Incentivizing Secondary Spectrum Trading: A Profit Perspective

by

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ABSTRACT

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With the ever-increasingly connected mobile devices, demand for mobile broadband service is likely to outstrip spectrum capacity in the near-term. Without action to address this spectrum crisis, service quality is likely to suffer and prices are likely to rise. Fortunately, recent studies show that a large part of licensed spectrum remains under-utilized, which should allow concepts such as dynamic spectrum access/sharing, open access, and secondary (spot or short-term) spectrum market to alleviate the crisis.

From the inception of the open access paradigm, it was clear that for it to work two issues must be adequately addressed: sensing and pricing. The first refers to the ability of a (secondary) device to accurately detect channel opportunity and more generally to acquire information on the spectrum environment. The second refers to mechanisms that provide license holders with the right incentives so that they will willingly allow access by unlicensed devices. In this thesis we examine both issues but in ways that are distinctly different from most of what has been done in the literature.

For the pricing issue, we formulate a contract design problem where a primary license holder wishes to profit from its excess spectrum capacity by selling it to po-

tential secondary users. It needs to determine how to optimally price the excess spectrum so as to maximize its profit, knowing that this excess capacity is stochastic in nature, does not come with exclusive access, and cannot provide deterministic service guarantees to a buyer. We adopt as a reference a traditional spectrum market where the buyer can purchase exclusive access with fixed/deterministic guarantees. We fully characterize the optimal solution in the cases where there is a single buyer type, and construct an algorithm that generates a set of contracts in the general case. When multiple primary holders exist, we develop a price competition model for the license holders selling on a secondary spectrum market. Standard results suggest that under full competition the equilibrium only exists when all sellers have zero profit. We introduce a regulator which can also be thought of as the sellers forming a coalition, whose role is to enable money transfer based on partial observations of the sellers' actions. We show that by proper design of the transfer mechanism, efficient equilibrium (profit-maximizing) can be achieved.

On the sensing front, a good channel model can greatly enhance the the ability of secondary devices to quickly detect spectrum availability and exploit instantaneous spectrum opportunities. We propose a spectrum utilization model which uses stochastic differential equations (SDE) to model dynamic scattering and multipath fading channels, in particular, Rayleigh-distributed stationary channels. The SDE model is in closed form, can generate spectrum dynamics as a temporal process, and is shown to provide very good fit for real spectrum measurement data. We show how synthetic spectrum data can be generated in a straightforward manner using this model to enable realistic simulation studies.

CHAPTER I

Introduction

1.1 Motivation

The first 1G service was launched in early 1980. In the two decades that followed (1990-2010) the worldwide mobile phone subscription grew from 12.4 million to over 4.6 billion [59]. Recent studies predict that the demand for mobile broadband - driven by devices like smart phones, such as the iPhone and Google's Android, and by newly connected devices, like the iPad and Amazon's Kindle, will increase 13-fold between 2012 and 2017 [17], with more than 10 billion mobile-connected devices by then.

With the ever-increasingly connected mobile devices, demand for mobile broadband service is likely to outstrip spectrum capacity in the near-term. Without action to address this spectrum crisis, service quality is likely to suffer and prices are likely to rise. Fortunately, recent studies show that a large part of licensed spectrum remain under-utilized: for instance, on average only about 5.2% of the available spectrum from 30MHz to 3GHz was being used at any given time in New York City, and the peak usage was only 13% [20]. A reasonable conjecture therefore is that unlicensed access of idle (but licensed) spectrum bands, commonly referred to as secondary spectrum access, would avert the impending crisis at least in the new-term. This idea has led to extensive research and development in recent years in such concepts as dynamic spectrum access/sharing, open access, and secondary (spot or short-term) spectrum market, see e.g., [5, 12].

1.2 Challenges

From the inception of the open access paradigm, it was clear that for it to work two issues must be adequately addressed: *pricing* and *sensing*. The first refers to mechanisms that provide license holders with the right incentives so that they may willingly allow access by unlicensed devices. The second refers to the ability of a (secondary) device to accurately detect channel opportunity and, more generally, to acquire information on the spectrum environment. These two aspects and the associated challenges are elaborated below.

Pricing The vision of large-scale secondary spectrum access will not be realized only through the availability of the enabling technology and the regulatory progress: secondary access cannot be merely rendered as a regulatory compulsion or even socially desirable, but must also be profitable for the license holders. It is unlikely that the FCC will compel the license holders to allow secondary access, but it can establish policies to incentivize such cooperation [60], such as creating a secondary spectrum market on which secondary users may lease or purchase spectrum usage from primary license holders.

In this scenario, the goods being traded is the excess capacity of the license holder. Note that the excess capacity is stochastic in nature, does not come with exclusive access, and cannot provide deterministic service guarantees to a buyer. Any incentive mechanism much be able to establish the benefit of purchasing non-guaranteed secondary spectrum. The competition among license holders must also be addressed; otherwise, the secondary market may be inefficient and the profit of the license holders might collapse to zero.

Sensing Communications over wireless media is generally much more error prone compared to their wireline counterparts due to the noisier, time-varying and often unpredictable nature of the wireless channel quality. Consequently, the modeling of error patterns in wireless communication has been an important tool for evaluating the performance of wireless communication and networking algorithms, both in analysis and in simulation. Research in recent years on dynamic (and open) spectrum access using cognitive radios (CR) [38] has further exemplified the importance of channel modeling: the ability of wireless devices to quickly detect spectrum availability and exploit instantaneous spectrum opportunities is greatly enhanced if they are equipped with a good channel model that accurately captures the stochastic behavior of the underlying channel conditions. This is reflected in our ability to derive better channel sensing and access decisions both in theory and in numerical experiment when good channels models are available. For this reason the sensing aspect of secondary spectrum access is also simultaneously treated the same as the channel modeling aspect.

Coupling between pricing and sensing While it may appear that channel modeling and pricing are two separate issues, they are actually closely connected. From the seller's point of view, in order to price the spectrum product appropriately they need to provide a good characterization of their own channel. Without an accurate description of the spectrum product, the price of the product cannot be justified because a buyer will not be able to determine whether the channel satisfies their needs. A precise description of the secondary spectrum product provided by the seller will enable the buyer to accurately calculate the amount needed to purchase, which increases the incentive of the buyer to participate in the secondary spectrum market. Thus, an accurate channel model will lead to accurate pricing which strengthens the incentives of both the primaries and the secondaries to participate in the secondary spectrum market.

1.3 Our Approach in Relation to Prior Work

Our assumption on the relation between the primary and secondary users can be classified in the hierarchical access model under the spectrum overlay model as described in [75]. In the hierarchical access structure, the primary users (licensees) open licensed spectrum to secondary users while limiting the interference perceived by primary users. Under the spectrum overlay model, the secondary users are restricted in when and where they can emit power. For example, the secondary users may only transmit when there exists a temporal white space in the spectrum. Under the limiting constraint, the primary users does not see the existence of the secondary users and its transmission quality is not affected at all. In this sense, when participating the secondary users absorb the entire risk of shared access, that os having non-guaranteed transmission service due to the strict constraint on transmission rights.

1.3.1 Contract Framework for Spectrum Market

There has been a number of mechanisms proposed to address the incentive issue, the most often used being the auction mechanism, see e.g., [70, 31, 40]. Auction is also the primary mechanism used in allocating spectrum on the primary market [35]. Under an auction, competing buyers submit bids to a license holder to obtain spectrum access. In selecting winning bids an auction can be designed to maximize the profit of the license holder [31], or to maximize social welfare [40], or some combination of both [31].

In this thesis we consider an alternative approach, that is based on *contracts* (posted price), to the trading of spectrum access on the secondary market. This is conceptually like the design of pricing plans by a cellular operator: it presents a potential user with a set of contract options, each consisting of parameters including the duration of the contract, discount on the device, number of free minutes per month, price per minute for those over the free limit, window of unlimited calling time, and so on. In coming up with these calling plans the operator typically studies carefully the types of callers it wants to attract and their calling patterns/habits; the subsequent plans are catered to these patterns with the objective of maximizing its revenue. A caller interested in entering into contract with the operator is expected to look through these plans and pick one that is the best suited for him/her needs.

Posted price focuses less on the competition among buyers, but more on the designs of products and prices on the part of the license holder to attract potential buyers. While the two mechanisms (auction and posted price) have been shown to generate equal profit under ideal conditions [58], they are suitable for different scenarios in general. The inherent cost of auction comes from setting up each individual auction. The processing of the bids, the wait time for the auction to end are all per auction-based additional costs to the seller [74]. On the contrary, posted price or contract is considered to have a one time cost related to the determination of the efficient price. This requires thorough understanding of the market which can take time and money to investigate [69]. The efficient price can be very hard to determine if the item is rare or the potential buyers are hard to identify. Not surprisingly, auction is suitable when the items are rare and the unit price is high. When the seller has a large quantity of identical items, posted price is the better option. As more and more license holders become interested in the secondary market trading smaller quantities for shorter duration of time compared to the primary wholesale market, we believe pricing schemes like the contracts studied in this paper offer a valid alternative to spectrum auctions.

1.3.2 Price Competition in Spectrum Market

As more and more license holders participate in the secondary market trading, price competition between the license holders is inevitable. When the license holders are competing against each other, the profit each license holder receives will be less than that from a monopoly market. However, this competition relationship in the secondary spectrum market is rarely discussed in the literature. When considering competition in markets, the often used models are the Bertrand and Cournot competition models. The Bertrand model shows that with just two sellers, the market reaches perfect competition and both sellers sell at marginal price. In the Cournot model, the results also show that the price approaches marginal price as the number of seller increases. In the real world, it is unlikely that any firm will sell at their marginal price. Modification of these two models aim to reflect the real market. For example, Bertrand-Edgeworth model assumes a production limit of firms in the Bertrand model. Various other factors can be incorporated in the model to avoid complete competition such as product differentiation, transport and search costs. The companies can also avoid competing with each other by colluding/side contracting.

1.3.3 Modeling of Spectrum Utilization

The desire to better understand spectrum utilization, especially in the presence of licensed primary users, has motivated a series of spectrum measurement studies published recently, see e.g., [56, 55, 41, 16]. These measurement studies, however, have not in general led to tools that can generate realistic spectrum utilization as a time process to evaluate spectrum sensing and access algorithms. In [52] a sequence of probability distributions of spectrum availability were derived using measurement data. However, these distributions capture only the average behavior of spectrum rather than describing spectrum activity as a process in time. Our goal in this dissertation is to construct stochastic models that can capture key properties of wireless channels that are important in evaluating opportunistic spectrum access schemes.

1.3.4 Portfolio Optimization of Secondary Spectrum

When multiple secondary spectrum are available in the market, the buyer may be able to combine multiple purchases of stochastic spectrum products to increase the quality of transmission, by taking into account the second order statistics of the spectrum products. In this thesis, we consider a buyer who purchases a portfolio of spectrum products to maximize the mean throughput while minimizing the variation of transmission throughput. Although secondary spectrum products, taken separately are unreliable and non-guaranteed, the combined quality may be significantly improved. The pricing can be dynamic depending on the dynamics of the quality of the secondary spectrum. The buyer in turn decides on the portfolio based on the instantaneous pricing and channel quality. In this case an accurate channel model becomes important for both the seller and the buyer so as to have good pricing and purchasing choices. Work most related to this includes [57], where they consider two possible metrics, the demand satisfaction rate constraint and the demand satisfaction probability constraint. We use the Sharpe ratio metric similar to [2] and extend the buyer's consideration problem to include pricing.

1.4 Overview of the Thesis and Main Contributions

In this section, we list the main contributions of this thesis.

- Contract Framework for Secondary Spectrum Market
  - We proposed a contract design framework where a primary license holder wishes to profit from its excess spectrum capacity by selling it to potential secondary users/buyers via designing a set of profitable contracts. We completely characterize the optimal solution in the cases where there is

a single buyer type, and when multiple types of buyers share a common, known channel condition. In the case when each type of buyers have different channel conditions we construct an algorithm that generates a set of contracts in a computationally efficient manner, and show that this set is optimal when the buyer types satisfy a monotonicity condition.

  - We generalize the contract design problem to a more general framework. The cost function of the buyer can be any function that is increasing in the money spent, while the quality constraint can be any function that can be mapped to the additional reference spectrum needed. The utility of the seller can be any form that is increasing in the price per bandwidth. We characterize the optimal solution where there is a single buyer type. In the case when more than one buyer types exist, we construct an algorithm that generates a set of contracts in a computationally efficient manner, and show that this set is optimal in the discretized grid when the buyer types satisfy a monotonicity condition.

- Price Competition in Secondary Spectrum Market

  - We introduce a competition model suitable for the secondary spectrum market. The model is a oligopoly model with multiple sellers competing in multiple markets. We first show that the market will result in full competition where equilibrium only exists when all sellers have zero profit. We then introduce a regulator who can facilitate a set of money transfer based on partial observations of the sellers actions. We show that by the introduction of this regulator, we can induce the market to have efficient (profit maximizing) equilibria. The conditions for designing a stable money transfer were characterized for cases of two-seller and multiple-seller cases, and how to achieve fair profit share is also discussed.

- Secondary Spectrum Utilization Model

  - We derive a stochastic differential equation (SDE) model to describe the secondary wireless channel power. We introduce a method to fit the SDE model to real spectrum measurement data and show that the SDE model represents the data very well under different measurement regimes. The SDE model can be used to generate synthesized sample paths whose entropy measure is consistent with the original measurement data. While we show that the 2-state GE model is a good choice when binary representation of the channel condition is sufficient, the SDE model is in general much more accurate and easier to use than an N-state model because we can derive an N-state model from the SDE model.

- A Portfolio Framework for Dynamic Channel Models

  - We consider buyers combine multiple secondary spectrum purchases (spectrum portfolio) to obtain better transmission quality. The quality is characterized by both the total transmission throughput and the variation of the total throughput. We first solve the buyer's problem of choosing the optimal spectrum portfolio under a budget constraint. Next, we introduce a reference market which sells guaranteed spectrum service and solve the buyer's problem again. Based on the result of the buyer's consideration, we find the optimal pricing plan which maximizes the seller's total revenue. If the seller has a cost per channel, we find the optimal pricing plan assuming both the cost and the pricing plan are proportional to the mean throughput on each spectrum channel.

1.5 Organization of the Thesis

The organization of the thesis is as follows. Profit of the license holder can be achieved if (1) the buyers have incentives to purchase the secondary spectrum (2) the competition between license holders result in a profitable market. For the first problem, we propose two frameworks that utilize guaranteed reference spectrum to show that buyers have incentive in participating in the secondary spectrum trading (Chapter II, III). For the second problem, we propose a modified Bertrand model to show the efficiency of the multiple secondary spectrum markets (Chapter IV). For sensing, we develop a continuous stochastic differential equation (SDE) model as an alternative to the commonly used channel model, particularly in the context of opportunistic and dynamic spectrum access (Chapter V). For connecting pricing frameworks with channel modeling, we propose a portfolio spectrum framework where the buyer dynamically purchases a portfolio of spectrum products to fulfill it's transmission needs (Chapter VI).

CHAPTER II

Trading Secondary Spectrum Through Contract Design

2.1 Introduction

In this chapter we consider an approach based on *contracts*, to the trading of spectrum access on the secondary market (see Section 2.7.4 on a discussion comparing the two mechanisms). This is conceptually like the design of pricing plans by a cellular operator: it presents a potential user with a set of contract options, each consisting of parameters including the duration of the contract, discount on the device, number of free minutes per month, price per minute for those over the free limit, window of unlimited calling time, and so on. In coming up with these calling plans the operator typically studies carefully the types of callers it wants to attract and their calling patterns/habits; the subsequent plans are catered to these patterns with the objective of maximizing its revenue. A caller interested in entering into contract with the operator is expected to look through these plans and pick one that is the best suited for him/her needs.

We adopt such a contract design approach in the context of the secondary spectrum market, where a license holder advertises a set of prices and service plans in the hope that a potential buyer will find one of them sufficiently appealing to enter into contract. The contracts are designed with the goal of maximizing the expected revenue of the license holder given a set of buyer *types* (more precisely defined in the next section).

To make the contracts appealing to a buyer, one must address the issue that the spectrum offered on the secondary (short-term) market is typically the excess capacity due to the primary license holder's own spectrum under-utilization. Its quality is therefore often uncontrolled and random, both spatially and temporally, and strongly dependent on the behavior of the primary users. The primary license holder can of course choose to eliminate the randomness by setting aside resources (e.g., bandwidth) exclusively for secondary users. This will however likely impinge on its current users and may not be in the interest of its primary business model. The alternative is to simply give non-exclusive access to secondary users for a fee, which allows the secondary users to *share* a certain amount of bandwidth simultaneously with its existing licensed users, but only under certain conditions on the primary traffic/spectrum usage. For instance, a secondary user is given access but can only use the bandwidth if the current activity by the licensed users is below a certain level, e.g., as measured by received SNR, the so-called spectrum overlay. Many spectrum sharing schemes proposed in the literature fall under this scenario, see e.g., [47, 50, 76, 4].

In this case a secondary user pays (either in the form of money or services in return) to gain spectrum access but not for guaranteed use of the spectrum. This presents a challenge to both the primary and the secondary users: On one hand, the secondary user must assess its needs and determine whether the uncertainty in spectrum quality is worth the price asked for and what level of uncertainty can be tolerated. On the other hand, the primary must decide how stochastic service quality should be priced so as to remain competitive against guaranteed (or deterministic) services which the secondary user may be able to purchase from a traditional market or a different primary license holder.

To address this challenge we adopt a reference point in the form of a traditional spectrum market from where a secondary user can purchase guaranteed service, i.e., exclusive access rights to certain bandwidth, at a fixed price per unit. This makes it possible for the secondary user to reject the offer from the primary if it is risk-averse or if the primary's offer is not attractive. This also implies that the price per unit of bandwidth offered by the primary user must reflect its stochastic quality.

Work most relevant to the study presented in this chapter includes [24, 57, 43, 32]. In [24] a contract problem is studied where the secondary users help relay primary user's data and in return are allowed to send their own data. In [57] an optimal portfolio problem is studied, where a secondary user can purchase a bundle of different stochastic channels, with the price of each already determined, and seeks to find the optimal purchase. In [43] a network revenue management problem is studied, where the customers arrive according to a Poisson process and the performance of a class of certainty-equivalent heuristic control policies was studied. In [32], spectrum trading is modeled as a monopoly market where the primary determines a price-quality contract. While our problem setting bears similarity to that considered in [32], there are several major differences, the chief of which is the fact that our model is not monopolistic due to the existence of a traditional market (exclusive access) mentioned above, that serves as a reference for the value of spectrum products offered on the secondary market (non-exclusive access). In addition, we model different buyer types by their required bandwidth, service quality and loss tolerance. As a result the types can only be partially ordered.

Main contributions of this chapter are as follows:

1. We formulate a contract design problem where the spectrum license holder seeks to sell his excess bandwidth to potential buyers. The model captures the following essential features: (1) excess bandwidth on the secondary spectrum market often comes with non-exclusive use and therefore highly uncertain channel con-

ditions; (2) incentives are built in for both the seller and the buyer to conduct spectrum trading on the secondary market.

2. We fully characterize the optimal set of contracts the seller should provide in the case of a single or two types of buyers, and when multiple types of buyers share the same channel condition due to primary user activities.

3. When there are multiple types of buyers and each experiences different channel conditions, we construct a computationally efficient algorithm and show that the set of contracts it generates is optimal when the buyer types satisfy a monotonicity condition.

4. When the spectrum holder has limited amount of bandwidth, we discuss three different scenarios and show how to modify our algorithm accordingly.

The remainder of the chapter is organized as follows. We present the contract design problem in Section 2.2. Section 2.3 characterizes the utility region and the optimal contract in the single buyer case. Section 2.5 deals with the case when the channel condition is common knowledge, while Section 2.6 focuses on the case when channel conditions are private knowledge. Discussion is given in Section 2.7.2, 2.7.4 and 2.7.3 and numerical results in Section 2.8.

2.2 Model and Assumptions

In this section we describe in detail the models for the two parties under the contract framework: the seller and the buyer, and their considerations in designing and accepting a contract, respectively. We also illustrate a basic idea underlying our model to capture the value of secondary spectrum service, which is random and non-guaranteed in nature, by using guaranteed service as a reference.

2.2.1 The Seller

There are two parties to a contract, the seller and the buyer. The *seller* is also referred to as the owner or the primary license holder, who uses the spectrum to provide business and service to its *primary users*, and carry *primary traffic*. He is willing to sell underutilized bandwidth he has as long as it generates positive profit and does not impact negatively his primary business. We will assume that the seller can pre-design up to M contracts and announce them to potential buyers.

2.2.2 The contract

Each contract is in the form of a pair of real numbers (x, p), where $x \in R^+$ and $p \in R^+$.

- x is the amount of bandwidth they agree to trade on (i.e., access to this amount of bandwidth is given from the seller to buyer).
- p is the price per unit of x; thus a total of xp is paid to the seller if the buyer purchases this contract.

The seller's profit or utility from contract (x, p) is given as

$$U(x, p) = x(p - c)$$

where c is a predetermined constant that takes into account the operating cost of the seller. We will assume that any contract the seller presents must be such that $p > c$; that is, the seller will not sell at a loss. If none of the contracts is accepted by the buyer, the *reserve utility* of the owner is defined by $U(0, 0) = 0$.

2.2.3 A reference market of fixed/deterministic service or exclusive use

We next consider what a contract specified by the pair (x, p) means to a potential buyer. To see this, we will assume that there exists a traditional (as opposed to this emerging, secondary) market from where the buyer can purchase services with fixed or deterministic guarantees. What this means is that the buyer can purchase *exclusive* use of certain amount of bandwidth, which does not have to be shared with other (primary) users. This serves as an alternative to the buyer, and is used in our model as a point of reference. We will not specify how the price of exclusive use is set, and will simply normalize it to be unit price per unit of bandwidth (or per unit of transmission rate). The idea is that given this alternative, the seller cannot arbitrarily set his price because the buyer can always walk away and purchase from this traditional market. This traditional market will also be referred to as the *reference* market, and the service it offers as the *fixed* or *deterministic* service. Our model allows a buyer to purchase from both markets should that be in the interest of the buyer. Note that even though we have assumed a single seller model, this is not a monopoly because of the existence of this reference market. However, we do not explicitly model the competition between multiple sellers on the secondary market, which remains an interesting subject of future study.

2.2.4 The buyer's consideration

When the set of M contracts are presented to a buyer, his choices are (1) to choose one of the contracts and abide by its terms, (2) to reject all contracts and go to the traditional market, and (3) to purchase a certain combination from both markets. The buyer's goal is to minimize his purchasing cost as long as certain quality constraints are satisfied.

While the framework presented here applies to any meaningful quality constraint, to make our discussion concrete below we will focus on a loss constraint. Suppose the buyer chooses to purchase y units of fixed service from the reference market together with a contract (x, p). Then its constraint on expected loss of transmission can be expressed as:

$$E[(q - y - xB)^+] \leq \epsilon \text{ , where}$$

- q is the amount of data/traffic the buyer wishes to transmit.
- $B \in \{0, 1\}$ is a binary random variable denoting the quality of the channel for this buyer. We will denote $b := P(B = 1)$.
- ϵ is a threshold on the expected loss acceptable to the buyer.
- y is the amount of bandwidth the buyer purchases additionally from the reference market; its price is 1 per unit bandwidth.

Note that quantities x, y and q are of the same unit; this unit can be bit (total amount of transmission), or rate (bits per second), and so on. Here we have adopted a simplifying assumption that the purchased service (in the amount of x) is either available in the full amount (when $B = 1$) or unavailable (when $B = 0$), with xb being the expected availability. If the contract duration is comparable to the time constant of the primary user activity (e.g., peak vs. off-peak hours) then this model captures the spectrum condition at the time of contract signing. More sophisticated models can be adopted here, by replacing xB with another random variable $X(x)$ denoting the random amount of data transmission the buyer can actually realize. Although the technical details will become different, the basic ideas are the same. More is discussed on how to incorporate a general model of B in Chapter III.

With a purchase of $(y, (x, p))$, the buyer's cost is given by $y + xp$. The cost of the contract (x, p) to this buyer is given by the value of the following minimization problem:

$$C(x,p) = \underset{y}{\text{minimize}} \quad y + xp \tag{2.1}$$

$$\text{subject to} \quad E[(q - y - xB)^+] \leq \epsilon \tag{2.2}$$

That is, to assess how much this contract actually costs him, the buyer has to consider how much additional fixed service he needs to purchase to fulfill his needs.

The buyer can always choose to not enter into any of the presented contracts and only purchase from the traditional market. In this case, his cost is given by the value of the following minimization problem:

$$C(0,0) = \underset{y}{\text{minimize}}\ y, \text{ subject to } E[(q-y)^+] \leq \epsilon$$

Since every term is deterministic in the above problem, we immediately conclude that $C(0,0) = q - \epsilon$, which will be referred to as the *reserve price* of the buyer. It is natural to assume that any buyer must be such that $q \geq \epsilon$, for otherwise the buyer does not need to perform any transmission as it can tolerate the loss of all of its data.

In deciding whether to accept a given contract (x,p), the buyer has to consider (1) whether the contract would satisfy its quality (loss) constraint, and (2) whether there is an incentive to enter into this contract, i.e., whether the cost of this contract is no higher than the reserve price. The latter is also referred to as the *individual rationality* (IR) constraint, $C(x,p) \leq C(0,0) = q - \epsilon$. Any contract that satisfies both constraints of a buyer is referred to as *acceptable* to that buyer.

If a buyer accepts one of the contracts, the two sides come to an agreement and have to follow the contract up to a predetermined period of time. We will leave this duration unspecified as it does not affect our analysis under the current model assuming the buyer's need is to transmit a certain amount of data over the entire contract period. However, the binary channel model would be more reasonable if the contract considered is short term.

2.2.5 Buyer types and informational constraints

We will assume that a potential buyer may be one of a number of different *types*; each type is characterized by a unique triple (q, ϵ, b), which is a buyer's *private information*. That is, a type is characterized by its transmission needs (amount q to be transferred and loss requirement ϵ), as well as its perceived spectrum/channel quality (b). Throughout the chapter we will assume that a type (q, ϵ, b) is such that there exists a contract with $p > c$ acceptable to the buyer, for otherwise the seller has no incentive to sell.

We will further assume two cases, where b is common to all types and where b may be different for different types. The first case models the scenario where buyers are relatively homogeneous and their perceived channel quality is largely determined by the primary user traffic reflected in b. In this case it is also natural to assume that b is known to the seller. The second case models the scenario where buyers may differ significantly in their location, quality of transceiver devices, and so on, which leads to different perceived channel quality, which is only known to a buyer himself.

The seller is assumed to know the distribution of the buyer types but not the actual type of a particular buyer. The buyer types and their distribution may be estimated from the seller's past experience. Specifically, we will assume there are K types of buyers, and a buyer is of type i with probability r_i and is given by the triple (q_i, b_i, ϵ_i). In subsequent sections we proceed in the following sequence: (1) single user type, (2) multiple user types; common b, and (3) multiple user types; different and private b.

2.3 Optimal contract for a single buyer type

We begin by considering the case where there is only one type of buyer (q, ϵ, b). Through this simplified scenario we will introduce a number of concepts key to our analysis and obtain some basic understanding of the nature of this problem.

Under our assumption that the seller knows the buyer type distribution, having a single type (i.e., a singleton distribution) essentially means that the triple (q, ϵ, b) is known to the seller. Denote by $T = \{(x, p) : C(x, p) \le C(0, 0)\}$ the set of all acceptable contracts for the buyer, or the *acceptance region*. This is characterized by the next result.

Theorem 2.1. *When $q(1-b) \le \epsilon$, the buyer accepts a contract (x, p) iff*

$$p \le \begin{cases} b & \text{if } x \le \frac{q-\epsilon}{b} \\ \frac{q-\epsilon}{x} & \text{if } x > \frac{q-\epsilon}{b} \end{cases} . \tag{2.3}$$

When $q(1-b) > \epsilon$, the buyer accepts the contract iff

$$p \le \begin{cases} b & \text{if } x \le \frac{\epsilon}{1-b} \\ \frac{b\epsilon}{x(1-b)} & \text{if } x > \frac{\epsilon}{1-b} \end{cases} . \tag{2.4}$$

The above theorem can be proved for each of the cases listed above. For brevity below we only show the proof for the sufficient condition under $q(1-b) \le \epsilon$ for the first case in Eqn (2.3); other cases can be done using similar arguments.

Lemma 2.2. *When $q(1-b) \le \epsilon$, the buyer accepts the contract (x, p) if $x \le \frac{q-\epsilon}{b}$ and $p \le b$.*

Proof. If both the IR constraint and the loss constraint are satisfied under the stated conditions, then the buyer accepts the contract. Below we check these two constraints. Let the buyer supplement this contract with an additional purchase of $y = q - \epsilon - xp$

deterministic service. Note that $y \geq 0$ under the stated conditions. The total cost of this contract to the buyer is then given by:

$$C(x,p) = y + xp = q - \epsilon - xp + xp = q - \epsilon = C(0,0).$$

The IR constraint is therefore satisfied. The buyer's loss under this combination of purchases is given by:

$$\begin{aligned}
E[(q-y-xB)^+] &= (q-y-x)^+ b + (q-y)^+(1-b) \\
&= (\epsilon + xp - x)^+ b + (\epsilon + xp)(1-b) \\
&= \begin{cases} (\epsilon + xp)(1-b) \leq (\epsilon + b\frac{q-\epsilon}{b})(1-b) \\ \ = q(1-b) \leq \epsilon, \text{ if } \epsilon + x(p-1) \leq 0 \\ (\epsilon + x(p-1))b + (\epsilon + xp)(1-b) \\ \ = \epsilon + x(p-b) \leq \epsilon, \text{ if } \epsilon + x(p-1) > 0 \end{cases}
\end{aligned}$$

Thus the loss constraint is also satisfied. □

The two acceptance regions given by Theorem 2.1 are illustrated in Figs. 2.1. Any contract that falls below the boundary is acceptable to the buyer. The two cases have the following interpretations. In the first case ($q(1-b) \leq \epsilon$), the quality of the stochastic channel is sufficiently good such that the loss constraint (3.2) may be met without any purchase of the deterministic channel. In this case the buyer is willing to spend up to the entire reserve price $C(0,0) = q - \epsilon$ on the contract. In the second case ($q(1-b) > \epsilon$), the quality of the stochastic channel is such that no matter how much is purchased, some deterministic channel is needed ($y > 0$) in order to satisfy the loss constraint (note $xp \leq \frac{b\epsilon}{1-b} < q - \epsilon$ because $q(1-b) > \epsilon$). Consequently, in the first case, further purchase from the reference market is needed only if the contract has $x < x^*$, whereas in the second case, the buyer always has to purchase from the reference market to satisfy the loss constraint. This observation holds throughout the

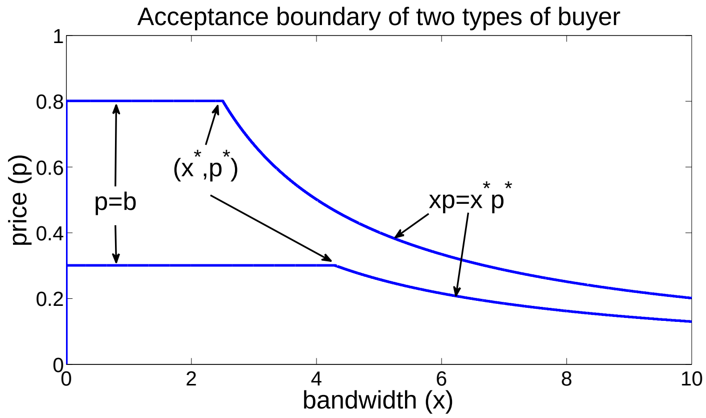

Figure 2.1: The upper curve is when $q(1-b) < \epsilon$ ($q = 5, b = 0.8, \epsilon = 3$), the lower curve is when $q(1-b) > \epsilon$ ($q = 5, b = 0.3, \epsilon = 3$)

chapter including when we introduce multiple buyer types.

For a given buyer type (q, ϵ, b), the seller can choose any point in the corresponding acceptance region T to maximize its utility: $\max_{(x,p)\in T} U(x, p)$. We next show that the optimal contract for the seller is given by the "knee" (the intersection point where the straight line meets the curve) on the boundary of the acceptance region, denoted as (x^*, p^*).

Theorem 2.3. *The optimal contract for the seller is the intersection point (x^*, p^*) on the acceptance region boundary of the buyer.*

Proof. We prove the optimality in two steps. First we show that the seller's utility is strictly increasing in p which implies that the optimal contract must be such that (2.3) and (2.4) hold with strict equality. Then we show that the intersection point is strictly better than any other point on the boundary. For any $x > 0$ and $\forall p' > p$, we have

$$U(x, p') = x(p' - c) > x(p - c) = U(x, p).$$

Thus $U(x, p)$ is strictly increasing in p. For any $x < x^*$ (points on the straight line) we have

$$U(x^*,p^*) = x^*(p^*-c) > x(p^*-c) = U(x,p^*),$$

which used the fact that $p^* > c$. (Recall we have assumed that for any buyer there must exist a contract with $p > c$ that it finds acceptable. This implies such a point must be within the acceptance region, which in turn implies that we must have $p^* > c$ since $p^* \geq p$, $\forall p$ in the region.) For any pair (x,p) such that $xp = x^*p^*$ and $x > x^*$ (points on the curve),

$$U(x,p) = x(p-c) = x^*p^* - xc > x^*(p^*-c) = U(x^*,p^*).$$

Thus $U(x^*,p^*)$ is strictly greater than any point $U(x,p)$ on the boundary. $\square$

Once the seller determines the optimal contract and presents it to the buyer, the buyer will accept because it satisfies both the loss and the IR constraints. It can be easily shown that the buyer's cost in accepting is exactly $C(0,0)$. Note that technically since the cost of the contract is exactly equal to the reserve price, the buyer is indifferent between getting only deterministic service and getting a mix of both types of services. In practice the seller can always lower the unit price p^* by an arbitrarily small amount to provide a positive incentive so that the buyer will accept the contract. For this reason even though the costs are equal, for simplicity we will assume that the buyer will accept this contract. For the same reason, we will also assume that when there exist multiple contracts of equal cost to the buyer, the seller can always induce the desired choice from the buyer by introducing a small difference to the desired contract. We have now a complete characterization of the contract design for a single type of buyer.

