\{F\}$-modules is at least $d+1$. Since we have already shown it is bounded by $d+1$, this completes the proof that the global dimension of right $R\{F\}$-modules is exactly $d+1$. $\square$

We can use the method in the proof of Theorem IV.12 to compute some Ext groups in the category of right $R\{F\}$-modules. Below we give an example which is a key ingredient when we show that the global dimension of F_R-modules is $d+1$.

Example IV.13. Let R be an F-finite regular ring of dimension d such that there exists a canonical module ω_R with $F^!\omega_R \cong \omega_R$. Let $\omega_R^\infty := \oplus_{j\in\mathbb{Z}}\omega_R z_j$ be an infinite direct sum of copies of ω_R. We give ω_R^∞ a right $R\{F\}$-module structure by setting $\tau\colon \omega_R^\infty \to F^!(\omega_R^\infty) \cong \omega_R^\infty$ such that

$$\tau(yz_j) = yz_{j+1}$$

for every $y \in \omega_R$. It is clear that ω_R^∞ is in fact a unit right $R\{F\}$-module, and it corresponds to the F_R-module R^∞ described in Example IV.10 under Theorem IV.9.

Lemma IV.14. *With the same notations as in Example IV.13, we have*

$$\mathrm{id}_{R\{F\}}\, \omega_R^\infty = d+1. \tag{4.8}$$

Proof. We first notice that the right $R\{F\}$-module structure on ω_R defined by $\omega_R \cong F^!\omega_R$ induces a canonical map $\phi\colon \omega_R^{(1)} \to \omega_R$, which is a generator of the free $R^{(1)}$-module $\mathrm{Hom}_R(\omega_R^{(1)}, \omega_R) \cong R^{(1)}$. That is, any map in $\mathrm{Hom}_R(\omega_R^{(1)}, \omega_R)$ can be expressed as $\phi(r^{(1)}\cdot -)$ for some $r^{(1)}\in R^{(1)}$. This fact is well-known, for example see Lemma 7.1 in [56].

Next we fix $x_1,\dots,x_d$ a regular sequence in R. We note that

$$\widetilde{\phi}:=\phi((x_1^{(1)}\cdots x_d^{(1)})^{p-1}\cdot -)\in \mathrm{Hom}_R(\omega_R^{(1)},\omega_R)$$

satisfies $\widetilde{\phi}((x_1^{(1)},\dots,x_d^{(1)})\omega_R^{(1)})\subseteq (x_1,\dots,x_d)\omega_R$, so it induces a map

$$(\omega_R/(x_1,\dots,x_d)\omega_R)^{(1)}\to \omega_R/(x_1,\dots,x_d)\omega_R.$$

That is, $\widetilde{\phi}$ gives $\omega_R/(x_1,\dots,x_d)\omega_R$ a right $R\{F\}$-module structure. It is clear that we can lift this map $\widetilde{\phi}$ to the Koszul complex $K_\bullet(x_1,\dots,x_d;\omega_R)$ as follows:

$$\begin{tikzcd}
0 \arrow[r] & \omega_R^{(1)} \arrow[r] \arrow[d, dotted, "\phi"] & \cdots \arrow[r] & (\omega_R^d)^{(1)} \arrow[r] \arrow[d, dotted] & \omega_R^{(1)} \arrow[r] \arrow[d, "\widetilde{\phi}"] & (\omega_R/(x_1,\dots,x_d)\omega_R)^{(1)} \arrow[r] \arrow[d, "\widetilde{\phi}"] & 0 \\
0 \arrow[r] & \omega_R \arrow[r] & \cdots \arrow[r] & \omega_R^d \arrow[r] & \omega_R \arrow[r] & \omega_R/(x_1,\dots,x_d)\omega_R \arrow[r] & 0
\end{tikzcd}$$

Chasing through the diagram, one can check that the induced map on the last spot of the above commutative diagram is exactly the map ϕ (the generator of $\mathrm{Hom}_R(\omega_R^{(1)},\omega_R)$).

Now we apply (4.3) to $M=\omega_R/(x_1,\dots,x_d)\omega_R$ with structure map $\widetilde{\phi}$ and let the first line in (4.3) be the Koszul complex $K_\bullet(x_1,\dots,x_d;\omega_R)$. The above argument shows that the induced right $R\{F\}$-module structure on $P_d=\omega_R$ is given by the canonical map ϕ: $\omega_R^{(1)}\to\omega_R$ (i.e., it corresponds to $\omega_R\cong F^!\omega_R$). As in Theorem IV.12, the mapping cone of $C_\bullet\to D_\bullet$ is a right $R\{F\}$-projective resolution of $\omega_R/(x_1,\dots,x_d)\omega_R$ of length $d+1$, and the tail of this resolution is

$$0\to\omega_R^{(1)}\otimes_R R\{F\}\xrightarrow{h}\omega_R\otimes_R R\{F\}\oplus(\omega_R^d)^{(1)}\otimes_R R\{F\}\to\cdots \tag{4.9}$$

where we have

$$h(y^{(1)}\otimes F^i)=(-1)^d(y\otimes F^{i+1}-\phi(y^{(1)})\otimes F^i)\oplus(x_1^{(1)}y^{(1)},\dots,x_d^{(1)}y^{(1)})\otimes F^i \tag{4.10}$$

for every $y \in \omega_R$. Now, when we apply $\mathrm{Hom}_{R\{F\}}(-, \omega_R^\infty)$ to (4.9) and identify $\mathrm{Hom}_R(-, \omega_R^\infty) = \mathrm{Hom}_{R\{F\}}(- \otimes_R R\{F\}, \omega_R^\infty)$, we get

$$0 \leftarrow \mathrm{Hom}_R(\omega_R^{(1)}, \omega_R^\infty) \xleftarrow{h^\vee} \mathrm{Hom}_R(\omega_R, \omega_R^\infty) \oplus \mathrm{Hom}_R(\omega_R^{(1)}, \omega_R^\infty)^d \leftarrow \cdots . \tag{4.11}$$

Since $\mathrm{Hom}_R(\omega_R^{(1)}, \omega_R) \cong R^{(1)}$ and $\mathrm{Hom}_R(\omega_R, \omega_R) = R$, we can rewrite (4.11) as

$$0 \leftarrow \oplus_{j\in\mathbb{Z}} R^{(1)} \xleftarrow{h^\vee} (\oplus_{j\in\mathbb{Z}} R) \oplus (\oplus_{j\in\mathbb{Z}} R^{(1)})^d \leftarrow \cdots .$$

After a careful computation using (4.10) and the right $R\{F\}$-module structure of ω_R^∞, we have

$$h^\vee(\{s_j\} \oplus (\{t_{1j}^{(1)}\}, \ldots, \{t_{dj}^{(1)}\})) = \{(-1)^d((s_j^{(1)})^p - s_{j-1}^{(1)}) + \sum_{i=1}^{d} x_i^{(1)} t_{ij}^{(1)}\}_{j\in\mathbb{Z}} \tag{4.12}$$

where $\{s_j\}$ denotes an element in $\oplus_{j\in\mathbb{Z}} R$ and $(\{t_{1j}^{(1)}\}, \ldots, \{t_{dj}^{(1)}\})$ denotes an element in $(\oplus_{j\in\mathbb{Z}} R^{(1)})^d$. The key point here is that $h^\vee$ is *not* surjective. To be more precise, I claim $(-1)^{d-1} z_0 = (\ldots, 0, (-1)^{d-1}, 0, 0, \ldots)$ (i.e., the element in $\oplus_{j\in\mathbb{Z}} R^{(1)}$ with 0-th entry $(-1)^{d-1}$ and other entries 0) is not in the image of $h^\vee$. This is because $\sum_{i=1}^d x_i^{(1)} t_{ij}^{(1)}$ can only take values in $\oplus_{j\in\mathbb{Z}}(x_1^{(1)}, \ldots, x_d^{(1)})$, so if $z_0 \in \mathrm{im}\, h^\vee$, then mod $\oplus_{j\in\mathbb{Z}}(x_1^{(1)}, \ldots, x_d^{(1)})$, we know by (4.12) that $(\overline{s_j}^{(1)})^p - \overline{s_{j-1}}^{(1)} = 0$ for $j \neq 0$ and $(\overline{s_0}^{(1)})^p - \overline{s_{-1}}^{(1)} = -1$. And it is straightforward to see that a solution $\{s_j\}_{j\in\mathbb{Z}}$ to this system must satisfy $\overline{s_j} = 0$ when $j \geq 0$ and $\overline{s_j} = \overline{1}$ when $j < 0$ where $\overline{s}$ denotes the image of $s \in R$ mod $(x_1, \ldots, x_d)$. So there is no solution in $\oplus_{j\in\mathbb{Z}} R$, since $\overline{s_j} = \overline{1}$ for every $j < 0$ implies there has to be infinitely many nonzero s_j.

Hence, we have

$$\mathrm{Ext}^{d+1}_{R\{F\}}(\omega_R/(x_1, \ldots, x_d)\omega_R, \omega_R^\infty) \cong \mathrm{coker}\, h^\vee \neq 0. \tag{4.13}$$

Combining (4.13) with Theorem IV.12 completes the proof of the Lemma. $\square$

Remark IV.15. One might hope that $\mathrm{id}_{R\{F\}}\,\omega_R = d+1$ by the same type computation used in Lemma IV.14. But there is a small gap when doing this. The problem is, when we apply $\mathrm{Hom}_{R\{F\}}(-,\omega_R)$ to (4.9) and compute $\operatorname{coker} h^\vee$, we get

$$\text{(4.14)}\qquad \mathrm{Ext}^{d+1}_{R\{F\}}(\omega_R/(x_1,\ldots,x_d)\omega_R,\omega_R) = \operatorname{coker} h^\vee \cong \frac{R}{(x_1,\ldots,x_d)+\{r^p-r\}_{r\in R}}.$$

So if the set $\{r^p-r\}_{r\in R}$ can take all values of R (this happens, for example when $(R,\mathfrak{m})$ is a complete regular local ring with algebraically closed residue field, see Remark IV.24), then $\mathrm{Ext}^{d+1}_{R\{F\}}(\omega_R/(x_1,\ldots,x_d)\omega_R,\omega_R) = 0$. So we cannot get the desired result in this way. However, we do get from (4.14) that if $(R,\mathfrak{m},K)$ is an F-finite regular local ring with $K\cong R/\mathfrak{m}$ a *finite* field, then $\mathrm{id}_{R\{F\}}\,R = d+1$.

Now we prove our main result. We start by proving that the Ext groups are the same no matter whether one computes in the category of unit right $R\{F\}$-modules or the category of right $R\{F\}$-modules. We give two proofs of this result, the second proof, in fact, proves a stronger result.

Theorem IV.16. *Let R be an F-finite regular ring of dimension d such that there exists a canonical module ω_R with $F^!\omega_R\cong\omega_R$. Let M, N be two unit right $R\{F\}$-modules. Then we have $\mathrm{Ext}^i_{uR\{F\}}(M,N)\cong\mathrm{Ext}^i_{R\{F\}}(M,N)$ for every i. In particular, the category of unit right $R\{F\}$-modules and the category of F_R-modules have finite global dimension $\leq d+1$.*

Proof. First we note that by Theorem IV.12 and Theorem IV.9, it is clear that we only need to show $\mathrm{Ext}^i_{uR\{F\}}(M,N)\cong\mathrm{Ext}^i_{R\{F\}}(M,N)$ for M, N two unit right $R\{F\}$-modules. Below we give two proofs of this fact.

First proof: We use Yoneda's characterization of Ext^i (*cf.* Chapter 3.4 in [64]). Note that this is the same as the derived functor Ext^i whenever the abelian category has enough injectives or enough projectives, hence holds for both the category of unit right $R\{F\}$-modules and the category of right $R\{F\}$-modules (unit right $R\{F\}$-modules has enough injectives by Theorem IV.1 and Theorem IV.9). An element in $\mathrm{Ext}^i_{uR\{F\}}(M,N)$ (resp. $\mathrm{Ext}^i_{R\{F\}}(M,N)$) is an equivalence class of exact sequences of the form

$$\xi : 0 \to N \to X_1 \to \cdots \to X_i \to M \to 0$$

where each X_i is a unit right $R\{F\}$-module (resp. right $R\{F\}$-module) and the maps are maps of unit right $R\{F\}$-modules (resp. maps of right $R\{F\}$-modules). The equivalence relation is generated by the relation $\xi_X \sim \xi_Y$ if there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\cong"] & X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_i \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & N \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_i \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

From this characterization of Ext^i it is clear that we have a well-defined map

$$\iota : \mathrm{Ext}^i_{uR\{F\}}(M,N) \to \mathrm{Ext}^i_{R\{F\}}(M,N)$$

taking an equivalence class of an exact sequence of unit right $R\{F\}$-modules to the same exact sequence but viewed as an exact sequence in the category of right $R\{F\}$-modules.

Conversely, if we have an element in $\mathrm{Ext}^i_{R\{F\}}(M,N)$, say ξ, we have an exact sequence of right $R\{F\}$-modules, this induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\cong"] & X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_i \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & F^!(N) \arrow[r] \arrow[d, "\cong"] & F^!(X_1) \arrow[r] \arrow[d] & \cdots \arrow[r] & F^!(X_i) \arrow[r] \arrow[d] & F^!(M) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & (F^!)^2(N) \arrow[r] \arrow[d, "\cong"] & (F^!)^2(X_1) \arrow[r] \arrow[d] & \cdots \arrow[r] & (F^!)^2(X_i) \arrow[r] \arrow[d] & (F^!)^2(M) \arrow[r] \arrow[d, "\cong"] & 0 \\
& & & & & &
\end{tikzcd}$$

Taking direct limits for columns and noticing that M, N are unit right $R\{F\}$-modules, we get a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\cong"] & X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_i \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & N \arrow[r] & \varinjlim (F^!)^e(X_1) \arrow[r] & \cdots \arrow[r] & \varinjlim (F^!)^e(X_i) \arrow[r] & M \arrow[r] & 0
\end{tikzcd}
\tag{4.15}
$$

Since the functor $F^!(-)$ and the direct limit functor are both exact, the bottom sequence is still exact, and hence it represents an element in $\mathrm{Ext}^i_{uR\{F\}}(M,N)$ (note that each $\varinjlim (F^!)^e(X_j)$ is a unit right $R\{F\}$-module by Remark IV.8). We call this element ξ'. Then we have a map

$$\eta : \mathrm{Ext}^i_{R\{F\}}(M,N) \xrightarrow{\xi \mapsto \xi'} \mathrm{Ext}^i_{uR\{F\}}(M,N).$$

This map is well-defined because it is easy to check that if $\xi_1 \sim \xi_2$, then we also have $\xi_1' \sim \xi_2'$. It is also straightforward to check that ι and η are inverses of each other. Obviously $\eta \circ \iota([\xi]) = [\xi]$ and $\iota \circ \eta([\xi]) = [\xi'] = [\xi]$, where the last equality is by (4.15) (which shows that $\xi \sim \xi'$, and hence they represent the same equivalence class in $\mathrm{Ext}^i_{R\{F\}}(M,N)$).

