Krieger's Finite Generator Theorem for Ergodic Actions of Countable Groups

by

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Abstract

For an ergodic probability-measure-preserving action $G \curvearrowright (X, \mu)$ of a countable group G, we define the Rokhlin entropy $h^{\mathrm{Rok}}_G(X, \mu)$ to be the infimum of the Shannon entropies of countable generating partitions. It is known that for free ergodic actions of amenable groups this notion coincides with classical Kolmogorov–Sinai entropy. It is thus natural to view Rokhlin entropy as a close analogue to classical entropy. Under this analogy we prove that Krieger's finite generator theorem holds for all countably infinite groups. Specifically, if $h^{\mathrm{Rok}}_G(X, \mu) < \log(k)$ then there exists a generating partition consisting of k sets. Using this result, we study the properties of Rokhlin entropy as an isomorphism invariant and investigate the still unsolved isomorphism problem for Bernoulli shifts. Under the assumption that every countable group admits a free ergodic action of positive Rokhlin entropy, we prove that Bernoulli shifts having base spaces of unequal Shannon entropy are non-isomorphic and that Gottschalk's surjunctivity conjecture and Kaplansky's direct finiteness conjecture are true.

CHAPTER I

Introduction

1.1 Background

Let (X, μ) be a standard probability space, meaning X is a standard Borel space and μ is a Borel probability measure. Let G be a countably infinite group, and let $G \curvearrowright (X, \mu)$ be a probability-measure-preserving (p.m.p.) action. For a collection $\mathcal{C}$ of Borel subsets of X, we let $\sigma\text{-alg}_G(\mathcal{C})$ denote the smallest G-invariant σ-algebra containing $\mathcal{C} \cup \{X\}$ and the null sets. A Borel partition α is *generating* if $\sigma\text{-alg}_G(\alpha)$ is the entire Borel σ-algebra $\mathcal{B}(X)$. For finite $T \subseteq G$ we write α^T for the join of the translates $t \cdot \alpha$, $t \in T$, where $t \cdot \alpha = \{t \cdot A : A \in \alpha\}$. The *Shannon entropy* of a countable Borel partition α is

$$\mathrm{H}(\alpha) = \sum_{A \in \alpha} -\mu(A) \cdot \log(\mu(A)).$$

If β is a partition with $\mathrm{H}(\beta) < \infty$, then the *conditional Shannon entropy* of α relative to β is

$$\mathrm{H}(\alpha \mid \beta) = \mathrm{H}(\alpha \vee \beta) - \mathrm{H}(\beta).$$

We write $\beta \leq \alpha$ if β is coarser than α. A *probability vector* is a finite or countable ordered tuple $\bar{p} = (p_i)$ of positive real numbers which sum to 1 (a more general definition will appear in Chapter II). We write $|\bar{p}|$ for the length of $\bar{p}$ and $\mathrm{H}(\bar{p}) = \sum -p_i \cdot \log(p_i)$ for the Shannon entropy of $\bar{p}$.

Generating partitions are frequently encountered in the study of entropy theory. If G is a countable amenable group and $G \curvearrowright (X, \mu)$ is a p.m.p. action, then the classical Kolmogorov–Sinai entropy of the action is defined as

$$h_G(X, \mu) = \sup_{\substack{\beta \\ \text{finite partition}}} \inf_{\substack{T \subseteq G \\ T \text{ finite}}} \frac{1}{|T|} \cdot \mathrm{H}(\beta^T),$$

and the supremum is achieved by generating partitions β. Generating partitions are powerful objects in the study of entropy. They not only simplify entropy computations, but also play critical roles in the proofs of some key results such as Sinai's factor theorem and Ornstein's isomorphism theorem. Furthermore, they are not simply a tool in this setting, but rather are intimately tied to entropy as revealed by the following fundamental theorems of Rokhlin and Krieger.

Theorem (Rokhlin's generator theorem [41], 1967)**.** *If $\mathbb{Z} \curvearrowright (X, \mu)$ is a free ergodic p.m.p. action then its entropy $h_{\mathbb{Z}}(X, \mu)$ satisfies*

$$h_{\mathbb{Z}}(X, \mu) = \inf \Big\{ \mathrm{H}(\alpha) \, : \, \alpha \textit{ is a countable generating partition} \Big\}.$$

Theorem (Krieger's finite generator theorem [35], 1970)**.** *If $\mathbb{Z} \curvearrowright (X, \mu)$ is a free ergodic p.m.p. action and $h_{\mathbb{Z}}(X, \mu) < \log(k)$ then there exists a generating partition α consisting of k sets.*

Both of the above theorems were later superseded by the following result of Denker.

Theorem (Denker [14], 1974)**.** *If $\mathbb{Z} \curvearrowright (X, \mu)$ is a free ergodic p.m.p. action and $\bar{p}$ is a finite probability vector with $h_{\mathbb{Z}}(X, \mu) < \mathrm{H}(\bar{p})$, then for every $\epsilon > 0$ there is a generating partition $\alpha = \{A_0, \ldots, A_{|\bar{p}|-1}\}$ with $|\mu(A_i) - p_i| < \epsilon$ for every $0 \leq i < |\bar{p}|$.*

Grillenberger and Krengel [21] obtained a further strengthening of these results which roughly says that, under the assumptions of Denker's theorem, one can control the joint distribution of α and finitely many of its translates. In particular, they showed that under the assumptions of Denker's theorem there is a generating partition α with $\mu(A_i) = p_i$ for every $0 \leq i < |\bar{p}|$.