We will now introduce the concept of an *equal-cost line* of a buyer. Consider a contract (x',p'). Denote by $P(x',p',x)$ a price such that the contract $(x, P(x',p',x))$

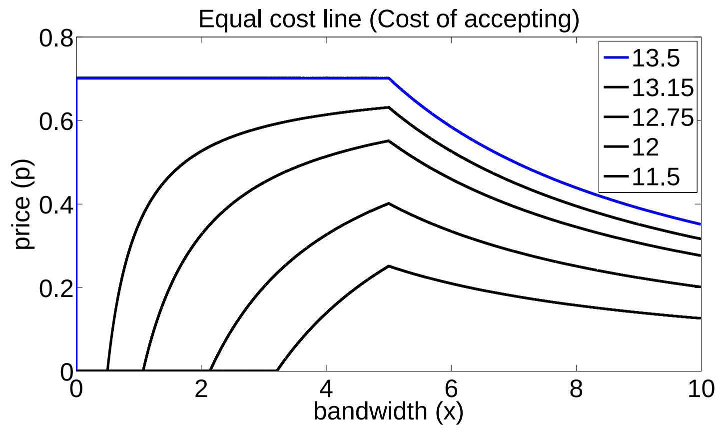

Figure 2.2: Example of equal cost lines

has the same cost as contract (x', p') to a buyer.

Definition 2.4. The equal-cost line E of a buyer of type (q, ϵ, b) is the set of contracts within the buyer's acceptance region T that are of equal cost to the buyer. Thus $(x, p) \in E$ if and only if $p = P(x', p', x)$ for some other $(x', p') \in E$. The cost of this line is given by $C(x', p')$, $\forall (x', p') \in E$.

It should be clear that there are many equal-cost lines, each with a different cost. Figure 2.2 shows an example of a set of equal-cost lines. The next theorem gives a precise expression for the equivalent price that characterizes an equal-cost line.

Theorem 2.5. *For a buyer of type (q, ϵ, b) with an intersection point (x^*, p^*) on its acceptance region boundary, and given a contract (x', p'), an equal-cost line consists of all contracts $(x, P(x', p', x))$ such that*

$$P(x', p', x) = \begin{cases} b - \frac{x'}{x}(b - p') & \text{if } x, x' \leq x^* \\ x'p'/x & \text{if } x, x' \geq x^* \\ (b(x^* - x') + x'p')/x & \text{if } x' < x^* < x \\ b - (x^*b - x'p')/x & \text{if } x < x^* < x' \end{cases}$$

Proof. We will prove this for the case $q(1 - b) \leq \epsilon$; the other case can be shown with similar arguments and is thus omitted for brevity. In this case $x^* = \frac{q-\epsilon}{b}$. When $x, x' \leq x^*$, without buying deterministic service the loss is given by

$$
\begin{aligned}
E[(q - xB)^+] &= (q-x)^+ b + q(1-b) \\
&= (q-x)b + q(1-b) = q - xb \geq \epsilon,
\end{aligned}
$$

where the second equality is due to the fact that $q(1-b) \leq \epsilon \Rightarrow \frac{q-\epsilon}{b} \leq q \Rightarrow x \leq \frac{q-\epsilon}{b} \leq q$. The incentive for the buyer is to purchase y such that the loss is just equal to ϵ.

$$
\begin{aligned}
E[(q - y - xB)^+] &= (q-y-x)b + (q-y)(1-b) \\
&= q - y - xb = \epsilon \ .
\end{aligned}
$$

The first equality follows from the fact that $q(1-b) \leq \epsilon$, which implies both $(q - y - x) \geq 0$ and $(q-y) \geq 0$. This is true for both (x, p) and (x', p'). Since (x, p) is on the equal cost line $E_{x',p'}$, we know that $C(x, p) = C(x', p')$. We also know that $C(x, p) = y + xp$ and $C(x', p') = y' + x'p'$,

$$
C(x,p) = q - \epsilon - xb + xp = q - \epsilon - x'b + x'p' = C(x', p') \ .
$$

Rearranging the second equality such that p is a function of x, x', p' immediately gives the result. When $x, x' > x^*$, x (x') alone is sufficient to achieve the loss constraint. For $C(x, p) = C(x', p')$ we must have $x'p' = xp$, resulting in the second branch. The third and fourth branch can be directly derived from the first two branches. When $x > x^* > x'$ $(x' > x^* < x)$, we first find the equivalent price at x^* by the first branch (second branch), and then use the second branch (first branch) to find $P(x', p', x)$. This gives the third branch (fourth branch) $\square$

The form of the equal-cost line is the same regardless whether $q(1-b) \leq \epsilon$ or $q(1-b) > \epsilon$. Note that every contract below an equal-cost line is strictly preferable to a contract on the line for the buyer. This is an observation we will use in subsequent sections. We end this section with a property of the equivalent price we will use later.

Lemma 2.6. *$P(x', p', x)$ is strictly increasing in p' when $x' > 0$.*

This lemma is easily shown by noting $C(x', p') = y + x'p'$, where y is only a function of x'. Thus, $p > p'$ implies $C(x', p) > C(x', p')$ when $x' > 0$.

2.4 Multiple buyer types: preliminaries

We now consider K types of buyers indexed by $i = 1, 2, \cdots, K$, each defined by the triple (q_i, ϵ_i, b_i) with an associated acceptance region T_i. We will use the notation

$$max_i = (x_i^*, p_i^*) = \text{argmax}_{(x,p)\in T_i} U(x, p)$$

to denote the optimal contract if type i were the only type existing. Similarly, we will use $C_i(x, p)$ to denote the cost to a type-i buyer for accepting contract (x, p).

A buyer is of type i with probability r_i. We assume that the seller knows only this distribution of types but not the actual type of a given buyer. Consequently it has to design the contracts in a way that maximizes its expected payoff. Since the payoff is measured in expectation, it turns out that it does not matter whether the seller is faced with a single buyer or multiple buyers as long as they are drawn from the same, known type distribution and the seller has sufficient bandwidth to honor its contracts. For this reason throughout our discussion we will take the view of a single buyer drawn from a certain type distribution. In Section 2.7.2 we discuss the case when the seller has limited bandwidth to trade.

Consider a set of contracts $\mathbb{C} = \{(x_1, p_1), ..., (x_K, p_K)\}$ designed by the seller with the intention that a buyer of type i prefers (x_i, p_i). This is true iff $C_i(x_i, p_i) \leq C_i(x_j, p_j), \forall j \neq i$. Let $R_i(\mathbb{C})$ denote the contract that a type-i buyer selects given a set $\mathbb{C}$. Then $R_i(\mathbb{C}) = \text{argmin}_{(x,p)\in\mathbb{C}}\, C_i(x, p)$ and the seller's expected utility for a given $\mathbb{C}$ is $E[U(\mathbb{C})] = \sum_i U(R_i(\mathbb{C}))r_i$. Note that there is no point in offering more than K contracts. In the case of more than K contracts offered, there will always be a contract not taken by any buyer type.

2.5 Multiple buyer types: Common channel condition

In this section we consider the case where different types share the same channel condition $b_i = b, i = 1, \cdots, K$, which is also known to the seller. As mentioned earlier, this models the case where the condition is primarily determined by the seller's primary user traffic. An example of the acceptance regions of three buyer types are shown in Figure 2.3. Note that max_i's need not be ordered in i; however, in the interest of simplicity in presentation, we will reindex them in ascending order of the x_i^*s for the remainder of this section. There are two possible cases: (1) the seller can announce as many contracts as he likes ($M = K$); (2) the seller is limited to at most $M < K$ contracts. Below we fully characterize the optimal contract set in both cases.

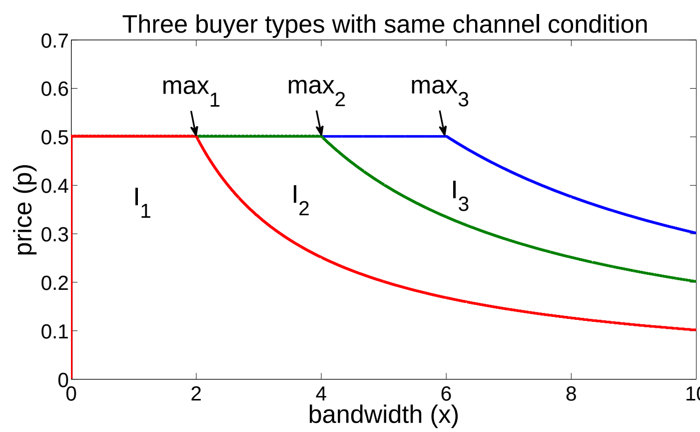

Figure 2.3: Three buyer types with common b

Theorem 2.7. *When $M = K$, the contract set that maximizes the seller's profit is $(max_1, max_2, ..., max_K)$.*

As shown in Figure 2.3, with a constant b, the intersection points of all acceptance regions are on the same line $p = b$. For a buyer of type i, all points to the left of max_i on this line cost the same as max_i, and all points to its right are outside the buyer's acceptance region. Therefore the type-i buyer will select the contract max_i given this contract set (see earlier discussion on how the seller can always incentivize this contract over others with equal cost). Since this is the best the seller can do with a type-i buyer (see Theorem 2.5) this set is optimal for the seller. It is also relatively straightforward to obtain a similar results in the case of $M < K$ given next.

Lemma 2.8. *When $M < K$ and $\forall b_i = b$, the optimal contract set is a subset of $(max_1, ..., max_K)$.*

Proof. Assume the optimal contract $\mathbb{C}$ is not a subset of $(max_1, ..., max_K)$. Then it must consists of some contract points from at least one of the I_i regions as demonstrated in Figure 2.3. Let these contracts be $A_i \subset I_i$ and $\bigcup_i A_i = \mathbb{C}$. For each non-empty A_i, we replace it by the contract max_i and call this new contract set $\mathbb{C}'$. The proof is to show that this contract set generates profit at least as large as the original one. For each type-i buyer that picked some contract $(x, p) \in A_j$ from the optimal contract $\mathbb{C}$, it must had a type greater than or equal to j otherwise (x, p) is not in its acceptance region. In the contract set $\mathbb{C}'$, type-i will now pick max_j or max_l with $l > j$. The choice of each possible type of buyer picks from $\mathbb{C}'$ is at least as profitable as the one they picked from $\mathbb{C}$. Thus, the expected profit of $\mathbb{C}'$ is at least as good as $\mathbb{C}$. □

This lemma suggests the following iterative way of finding the optimal contract set without having to solve what would seem like a combinatorial problem. Define function $g(m, i)$ as the maximum expected profit for the seller by picking contract max_i and selecting optimally $m - 1$ contracts from the set $(max_{i+1}, ..., max_K)$. Note that if we include max_i and max_j $(i < j)$ in the contract set but nothing else in between i and j, then a buyer of type l ($i \leq l < j$) will pick contract max_i. These types contribute to an expected profit of $x_i^*(b-c)\sum_{l=i}^{j-1} r_l$. At the same time, no types below i will select max_i (as it is outside their acceptance regions), and no types at or above j will select max_i (as for them max_j is preferable).

The function $g(m,i)$ can be recursively obtained as follows:

$$g(m,i) = \max_{j:i<j\leq K-m+2} g(m-1,j) + x_i^*(b-c)\sum_{l=i}^{j-1} r_l,$$

with the boundary condition $g(1,i) = x_i^*(b-c)\sum_{l=i}^{K} r_l$.

Finally, it should be clear that the maximum expected profit for the seller is given by $\max_{1\leq i\leq K} g(M,i)$, and the optimal contract set can be determined by going backwards: first determine $i_M^* = \arg\max_{1\leq i\leq K} g(M,i)$, then $i_{M-1}^* = \arg\max_{1\leq i\leq K-1} g(M-1,i)$, and so on. In computing the set of MK values of $g(m,i)$, we note that each can be computed in less than K steps if $g(m-1,i), i = 1,...,K$ is already known. These values can therefore be computed in an increasing order, resulting in a complexity of $O(K^2M)$. By comparison a brute force search on K choose M possible contract sets is exponential.

Theorem 2.9. *The set $\{max_{i_1^*}, max_{i_2^*}, \cdots, max_{i_M^*}\}$ obtained using the above procedure is optimal and its expected profit is given by $g(M, i_M^*)$.*

2.6 Multiple buyer types: Private channel condition

We now consider multiple buyer types each with a different channel condition b_i, $i = 1, \cdots, K$. We will start with the special case of $K = 2$ and characterize the optimal contracts in this case. Using these results we then construct an algorithm to compute a set of contracts for the case of $K \geq 2$.

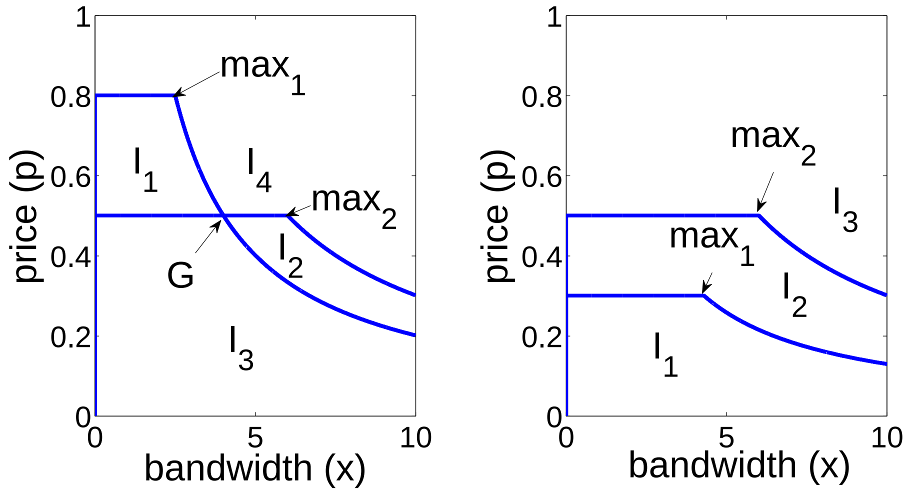

Figure 2.4: (left) $max_1 \notin T_2$ and $max_2 \notin T_1$; (right) $max_1 \in T_2$

2.6.1 Two buyer types: $K = 2$

Consider two buyer types (q_i, ϵ_i, b_i), $i = 1, 2$, with probability r_i, $r_1 + r_2 = 1$. We first consider the case that the seller is limited to one contract: $M = 1$.

Theorem 2.10. *The optimal contract when $K = 2$ and $M = 1$ is as follows:*

1. *If $max_1 \notin T_2$ and $max_2 \notin T_1$,*

$$optimal = \begin{cases} max_1 & \textit{if } r_1 U(max_1) \geq r_2 U(max_2) \\ & \textit{and } r_1 U(max_1) \geq U(G) \\ max_2 & \textit{if } r_2 U(max_2) \geq r_1 U(max_1) \\ & \textit{and } r_2 U(max_2) \geq U(G) \\ G & \textit{if } U(G) \geq r_2 U(max_2) \\ & \textit{and } U(G) \geq r_1 U(max_1) \end{cases}$$

where G denotes the intersecting point between acceptance region boundaries of the two types.

2. If $max_1 \in T_2$*.*

$$optimal = \begin{cases} max_1 & \text{if } U(max_1) \geq r_2 U(max_2) \\ max_2 & \text{if } r_2 U(max_2) \geq U(max_1) \end{cases}$$

3. If $max_2 \in T_1$*.*

$$optimal = \begin{cases} max_2 & \text{if } U(max_2) \geq r_1 U(max_1) \\ max_1 & \text{if } r_1 U(max_1) \geq U(max_2) \end{cases}$$

The above result is illustrated in Figure 2.4 and can be argued by showing the profit of every contract in a particular region (such as I_1) is no greater than some specific contract (such as max_1). Take the case $max_1 \notin T_2$ and $max_2 \notin T_1$ for example, any point in I_3 is suboptimal to point G because any contract in I_3 is acceptable by both types of buyers, but G has a strictly higher profit than any other point in I_3.

We now consider the case $M = 2$. We shall see that providing multiple contracts can help the obtain higher profits.

Theorem 2.11. *In the case of* $M = 2$*,* $max_1 \notin T_2$ *and* $max_2 \notin T_1$*, the optimal contract set is* $\{max_1, max_2\}$*.*

Proof. The set $\mathbb{C} = \{max_1, max_2\}$ gives an expected payoff of

$$\begin{aligned} E[U(\mathbb{C})] &= r_1 U(R_1(\mathbb{C})) + r_2 U(R_2(\mathbb{C}))) \\ &= r_1 U(R_1(max_1)) + r_2 U(R_2(max_2)). \end{aligned}$$

The second equality holds because $max_1 \notin T_2$ and $max_2 \notin T_1$ and thus type i will pick max_i. Suppose $\mathbb{C}$ is not the optimal set of 2 contracts, then there must exists some $\mathbb{C}' = \{(x_1, p_1), (x_2, p_2)\}$ such that

$$\begin{aligned}
E[U((\mathbb{C}'))] &= r_1 U(R_1(x_1, p_1)) + r_2 U(R_2(x_2, p_2)) \\
&> E[U(\mathbb{C})] \\
&= r_1 U(R_1(max_1)) + r_2 U(R_2(max_2))
\end{aligned}$$

This implies either $U(R_1(x_1, p_1)) > U(R_1(max_1))$, or $U(R_2(x_2, p_2)) > U(R_2(max_2))$, or both, all of which contradict the definition of max_i. Thus, $\{max_1, max_2\}$ is the optimal contract set. $\square$

The proof as well as the intuition behind the above result are straightforward. The next case, $M = 2$, $max_1 \in T_2$ or $max_2 \in T_1$, is more complicated. Without loss of generality, we will assume that the type-1 buyer has a smaller b_1 ($b_1 \leq b_2$), thus $max_1 \in T_2$. We first determine the optimal contract when $x_1^* \leq x_2^*$; this result is then used for the case when $x_1^* > x_2^*$. Without loss of optimality we consider only contract pairs $\{(x_1, p_1), (x_2, p_2)\}$ where type-i buyer picks (x_i, p_i) instead of the other one.

To find the optimal contract, we 1) first show that for each (x_1, p_1) we can express the optimal (x_2, p_2) in terms of x_1 and p_1; 2) then we show that (x_1, p_1) must be on the boundary of T_1 with $x_1 \leq x_1^*$; 3) using 1) and 2) we optimize the expected profit over possible choices of x_1.

Lemma 2.12. *When $K = 2$, if $max_1 \in T_2$ and $x_1^* \leq x_2^*$, then given a contract for type-1 (x_1, p_1), the optimal contract for type-2 must be $(x_2^*, P_2(x_1, p_1, x_2^*))$.*

Proof. Given a contract (x_1, p_1), the feasible region for the contract of type-2 buyer is the area below $P_2(x_1, p_1, x)$ as defined in Theorem 2.5 (see Figure 2.5). Since the seller's profit is increasing in both p and x, the contract that generates the highest profit is at $x_2 = x_2^*$ and $p_2 = P_2(x_1, p_1, x_2^*)$. $\square$

Lemma 2.13. *When $K = 2$, if $max_1 \in T_2$ and $x_1^* \leq x_2^*$, an optimal contract for*

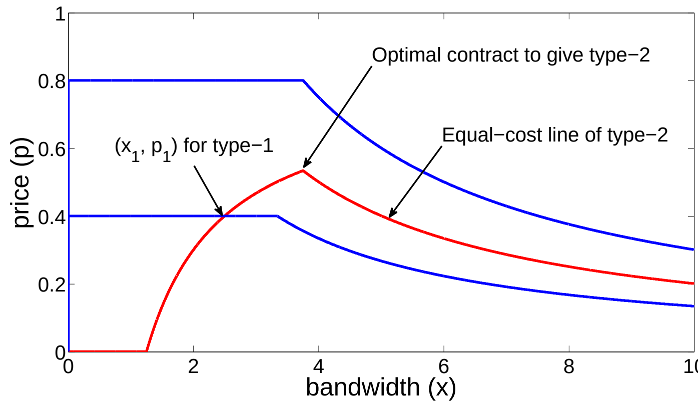

Figure 2.5: The regions to distinguish type-2 given (x_1, p_1)

type-1 must be $p_1 = b_1$ *and* $x_1 \leq x_1^*$.

Proof. Assume the optimal contract has $(x_1, p_1) \in T_1$ and given some $\delta > 0$ we still have $(x_1, p_1 + \delta) \in T_1$. By noticing that both $U(x, p)$ and $P(x, p, x')$ are increasing in p. We know that both $U(x_1, p_1 + \delta)$ and $U(x_2^*, P_2(x_1, p_1 + \delta, x_2^*)))$ are strictly larger than $U(x_1, p_1)$ and $U(x_2^*, P_2(x_1, p_1, x_2^*)))$. This contradicts the assumption that it was optimal before, thus, we know that the optimal contract for (x_1, p_1) must be on the two lines (the upper boundary of T_1) defined in Theorem 2.1. Then we exclude the possibility of having optimal contract with $x_1 > x_1^*$. If $x_1 > x_1^*$, we can move $(x_1, x_1^* b_1 / x_1)$ to (x_1^*, b_1). This will increase the profit from type-1, leaving the profit from type-2 unchanged. □

Using Lemmas 2.12, 2.13 and Theorem 2.5, the expected profit can be expressed as follows.

$$
\begin{aligned}
E[U(\mathbb{C})] &= r_1 U(x_1, p_1) + r_2 U(x_2, P_2(x_1, p_1, x_2^*)) \\
&= r_1 U(x_1, b_1) + r_2 U(x_2^*, b_2 - \frac{x_1}{x_2^*}(b_2 - b_1)) \\
&= r_1 x_1 (b_1 - c) + r_2 x_2^* (b_2 - \frac{x_1}{x_2^*}(b_2 - b_1) - c) \\
\frac{\partial E[U(\mathbb{C})]}{\partial x_1} &= r_1 (b_1 - c) - r_2 (b_2 - b_1)
\end{aligned}
$$

The x_1 achieving the optimal contract $\mathbb{C}$ is given by,

$$x_1 = \begin{cases} 0 & \text{if } r_1(b_1 - c) - r_2(b_2 - b_1) < 0 \\ x_1^* & \text{if } r_1(b_1 - c) - r_2(b_2 - b_1) > 0 \end{cases}$$

$$\mathbb{C} = \begin{cases} \{max_2\} \text{ if } r_1(b_1 - c) - r_2(b_2 - b_1) < 0 \\ \{max_1, (x_2^*, b_2 - \frac{x_1^*}{x_2^*}(b_2 - b_1))\} \text{ o.w.} \end{cases}$$

This result shows two operating regimes: 1) When $\frac{r_1}{r_2} < \frac{b_2 - b_1}{b_1 - c}$, type-2 is more profitable and the seller will offer max_2. In this case there is no way to offer another contract for type-1 without affecting the behavior of type-2. Consequently, the seller only offers one contract. 2) When $\frac{r_1}{r_2} > \frac{b_2 - b_1}{b_1 - c}$, type-1 is more profitable and the seller will offer max_1. After choosing max_1, the seller can also choose $(x_2^*, b_2 - \frac{x_1^*}{x_2^*}(b_2 - b_1))$ for the type-2 buyer without affecting the type-1 buyer's choice. As a result, the seller offers a pair of contracts to get the most profit.

The optimal contract for $x_1^* > x_2^*$ can be determined with a similar argument. Again, we can prove that the optimal contract must have $p_1 = b_1$ and $x_1 \leq x_1^*$. The difference is that when $x_1^* > x_2^*$, the expression for $(x_2^*, P_2(x_1, p_1, x_2^*))$ has two cases depending on whether $x_1 > x_2^*$ or $x_1 \leq x_2^*$.

$$\frac{\partial E[U(\mathbb{C})]}{\partial x_1} = \begin{cases} r_1(b_1 - c) - r_2(b_2 - b_1) & \text{if } x_1 \leq x_2^* \\ r_1(b_1 - c) + r_2 b_1 & \text{if } x_1 > x_2^* \end{cases}$$

To summarize, when $r_1(b_1 - c) - r_2(b_2 - b_1) > 0$, $E[R(\mathbb{C})]$ is strictly increasing in x_1 and we know that $x_1 = x_1^*$ maximizes the expected profit. When $r_1(b_1 - c) - r_2(b_2 - b_1) < 0$, $E[R(\mathbb{C})]$ is decreasing in x_1 if $x_1 \in [0, x_2^*]$ and increasing in x_1 if $x_1 \in [x_2^*, x_1^*]$. We can only conclude that either $x_1 = 0$ or $x_1 = x_1^*$ maximizes the expected profit.

$$x_1 = \begin{cases} 0 \text{ or } x_1^* & \text{if } r_1(b_1 - c) - r_2(b_2 - b_1) < 0 \\ x_1^* & \text{if } r_1(b_1 - c) - r_2(b_2 - b_1) > 0 \end{cases}$$

$$\mathbb{C} = \begin{cases} max_2/\{max_1, (x_2^*, \frac{x_1^* b_1}{x_2^*})\} & \text{if } r_1(b_1 - c) - r_2(b_2 - b_1) < 0 \\ \{max_1, (x_2^*, \frac{x_1^* b_1}{x_2^*})\} & \text{if } r_1(b_1 - c) - r_2(b_2 - b_1) > 0 \end{cases}$$

In the first condition, we can calculate the expected profit of the two contract sets and pick the one with the higher profit.

2.6.2 K buyer types, $K > 2$

The previous section gives the explicit solution to the contract design problem when $K = 2$. When $K > 2$ we no longer have explicit solutions; even numerically searching for the optimal contract set becomes very complicated. For instance, even if we assume that both x and p are from discrete sets, with X and P possible values, respectively, the search must be done over the space of all possible sets of K different contracts, on the order of $(XP)^K$. In general X and P both take on real values, making the search space uncountable. In order to reduce the complexity we will need to exploit special properties of the problem. We first reindex the buyer types such that $b_1 \leq ... \leq b_K$. Then under certain conditions we will determine a procedure which finds the optimal contract. In the remainder of this section, we will assume the seller can design up to K contracts.

Definition 2.14. The buyer types are said to satisfy a monotonicity condition (MC), if $\forall i, j$, $b_i \leq b_j$ implies $x_i^* \leq x_j^*$.

Thus when the types are ordered $b_1 \leq ... \leq b_K$, we have $x_1^* \leq ... \leq x_K^*$. This monotonicity condition (MC) says that the amount a buyer willing to buy is strictly increasing in the quality it gets from buying the secondary spectrum. This condition leads to special properties which allows us to construct simpler ways to find the optimal contracts.

Theorem 2.15. *When the MC is satisfied, $b_i \leq b_j$ and $x < x'$ implies $P_i(x', p', x) \geq P_j(x', p', x)$.*

Proof.

Case 1. $x' \leq x_i^* \leq x_j^*$

When $x' \leq x_i^* \leq x_j^*$, both types have equal utiliy line of the same form.

$$\begin{aligned} P_i(x', p', x) &= b_i - \frac{q_i - \epsilon_i - C_i(x', p')}{x} \\ P_i(x', p', x) &= b_j - \frac{q_j - \epsilon_j - C_j(x', p')}{x} \end{aligned} \tag{2.5}$$

By exactly the same argument as in Theorem. 2.16 we can find out that. $\frac{\partial P_j(x',p',x)}{\partial x} \geq \frac{\partial p_i(x',p',x)}{\partial x}$, and thus,

$$P_i(x', p', x) \leq P_j(x', p', x) \; \forall x_i^* \geq x \geq x'$$

When $x_i^* < x < x_j^*$, while $P_j(x', p', x)$ still follows the same formula (Equation. 2.5), $P_i(x', p', x)$ starts to decrease by following the line $P_i(x', p', x) = x' P_i(x', p', x_i^*)/x$. Thus,

$$P_i(x', p', x) \leq P_j(x', p', x) \; \forall x_i^* \leq x \leq x_j^*$$

When $x > x_j^*$, both i, j follow the form $P(x', p', x) = P(x', p', x_j^*)/x$. But $P_i(x', p', x_j^*) \leq P_j(x', p', x_j^*)$, they never cross and $P_j(x', p', x) \geq P_i(x', p', x) \; \forall x > x_j^*$.

Case 2. $x_i^* < x' < x_j^*$

When $x_i^* < x' < x < x_j^*$ they are of the form,

$$\begin{aligned} P_i(x', p', x) &= \frac{x'p'}{x} \\ P_j(x', p', x) &= b_j - \frac{q_j - \epsilon_j - C_j(x', p')}{x} \end{aligned}$$

respectively. By the same argument as in Theorem. 2.16, P_i is decreasing while P_j is increasing. Thus, $P_i(x', p', x_j^*) \leq P_j(x', p', x_j^*)$. When $x > x_j^*$,

$$P_i(x', p', x) = \frac{x_j^* P_i(x', p', x_j^*)}{x}$$
$$P_j(x', p', x) = \frac{x_j^* P_j(x', p', x_j^*)}{x}$$

Since $P_i(x', p', x_j^*) < P_j(x', p', x_j^*)$ we know that $P_i(x', p', x) < P_j(x', p', x)$ $\forall x > x_j^*$.

Case 3. $x' > x_j^* > x_i^*$

When $x > x_j^*$, both types have equal cost line as $xp = x'p'$. Thus, $P_i(x', p', x) = P_j(x', p', x)$ $\forall x > x_j^*$. □

Theorem 2.16. *When the MC is satisfied, $b_i \leq b_j$ and $x > x'$ implies $P_i(x', p', x) \leq P_j(x', p', x)$.*

Proof.

Case 1. $x' \leq x_i^* \leq x_j^*$

When $x' \leq x_i^*$ and $x' \leq x_j^*$, the equal cost lines for $x < x'$ are of the form,

$$P_i(x', p', x) = b_i - \frac{q_i - \epsilon_i - \delta_i}{x}$$
$$P_j(x', p', x) = b_j - \frac{q_j - \epsilon_j - \delta_j}{x}$$

where we let $\delta_i = C_i(x', p')$ and $\delta_j = C_j(x', p')$. Take the derivatives with respect to

x.

$$
\begin{aligned}
\frac{\partial P_i(x',p',x)}{\partial x} &= (q_i-\epsilon_i-\delta_i)x^{-2} \\
\frac{\partial P_j(x',p',x)}{\partial x} &= (q_j-\epsilon_j-\delta_j)x^{-2}
\end{aligned}
$$

By definition, $P_i(x',p',x') = p' = P_j(x',p',x')$,

$$
p' = b_i - \frac{q_i-\epsilon_i-\delta_i}{x'} = b_j - \frac{q_j-\epsilon_j-\delta_j}{x'}
$$

Considering $b_i < b_j$, we know that $q_j-\epsilon_j-\delta_j > q_i-\epsilon_i-\delta_i$. Which implies $\frac{\partial P_j(x',p',x)}{\partial x} \geq \frac{\partial P_i(x',p',x)}{\partial x}$, and thus $P_i(x',p',x) \geq P_j(x',p',x)$, $\forall x < x'$.

Case 2. $x_i^* \leq x' \leq x_j^*$

The equal cost lines are,

$$
\begin{aligned}
P_i(x',p',x) &= \begin{cases} \frac{x'p'}{x} & x_i^* \leq x \leq x' \\ b_i - \frac{q_i-\epsilon_i-\delta_i}{x} & x \leq x_i^* \end{cases} \\
P_j(x',p',x) &= \quad b_j - \frac{q_j-\epsilon_j-\delta_j}{x} \qquad x \leq x'
\end{aligned}
$$

Where $\delta_i = C_i(x',p')$ and $\delta_j = C_j(x',p')$. Taking the derivatives,

$$
\begin{aligned}
P_i'(x',p',x) &= \begin{cases} -x'p'x^{-2} < 0 & x^{i*} \leq x \leq x' \\ (q_i-\epsilon_i-\delta_i)x^{-2} > 0 & x \leq x^{i*} \end{cases} \\
P_j'(x',p',x) &= \quad (q_j-\epsilon_j-\delta_j)x^{-2} > 0 \qquad x \leq x'
\end{aligned}
$$

This implies $P_i(x',p',x) > P_j(x',p',x)$, $\forall x\ x_i^* \leq x \leq x'$.

$$
\begin{aligned}
P_i(x',p',x_i^*) &= b_i - \frac{q_i-\epsilon_i-\delta_i}{x_i^*} \\
&> P_j(x',p',x_i^*) = b_j - \frac{q_j-\epsilon_j-\delta_j}{x_i^*}
\end{aligned}
$$

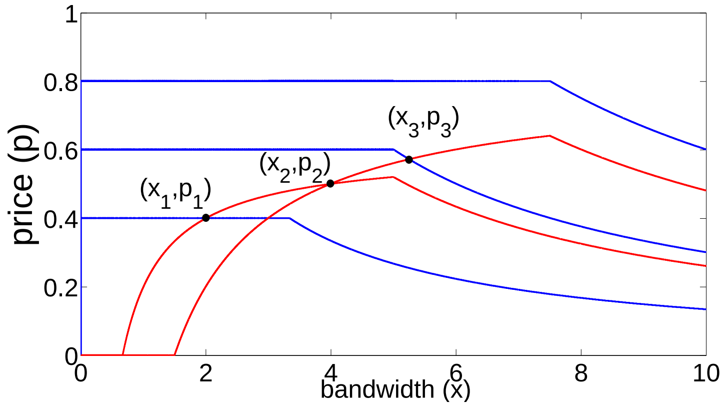

Figure 2.6: Example of a possible optimal contract

Since $b_i < b_j$, we conclude that $q_j - \epsilon_j - \delta_j \geq q_i - \epsilon_i - \delta_i$. Which indicates that $\frac{\partial P_j(x',p',x)}{\partial x} \geq \frac{\partial p_i(x',p',x)}{\partial x}$ and $P_i(x', p', x) \geq P_j(x', p', x)$, $\forall x \leq x_i^*$.