Second proof: By Theorem IV.1 and Theorem IV.9 we know that the category of unit right $R\{F\}$-module has enough injectives. Now we show that every injective object in the category of unit right $R\{F\}$ modules is in fact injective in the category of right $R\{F\}$-modules. To see this, let I be a unit right $R\{F\}$-injective module. It is enough to show that whenever we have $0 \to I \to W$ for some right $R\{F\}$-module

W, the sequence splits. But $0 \to I \to W$ induces the following diagram:

$$\begin{array}{ccccc}
0 & \longrightarrow & I & \longrightarrow & W \\
 & & \Big\downarrow \cong & & \Big\downarrow \\
0 & \longrightarrow & F^!(I) & \longrightarrow & F^!(W) \\
 & & \Big\downarrow \cong & & \Big\downarrow \\
0 & \longrightarrow & (F^!)^2(I) & \longrightarrow & (F^!)^2(W) \\
 & & \Big\downarrow \cong & & \Big\downarrow
\end{array}$$

Taking the direct limit for the columns we get

$$\begin{array}{ccccc}
0 & \longrightarrow & I & \longrightarrow & W \\
 & & \Big\downarrow \cong & & \Big\downarrow \\
0 & \longrightarrow & I & \longrightarrow & \varinjlim (F^!)^e(W)
\end{array} \tag{4.16}$$

We still have exactness because the functor $F^!(-)$ and the direct limit functor are both exact. We also note that $\varinjlim (F^!)^e(W)$ is a unit right $R\{F\}$-module by Remark IV.8. Now, since I is injective in the category of unit right $R\{F\}$-modules, we know that the bottom map $0 \to I \to \varinjlim (F^!)^e(W)$ splits as a map of unit right $R\{F\}$-modules, so it also splits as a map of right $R\{F\}$-modules. But now, composing with the commutative diagram (4.16) shows that the map $0 \to I \to W$ splits as a map of right $R\{F\}$-modules.

Now we can show that $\mathrm{Ext}^i_{uR\{F\}}(M, N) \cong \mathrm{Ext}^i_{R\{F\}}(M, N)$ as follows. One can take an injective resolution of N in the category of unit right $R\{F\}$-modules:

$$0 \to N \to I_0 \to I_1 \to \cdots\cdots . \tag{4.17}$$

By the above argument this can be also viewed as an injective resolution in the category of right $R\{F\}$-modules. Since applying $\mathrm{Hom}_{R\{F\}}(M, -)$ and $\mathrm{Hom}_{uR\{F\}}(M, -)$ to (4.17) are obviously the same, we know that $\mathrm{Ext}^i_{uR\{F\}}(M, N) \cong \mathrm{Ext}^i_{R\{F\}}(M, N)$. $\square$

Theorem IV.17. *Let R be an F-finite regular ring of dimension d such that there exists a canonical module ω_R with $F^!\omega_R \cong \omega_R$. Then the category of unit right $R\{F\}$-modules and the category of Lyubeznik's F_R-modules both have finite global dimension $d+1$.*

Proof. By Theorem IV.9, it suffices to show that the category of unit right $R\{F\}$-modules has finite global dimension $d+1$. By Theorem IV.16, we know that the global dimension is at most $d+1$.

Now let ω_R^∞ be the unit right $R\{F\}$-module described in Example IV.13. If the global dimension is $\leq d$, then we know that ω_R^∞ has a unit right $R\{F\}$-injective resolution of length $d' \leq d$:

$$0 \to \omega_R^\infty \to I_0 \to I_1 \to \cdots \to I_{d'} \to 0 \tag{4.18}$$

But by the argument in the second proof of Theorem IV.16, we know that each I_j is injective in the category of right $R\{F\}$-modules. So (4.18) can be viewed as an injective resolution of ω_R^∞ in the category of right $R\{F\}$-modules. And hence $\mathrm{id}_{R\{F\}}\,\omega_R^\infty \leq d$, which contradicts Lemma IV.14. □

Remark IV.18. It is clear from Theorem IV.9 and the above proof of Theorem IV.17 that

$$\mathrm{id}_{F_R}\,R^\infty = \mathrm{id}_{uR\{F\}}\,\omega_R^\infty = d+1.$$

4.3 Finite and non-finite results on $\mathrm{Ext}^1_{F_R}$

In this section we study the group $\mathrm{Ext}^1_{F_R}(M,N)$ when M, N are F_R-finite F_R-modules. We prove that when $(R,\mathfrak{m},K)$ is a regular local ring, $\mathrm{Ext}^1_{F_R}(M,N)$ is finite when $K=R/\mathfrak{m}$ is separably closed and M is supported only at $\mathfrak{m}$. However, we provide examples to show that in general $\mathrm{Ext}^1_{F_R}(M,N)$ is not necessarily a finite set. We begin with some lemmas.

Lemma IV.19 (*cf.* Proposition 3.1 in [41])**.** *Let S be a regular ring of characteristic $p>0$ and let $R \to S$ be a surjective homomorphism with kernel $I \subseteq R$. There exists an equivalence of categories between F_R-modules supported on $\operatorname{Spec} S = V(I) \subseteq \operatorname{Spec} R$ and F_S-modules. Under this equivalence the F_R-finite F_R-modules supported on $\operatorname{Spec} S = V(I) \subseteq \operatorname{Spec} R$ correspond to the F_S-finite F_S-modules.*

Lemma IV.20 (*cf.* Theorem 4.2(c)(e) in [28])**.** *Let K be a separably closed field. Then every F_K-finite F_K-module is isomorphic with a finite direct sum of copies of K with the standard F_K-module structure. Moreover, $\mathrm{Ext}^1_{F_K}(K,K)=0$.*

Lemma IV.21. *Let $(R,\mathfrak{m},K)$ be a regular local ring with K separably closed. Then every F_R-finite F_R-module supported only at $\mathfrak{m}$ is isomorphic (as an F_R-module) with a finite direct sum of copies of $E=E(R/\mathfrak{m})$ (where E is equipped with the standard F_R-module structure). Moreover, $\mathrm{Ext}^1_{F_R}(E,E)=0$.*

Proof. This is clear from Lemma IV.19 (applied to $S=K$ and $I=\mathfrak{m}$) and Lemma IV.20 because it is straightforward to check that the standard F_R-module structure on E corresponds to the standard F_K-module structure on K via Lemma IV.19. □

Theorem IV.22. *Let $(R,\mathfrak{m},K)$ be a regular local ring such that K is separably closed and let M, N be F_R-finite F_R-modules. Then $\mathrm{Ext}^1_{F_R}(M,N)$ is finite if M is supported only at $\mathfrak{m}$.*

Proof. Since K is separably closed, by Lemma IV.21 we know that M is a finite direct sum of copies of E in the category of F_R-modules. So it suffices to show that $\mathrm{Ext}^1_{F_R}(E,N)$ is finite. For every exact sequence of F_R-finite F_R-modules

$$0 \to N_1 \to N_2 \to N_3 \to 0,$$

the long exact sequence for Ext gives

$$\mathrm{Ext}^1_{F_R}(E, N_1) \to \mathrm{Ext}^1_{F_R}(E, N_2) \to \mathrm{Ext}^1_{F_R}(E, N_3).$$

So we immediately reduce to the case that N is simple (since R is local, every F_R-finite F_R-module has finite length by Theorem 3.2 in [41]).

We want to show that $\mathrm{Ext}^1_{F_R}(E, N)$ is finite when N is simple. There are two cases: $\mathrm{Ass}_R(N) = \mathfrak{m}$ or $\mathrm{Ass}_R(N) = P \neq \mathfrak{m}$. If $\mathrm{Ass}_R(N) = \mathfrak{m}$, then $N \cong E$ as F_R-modules by Lemma IV.21. So $\mathrm{Ext}^1_{F_R}(E, N) = \mathrm{Ext}^1_{F_R}(E, E) = 0$ by Lemma IV.21.

If $\mathrm{Ass}_R(N) = P \neq \mathfrak{m}$, by Yoneda's characterization of Ext groups, it suffices to show that we only have a finite number of isomorphism classes of short exact sequences

$$0 \to N \to L \to E \to 0$$

of F_R-modules. We first show the number of choices of isomorphism classes for L is finite. Say $\mathrm{Ass}_R(N) = P \neq \mathfrak{m}$, we have $P \in \mathrm{Ass}_R(L) \subseteq \{P, \mathfrak{m}\}$. If $\mathrm{Ass}_R(L) = \{P, \mathfrak{m}\}$, then $H^0_{\mathfrak{m}}(L) \neq 0$ and it does not intersect N. So $H^0_{\mathfrak{m}}(L) \oplus N$ is an F_R-submodule of L. Hence we must have $L \cong H^0_{\mathfrak{m}}(L) \oplus N \cong E \oplus N$ since L has length 2 as an F_R-module. If $\mathrm{Ass}_R(L) = \{P\}$, we can pick $x \in \mathfrak{m} - P$. Localizing at x gives a short exact sequence

$$0 \to N_x \to L_x \to E_x \to 0.$$

But $E_x = 0$, so we get $N_x \cong L_x$ as F_R-module. Since x is not in P, we have $L \hookrightarrow L_x$ as F_R-module. That is, L is isomorphic to an F_R-submodule of L_x, hence is isomorphic to an F_R-submodule of N_x. But N_x is F_R-finite by Proposition 2.9(b) in [41], so it only has finitely many F_R-submodules by Theorem IV.2. This proves that the number of choices of isomorphism classes for L is finite.

Because the number of choices of isomorphism classes for L is finite, and for each

F_R-finite F_R-module L, $\mathrm{Hom}_{F_R}(N, L)$ is always finite by Theorem IV.2. It follows that the number of isomorphism classes of short exact sequences $0 \to N \to L \to E \to 0$ is finite. $\square$

If M is an F_R-module with structure morphism θ_M, for every $x \in M$ we use x^p to denote $\theta_M^{-1}(1 \otimes x)$. Notice that when $M = R$ with the standard F_R-module structure, this is exactly the usual meaning of x^p. We let G_M denote the set $\{x^p - x | x \in M\}$. It is clear that G_M is an abelian subgroup of M.

Theorem IV.23. *Let R be a regular ring. Giving R the standard F_R-module structure, we have $\mathrm{Ext}^1_{F_R}(R, M) \cong M/G_M$ as an abelian group for every F_R-module M.*

Proof. By Yoneda's characterization of Ext groups, an element in $\mathrm{Ext}^1_{F_R}(R, M)$ can be represented by an exact sequence of F_R-modules

$$0 \to M \to L \to R \to 0.$$

It is clear that $L \cong M \oplus R$ as R-module. Moreover, one can check that the structural isomorphism θ_L composed with $\theta_M^{-1} \oplus \theta_R^{-1}$ defines an isomorphism

$$M \oplus R \xrightarrow{\theta_L} F(M) \oplus F(R) \xrightarrow{\theta_M^{-1} \oplus \theta_R^{-1}} M \oplus R$$

which sends (y, r) to $(y + rz, r)$ for every $(y, r) \in M \oplus R$ and for some $z \in M$. Hence, giving a structural isomorphism for L is equivalent to giving some $z \in M$. That is, θ_L is determined by an element $z \in M$. Two exact sequences with structure isomorphism θ_L, θ'_L are in the same isomorphism class if and only if there exists a map $g\colon L \to L$, sending (y, r) to $(y + rx, r)$ for some $x \in M$ such that

$$(1 \otimes g) \circ \theta_L = \theta'_L \circ g.$$

Now we apply $\theta_M^{-1} \oplus \theta_R^{-1}$ on both side. If θ_L, θ'_L are determined by z_1 and z_2 respectively, a direct computation gives that

$$(\theta_M^{-1} \oplus \theta_R^{-1}) \circ (1 \otimes g) \circ \theta_L(y, r) = (y + rz_1 + rx^p, r)$$

while

$$(\theta_M^{-1} \oplus \theta_R^{-1}) \circ \theta'_L \circ g(y, r) = (y + rz_2 + rx, r).$$

So θ_L and θ'_L are in the same isomorphism class if and only if there exists $x \in M$ such that

$$z_2 - z_1 = x^p - x.$$

So $\mathrm{Ext}^1_{F_R}(R, M) \cong M/G_M$ as an abelian group. $\square$

Before we use Theorem IV.23 to study examples, we make the following remark.

Remark IV.24.

1. Let R be a regular ring which is *F-finite* and *local*. By Theorem IV.9, we can identify the category of F_R-modules with the category of unit right $R\{F\}$-modules ($\omega_R = R$ is unique). And by Theorem IV.16, we can compute $\mathrm{Ext}^1_{F_R}(R, M) \cong \mathrm{Ext}^1_{uR\{F\}}(R, M) \cong \mathrm{Ext}^1_{R\{F\}}(R, M)$ by taking the right $R\{F\}$-projective resolution of R and then applying $\mathrm{Hom}_{R\{F\}}(-, M)$. Note that one right $R\{F\}$-projective resolution of R is given by

$$0 \to R^{(1)} \otimes_R R\{F\} \to R\{F\} \to R \to 0$$

as in Lemma IV.11.

2. Let $(R, \mathfrak{m})$ be a strict Henselian local ring (e.g., this holds when $(R, \mathfrak{m})$ is a complete local ring with separably closed residue field). The Artin-Schreier sequence

$$0 \to \mathbb{F}_p \to R \xrightarrow{x^p - x} R \to 0$$

is exact in the Zariski topology, which shows that $G_R = R$, and hence $\mathrm{Ext}^1_{F_R}(R,R) = 0$ when R is a strict Henselian local ring. In particular, applying this to $R = K$ a separably closed field, we recover Lemma IV.20.

Now we give some examples to show that, in general, $\mathrm{Ext}^1_{F_R}(R,M) \cong M/G_M$ is not necessarily finite, even in simple cases.

Example IV.25. Let $R = k(t)$ or $k[t]_{(t)}$ with k an algebraically closed field. We will prove that $\mathrm{Ext}^1_{F_R}(R,R)$ is infinite in both cases. By Theorem IV.23, it suffices to show that for $a, b \in k$ ($a, b \neq 0$ in the second case), $\dfrac{1}{t-a}$ and $\dfrac{1}{t-b}$ are different in R/G_R whenever $a \neq b$. Otherwise there exists $\dfrac{h(t)}{g(t)} \in R$ with $h(t), g(t) \in k[t]$ ($g(t)$ is not divisible by t in the second case) and $\gcd(h(t), g(t)) = 1$ such that

$$\frac{1}{t-a} - \frac{1}{t-b} = \frac{h(t)^p}{g(t)^p} - \frac{h(t)}{g(t)}$$

which gives

$$\frac{a-b}{t^2-(a+b)t+ab} = \frac{h(t)^p - h(t)\cdot g(t)^{p-1}}{g(t)^p}. \tag{4.19}$$

Since $\gcd(h(t), g(t)) = 1$, $\gcd(h(t)^p - h(t)\cdot g(t)^{p-1}, g(t)^p) = 1$. So from (4.19) we know that $g(t)^p | (t^2 - (a+b)t + ab)$. This is clearly impossible.