Case 3. $x' \geq x_j^* \geq x_i^*$

When $x \geq x_j^* \geq x_i^*$, the equal cost line of both types follow $x'p' = xp$. Thus, $P_i(x', p', x_j^*) = P_j(x', p', x_j^*)$. Then the case falls into Case 2 and $P_i(x_j^*, P_j(x', p', x_j^*), x) \geq P_j(x_j^*, P_j(x', p', x_j^*), x)$, $\forall x < x_j^*$. □

Lemma 2.17. *When the MC is satisfied, the optimal contract such that type i buyer picks* (x_i, p_i) *for all i must have* $x_1 \leq ... \leq x_K$.

Proof. Let (x_i, p_i) denote the contract designed for the type i buyer. Consider now the contract for the type j buyer where $b_j < b_i$ and $x_j > x_i$. From Theorem 2.16 we know that $P_j(x_i, p_i, x_j) \leq P_i(x_i, p_i, x_j)$ when the MC is satisfied. This implies that whatever p_j we determined, if the type j buyer prefers (x_j, p_j) over (x_i, p_i) then the type i buyer must think the same way. From the IC constraint, the type j buyer has to prefer the (x_j, p_j) over (x_i, p_i). Thus, we must have $x_j \leq x_i$ in the optimal contract where each type of buyer selects its own designated contract. □

Lemma 2.18. *When the MC is satisfied, the optimal contract must have* $x_i \leq x_i^*$ $\forall i = 1...K$.

Proof. Proof by contradiction. Consider some optimal contract having $x_i > x_i^*$, we show that replacing $x_i = x_i^*$ is actually better. By Theorem 2.19, we know that $p_i = P_i(x_{i-1}, p_{i-1}, x_i)$ and by definition of P_i it is better off to the seller by providing x_i^* instead if we only consider the profit from the type i buyer. Now, by Theorem 2.15 $P_{i+1}(x_i, p_i, x_i^*) \leq P_i(x_i, p_i, x_i^*)$. Also, because $P_i(x', p', x)$ is a strictly increasing function in p'. The price p_{i+1} is strictly higher for assigning x_i^* instead of x_i. This results in every p_j $j > i$ is strictly increased and the payoff change must be positive. The only question is whether we can assign x_i^* without affecting the contracts (x_j, p_j) $j < i$. The answer is if $\forall j < i$ $x_j \leq x_j^*$ we can do it. By mathematical induction, we can again prove that for all $i = 1...K$ $x_i \leq x_i^*$. An example is illustrated in Figure 2.6. □

This result allows us to restrict our search for the optimal contract to the set where $x_i \leq x_i^*$. We can further simplify our search by expressing the values p_i, $\forall i = 1...K$ as functions of x_i $\forall i = 1...K$, by the following theorem.

Theorem 2.19. *Given a set $x_1 \leq ... \leq x_K$, define $(x_0, p_0) = (0, 0)$ and find the contracts $(x_i, p_i) = (x_i, P_i(x_{i-1}, p_{i-1}, x_i))$ in the order $i = 1...K$. When the MC is satisfied this procedure produces a contract set that maximizes the seller's profit, where each type-i buyer accepts (x_i, p_i).*

Proof. a) Each buyer of type i picks (x_i, p_i).
Induction hypothesis: At each step, when we pick contract (x_i, p_i) $\forall i = 0...K$, each buyer type-j with $j < i$ prefers contract (x_j, p_j) and each buyer type-j with $j \geq i$ prefers contract (x_i, p_i).

1. Base Case: When picking $(x_0, p_0) = (0, 0)$, it is clear that each buyer type is greater than 0 and each buyer prefers the only contract that is the same as not buying.

2. Assume the induction hypothesis is true when picking (x_i, p_i), we will show that the hypothesis is also true for (x_{i+1}, p_{i+1}). Assume the hypothesis is true for step i means we have determined the contracts $((x_1, p_1), ..., (x_i, p_i))$ and a type-j buyer $(j \leq i)$ prefers (x_j, p_j) over other contracts, while a type-j buyer $(j > i)$ prefers the ith contract over all contracts. By Theorem. 2.16 and $x_{i+1} > x_i$,

$$\forall j \leq i, p_{i+1} = P_{i+1}(x_i, p_i, x_{i+1}) \geq P_j(x_i, p_i, x_{i+1})$$

The contract (x_{i+1}, p_{i+1}) is above the equal cost line of the contract (x_i, p_i) for buyer type less than or equal to i. Which means they prefer the ith contract over the $i + 1$th contract. But from step i, they prefer their own contract over existing contracts. Thus, buyer j $(j \leq i)$ prefers (x_j, p_j) over all contracts. By Theorem. 2.15 and $x_{i+1} > x_i$,

$$\forall j \geq i + 1, p_{i+1} = P_{i+1}(x_i, p_i, x_{i+1}) \leq P_j(x_i, p_i, x_{i+1})$$

Thus, the contract (x_{i+1}, p_{i+1}) is below the equal cost line of the contract (x_i, p_i) for buyer type $j > i$ and they prefer (x_{i+1}, p_{i+1}) over (x_i, p_i). But from step i, they prefer the (x_i, p_i) contract over all existing contracts. This shows that the hypothesis is true for step $i + 1$.

3. By Mathematical Induction, the hypothesis is true for all $i \leq K$.

b) This process results in the highest profit.

Since the $x_i's$ are fixed, the only way one could increase the buyer's profit is to increase one of the p_i's. We will show that this is not possible. Assume there exists some contract with the contract set $(x_1, p'_1)...(x_K, p'_K)$ with some $p'_i > p_i$, by the increasing property of P_i (Lemma 3.4) we need $p'_{i-1} > p_{i-1}$ to insure that type-i buyer picks (x_i, p'_i). By induction, we can show that it must be that $(p'_1 > p_1)$. Since $p_1 = b_1$,

(x_1, p_1) is already on the boundary of acceptance region of the type-1 buyer. Thus, any contract with some $p_i' > p_i$ is not a contract where each buyer accepts its own designated contract. $\square$

Figure 2.6 shows an example of applying this theorem with three buyer types: given $x_1 = 2$, $x_2 = 4$, $x_3 = 6$, p_i is sequentially determined on the equal-cost line of the previous contract. With Lemma 2.18, the equal cost line can be restricted to the form $P_i(x_{i-1}, p_{i-1}, x_i) = b_i - \frac{x_{i-1}}{x_i}(b_i - p_{i-1})$. The expected profit of the seller can now be expressed as:

$$\begin{aligned} E[R(\mathbb{C})] &= \max_{x_1,..,x_K} r_1 x_1(b_1 - c) + ... + r_i x_i(p_i - c) + ... + r_K x_K(p_K - c) \\ &= \max_{x_1 \leq ... \leq x_K} r_1 x_1(b_1 - c) + ... + r_i x_i(P_i(x_{i-1}, p_{i-1}, x_i) - c) + ... \\ &\qquad + ... + r_K x_K(P_K(x_{K-1}, p_{K-1}, x_K) - c) \end{aligned}$$

By plugging in the values of $p_i = P_i(x_{i-1}, p_{i-1}, x_i) = b_i - \frac{x_{i-1}}{x_i}(b_i - p_{i-1})$ recursively. Each term in the optimization problem can be simplified to

$$r_i x_i(p_i - c) = r_i(x_i(b_i - c) - \sum_{j=1}^{i-1} x_j(b_{j+1} - b_j))$$

By simplifying and separate the terms with respect to x_i, the expected profit of the seller can be expressed as,

$$E[R(\mathbb{C})] = \max_{x_1 \leq ... \leq x_K} \sum_{i=1}^{K} x_i[r_i(b_i - c) - (b_{i+1} - b_i) \sum_{j=i+1}^{K} r_j]$$

Firstly, we observe that the above expression is linear in every x_i. Thus differentiating with respect to x_i we get a constant:

$$\frac{\partial E[R(\mathbb{C})]}{\partial x_i} = r_i(b_i - c) - \sum_{j=i+1}^{K} r_j(b_{i+1} - b_i)$$

Secondly, because the term $\frac{\partial E[R(\mathbb{C})]}{\partial x_i}$ does not depend on any x_j, the optimizer can be easily determined. When $\frac{\partial E[R(\mathbb{C})]}{\partial x_i} > 0$ we want to make x_i as large as possible ($\leq x_i^*$); when $\frac{\partial E[R(\mathbb{C})]}{\partial x_i} < 0$ we want to make x_i as small as possible. This leads us to the following algorithm which finds the optimal set of $(x_1, ..., x_K)$. The variable LD (Last Determined) below is used to keep track of the last type for which we have already determined its value.

Algorithm 1 Optimal contract under monotonicity condition

Let $x_K \leftarrow x_K*$, $LD \leftarrow K$ ▷ because $\frac{\partial E[R(\mathbb{C})]}{\partial x_K} > 0$
for $i = K-1 \rightarrow 1$ **do**
  $\pi \leftarrow (b_i - c)\sum_{j=i}^{LD-1} r_j - (b_{i+1} - b_i)\sum_{j=LD}^{K} r_j$
  if $\pi > 0$ **then**
    $\forall i \leq j < LD, x_j \leftarrow x_i^*$
    $LD \leftarrow i$
  else if $i = 1$ **then**
    $\forall 1 \leq j < LD, x_j \leftarrow 0$
  end if
end for

This algorithm works as follows: We start from determining the value of x_K, then we determine x_{K-1} and so on all the way to x_1. At step i we take the derivative with respect to x_i. If it is better to maximize it, we assign it to be x_i^*. If it is better to minimize it, we push the value to x_{i-1} (which we have not determined). However, we have to add the probability of occurrence r_i to the value (x_{i-1}) we pushed to so that it reflects the weight of occurrence when determining the value x_{i-1}. Once we determined the value for some x_i, every x_j previously pushed to it will be assigned the same value.

Together with Theorem 2.19 the above algorithm produces a set of (x_i, p_i)'s that's optimal under the monotonicity condition. This algorithm takes exactly K steps to find the optimal contract set. While calculating the $\sum r_i$ might also take K steps, with careful calculation the method can still be completed in $O(K)$ time. By comparison, an exhaustive search method will take $O((XP)^K)$ time to find the optimal contract even if we discretize the search space of x and p with X and P possible values. When x and p are continuous, an exhaustive search might not even be possible.

2.7 Discussions

2.7.1 More general models of channel quality B

Although some of the analysis in this chapter relies on B being a binary random variable, most of our definitions can be easily generalized to any random variable, such as the acceptance region, equal-cost line and max_i are general to any B. Take for instance the notion of acceptance region. Consider any random variable B with support $[0, 1]$. By the definition of $C(x, p)$, the reserved cost $C(0, 0) = q - \epsilon$ is unchanged. The acceptance region of a single buyer type can still be calculated using $T = \{(x, p) : C(x, p) \leq C(0, 0)\}$ with the boundary being $f(x) = \max_p C(x, p) \in T$. It is also easy to show that the optimal contract for the buyer must be on this boundary, thus $optimal = \max_x U(x, f(x))$. Similarly, the equal-cost line will continue to be strictly ordered according to the price p. With these set calculated explicitly, the same process of contract selection can be used. For example, if under some ordered conditions of B, the equal-cost lines can be shown to satisfy Theorem 2.15, 2.16; then a process similar to Algorithm 1 can be applied to the problem. An extension of the contract model which considers general utility functions and channel random variables is discussed in Chapter III.

2.7.2 A seller with limited resource

Our analysis so far has been based on the assumption that the seller has sufficient bandwidth to fulfill all accepted contracts. We now discuss what happens when the seller's resources are limited. In the full information case when the seller knows the type of each of a group of potential buyers, it will extract the most by offering (b_i, x_i^*)

to a buyer of type i. Under a resource constraint, because the seller can offer any $0 < x < x_i^*$ (when p is set to b_i), this becomes a form of the continuous (fractional) knapsack problem [21].

When buyer types are private information, we consider 3 possible scenarios and methods to determine the optimal contract solution by modifying Algorithm 1 in Section 3.4. We assume at most $\bar{X}$ bandwidth can be sold and the monotonicity condition is satisfied for simplicity.

Case 1: The seller knows that there is only one buyer, does not know its type, but knows the distribution of the type. This is the same as the case of $K > 2$ under MC condition except that the maximum bandwidth sold is limited by $\bar{X}$. If we have $\forall x_i^* \leq \bar{X}$, then Algorithm 1 works without modification. But if for some i, $x_i^* > \bar{X}$, then the algorithm no long works. Note that in determining the optimal set x_i, Algorithm 1 does not explicitly determine the value of each x_i but only whether we need to push the x_i value bigger or smaller. Also the analysis does not rely on the actual values of x_i^*, but only that $\forall i < j, x_i^* \leq x_j^*$. This discussion leads to the next result.

Corollary 2.20. *Let* $\forall i, \hat{x_i^*} = min(x_i^*, \bar{X})$*, then running Algorithm 1 on the set* $(b_i, \hat{x_i^*})$ *will result in the optimal contract for limited bandwidth* $\bar{X}$ *with a single customer.*

Case 2: The seller knows that there are N_i of each possible type, but cannot distinguish between the different types. Letting $r_i = N_i$, Algorithm 2 finds the optimal contract when there is insufficient bandwidth. Note that this algorithm is similar to Algorithm 1 with two differences: 1) it replaces the distribution in Algorithm 1 by the actual number of buyers of each type. 2) it designs contracts for higher buyer types. Changing the distribution of buyers to actual number of buyers will not change the optimality of Algorithm 1 if the bandwidth is sufficient. If bandwidth is insufficient, because an optimal contract must have $p_i \geq p_j$ for $b_i \geq b_j$, it is preferable to keep higher buyer types. The step i where the algorithm breaks is the cutoff type that should be accepted; any type smaller will not be considered in the contract. All previous types pushed to the same values of this cutoff type are then recalculated such that the bandwidth amount satisfies the constraint ($\bar{X}$). The price determining process ($p_{i+1} = P_i(x_i), p_i, x_{i+1})$) is then applied on this set, with price $p_i = b_i$ as the first contract.

Algorithm 2 Limited resource

```
Let x_K ← x_K*, LD ← K                                   ▷ because ∂E[R(ℂ)]/∂x_K > 0
for i = K − 1 → 1 do
    π ← (b_i − c) Σ_{j=i}^{LD−1} r_j − (b_{i+1} − b_i) Σ_{j=LD}^{K} r_j
    if π > 0 then
        ∀i ≤ j < LD, x_j ← x_i^*
        if Σ_{j=LD}^{K} x_j ≥ X̄ then
            FLAG ← true
            break
        end if
        LD ← i
    else if i = 1 then
        ∀1 ≤ j < LD, x_j ← 0
    end if
end for
if FLAG then
    ∀i ≤ k < LD, x_k ← ((Σ_{j=i}^{K} x_j) − X̄) / (Σ_{j=i}^{LD−1} N_j)
end if
```

Case 3: Users arrive as a Poisson random process. This is a case that is similar to that studied in [43], where it is shown that repeatedly solving the expected version of the stochastic optimization problem will result in a policy with expected revenue lost upper bounded by a constant which is independent of the size ($\bar{X}$) of the problem. Notice that Case 2 is exactly the expected version of this stochastic optimization problem, thus, we can again use Algorithm 2 to solve the problem.

2.7.3 Learning buyer types

We have assumed in our analysis that the seller knows a priori the buyer distribution which is discrete. If this distribution is unknown, it can be obtained through online learning. Consider a stream of arriving buyers and a seller offering contracts designed not only to make profit (exploit) but also to learn the buyer type distribution (explore) by observing whether the contract is accepted or rejected. This can be cast as a multi-armed bandit problem with an independent reward process (assuming buyers are independently drawn from a distribution), and potentially a continuum of arms (each contract is an arm under this model). Algorithms exist in the literature that produce sublinear regret (defined as the profit difference between the best single contract and the algorithm) in time [8], and logarithmic regret in time when the number of arms is finite [7]. Although the continuum contract (arm) space might seem a challenge, we note that Algorithm 1 always generates a set of contracts with x_is a subset of $\{x_1^*, x_2^* ... x_K^*\}$. From Theorem 2.19, if we know the set of x_is, we can explicitly determine the optimal price. Thus, there are only 2^K possible contracts that can be optimal. Using this observation, one can construct a learning algorithm like that in [8] to achieve logarithmic regret.

2.7.4 Comparing to auction

Auction has been used extensively for the allocation of spectrum on the traditional, wholesale market, and has been proposed for the secondary market as well, see e.g., [70, 31, 40]. Auction is a mechanism aimed at extracting profit from the sale of rare goods for which potential buyers' valuation is unknown and can be very hard to obtain. The contract mechanism studied in this chapter may be viewed as a form of sale by *posted price*. Compared to auction, posted price is more often used in the sale of multiple (and potentially large quantity of) similar goods, the valuation of which is obtained through market research [69]. Since the cost spent on market research can be amortized over multiple goods, posted price sale can be more efficient than auction which incurs cost in conducting each single auction [74]. It has been shown that under ideal conditions the two are equivalent in profit generation [58]. As more and more license holders become interested in the secondary market trading smaller quantities for shorter duration of time compared to the primary wholesale market, we believe pricing schemes like the contracts studied in this chapter offer a valid alternative to spectrum auctions.

2.8 Numerical Evaluation

In this section, we compare the performance of contracts generated by the following methods under limited and unlimited resource constraints.

1. The optimal set of M contracts (denoted OPT(M) in the figures): Finding this set is done by an exhaustive search over a set of discretized values x and p as an approximation of the uncountable choices (the step size for x is 0.5 and the step size for p is 0.1). As discussed earlier in Section 2.6.2, the complexity increases exponentially in M. This restricts us to run at most $M = 2$ in our evaluation.

2. The algorithm we introduced in the previous section (Algorithm 1 (Algorithm 2) in the unlimited (limited) resource setting, denoted as ALG1 (ALG2) in the figures): As previously shown, ALG1/ALG2 is optimal when the monotonicity condition holds. Since the complexity of this algorithm increases only linearly in M, M can be on the order of thousands in our numerical evaluation.

3. A K-choose-1 method (denoted as MAX in the figures): This is the method that selects the contract with the highest expected profit over the set $\{max_1, \max_2, \cdots, \max_K\}$: $\underset{max_i, i=1...K}{\text{maximize}}\ E[U(max_i)]$. This is done by checking all (b_i, x_i^*) pairs; the complexity increases linearly in M.

2.8.1 Unlimited resource

The experiments are run by increasing $K = 1...7$. For each K value the parameters $(q_i, b_i, \epsilon_i, r_i)$ are independently and randomly generated from uniform distributions

($b_i \in [0,1]$, $q_i \in [0,10]$, $\epsilon_i \in [0,2]$ and $r_i \in [0,1]$ but normalized such that $\sum r_i = 1$) For each K we record the average (in expected profit) over 12000 randomly generated cases; these are plotted in Figure 2.7. We repeat the same but only for cases that satisfy the monotonicity condition; results are shown in Figure 2.8.

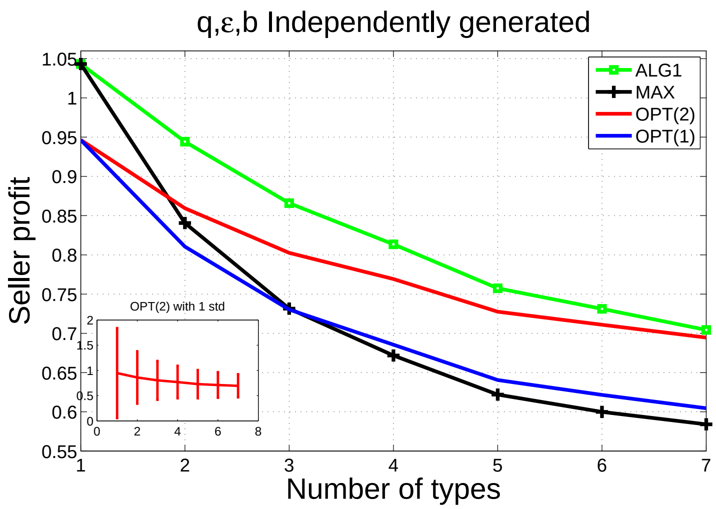

Figure 2.7: Simulation results of the sellers profit versus different contracts in the general case. (The inset is the standard deviation of OPT(2))

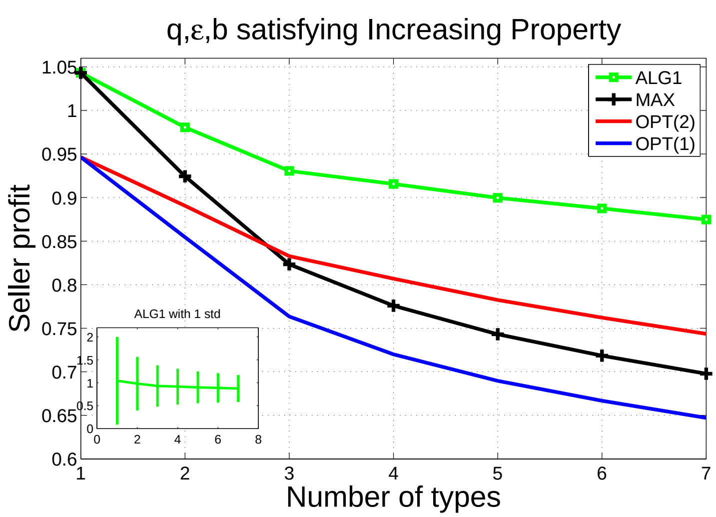

Figure 2.8: Simulation results of the sellers profit versus different contracts when increasing property holds. (The inset is the standard deviation of ALG1)

Our observations are as follows. Being able to use more contracts is always better as expected (i.e., OPT(1) $\leq$ OPT(2) in all cases). When the monotonicity condition holds, ALG1 is optimal and thus outperforms all other algorithms. When $K = 1, 2$ OPT(2) should have been optimal but it falls below ALG1 due to the discretization error. When $K > 2$, ALG1 further has the advantage of being able to use more than 2 contracts. Recall that MAX is the optimal contract when the seller knows exactly the type; thus, MAX is optimal when $K = 1$ and outperforms exhaustive search because it does not suffer from discretization error. In the general case when the monotonicity does not necessarily hold, although ALG1 is not always optimal it still outperforms both OPT(1) and OPT(2). Finally, when there are more possible buyer types (as K increases), the maximum expected profit decreases because it is harder to put all the contracts right on the buyers' acceptance boundaries while still satisfying the incentive compatibility condition.

We show the standard deviation for ALG1 under the monotonicity condition and OPT(2) under the general case in Figures 2.7 and 2.8, respectively. Other cases are similar and thus not shown for better readability. We see that the deviation is decreases as the number of buyer types increases. This is because the amount of profit depends on the realization of the buyer types (q, b, ϵ). With fewer buyer types, the profit changes heavily depends on the realization, e.g., a type with very low channel quality can lead to low profit. With more buyer types, the profit is averaged out over the buyer distribution and thus has a smaller variation.

In Figures 2.9, 2.10 and 2.11 we show the results for averaging over 12000 cases satisfying the monotonicity condition. In Figure 2.9, we show the buyers gain over not accepting any contract. It shows that as more buyer types exist, the buyer's average gain increases as expected. In Figure 2.10, we show the sum of the buyers' and the seller's gains. We see that only in the case of ALG1, the total utility remains constant as the number of types increases. This shows that ALG1 generates contracts that are more socially optimal. In Figure 2.11, we show the portion of buyers accepting one of the contracts. We observe that as the number of buyer types increases, a larger

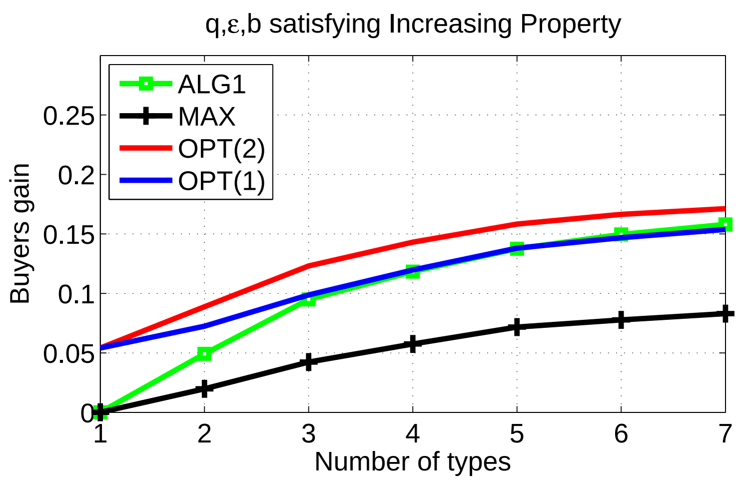

Figure 2.9: Buyer utility

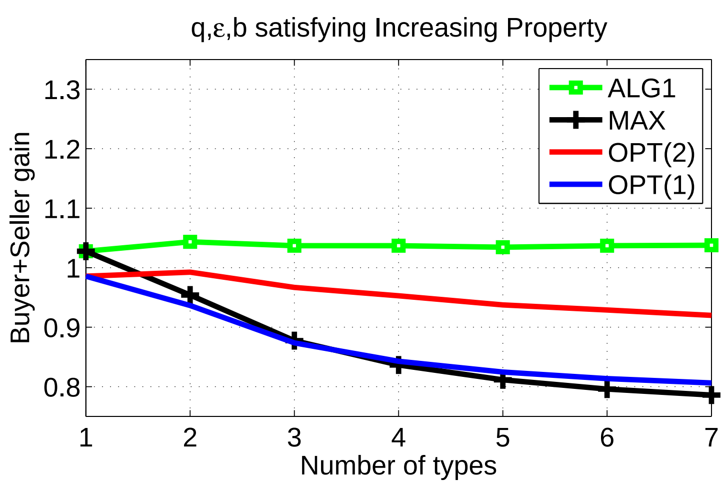

Figure 2.10: Total utility of buyer and seller

portion of buyers walk away from all contracts. Note that ALG1 has the highest participation rate.

2.8.2 Limited resource

We next perform the same experiments under the limited resource condition. The simulations are done with randomly generated buyer types *not* satisfying the monotonicity condition. Algorithm 2 is used to replace Algorithm 1. The possible buyer type is fixed at $K = 3$ with 3 buyers of each type. We change the x-axis to the resource limit and test it from insufficient bandwidth to sufficient bandwidth.

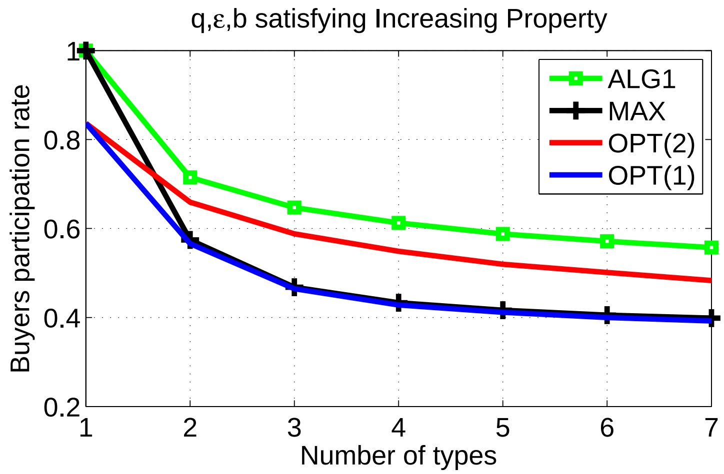

Figure 2.11: Buyer participation rate

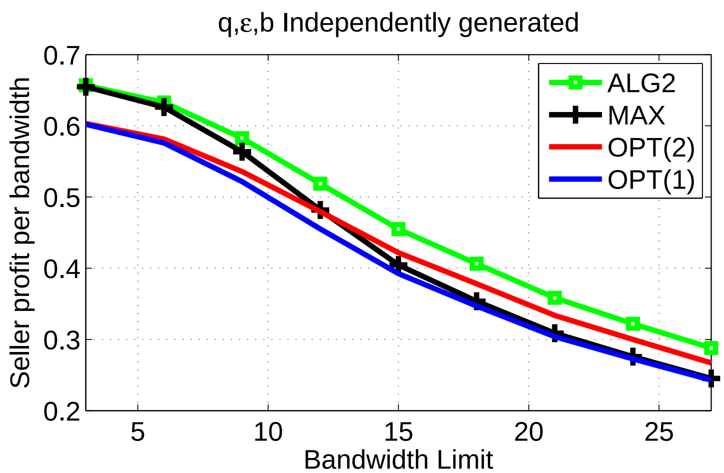

Figure 2.12: Seller profit per bandwidth limit

Figure 2.12 shows the seller's profit *per unit of bandwidth* (y-axis) as a function of its bandwidth limit (x-axis), while Figure 2.13 shows the amount left unsold. We see that when the seller has very limited amount of bandwidth, it can sell all of it and enjoys a high unit profit. When it has more bandwidth than the purchasing need, its unit profit drops. This happens for two reasons: 1) When it has little to sell, the seller tends to target the higher type that accepts the contract at higher prices. When it has more, the seller wants to sell more. In this case, it will have to sell to lower buyer types which only accept at lower prices. 2) When there is a surplus of supply, bandwidth left unsold generates no profit. Also from Figure 2.12, we see that

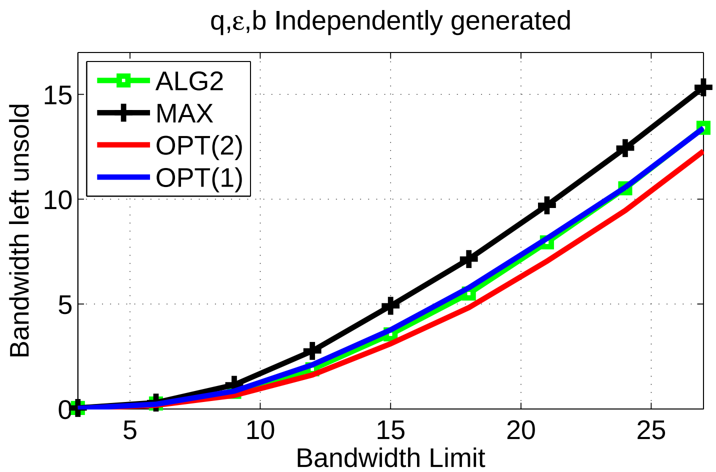

Figure 2.13: Amount of bandwidth left

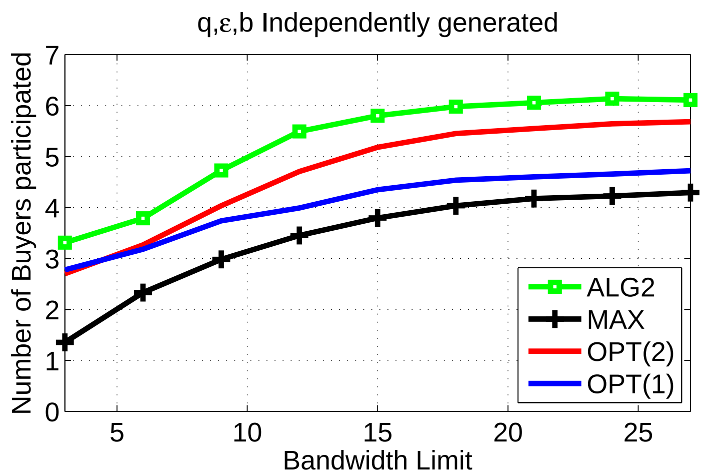

Figure 2.14: Number of buyers participated

ALG2 generates the most profit over all other methods considered. In Figure 2.14 we observe that ALG2 acquires the most number of buyers to the contract. Although there is a total of 9 buyers (3 buyers of each of 3 types), all methods on average sell to much fewer than 9 in the sufficient bandwidth region (4 to 6 buyers). This is explained by our earlier analysis (in the unlimited case) where it is shown that it may be in the seller's interest to not sell to the smaller buyer types in order to increase profit from the higher types.

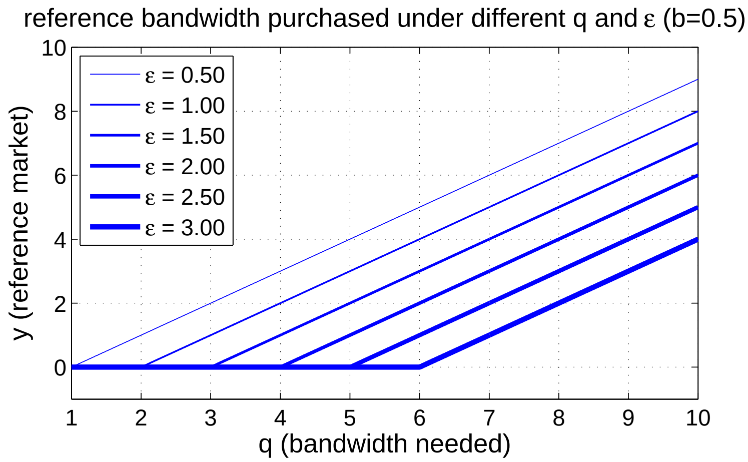

Figure 2.15: Amount of bandwidth purchased from the reference market

2.8.3 Bandwidth purchased from the reference market

We end this section by considering the amount of bandwidth the buyer needs to purchase from the reference market, shown in Figure 2.15 as a function of the transmission need q and different tolerance ϵ. Here we assume a common channel condition where the seller can sell at the optimal contract (x^*, b). We fix the channel quality at $b = 0.5$ and vary the other quantities. We can see that for each ϵ, the purchased bandwidth is 0 while q is small. When q increases, the amount needed starts to increase. Note that this is the minimum amount of guaranteed service the buyer has to purchase regardless of how much secondary bandwidth already purchased (see discussion after Lemma 1).