Example IV.26. Let $(R, \mathfrak{m}, K)$ be a regular local ring of dimension $d \geq 1$. Let $E = E(R/\mathfrak{m})$ be the injective hull of the residue field. We will show that $\mathrm{Ext}^1_{F_R}(R,E)$ is not zero and is in fact infinite when K is infinite. In particular, $E = E(R/\mathfrak{m})$, though injective as an R-module, is *not* injective as an F_R-module (with its standard F_R-structure) when $\dim R \geq 1$.

Recall that $E = \varinjlim_n \dfrac{R}{(x_1^n, \ldots, x_d^n)}$. So every element z in E can be expressed as $(r; x_1^n, \ldots, x_d^n)$ for some $n \geq 1$ (which means z is the image of r in the n-th piece in this direct limit system). By Theorem IV.23, $\mathrm{Ext}^1_{F_R}(R,E) \cong E/G_E$. I claim that any two different socle elements u_1, u_2 are different in E/G_E. For if this were not true, we would have:

$$u_1 - u_2 = z^p - z \tag{4.20}$$

in E. Since $u_1 - u_2$ is a nonzero element in the socle of E, we may write $u_1 - u_2 = (\lambda; x_1, \ldots, x_d)$ for some $\lambda \neq 0$ in K. Say $z = (r; x_1^n, \ldots, x_d^n)$ with n minimum. Then (4.20) gives

$$(r; x_1^n, \ldots, x_d^n) = (\lambda; x_1, \ldots, x_d) + (r^p; x_1^{np}, \ldots, x_d^{np}).$$

This yields

$$r^p + \lambda(x_1 \cdots x_d)^{np-1} - r(x_1 \cdots x_d)^{np-n} \in (x_1^{np}, \ldots, x_d^{np}). \tag{4.21}$$

If $n = 1$, then $0 \neq z \in \mathrm{socle}(E)$, hence r is a nonzero unit in R. But (4.21) shows that $r^p \in (x_1, \ldots, x_d)$ which is a contradiction.

If $n \geq 2$, we have $np-1 \geq np-n \geq p$. We know from (4.21) that for every $1 \leq i \leq d$, we have $r^p \in (x_1^{np}, \ldots, x_{i-1}^{np}, x_i^p, x_{i+1}^{np}, \ldots, x_d^{np})$. Hence $r \in (x_1^n, \ldots, x_{i-1}^n, x_i, x_{i+1}^n, \ldots, x_d^n)$ for every $1 \leq i \leq d$. Taking their intersection, we get that $r \in (x_1 \cdots x_d, x_1^n, \ldots, x_d^n)$. That is, mod $(x_1^n, \ldots, x_d^n)$, we have $r = (x_1 \cdots x_d)r_0$. But then we have $z = (r_0; x_1^{n-1}, \ldots, x_d^{n-1})$ contradicting our choice of n.

Therefore we have proved that any two different socle elements u_1, u_2 are different in $\mathrm{Ext}^1_{F_R}(R, E) = E/G_E$. This shows that $\mathrm{Ext}^1_{F_R}(R, E) \neq 0$ and is infinite when K is infinite.

CHAPTER V

Lech's Conjecture

5.1 Lech's Conjecture and Generalized Lech's Conjectures

We begin with the long standing open question of Lech [39] and [40]:

Conjecture V.1 (Lech's Conjecture)**.** *Let* $(R, \mathfrak{m}, K) \to (S, \mathfrak{n}, L)$ *be a flat local extension of local rings. Then* $e_R \leq e_S$*.*

This (seemingly simple) conjecture turns out to be supremely hard. It has now stood for over fifty years, and remains open in almost all cases, with the best partial results still those proved in Lech's original two papers [39] and [40]. There it was pointed out that the conjecture can be reduced to the case $\dim R = \dim S$ and the conjecture was proved in the following cases

1. R has dimension ≤ 2.

2. $S/\mathfrak{m}S$ is a complete intersection.

With the development of the theory of *linear maximal Cohen-Macaulay modules* (also known as *Ulrich modules*, see [25]), Lech's Conjecture V.1 can be proved in some new cases (basically for rings which admit Ulrich modules). Hanes improved this treatment and, combined with some characteristic $p > 0$ methods, proved Lech's Conjecture in some more cases in [16]. Hanes's results are mostly for *standard graded* rings $(R, \mathfrak{m}, K)$, which means that R is an $\mathbb{N}$-graded K-algebra generated over K by degree 1 forms. In sum, Conjecture V.1 was proved when

1. (Herzog, Ulrich, Backelin [25]) R is a Cohen-Macaulay ring of *minimal multiplicity*, i.e., $e_R = \operatorname{edim} R - \dim R + 1$ where $\operatorname{edim} R$ denote the embedding dimension of R.

2. (Herzog, Ulrich, Backelin [25]) R is a *strict complete intersection*, i.e., $gr_{\mathfrak{m}}R$ is a complete intersection.

3. (Hanes [16]) R is a 3-dimensional standard graded K-algebra with K a perfect field of characteristic $p > 0$.

4. (Hanes [17]) $(R, \mathfrak{m})$, $(S, \mathfrak{n})$ are both standard graded K-algebras and the map $R \to S$ sends a minimal reduction of $\mathfrak{m}$ to homogeneous elements in S.

We note that in general, having minimal multiplicity and being a strict complete intersection are quite strong conditions. For example, a hypersurface has minimal multiplicity if and only if its multiplicity is less than or equal to 2. And not all complete intersections are strict complete intersections. We will explain these conditions in detail in Section 4.

B. Herzog has many partial results on Conjecture V.1 when we put various conditions on the closed fibre $S/\mathfrak{m}S$. Some generalize and recover Lech's original result when $S/\mathfrak{m}S$ is a complete intersections. We refer to [22] and [24] for details. However, so far as we know Lech's Conjecture is open as long as $\dim R \geq 3$. In fact, to the best of our knowledge it is open if R is a 3-dimensional Cohen-Macaulay ring in equal characteristic $p > 0$.

We want to attack Conjecture V.1 in a new and different way. More specifically, we want to study the following stronger conjectures which we call them "Generalized

Lech's Conjectures". These are open questions, and are interesting in their own right. What's more, we will show that all of them imply Lech's Conjecture (some under Cohen-Macaulay conditions).

Conjecture V.2 (Generalized Lech's Conjecture)**.** *Let $(R, \mathfrak{m}, K) \to (S, \mathfrak{n}, L)$ be a local map of local rings such that $fd_R S < \infty$. Then $e_R \leq e_S$.*

Conjecture V.3 (Cyclic Generalized Lech's Conjecture)**.** *Let $(R, \mathfrak{m}, K)$ be a local ring and I be an ideal in R such that $pd_R R/I < \infty$. Then $e_R \leq e_{R/I}$.*

Conjecture V.4 (Weakened Generalized Lech's Conjecture)**.** *Let $(R, \mathfrak{m}, K)$ be a local ring and I be an ideal in R such that $pd_R R/I < \infty$ and $l_R(R/I) < \infty$. Then $e_R \leq l_R(R/I)$.*

5.2 Structure of local maps

In this section we will show the relations among the Generalized Lech's Conjectures. We will show Conjecture V.3 $\Longleftrightarrow$ Conjecture V.2 $\Longrightarrow$ Conjecture V.1 holds in general, and Conjecture V.4 $\Longleftrightarrow$ Conjecture V.3 $\Longleftrightarrow$ Conjecture V.2 $\Longrightarrow$ Conjecture V.1 for Cohen-Macaulay couples (we note that the conditions in Conjecture V.4 imply that R is Cohen-Macaulay by the New Intersection Theorem [53]).

We will use the following structure theorem of local maps from [4] throughout. In particular, Conjecture V.3 $\Longleftrightarrow$ Conjecture V.2 will follow easily from it.

Theorem V.5 (Cohen Factorization [4])**.** *Any local homomorphism $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ with S complete can be factored as $(R, \mathfrak{m}) \to (T, \mathfrak{n}) \to (S, \mathfrak{n})$ such that $(R, \mathfrak{m}) \to (T, \mathfrak{n})$ is flat with $T/\mathfrak{m}T$ regular and $(T, \mathfrak{n}) \to (S, \mathfrak{n})$ is surjective.*

We also need the following theorem repeatedly throughout the sequel. For a proof we refer to Matsumura's book [46].

Theorem V.6. *Let $(R, \mathfrak{m}, K) \to (T, \mathfrak{n}, L)$ be a flat local map with $T/\mathfrak{m}T$ regular. Let M be a finitely generated T-module which is flat over R. If $x_1, \ldots, x_d$ is part of a regular system of parameters in $T/\mathfrak{m}T$, then it is a regular sequence on M. Moreover $M/(x_1, \ldots, x_d)M$ is faithfully flat over R.*

We will need the following result throughout. It is proved in [4]. We include a proof here without using spectral sequences.

Theorem V.7. *Let $(R, \mathfrak{m}, K) \to (T, \mathfrak{n}, L)$ be a flat local map with $T/\mathfrak{m}T$ regular and let M be a finitely generated T-module such that $fd_R M < \infty$. Then $pd_T M \leq fd_R M + \dim T/\mathfrak{m}T$. In particular, $pd_T M < \infty$.*

Proof. Set $\dim T/\mathfrak{m}T = d_1$, $fd_R M = d_2$. We want to show $pd_T M \leq d_1 + d_2$. We have the following exact sequence:

$$0 \to Syz_T^{d_2} M \to T^{n_{d_2-1}} \to \cdots \to T^{n_1} \to T^{n_0} \to M \to 0.$$

Because T is flat over R and $fd_R M = d_2$, $Syz_T^{d_2} M$ is a flat R-module. Now we resolve $Syz_T^{d_2} M$ over T. We get:

$$(5.1) \qquad 0 \to Syz_T^{d_1+d_2} M \to T^{m_{d_1-1}} \to \cdots \to T^{m_1} \to T^{m_0} \to Syz_T^{d_2} M \to 0.$$

In the above resolution all modules are flat over R, so (5.1) will remain exact after we apply $- \otimes_T T/\mathfrak{m}T$, since this is the same as $- \otimes_R R/\mathfrak{m}$. Hence, after tensoring with $T/\mathfrak{m}T$, (5.1) will give a free resolution of $Syz_T^{d_2} M/\mathfrak{m} Syz_T^{d_2} M$ over $T/\mathfrak{m}T$ (since $T/\mathfrak{m}T$ is a regular local ring of dimension d_1). This means $Syz_T^{d_1+d_2} M \otimes_T T/\mathfrak{m}T$ is free over $T/\mathfrak{m}T$. So it suffices to show that if M is flat over R and $M/\mathfrak{m}M$ is free over $T/\mathfrak{m}T$ then M is free over T.

Pick $x_1, \ldots, x_{d_1}$ a regular system of parameters of $T/\mathfrak{m}T$, by Theorem V.6 we know that $x_1, \ldots, x_{d_1}$ is a regular sequence on both T and M and $T/(x_1, \ldots, x_{d_1})T$ and $M/(x_1, \ldots, x_{d_1})M$ are both faithfully flat over R.

Take a free resolution of $M/(x_1,\dots,x_{d_1})M$ over $T/(x_1,\dots,x_{d_1})T$. We can view it as a flat resolution of $M/(x_1,\dots,x_{d_1})M$ over R. Moreover, we can view $-\otimes_{T/(x_1,\dots,x_{d_1})T} L$ as $-\otimes_R K$ when applied to the resolution. So we have:

$$\mathrm{Tor}_i^{T/(x_1,\dots,x_{d_1})T}(M/(x_1,\dots,x_{d_1})M, L) = \mathrm{Tor}_i^R(M/(x_1,\dots,x_{d_1})M, K) = 0.$$

So $M/(x_1,\dots,x_{d_1})M$ is flat (equivalently, free) over $T/(x_1,\dots,x_{d_1})T$. But since $x_1,\dots,x_{d_1}$ is a regular sequence on both T and M by Theorem V.6, we know that $\mathrm{Tor}_i^T(T/(x_1,\dots,x_{d_1})T, M) = 0$. By the local criterion of flatness [46], M is flat (equivalently, free) over T. This finishes the proof. □

We also need the following well-known simple lemma throughout.

Lemma V.8 (*cf.* Corollary 4 in [61])**.** *Let $(R,\mathfrak{m},K)$ be a local ring and $x_1,\dots,x_s$ is a regular sequence in R, then $e_{R/(x_1,\dots,x_s)} \geq e_R$.*

Proposition V.9. *Conjecture V.3 and Conjecture V.2 are equivalent. Hence both will imply Conjecture V.1.*

Proof. It is obvious that Conjecture V.2 will imply Conjecture V.3. For the converse, let $R \to S$ be a local map such that $fd_R S < \infty$. We first complete R and S, this will not change e_R and e_S, and we still have $fd_{\widehat{R}}\widehat{S} < \infty$. So we can assume both R and S are complete. By Theorem V.5, we can factor the map into $R \to T \to S$. Now apply Theorem V.7 to $M = S$, we get that $pd_T S < \infty$. Since we assume Conjecture V.3 is known, we have $e_S \geq e_T$. But $e_R \leq e_T$ because it is easy to see that if $R \to T$ is faithfully flat with regular fibres, then there is an induced map from $gr_{\mathfrak{m}}R \to gr_{\mathfrak{n}}T$ which is faithfully flat (this is called *tangentially flat*, see [23] for details). □

Remark V.10. 1. Lemma V.8 holds under the weaker condition that $x_1,\dots,x_s$ is part of a system of parameters, and we have $e_{R/(x_1,\dots,x_s)} \geq (\prod_{i=1}^s \mathrm{ord}(x_i))e_R$

where $\mathrm{ord}(f)$ denote the $\mathfrak{m}$-adic order of f, i.e., the largest number a such that $f \in \mathfrak{m}^a$ (see Corollary 4 in [61] for an elementary proof of this result). Also notice that Lemma V.8 tells us immediately that Conjecture V.3 holds if I is generated by a regular sequence.

2. When $R \to T$ is faithfully flat with regular closed fibre, one can actually show that $e_R = e_T$. Because we may complete R and T hence assuming both of them are complete local rings. Let $x_1, \dots, x_d$ be a regular system of parameters in $T/\mathfrak{m}T$, we have $T/(x_1, \dots, x_d)T$ is faithfully flat over R (by Theorem V.6) whose fibre is a field. So $e_R = e_{T/(x_1,\dots,x_d)T} \geq e_T$ where the last inequality is by V.8. On the other hand, let $Q = \mathfrak{m}T$ which is prime in T. We know $(R, \mathfrak{m}) \to (T_Q, QT_Q)$ is faithfully flat whose closed fibre is a field. So $e_R = e_{T_Q} \leq e_T$ where the last inequality is by the localization formula for multiplicities (see [40] or [5]).

Proposition V.11. *Let $(R, \mathfrak{m}, K)$ be a Cohen-Macaulay local ring. Assume Conjecture V.4 is true for R. If I is an ideal in R such that $pd_R R/I < \infty$ and R/I is Cohen-Macaulay, then $e_R \leq e_{R/I}$.*

Proof. First we may assume that K is infinite, since we can replace R by $R(t) = R[t]_{\mathfrak{m}R[t]}$, this will not change the multiplicities and will preserve $pd_R R/I < \infty$ and Cohen-Macaulayness of R/I. Now we pick a linear reduction $(x_1, \dots, x_s)$ of $\mathfrak{m}(R/I)$. Since R/I is Cohen-Macaulay, $x_1, \dots, x_s$ is a regular sequence on R/I, hence

$$pd_R R/(I + (x_1, \dots, x_s)) < \infty, l_R(R/(I + (x_1, \dots, x_s)) < \infty.$$

So

$$e_{R/I} = l_R(R/(I + (x_1, \dots, x_s)) \geq e_R$$

where the last inequality is because we assume Conjecture V.4 is known for R. $\square$

Proposition V.12. *For Cohen-Macaulay couples, Conjecture V.4 and Conjecture V.2 are equivalent.*

Proof. This is clear from Proposition V.9 and Proposition V.11. □

Remark V.13. 1. Note that in spite of Proposition V.9 and Proposition V.12, we do not know the equivalence of Conjecture V.4, Conjecture V.3, and Conjecture V.2 for any specific R (i.e., it is not clear that Conjecture V.4 or even Conjecture V.3 for a specific R will imply that V.2 is true for this R). In the proofs we pass to T, which is a different ring.

2. Nevertheless, Proposition V.11 does tell us that Conjecture V.4 and the Conjecture V.3 are equivalent for any specific (Cohen-Macaulay) R with R/I Cohen-Macaulay.

3. The original Lech's Conjecture, Conjecture V.1, can be reduced to the case that $\dim R = \dim S$ (see [39]). In order to prove Conjecture V.1 for R Cohen-Macaulay, we can assume S is Cohen-Macaulay for free. Combining this with Proposition V.12, we get that the Weakened Generalized Lech's Conjecture (=Conjecture V.4) will imply Lech's Conjecture (=Conjecture V.1) if R is Cohen-Macaulay (with no further assumptions on S).

We end this section with three important theorems (which are well-known to experts) on modules of finite projective dimension. These theorems will be used repeatedly throughout the article. For proofs we refer to [2], [8] and [51].

Theorem V.14. *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module such that $pd_R M < \infty$ and $\operatorname{Ann}_R M \neq 0$. Then $\operatorname{Ann}_R M$ contains a nonzerodivisor.*

Theorem V.15. *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module* *such that* $pd_R M < \infty$. *If* N *is an MCM over* R, *then* $\operatorname{Tor}_i^R(M,N) = 0$ *for every* $i \geq 1$.

Theorem V.16. *Let* $(R,\mathfrak{m})$ *be a local ring and* M *be a finitely generated* R*-module such that* $pd_R M < \infty$. *If* $x \in \operatorname{Ann}_R M$ *is such that* $x \notin \mathfrak{m}^2 \cup_{P \in \operatorname{Ass}_R R} P$ *(i.e.,* x *is a nonzerodivisor and* $x \notin \mathfrak{m}^2$*), then* $pd_{R/x} M < \infty$.

5.3 First examples of the Generalized Lech's Conjectures

In this section we look at examples of the Generalized Lech's Conjectures. We show that the Cyclic Generalized Lech's Conjecture (=Conjecture V.3) is true when R is a standard graded K-algebra and I is a homogeneous ideal, or when R/IR is obtained from a flat base change of a regular local ring A and an ideal I in A in equal characteristic. These are based on the classical results on Hilbert functions and Serre's reduction to the diagonal.

We start by proving the graded case. We first recall that for a finitely generated $\mathbb{Z}$-graded module M over an $\mathbb{N}$-graded K-algebra R, the Hilbert series $H_M(t)$ is defined to be

$$H_M(t) = \sum_i \dim_K(M_i)t^i$$

where M_i denote the degree i-piece of M. Since we require M be finitely generated, $H_M(t)$ is a well-defined element in $\mathbb{Z}((t)) = \mathbb{Z}[[t]][t^{-1}]$. When R is standard graded, it is well known that

$$H_M(t) = \frac{h_M(t)}{(1-t)^{d_M}}$$

where d_M is the dimension of M and $h_M(t)$ is a Laurent polynomial in $\mathbb{Z}[t,t^{-1}]$ with $e_R(M) = h_M(1)$.

Theorem V.17 (*cf.* Lemma 7 in [3] or [52])**.** *Let* $(R,\mathfrak{m},K)$ *be a standard graded* K*-algebra with* $\mathfrak{m}$ *the irrelevant ideal. Let* M*,* N *be finitely generated* $\mathbb{Z}$*-graded* R*-*

modules. Then $\chi^R(M,N)(t) = \sum_i (-1)^i H_{\mathrm{Tor}_i^R(M,N)}(t)$ *is a well defined element of* $\mathbb{Z}((t))$ *and we have an equality of formal Laurent series:*

$$\chi^R(M,N)(t) = \frac{H_M(t)H_N(t)}{H_R(t)}.$$

The result below is already known [3], but we give a short proof for completeness.

Proposition V.18 (*cf.* Proposition 8 in [3])**.** *Let* $(R, \mathfrak{m}, K)$ *be a standard graded* K*-algebra with* $\mathfrak{m}$ *the irrelevant ideal. Let* I *be a homogeneous ideal of* R *such that* $pd_R R/I < \infty$*. Then* e_R *divides* $e_{R/I}$*. In particular* $e_R \leq e_{R/I}$*.*

Proof. We know that $H_R(t) = \dfrac{h_R(t)}{(1-t)^d}$, $H_{R/I} = \dfrac{h_{R/I}(t)}{(1-t)^{d'}}$ where $d = \dim R$, $d' = \dim R/I$. Such that $h_R(t)$, $h_{R/I}(t)$ are Laurent polynomials in $\mathbb{Z}[t, t^{-1}]$ with $h_R(1) = e_R$, $h_{R/I}(1) = e_{R/I}$. Apply Theorem V.17 to $M = R/I$ and $N = K$ we get:

$$\chi^R(R/I, K)(t) \cdot h_R(t) = h_{R/I}(t) \cdot (1-t)^{d-d'}.$$

Since $pd_R R/I < \infty$, each $\mathrm{Tor}_i^R(R/I, K)$ is a finite dimensional graded K-vector space and $\mathrm{Tor}_i^R(R/I, K) = 0$ for $i > pd_R R/I$, it follows that $\chi^R(R/I, K)(t)$ is a Laurent polynomial. Since $d - d' \geq 0$ and $h_R(1) \neq 0$, we know that $h_R(t)$ divides $h_{R/I}(t)$ in $\mathbb{Z}[t, t^{-1}]$ by the unique factorization in $\mathbb{Z}[t, t^{-1}]$. In particular, $e_R = h_R(1)$ divides $e_{R/I} = h_{R/I}(1)$. $\square$

Remark V.19. In the non-graded case, one cannot expect to prove e_R divides $e_{R/I}$ when $pd_R R/I < \infty$. Here is a simple counter-example: take $R = k[[t^2, t^3]]$ and $I = (t^3)$. R is a one-dimensional complete local domain and I is generated by a nonzerodivisor, so $pd_R R/I < \infty$ is satisfied. However, one can check easily that $e_R = 2$ while $e_{R/I} = l_R(R/I) = 3$.

A typical example that one can write down of an ideal I of R such that $pd_R R/I < \infty$ is that R is flat over some regular local ring A and $I = I_0 R$ where I_0 is an ideal in A. We will show that, in equal characteristic, for this class of examples, $e_{R/I} \geq e_R$ always holds. The proof is a modification of Herzog's argument in [22]. It is essentially Serre's trick of reduction to the diagonal.

Theorem V.20. *Let $(A, \mathfrak{m}_0, K_0)$ be a regular local ring of equal characteristic. Let $(A, \mathfrak{m}_0, K_0) \to (R, \mathfrak{m}, K)$ be a flat local extension. Let I be an arbitrary ideal in A. Then we have $e_{R/IR} \geq e_R \cdot e_{A/I}$, in particular $e_{R/IR} \geq e_R$ so Conjecture V.3 holds in this case.*

Proof. Without loss of generality we may assume both A and R are complete, since completion does not affect multiplicity. By Theorem V.5, the map $A \to R$ factors through

$$(A, \mathfrak{m}_0, K_0) \to (T, \mathfrak{n}, K) \xrightarrow{\pi} (R, \mathfrak{m}, K)$$

where $A \to T$ is flat with $T/\mathfrak{m}_0 T$ regular, and $T \xrightarrow{\pi} R$ is surjective.

Since A is a complete regular local ring, so is T (we can complete T if necessary). Say $T \cong K[[x_1, \dots, x_n]]$. By Serre's reduction to the diagonal, we know that

$$R/IR \cong T/IT \otimes_T R \cong \frac{(T/IT)\widehat{\otimes}_K R}{(1 \otimes \pi(x_i) - \overline{x_i} \otimes 1, 1 \leq i \leq n)}$$

where $\overline{x_i}$ denotes the image of x_i in T/IT.

Notice that both R and T are flat over A, this implies that

$$\operatorname{ht} IT = \operatorname{ht} IR = \operatorname{ht} I,$$

so we have

$$\dim R - \dim R/IR = \operatorname{ht} I = \dim T - \dim T/IT$$

$$\dim R/IR = \dim T/IT + \dim R - n.$$

Since the dimension of $(T/IT)\widehat{\otimes}_K R$ is $\dim T/IT + \dim R$, the above shows that $\{1 \otimes \pi(x_i) - \overline{x_i} \otimes 1\}_{i=1}^n$ are part of a system of parameters in $(T/IT)\widehat{\otimes}_K R$. By Remark

V.10 (1), we know that

$$e_{R/IR} \geq e_{(T/IT)\widehat{\otimes}_K R} = e_{T/IT} \cdot e_R \tag{5.2}$$

where the last equality is because R and T/IT are homomorphic image of T, so we have a natural bijective map $gr_{\mathfrak{n}}(T/IT) \otimes_K gr_{\mathfrak{m}} R \to gr_{\mathfrak{n}_0}((T/IT)\widehat{\otimes}_K R)$, where $\mathfrak{n}_0$ denote the maximal ideal of $(T/IT)\widehat{\otimes}_K R$. Now just notice that $e_{T/IT} = e_{A/I}$ because $A \to T$ is flat with $T/\mathfrak{m}_0 T$ regular (see Remark V.10). $\square$

Remark V.21. We don't know how to prove the above theorem if A does not contain a field.

5.4 Rings of minimal multiplicity and strict complete intersections

In this section we show that Conjecture V.4 is true when R is either a Cohen-Macaulay ring of minimal multiplicity or a strict complete intersection. In view of the results in section 2.2, this generalizes the results on Lech's Conjecture. We also prove that the existence of a sequence of MCM with reduction degree approaching 1 (in the sense of Hanes [16]) will imply Conjecture V.2 when the target ring is a Cohen-Macaulay domain. The results in this section strongly suggest that the existence of Ulrich modules should imply Conjecture V.4 or even Conjecture V.2, and should have surprising consequences about modules of finite length and finite projective dimension.

We first recall that if $(R, \mathfrak{m})$ is a local ring and M is an MCM over R, then we always have $e_R(M) \geq \nu_R(M)$. We begin with some basic definitions.

Definition V.22. An MCM M over $(R, \mathfrak{m})$ is called a linear MCM (or an Ulrich module) if $e_R(M) = \nu_R(M)$. If $(x_1, \ldots, x_d)$ is a minimal reduction of $\mathfrak{m}$, then M is a linear MCM if and only if $(x_1, \ldots, x_d)M = \mathfrak{m}M$.

Definition V.23. A Cohen-Macaulay ring $(R, \mathfrak{m})$ of dimension d is said to have *minimal multiplicity* if $e_R = \operatorname{edim}(R) - d + 1$ (note that for a Cohen-Macaulay ring $(R, \mathfrak{m})$ we always have $e_R \geq \operatorname{edim}(R) - d + 1$).

Definition V.24. A local ring $(R, \mathfrak{m})$ which is a quotient of a regular local ring is called a strict complete intersection if $gr_{\mathfrak{m}}R$ is a complete intersection (in particular hypersurfaces and standard graded complete intersections are strict complete intersections). Strict complete intersections are complete intersections [25].

Remark V.25.

1. Let $(R, \mathfrak{m}) = (S, \mathfrak{m})/(f_1, \ldots, f_n)$ be a complete intersection (here $(S, \mathfrak{m})$ is a regular local ring). Then it is well known that $e_R \geq \prod_{i=1}^{n} \operatorname{ord}(f_i)$ where $\operatorname{ord}(f)$ denote the $\mathfrak{m}$-adic order of f, i.e., the largest number a such that $f \in \mathfrak{m}^a$. When R is a strict complete intersection, it is easy to check that $e_R = \prod_{i=1}^{n} \operatorname{ord}(f_i)$, see [25] or Corollary 4 in [61].

2. Not all complete intersections are strict complete intersections. For example, let $R = k[[x, y, z]]/(x^2 - y^3, xy - z^3)$. One can check that $R \cong k[[t^5, t^6, t^9]]$ with t^5 a minimal reduction of the maximal ideal. It is straightforward to check that $e_R = 5$ while $\operatorname{ord}(x^2 - y^3) \cdot \operatorname{ord}(xy - z^3) = 4$. So R is not a strict complete intersection.

Theorem V.26. *Let $(R, \mathfrak{m}, K)$ be a Cohen-Macaulay local ring with minimal multiplicity and I be an ideal in R such that $pd_R R/I < \infty$ and $l_R(R/I) < \infty$. Then $e_R \leq l_R(R/I)$. Moreover, if we assume K is infinite, then $e_R = l_R(R/I)$ if and only if I is a minimal reduction of $\mathfrak{m}$.*

Proof. We may assume $d = \dim R > 0$, since otherwise the only ideal of finite projective dimension is 0 and the result is obvious.