2.9 Conclusion

We considered a contract design problem where a primary license holder wishes to profit from its excess spectrum capacity by selling it to potential secondary users/buyers via designing a set of profitable contracts. We completely characterize the optimal solution in the cases where there is a single buyer type, and when multiple types of buyers share a common, known channel condition. In the case when each type of buyers have different channel conditions we construct an algorithm that generates a set of contracts in a computationally efficient manner, and show that this set is optimal when the buyer types satisfy a monotonicity condition.

CHAPTER III

Generalization of Secondary Trading Using Contracts

3.1 Introduction

In this chapter, we extend the method developed in Chapter II to a more general framework where the buyers and the seller are given by more general utility functions. In particular, we will show that concepts such as the equal-cost line introduced in Chapter II can be applied beyond the set of specific utility functions assumed. In what follows we will first restate the problem in a more generalized framework in Section 3.2, and then analyze the single buyer type in Section 3.3 and characterize the properties of the optimal contract. We derive the equivalent condition in the multiple buyer case such that the sequential process introduced in Chapter II produces the optimal contract in Section 3.4.

3.2 Model and Assumptions

In this section, we describe in detail the generalized model. Although the utility functions are different, some of the key concepts such as how the two parties consider their choices follow the assumption of the previous chapter. Thus, we will not repeat all the definitions that can be extended here.

3.2.1 The Seller

The seller's profit or utility from contract (x, p) is given as

$$U(x, p, c)$$

where c is a predetermined constant that takes into account the operating cost of the seller. We will keep this function unspecified and only assume that the utility of the seller is increasing in the price of the contract p when keeping the amount of bandwidth sold x at a fixed value. If none of the contracts is accepted by the buyer, the *reserve utility* of the owner is given by $U(0, 0, c)$. We will assume that any contract the seller presents must be such that $U(x, p, c) \geq U(0, 0, c)$, i.e., the seller will not sell at a loss.

3.2.2 The buyer's consideration

For a given contract pair (x, p) where x is the amount of secondary bandwidth sold and p is the price per bandwidth sold, the total payment to the primary is xp. Suppose in addition to this contract, the buyer purchases y units of guaranteed bandwidth from the reference market. Let $w(xp + y)$ denote the cost function of purchasing contract (x, p) and y guaranteed channels. We do not specify the function $w(.)$ but only assume that it is increasing in the value $xp + y$ (the total money spent).

The buyer has a constraint on its transmission quality, given by an indicator function $Q(x, y)$ that characterizes different buyer types:

$$Q(x, y) = \begin{cases} 0 & \text{the combination } (x, y) \text{ does not satisfy the buyer's needs} \\ 1 & \text{the combination } (x, y) \text{ satisfies the buyer's needs} \end{cases}$$

The function $Q(x, y)$ captures the buyer's transmission needs and can take into account the variability of the secondary channel. Subsequently this function will also be referred to as the buyer's *type*. $Q(x, y)$ can be of any form, but is assumed to be such that for any (x, y) pair that satisfies the buyer's quality of service requirements, (x', y') must also be satisfactory if $x' \geq x$ and $y' \geq y$, i.e., getting more bandwidth of either kind will not make the requirements harder to meet.

The buyer's consideration is given by the following minimization problem where we use $C(x, p, Q)$ to denote the cost for the buyer if he chooses to accept the contract (x, p).

$$\begin{aligned} C(x, p, Q) = \underset{y}{\text{minimize}} \quad & w(y + xp) && (3.1) \\ \text{subject to} \quad & Q(x, y) = 1 && (3.2) \end{aligned}$$

To assess how much this contract actually costs him, the buyer has to consider how much additional fixed service he needs to purchase to fulfill his needs. The buyer can always choose to not enter into any of the presented contracts and only purchase from the traditional market. In this case, his cost is given by the value of the following minimization problem:

$$\begin{aligned} C(0, 0, Q) = \underset{y}{\text{minimize}} \quad & w(y) && (3.3) \\ \text{subject to} \quad & Q(0, y) = 1 && (3.4) \end{aligned}$$

In deciding whether to accept a given contract (x, p), the buyer has to consider (1) whether the contract would satisfy its quality (loss) constraint, and (2) whether there is an incentive to enter into this contract, i.e., whether the cost of this contract is no higher than the reserve price. The first can always be achieved by purchasing more reference spectrum. The second is also referred to as the *individual rationality* (IR) constraint, $C(x, p, Q) \leq C(0, 0, Q)$. Any contract that satisfies both constraints of a buyer is referred to as *acceptable* to that buyer.

Considering the form of the buyer's optimization problem, the buyer chooses the minimum amount of y to minimize cost. Let $y(x,Q)$ denote the minimum additional y that needs to be purchased given the contracted amount x for a buyer of type Q. Note that this does not depend on the price p but only depends on the value x and the buyer type Q.

$$y(x,Q) = \underset{y}{argmin}\ Q(x,y) = 1$$

Thus, we can rewrite $C(x,p,Q) = w(xp + y(x,Q))$ as the minimum total cost of buyer type Q when accepting contract (x,p). The function $y(x,Q)$ depends on the buyer type Q function, but for a reasonable loss constraint $y(x,Q)$ should be non-increasing in the value x, i.e., the buyer should need less from the reference market if he purchases more from the secondary market.

Example 3.1. Taking the model in Chapter II, the function $Q(x,y)$ would be

$$Q(x,y) := I(E[(q - y - xB)^+] - \epsilon \leq 0)\ , \tag{3.5}$$

where B is the binary random variable with probability b of being 1. The constants (q,b,ϵ) represent the different types of buyer under the same form of quality constraint. The function $y(x,Q)$ would be as follows.

When $q(1-b) \leq \epsilon$,

$$y(x,Q) = \begin{cases} q - \epsilon - bx & \text{if } x \leq x^* \\ 0 & \text{if } x^* \end{cases} .$$

When $q(1-b) > \epsilon$,

$$y(x,Q) = \begin{cases} q - \epsilon - bx & \text{if } x \leq x^* \\ q - \frac{\epsilon}{(1-b)} & \text{if } x > x^* \end{cases} .$$

3.2.3 Buyer types and informational constraints

We will still assume that a potential buyer may be one of a number of different types; each type is characterized by a unique function Q, which is a buyer's *private information*. The seller is assumed to know the distribution of the buyer types but not the actual type of a particular buyer. The buyer types and their distribution may be estimated from the seller's past experience. Specifically, we will assume there are K types of buyers, and a buyer is of type i with probability r_i and is given by the set Q_i. In subsequent sections we will first discuss the case of a single user type, then the case of multiple types.

3.3 Optimal contract for a single buyer type

We begin by considering the case where there is only one type of buyer Q. Denote by $T_Q = \{(x, p) : C(x, p, Q) \leq C(0, 0, Q)\}$ the set of all acceptable contracts for the buyer, or the *acceptance region*. All possible contracts can be represented by a point on the x-p plane and the set T_Q consists of an area on the x-p plane. Recall that the buyer will only accept a contract if the cost of accepting the contract is less than or equal to not accepting the contract. For a fixed x, because the cost of the buyer is increasing in p, the highest price p such that the buyer will accept the contract is

$$w(xp + y(x)) = w(0 + y(0)) \ .$$

We can express the highest price $t(x, Q) = \frac{y(0,Q)-y(x,Q)}{x}$ of an amount of bandwidth x for a type Q. Let $t(x, Q)$ denote the upper boundary of the area T_Q on the x-p plane and all points below $p < t(x, Q)$ (contracts having lower price) are acceptable to the buyer.

Similar to the previous chapter, we can derive the *equal-cost line* of a buyer.

Definition 3.2. The equal-cost line E of a buyer of type Q is the set of contracts within the buyer's acceptance region T that are of equal cost to the buyer. Thus $(x, p) \in E$ if and only if $p = P_Q(x', p', x)$ for some other $(x', p') \in E$. The cost of this line is given by $C(x', p')$, $\forall (x', p') \in E$.

Using the function $y(x, Q)$ introduced in Section 3.2 we have:

$$\begin{aligned}
&C(x,p,Q) = C(x',p',Q) \\
&\Leftrightarrow w(xp+y) = w(x'p'+y') \\
&\Leftrightarrow xp + y(x,Q) = x'p' + y(x',Q) \\
&\Leftrightarrow P_Q(x,p,x') = \frac{xp + y(x,Q) - y(x',Q)}{x'}
\end{aligned}$$

Among the set of equal-cost lines, the line $P_Q(0, 0, x) = t(x, Q)$ is equivalent to the acceptance boundary as defined previously.

For a given buyer type (Q), the seller can calculate the region T_Q and choose any point in the corresponding acceptance region T_Q to maximize its utility: $\max_{(x,p)\in T_Q} U(x, p, c)$, since $U(x, p, c)$ represents the seller's profit obtained from selling the contract (x, p). We will assume that the utility is increasing in p for a fixed x. We can show some of the properties of the optimal contract in the single buyer case.

Lemma 3.3. *The contract that maximizes the primaries profit is on the boundary* $t(x, Q)$.

This is based on the fact that $U(x, p, c)$ is increasing in p. We denote this maximum point as max_Q for a buyer type Q. Once the seller determines the optimal contract and presents it to the buyer, the buyer will accept because it satisfies both the loss and the IR constraints. The buyer's cost in accepting is exactly the reserved cost $C(0, 0, Q)$ because the contract is on the boundary. Every contract below an equal-cost line is strictly preferable to a contract on the line for the buyer. This observation is true as long as $w(.)$ is an increasing function.

Lemma 3.4. *As p' increases, $P(x', p', x)$ is strictly increasing in the buyer cost.*

This lemma is easily shown by noting $C(x', p', Q) = w(y(x') + x'p')$. Thus, $p > p'$ implies $C(x', p, Q) > C(x', p', Q)$.

Lastly, after the seller determines the form of the equal-cost line of the buyers the cost related to each line is no longer important. This is because in designing the contracts, the seller only has to know which contract is more preferable to the buyer. Also, the profit of the seller only depends on whether the buyer accepts the contract but not on the type of the buyer. Thus, the buyer's cost function $w(.)$ can be of any form which does not affect any of our analysis further on.

3.4 Multiple buyer types

We discuss two cases in this section: (1) the seller can only give out one contract (2) the seller can give out as many different contracts needed.

3.4.1 Single Contract

When the seller can only give out one contract, there are two factors that affect the total profit generated from this single contract. The first is the profit contract (x, p) generates if it is chosen which is $U(x, p, c)$. The second is the probability that contract (x, p) will be selected; this depends on the buyer types and the distribution of the buyer types. The factors depend on the actual realization of the seller/buyer type and distribution, but we can characterize some properties of the single best contract.

Lemma 3.5. *The optimal single contract is on the acceptance boundary of one of the buyer types.*

Proof. Let's assume that the boundaries of each buyer type divides the plane x-p plane into n areas. Let's label these regions as $G_1, ..., G_n$. For each G_i, it must be either $G_i \subset T_j$ or $G_i \subset T_j^C$ for all i, j. (where T_j^C denotes the complement of the region T_j) Thus, all contracts inside a region G_i are accepted by the same set of buyer types and not accepted by the other buyer types. Since inside each region G_i, all contracts are accepted with the same probability, the only factor is $U(x, p, c)$. The contract that has the highest $U(x, p, c)$ must be on the boundary of G_i which implies it is on the boundary of one T_j. This completes the proof that the optimal single contract must be on the acceptance boundary of one of the buyer types. □

3.4.2 Multiple Contracts

3.4.2.1 $max_i \notin T_j$ for all i, j

Under the condition $max_i \notin T_j$ for all i, j, each buyer type prefers different types of contracts and this does not conflict with the seller's profit. The optimal set of contracts will be $\{max_1, ..., max_K\}$. Because $max_i \notin T_j$ for all i, j, each buyer type i considers other max_j as unacceptable contracts. Thus, the seller can use this set $\{max_1, ..., max_K\}$ and get the profit as if first knowing the buyer type and giving out the corresponding contract.

3.4.2.2 Generalized Monotonicity Condition

For multiple buyer types in the generalized utility function for buyer and sellers, we define a similar monotonicity condition as in Chapter II. We say that the monotonicity condition (MC2) is satisfied if we can find an ordering of the buyer types $1, 2, ..., K$ such that the following property is true.

Definition 3.6. The buyer types are said to satisfy a monotonicity condition (MC2), if $\forall i > j, x < x'$, $P_i(x', p', x) < P_j(x', p', x)$ and $\forall i > j, x > x'$, $P_i(x', p', x) > P_j(x', p', x)$

This monotonicity condition (MC2) says that there exists an ordering in the buyer types such that higher types have higher valuation for more bandwidth purchased.

It can be shown that MC2 is implied by the monotonicity condition (MC) for the specific utility function in Chapter II. If MC2 is satisfied, then Lemma 2.17 says that the optimal contract set must have the x_i's ordered. Theorem 2.19 also follows through because we have assumed that the seller's profit is increasing in p and from Lemma 3.4 the equal-cost lines are strictly increasing in the buyer's cost.

By Theorem 2.19, the generalized problem can be reduced from the space $\{x_i, p_i\}$ to the ordered space $\{x_1, ..., x_K\}$, where we can write the maximization problem as follows.

$$
\begin{aligned}
E[R(\mathbb{C})] &= \max_{x_1,..,x_K} r_1 U(x_1, p_1) ... + r_K U(x_K, p_K) \qquad (3.6)\\
&= \max_{x_1 \leq ... \leq x_K} r_1 U(x_1, p_{b_1}(0, 0, x_1)) + ... + r_K U(x_K, p_{b_K}(x_{K-1}, p_{K-1}, x_K)) \ .
\end{aligned}
$$

Since we prefer not to specify the function $w(.)$, we cannot proceed as in Chapter II. However, we note that the procedure in Theorem 2.19 has a special property that if we fix the contract (x_i, p_i) for buyer type i, then the optimal set of contracts for buyer type $i+1, ..., K$ does not depend on the optimal contract set of the buyer types below i. Thus we discretize the x-p plane into a grid where X and P denote the number of x and p values. We can utilize a dynamic programming method to reduce the complexity of finding the optimal contract set. Let $g(x, p, i)$ denote the maximum profit from giving out contract (x, p) to buyer type i and the optimal contract set to the buyer types above i. We can easily calculate the values of $g(x, p, i-1)$ for the entire x-p plane with the following method. For each grid $g(x, p, i-1)$, since we are giving out (x, p) for the buyer type $i-1$, we know that the optimal contract for the buyer type i must be of the form $(x', P_i(x, p, x'))$ from Theorem 2.19. Thus, we only need to compare at most X different values to determine $g(x, p, i-1)$. After determining all (x, p) values for $g(x, p, i-1)$ we can repeat the same process for $g(x, p, i-2)$. Thus, the complexity of the algorithm is KX^2P which is polynomial in the number of buyer types compared to the $(XP)^K$ of the brute force search.

Algorithm 3 Optimal contract using Dynamic Programming

```
for k = K → 1 do
    for i = 1 → X do
        for j = 1 → P do
            if (x(i), p(j)) ∉ T_k then
                g(i, j, k) ← 0
            else if k == K then
                g(i, j, k) ← r_k U(X(i), P(j))
            else
                g(i, j, k) ← r_k U(X(i), P(j)) + max_{i'>i} g(i', [P_{k+1}(x(i), p(j), x(i'))], k + 1)
            end if
        end for
    end for
end for
return max_{i,j} g(i, j, 1)
```

We give the pseudo code (Alg. 3) of the described algorithm. The x, p values are discretized to X, P number of values, and we can iterated them from small to large by $1, ..., X$ $(1, ..., P)$. Here $x(i)$ and $p(j)$ denote the transformation from integer to the actual real number. We use $[p]$ to denote the mapping from a real number p to the index of the closest discretized value smaller than p. The algorithm returns the maximum profit achievable, and the contract set can be determined by back tracking.

Below we list the main differences of the generalized contract design framework compared with Chapter II.

- Buyer's quality constraint:

  In this chapter the buyer's quality constraint can be any reasonable indicator function $Q(x, y)$ that takes x (the amount of secondary bandwidth) and y (the amount of reference bandwidth) as input. The randomness of the secondary spectrum can be represented by any type of random variable described by the indicator function. Since quality should increase by purchasing more bandwidth, the indicator function is assumed to be increasing in both x and y.

- Buyer's cost function:

  In Chapter II, the buyer's utility function was the total amount of money spent $xp+y$. In this chapter, we generalized the buyer's cost function to any function increasing in the total amount of money spent $w(xp+y)$.

- Monotonicity condition:

  We proposed a generalization of the monotonicity condition defined by the equal-cost lines. The monotonicity condition requires that we can find an ordering of the buyer types where higher types have higher valuation for more bandwidth.

- Seller's profit function:

  In Chapter II, the seller's profit function was restricted to a linear form $U(x,p) = x(p-c)$. The optimal contract could be derived analytically under the monotone condition. In this chapater, we generalized the seller's profit function to any function that is increasing in both x and p. Under the generalized monotone condition, we obtain the optimal contract using a dynamic programming approach.

3.5 Probability of Loss Example

In this section, we show how to apply the general framework derived in this chapter to specific buyer/seller utility functions. Consider a buyer who may not have a constraint on the expected transmission loss but on the probability of loss. This could be the case when partial reception of a packet is not accepted so that an entire packet needs to be retransmitted due to error, or when there is a strict delay requirement so that lost packets are not retransmitted (e.g., in the case of real-time streaming). In such cases the probability of loss is a more relevant measure. We express the probability of loss constraint by the following expression:

$$Q_i(x, y) = I(P(q_i - y - xb_iB > 0) < \epsilon_i), \tag{3.7}$$

where B is an uniform random variable between 0 and 1. The constant $b \in [0, 1]$ denotes the channel condition (defined by the buyer type). By purchasing x units of secondary spectrum, the buyer gets a throughput of xb_iB. q_i is the transmission needed and $q_i - y - xb_iB$ is the amount left untransmitted.

We first derive the function $y(x)$:

$$P(q - xBb - y > 0) \leq \epsilon \Leftrightarrow P(B < \frac{q - y}{xb}) \leq \epsilon \tag{3.8}$$

$$\Leftrightarrow \frac{q - y}{xb} \leq \epsilon \; . \tag{3.9}$$

For the buyer, the minimum amount of additional purchase y is,

$$y(x) = max(q - xb\epsilon, 0) \tag{3.10}$$

That is, if $x > \frac{q}{b\epsilon}$ there is no need to buy additional reference bandwidth. If $x < \frac{q}{b\epsilon}$, the buyer purchases $q - xb\epsilon$. The equal-cost line has two different forms depending on the different cases.

- Case: $x > \frac{q}{b\epsilon}$

  In this region $y = 0$, the cost of the buyer is of the form $u(xp+y) = u(x'p'+y') \Leftrightarrow xp + y = x'p'$. The equal-cost line takes the form $p(x, p, x') = xp/x'$.

- Case: $x \leq \frac{q}{b\epsilon}$

  In this region $y = q - xb\epsilon$. $u(xp+y) = u(x'p'+y') \Leftrightarrow xp+q-xb\epsilon = x'p'+q-x'b\epsilon$ The equal-cost line takes the form $p(x, p, x') = \frac{b\epsilon - p}{x'}x + b\epsilon$.

To verify whether the monotonicity condition holds, we can take the derivative of the equal-cost line with respect to x'. For $x > \frac{q}{b\epsilon}$,

$$p'(x, p, x') = \frac{b\epsilon - p}{x'^2} x. \tag{3.11}$$

The derivative of the equal-cost line with respect to x' is always increasing in x' and for larger $b\epsilon$ the slope is greater. Thus, the higher type here is defined as $b\epsilon$. For $x \leq \frac{q}{b\epsilon}$, the equal cost line is the same for all types, thus, we need that the region for the second case be larger for higher types. In conclusion, for the buyer types to have an ordering, we need the following property.

Proposition 3.7. *If there exists an ordering in the buyer types* $1..K$ *such that* $b_1\epsilon_1 \leq ... \leq b_K\epsilon_K$ *and* $q_1/b_1\epsilon_1 \leq ... \leq q_K/b_K\epsilon_K$.

If Proposition 3.7 holds, then we know that Algorithm 3 is optimal. We will use the function $U(x, p) = (x(p - 0.1))^{0.5}$ as the utility function of the seller in our simulation. Note that the buyer's cost function $w(x, p)$ can be of any form as long as it is increasing in p. Similar to the previous section, we compare contracts derived from different methods under the monotonicity condition.

1. The optimal set of M contracts (denoted OPT(M) in the figures): Finding this set is done by an exhaustive search over a set of discretized values x and p as an approximation of the uncountable choices (the step size for x is 0.5 and the step size for p is 0.1).

2. The algorithm we introduced in the previous section: Algorithm 3

3. A K-choose-1 method (denoted as MAX in the figures): This is the method that selects the contract with the highest expected profit over the set $\{max_1, \cdots, \mathrm{max}_K\}$. This is done by checking all $(b_i\epsilon_i, q_i/b_i\epsilon_i)$ pairs; the complexity increases linearly in M.

The experiments are run by increasing $K = 1...7$. For each K value the parameters $(q_i, b_i, \epsilon_i, r_i)$ are independently and randomly generated from uniform distributions ($b_i \in [0, 1]$, $q_i \in [0, 10]$, $\epsilon_i \in [0, 2]$ and $r_i \in [0, 1]$ but normalized such that $\sum r_i = 1$) For each K we record the average (in expected profit) over 12000 cases that satisfy the monotonicity condition; results are shown in Figure 3.1. Our observations are as follows. Being able to use more contracts is always better as expected (i.e., OPT(1) $\leq$ OPT(2) in all cases). ALG is optimal under the monotonicity constraint and thus outperforms all other algorithms. OPT(1) is optimal for $K = 1$ and OPT(2) is optimal for $K = 1, 2$. Because both OPT and ALG work on the discretized plane, ALG does not have the advantage as in the previous chapter. When there are more possible buyer types (as K increases), the maximum expected profit decreases because it is harder to put all the contracts right on the buyers' acceptance boundaries while still satisfying the incentive compatibility condition. Lastly, max_i is not optimal for $K = 1$ because it is approximated to the closest discretized grid.

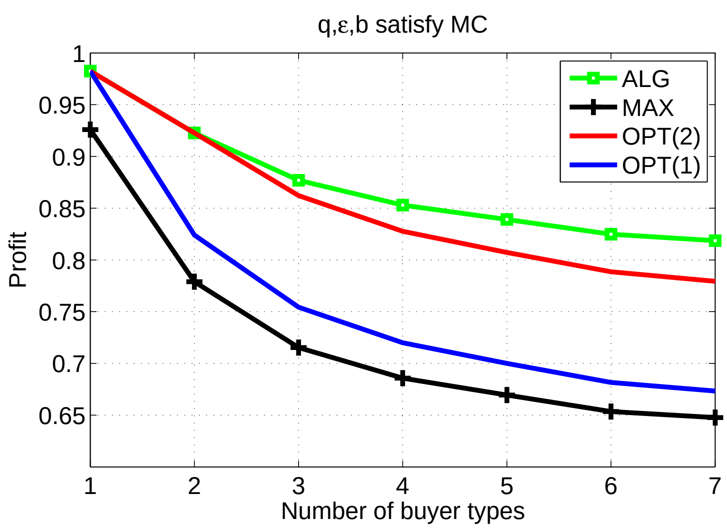

Figure 3.1: Simulation results of the sellers profit versus different contracts satisfying monotonicity condition

3.6 Conclusion

In this chapter, we generalized the contract design problem (Chapter II) where a primary license holder wishes to profit from its excess spectrum capacity by selling it to potential secondary users/buyers via designing a set of profitable contracts. The cost function of the buyer can be any function that is increasing in the money spent, while the quality constraint can be any function that can be mapped to the additional reference spectrum needed. The utility of the seller can be any form that is increasing in the price per bandwidth. We characterize the optimal solution where there is a single buyer type. In the case when more than one buyer types exist, we construct an algorithm that generates a set of contracts in a computationally efficient manner, and show that this set is optimal in the discretized grid when the buyer types satisfy a monotonicity condition.

CHAPTER IV

A Regulated Oligopoly Multi-Market Model for Secondary Spectrum Trading

4.1 Introduction

In this chapter, we examine the effectiveness of posted price sale mechanism in the context of secondary spectrum trading. Due to spectrum under-utilization and the emergence of wireless applications requiring a variety of spectrum products in terms of time duration and bandwidth (e.g., Internet of Things, body-area networks, etc.), it is increasingly likely that primary spectrum holders will trade unused spectrum in small pieces, both in terms of bandwidth and time duration. At the same time, as more applications turn to the secondary market to purchase spectrum, the valuation of spectrum products also becomes easier to determine. For these reasons, holding auction for each piece may no longer be the only choice or a good choice, and posted sale becomes a viable alternative.

A number of studies have looked into spectrum trading using contract design by a primary holder. In [25] a contract based framework was proposed for cooperative spectrum sharing, whereby the primary offers certain amount of spectrum access to secondary users in exchange for help in relaying data for the primary. Both [32, 63] studied a monopoly market where a single primary designs contracts that maximize its profit when selling to different secondary types each having different communication needs. [46] considered the contract between a single primary and a single secondary and examined different outcomes when either dominates the contract form. In [77], the interference between simultaneous secondary buyers was considered in the model, and heuristics were introduced in solving the profit maximization and spectrum allocation problem.

However, in most of existing works the competition among sellers is rarely discussed with the notable exception of [26], where a two-stage game in a duopoly setup was studied. Under this model, the two primaries choose the amount of spectrum they will lease in the first stage, and then compete in the second stage. Their results show that there exists a threshold in the difference of their respective leasing costs which determines where either player will remain in the market.

Our goal in this study is to analyze the resulting spectrum market where multiple sellers participate in the posted sale of multiple spectrum products, each catered to the needs of a different type of secondary user, and determine whether profit-efficient equilibria can be achieved.

4.1.1 Our approach and modeling perspective

We consider the sellers' competition with a goal of extracting profit. In this sense our analysis takes the view of the primary license holders and seeks to understand how they can be incentivized to participate in secondary spectrum trading.

When considering competition in markets, the often used models are the Bertrand and Cournot competition models [10, 22]. The Bertrand model shows that with just two sellers, the market reaches perfect competition and both sellers sell at the marginal price. Specifically, it assumes that competing firms produce a homogeneous product; thus products from different firms are interchangeable, the result being that customers will always purchase from the firm that sets the lowest price. It follows that the only equilibrium point is when all firms set their price at the unit cost of production. Under the Cournot model, firms compete by choosing the amount of output they produce. Although they can choose their production quantity at will, the total amount all firms produce affects the market price of that product, the result being that the price approaches the marginal price as the number of sellers increases.

In reality we do not often see perfect competition where firms sell at marginal prices. Modification of these models thus typically aim to reflect the real market. For example, Bertrand-Edgeworth model [64] assumes a production limit of firms in the Bertrand model. Various other factors can also be incorporated to avoid perfect competition such as product differentiation, transport and search costs. Firms can also avoid competing with each other by colluding/side contracting, which has often been shown to improve the outcome, i.e., increase the sellers' collective profit. Examples include Coase [19], which showed that bargaining leads to an efficient (profit-maximizing) outcome in a trade with fully symmetric information and no transaction cost, Jackson [42], which considered a two stage game where by firms first agree on utility transfers that effectively rewrite the payoff functions, and then play the price competition game in the second stage, and Ferreia[29], which considered cross-ownership as a form of side contracting. Other examples can be found in [30, 28, 51].

Following this line of thought, in this chapter we consider the setting where multiple primary spectrum holders (also referred to as sellers) compete with each other on the secondary market over multiple spectrum products. This market in its unregulated form is inefficient: all primaries will sell at marginal prices as discussed above. We introduce a regulator who coordinates the sellers so that they avoid competition by each focusing on different spectrum products/markets. This may be equivalently viewed as forming an alliance/association of sellers who agree to abide by certain rules without violating privacy and individual rationality. Specifically, we propose a money transfer scheme where the seller transfers part of its profit each time it completes a transaction to other sellers, resulting in a profit sharing mechanism. The regulator is not assumed to fully observe the sellers' strategies (the details of the transactions that have occurred such as the price, the bandwidth or duration sold, etc.), but only assumed to know each time that a transaction has occurred. In other words, its role is to simply to register/certify each transaction and facilitate the money transfer that follows.

Under this model, we will discuss the conditions under which such a money exchange scheme could enforce efficient equilibria, i.e., profit maximizing. We will specify the equilibrium region for both the special of two sellers and the more general case of multiple sellers. In the first case we also identify the fairness region (in terms of profit sharing) within the equilibrium region.

There is an interesting connection between our model and the class of coalition games, which has been used in spectrum access context, see e.g., [61] that modeled the collaborative spectrum sensing problem as a coalition game and showed that through coalition, the secondary users can greatly reduce the average miss probability. Under our model, the presence of the regulator may be viewed as forcing a coalition, though ours is a non-cooperative game while coalition games belong to the family of cooperative games. Moreover, since any kind of competition will result in zero profit for all sellers, there is no other efficient equilibrium other than the grand coalition in our context.

The remainder of this chapter is organized as follows. We introduce the model for spectrum market in Section 4.2. In Section 4.3 we show that with two sellers and two buyer types, the equilibrium only exists when price equals marginal cost. With the introduction of a regulator, we show that an efficient equilibrium with fairness can be achieved in certain conditions and fully characterize the conditions. In Section 4.4, we extend the result of Section 4.3 to more than two buyer types. In Section 4.5 we characterize the conditions for an efficient equilibrium to exist in the multiple seller case.

4.2 Model

The model we present next is similar to the Bertrand model but extended to multiple products catered to different buyer types with different spectrum needs. We assume the sellers all have sufficient supply to support the buyers' needs. Under this assumption, similar to the Bertrand model the result would end in a price competition and the only equilibrium point is when all sellers sell at their marginal prices. In order to move the equilibrium to a more efficient (profitable) point, we introduce a regulator who can force money transfers among sellers. This transfer is only based on the occurrence of each transaction but not on the details of the transaction; thus the resulting game is one of partial information.

4.2.1 Sellers

There are K sellers, each with sufficient spectrum resource to supply all products if they want to. We assume there are N different types of buyers – as each type seeks a distinct spectrum product, a buyer type is equated to a product in our exposition – each having different spectrum needs. The different spectrum products can differ in their leasing durations, bandwidths or access rights. We assume that this set of products are known and well defined. A seller's strategy then concerns the price at which it sets for each product. Formally, its strategy space on R^N is defined by the set of prices corresponding to each spectrum product. The sellers' goal is to maximize the total profit generated from all products. If a spectrum product i is sold at price p_i, then its profit is $p_i - c_i$ where c_i is the unit cost of the i-th product.

4.2.2 Buyers

There are N types of buyers each corresponding to a spectrum product. A buyer i will always choose to buy from the seller who offers the lowest price for product i given that it is below some M_i. This amount M_i reflects buyer i's price tolerance/upper limit, beyond which the buyer will simply walk away. M_i can also be viewed as the monopoly price for if there is only one seller, then the optimal price for the seller would be M_i. Since there is a cost c_i for each product i sold, the monopoly profit from buyer type i is defined as $\Theta_i = M_i - c_i$. Let r_i denote the number of buyers of type i among the buyer population. Equivalently r_i can also be the probability of a random arriving buyers being of type i; this will not affect our analysis. Note that the values M_i and r_i are market information assumed known to the seller prior to entering the market (this would be part of the market research done by the seller mentioned earlier).

Throughout the chapter we will also often refer to a particular buyer type as a distinct "market" featured with a distinct spectrum product, whenever there is no ambiguity. This should not be confused with the more generic use of the word "market" as in spectrum market.

4.2.3 Regulator

We define a third party in the sellers' game, referred to as the regulator. This regulator need not be imposed by entities outside the group of sellers; it could be self-imposed by an alliance or coalition of sellers sharing the common goal of profit maximization. The regulator can enforce money transfer based on partial information of the actions of the sellers. Specifically, the regulator observes a signal each time a transaction takes place (a buyer completing the purchasing of a spectrum product from a seller). This signal contains no information of which product was sold and what price it was sold at. The money transfer takes the following form. When seller

i sells a product, he has to give another sellers j an amount t_{ij} (e.g., in dollars), for each $j \neq i$. This amount t_{ij} is a real nonnegative number. Although we have set up the regulator as a third party in the game, the role of the regulator can also be viewed as a collusion between the sellers.

4.2.4 Efficiency

The intention in introducing the regulator is to force the sellers to avoid competition and attain higher profits. In this context, efficiency is measured by the total profit of all sellers. Accordingly, at an efficient equilibrium the price that a buyer pays for is the same as one commanded in a monopoly market. Thus, an efficient equilibrium in our formulation maximizes the total profit of all sellers.