If $I \subseteq \mathfrak{m}^2$, then $l_R(R/I) \geq l_R(R/\mathfrak{m}^2) = \operatorname{edim} R + 1 > \operatorname{edim} R - d + 1 = e_R$ where the last equality is by definition of minimal multiplicity. If $I \nsubseteq \mathfrak{m}^2$, by prime avoidance, we can pick some $x_1 \in I$ but $x_1 \notin \mathfrak{m}^2 \cup_{P \in \operatorname{Ass} R} P$. We can kill x_1 and replace R, I by $R_1 = R/x$, $I_1 = IR_1$. We have $\operatorname{edim} R_1 = \operatorname{edim} R - 1$, $\dim R_1 = d - 1$, $l_R(R/I) = l_{R_1}(R_1/I_1)$ and $pd_{R_1} R_1/I_1 < \infty$ by Theorem V.16. Now if $I_1 \subseteq \mathfrak{m}_1^2$ (where $\mathfrak{m}_1 = \mathfrak{m}R_1$), then $l_R(R/I) = l_{R_1}(R_1/I_1) \geq \operatorname{edim} R_1 + 1 = \operatorname{edim} R - 1 + 1 \geq \operatorname{edim} R - d + 1 = e_R$. If $I_1 \nsubseteq \mathfrak{m}_1^2$, we pick some $x_2 \in I_1$ but $x_2 \notin \mathfrak{m}_1^2 \cup_{P \in \operatorname{Ass} R_1} P$. Now we kill x_2 and similarly we get R_2, I_2, $\mathfrak{m}_2$ and we repeat the above process.

Note that, each time we kill an x_j, the length of R/I stays the same (that is, $l(R/I) = l(R_j/I_j)$ for every j) while the dimension and the embedding dimension both drop by 1. If the process stops after r steps with $r < d$, which means, $I_r \subseteq \mathfrak{m}_r^2$. Then we have that $l_R(R/I) = l_{R_r}(R_r/I_r) \geq \operatorname{edim} R_r + 1 = \operatorname{edim} R - r + 1 > \operatorname{edim} R - d + 1 = e_R$ (in particular $l_R(R/I) = e_R$ cannot occur in this case). Otherwise we will stop after d steps and we get an Artinian ring R_d with an ideal I_d of finite projective dimension, such an ideal must be 0. So in this case, $l_R(R/I) = l_{R_d}(R_d) \geq \operatorname{edim} R_d + 1 = \operatorname{edim} R - d + 1 = e_R$. Moreover, in this case the original I must be generated by a regular sequence, namely $I = (x_1, \ldots, x_d)$. If equality occurs, we know that $e(I) = l_R(R/I) = e_R$ hence I must be a linear reduction of $\mathfrak{m}$. $\square$

In order to prove Conjecture V.4 for strict complete intersections, we need the following result proved in [25].

Theorem V.27 (*cf.* Theorem 1.2 and Theorem 2.1 in [25])**.** *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay ring and let $f \in \mathfrak{m}^d$ be a nonzero divisor. Then there exists some integer s such that there exists a chain of R/f-modules: $0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_d \cong (R/f)^s$, such that:*

1. $U_{i-1} \subseteq \mathfrak{m}U_i$.

2. *Each $M_i = U_i/U_{i-1}$ is an MCM over R/f, with finite projective dimension over R (actually they all have projective dimension 1), and is minimally generated by s elements.*

Theorem V.28. *Let $(S, \mathfrak{m}, K)$ be a regular local ring with K an infinite field and let $(R, \mathfrak{m}) \cong (S, \mathfrak{m})/(f_1, \ldots, f_n)$ be a complete intersection. Let $d_i = \operatorname{ord}(f_i)$ be the $\mathfrak{m}$-adic order of f_i. Let W be an R-module of finite length and finite projective dimension. Let $d_R = \prod_{i=1}^n d_i$. We have*

1. *There exists some h and a chain of R-modules*

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{d_R} \cong R^h$$

such that each $M_i = U_i/U_{i-1}$ is an MCM over R and minimally generated by h elements.

2. *$l_R(W) \geq d_R$, and $l_R(W) = d_R$ if and only if $W \cong R/I$ for I a minimal reduction of $\mathfrak{m}$ (and in this case we have $l_R(W) = d_R = e_R$).*

Proof. We use induction on n. When $n = 0$, $R \cong S$ is regular and all the conclusion are obvious. Now let $T = (S, \mathfrak{m})/(f_1, \ldots, f_{n-1})$, so $(R, \mathfrak{m}) = (T, \mathfrak{m})/f_n$. By induction we know the result for T.

By Theorem V.27 there exits a chain of R-modules:

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{d_n} \cong R^s \tag{5.3}$$

such that each $M_i = U_i/U_{i-1}$ is an MCM over R, with finite projective dimension over T (actually projective dimension 1), and is minimally generated by s elements.

Let N be any MCM over T, apply $\otimes_T N$ to (5.3), we get

$$0 = U_0 \otimes_T N \subseteq U_1 \otimes_T N \subseteq \cdots \subseteq U_{d_n} \otimes_T N \cong (R \otimes_T N)^s \tag{5.4}$$

We still have "$\subseteq$" because $\mathrm{Tor}_1^T(N, M_i) = 0$ (note that $pd_T M_i < \infty$ and N is an MCM over T and we apply Theorem V.15). Moreover, $\widetilde{M_i} = M_i \otimes_T N \cong \dfrac{U_i \otimes_T N}{U_{i-1} \otimes_T N}$ is still an MCM over R, this is because we can write down the resolution of M_i over T, and tensor it with N, we get a resolution of $\widetilde{M_i}$ by N by Theorem V.15 and we can compute the depth to check that each $\widetilde{M_i}$ is an MCM over R.

We first prove (1). By the induction hypothesis we know that T^h has a filtration of length d_T with successive quotients N_i, each is an MCM over T, minimally generated by h elements. So we know that $(R \otimes_T T^h)^s = R^{hs}$ has a filtration of length d_T with successive quotients $(R \otimes_T N_i)^s$. But by (5.4), each $(R \otimes_T N_i)^s$ has a filtration of length d_n with MCM quotients, each minimally generated by $\nu(N_i) \cdot s = hs$ elements. Combining these filtrations we know that R^{hs} has a filtration with the desired property.

Next we prove (2). We apply $- \otimes_R W$ to (5.4) to get:

(5.5)

$$0 = (U_0 \otimes_T N) \otimes_R W \subseteq (U_1 \otimes_T N) \otimes_R W \subseteq \cdots \subseteq (U_{d_n} \otimes_T N) \otimes_R W \cong (R \otimes_T N \otimes_R W)^s.$$

We still have "$\subseteq$" because $\mathrm{Tor}_1^R(W, \widetilde{M_i}) = 0$ (note that $pd_R W < \infty$ and each $\widetilde{M_i}$ is an MCM and we may apply Theorem V.15). Computing length in (5.5), we get:

$$\begin{aligned} s \cdot l_R(N \otimes_T W) &= l_R(R \otimes_T N \otimes_R W)^s = \sum_{i=1}^{d_n} l_R(\widetilde{M_i} \otimes_R W) \\ &\geq \sum_{i=1}^{d_n} l_R(\widetilde{M_i} \otimes_R K) = \sum_{i=1}^{d_n} l_R(M_i \otimes_T N \otimes_R K) \\ &= \sum_{i=1}^{d_n} l_R(\frac{M_i}{\mathfrak{m}M_i} \otimes \frac{N}{\mathfrak{m}N}) = \sum_{i=1}^{d_n} \nu(M_i) \cdot \nu(N) \\ &= d_n \cdot s \cdot \nu(N). \end{aligned}$$

After cancelling "s", we obtain

$$l_R(N \otimes_T W) \geq d_n \cdot \nu(N) \tag{5.6}$$

for every MCM N over T. But by induction we know there exists h such that T^h has a filtration

$$0 = V_0 \subseteq V_1 \subseteq \cdots\cdots \subseteq V_{d_T} = T^h \tag{5.7}$$

such that $N_i = V_i/V_{i-1}$ is an MCM over T and minimally generated by h elements. Hence we have

$$l_R(T^h \otimes_T W) = \sum_{i=1}^{d_T} l_R(N_i \otimes_T W) \geq \sum_{i=1}^{d_T} d_n \cdot \nu(N_i) = d_T \cdot d_n \cdot h. \tag{5.8}$$

Cancelling "h" we get $l_R(W) \geq d_T \cdot d_n = d_R$.

Finally, suppose we have $l_R(W) = d_R$ (that is, equality occurs). This means, in (5.8), equality occurs, which means that for each N that occurs as an N_i in (5.7), the inequality in (5.6) is an equality. We trace back the inequalities in the length computation of (5.5). We know that $l_R(\widetilde{M_i} \otimes_R W) = l_R(\widetilde{M_i} \otimes_R K)$ for each $\widetilde{M_i}$. The key point is, at least we know there exist an MCM M over R such that $l_R(M \otimes_R W) = l_R(M \otimes_R K)$.

From this we know W is cyclic, since otherwise $W/\mathfrak{m}W \cong K^n$, with $n > 1$. Then we will have $M \otimes_R W \twoheadrightarrow M \otimes_R K^n$, so $l_R(M \otimes_R W) \geq n \cdot l_R(M \otimes_R K) > l_R(M \otimes_R K)$, which is a contridiction. To see $W = R/I$ for some I minimal reduction of $\mathfrak{m}$, we first note that $I \nsubseteq \mathfrak{m}^2$, since otherwise we have

$$l_R(M \otimes_R W) = l_R(M/IM) \geq l_R(M/\mathfrak{m}^2 M) > l_R(M/\mathfrak{m}M) = l_R(M \otimes_R K)$$

which is a contradiction. So we may pick $y_1 \in I$ but $y_1 \notin \mathfrak{m}^2 \cup_{P \in Ass_R R} P$ by prime avoidence. Now we kill y_1 to get that $\overline{R} = R/y_1 R$ and $\overline{I} = I\overline{R}$ with $pd_{\overline{R}} \overline{R}/\overline{I} < \infty$, and $\overline{M} = M/y_1 M$ is still an MCM over $\overline{R}$ with $l_{\overline{R}}(\overline{M} \otimes_{\overline{R}} \overline{R}/\overline{I}) = l_{\overline{R}}(\overline{M} \otimes_{\overline{R}} K)$. By the same argument as above we know that $\overline{I} \nsubseteq \mathfrak{m}^2\overline{R}$ and we repeat the above process: that is, we find some $y_2 \notin \mathfrak{m}^2\overline{R} \cup_{P \in Ass_{\overline{R}} \overline{R}} P$ and kill y_2 and so on.

When we do this process $d = \dim R$ times, we get an Artinian ring with a proper ideal of finite projective dimension. Such an ideal must be 0. So the original ideal I must be generated by a regular sequence in R: the elements $y_1, y_2, \ldots, y_d$. So we have

$$e_R \leq e(I) = l_R(R/I) = d_R \leq e_R \tag{5.9}$$

where we have the last inequality because we always have $d_R = \prod_{i=1}^{n} d_i \leq e_R$ as $f_1, \ldots, f_n$ is a regular sequence in S (see Remark V.25). Hence, we must have all equalities in (5.9), so I must be a minimal reduction of $\mathfrak{m}$ in R. □

Corollary V.29. *Let $(R, \mathfrak{m}, K)$ be a strict complete intersection and I be an ideal in R such that $\mathrm{pd}_R R/I < \infty$ and $l_R(R/I) < \infty$. Then $e_R \leq l_R(R/I)$.*

Proof. Since we may pass from R to $R[t]_{\mathfrak{m}R[t]}$, we may assume K is infinite. The result follows immediately from Theorem V.28 applied to $W = R/I$ and the fact that $d_R = e_R$ for strict complete intersections (see Remark V.25). □

Since Cohen-Macaulay rings of minimal multiplicity and strict complete intersections are shown to admit linear MCM [25]. Theorem V.26 and Corollary V.29 make it natural to expect that the existence of linear MCM should imply Conjecture V.4, or even Conjecture V.2 for Cohen-Macaulay couples. At this point we don't have a proof of this, but we can handle the case when the target ring is assumed to be a Cohen-Macaulay domain and in fact we can prove a much stronger result.

Lemma V.30. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and I be an ideal in R such that $\mathrm{pd}_R R/I < \infty$ and R/I is Cohen-Macaulay. Let M be an MCM over R, then M/IM is an MCM over R/I.*

Proof. Let $0 \to R^{n_k} \to \cdots \to R^{n_1} \to R^{n_0} \to R/I \to 0$ be a minimal resolution of R/I over R. Tensoring with M we get: $0 \to M^{n_k} \to \cdots \to M^{n_1} \to M^{n_0} \to M/IM \to 0$.

This is still exact by Theorem V.15. So we have $\operatorname{depth}(M/IM) + k \geq \operatorname{depth} M = \dim R = \dim(R/I) + k$ (where the last equality we use R/I is Cohen-Macaulay), so $\operatorname{depth}(M/IM) \geq \dim R/I$. So we must have "=", hence M/IM is an MCM over R/I. □

Definition V.31 ([16]). A sequence of MCM $\{M_i\}_{i\geq 0}$ over $(R, \mathfrak{m})$ is said to have reduction degrees approaching t if for some minimal reduction I of $\mathfrak{m}$,

$$\lim_{i\to\infty} \frac{l_R((\mathfrak{m}^t + I)M_i/IM_i)}{l_R(M_i/IM_i)} = 0.$$

In particular, $\{M_i\}_{i\geq 0}$ have reduction degrees approaching 1 if and only if

$$\lim_{i\to\infty} \frac{\nu_R(M_i)}{e_R(M_i)} = 1.$$

Theorem V.32. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local map of local rings such that $fd_R S < \infty$ and S is a Cohen-Macaulay domain. If there exists a sequence of MCM $\{M_i\}$ over R with reduction degrees approaching* 1*. Then $e_R \leq e_S$.*

Proof. We may assume both R and S are complete. By Theorem V.5 there is a factorization $(R, \mathfrak{m}) \to (T, \mathfrak{n}) \to (S, \mathfrak{n})$ such that $R \to T$ is flat with $T/\mathfrak{m}T$ regular and $T \to S$ is surjective. Since S is a Cohen-Macaulay domain, it follows from Auslander's Zerodivisor Theorem and the New Intersection Theorem (see [53] and [54]) that T is also a Cohen-Macaulay domain. Since $R \to T$ is flat with $T/\mathfrak{m}T$ regular, it follows that R is also a Cohen-Macaulay domain. We may replace $(R, \mathfrak{m})$ by $(T, \mathfrak{n})$ and replace $\{M_i\}$ by $\{M_i \otimes_R T\}$. The conditions on $\{M_i\}$ still hold because both the multiplicity and least number of generators remain the same when we pass from $\{M_i\}$ to $\{M_i \otimes_R T\}$, and we also know that $e_R = e_T$ by Remark V.10. So without loss of generality we may assume R, S are both Cohen-Macaulay domains and $S \cong R/P$ with $pd_R S < \infty$ by Theorem V.7.