4.3 2 sellers, 2 buyer types

4.3.1 Unregulated

We begin with a simplified version of the spectrum market with only 2 sellers and 2 buyer types, of population r_1, r_2, and monopoly profit Θ_1, Θ_2, respectively. As already discussed in relation to the Bertrand model, under perfect competition, the market will not exist with both sellers driven to selling for zero profit.

We next show that with some constraint on the sellers' strategy space we can achieve efficient (positive profit) equilibria. Specifically, assume that each seller can only set one of the product prices below M_i meaning that they each can only choose one of the products to sell. Let seller 1 (2) be assigned to take product 1 (2). Let's also include a third product which can attract both buyer types with less profit Θ_3, i.e., $(r_1 + r_2)\Theta_3 < r_1\Theta_1 + r_2\Theta_2$. For this single-contract setup, deviation is not profitable for seller 1 if,

$$r_1\Theta_1 \geq r_2\Theta_2 \tag{4.1}$$

$$r_1\Theta_1 \geq (r_1 + r_2)\Theta_3 \tag{4.2}$$

Equation (4.1) is for seller 1 to not take the market of seller 2, Eqn. (4.2) is for seller 1 to not choose to acquire both types of buyers by the third product (note that it will not make sense for seller 1 to offer both product 1 and product 3 as a buyer, no matter the type, will prefer product 3 as it is cheaper). Similarly,

$$r_2\Theta_2 \geq r_1\Theta_1 \tag{4.3}$$

$$r_2\Theta_2 \geq (r_1 + r_2)\Theta_3 \tag{4.4}$$

would ensure seller 2 does not deviate. We don't have to consider other cases because if these conditions are satisfied, any price lower will not be beneficial to offer. For these 4 equations to be satisfied, the set of parameters much satisfy the following condition:

$$r_1\Theta_1 = r_2\Theta_2 \geq (r_1 + r_2)\Theta_3\ . \tag{4.5}$$

If we assume $r_1 = r_2 = 0.5$, $\Theta_3 = 1$ and plot it on the Θ_1-Θ_2 plane, then the only values for Θ_1, Θ_2 that satisfy this condition lie on the 45 degrees line starting from $\Theta_1 = \Theta_2 = 2$. This will also be referred to as the *stability or stable region* of these parameters. This example suggests that when each seller is limited to selling only one product, it is possible for the market to exist whereby the sellers make non-zero profit. However, such existence depends on very restrictive selections of the problem parameters, e.g., a line out of a 2D plan in this example. In other words, it is all but impossible for sellers to not compete, or to make a profit, in an unregulated environment.

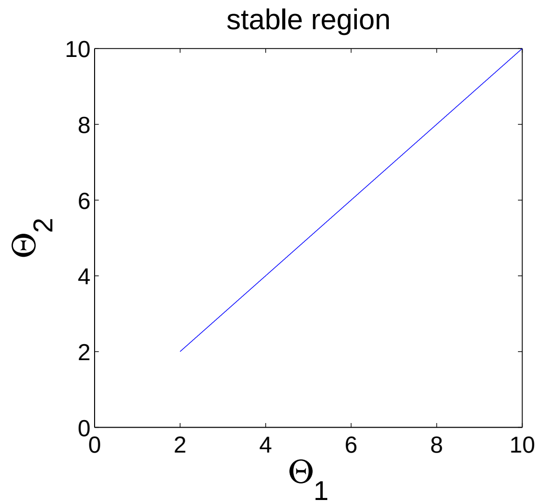

Figure 4.1: Stable region without regulation

4.3.2 With Regulation

Suppose that a seller, upon each completed sale, pays a certain amount of money to its rival. Let's denote by t_1 (t_2) the money given from seller 1 to 2 (2 to 1) when seller 1 (2) sells. Then the incentive compatibility condition for staying with its assigned market/product is rewritten as follows for seller 1:

$$r_1(\Theta_1 - t_1) + r_2 t_2 \geq r_2(\Theta_2 - t_1) \tag{4.6}$$

$$r_1(\Theta_1 - t_1) + r_2 t_2 \geq r_1(\Theta_1 - t_1) + r_2(\Theta_2 - t_1) \tag{4.7}$$

$$r_1(\Theta_1 - t_1) + r_2 t_2 \geq (r_1 + r_2)(\Theta_3 - t_1) \tag{4.8}$$

$$r_1(\Theta_1 - t_1) + r_2 t_2 \geq r_2 t_2 \tag{4.9}$$

and for seller 2:

$$r_2(\Theta_2 - t_2) + r_1 t_1 \geq r_1(\Theta_1 - t_2) \tag{4.10}$$

$$r_2(\Theta_2 - t_2) + r_1 t_1 \geq r_1(\Theta_1 - t_2) + r_2(\Theta_2 - t_2) \tag{4.11}$$

$$r_2(\Theta_2 - t_2) + r_1 t_1 \geq (r_1 + r_2)(\Theta_3 - t_2) \tag{4.12}$$

$$r_2(\Theta_2 - t_2) + r_1 t_1 \geq r_1 t_1 \tag{4.13}$$

Note that we discarded the constraint where each seller can only sell to one buyer type. Thus, there are 4 different scenarios for each seller: (i) to switch to the other seller's market (Eqn. (4.6)); (ii) to take the other seller's market (Eqn. (4.7)); (iii) to switch to the third product (Eqn. (4.8)); and (4) to give up its own market and just receive money from the other seller (Eqn. (4.9)). We want to show that by choosing appropriate t_1 and t_2, we can make staying with the assigned market the best strategy of both sellers. Consider the extreme case where $t_1 = \Theta_1$ and $t_2 = \Theta_2$, then Eqns. (4.6), (4.7), (4.10) and (4.11) are satisfied.

$$r_1(\Theta_1 - \Theta_1) + r_2\Theta_2 \geq r_2(\Theta_2 - \Theta_1) \tag{4.14}$$

$$r_2(\Theta_2 - \Theta_2) + r_1\Theta_1 \geq r_1(\Theta_1 - \Theta_2) \tag{4.15}$$

Also, note that Eqns. (4.8) and (4.12) can be rearranged as follows where maximizing t_1 and t_2 makes the inequality the least binding/restrictive.

$$r_2(t_1 + t_2) \geq (r_1 + r_2)\Theta_3 - r_1\Theta_1 \tag{4.16}$$

$$r_1(t_1 + t_2) \geq (r_1 + r_2)\Theta_3 - r_2\Theta_1 \tag{4.17}$$

Θ_1 and Θ_2 are the largest values t_1 and t_2 can be. This is because they would rather not give any contract if $t_i > \Theta_i$ and Eqns. (4.9) and (4.13) will not be satisfied. This means that if setting $t_1 = \Theta_1, t_2 = \Theta_2$ cannot allow all equations be satisfied, any other values of t_1, t_2 cannot allow the equations be satisfied. By setting $t_1 = \Theta_1, t_2 = \Theta_2$, we know that the following equations are the conditions to check whether it is possible to have any money transfer to cause both sellers to follow the assignment:

$$r_1\Theta_1 - (r_1 + r_2)\Theta_3 + r_2(\Theta_1 + \Theta_2) \geq 0 \tag{4.18}$$

$$r_2\Theta_2 - (r_1 + r_2)\Theta_3 + r_1(\Theta_1 + \Theta_2) \geq 0 \tag{4.19}$$

Solving for these two inequalities we have,

$$\Theta_2 \geq max(\Theta_3 - \frac{r_1}{r_1 + r_2}\Theta_1, \frac{r_1 + r_2}{r_2}(\Theta_3 - \Theta_1)) \tag{4.20}$$

Taking the same example as in the previous subsection, $r_1 = r_2 = 0.5$ and $\Theta_3 = 1$, we get $\Theta_2 \geq max(1 - 0.5\Theta_1, 2(1 - \Theta_1))$ as shown in Fig. 4.2. The stable region now contains all points above both lines.

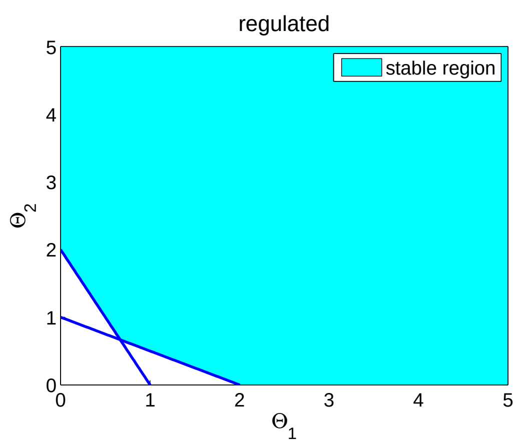

Figure 4.2: Stable region under regulation

Comparing Figures 4.1 and 4.2, we observe that the stable region expanded from a line to a plane, not to mention the elimination of the one-contract constraint. In Figure 4.1, only the Θ values exactly on the line were possible for the market to exist with positive profit, significantly limiting the type of spectrum products the market can profitably sustain. By contrast, Figure 4.2 suggests that for a majority of the spectrum products there exists a money transfer mechanism that can separate the markets between the sellers and enable positive and indeed, maximum profit.

4.3.3 Fairness

We have shown that the stable region can be expanded by introducing a regulating money exchange without knowing what the sellers actually did. However, we have not specified any constraints on the resulting profit share. It is conceivable that sellers will only agree to this money transfer scheme if the profit earned is fair in some sense. In what follows we consider not only the stability region but also the region where fairness is achieved. Without loss of generality, we will assume $r_1 \geq r_2$. Let's consider the additional fairness condition where both sellers obtain the same profit under the money transfer t_1, t_2:

$$r_1(\Theta_1 - t_1) + r_2 t_2 = r_2(\Theta_2 - t_2) + r_1 t_1 \tag{4.21}$$

The reason for using equality as the fairness criteria stems from the assumption that all sellers have enough supply on their own so they have the same profit extraction power.

Similar as before, if we maximize both t_1 and t_2 then Eqns. (4.6-4.13) become less restrictive. Previously the maximums were Θ_1, Θ_2; however, now we cannot simply use the maximum because t_1 and t_2 are coupled. We consider 2 cases.

1. $\Theta_2 \leq \frac{r_1}{r_2}\Theta_1$: In this case, if we set $t_2 = \Theta_2$, then $t_1 = \Theta_1/2 + \frac{r_2}{2r_1}\Theta_2 \leq \Theta_1$ from Eqn. (4.21). Since $t_1 \leq \Theta_1$, this is a valid choice that maximizes t_1 and t_2. Using Eqns. (4.6) and (4.10), the following conditions have to be satisfied:

$$\frac{1 + r_1/r_2}{1 - r_2/r_1}\Theta_1 \geq \Theta_2 \geq \frac{1}{(2 + r_1/r_2)}\Theta_1 \ . \tag{4.22}$$

Because we assumed $r_1 \geq r_2$, we have $\frac{1+r_1/r_2}{1-r_2/r_1}\Theta_1 \geq \frac{r_1}{r_2}\Theta_1$. The region is thus given by the following condition:

$$\frac{r_1}{r_2}\Theta_1 \geq \Theta_2 \geq \frac{1}{(2 + r_1/r_2)}\Theta_1 \ . \tag{4.23}$$

2. $\Theta_2 \geq \frac{r_1}{r_2}\Theta_1$: Similarly we set $t_1 = \Theta_1$ and obtain $t_2 = \Theta_2/2 + \frac{r_1}{2r_2}\Theta_1 \leq \Theta_2$ from Eqn. (4.21). Since $t_2 \leq \Theta_2$, this is a valid choice that maximizes t_1 and t_2. Using Eqns. (4.6) and (4.10), we find the following condition:

$$(2 + r_1/r_2)\Theta_1 \geq \Theta_2 \geq \frac{1 - r_1/r_2}{1 + r_2/r_1}\Theta_1 \tag{4.24}$$

Because $\frac{1-r_1/r_2}{1+r_2/r_1} \leq 0$, the right hand side of the inequality is always satisfied. We can conclude that fairness can be achieved in the region

$$(2 + r_1/r_2)\Theta_1 \geq \Theta_2 \geq \frac{r_1}{r_2}\Theta_1 \ . \tag{4.25}$$

Combining Eqns. (4.23) and (4.25), we conclude that the region where fairness is achievable is,

$$(2 + r_1/r_2)\Theta_1 \geq \Theta_2 \geq \frac{1}{(2 + r_1/r_2)}\Theta_1 \ . \tag{4.26}$$

Next consider the condition given by Eqns. (4.8) and (4.12).

1. $\Theta_2 \leq \frac{r_1}{r_2}\Theta_1$: Let $t_1 = \Theta_1/2 + \frac{r_2}{2r_1}\Theta_2$, $t_2 = \Theta_2$,

$$\Theta_2 \geq \frac{(r_1 + r_2)\Theta_3 - r_1\Theta_1/2}{3r_2/2 + r_1} \tag{4.27}$$

$$\Theta_2 \geq \frac{(r_1 + r_2)\Theta_3 - (r_1 + r_2/2)\Theta_1}{r_2 + r_2^2/(2r_1)} \tag{4.28}$$

2. $\Theta_2 \geq \frac{r_1}{r_2}\Theta_1$: Let $t_1 = \Theta_1$, $t_2 = \Theta_2/2 + \frac{r_1}{2r_2}\Theta_1$,

$$\Theta_2 \geq \frac{(r_1+r_2)\Theta_3 - (3r_1/2 + r_2)\Theta_1}{r_2/2} \tag{4.29}$$

$$\Theta_2 \geq \frac{(r_1+r_2)\Theta_3 - (r_1 + r_1^2/(2r_2)\Theta_1}{r_2 + r_1/2} \tag{4.30}$$

Eqns. (4.26) and (4.27-4.30) characterize the entire region where fairness is achievable by at least one (t_1,t_2) pair that also results in an efficient equilibrium.

Using the same example $r_1 = r_2$ and $\Theta_3 = 1$, we plot the region in Figure 4.3. Here the solid lines correspond to Eqn. (4.26). Between the two solid lines is the

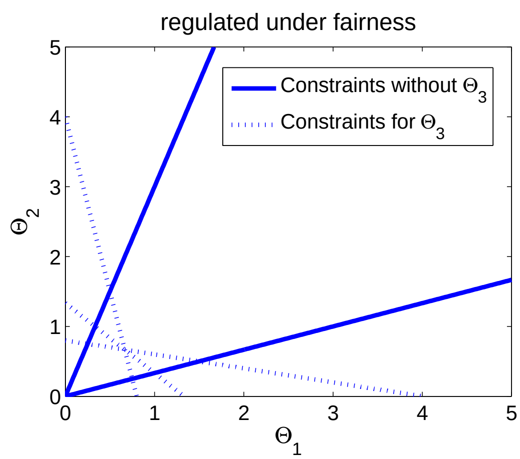

Figure 4.3: Fairness region

area where both fairness and efficiency can be achieved through money transfer. The dashed lines correspond to Eqns. (4.27-4.30). If there exists the third product Θ_3, then the Θ_1 and Θ_2 values have to be above these lines. In Fig. 4.4 we further show how the regions compare between efficient equilibrium and fair and efficient equilibrium.

In this section, we showed that an efficient market can exist by introducing a regulator. The money transfer provides the incentive for a seller not to steal her rival's market in two aspects: (i) money in-flow from the rival whenever she completes a transaction, and (ii) additional money out-flow to the rival when the seller completes

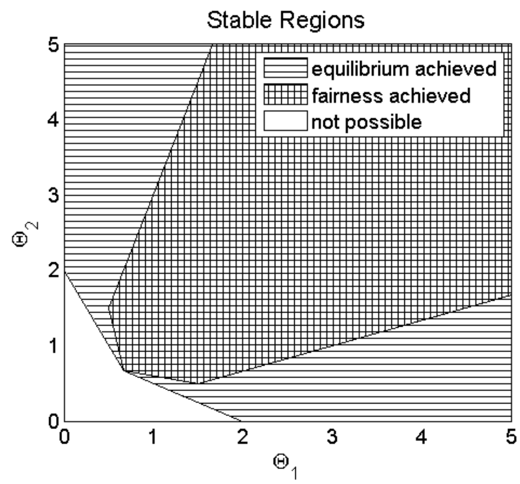

Figure 4.4: Comparing different regions

more transactions. We have shown that the stable/equilibrium region is maximized when we maximize the money transfer. This is not to suggest that sellers should give away all proceeds from a transaction; this is only done so as to reveal the largest stable region. When conditions are less strict (given by the Θ_1, Θ_2 values), the money transfer amount can be reduced. We compute the minimum transfer required for each point on the Θ_1-Θ_2 plane and Fig. 4.5 shows the ratio between the minimum transfer over the total profit ($\frac{t_1+t_2}{\Theta_1+\Theta_2}$). Note that this value is 1 at the boundary of the achievable stable region and decreases in the middle where the value becomes 0.5, meaning that half of the money earned is transferred.

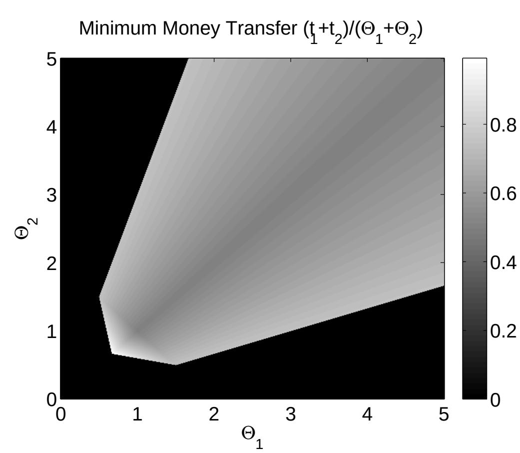

Figure 4.5: Minimum transfer for fairness

4.4 More than 2 buyer types

We now extend the result from the previous section and assume that there are more than 2 buyer types but still with 2 sellers. The first problem we have to address is how the buyer types might be divided among the sellers so that they can avoid competition by selling in different markets as previously done.

4.4.1 Predetermined contract dividing sets

Theoretically, we could assign all contract types to one of the seller (say seller 1) and then transfer profit to the other seller (seller 2). Assume the contracts will generate profits $\Theta_1, ...\Theta_N$ respectively with type distribution $r_1, ..., r_N$. We can let seller 1 have all the contracts and let $t_2 = \infty$ such that seller 2 will never try to sell any spectrum product. By assigning $t_1 = \frac{\sum_{i=1}^{N} \Theta_i r_i / 2}{\sum_{i=1}^{N} r_i}$ the profit is divided equally among the sellers. However, it is hardly reasonable or practical to let one seller do all the work while the other does nothing and just collects money. We thus consider whether it is possible to achieve an equilibrium if we have a pre-assigned dividing rule for the contracts. Assume it is already determined that seller 1 should sell to buyer types $11, 12, ..., 1N_1$ with profit $\Theta_{11}, \Theta_{12}, ..., \Theta_{1N_1}$ and seller 2 should sell buyer types $21, 22, ..., 2N_2$ to $\Theta_{21}, \Theta_{22}, ..., \Theta_{2N_2}$. Where N_1, N_2 are 2 positive integers with $N_1 + N_2 = N$ The buyer distribution is $r_{11}, r_{12}, ..., r_{1N_1}$ and $r_{21}, r_{22}, ..., r_{2N_2}$ respectively. Again, the transfer payment is incurred when a seller sells a spectrum product. The transfer amounts are t_1/t_2 from seller 1/2 to seller 2/1. The following equations are the profit of both sellers if they stick to the assigned contract allocations:

$$\sum_{i=1}^{N_1} r_{1i}(\Theta_{1i} - t_1) + \sum_{i=1}^{N_2} r_{2i} t_2 \tag{4.31}$$

$$\sum_{i=1}^{N_2} r_{2i}(\Theta_{2i} - t_2) + \sum_{i=1}^{N_1} r_{1i} t_1 \tag{4.32}$$

If we write out all the deviations possible for each seller, there will be $2^{N_1}2^{N_2}$ possible strategies for each seller. We can however reduce the action space by the following Lemma.

Lemma 4.1. *Seller 1/2 will follow the contract assignment if and only if it is neither valuable to drop one of her own contacts nor valuable to add one of her rival's contracts to her own contract set.*

Proof. Changing from a set of contracts to another set of contracts can be viewed as dropping some contracts and adding some contracts. The effects of these changes are linear in the profit; thus, if changing the whole set is profitable then there must be at least one profitable single change action. That also means if none of these changes increases the total profit, the total profit cannot be increased by any combination of these changes. □

Lemma 4.1 says that if we want to check whether the assigned contract is a Nash equilibrium, we only have to check dropping or adding one contract; we don't have to verify all possible strategies for the seller.

1. Drop a contract: The condition for seller 1 to not drop a contract $1j$ is

$$\begin{aligned}&\sum_{i=1}^{N_1} r_{1i}(\Theta_{1i}-t_1)+\sum_{i=1}^{N_2} r_{2i}t_2 \geq \\ &\sum_{i=1}^{N_1} r_{1i}(\Theta_{1i}-t_1)-r_{1j}(\Theta_{1j}-t_1)+\sum_{i=1}^{N_2} r_{2i}t_2\end{aligned}$$

   which can be simplified to $\Theta_{1j} \geq t_1$. This means that seller 1 should needs to have positive profit from selling this contract.

2. Adding a contract: The condition for seller 1 to not want to add a contract $2j$

is

$$\sum_{i=1}^{N_1} r_{1i}(\Theta_{1i} - t_1) + \sum_{i=1}^{N_2} r_{2i} t_2 \geq$$
$$\sum_{i=1}^{N_1} r_{1i}(\Theta_{1i} - t_1) + r_{2j}(\Theta_{2j} - t_1) + \sum_{i=1}^{N_2} r_{2i} t_{2i} - r_{2j} t_2$$

which can be simplified to $t_2 \geq \Theta_{2j} - t_1$.

As previously mentioned, we analyze these condition as if a seller could take over the rival's market at the exact same price set by the rival, while in reality in order to steal from the other's market a seller needs to offer slightly lower prices. However, the intention is that if the seller has no incentive to take over the rival's market at the same price, then she will have no incentive at any lower prices. Repeating the same analysis for seller 2, the conditions that need to be satisfied such that an equilibrium assignment exists are

$$\Theta_{1j} \geq t_1, \forall j = 1, ..., N_1 \tag{4.33}$$

$$\Theta_{2j} \geq t_2, \forall j = 1, ..., N_2 \tag{4.34}$$

$$t_2 \geq \Theta_{2j} - t_1, \forall j = 1, ..., N_2 \tag{4.35}$$

$$t_1 \geq \Theta_{1j} - t_2, \forall j = 1, ..., N_1 \tag{4.36}$$

If we just consider Eqns. (4.35) and (4.36), maximizing both t_1 and t_2 will relax these two constraints. However, t_1, t_2 are restricted by Eqns. (4.33) and (4.33). Let $t_1 = \underline{\Theta}_1 = \min_{i=1,\ldots,N_1} \Theta_{1i}$ and $t_2 = \underline{\Theta}_2 = \min_{i=1,\ldots,N_2} \Theta_{2i}$ be the maximum values of t_1 and t_2. Then constraints Eqns. (4.35) and (4.36) can be simplified to:

$$\underline{\Theta}_2 \geq \Theta_{2j} - \underline{\Theta}_1, \forall j = 1, ..., N_2 \tag{4.37}$$

$$\underline{\Theta}_1 \geq \Theta_{1j} - \underline{\Theta}_2, \forall j = 1, ..., N_1 \ . \tag{4.38}$$

Let $\overline{\Theta}_1 = \max_{i=1\ldots N_1} \Theta_{1i}$ and $\overline{\Theta}_2 = \max_{i=1\ldots N_2} \Theta_{2i}$. Then these constraints can be further simplified to:

$$\underline{\Theta}_2 \geq \overline{\Theta}_2 - \underline{\Theta}_1 \tag{4.39}$$

$$\underline{\Theta}_1 \geq \overline{\Theta}_1 - \underline{\Theta}_2 \tag{4.40}$$

or

$$\underline{\Theta}_1 + \underline{\Theta}_2 \geq \max(\overline{\Theta}_1, \overline{\Theta}_2) = \max \Theta \tag{4.41}$$

where $\max \Theta$ is the highest profit among all contracts. In conclusion, the predetermined assignment of buyer types to the 2 sellers has to satisfy the condition where the sum of the least profitable customer of each seller has to be greater than the profit from the most profitable type. If this is satisfied, then an equilibrium where both sellers follow the assignment is possible.

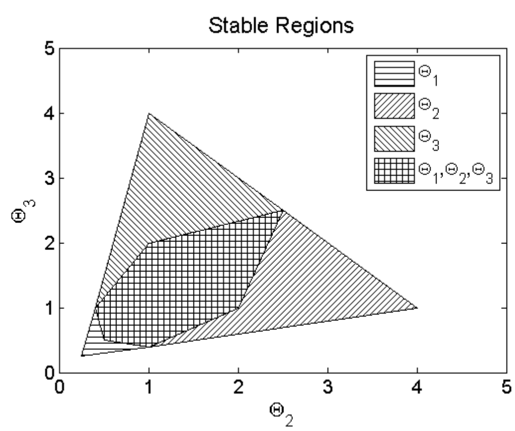

Figure 4.6: Stable regions for Θ_i separated to one of the sellers

In Fig. 4.6 we plot the regions where money exchange can force an efficient and equal-share equilibrium in a 3-buyer type case. Here we fix $\Theta_1 = 1$ and vary Θ_2 and Θ_3. Since there are 2 sellers dividing the 3 types, one of the sellers is assigned to one buyer while the other is assigned to two buyers. We plot the possible regions of the 3 different assignments. The plot indicates which buyer type is assigned to be the only buyer type for one of the sellers. The triangle is the area where there exists an assignment that equal share is possible. The 3 assignments are all possible in the middle area while only one assignment is possible in each corner of the triangle.

4.4.2 Proportional fairness

Now we generalize the results where we consider whether it is possible to achieve a stable money transfer where the profit is divided among the sellers such that seller 1 gets p times the profit of seller 2. The proportional fairness is written as

$$\begin{aligned} & \sum_i r_{1i}(\Theta_{1i} - t_1) + \sum r_{2j} t_2 \\ = & \; p(\sum_j r_{2j}(\Theta_{2j} - t_2) + \sum r_{1i} t_1) \end{aligned} \tag{4.42}$$

Since t_1 and t_2 are coupled by Eqn. (4.42), only one of them can be set to $\underline{\Theta_i}$. Thus, we have 2 cases,

1. $\sum_i r_{1i}(\Theta_{1i} - \underline{\Theta_1}) + \sum r_{2j}\underline{\Theta_2} > p(\sum_j r_{2j}(\Theta_{2j} - \underline{\Theta_2}) + \sum r_{1i}\underline{\Theta_1})$. Under this case, t_1 reaches $\underline{\Theta_1}$ first when we increase t_1 and t_2. Thus, t_2 cannot be set to $\underline{\Theta_2}$.

$$t_1 = \underline{\Theta_1} \tag{4.43}$$

$$t_2 = \frac{p \sum r_{2j}\Theta_{2j} - \sum r_{1i}(\Theta_{1i} - (p+1)\underline{\Theta_1})}{(p+1)\sum r_{2j}} \tag{4.44}$$

   t_2 is determined by Eqn. (4.42). For this to be a valid choice, the conditions in Eqns. (4.35) and (4.36) have to hold. That is $t_1 + t_2 > \overline{\Theta}$.

2. $\sum_i r_{1i}(\Theta_{1i} - \underline{\Theta_1}) + \sum r_{2j}\underline{\Theta_2} < p(\sum_j r_{2j}(\Theta_{2j} - \underline{\Theta_2}) + \sum r_{1i}\underline{\Theta_1})$. Similarly, we

have

$$t_1 = \frac{\sum r_{1i}\Theta_{1i} - \sum r_{2j}(p\Theta_{2j} - (p+1)\underline{\Theta_2})}{(p+1)\sum r_{1i}} \tag{4.45}$$

$$t_2 = \underline{\Theta_2} \tag{4.46}$$

and the same condition $t_1 + t_2 > \overline{\Theta}$ has to hold to guarantee that an equilibrium is feasible.

The calculated t_1 and t_2 will maximize the stable region; this means that if these values cannot result in an equilibrium where both sellers follow the assignment, there will be no other money transfer that can achieve an equilibrium.

4.4.3 Limited bandwidth/supply

Our assumption so far has been that both sellers have unlimited bandwidth to compete in all markets at the same time. If the sellers have limited bandwidth, their strategy spaces become subsets of the unlimited bandwidth condition. Thus, if they do not deviate with unlimited bandwidth, they will not deviate when they have bandwidth limit. Thus, we can apply our results to the following setup.

Assume that the bandwidth of the 2 sellers are limited at X_1 and X_2, respectively, and they want to share profit proportional to the bandwidth each of them has. Since the bandwidth is limited, if they want to collaborate they will only sell to the set of markets with the highest profit per bandwidth. From a centralized view point, a water filling algorithm can find the optimal set of markets in which they should sell. Then, the problem becomes a subset sum problem to determine two subsets where each subset of market requires X_1 and X_2 bandwidth. The subset sum problem is known to be NP-complete. However, there are polynomial time approximating algorithms that can get close to optimal solutions, see e.g., [33]. After determining the subset, we can use Eqns. (4.43-4.46) to check whether there exists money exchange that is stable and achieves proportional fairness. If there is such a point, we then also know what value the money transfer should be from the same set of equations.

This mechanism however requires the sellers to report truthfully about their bandwidth (X_1 and X_2) at the beginning. There are a number of ways to address this requirement.

1. We can show that the sellers do not have incentives to under report. Since the share of the profit is proportional to the amount of bandwidth they reported, if a seller under reports, she will get less profit for sure. The only possible scenario is to under report and then sell the excess bandwidth to buyers not assigned to her. Since the algorithm already assigns the most profitable buyer types to either seller and there is no incentive to sell to the rival's buyer types, the only possible buyers are the ones that are not assigned, which means they present lower profits. Thus, there is no incentive to under report the bandwidth.

2. Since the profit share is proportional to the bandwidth limit each of them reports, the sellers may hope to increase her profit share by reporting higher than the true bandwidth. However, this can be easily caught because the actual transactions occurred will not match the assigned amount of contracts.

3. Another way of achieving truthful sharing is to use a VCG-like mechanism. By designing the shared portion to be only related to the rival's reported value. However, without other side information (such as the sum of $X_1 + X_2$) the result will become equally dividing the profit.

4.5 Multiple Sellers, Multiple Buyer Types

We now consider the more general case of multiple sellers with multiple buyer types. Let the sellers be $i = 1, ..., K$, each assigned with N_i buyer types indexed by $i1, i2, ..., iN_i$. With regulation, a seller is forced to transfer t_{ij} to seller j if she completed a transaction. Accordingly, the profit of seller i can be written as follows:

$$\sum_{l=1}^{N_i} r_{il}(\Theta_{il} - \sum_{j \neq i} t_{ij}) + \sum_{j \neq i} \sum_{l=1}^{N_j} r_{jl} t_{ji} \ . \tag{4.47}$$

Similar as in the previous sections, we only need to consider two types of deviations, i.e., adding an additional contract or removing a contract. The condition for a buyer/contract allocation rule being a Nash equilibrium for seller i is:

$$\Theta_{il} - \sum_{j \neq i} t_{ij} \geq 0, \ \forall l = 1, ..., N_i \tag{4.48}$$

$$t_{ji} \geq \Theta_{jl} - \sum_{m \neq i} t_{im}, \ \forall j \neq i, \forall l = 1, ..., N_j \tag{4.49}$$

The most restrictive condition in the first equation is the smallest Θ_{il}. We define $\underline{\Theta_i} = \min_{l=1,...,N_i} \Theta_{il}$ and observe that $\sum_{j \neq i} t_{ij} \leq \underline{\Theta_i}$ is a necessary condition. Substituting the maximum $\underline{\Theta_i}$ into the second equation,

$$t_{ji} \geq \Theta_{jl} - \underline{\Theta_i}, \ \forall j \neq i, \forall l = 1, ..., N_j \ . \tag{4.50}$$

Similarly, t_{ji} has to be at least $\overline{\Theta_j} - \underline{\Theta_i}$ for Eqn. (4.49) to be satisfied for all l. But we have to check again whether the first equation is satisfied, which is

$$\Theta_{il} - \sum_{j \neq i} (\overline{\Theta_i} - \underline{\Theta_j}) \geq 0, \ \forall l = 1, ..., N_i \ . \tag{4.51}$$

Rearranging the above equation and taking account for all $l = 1, ..., N_i$, we conclude that,

$$\underline{\Theta_i} \geq \sum_{j \neq i} (\overline{\Theta_i} - \underline{\Theta_j}), \ \forall i \tag{4.52}$$

is the necessary and sufficient condition such that there exists money transfers for the regulation to ensure an efficient equilibrium corresponding to the contract allocation. The money transfer from seller i to j, $t_{ji} = \overline{\Theta_j} - \underline{\Theta_i}$ will maximize the stable region. By the condition Eqn. (4.52) above, we notice a few factors that affect whether a stable money transfer is possible: (i) large $\underline{\Theta_i}$ is better; (ii) small $\overline{\Theta_i}$ is better; (iii) less number of sellers K is better. Combing (i) and (ii) we see that the it would be desirable for most of the products to have profits close to each other and the products assigned to each seller to have similar profits. From (iii) we note that as the number of sellers increases, the summation on the right hand side of Eqn. (4.52) increases.

4.5.1 Limiting the money transfer $t_{ij} = t_{ij'}$

In this subsection, we consider the special case of limiting the money transfer to be identical $t_{ij} = t_{ij'}$ $\forall j, j'$, $\forall i$. We get a similar set of equations on individual rationality:

$$\Theta_{il} - t_i \geq 0, \ \forall i, \forall l = 1, ..., N_i \tag{4.53}$$

$$t_j/(K-1) \geq \Theta_{jl} - t_i, \ \forall j \neq i, \forall l = 1, ..., N_j \tag{4.54}$$

As before we get $t_i \leq \underline{\Theta_i}$ and taking it into the second equation gives us

$$\underline{\Theta_j}/(K-1) \geq \Theta_{jl} - \underline{\Theta_i}, \ \forall j \neq i, \forall l = 1, ..., N_j \tag{4.55}$$

Taking into account all possible l, we can rearrange and get the following necessary and sufficient condition.

$$\underline{\Theta_i} \geq (K-1)(\overline{\Theta_i} - \underline{\Theta_j}), \ \forall i, \forall j \neq i \tag{4.56}$$

Not surprisingly, the stability region shrinks compared to the previous subsection. As we discussed in the previous section, the conditions are satisfied if profits of different products are close to each other and the number of sellers are small.

4.5.2 Multiple seller each with one buyer type

From the previous results, we immediately obtain the result for the special case where each seller is assigned with exactly one buyer type. The results can be obtained by substituting all $\overline{\Theta}$ and $\underline{\Theta}$ by Θ. The conditions are as follows,

$$\Theta_i \le \frac{\sum_{j \ne i} \Theta_j}{K-2} \ \forall i \tag{4.57}$$

$$\Theta_i \le \frac{K-1}{K-2} \Theta_j \ \forall i, j \tag{4.58}$$

Equation (4.57) is for any t_{ij}, while Eqn. (4.58) is for restricting $t_{ij} = t_{ij'}$. We can see that similar to the two-seller case, the conditions are that the profit differences are not too far.

4.5.3 Multiple sellers with fairness

If we would also like to achieve fairness in a multiple seller setting, this would require an additional set of equations to be satisfied. For example, if the proportional fairness is that the profit ratio is $p_1 : p_2 ... : p_{K-1} : p_K$. This would introduce a set of equations given by

$$\frac{\sum_{l=1}^{N_i} r_{il}(\Theta_{il} - \sum_{j \ne i} t_{ij}) + \sum_{j \ne i} \sum_{l=1}^{N_j} r_{jl} t_{ji}}{\sum_{l=1}^{N_{i'}} r_{i'l}(\Theta_{i'l} - \sum_{j \ne i} t_{i'j}) + \sum_{j \ne i'} \sum_{l=1}^{N_j} r_{jl} t_{ji'}} = \frac{p_{i'}}{p_i}$$

We note that combined with the set of conditions to not drop a contract or add a contract (Eqns. (4.48), (4.49)), this is a feasibility problem of whether there exists a set $\{t_{ij}\}$ that can satisfy all constraints. Although there are no polynomial algorithm that can solve this problem, this is a simplified version of a linear programming problem for which we do not have the maximization term. There are many existing algorithms that could be used to solve this problem, see e.g., [23].

4.6 Conclusion

In this chapter we introduced a competition model suitable for the secondary spectrum market. We first show that the market will result in full competition where equilibrium only exists when all sellers have zero profit. We then introduce a regulator who can facilitate a set of money transfer based on partial observations of the sellers actions. We show that by the introduction of this regulator, we can induce the market to have efficient (profit maximizing) equilibria. The conditions for designing a stable money transfer were characterized for cases of two-seller and multiple-seller cases, and how to achieve fair profit share is also discussed.

CHAPTER V

Data-Driven Channel Modeling Using Spectrum Measurement

5.1 Introduction

Dynamic spectrum access has been a subject of extensive study in recent years. The increasing volume of literature calls for better understanding of the characteristics of current spectrum utilization as well as better tools for analysis. A number of measurement studies have been conducted recently, revealing previously unknown features. On the other hand, analytical studies largely continues to rely on standard models like the two-state Markov (Gilbert-Elliot) model. In this chapter we present an alternative, stochastic differential equation (SDE) based spectrum utilization model that captures dynamic changes in channel conditions induced by primary users' activities.

The SDE model is in closed form, can generate spectrum dynamics as a temporal process, and is shown to provide very good fit for real spectrum measurement data. We show how synthetic spectrum data can be generated in a straightforward manner using this model to enable realistic simulation studies. Moreover, we show that the SDE model can be viewed as a more general modeling framework (continuous in time and continuous in value) than commonly used discrete Markovian models: it is defined by only a few parameters but can be used to obtain the transition matrix of any N-state Markov model. This is verified by comparing the 2-state GE model generated by the SDE model and that trained directly from the data. We show that the GE model is a good fit for the (quantized) data, thereby a fine choice when binary descriptions of the channel condition is sufficient. However, when high resolution (in channel condition) is needed, the SDE model is much more accurate than an N-state model, and is much easier to train and store.

5.1.1 Related work

One commonly used category of channel models that obtain the time process property is based on Markov chains, where each state often represents a different condition of the channel, with dynamic changes described by the state transition probability matrix. Sometimes a mixture of Markov models with other models is used to capture characteristics of the error patterns, see e.g., the chaotic maps [49] and the MTA [48].

Within this category, the Gilber-Elliott (GE) model [34, 27] is the simplest Markov model consisting of only two states. Perhaps due to its simplicity (and often the associated analytical tractability), the GE model is widely used as the underlying model for wireless channels both in analysis and in simulation. Under this model, the channel is given by a two-state Markov chain with state G (good) and B (bad), see Figure 5.6. In state G, transmission is assumed error-free, while in state B the channel has a probability h of transmitting the packet correctly[1]. These two states are used to model a burst-noise channel. A more general variation of this model includes a probability k (usually $k > h$) such that both good and bad states have a chance to generate an error bit.

A lot of studies have been conducted on these two-state Markov models. Mc-

[1]The GE model is very often used in a simplified version where $h = 0$, which is the version we focus on in this paper.

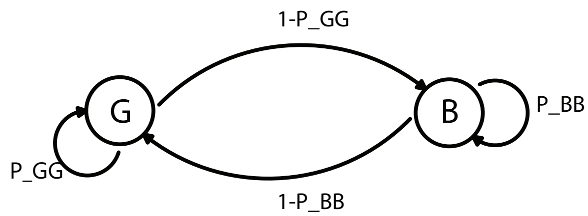

Figure 5.1: The Gilbert-Elliott (2-state) model

Dougall et al. [54] showed that at low SNR, the two-state Markov model does not generate an adequate frame error process because it lacks the ability to match higher-order block error statistics. Hartwell et al. [37] showed that using the higher-order state hidden Markov models provides a better fit of measured data than the traditional 2-state GE models. However, high-order state Markov models require high computational complexity to train the parameters of the Markov model. Yu et al. [73] proposed a four-state Markov model and showed how to analytically establish the transition probability. Konrad et al. [48] proposed a Markov based model aimed at capturing the non-stationary behaviors of wireless channels.

5.1.2 Our approach and main contributions

We introduce a stochastic differential equation (SDE) model derived partly based on the physics of electromagnetic wave propagation. This SDE model is continuous both in time and in value, and falls under the category of *diffusion models*, which is more commonly used in queuing analysis when dealing with large systems, e.g., those with heavy loads [36]. The main idea is that when queue sizes are large, the increments over a single discrete step become relatively small by comparison. Thus under such operating regimes, it is reasonable to model the discrete change in the queue occupancy by a continuous flow, resulting in diffusion models. The analytical advantage of using a diffusion model is that it is amenable to both transient and steady state analysis and can be used to derive queue size distributions which is hard to do using a discrete model when the state space is large. We shall see that the SDE model introduced here holds similar advantages over discrete, GE-type models. We therefore conclude that the SDE model serves as a valuable alternative to the commonly used GE-type models.

5.1.3 Organization

The remainder of the chapter is organized as follows. We introduce the SDE model and show how to estimate its parameters in Section 5.2. In Section 5.3 we verify the model by using spectrum measurement data from CRAWDAD [39] as well as our own study [16]. In Section 5.4 we show how to synthesize data from the trained SDE model and compare the spectral entropy of the synthesized data to the spectral entropy of the collected data. Then in Section 5.5 we show that we can obtain the 2-state GE model from the synthesized data and compare it with the GE model trained directly from the quantized measurement data. In Section 5.6 we compare the channel prediction performance of the SDE model and an N-state Markov model, both trained from the measurement data. We conclude the chapter in Section 5.7.

5.2 The SDE model

5.2.1 Constructing the model

The spectrum utilization model presented here uses stochastic differential equations (SDE) to model dynamic scattering and multipath fading channels, in particular, Rayleigh-distributed stationary channels. This is a technique developed and used in a number of studies, see e.g., [66, 13].

Specifically, our model is derived from a dynamic wireless channel model developed in [65] using similar techniques. Underlying this model is the assumption of either a single transmitter or many non-dominant transmitters stationary in space and in time. The model describes the complex signal received by a stationary receiver (thus with zero Doppler's effect) Building upon this work, our contribution lies in (1) extracting the received energy as a random process expressed as an SDE and the construction of the subsequent spectrum model, and (2) developing a method to estimate the unknown parameters of the model.

In this model the signal detected at a receiver is viewed as a collection of a large number of reflected waves, and thus exhibits a multipath propagation phenomenon. This makes the received signal's phase random and hard to predict, and can possibly lead to large fluctuation in the received power. Assuming that the received signal on each path is random, the model developed in [65] is based on a continuous time description of the scattered electric field received at a stationary receiver with multi-path reception along N paths, expressed as $\epsilon_t^{(N)} = \sum_{k=1}^{N} a_k \exp[i\varphi^{(k)}(t)]$, where a_k is the amplitude of the received signal along path k and i is the square root of -1. The phase factors $\exp[i\varphi^{(k)}(t)]$ are independent and uniformly distributed on a unit circle in the complex plane and for each t. In addition, it is assumed that the phase $\varphi^{(k)}(t)$ satisfies the following SDE: $d\varphi^{(k)}(t) = B^{\frac{1}{2}} dW^{(k)}(t)$, where $\varphi^{(k)}(0)$ are uniformly distributed on $[0, 2\pi)$, $W^{(k)}$ are independent Weiner processes, and B is a constant that represents the rate of change in the phase of the received signal. By integrating the above SDE it is readily seen that $\mathrm{VAR}(\varphi^{(k)}(t) - \varphi^{(k)}(0)) = Bt$.

Using stochastic calculus, it was established in [65] that the amplitude process is given by $\Psi(t) = I(t) + iQ(t)$, where $I(t)$ and $Q(t)$ are the in phase and quadrature components of the incoming waves received at time t, and can be represented by the following two SDEs:

$$dI(t) = -\frac{1}{2}BI(t)dt + \frac{\sqrt{2}}{2}\sigma B^{\frac{1}{2}} dW^{(I)}(t) \; ; \tag{5.1}$$

$$dQ(t) = -\frac{1}{2}BQ(t)dt + \frac{\sqrt{2}}{2}\sigma B^{\frac{1}{2}} dW^{(Q)}(t) \; , \tag{5.2}$$

with $I(0) = 0$, $Q(0) = 0$, and $W^{(I)}(t)$ and $W^{(Q)}(t)$ two independent standard Wiener processes. The parameter B makes these two SDEs *mean-reverting*, i.e., the process, in equilibrium, approaches the mean [9]. Such processes are also referred to as Ornstein-Uhlenbeck processes [9]. The parameter σ^2 represents the stationary magnitude of the scattering power averaged over an asymptotically large number of propagation paths [65], and is shown to be the asymptotic (in t) variance of ϵ_t and satisfies: $\sigma^2 = \sum_{k=1}^{\infty} a_k^2$, which is assumed to be finite, assuming no single path dominates: $\lim_{N\to\infty} \frac{a_j^2}{\sum_{k=1}^{N} a_k^2} = 0$.

The above summarizes what was developed in [65]. We now proceed to derive the power process received at the receiving antenna at time t. This is given by $R(t) = \sqrt{I^2(t) + Q^2(t)}$. Assuming processes (5.1) and (5.2), and using standard arguments from stochastic calculus [9], we have the following lemma:

Lemma 5.1. *Let $\bar{I}(t) = e^{\frac{1}{2}Bt} I(t)$ and $\bar{Q}(t) = e^{\frac{1}{2}Bt} Q(t)$. Then*

$$d\bar{I}(t) = \frac{\sqrt{2}}{2}\sigma B^{\frac{1}{2}} e^{\frac{1}{2}Bt} dW^{(I)}(t)\ , \tag{5.3}$$

$$d\bar{Q}(t) = \frac{\sqrt{2}}{2}\sigma B^{\frac{1}{2}} e^{\frac{1}{2}Bt} dW^{(Q)}(t). \tag{5.4}$$

Furthermore, both $\bar{I}(t)$ and $\bar{Q}(t)$ are normally distributed, with mean 0 and variance $\frac{\sigma^2}{2}(e^{Bt} - 1)$.

Proof. Let $f(x,t) = e^{\frac{1}{2}Bt}x$. Then $f_t = \frac{\partial f}{\partial t} = \frac{1}{2}Be^{\frac{1}{2}Bt}x$, $f_x = \frac{\partial f}{\partial x} = e^{\frac{1}{2}Bt}$ and $f_{xx} = \frac{\partial^2 f}{\partial^2 x} = 0$. Using Ito's formula (Theorem 4.2.3 [9]) and replacing x with $I(t)$, we have

$$\begin{aligned} d\bar{I}(t) &= f_t dt + f_x dI(t) + \frac{1}{2} f_{xx} (dI(t))^2 \\ &= \frac{1}{2}Be^{\frac{1}{2}Bt} I(t)dt + e^{\frac{1}{2}Bt}(-\frac{1}{2}BI(t)dt \\ &\quad + \frac{\sqrt{2}}{2}\sigma B^{\frac{1}{2}} dW^{(I)}(t)) \\ &= \frac{\sqrt{2}}{2}\sigma B^{\frac{1}{2}} e^{\frac{1}{2}Bt} dW^{(I)}(t) \end{aligned}$$

where we have substituted Eqn (5.1) for $dI(t)$. The second, Eqn (5.4), can be obtained by the same argument. Integrating, say the equation for $\bar{I}(t)$, we see that

$$\bar{I}(t) = \bar{I}(0) + \frac{\sqrt{2}}{2} B^{\frac{1}{2}} \sigma \int_0^t e^{\frac{1}{2}Bs} dW^{(I)}(s) \ .$$

Using the facts that the mean of the Ito's integral $\int_0^t e^{\frac{1}{2}Bs} dW^{(I)}(s)$ is zero and its variance $\int_0^t e^{Bs} ds$ we get the result. □

An immediate consequence of the above lemma is that

$$\bar{R}(t) = \sqrt{\bar{I}(t)^2 + \bar{Q}(t)^2} \tag{5.5}$$

has a Rayleigh distribution with parameter $\sqrt{\frac{\sigma^2}{2}(e^{Bt} - 1)}$. The main theorem is given as follows:

Theorem 5.2. *The power process $R(t)$ satisfies the following SDE:*

$$dR(t) = -\frac{BR(t)}{2} dt + \frac{1}{4} \frac{B\sigma^2}{R(t)} dt + \frac{1}{\sqrt{2}} \sigma B^{\frac{1}{2}} dW(t) \tag{5.6}$$

with $R(0) = r_0$, some constant, and W a standard Wiener process.

Proof. Consider $\bar{R}(t)$ as in Eqn (5.5) and note that $\bar{R}(t) = e^{\frac{1}{2}Bt} R(t)$. Now, consider the function $f(x, y) = \sqrt{x^2 + y^2}$, for which we have the following first and second order partial derivatives:

$$
\begin{aligned}
f_x &= \frac{x}{\sqrt{x^2+y^2}} \\
f_y &= \frac{y}{\sqrt{x^2+y^2}} \\
f_{xx} &= \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(\sqrt{x^2+y^2})^3} \\
f_{yy} &= \frac{1}{\sqrt{x^2+y^2}} - \frac{y^2}{(\sqrt{x^2+y^2})^3} \\
f_{xy} &= -\frac{xy}{(\sqrt{x^2+y^2})^3}
\end{aligned}
$$

Substituting the above into Ito's formula when differentiating (5.5), and using standard results on Wiener processes: $dW^{(I)}(t)^2 = dt$, $dW^{(Q)}(t)^2 = dt$, $dW^{(I)}(t)dW^{(Q)}(t) = 0$, we get

$$
d\bar{R}(t) = \frac{\bar{I}(t)d\bar{I}(t)}{\bar{R}(t)} + \frac{\bar{Q}(t)d\bar{Q}(t)}{\bar{R}(t)} + \frac{1}{4}\sigma^2 B e^{Bt}\frac{1}{\bar{R}(t)}dt. \tag{5.7}
$$

Consider the first two terms in the above expression. It is seen

$$
\frac{\bar{I}(t)d\bar{I}(t)}{\bar{R}(t)} + \frac{\bar{Q}(t)d\bar{Q}(t)}{\bar{R}(t)} = \frac{\sqrt{2}}{2}\sigma B^{\frac{1}{2}} e^{\frac{1}{2}Bt}\left[\frac{I(t)dW^{(I)}(t) + Q(t)dW^{(Q)}(t)}{R(t)}\right].
$$

We have that

$$
\frac{I(t)dW^{(I)}(t) + Q(t)dW^{(Q)}(t)}{R(t)} = \frac{1}{R(t)}[I(t), Q(t)]\begin{bmatrix} dW^{(I)}(t) \\ dW^{(Q)}(t) \end{bmatrix}
$$

and by the definition of $R(t)$

$$\frac{1}{R(t)^2}[I(t), Q(t)] \begin{bmatrix} I(t) \\ Q(t) \end{bmatrix} = 1 \ .$$

Therefore, using Theorem 8.4.2 of [9] we conclude that $\int_0^t \frac{I(s)dW^{(I)}(s)+Q(s)dW^{(Q)}(s)}{R(s)} ds$ has the same law as a Wiener process, denoted as $W(t)$, independent of $W^{(I)}$ and $W^{(Q)}$. This means that we can write

$$\frac{\bar{I}(t)d\bar{I}(t)}{\bar{R}(t)} + \frac{\bar{Q}(t)d\bar{Q}(t)}{\bar{R}(t)} = \frac{\sqrt{2}}{2}\sigma B^{\frac{1}{2}} e^{\frac{1}{2}Bt} dW(t) \ .$$

Substituting the above into (5.7) we obtain:

$$d\bar{R}(t) = \frac{\sqrt{2}}{2}\sigma B^{\frac{1}{2}} e^{\frac{1}{2}Bt} dW(t) + \frac{1}{4}\sigma^2 B e^{Bt} \frac{1}{\bar{R}(t)} dt. \tag{5.8}$$

Since $R(t) = e^{-\frac{1}{2}Bt}\bar{R}(t)$, we have

$$dR(t) = -\frac{1}{2}Be^{-\frac{1}{2}Bt}\bar{R}(t)dt + e^{-\frac{1}{2}Bt}d\bar{R}(t) \ . \tag{5.9}$$

Replacing $d\bar{R}(t)$ with (5.8) in the above equation gives the desired result. □

The power process we propose to use in this chapter is (5.6) with a modified mean reverting term:

$$dR(t) = \frac{B}{2}(\mu - R(t))dt + \frac{1}{4}\frac{B\sigma^2}{R(t)}dt + \frac{1}{\sqrt{2}}\sigma B^{\frac{1}{2}} dW(t) \ . \tag{5.10}$$

The additional term $\frac{B\mu}{2}dt$ on the RHS, which is now part of the mean reverting term, has the effect of steering the mean of the process to approximately μ (at $t \to \infty$). To summarize, our model given in (5.10) consists of three terms: first a "mean reverting" process (or the O-U process) with mean μ, the second the "radial" term, and the third a volatility term (together a Bessel process). Using the terminology of [1], we will call this process a *Radial Ornstein-Uhlenbeck process.*

Three unknown parameters uniquely define this model: μ, B, and σ. μ will be referred to as the mean. B will be referred to as the phase constant; it corresponds to the rate at which the received signal phase changes. σ^2 will be referred to as the power constant (not to be confused with the received power); it is the sum of signal magnitudes over all paths. The value $B\sigma^2$ determines the volatility of this process. In the next section, we will show how these three parameters can be estimated using spectrum measurement data for training.

5.2.2 Parameter estimation

In order to estimate the three unknowns μ, B and σ from real measurement data, we first rearrange terms in (5.10) to obtain the following:

$$\frac{dW(t)}{\sqrt{dt}} = \frac{\sqrt{2}}{\sigma B^{\frac{1}{2}}}\left\{\frac{dR(t)}{\sqrt{dt}} - \frac{B}{2}(\mu - R(t))\sqrt{dt} - \frac{B\sigma^2}{4R(t)}\sqrt{dt}\right\}. \tag{5.11}$$

Note that the left hand side of the above equation is now a zero-mean, unit-variance normally distributed random variable. The idea behind our parameter estimation procedure is to use real measurement data to generate data points corresponding to the right hand side of Eqn (5.11), and then match the first two (or more) sample moments to that of the 0-mean unit-variance normal distribution, thereby solving three unknowns (μ, B, σ).

Specifically, for a given frequency band our measurements are in the form of a time series of power readings, denoted as $\hat{R}(t_i)$, $i = 0, 1, 2, \cdots, N$. From these measurements we can now obtain successive differences between these readings, denoted as $d\hat{R}(t_i) = \hat{R}(t_i) - \hat{R}(t_{i-1})$, $i = 1, 2, \cdots, N$. We can also obtain the differences in sampling times, denoted as $dt_i = t_i - t_{i-1}$, $i = 1, 2, \cdots, N$. For our measurement data, sampling times are evenly spaced. Therefore in our case dt_i is treated as a constant.

Following this, the original measurement data may be viewed as a collection of N triples $(\hat{R}(t_i), d\hat{R}(t_i), dt_i)$, $i = 1, 2, \cdots, N$. Each such triple will now be referred to as a *sample* within the context of estimation and testing. From this collection of samples, we now select a random subset $\mathcal{N}_{est}$ of size N_{est} for estimation. We plug in each selected sample into the RHS of Eqn (5.11) and obtain the following data point $\hat{w}_i$ for $i \in \mathcal{N}_{est}$:

$$\hat{w}_i \quad = \quad \frac{\sqrt{2}}{\sigma B^{\frac{1}{2}}} \left\{ \frac{d\hat{R}(t_i)}{\sqrt{dt_i}} - \frac{B}{2}(\mu - \hat{R}(t_i))\sqrt{dt_i} - \frac{B\sigma^2}{4\hat{R}(t_i)}\sqrt{dt_i} \right\}. \tag{5.12}$$

This gives us a total of N_{est} data points $\{\hat{w}_i, i \in \mathcal{N}_{est}\}$, each a function of μ, B and σ. We also obtain an estimate of the mean by $\hat{\mu} = \frac{1}{N_{est}} \sum_{i \in \mathcal{N}_{est}} \hat{R}(t_i)$. These three unknown parameters can now be estimated by matching (1) the sample mean of the data set $\{\hat{w}_i, i \in \mathcal{N}_{est}\}$ to 0; (2) its sample variance to 1; (3) the parameter μ to the mean estimate $\hat{\mu}$; That is, the parameters are estimated by matching the first two sample moments to the first two moments of the 0-mean unit-variance normal distribution and matching the parameter μ to the estimated mean of the received power.

In our experiments, we obtain the estimates by solving the following minimization problem:

$$\min_{\mu, B, \sigma} (m_1 - 0)^2 + (m_2^{1/2} - 1)^2 + \frac{(\mu - \hat{\mu})^2}{\mu^2} + P + (m_3^{1/3} - 0)^2 + (m_4^{1/4} - 3^{1/4})^2$$

where m_i denotes the i-th sample moment of the data set $\{\hat{w}_i, i \in \mathcal{N}_{est}\}$, and 0 and 1 are the first two moments of the standard Normal distribution. P is a penalty term designed to penalize the minimization when the parameter B is negative or too close to zero (note that in the process $R(t)$, B is a positive term). The term is in the form of $P = \mathcal{C}(B-\delta)^2$ when $B < \delta$ and $P = 0$ when $B \geq \delta$ (where δ is a small number and $\delta, \mathcal{C} > 0$).

Once these parameters are estimated, we use the remaining $N - N_{est}$ samples for testing and model verification. This is done in a very similar way as in estimation. Specifically, the testing samples are also plugged into the RHS of Eqn (5.11). However, this time the computation is done with the estimated values of μ, B and σ. This gives us $N - N_{est}$ data points, also commonly referred to as the *residual of the test data*. The model verification test lies in checking whether the residual follows the standard normal distribution.

5.2.3 Analytical expression of SDE distribution

An immediate application of the SDE model is to derive received power distribution in a channel. As mentioned earlier, the SDE model belongs to the family of diffusion models. Diffusion models are often used in large-scale queuing systems as good alternatives to the discrete valued Markov chains. Specifically, by allowing the queue to have continuous values, the discrete valued Markov chain can be approximated by a diffusion model with the appropriate parameters. This makes it feasible to derive the steady state distribution of the queue analytically, which is otherwise impossible under a discrete model. Below we use similar techniques to obtain the steady state distribution of the power process.

For any stochastic differential equation of the form

$$dX_t = U(X_t, t)dt + \sqrt{2D(X_t, t)}dW_t, \tag{5.13}$$

the distribution of the process $f(x, t)$, where $f(x, t)$ denotes the probability $P(X_t =$

x), satisfies the following equality by using the Fokker-Planck equation [36]:

$$\frac{\partial}{\partial t} f(x,t) = -\frac{\partial}{\partial x}[U(x,t)f(x,t)] + \frac{\partial^2}{\partial x^2}[D(x,t)f(x,t)] \ . \tag{5.14}$$

In our SDE model, $R(t)$, the power process, takes the role of X_t in Eqn (5.13). Recalling for convenience the model:

$$dR(t) = \frac{B}{2}(\mu - R(t))dt + \frac{B\sigma^2}{4R(t)}dt + \frac{\sigma\sqrt{B}}{\sqrt{2}}dW(t) \ , \tag{5.15}$$

we then have the following mapping between (5.13) and (5.15):

$$\begin{aligned} U(X_t, t) &= \frac{B}{2}(\mu - X_t) + \frac{B\sigma^2}{4x} \\ D(X_t, t) &= \sigma^2 B/4 \end{aligned}$$

Assuming that the process reaches a steady state, i.e., $\frac{\partial}{\partial t} f(x,t) = 0$. Then integrating the right hand side and suppressing the argument t will give us,

$$\frac{B}{2}(\mu - x + \frac{\sigma^2}{2x})f(x) = \frac{\sigma^2 B}{4} f'(x) \ , \tag{5.16}$$

or

$$\frac{2}{\sigma^2}(\mu - x) + \frac{1}{x} = \frac{f'(x)}{f(x)} \ . \tag{5.17}$$

Integrating the above over x we get

$$\frac{2}{\sigma^2}(\mu x - x^2/2) + \log x = \log f(x) + c \ . \tag{5.18}$$

Rearranging to solve for $f(x)$ we get,

$$f(x) = c \cdot x \cdot \exp(\frac{2}{\sigma^2}(\mu x - x^2/2)) \ , \tag{5.19}$$

and the normalizing constant value $c = 1/\int_0^\infty x \cdot \exp(\frac{2}{\sigma^2}(\mu x - x^2/2))$. This gives us the complete description of the steady state distribution function of the power process $R(t)$.

5.3 Model verification using spectrum measurement data

The model verification uses spectrum data from two sources: (1) The first is our measurement study reported in [16], which was done over a period of multiple days continuously, and simultaneously at multiple locations. The resolution of the measurement is such that one energy reading (in μV) is produced for each band of width 200KHz, for roughly every 75 seconds of sweep time over the range of 20MHz to 3GHz. (2) The second is a dataset obtained from crawdad.org [39]. Compared to the first data set, this set consists of readings for a larger band (10MHz), but over a much smaller sweep time of roughly 80 nano seconds. These two sets of data thus represent two very different measurement regimes: the first has low time resolution (sampling rate) but high spectral resolution (narrow bandwidth) while the second has a much higher sampling rate but wider bandwidth.

From these two data sets we selected the bands of TV Digital and TV Analog for training and verifying the SDE model. The center frequencies of these bands are listed in table 5.1. For each data set (note that each set is now a collection of samples (triples) as described in the previous section), we randomly select 50% to be the estimation/training data set and the remaining 50% the testing data set[2].

To check whether the residual follows a standard normal distribution, and in

[2]Randomly selecting a set for estimation is a standard procedure in statistical analysis; the resulting estimation represents the data better in case the underlying process is non-stationary.

Data Set	MHz	Primary user	Source
1	518	TV	measurement study [16]
1	738	TV	measurement study [16]
1	1842	GSM	measurement study [16]
2	551	TV	CRAWDAD [39]
2	629	TV	CRAWDAD [39]
2	665	TV	CRAWDAD [39]

Table 5.1: Data sets for model verification

particular to check how far it is from the standard normal distribution, we employ the Quantile-Quantile(Q-Q) plot, a commonly used graphical statistical tool, see e.g, [11, 14, 18]. The quantiles are points taken at regular intervals from the cumulative distribution function (CDF) of a random variable. The p-quantile for a random variable X is the value x such that $\text{Prob}(X < x) = p$. A Q-Q plot shows the quantiles of the first data set against the quantiles of the second data set, and is therefore an intuitive (as well as visual) and efficient way to determine if two data sets follow a common distribution.

For two random data sets S_1 and S_2, the Q-Q plot is generated by first sorting each set in increasing order, and then sequentially placing points on the plot. The i-th point is placed at coordinates (s_i^1, s_i^2), where s_i^1 and s_i^2 are the values of the i-th data point in the sorted sets S_1 and S_2, respectively.

In order to check whether a data set follows a standard normal distribution, we will make the first data set the theoretical quantiles of the standard normal distribution and the second data set the residual of the test data on the Q-Q plot. If the points fall along a 45-degree reference line, then it is strong evidence that the residual follows the standard normal distribution. If the points fall along a line, but not the reference line, then this suggests that the residual follows normal distribution but is not exactly standard.

In Figure 5.2 we show results on the first data set [72]. As noted earlier, this data set has a much coarser time resolution, with 72 seconds between two consecutive

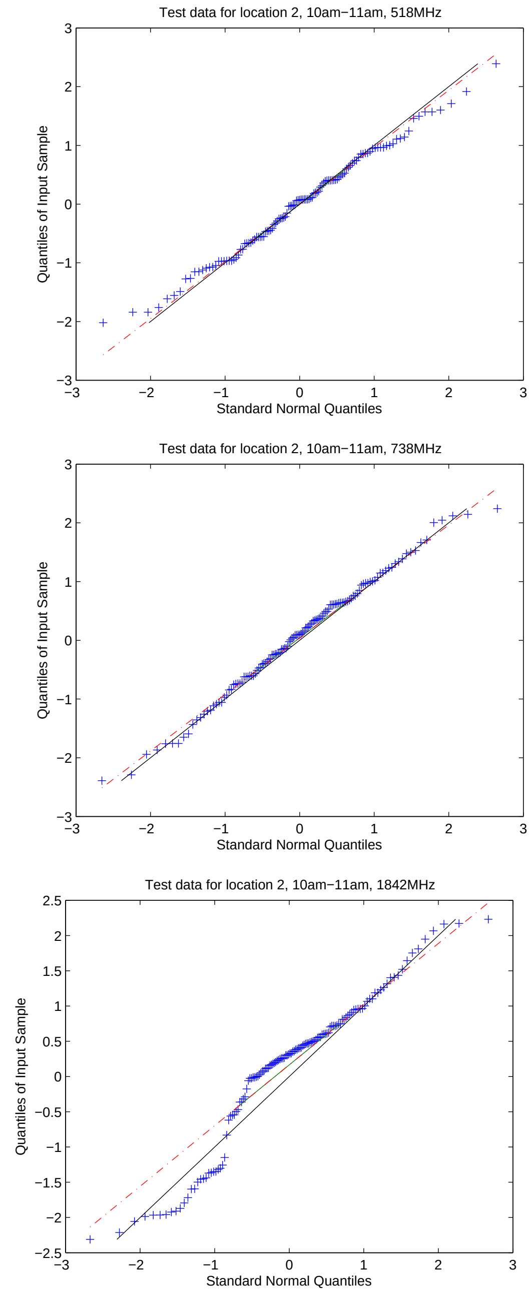

Figure 5.2: Q-Q plot, location 2, 10-11am: 518MHz (top), 738MHz (middle), and 1842MHz (bottom) from [72]

data points (as apposed to 78 nano seconds of the second data set). The dashed line represents the best linear fit of the points (the residual) – the closer the points are to the dashed line, the more normally distributed the residual is. The solid line represents the 45-degree reference line.