We have:

$$\begin{aligned} e_R &= \frac{e_R(M_i)}{\operatorname{rank}_R(M_i)} = \frac{\nu(M_i)}{\operatorname{rank}_R(M_i)} \cdot \frac{e_R(M_i)}{\nu(M_i)} \\ &= \frac{\nu(M_i/PM_i)}{\operatorname{rank}_R(M_i)} \cdot \frac{e_R(M_i)}{\nu(M_i)} \\ &\leq \frac{e_S(M_i \otimes_R S)}{\operatorname{rank}_R(M_i)} \cdot \frac{e_R(M_i)}{\nu(M_i)} \\ &= e_S \cdot \frac{\operatorname{rank}_S(M_i \otimes_R S)}{\operatorname{rank}_R(M_i)} \cdot \frac{e_R(M_i)}{\nu(M_i)} \end{aligned}$$

where the only inequality is because each M_i/PM_i is an MCM over R/P by Lemma V.30.

So if we can show $\operatorname{rank}_S(M \otimes_R S) = \operatorname{rank}_R(M)$ for all MCM M, then the above inequality will give $e_R \leq e_S \cdot \dfrac{e_R(M_i)}{\nu(M_i)}$. Let $i \to \infty$, by the assumption on the sequence $\{M_i\}$, we will get $e_R \leq e_S$.

Since $S = R/P$ and $pd_R S < \infty$, R_P is regular (because localizing the resolution of R/P over R will give a resolution of κ_P over R_P). Hence for any MCM M over R, M_P is R_P-free. So

$$\operatorname{rank}_S(M \otimes_R S) = \operatorname{rank}_{R/P}(M/PM) = \dim_{\kappa_P}(\frac{M_P}{PM_P}) = \operatorname{rank}_{R_P}(M_P) = \operatorname{rank}_R(M)$$

where the third equality is because M_P is R_P-free. □

Remark V.33.

1. Note that the condition in Theorem V.32 is trivially satisfied if R admits a linear MCM (we can take $M_i = M$ for all i). We also note that it is still an open question whether every Cohen-Macaulay ring has a linear MCM.

2. In [16], Hanes showed that if R is a 3-dimensional standard graded K-algebra with K a perfect field of characteristic $p > 0$, then R admits a sequence of MCM's $\{M_i\}$ with reduction degrees approaching 1.

5.5 Regularity defect

In this section we show that the regularity defect for local maps of rings of finite flat dimension does not decrease. This greatly generalized the corresponding result for flat local extensions as shown by Lech in [40]. Our approach here is quite different.

In [40], Lech showed that if $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a flat couple of local rings, then $\operatorname{edim}(R) - \dim(R) \le \operatorname{edim}(S) - \dim(S)$, i.e., the regularity defect of R is less than or equal to the regularity defect of S. In this section we extend this result. The key argument in the proof is very similar to the proof we used in Theorem V.26.

Theorem V.34. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local map with $fd_R S < \infty$, then we have:*

$$\operatorname{edim}(R) - \dim(R) \le \operatorname{edim}(S) - \dim(S). \tag{5.10}$$

Proof. We may assume both R and S are complete, we decompose the map $R \to S$ into $R \to T \to S$ where $R \to T$ flat with $T/\mathfrak{m}T$ regular and $T \to S$ is surjective by Theorem V.5. Moreover, $pd_T S < \infty$ by Theorem V.7.

First we consider $R \to T$, since this is flat, by Lech's result, we have:

$$\operatorname{edim}(R) - \dim R \le \operatorname{edim}(T) - \dim T \tag{5.11}$$

Next we consider $(T, \mathfrak{n}) \twoheadrightarrow (S, \mathfrak{n})$. Write $S = T/J$. I claim that:

$$\operatorname{edim}(T) \le \operatorname{edim}(S) + \operatorname{depth}_J T \tag{5.12}$$

This is easy if $J \subseteq \mathfrak{n}^2$, because in this case we have $\operatorname{edim}(S) = \operatorname{edim}(T)$, so (5.12) holds trivially. Now assume $J \nsubseteq \mathfrak{n}^2$. Since $pd_T(T/J) < \infty$, we know that J contains some nonzero divisor of T by Theorem V.14. So by prime avoidance, we may pick some $x \in J$ but $x \notin \mathfrak{n}^2 \cup_{P \in \operatorname{Ass} T} P$. Now we kill x and replace T by $\overline{T} = T/x$, J by $\overline{J} = J(T/x)$ and we still have $S = \overline{T}/\overline{J}$ with $pd_{\overline{T}} S < \infty$ by Theorem V.16. And once we do this, $\operatorname{edim}(T)$ will drop by 1 while $\operatorname{edim}(S)$ stays the same. But we can do this process for at most $\operatorname{depth}_J T$ times (we either end up with $J = 0$ or we stop at some point with $J \subseteq n^2$). This proves (5.12).

Since we always have $\operatorname{depth}_J T \leq \dim T - \dim(T/J) = \dim T - \dim S$, combined this with (5.12) we get:

$$\operatorname{edim}(T) - \dim T \leq \operatorname{edim}(S) - \dim S \tag{5.13}$$

Now (5.10) is clear from (5.11) and (5.13). □

We can modified the above proof to get another interesting result.

Proposition V.35. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local map with $fd_R S < \infty$, then we have:*

$$\operatorname{edim}(R) \leq \operatorname{edim}(S) + fd_R S$$

Proof. We may assume both R and S are complete, we decompose the map $R \to S$ into $R \to T \to S$ where $R \to T$ flat with $T/\mathfrak{m}T$ regular and $T \to S$ is surjective by Theorem V.5. Moreover, we have $pd_T S \leq fd_R S + \dim T/\mathfrak{m}T$ by Theorem V.7.

Consider $R \to T$, since this is flat with $T/\mathfrak{m}T$ regular, we can pick $x_1, \dots, x_d$ in $T/\mathfrak{m}T$ which forms a regular system of parameters. We know $R \to T/(x_1, \dots, x_d)T$ is still flat by Theorem V.6. So from Lech's result we know that

$$\operatorname{edim}(R) \leq \operatorname{edim}(T/(x_1, \dots, x_d)T) = \operatorname{edim}(T) - \dim T/\mathfrak{m}T \tag{5.14}$$

Now consider $T \twoheadrightarrow S$. By (5.1.3) we know $\operatorname{edim}(T) \leq \operatorname{edim}(S) + \operatorname{depth}_J T$. Since we always have $\operatorname{depth}_J T \leq pd_T(T/J)$ (see [47]). We have

$$\operatorname{edim}(T) \leq \operatorname{edim}(S) + pd_T S \leq \operatorname{edim}(S) + fd_R S + \dim T/\mathfrak{m}T \tag{5.15}$$

So combining (5.2.1) and (5.2.2), we get

$$\begin{aligned} \operatorname{edim}(R) &\leq \operatorname{edim}(T) - \dim T/\mathfrak{m}T \\ &\leq \operatorname{edim}(S) + fd_R S + \dim T/\mathfrak{m}T - \dim T/\mathfrak{m}T \\ &= \operatorname{edim}(S) + fd_R S. \end{aligned}$$

□

5.6 Lech's Conjecture for 3-dimensional Gorenstein rings

We will show that Conjecture V.4 is true for Cohen-Macaulay rings of dimension ≤ 2 or Gorenstein rings of dimension ≤ 3 (and in fact for a class of rings known as *numerically Roberts rings*: we will define this notion in the sequel) in equal characteristic $p > 0$. We use these results, combined with results on Hilbert-Kunz multiplicity, to show that Lech's Conjecture (=Conjecture V.1) is true for 3-dimensional Gorenstein rings of equal characteristic $p > 0$.

We begin with three lemmas which are characteristic free. After that we will assume R and S are local rings of equal characteristic $p > 0$ with perfect residue fields.

Lemma V.36. *$(R, \mathfrak{m}, K) \to (T, \mathfrak{n}, L)$ is a faithfully flat map with $T/\mathfrak{m}T$ regular. Let $I \subseteq J \subseteq R$ be two $\mathfrak{m}$-primary ideals. And let $x_1, \ldots, x_n \in T$ be a regular system of parameters in $T/\mathfrak{m}T$. Let q_1, q_2 be two sets of monomials in the x_i such that $q_1 \supseteq q_2$ and both contain some power of each x_i. Let Q_1 (resp., Q_2) be the set of all monomials in $\mathbb{Z}[X_1, \ldots, X_n]$ that can be generated by $X_1^{a_1} \cdots X_n^{a_n}$, corresponding to $x_1^{a_1} \cdots x_n^{a_n} \in q_1$ (resp, q_2). Then we have:*

$$l_T\left(\frac{T}{I + J(q_1) + (q_2)}\right) \geq l_R\left(\frac{R}{J}\right) \cdot \sharp\{\text{monomials} \in Q_1, \notin Q_2\} + l_R\left(\frac{R}{I}\right) \cdot \sharp\{\text{monomials} \notin Q_1\}$$

where the right hand side is the case $T \cong R[[x_1, \ldots, x_n]]$.

Proof. First of all we can kill I. Then $R/I \to T/IT$ is still faithfully flat with $T/\mathfrak{m}T$ regular and the lengths on both sides do not change. So we may assume $I = 0$ and R is an artinian local ring. Now we have:

$$(5.16)\qquad l_T(\frac{T}{J(q_1)+(q_2)}) = l_T(\frac{T}{(q_1)T}) + l_T(\frac{(q_1)T}{J(q_1)+(q_2)}) \geq l_T(\frac{T}{(q_1)T}) + l_T(\frac{(q_1)\overline{T}}{(q_2)\overline{T}})$$

where $\overline{T} \cong T/JT$. We have a $\geq$ because we have a natural surjection $\frac{(q_1)T}{J(q_1)+(q_2)} \twoheadrightarrow \frac{(q_1)+JT}{(q_2)+JT} \cong \frac{(q_1)\overline{T}}{(q_2)\overline{T}}$

Now I claim that $l_T(\frac{(q_1)\overline{T}}{(q_2)\overline{T}}) = l_R(\frac{R}{J}) \cdot \sharp\{\text{monomials} \in Q_1, \notin Q_2\}$. Take a filtration $(q_2)\overline{T} = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = (q_1)\overline{T}$, where each I_j is generated by one single monomial in the x_i that is not in I_{j-1} but after multiplying by any x_i, it is in I_{j-1}. Such a filtration has length exactly $\sharp\{\text{monomials} \in Q_1, \notin Q_2\}$. Since $x_1, \ldots, x_n$ is a regular sequence in $\overline{T}$ by Theorem V.6, each I_j/I_{j-1} is isomorphic to $\overline{T}/(x_1, \ldots, x_n)\overline{T}$. Hence $l_{\overline{T}}(I_j/I_{j-1}) = l_{\overline{T}}(\overline{T}/(x_1, \ldots, x_n)\overline{T}) = l_R(R/J)$, where the last equality is because $\overline{T}/(x_1, \ldots, x_n)\overline{T}$ is faithfully flat over R/J whose fibre is a field. This proves the claim.

Using the same trick one also shows that $l_T(\frac{T}{(q_1)T}) = l_R(R) \cdot \sharp\{\text{monomials} \notin Q_1\}$. Now go back to (5.16), which proves the lemma. $\square$

Lemma V.37. *Let N be an MCM of $(S, \mathfrak{n})$, we have:*

$$e_S(N) \geq \nu(N) + \nu(\mathfrak{n}N) - d \cdot \nu(N)$$

where $d = \dim S$.

Proof. We may assume S has infinite residue field. Let I be a minimal reduction of $\mathfrak{n}$, we have:

$$(5.17)\qquad e_S(N) = l(\frac{N}{IN}) = l(\frac{N}{\mathfrak{n}IN}) - l(\frac{IN}{\mathfrak{n}IN})$$

Since I is generated by a system of parameters in S and N is an MCM over S, we have $\frac{IN}{I^2N} \cong (\frac{N}{IN})^{\oplus d}$. Hence $\frac{IN}{\mathfrak{n}IN} \cong (\frac{N}{\mathfrak{n}N})^{\oplus d}$, so $l(\frac{IN}{\mathfrak{n}IN}) = d \cdot \nu(N)$. Substituting these into (5.17) we get:

$$e_S(N) = l(\frac{N}{\mathfrak{n}IN}) - d \cdot \nu(N) \geq l(\frac{N}{\mathfrak{n}^2N}) - d \cdot \nu(N) = \nu(N) + \nu(\mathfrak{n}N) - d \cdot \nu(N)$$

□

The following lemma is from Hanes' Thesis [16]. We give a proof here for completeness.

Lemma V.38 (*cf.* Proposition 4.2.3 in [16])**.** *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local map and let M be a finitely generated module over R. Then:*

$$\nu_S(\mathfrak{n}M') \geq \nu_R(\mathfrak{m}M) + (\nu_S(\mathfrak{n}) - \nu_R(\mathfrak{m})) \cdot \nu_R(M)$$

where $M' = S \otimes_R M$.

Proof. This is true for $M = R^n$ free (in this case we trivially have an equality). So by an induction, it suffices to prove the statement for $N = M/Ry$ where $y \in \mathfrak{m}M$, assuming that it is true for M. Since S is flat, we know that $N' = S \otimes_R N = M'/Sy$. Since $y \in \mathfrak{m}M$, we have $\nu_R(M) = \nu_R(N)$.

If $y \in \mathfrak{m}^2M$, then the image of y is in $\mathfrak{m}^2M' \subseteq \mathfrak{n}^2M'$, we have $\nu_R(\mathfrak{m}M) = \nu_R(\mathfrak{m}N)$ and $\nu_S(\mathfrak{n}M') = \nu_S(\mathfrak{n}N')$. All the terms do not change when we pass from M to N, so the conclusion holds for N. If $y \in \mathfrak{m}M - \mathfrak{m}^2M$, then $\nu_R(\mathfrak{m}N) = \nu_R(\mathfrak{m}M) - 1$. Since the image of y is in $\mathfrak{m}M' \subseteq \mathfrak{n}M'$, we know that $\nu_S(\mathfrak{n}N') \geq \nu_S(\mathfrak{n}M') - 1$. But $\nu_R(N) = \nu_R(M)$, the remaining terms do not change. Hence we see that the inequality continues to hold for N. □

From now until the end of this section, we always assume R and S are complete local rings (since completion will preserve all the properties we need and will not affect multiplicities) of equal characteristic $p > 0$ with perfect residue fields. We will use the notation $M^{(e)}$ to mean the R-module which is same as M as an abelian group but with R-module structure given by $r \cdot m = r^q m (q = p^e)$. We begin with a definition that comes from [38] and a few more lemmas.