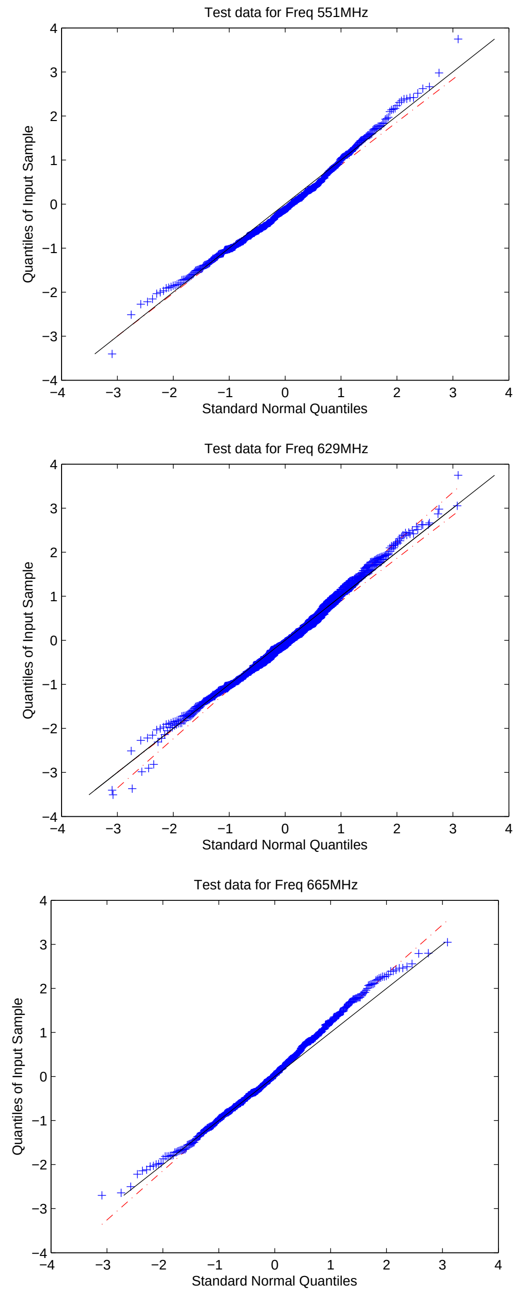

Figure 5.3: Q-Q plot, 551MHz (top), 629MHz (middle), 665MHz (bottom)

Figure 5.3 shows the above normality test results for the three frequencies in the second data set. It can be seen that in all three cases the points have very good linear fit, indicating strong normality of the residuals. In addition, all three dashed lines are very close to the reference line, indicating the residuals follow close-to-standard normal distributions.

We observe that the data acquired from CRAWDAD and in [72] both fit really well but the CRAWDAD data fits better because the residuals in Fig. 5.3 fall more linearly on the 45-degree reference line. The reason is likely due to the fact that the second data set has a better time resolution. For the first data set, the fit is also good with the exception of 1842MHz The reason that the model fails in this case is because when calibrating the data, we assumed that the whole set (Monday-Friday) comes from the same process which has the same three parameter values. Unfortunately, this set contains an out-lier – one of the days has a mean significantly lower than the other four (see Section 5.4 for more detail). Overall, these results verify that the SDE model can be applied to both sparse and fine sampling rates of data.

Along with the fitting result, Table 5.2 shows the values of the estimated parameters; these values can be used in synthesizing spectrum utilization data, described in the next section.

Data Set	MHz	μ	b	σ
1	518	0.3785	0.0391	0.0327
1	738	0.9661	0.0383	0.0732
1	1842	3488.0	0.0297	894.29
2	551	144.5060	1.2606e+07	93.1635
2	629	234.4078	1.5509e+07	124.3106
2	665	226.9313	1.7845e+07	145.8981

Table 5.2: Trained parameters

5.4 Synthesizing spectrum data

An important reason for developing any channel model is to provide a way to generate synthetic channel data (sample paths of energy levels in a channel) that are statistically close to variations observed in a real channel, so that one can easily generate a realistic "spectrum environment" in which to test and evaluate various algorithms and protocols. Below we show such synthetic data can be easily generated under our SDE model.

Taking Eqn (5.10) and integrating over a small interval ϵ,

$$\begin{aligned}
R(t+\epsilon) - R(t) &= \frac{B\mu}{2}(t+\epsilon-t) - \frac{B}{2}\int_{t}^{t+\epsilon} R(\tau)d\tau \\
&\quad + \frac{1}{4}B\sigma^2 \int_{t}^{t+\epsilon} \frac{1}{R(\tau)}d\tau \\
&\quad + \sigma\frac{1}{\sqrt{2}}B^{\frac{1}{2}}(W(t+\epsilon) - W(t)) \\
&\approx \frac{B\mu}{2}\epsilon - \frac{B}{2}R(t)\epsilon + \frac{1}{4}B\sigma^2\frac{1}{R(t)}\epsilon \\
&\quad + \sigma\frac{1}{\sqrt{2}}B^{\frac{1}{2}}(W(t+\epsilon) - W(t))
\end{aligned} \tag{5.20}$$

where the approximation holds when ϵ is sufficiently small. Assuming we start from some initial condition $R(t_o)$ at time t_o, we can generate a sequence of data $R(t_o + k\epsilon)$ at times $t_o + k\epsilon$ for $k = 1, 2, \cdots$ with time resolution (or time step) of ϵ as follows. Note that $W(t+\epsilon) - W(t)$ is normally distributed with zero mean and variance ϵ.

1. Generate a random sample from the 0-mean ϵ-variance normal distribution; denote it by $W(t_o + \epsilon) - W(t_o)$.

2. Take this sample value into Eqn (5.20), replacing the corresponding part in the last term on the RHS.

3. Compute the RHS, which gives the difference between $R(t_o+\epsilon)$ and $R(t_o)$, hence we have generated a value for $R(t_o + \epsilon)$.

4. Now repeat the above procedure indefinitely to produce a time series of desired length.

The end result of this procedure is a sequence of synthesized $R(t)$ values, representing a particular realization.

To verify the validity of the above approach, one needs to verify that sample paths (i.e., the synthesized $\tilde{R}(t)$ process) generated by the above synthesis procedure follow the same distribution as the actual sample paths collected (i.e., the measured $\hat{R}(t)$ process). However, directly comparing sample paths is not a very meaningful exercise. It is also not clear how to extract the underlying distribution from these sample paths since we are dealing with random processes.

To overcome these difficulties, we take the following approach. We first use the synthesis procedure to generate a large number of sample paths and from these we calculate the $x\%$ quantiles for each time t, where $x = 5, 40, 60, 95$. We then inspect how well the real measurement traces fit into these quantiles.

Figure 5.4 shows this comparison result for the one-hour traces collected during 10-11am at location 2. The synthetic sample paths are generated using parameters estimated by randomly selecting 50% of the data collected during this hour on the first five days (Monday-Friday), and the actual sample path is whole set of data collected during this hour. In these figures the dense curves represent (from bottom up) the 5%, 40%, 60%, and 95% quantiles from the synthetic data[3]. The narrow lines running in between are where the real data points lie. In the case of 518MHz and 738MHz, almost all the real observations are within the range between $5-95\%$ quantiles. This shows that the synthesized data and the collected data are more or less consistent.

In the case of 1842MHz, there is one path of data lying below the 5% quantile. This observation tells us that the synthetic data does not describe the data correctly, which agrees with the Q-Q plot analysis in the previous chapter (Figure 5.2). The reason that the model fails is because when calibrating the data, we assumed that the whole set(Monday-Friday) comes from the same process which has the same three parameters. Unfortunately, one of the days has a mean obviously lower than the other four. This tells us that we have to be cautious when the Q-Q plot analysis does not result in straight line.

[3]These quantile curves are much denser (higher temporal resolution) than the actual measurement because in synthesis we are able to choose very small time steps ϵ, at 1 second; indeed the smaller this step the more accurate the approximation in (5.20). On the other hand, the actual measurement has a time resolution around one reading per 70 seconds, as mentioned earlier.

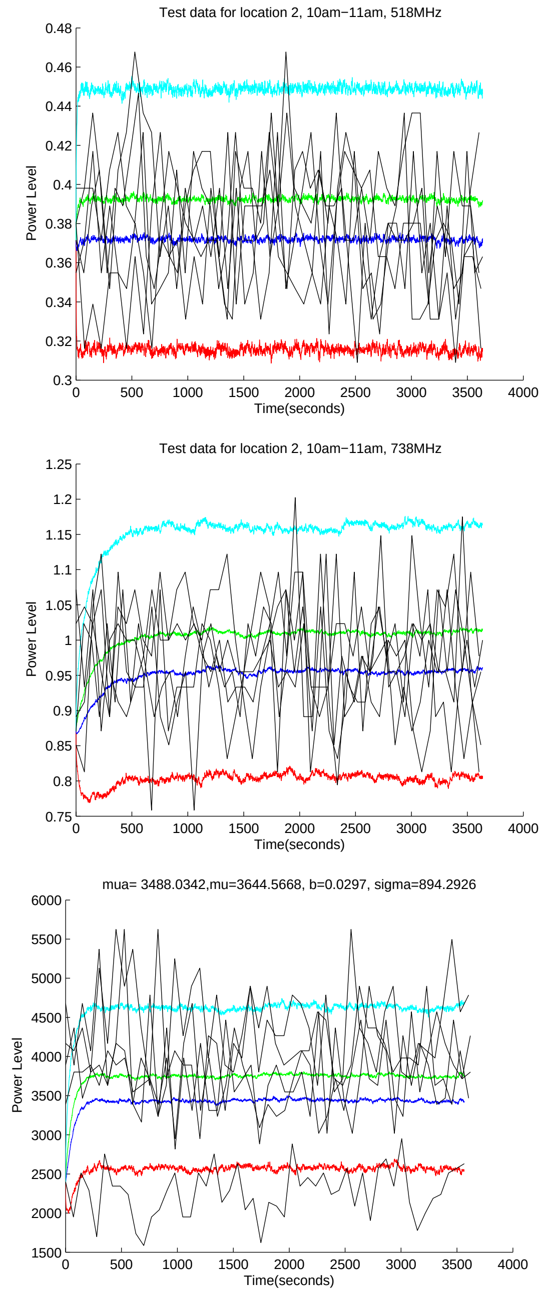

Figure 5.4: Location 2, 10-11am, 518MHz (top), 738MHz (middle) and 1842MHz(bottom). In each figure the dense curves represent (from bottom up) the 5%, 40%, 60%, and 95% quantiles from the synthetic data.

We can also compare the entropy of the two sample paths as an additional, indirect means of validating the SDE model.. The experiment is done by choosing a sliding window of size 10000 (samples). Within this window, we compute the power spectral density of the samples, from which we then compute the entropy of the samples within this window. We do this over the entire sequence with overlapping of 50% between windows.

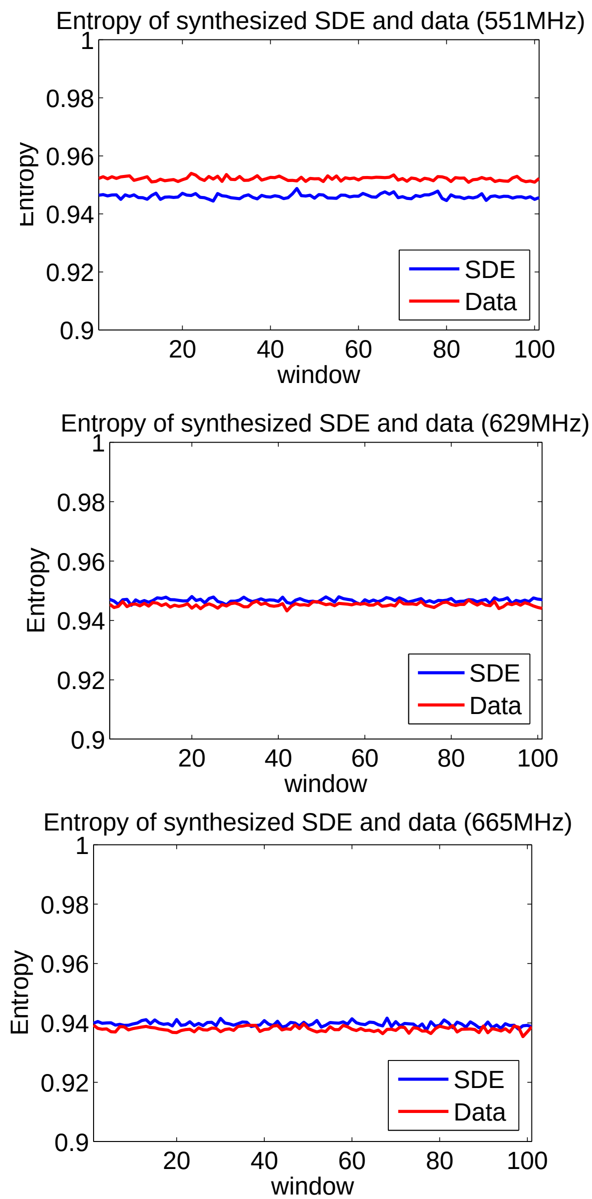

Figure 5.5: Entropy of synthesized SDE and data (551MHz top, 629MHz middle, 665MHz bottom)

Figure 5.5 shows the results of the above comparison with measurement data. It can be seen that the entropy measures of the two are very close in all cases (they also both remain steady throughout the entire sequence).

5.5 SDE as a generalization of the GE model

In this section we show that the SDE model can be used to generate a 2-state GE model; furthermore, this GE model is virtually identical to that generated directly from the data. This in some sense validates the synthetic data generated by the SDE model through yet another means, and at the same time shows that the SDE model may be viewed as a more general modeling framework.

5.5.1 Generating the GE model

Given a sample path (measured or synthesized) of continuous power levels, the discrete GE model may be generated by quantizing/discretizing the data into binary values (where 0 represents a bad channel condition/state and 1 a good channel condition/state). Using guidelines from [15], this discretization is done by viewing the channel as occupied when the received power exceeds the minimum received power (observed in the same sample path) by more than 3dB. Both the measurement data and the synthetic data are already discrete in time, but can be easily down-sampled to obtain a desired time resolution.

We then compute the ratio between how many times BB and BG (respectively GG and GB) transitions occurred over the quantized sample paths, and use this as the estimate for the transition probability ratio P_{BB}/P_{BG} (respectively P_{GG}/P_{GB}). This ratio together with the fact the $P_{BG} = 1 - P_{BB}$ (respectively $P_{GB} = 1 - P_{GG}$) lead to the values of the transition probabilities for the 2-state GE model, illustrated in Figure 5.6.

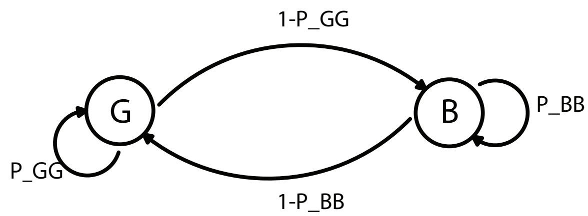

Figure 5.6: The Gilbert-Elliott (2-state) model

5.5.2 Comparison between data-generated GE and SDE-generated GE

The following experiment is performed on the sample paths collected at 551, 629, and 665 MHz, respectively:

1. Train the parameters of the SDE using this data set.
2. Synthesize data from this trained SDE model.
3. Quantize the synthesized data trace to 0-1 binary value as described above.
4. Obtain a GE model from the discretized synthetic data also as described above.
5. Obtain a second GE model directly from the original set of sample paths, similarly quantized.

We then compare these two GE models, the idea being that if the two are similar, then the SDE model can also be used to generate a valid GE model. The comparison results are listed in Table 5.3.

Data Set	MHz	GE from data		GE from SDE	
		P_{BG}	P_{GG}	P_{BG}	P_{GG}
1	551	0.6489	0.6494	0.6350	0.6201
2	629	0.6506	0.6499	0.6427	0.6221
3	665	0.6151	0.6439	0.6615	0.5970

Table 5.3: Data sets for model verification

We see that the two GE models share very similar parameters. This indicates two things. Firstly, since the two GE models are generated from the synthetic and the measurement data, respectively, this similarity suggests that the synthetic and the measurement data are very similar in nature, thereby validating the SDE model as a means of generating realistic synthetic data. Secondly, this means that we can use the SDE model to generate essentially equivalent GE models without having to rely on the original measurement data, i.e., once the SDE model has been trained, we only need to record the three parameters μ, b, σ for future use. In this sense the SDE model may be viewed as a more general modeling framework.

It is also worth noting that the GE model, i.e., its parameters, can be calculated directly from the SDE model, given the threshold, Th, used to quantize the data. Specifically, since we know the steady state distribution from Section 5.2.3, by observing that the SDE model consists of a deterministic term and a normal distribution, we can obtain the following expression for the transition probability:

$$P_{GG} = \frac{\int_0^{Th} f(x)\Phi(\frac{\sqrt{2}}{\sigma\sqrt{B}}(\frac{Th-x}{dt} + \frac{B}{2}(x-\mu) - \frac{B\sigma^2}{4x}))dx}{\int_0^{Th} f(x)dx}, \tag{5.21}$$

i.e., P_{GG} is calculated by integrating the distribution of the power process being below the threshold multiplied by the probability that the next step falls below the threshold. The same can be done for P_{BB}. This method allows us to directly obtain the GE model from the SDE model without having to go through the synthetic data. This method can be extended to any N-state Markov model and will be used in Section 5.6.

5.5.3 Fitting performance of the GE model

Following the previous section, it would be natural to question how well the GE model fits the data if we only consider the quantized, binary description of the channel.

In this section we examine this by comparing (1) the autocorrelation of the sample paths generated by the GE model and the quantized actual data, and (2) the run length distribution of error and error-free runs/sequences observed in these samples.

We first compare the autocorrelation and power spectrum density of the actual data (data set 1, quantized) and the sample paths generated by the GE model. The estimated GE transition probabilities from good state to bad state and good state staying in good state are $P_{GB} = 0.3506$ and $P_{GG} = 0.6494$. The transition probabilities starting from the bad state are $P_{BB} = 0.3511$ and $P_{BG} = 0.6489$. Results for the autocorrelation and power spectral density are shown in Figure 5.7 and 5.8.

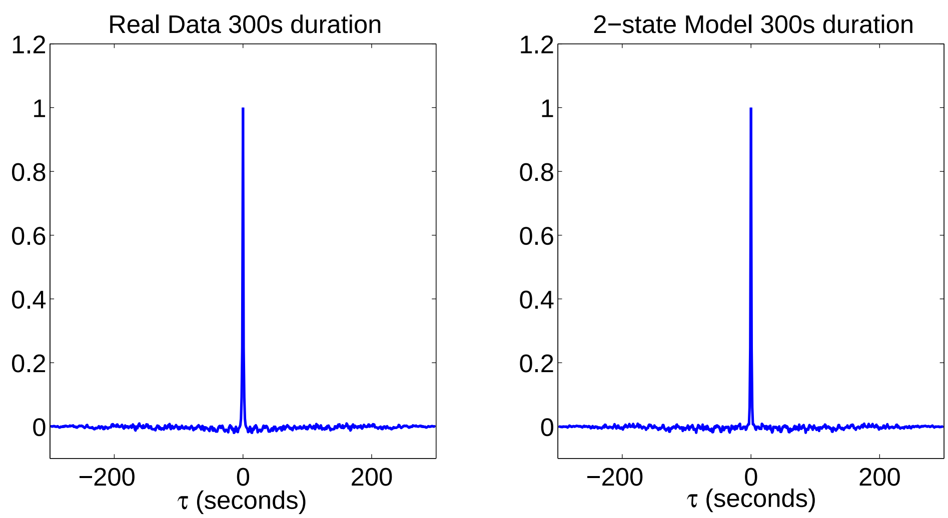

Figure 5.7: The autocorrelation of real data and the GE model over a 300-second duration.

The figures show that the autocorrelation and power spectral density of the GE model matches the actual discretized data samples very well with similar maximum values and similar shapes.

Next we examine the length of consecutive available states (consecutive ‘1’s) and consecutive unavailable states (consecutive ’0’s). Figure 5.9 (left) shows the distribution of consecutive runs of availability. The x-axis is the run length plotted in logarithm scale, and the y-axis is the proportion of runs of that length. We observe

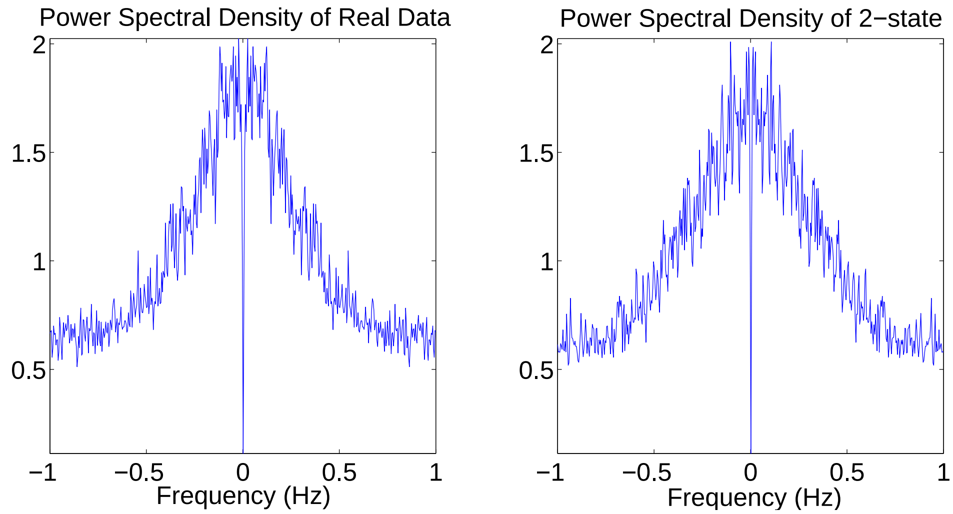

Figure 5.8: The power spectral density of real data and the GE model over a 300-second duration.

that the GE model generates run lengths of both error and error-free close to the actual data.

The autocorrelation and the error length show that the GE model fits the actual data really well, suggesting that the GE model is a good choice when the discretized binary representation of the channel condition is sufficient. In other words, if an application only needs to observe whether the channel is in a good quality, the GE model is more or less adequate. This however is no longer the case if we require higher resolution. In the next section, we demonstrate that when we need more information than just discretized binary values, the SDE model provides much more richness.

5.6 Predictive performance of the SDE model

One obvious advantage of having a continuous model like SDE is the richness of the data it provides compared to a discrete (esp. binary) model like GE. In this section we take a closer look at this aspect within the context of the respective model's ability to predict channel conditions. In many applications including channel-aware transmission scheduling [3] and opportunistic spectrum access [6, 68], collecting and

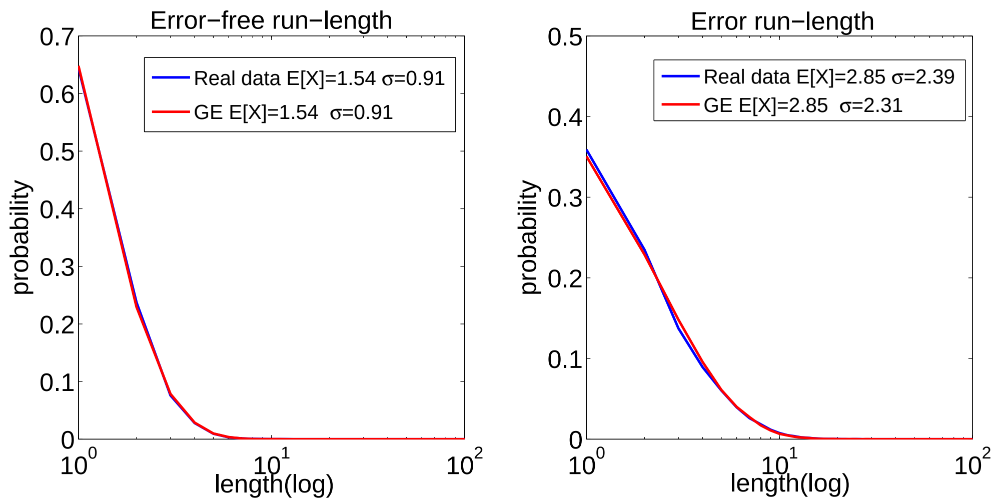

Figure 5.9: Comparison between the SDE and GE models: consecutive available run length (left) and consecutive occupied run length (right)

predicting channel condition (or channel side information (CSI) in some contexts) is often critical to the effectiveness of the overall mechanism. For these applications, we often wish to obtain information more than just the binary value representing whether the channel is occupied. This is so that we could, for instance, more accurately calculate the SNR from the interference power to estimate the achievable transmission throughput.

Below we examine a channel model's ability to predict, in a discrete-time setting, the received power of the next step given the current power level. We do so for the SDE model, and an N-state Markov model, a more general version of the 2-state GE model which allows us to adjust the resolution (the size of the state space N) and the corresponding quantization error.

We use the data-generated SDE model as a channel prediction model and compare it with the prediction of an N-state Markov model. Since both models have Markovian behavior, knowing the current state is sufficient to predict the next state. The error of the prediction model is defined as the absolute value of the difference between the prediction and the actual value. Given the past values of a process, the SDE model satisfies the Markovian property and predicts the next time step purely on the current power value:

$$\begin{aligned} R(t+\epsilon) \quad \approx \quad & R(t) + \frac{B\mu}{2}\epsilon - \frac{B}{2}R(t)\epsilon + \frac{1}{4}B\sigma^2\frac{1}{R(t)}\epsilon \\ & +\sigma\frac{1}{\sqrt{2}}B^{\frac{1}{2}}(W(t+\epsilon) - W(t)) \ . \end{aligned} \tag{5.22}$$

Notice that only the last term contributes to a stochastic value with mean 0 and variance ϵ; all other terms are deterministic. If our metric is the absolute loss |prediction value − actual value|, then the best prediction is to predict the expected value. The expected value is obtained by discarding the stochastic terms and leaving only the deterministic terms. Thus, in this comparison we will use eqn. 5.23 as the SDE model prediction.

$$R(t+\epsilon) = R(t) + \frac{B\mu}{2}\epsilon - \frac{B}{2}R(t)\epsilon + \frac{1}{4}B\sigma^2\frac{1}{R(t)}\epsilon \tag{5.23}$$

For the N-state Markov model, we take the first 200000 sample points from the CRAWDAD data for training. The N-state Markov model requires a quantization of different power levels. The quantization level is determined by dividing the sorted 200000 sample points into N levels; all points are quantized into the mean value of the level it is placed in. We can then construct the N-state transition probability matrix from the transitions of the 200000 data points between the quantile levels.

It is worth noting that the SDE model only requires several hundred of sample points for training to obtain the 3 parameters (μ, b, σ) while the N-state Markov model requires at least two orders of magnitude (100x) more in the number of training samples. Also, the N-state Markov model requires $N \times N$ storage to hold the transition probability matrix.

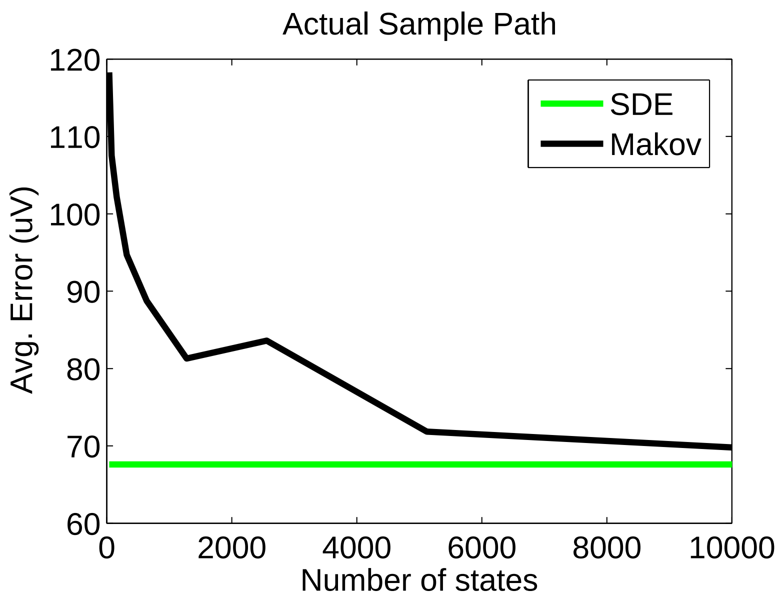

Figure 5.10: Average error obtained using Markov models of different number of states compared with SDE

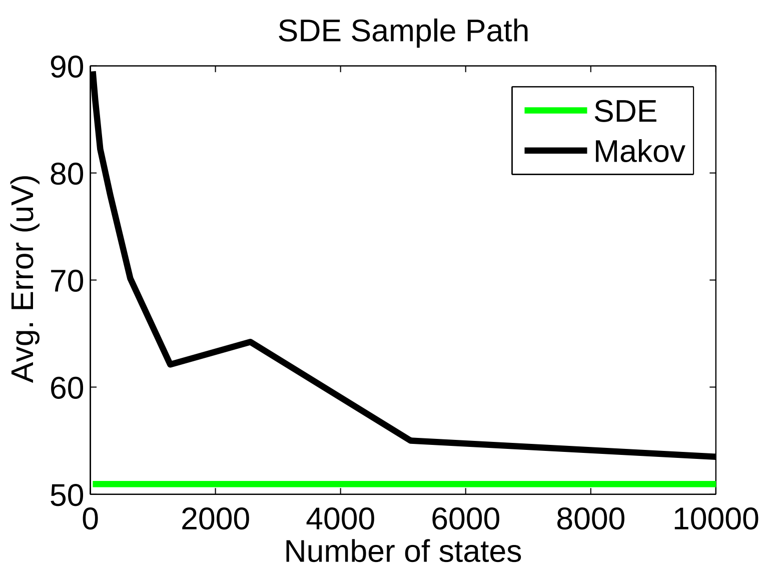

Figure 5.11: Average error obtained using Markov models of different number of states compared with SDE

Figure 5.10 shows the average prediction error of the N-state Markov model for $N \in [1, 10000]$. Compared with the SDE model, we can see that the N-state Markov model is not very useful in prediction when N is small, but can achieve predictions close to the SDE model by trading space for accuracy. Figure 5.11 shows the same result but replacing CRAWDAD data by SDE generated sample path. As expected, since it is actually generated from the SDE model, the SDE model predicts the power levels even more accurately. This error value is actually the average deviation of the last term in equation 5.22.

We next show the prediction performance over multiple time steps. We fix the number of states for the Markov model, but vary the prediction steps n from 1 to 37. The prediction of the Markov model is done by multiplying the transition matrix of the Markov model by it self for n times. Assuming the trained transition matrix is correct, this will give the correct n step transition probability matrix. Prediction under the SDE is done by recursively plugging in Equation 5.23 for n times. The results are shown in Fig. 5.12.

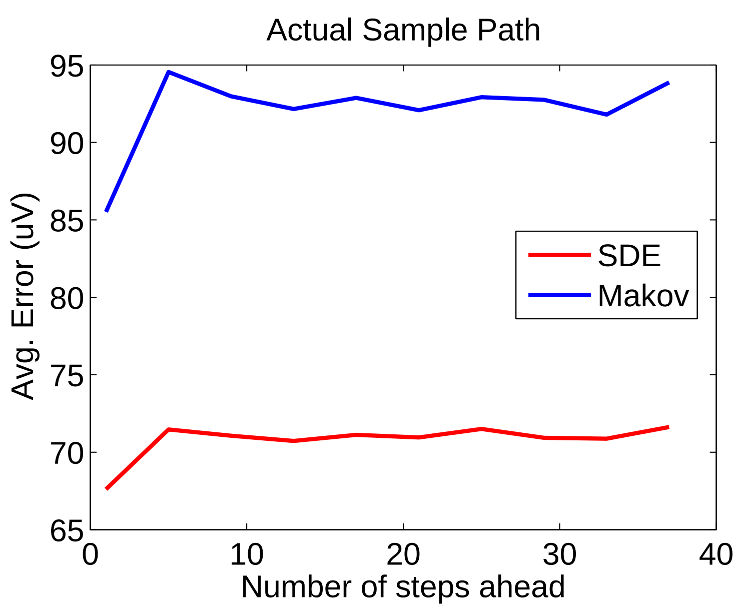

Figure 5.12: Average error obtained using a 1000-state Markov model and SDE for n-step prediction.

We see that the SDE clearly outperforms under all step sizes. The error accumulates when we increase the prediction steps, but this saturates to a near-constant value when the step is larger than 5. This is most likely because after 5 steps the dependency of the future power level on the current value becomes negligible, so that the prediction of both models become the steady state value.

It should be mentioned that one of the main reasons that contributed to the error under the Markov prediction model is the training process it takes to obtain the transition matrix – the measurement data we have appears insufficient in volume for this purpose when N is large. We thus conduct the following, improved experiment. We first use the SDE model to directly calculate the transition probabilities as shown earlier, so as to minimize this training error. This would be equivalent to training the Markov model using very long data traces. We then compare the prediction performance of the SDE and this more accurate N-state model over a synthetic sample path; the results are shown in Figure 5.13. We see that the error of the Markov model is reduced even with small state space N. Note that when N is sufficiently large, the Markov model can predict slightly better than the SDE does. This is because under the Markov model we are truly calculating the probability of multi-step transitions whereas under the SDE model we recursively calculate the mean which is only an estimate. The improvement is, however, very minor, and comes at enormous computational expense[4]

We end this section by highlighting a further advantage of the SDE model, which is flexibility. This can be seen in two aspects: (1) suppose the training data have measurements measured at nonuniform or random times. In this case it would be hard or even impossible to use this data to train a simple N-state Markov model because the transition of the Markov model is at a fixed time step. (2) If the we want to change the discrete time unit of the Markov model, we would need to either retrain the model (in the case of upsampling), or perform matrix multiplication (in the case of down-sampling and only feasible when the new time unit is an integer

[4]Again, note that we actually had to use the SDE model to generate the N-state model for lack of sufficient training, so this comparison would not have been possible without the SDE model!