Definition V.39. We say a Cohen-Macaulay local ring $(R, \mathfrak{m}, K)$ is a *numerically Roberts ring* if for any R-module M of finite length and finite projective dimension, $\lim_{e\to\infty} \dfrac{l_R(F^e_R(M))}{p^{de}} = l_R(M)$ where $d = \dim R$. In particular, $l_R(R/J) = e_{HK}(J, R)$ for all $\mathfrak{m}$-primary ideal J in R with $pd_R R/J < \infty$.

Remark V.40. In fact, numerically Roberts ring can be defined in great generality, even in mixed characteristics: see Definition 6.1 in [38]. However, Theorem 6.4 in [38] showed that when R is Cohen-Macaulay of equal characteristic $p > 0$, Definition V.39 is equivalent to the general definition. We refer to [38] for more details.

The following lemma gives the first examples of numerically Roberts rings. It is due to Dutta [7]. See [38] for generalizations and more examples.

Lemma V.41 ([7], [38])**.** *Let $(R, \mathfrak{m}, K)$ be a local ring, assume either*

1. *R is a complete intersection*
2. *R is a Gorenstein ring of dimension* 3
3. *R is a Cohen-Macaulay ring of dimension* ≤ 2

Then R is a numerically Roberts ring.

In [38] Remark 6.13, it is stated that if $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is flat local homomorphism with $S/\mathfrak{m}S$ a complete intersection and $S/\mathfrak{n}$ is a finite algebraic extension of $R/\mathfrak{m}$. Then, R is a numerically Roberts ring if and only if S is a numerically Roberts ring.

However the proof is omit there. The following lemma is a very special case and we give a short proof in our setting.

Lemma V.42. *If $(R, \mathfrak{m}, K)$ is a numerically Roberts ring then $T = R[[x]]$ is also a numerically Roberts ring.*

Proof. Let M be a module of finite length and finite projective dimension over $R[[x]]$ (and hence over R). We have ($q = p^e$)

$$\begin{aligned} F^e_{R[[x]]}(M) &\cong M \otimes_{R[[x]]} R[[x]]^{(e)} \\ &\cong M \otimes_{R[[x]]} R^{(e)}[[x]] \otimes_{R^{(e)}[[x]]} R[[x]]^{(e)} \end{aligned}$$

Since $R[[x]]^{(e)}$ is free of rank q over $R^{(e)}[[x]]$, $l(F^e_{R[[x]]}(M)) = q \cdot l(M \otimes_{R[[x]]} R^{(e)}[[x]])$. But $M \otimes_{R[[x]]} R^{(e)}[[x]] \cong M \otimes_R R^{(e)}$ (as R-modules). So $l(F^e_{R[[x]]}(M)) = q \cdot l(F^e_R(M))$. Therefore $\lim\limits_{e\to\infty} \dfrac{l_R(F^e_{R[[x]]}(M))}{p^{(d+1)e}} = \lim\limits_{e\to\infty} \dfrac{F^e_R(M)}{p^{de}} = l(M)$. This finishes the proof. $\square$

The next lemma is also proved for general numerically Roberts rings in Remark 6.9 in [38]. We give a proof in our setting for completeness. The proof here is entirely different from the one in [38] since we are in equal characteristic $p > 0$. The proof is a slight modification of the one in [48] and it is essentially the idea from [7].

Lemma V.43. *If $(R, \mathfrak{m}, K)$ is a numerically Roberts ring and $y_1, \ldots, y_t$ is a regular sequence in R, then $T = R/(y_1, \ldots, y_t)R$ is also a numerically Roberts ring.*

Proof. By induction we immediately reduce to the case that $T = R/yR$ where y is a nonzero divisor in R. I want to show R is numerically Roberts implies T is numerically Roberts.

Let $d = \dim R$, then $\dim T = d - 1$. Let M be a module of finite length and finite projective dimension over T, then obviously M also has finite projective dimension (and finite length) over R. Consider the filtration of $R/y^{p^e}R$ by p^e copies of R/yR, that is, a set of short exact sequences of the form:

$$0 \to N_1 \to R/y^{p^e}R \to R/yR \to 0$$

$$0 \to N_2 \to N_1 \to R/yR \to 0$$

$$\dots\dots$$

$$0 \to R/yR \to N_{p^e-1} \to R/yR \to 0$$

Tensoring the above sequences over R with $F_R^e(M)$ gives a set of long exact sequences of $\mathrm{Tor}_i^R(F_R^e(M), -)$, $i \geq 0$. Using the isomorphisms $F_R^e(M) \otimes_R R/y^{p^e}R \cong F_R^e(M)$ and $F_R^e(M) \otimes_R R/yR \cong F_T^e(M)$, one can write these sequences as follows:

$$\to F_R^e(M) \otimes_R N_1 \to F_R^e(M) \to F_T^e(M) \to 0$$

$$\to F_R^e(M) \otimes_R N_2 \to F_R^e(M) \otimes_R N_1 \to F_T^e(M) \to 0$$

$$\dots\dots$$

$$\to F_T^e(M) \to F_R^e(M) \otimes_R N_{p^e-1} \to F_T^e(M) \to 0.$$

Computing lengths gives the inequalities:

$$\begin{aligned} l(F_R^e(M)) &\leq l(F_T^e(M)) + l(F_R^e(M) \otimes N_1) \\ &\leq l(F_T^e(M)) + l(F_T^e(M)) + l(F_R^e(M) \otimes N_2) \\ &\dots\dots \\ &\leq p^e l(F_T^e(M)). \end{aligned}$$

Hence we get

$$l_T(M) = l_R(M) = \lim_{e\to\infty} \frac{l(F_R^e(M))}{p^{de}} \leq \lim_{e\to\infty} \frac{p^e l(F_T^e(M))}{p^{de}} = \lim_{e\to\infty} \frac{l(F_T^e(M))}{p^{(d-1)e}}. \tag{5.18}$$

Let $\underline{z} = z_1, \ldots, z_{d-1}$ be a maximal T-regular sequence in the annihilator of M, and consider a short exact sequence:

$$0 \to Q \to (T/\underline{z}T)^k \to M \to 0$$

Since $\operatorname{Tor}_i^T(M, T^{(e)}) = 0$ for all $i > 0$ and $e > 0$ by Theorem V.15 (because each $T^{(e)}$ is an MCM), tensoring this sequence with $T^{(e)}$ gives an exact sequence:

$$0 \to F_T^e(Q) \to F_T^e((T/\underline{z}T)^k) \to F_T^e(M) \to 0.$$

Computing lengths, we get:

$$\frac{l(F_T^e(M))}{p^{e(d-1)}} + \frac{l(F_T^e(Q))}{p^{e(d-1)}} = \frac{l(F_T^e((T/\underline{z}T)^k))}{p^{e(d-1)}}.$$

Since $\underline{z}$ is a T-sequence, the right hand side is just $k \cdot l(T/\underline{z}T)$, which equals $l(M) + l(Q)$. We therefore have

$$\lim_{e\to\infty} \frac{l(F_T^e(M))}{p^{e(d-1)}} + \lim_{e\to\infty} \frac{l(F_T^e(Q))}{p^{e(d-1)}} = l(M) + l(Q).$$

But both M and Q have finite length and finite projective dimension over T, so $\lim\limits_{e\to\infty} \dfrac{l(F_T^e(M))}{p^{e(d-1)}} \geq l(M)$ and $\lim\limits_{e\to\infty} \dfrac{l(F_T^e(Q))}{p^{e(d-1)}} \geq l(Q)$ by (5.18), this forces

$$\lim_{e\to\infty} \frac{l(F_T^e(M))}{p^{e(d-1)}} = l_T(M).$$

So T is a numerically Roberts ring. $\square$

The following lemma will also be used. Its proof is straightforward from Lemma V.36.

Lemma V.44. *Let $(R, \mathfrak{m}, K) \to (T, \mathfrak{n}, L)$ be a flat local map such that $T/\mathfrak{m}T$ is regular, then we have:*

1. *$e_{HK}(\mathfrak{n}^2, T) \geq e_{HK}(\mathfrak{m}^2, R)$.*

2. $e_{HK}(\mathfrak{m}T+\mathfrak{n}^2,T) \geq (\dim T/\mathfrak{m}T+1)\cdot e_{HK}(R)$.

Proof. Assume $\dim T/\mathfrak{m}T = n$, let $x_1,\ldots,x_n$ be a regular system of parameters in $T/\mathfrak{m}T$ and let $(T_0,\mathfrak{n}_0,K) = R[[x_1,\ldots,x_n]]$.

Apply Lemma V.36 with $I = (\mathfrak{m}^2)^{[q]}$, $J = \mathfrak{m}^{[q]}$, $q_1 = (x_1^q,\ldots,x_n^q)$ and $q_2 = (x_1^q,\ldots,x_n^q)^2$ for every $q = p^e$, We get $l_T(\frac{T}{(\mathfrak{n}^2)^{[q]}}) \geq l_{T_0}(\frac{T_0}{(\mathfrak{n}_0^2)^{[q]}})$ for every $q = p^e$. Hence we have:

$$e_{HK}(\mathfrak{n}^2,T) \geq e_{HK}(\mathfrak{n}_0^2,T_0) \geq e_{HK}(\mathfrak{m}^2,R)$$

where the last inequality is by a direct computation in $R[[x_1,\ldots,x_n]]$. This finish the proof of (1).

Apply Lemma V.36 with $I = (\mathfrak{m})^{[q]}$, $J = \mathfrak{m}^{[q]}$, $q_1 = (x_1^q,\ldots,x_n^q)$ and $q_2 = (x_1^q,\ldots,x_n^q)^2$ for every $q = p^e$, we get $l_T(\frac{T}{(\mathfrak{m}T+\mathfrak{n}^2)^{[q]}}) \geq l_{T_0}(\frac{T_0}{(\mathfrak{m}T_0+\mathfrak{n}_0^2)^{[q]}})$ for every $q = p^e$. Hence we have:

$$e_{HK}(\mathfrak{m}T+\mathfrak{n}^2,T) \geq e_{HK}(\mathfrak{m}T_0+\mathfrak{n}_0^2,T_0) = (n+1)\cdot e_{HK}(R) = (\dim T/\mathfrak{m}T+1)\cdot e_{HK}(R)$$

where the second equality is by a direct computation in $R[[x_1,\ldots,x_n]]$. This finishes the proof of (2). $\square$

Proposition V.45. *Let $(R,\mathfrak{m},K)$ be a numerically Roberts local ring of dimension ≤ 3. Let I be an $\mathfrak{m}$-primary ideal in R such that $pd_R R/I < \infty$, then $e_R \leq l_R(R/I)$, i.e., the Weakened Generalized Lech's Conjecture (= Conjecture V.4) is true in this case.*

Proof. By Definition V.39, we have $l_R(R/I) = e_{HK}(I,R)$. Let $\dim R = d \leq 3$, if $I \subseteq \mathfrak{m}^2$, we have:

$$e_{HK}(J,R) \geq e_{HK}(\mathfrak{m}^2,R) \geq \frac{2^d}{d!}e_R \geq e_R$$

where the lase inequality is because $d \leq 3$.

If $I \nsubseteq \mathfrak{m}^2$, by prime avoidance we can pick $x \in I$ but $x \notin \mathfrak{m}^2 \cup_{P \in \mathrm{Ass}_R R} P$. Now we can kill x and still have $pd_R R/I < \infty$ by Theorem V.16. Also notice that when we kill x, $l_R(R/I)$ remains the same while e_R may only get larger by Lemma V.8. So we are done by induction on the dimension (when $\dim R = 0$ the statement is trivial). $\square$

Now we can prove the main theorems of this section.

Theorem V.46. *Let $(R, \mathfrak{m}, K)$ be a 3-dimensional Cohen-Macaulay ring and $(R, \mathfrak{m}, K) \to (S, \mathfrak{n}, L)$ be a flat local map of the same dimension. If $e_{HK}(R) \leq \frac{1}{3} e_R$, then $e_R \leq e_S$.*

Proof. Apply Lemma V.37 to $N = S \otimes_R R^{(e)}$ and notice that $\nu_S(N) = \nu_S(S \otimes_R R^{(e)}) = \nu_R(R^{(e)})$ and $\nu_S(\mathfrak{n}N) \geq \nu_R(\mathfrak{m}R^{(e)}) + (\nu_S(\mathfrak{n}) - \nu_R(\mathfrak{m})) \cdot \nu_R(R^{(e)})$ (by Lemma V.38 applied to $M = R^{(e)}$), we get:

$$e_S(S \otimes_R R^{(e)}) \geq \nu_R(R^{(e)}) + \nu_R(\mathfrak{m}R^{(e)}) + (\nu_S(\mathfrak{n}) - \nu_R(\mathfrak{m})) \cdot \nu_R(R^{(e)}) - 3 \cdot \nu_R(R^{(e)}) \tag{5.19}$$

Notice that when $e \to \infty$,

$$\nu(R^{(e)}) \to e_{HK}(R) \cdot q^3$$

$$\nu(R^{(e)}) + \nu(\mathfrak{m}R^{(e)}) \to e_{HK}(\mathfrak{m}^2, R) \cdot q^3$$

$$e_S(S \otimes_R R^{(e)}) = e_S \cdot q^3$$

We give a short explanation of the last line: This is clear if R is a domain, as in this case we have $e_S(S \otimes_R R^{(e)}) = e_S \cdot \mathrm{rank}_R(R^{(e)}) = e_S \cdot q^3$. In the general case, we can give R a prime filtration with factors R/P_i. Then we know that $R^{(e)}$ has a filtration with factors $(R/P_i)^{(e)}$, which has rank $q^{\dim R/P_i}$ over R/P_i. Since S is flat over R, we know that $\dim S/P_iS = \dim R/P_i$. In particular,

$$e_S(S \otimes_R R^{(e)}) = \sum_{\dim R/P_i = 3} e_S(S \otimes_R (R/P_i)^{(e)}) = q^3 \cdot \sum_{\dim R/P_i = 3} e_S(S \otimes_R (R/P_i)) = q^3 \cdot e_S.$$

Now we divide both side of (5.19) by q^3, we get:

$$e_S \geq e_{HK}(\mathfrak{m}^2, R) + (\nu_S(\mathfrak{n}) - \nu_R(\mathfrak{m}) - 3) \cdot e_{HK}(R). \tag{5.20}$$

If $\nu_S(\mathfrak{n}) - \nu_R(\mathfrak{m}) \leq 1$, then $e_R \leq e_S$ by a result of Lech (see [40]). Otherwise we have $\nu_S(\mathfrak{n}) - \nu_R(\mathfrak{m}) \geq 2$, and from (5.20) we get:

$$\begin{aligned} e_S &\geq e_{HK}(\mathfrak{m}^2, R) - e_{HK}(R) \geq \frac{1}{3!} e(\mathfrak{m}^2, R) - e_{HK}(R) \\ &= \frac{2^3}{3!} e_R - e_{HK}(R) = e_R + (\frac{1}{3} e_R - e_{HK}(R)) \\ &\geq e_R, \end{aligned}$$

where the last inequality is by assumption $e_{HK}(R) \leq \dfrac{1}{3} e_R$. $\square$

Theorem V.47. *Let $(R, \mathfrak{m}, K)$ be a 3-dimensional Gorenstein ring and $(R, \mathfrak{m}, K) \to (S, \mathfrak{n}, L)$ be a local map with $fd_R S < \infty$ and S is Cohen-Macaulay (note this is satisfied if $(R, \mathfrak{m}, K) \to (S, \mathfrak{n}, L)$ is a flat local map of the same dimension). If $e_{HK}(R) \geq \dfrac{1}{3} e_R$, then $e_R \leq e_S$.*

Proof. Using Cohen Factorizations Theorem V.5 we factor $R \to S$ into $R \to T \to S$ where $R \to T$ is flat with $T/\mathfrak{m}T$ regular and $S = T/J'$. Moreover, we can actually write down explicitly what T is in this case.