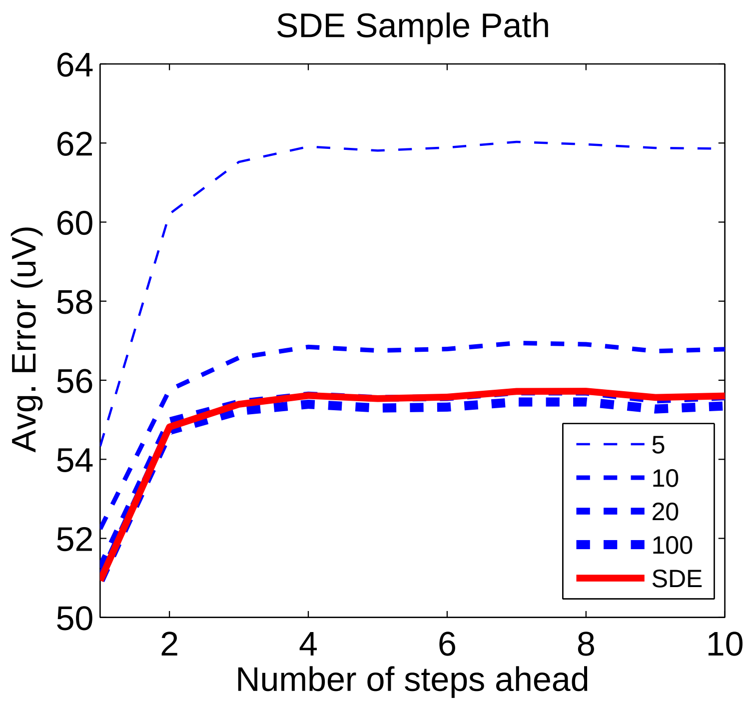

Figure 5.13: Markov model obtained from SDE model

multiple of the current one). In contrast, the SDE model does not require a uniformly-sampled training data. Recall that in Section 5.2.2, the training is done with the triple $(\hat{R}(t_i), d\hat{R}(t_i), dt_i)$ where dt_i can be any real number. In addition, once we obtain the SDE model it is very easy to select/adjust the desired time resolution by choosing the Brownian motion term with the corresponding ϵ in Equation 5.20 without having to retrain the model.

5.7 Conclusion

In this chapter we introduced a stochastic differential equation (SDE) model to describe the secondary wireless channel power and compared it with N-state Markov models. We show that the SDE model fits spectrum measurement data very well under different measurement regimes. The SDE model can easily generate synthesized sample paths whose entropy measure is consistent with the original measurement data. The SDE model is a more general modeling framework that can be used to obtain an N-state Markov model. While we show that the 2-state GE model is a good choice when binary representation of the channel condition is sufficient, the SDE model is in general much more accurate and easier to use than an N-state model.

CHAPTER VI

Trading Secondary Spectrum Through Spectrum Portfolio

6.1 Introduction

In Chapter II and III, the buyer only considers purchasing one type of secondary spectrum product and the rest from the reference market to satisfy the transmission quality constraint. When multiple secondary spectrum products are available on the market, the buyer may be able to combine multiple purchases of stochastic spectrum products to increase the quality of transmission. In this chapter, we consider a buyer who purchases a portfolio of spectrum products to maximize the mean throughput while minimizing the variation of transmission throughput. The buyer's consideration will rely on the understanding of the channel statistics and the channel statistics can change when more information of the channel is gathered. A precise channel dynamics model will be important for the buyer and seller to have good purchase and pricing plans.

6.2 Model

In this section we describe in detail the models for the two parties under the framework: the seller and the buyer, and their considerations in designing and choosing the purchase, respectively. The basic idea underlying our model is that the buyer purchases a mixture of secondary spectrum products, which are random and non-guaranteed. By a careful selection of the different secondary spectrum products, the buyer minimizes the variation of the service by considering the statistics of each spectrum product. Another important idea is the reference spectrum which provides guaranteed spectrum product, this gives the buyer an alternative if the secondary spectrum products are all undesirable or overpriced.

6.2.1 The Seller

There are two parties, the seller and the buyer. The *seller* who uses the spectrum to provide business and service to its *primary users*, and carry *primary traffic*. The seller is willing to sell underutilized bandwidth as long as it generates positive profit and does not impact negatively it's primary business. We will assume that the seller has N types of spectrum products each running a different primary service. Based on the primary services running on each type of spectrum, the throughput obtained by the buyer is different. The seller measures the statistics of each channel and announces the measurements; in this case we will primarily consider the first and second order statistics of the throughput (the mean and the variance of throughput).

The seller can decide a price corresponding to each type of spectrum product, the goal is to obtained as much profit as possible from the secondary market. The seller has to design the prices carefully, let the vector $\mathbf{C} = [C_1...C_N]$ denote the price vector of the spectrum and $\mathbf{G} = [G_1...G_N]$ denote the mean throughput of the spectrum. $\mathbf{\Sigma}$ denotes the correlation matrix of the N spectrum products in terms of their throughputs. We will assume that the throughput obtained from the spectrum products are independent random variables so $\mathbf{\Sigma}$ would be a diagonal matrix. The analysis in this chapter is not necessarily limited to this independent assumption and we will mention when it can be applied to dependent spectrum setups.

6.2.2 The Buyer

The buyer purchases a portfolio of different secondary spectrum products for it's own use. The buyer has a fixed amount of money and wants to maximize the mean throughput while minimizing the variation of the throughput obtained from the spectrum products. Assume the random variable X denotes the throughput of the sum throughput of the spectrum portfolio purchased. The utility of the buyer is captured by the following expression:

$$\max E[X] - \eta Var[X],$$

where η represents the risk the buyer is willing to take. For risk averse buyer, η will be a large number. For a risk neutral buyer, η would be 0 and the buyer only cares about maximizing expected throughput. Let the vector $\mathbf{P} = [P_1 ... P_N]$ denote the amount of each channel purchased by the buyer. Then $E[X] = \mathbf{P}^T\mathbf{G}$ and $Var[X] = \mathbf{P}^T\mathbf{\Sigma}\mathbf{P}$. The budget constraint of the buyer is denoted by a scalar t. We assume the buyer will spend all of the money so that $\mathbf{P}^T\mathbf{C} = t$.

6.2.3 Reference Spectrum

We include a reference spectrum market as a third party to the buyer and seller. The buyer can choose any portion of the budget t to purchase from the reference market and spend the rest in the secondary market. The spectrum bought from the reference market is assumed to have exclusive access right and will generate a fixed amount of throughput with 0 variance. The reference spectrum only serves as a comparison to find the fair price of the secondary spectrum, thus, we normalize it to unit price per bandwidth.

6.3 Analysis

We first consider the problem where the buyer only purchases from the secondary spectrum market given prices set by the seller. Based on the result, we find the optimal portion that the buyer will put into the secondary spectrum market, which allows us to derive the pricing strategy of the seller.

6.3.1 Buyer's consideration without the reference channel

The buyer's consideration can be written as an optimization problem over the vector $\mathbf{P}$, while subject to the budget constraint.

$$\max_{\mathbf{P}} \mathbf{P}^T\mathbf{G} - \eta\mathbf{P}^T\mathbf{\Sigma}\mathbf{P} \tag{6.1}$$

$$\text{subject to } \mathbf{P}^T\mathbf{C} = t \tag{6.2}$$

$$P_i \geq 0, \tag{6.3}$$

where each P_i has to be nonnegative. If we first ignore the inequality constraint, the problem can be solved by standard matrix calculus where the solution is as follows,

$$\mathbf{P} = \frac{1}{\eta}\mathbf{\Sigma}^{-1}[\mathbf{G} + \mathbf{C}(\frac{\eta t - \mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{G}}{\mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{C}})] \tag{6.4}$$

If there are entries less than zero, the solution can be obtained by iterative elimination of entries less than zero then reapplying Equation 6.4 [53]. The channels that are left un-eliminated depends on the amount of money t the buyer wants to purchase. Note that the iterative elimination process does not require the spectrum products being independent.

6.3.2 Buyer's consideration with the reference channel

We now include the reference market, which sells guaranteed spectrum with price 1 and 1 unit power per bandwidth in the buyer's problem. Consider that the buyer uses a portion of $t - t_0$ to invest in the reference market. The buyer's problem is to choose the optimal amount between the reference market and the secondary market. Now we will assume that the channels have independent quality and the covariance matrix is a diagonal matrix. In this case, $\mathbf{\Sigma}^{-1}$ is a diagonal matrix with diagonal entries being the inverse of each variance and are nonnegative. The entry P_i is negative only if $G_i - C_i(\frac{\eta t_0 - \mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{G}}{\mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{C}})$ is negative. Note that because this expression is increasing in t_0, the channels uneliminated will not be eliminated when we increase t_0. The buyer's problem can be expressed as the following maximization problem where $\mathbf{P} = \frac{1}{\eta}\mathbf{\Sigma}^{-1}[\mathbf{G} + \mathbf{C}(\frac{\eta t - \mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{G}}{\mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{C}})]$.

$$\max_{t_0} \mathbf{P}^T\mathbf{G} + (t - t_0) - \eta\mathbf{P}^T\mathbf{\Sigma}\mathbf{P} \tag{6.5}$$

$$= \max_{t_0} \frac{1}{\eta}[\mathbf{G} + \mathbf{C}(\frac{\eta t_0 - \mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{G}}{\mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{C}})]^T\mathbf{\Sigma}^{-1}\mathbf{G} + t - t_0 \tag{6.6}$$

$$- \eta[\mathbf{G} + \mathbf{C}(\frac{\eta t_0 - \mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{G}}{\mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{C}})]^T\mathbf{\Sigma}^{-1}\mathbf{\Sigma}\mathbf{\Sigma}^{-1}[\mathbf{G} + \mathbf{C}(\frac{\eta t_0 - \mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{G}}{\mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{C}})] \tag{6.7}$$

$$= \max_{t_0} \frac{1}{\eta}[\mathbf{G} + \mathbf{C}(\frac{\eta t_0 - \mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{G}}{\mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{C}})]^T\mathbf{\Sigma}^{-1}\mathbf{G} + t - t_0 \tag{6.8}$$

$$- \eta[\mathbf{G} + \mathbf{C}(\frac{\eta t_0 - \mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{G}}{\mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{C}})]^T\mathbf{\Sigma}^{-1}[\mathbf{G} + \mathbf{C}(\frac{\eta t_0 - \mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{G}}{\mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{C}})] \tag{6.9}$$

$$= \max_{t_0} \frac{\eta}{\mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{C}}(t_0^2 - \frac{1}{\eta}\mathbf{C}^T\mathbf{\Sigma}^{-1}(\mathbf{G} - \mathbf{C})t_0 + ...) + ..., \tag{6.10}$$

which has a quadratic form and the t_0 that maximizes the expression can be calculated as follows,

$$t_0 = \frac{1}{2\eta}\mathbf{C}^T\mathbf{\Sigma}^{-1}(\mathbf{G} - \mathbf{C}). \tag{6.11}$$

If $t_0 > t$, the buyer will invest all money t in the secondary spectrum market and the iterative elimination process Equation 6.4 will determine the optimal portfolio. If $t_0 \le t$, the buyer will only invest t_0 in the secondary spectrum market and the remaining $t - t_0$ will be invested in the reference market. Note that in the latter case, the money $P_i C_i$ invested in each spectrum product be at the value such that the marginal gain of investing in the ith product equal to the gain of investing in the reference market. In this case, the choices for each spectrum product are decoupled and are only compared with the price of the reference spectrum. We will see this in the next section where the pricing of each spectrum product can be decided independently.

6.3.3 Optimal pricing for the seller

We assume the primary knows the buyer's preference and wants to maximize total revenue obtained from the buyer. Because t_0 is exactly the amount the buyer spends in the secondary market, the seller wants to maximize t_0 with the choice of $\mathbf{C}$. (The revenue is capped at $t = t_0$ because the buyer has budget t.)

$$\max_{\mathbf{C}} \frac{1}{2\eta} \mathbf{C}^T \Sigma^{-1} (\mathbf{G} - \mathbf{C}) \tag{6.12}$$

Because $\mathbf{\Sigma}$ is a diagonal matrix, the amount each spectrum product contributed to the seller's revenue is additive and does not depend on the other spectrum products. In order to maximize the revenue, the seller will set the price such that all products are not eliminated from the buyers choice. The maximum choice is $\mathbf{C} = \mathbf{G}/2$ where the total revenue is $\frac{\mathbf{G}^T \mathbf{\Sigma}^{-1} \mathbf{G}}{8\eta}$. We substitute this result into the buyer's consideration and verify that each $P_i > 0$. For any constant $0 \leq b < 1$, if $\mathbf{C} = b\mathbf{G}$,

$$\mathbf{P} = \frac{1}{\eta}\mathbf{\Sigma}^{-1}[\mathbf{G} + (\frac{\eta t - \mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{G}}{\mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{C}})] = \frac{1}{\eta}\mathbf{\Sigma}^{-1}[\mathbf{G} + b\mathbf{G}(\frac{\eta t - b\mathbf{G}^T\mathbf{\Sigma}^{-1}\mathbf{G}}{b^2\mathbf{G}^T\mathbf{\Sigma}^{-1}\mathbf{G}})] \tag{6.13}$$

$$= \frac{1}{\eta}\mathbf{\Sigma}^{-1}[\mathbf{G} - \mathbf{G} + \frac{\eta t}{b\mathbf{G}^T\mathbf{\Sigma}^{-1}\mathbf{G}}] \tag{6.14}$$

$$= \frac{1}{\eta}\mathbf{\Sigma}^{-1}[\mathbf{G}\frac{\eta t}{b\mathbf{G}^T\mathbf{\Sigma}^{-1}\mathbf{G}}] \tag{6.15}$$

$$\geq 0. \tag{6.16}$$

This says that if the seller designs the price of each channel proportional to the mean throughput, the buyer will have an incentive to purchase non-zero amount of each secondary spectrum product.

6.3.4 Optimal pricing for seller with channel cost

If the primary has a cost per bandwidth for each channel $\bar{\mathbf{C}}$, then the maximum profit can be expressed as.

$$\begin{aligned}
\max_{\mathbf{C}} \mathbf{P}^T(\mathbf{C} - \bar{\mathbf{C}}) &\Leftrightarrow \max_{\mathbf{C}} (\frac{1}{\eta}\mathbf{\Sigma}^{-1}[\mathbf{G} + \mathbf{C}(\frac{\eta t_0 - \mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{G}}{\mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{C}})])^T(\mathbf{C} - \bar{\mathbf{C}}) \\
&\Leftrightarrow \max_{\mathbf{C}} (\frac{1}{\eta}\mathbf{\Sigma}^{-1}[\mathbf{G} + \mathbf{C}(\frac{\eta\frac{1}{2\eta}\mathbf{C}^T\mathbf{\Sigma}^{-1}(\mathbf{G} - \mathbf{C}) - \mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{G}}{\mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{C}})])^T(\mathbf{C} - \bar{\mathbf{C}}) \\
&\Leftrightarrow \max_{\mathbf{C}} \frac{\mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{C}\mathbf{C}^T\mathbf{\Sigma}(\mathbf{G} - \mathbf{C} + \bar{\mathbf{C}}) - 2\mathbf{C}^T\mathbf{\Sigma}\mathbf{C}\mathbf{G}^T\mathbf{\Sigma}\bar{\mathbf{C}} + \mathbf{C}^T\mathbf{\Sigma}\bar{\mathbf{C}}\mathbf{C}^T\mathbf{\Sigma}\mathbf{G}}{2\eta\mathbf{C}^T\mathbf{\Sigma}^{-1}\mathbf{C}}
\end{aligned}$$

This expression is non-convex and t_0 depends on the design of $\mathbf{C}$. Assuming that the cost $\bar{\mathbf{C}} = a\mathbf{G}$ is a fraction of the mean $\mathbf{G}$ and the seller designs $\mathbf{C} = b\mathbf{G}$. Then we can simplify the problem to,

$$\max_b \frac{b^3\mathbf{G}^T\mathbf{\Sigma}^{-1}\mathbf{G}\mathbf{G}^T\mathbf{\Sigma}(\mathbf{G}-b\mathbf{G}+a\mathbf{G})-2b^2\mathbf{G}^T\mathbf{\Sigma}\mathbf{G}\mathbf{G}^T\Sigma a\mathbf{G}+b^2a\mathbf{G}^T\mathbf{\Sigma}\mathbf{G}\mathbf{G}^T\mathbf{\Sigma}\mathbf{G}}{2\eta b^2\mathbf{G}^T\mathbf{\Sigma}^{-1}\mathbf{G}} \tag{6.17}$$

$$\Leftrightarrow \max_b \frac{b^3(1-b+a)-2b^2a+b^2a}{2\eta b^2}\mathbf{G}^T\mathbf{\Sigma}^{-1}\mathbf{G} \tag{6.18}$$

$$\Leftrightarrow \max_b \frac{-b^2+(1+a)b-a}{2\eta}\mathbf{G}^T\mathbf{\Sigma}^{-1}\mathbf{G} \tag{6.19}$$

Since $\mathbf{C} = b\mathbf{G}$, the buyer will purchase from all channels. Under the assumption, the optimal pricing plan is $\mathbf{C} = \frac{1+a}{2}\mathbf{G}$ and the total profit is $(\frac{1-a}{4})^2\frac{\mathbf{G}^T\mathbf{\Sigma}^{-1}\mathbf{G}}{2\eta}$. As one would expect, the profit is higher with low cost (small a), small η, large $\mathbf{G}$ (high channel mean throughput) and small $\mathbf{\Sigma}$ (good quality spectrum).

6.4 Simulation

In this section, we test the resulting throughput of the buyer's portfolio derived in this chapter. The seller has 10 channels each modeled by an independent Gaussian random variable with mean ($\mu = 144$) $\pm$ 50% and standard deviation 15 $\pm$ 50% uniformly generated. The price per channel is set to 90 $\pm$ 10% of the channel mean throughput. The buyer is assumed to have 100 units of money, which can purchase 111 units of throughput from the secondary market on average. The reference market is set at 1 unit price per bandwidth so the buyer can get 100 units of throughput at 0 variation. The parameter η is set to 1 in this particular case.

In Fig. 6.1 we show the throughput obtained by the portfolio optimization proposed in this chapter compared with an uniform purchasing of each channel. As shown in Fig. 6.1, the throughput bought from the portfolio has a smaller variation compared with the uniform purchasing. The variance of the portfolio method obtains a mean 117 and standard deviation 1.37 while the uniform purchase obtains mean 113

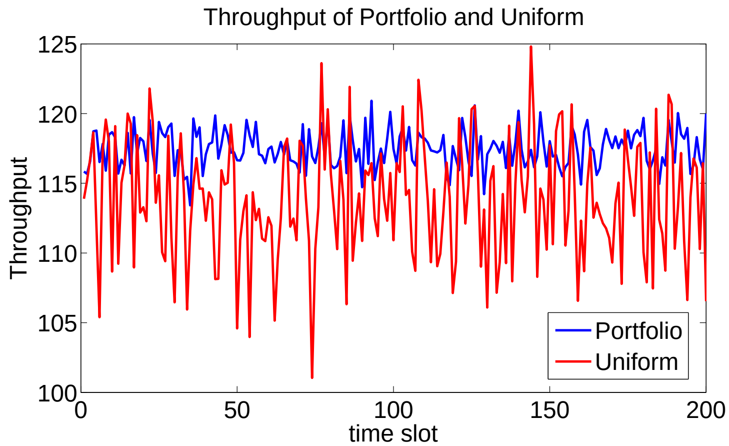

Figure 6.1: Independent Gaussian random variables

and standard deviation 4.2. This shows that the portfolio method effectively reduces the risk.

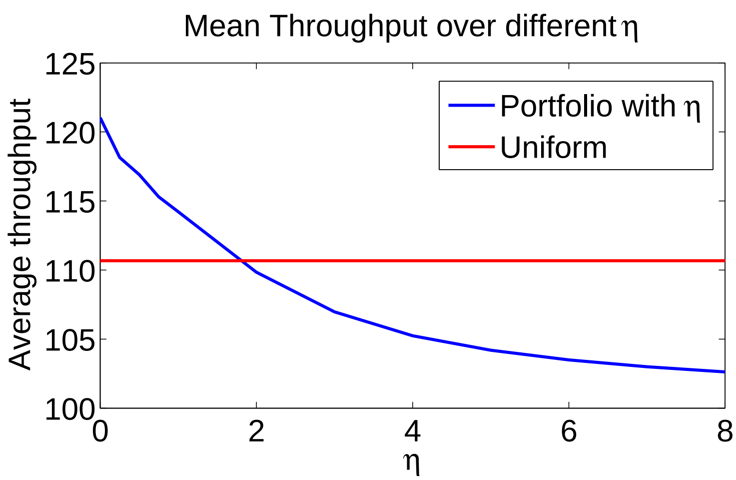

Figure 6.2: Mean throughput for different η

Next, we plot the mean and variance of the obtained throughput over different values of $\eta = [0, 8]$. The results are shown in Figure 6.2 and 6.3. We can see that with small η, the portfolio acquires more throughput because the variance term is weighted with a small value. However, the standard deviation of the portfolio is very high. This would be the case when the buyer only cares about the expected throughput. On the other hand, when $\eta = 8$, the average throughput is low and the standard deviation is also very close to zero. This would be the case of a risk averse buyer.

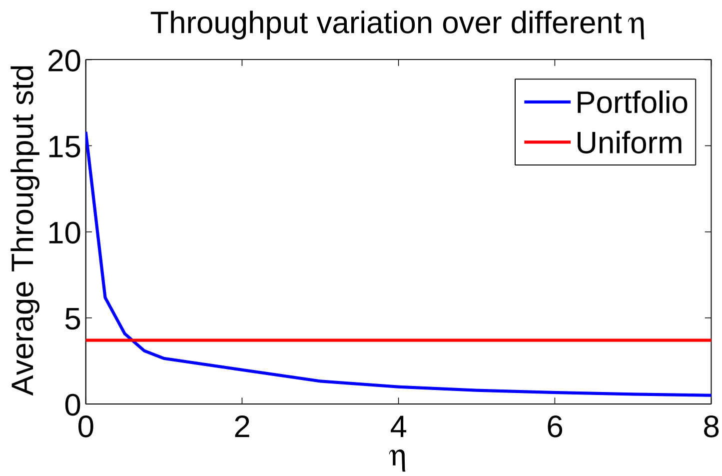

Figure 6.3: Throughput variance for different η

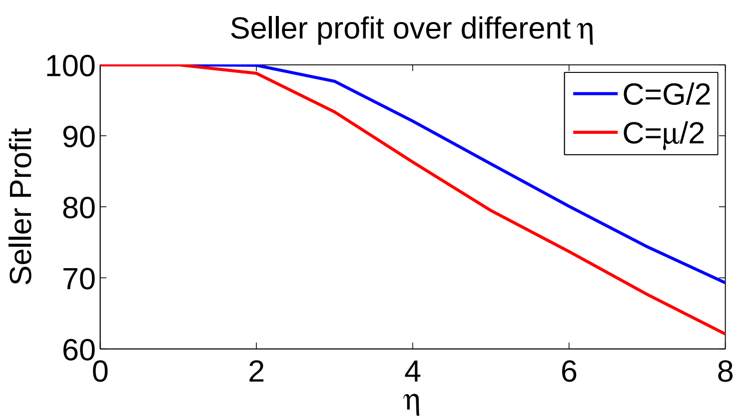

Figure 6.4: Seller's profit for different η

We also test the efficiency of the proposed pricing plan for the seller. For independent channels, the optimal price of each channel is $C = G/2$. We keep the simulation setups the same, but set $C = G/2$. We compare the result with an uniform price setting $C_i = \mu/2 = 72$ shown in Figure 6.4. When η is low, the buyer will prefer to spend all the money in the secondary market because the prices are lower. Thus, both pricing plans receive 100% of the buyer's money. When η is high, we can see that there is a gap between the optimal pricing plan (blue) compared with the suboptimal pricing plan (red).

6.4.1 Prospect Theory

Prospect theory [44, 67, 45] is a method to describe the way people choose between probabilistic alternatives that involve risk. In particular, individuals decide which outcomes they consider equivalent, set a reference point and then consider lesser outcomes as losses and greater ones as gains. We use the prospect theory utility function to model the risk-averse buyer. To be more precise, let x be the value compared to the reference point a common utility function is as follows,

$$v(x) = \begin{cases} x^{\beta} & x \geq 0 \\ \lambda(-x)^{\beta} & x < 0 \end{cases} \tag{6.20}$$

The utility changes most rapidly around the reference point which the person considers the normal outcome. In our problem, we consider investing all money in the reference market as the reference point. Spending 100 will result in guaranteed 100 throughput, thus, the gain (or loss) is $x = X - 100$. We let $\beta = 0.5$ and vary λ from 1 to 4. (higher λ means the buyer is more loss averse)

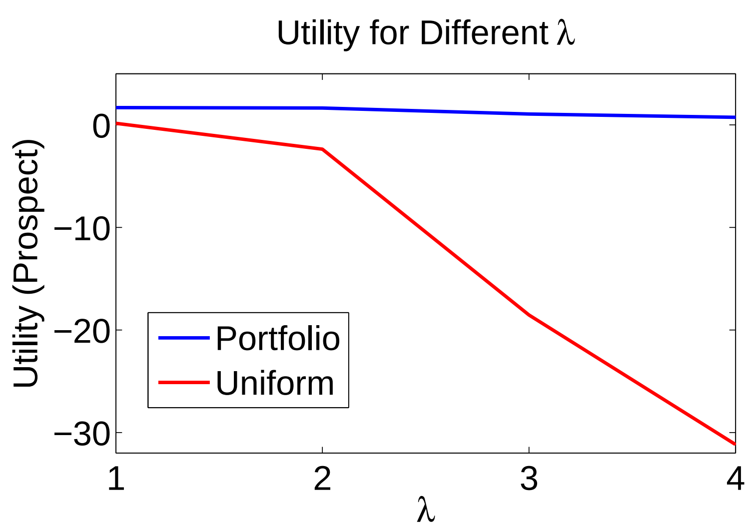

Figure 6.5: Buyer's utility for different λ

Figure 6.5 shows the result. Because the channel set selected by the proposed portfolio method has a smaller standard deviation, it performs consistently regardless of how bad the buyer considers the loss. On the contrary, a uniform channel selection would have lower utility when λ is increased.

6.4.2 Dynamic Spectrum Access

Lastly, we apply the portfolio selection method to a dynamic channel model. We use the stochastic differential equation (SDE) model developed in Chapter V, where the SDE channel statisitcs of the next state depends on the current channel observation. The SDE model can be discretized to discrete time slots and each SDE channel is driven by an independent Brownian motion. Based on the channel condition and the price of each channel, the user can optimize the allocation power on these channels to get a higher combination of throughput. We will fix the price to the optimal pricing $C = G/2$. In particular use a SDE model with $\mu = 144 \pm 20\%$, $b = 1.26e + 07 \pm 20\%$ and $\sigma = 93 \pm 20\%$.

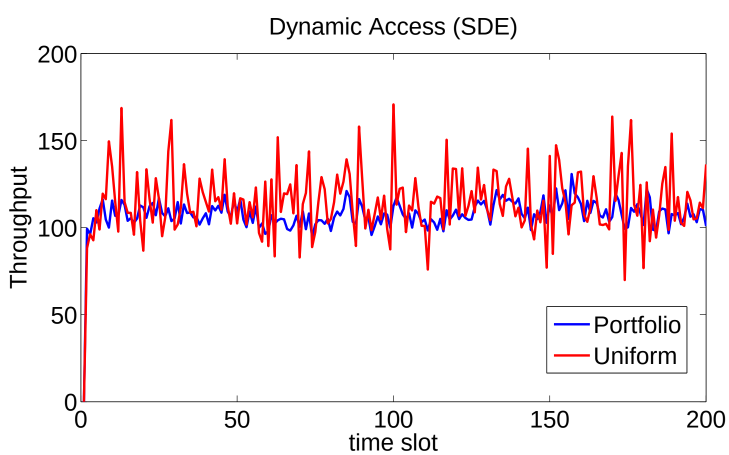

Figure 6.6: SDE channels

The results are shown in Figure 6.6. We compare the proposed portfolio method with an uniform allocation on all channels. As shown in Figure 6.6, the proposed portfolio method obtains a combined throughput with smaller variation compared to the uniform channel selection.

6.5 Related Work

The work in this chapter uses the Sharpe ratio metric [2] for the buyer's consideration problem. The mean-variance based analysis is often used in finance because the variance of return is as important as the mean returns [62]. Work most related to that presented here includes [71] where the authors considered the QoS management in cognitive radio using portfolio selection theory. The main difference is that we include the pricing of channel products in the secondary users' consideration and include a reference market which sells guaranteed spectrum. We also consider the optimal pricing plans for the primary user. In [57], Muthuswamy et. al. considered two different metrics, the demand satisfaction rate constraint and the demand satisfaction probability constraint for the buyer's objective (we borrowed these two metrics as the expected loss and probability of loss constraint considered in Chapter II and III). The authors showed that the buyer's objective is a convex problem which can be solved numerically.

6.6 Conclusion

In this chapter, we consider the buyer's problem of combining multiple secondary spectrum to obtained more stable transmission. The buyer maximizes a combination of mean throughput and negative of throughput variance. We solve the buyer's optimization problem with and without the reference market. Base on the result of the buyer's consideration, we find the optimal pricing plan for the seller if the seller maximizes total revenue. If the seller has a cost per channel, we find the optimal pricing plan assuming both the cost and the pricing plan are proportional to the mean throughput.

CHAPTER VII

Conclusion

In this dissertation, we studied the pricing and sensing issues that arise in association with concepts such as dynamic spectrum access/sharing, open access, and secondary (spot or short-term) spectrum markets. For the pricing issue we proposed a contract design framework, and then studied two related problems: portfolio design and oligopoly market. For the sensing issue we proposed a spectrum utilization model which uses stochastic differential equations.

Specifically, we first formulate a contract design problem where a primary license holder wishes to profit from its excess spectrum capacity by selling it to potential secondary users. It needs to determine how to optimally price the excess spectrum so as to maximize its profit, knowing that this excess capacity is stochastic in nature and cannot provide deterministic service guarantees to a buyer. We adopt as a reference a traditional spectrum market where the buyer can purchase exclusive access with fixed/deterministic guarantees. The model captures the following essential features: (1) excess bandwidth on the secondary spectrum market often comes with non-exclusive use and therefore highly uncertain channel conditions; (2) incentives are built in for both the seller and the buyer to conduct spectrum trading on the secondary market. We fully characterize the optimal set of contracts the seller should provide in the case of a single buyer. When there are multiple types of buyers and each experiences different channel conditions, we construct a computationally efficient algorithm and show that the set of contracts it generates is optimal when the buyer types satisfy a monotonicity condition.

When multiple primary holders exist, we formulate a price competition model for the primary licensees selling on a secondary spectrum market. Standard results suggest that under full competition the equilibrium only exists when all sellers have zero profit. We introduce a regulator which can also be thought of as the sellers forming a coalition, whose role is to enable money transfer based on partial observations of the sellers' actions. We show that by proper design of the transfer mechanism, efficient equilibrium (profit-maximizing) can be achieved.

For the sensing issue, we propose a spectrum utilization model which uses stochastic differential equations (SDE) to model dynamic scattering and multipath fading channels, in particular, Rayleigh-distributed stationary channels. The SDE model is in closed form, can generate spectrum dynamics as a temporal process, and is shown to provide very good fit for real spectrum measurement data. We use real data collected from spectrum measurement studies to verify the SDE model and it is shown to fit the data very well. By using this model we can synthesize sample paths (temporal power process) of a wireless channel, thereby creating a realistic spectrum environment which can be used for simulation studies. The SDE model can be used to generate the 2-state Markov (GE) model (and also can generate an N-state Markov models) through time-discretization and value-quantization. This SDE-generated GE model also matches closely the GE model generated directly from the quantized data. Therefore the SDE model may be viewed as a continuous generalization of the discrete GE model (and more broadly a discrete N-state model), and while the former can be used to obtain the latter the reverse is not true. The SDE model is defined by only 3 parameters and is thus very easy and inexpensive to train with much less data compared to a discrete N-state model. It is also much more robust to imperfections in the data, e.g., when samples are not exactly collected at constant intervals. Furthermore, once the SDE model is trained, it can be used to at any desired time resolution due to its continuous-time nature, whereas an N-state Markov model would need to be retrained if one wants to reduce the size of the discrete time step (i.e., increase the time resolution).

Next, we consider the problem where the buyer purchases a portfolio of secondary spectrum services. By combining multiple secondary spectrum purchases with different randomness, the quality of transmission over the combined spectrum can be improved. The seller has a number of different spectrum channels each running its own primary service. Based on the different services, each channel has different quality measures when sold the secondary user. The seller can decide the price per bandwidth of each channel. The buyer's utility metric is a combination of mean throughput weighted with throughput variance of the portfolio purchased. We again use a reference market that sells guaranteed service to the buyer, with which a buyer can compute its optimal portfolio. Based on the knowledge of the buyer's optimal portfolio selection, we show how to calculate the optimal pricing for each secondary spectrum channel.

In Chapter V, we developed an accurate model for describing the usage of primary users viewed by the secondaries in the same channel. To connect pricing with the sensing model, we used the SDE model in Chapter VI for the channel condition in testing the performance of the channel portfolio selection as a spectrum access policy. It would be interesting to see how the incentives of the secondary users are affected if we replace the binary/uniform random variable models we used in Chapter II-III with the SDE model. The SDE model can be used in different forms that are suitable for different needs. One approach would be using it directly in the continuous time continuous valued form. If the stochastic differential equation is too hard to analyze in the differential form, the other approach would be to use the N-state Markov model derived from the SDE. The Markov model is easier to analyze and inheres the same characteristics for state evolution of the SDE model.

Describing the incentives of the secondary user will require the channel access policies used by the user because we need to know the policy in order to estimate the utility obtained by the user. The SDE model has the Markovian property because knowing the current state, the next time step of the channel does not depend on the previous states of the channel. Thus, we know that there exists a state based strategy that is optimal. If we could find the optimal channel access policy, we can better estimate the throughput/utility obtained by the secondary user in the SDE model. Then, the incentive studies of the secondary user in the SDE model would be more meaningful and accurate.

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