We have $K[[x_1, \ldots, x_n]] \twoheadrightarrow R$ and $L[[y_1, \ldots, y_m]] \twoheadrightarrow S$ by Cohen's structure theorem (where K and L are (unique) coefficient fields of R and S). Since K is perfect, we have a natural map $K \hookrightarrow L$ and a map $L[[x_1, \ldots, x_n]] \to S$ where the x_i are mapped to S under the composite $K[[x_1, \ldots, x_n]] \twoheadrightarrow R \to S$. So we get $R_1 \cong R \otimes_{K[[x_1, \ldots, x_n]]} L[[x_1, \ldots, x_n]] \to S$, and we can simply take $T \cong R_1[[y_1, \ldots, y_m]]$.

It is clear that $R_1 \cong R \otimes_{K[[x_1, \ldots, x_n]]} L[[x_1, \ldots, x_n]]$ is a flat local extension of R whose closed fibre $R_1/\mathfrak{m}R_1$ is a field L. So R_1 is also a 3-dimensional Gorenstein ring with perfect residue field L. In particular R_1 is a numerically Roberts ring by Lemma V.41. So $T \cong R_1[[y_1, \ldots, y_m]]$ is a numerically Roberts ring by Lemma V.42.

We know that $pd_T S < \infty$ by Theorem V.7. Moreover, since we assume that S is Cohen-Macaulay, we may pick a minimal reduction $a_1, \ldots, a_r$ of $\mathfrak{n}$ (we may always assume that the residue field is infinite). Therefore $e_S = l_T(T/J)$ where $J = J' + (a_1, ..., a_r)$ and we still have $pd_T T/J < \infty$ (same as the proof of Proposition V.11). Hence we have reduced the original problem to the following:

(**) Let $(R, \mathfrak{m}) \to (T, \mathfrak{n})$ be a flat local map such that $T/\mathfrak{m}T$ is regular. Assume R is a Gorenstein ring of dimension 3 with $e_{HK}(R) \geq \dfrac{1}{3} e_R$ and T is a numerically Roberts ring. If J is an $\mathfrak{n}$-primary ideal of T such that $pd_T T/J < \infty$, then $l_T(T/J) \geq e_R$.

First I claim we may assume $J \subseteq \mathfrak{m}T + \mathfrak{n}^2$. If $J \not\subseteq \mathfrak{m}T + \mathfrak{n}^2$, since J contains a nonzero divisor by Theorem V.14, there exists $x \in J$ such that $x \notin (\mathfrak{m}T + \mathfrak{n}^2) \cup_{P \in AssT} P$ by prime avoidance. If we kill x, we get $\overline{T} = T/x$, $\overline{J} = J\overline{T}$, which satisfy:

1. $(R, \mathfrak{m}) \to (\overline{T}, \overline{\mathfrak{n}})$ is still flat (by Theorem V.6) with $\overline{T}/m\overline{T}$ regular (since $x \notin \mathfrak{m}T + \mathfrak{n}^2$!).

2. $pd_{\overline{T}} \overline{T}/\overline{J}$ is still finite by Theorem V.16.

3. $\overline{T}$ is still a numerically Roberts ring by Lemma V.43.

4. $l_{\overline{T}}(\overline{T}/\overline{J})$ and assumptions on R don't change.

So all conditions and the conclusion we want to prove in (**) descend.

Now we prove (**) assuming $J \subseteq \mathfrak{m}T + \mathfrak{n}^2$. We study $T/\mathfrak{m}T$ in 3 cases:

1. $\dim T/\mathfrak{m}T \leq 1$ and $J \subseteq \mathfrak{n}^2$

$$l_T(T/J) = e_{HK}(J, T) \geq e_{HK}(\mathfrak{n}^2, T) \geq e_{HK}(\mathfrak{m}^2, R) \geq \frac{2^3}{3!} e_R > e_R$$

where the first equality is because T is a numerically Roberts ring, and the third inequality is by Lemma V.44.

2. $\dim T/\mathfrak{m}T \leq 1$ and $J \nsubseteq \mathfrak{n}^2$

   We can pick $x \in J$ but $x \notin \mathfrak{n}^2 \cup_{P \in AssT} P$ by prime avoidance. We can kill x and still have $pd_{\overline{T}}\overline{T}/\overline{J} < \infty$ by Theorem V.16 and by Lemma V.43, $\overline{T}$ is a numerically Roberts ring (but $\overline{T}/\mathfrak{m}\overline{T}$ may not be regular anymore since x may be in $\mathfrak{m}T + \mathfrak{n}^2$). We can keep killing such elements x until we get to $\overline{J} \subseteq \overline{\mathfrak{n}}^2$ in $\overline{T}$. Since we kill at least one x, we have $h = \dim \overline{T} \leq 3$. So we have:

   $$l_T(T/J) = l_{\overline{T}}(\overline{T}/\overline{J}) = e_{HK}(\overline{J}, \overline{T}) \geq e_{HK}(\overline{\mathfrak{n}}^2, \overline{T}) \geq \frac{2^h}{h!}e_{\overline{T}} \geq \frac{2^h}{h!}e_T = \frac{2^h}{h!}e_R \geq e_R$$

   where the second equality is because $\overline{T}$ is a numerically Roberts ring (by Lemma V.43) and the last inequality is because $h \leq 3$ (also note that $e_{\overline{T}} \geq e_T$ by Lemma V.8 and $e_T = e_R$ by Remark V.10).

3. $\dim T/\mathfrak{m}T \geq 2$

   $$\begin{aligned} l_T(T/J) &= e_{HK}(J, T) \\ &\geq e_{HK}(\mathfrak{m}T + \mathfrak{n}^2, T) \\ &\geq (\dim T/\mathfrak{m}T + 1)e_{HK}(R) \\ &\geq 3 \cdot e_{HK}(R) \geq e_R \end{aligned}$$

   where the first equality is because T is a numerically Roberts ring, the second inequality is because $J \subseteq \mathfrak{m}T + \mathfrak{n}^2$, the third inequality is by Lemma V.44, and the last inequality is by assumption $e_{HK}(R) \geq \frac{1}{3}e_R$.

$\square$

Corollary V.48. *Lech's Conjecture (=Conjecture V.1) is true if R is a 3-dimensional Gorenstein ring of equal characteristic $p > 0$ with perfect residue field.*

Proof. Notice that if $R \to S$ is a flat local map of the same dimension and R is a 3-dimensional Gorenstein ring of equal characteristic $p > 0$ and the residue field of S is also perfect, then the result follows from Theorem V.46 and Theorem V.47. But the general case can be reduced to the case that $\dim R = \dim S$ (see [39]) and the residue field of S is algebraically closed (because we can always enlarge the residue field of S without changing the other hypothesis). $\square$

Remark V.49.

1. By standard method of reduction to characteristic $p > 0$, Lech's Conjecture can be proved if R is a 3-dimensional Gorenstein ring of equal characteristic 0.

2. We don't know whether it is true that for any flat local extension $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ such that the closed fibre $S/\mathfrak{m}S$ is a field, R is a numerically Roberts ring implies S is a numerically Roberts ring (in fact, we don't even know whether it is true that being numerically Roberts is preserved under localization, see [38]). If this is true, then our methods should work for all 3-dimensional Cohen-Macaulay numerically Roberts rings in equal characteristic (so Lech's Conjecture will be proved for such class of rings).

3. Since numerically Roberts rings can be defined in a characteristic-free way [38], it is quite natural to ask that whether for a (probably mixed characteristic) Cohen-Macaulay numerically Roberts local ring $(R, \mathfrak{m})$ of dimension ≤ 3 and an $\mathfrak{m}$-primary ideal I in R such that $pd_R R/I < \infty$, we have $e_R \leq l_R(R/I)$? At this point we don't know this.

5.7 Related open questions

In this section we propose two interesting related problems that are closely related to the series of conjectures (Conjecture V.1, Conjecture V.2, Conjecture V.3 and

Conjecture V.4) introduced in section 2.1 and we provide some partial answers.

Question V.50. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d and let M be an R-module such that $l_R(M) \leq \infty$, $pd_R(M) < \infty$. Then is it true that $l_R(M) \geq e_R$? (or at least in equal characteristic $p > 0$, $l_R(M) \geq e_{HK}(R)$?)*

Remark V.51.

1. Obviously this is a strong version of the Weakened Generalized Lech's Conjecture (= Conjecture V.4), since it deals with all modules of finite length and finite projective dimension, not only cyclic ones. We don't know whether they are equivalent or not.

2. The graded case follows from Theorem V.18 and the strict complete intersection case follows from Corollary V.29 (just replace R/I by M in the proofs). However we don't know whether it holds for Cohen-Macaulay rings of minimal multiplicities.

3. In equal characteristic $p > 0$, it is not hard to see that for a numerically Roberts ring $(R, \mathfrak{m}, K)$, we always have $l_R(M) \geq e_{HK}(R)$. Because in this case, we have

$$l_R(M) = \lim_{e\to\infty} \frac{l_R(F_R^e(M))}{p^{de}} \geq \lim_{e\to\infty} \frac{l_R(F_R^e(K))}{p^{de}} = e_{HK}(R).$$

Another case such that Question V.50 has a positive answer is when R has dimension 1 (in all characteristic). We give a short proof below. We don't know the answer even if R is a Cohen-Macaulay ring of dimension 2.

Theorem V.52. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension 1 and let M be an R-module such that $l_R(M) \leq \infty$, $pd_R(M) < \infty$. Then $l_R(M) \geq e_R$.*

Proof. Since R has dimension 1, $pd_R M = 1$. So we have

$$0 \to R^n \xrightarrow{A} R^n \to M \to 0$$

where A is some n by n matrix with entries in $\mathfrak{m}$. But it is well known that in this case we have $\det(A)$ is a nonzero divisor in R and $l_R(M) = l_R(R/\det(A))$ (for example, see Appendix A2 of [14]). So we have $l_R(M) = l_R(R/\det(A)) \geq e_R$ by Lemma V.8. □

In [39], a weaker result on multiplicities of flat local extensions was proved. Namely if $R \to S$ is faithfully flat, then $e_R \leq d! \cdot e_S$ where $d = \dim R$. So in view of the our discussions and results on the generalized conjectures, it is quite natural to ask the following weaker question

Question V.53. *Let $(R, \mathfrak{m})$ be a local ring of dimension d and let $S = R/I$ such that $pd_R S < \infty$. Then is it true that $e_R \leq d! \cdot e_S$?*

Remark V.54. When both R and S are Cohen-Macaulay, we can reduce the question to the case that $S = R/I$ for some $\mathfrak{m}$-primary ideal (see the proof of Proposition V.11). So if we assume R is a numerically Roberts ring of equal characteristic $p > 0$, then we would have

$$d! \cdot e_S = d! \cdot l_R(R/I) = d! \cdot e_{HK}(I) \geq d! \cdot e_{HK}(R) \geq e_R.$$

This shows that Question V.53 has a positive answer when both R, S are Cohen-Macaulay, and R is a numerically Roberts ring of equal characteristic $p > 0$.

Moreover, the following theorem strongly suggest that Question V.53 should have a positive answer, at least when both R and S are Cohen-Macaulay.

Theorem V.55. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local map of local rings of equal characteristic $p > 0$ with $fd_R S < \infty$. If S is a Cohen-Macaulay domain, then $e_R \leq d! \cdot e_S$, where $d = \dim R$.*

Proof. Using Cohen Factorizations (Theorem V.5) we factor $R \to S$ into $R \to T \to S$ where $R \to T$ is flat with $T/\mathfrak{m}T$ regular and $S = T/P$ with $pd_T S < \infty$ by Theorem V.7.

Since S is a Cohen-Macaulay domain, it follows from Auslander's Zerodivisor Theorem and the New Intersection Theorem (see [53] and [54]) that T is also a Cohen-Macaulay domain. Since $R \to T$ is flat with $T/\mathfrak{m}T$ regular, it follows that R is also a Cohen-Macaulay domain. Hence $R^{(e)}$ is an MCM over R for every e. Let $M_e = R^{(e)} \otimes_R T$. Recall that if M is an MCM over T, then M/PM is an MCM over $S = T/P$ with $\mathrm{rank}_S(M/PM) = \mathrm{rank}_T(M)$ by Lemma V.30 and the proof of Theorem V.32. Now we have:

$$\begin{aligned} e_S = e_{T/P} &= \frac{e_S(M_e/PM_e)}{\mathrm{rank}_S(M_e/PM_e)} \geq \frac{\nu_S(M_e/PM_e)}{\mathrm{rank}_S(M_e/PM_e)} \\ &= \frac{\nu_T(M_e)}{\mathrm{rank}_T(M_e)} = \frac{\nu_T(R^{(e)} \otimes_R T)}{\mathrm{rank}_T(R^{(e)} \otimes_R T)} \\ &= \frac{\nu_R(R^{(e)})}{\mathrm{rank}_R(R^{(e)})} = \frac{l_R(R/\mathfrak{m}^{[q]})}{q^d} \end{aligned}$$

where the first equality we use that M_e/PM_e is an MCM over S (since M_e is an MCM over T), the only inequality we use that $e_S(N) \geq \nu_S(N)$ for any MCM N over S, the equality on the second line we use that $\mathrm{rank}_S(M_e/PM_e) = \mathrm{rank}_T(M_e)$.

If we let $q \to \infty$, we immediately get

$$d! \cdot e_S = d! \cdot e_{T/P} \geq d! \cdot e_{HK}(R) \geq e_R.$$

$\square$

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