Over the years, Krieger's theorem acquired much fame and underwent various generalizations. In 1972, Katznelson and Weiss [24] outlined a proof of Krieger's theorem for free ergodic actions of $\mathbb{Z}^d$. Roughly a decade later, Šujan [50] stated Krieger's theorem for amenable groups but only outlined the proof. The first proof for amenable groups to appear in the literature was obtained in 1988 by Rosenthal [42] who proved Krieger's theorem under the more restrictive assumption that $h_G(X, \mu) < \log(k-2) < \log(k)$. This was not improved until 2002 when Danilenko and Park [13] proved Krieger's theorem for amenable groups under the assumption $h_G(X, \mu) < \log(k-1) < \log(k)$. It is none-the-less a folklore unpublished result that Krieger's theorem holds for amenable groups, i.e. if $G \curvearrowright (X, \mu)$ is a free ergodic p.m.p. action of an amenable group and $h_G(X, \mu) < \log(k)$ then there is a generating partition consisting of k sets. Our much more general investigations here yield this as a consequence. We believe that this is the first explicit proof of this fact. Rokhlin's theorem was generalized to actions of abelian groups by Conze [12] in 1972 and was just recently extended to amenable groups by Seward and Tucker-Drob [48]. Specifically, if $G \curvearrowright (X, \mu)$ is a free ergodic p.m.p. action of an amenable group then the entropy $h_G(X, \mu)$ is equal to the infimum of $\mathrm{H}(\alpha)$ over all countable generating partitions α. Denker's theorem on the other hand has not been extended beyond actions of $\mathbb{Z}$.

Outside of the realm of amenable groups, a new entropy theory is beginning to emerge. Specifically, Bowen [6] recently introduced the notion of sofic entropy for p.m.p. actions of sofic groups, and his definition was improved and generalized by Kerr

[27] and Kerr–Li [29]. We remind the reader that the class of sofic groups contains the countable amenable groups, and it is an open question whether every countable group is sofic. Sofic entropy extends classical entropy, as when the acting sofic group is amenable the two notions coincide [7, 30]. Generating partitions continue to play an important role in this theory, as sofic entropy is easier to compute when one has a finite generating partition. Bowen [6, 8] has extended much of Ornstein's isomorphism theorem to this new setting, however the status of Sinai's factor theorem and Ornstein theory are unknown, and new techniques for generating partitions must be developed in order to move forward. Additionally, the following questions remain open.

Question I.1.

(1) Does sofic entropy satisfy Rokhlin's generator theorem when the sofic entropy is not $-\infty$?

(2) Does sofic entropy satisfy Krieger's finite generator theorem when the sofic entropy is not $-\infty$?

The most well known application of entropy is the classification of Bernoulli shifts over $\mathbb{Z}$ up to isomorphism. This application in fact lies at the root of its conception by Kolmogorov in 1958 [33, 34]. Bernoulli shifts were classified over $\mathbb{Z}$ by Ornstein in 1970 [36, 37], over amenable groups by Ornstein–Weiss in 1987 [39], and recently classified over many sofic groups by Bowen [6, 8] and Kerr–Li [31]. Nevertheless, the following fundamental problem has not yet been settled.

Question I.2. For every countably infinite group G, are the Bernoulli shifts (L^G, λ^G) classified up to isomorphism by the Shannon entropy $\mathrm{H}(L, \lambda)$ of their base space?

In summary, generating partitions played a critical role in classical entropy theory and need to be further studied in the non-amenable setting for the development of sofic entropy theory. Additionally, if non-sofic groups exist then a new entropy-style invariant may be needed in order to complete the classification of Bernoulli shifts. Drawing motivation from these issues, we introduce the following natural isomorphism invariant. For an ergodic p.m.p. action $G \curvearrowright (X, \mu)$ we define the *Rokhlin entropy* as

$$h_G^{\mathrm{Rok}}(X, \mu) = \inf \Big\{ \mathrm{H}(\alpha) \, : \, \alpha \text{ is a countable Borel generating partition} \Big\}.$$

This invariant is named in honor of Rokhlin's generator theorem. For free ergodic actions of amenable groups, Rokhlin's generator theorem [48] says that Rokhlin entropy is identical to classical entropy. Thus, Rokhlin entropy may be viewed as a close analogue to entropy.

1.2 The main theorem

Our main theorem is the following generalization of Krieger's finite generator theorem.

Theorem I.3. *Let G be a countably infinite group acting ergodically, but not necessarily freely, by measure-preserving bijections on a non-atomic standard probability space (X, μ). If $\bar{p} = (p_i)$ is any finite or countable probability vector with $h_G^{\mathrm{Rok}}(X, \mu) < \mathrm{H}(\bar{p})$, then there is a generating partition $\alpha = \{A_i \, : \, 0 \leq i < |\bar{p}|\}$ with $\mu(A_i) = p_i$ for every $0 \leq i < |\bar{p}|$.*

This theorem supersedes previous work of the author in [46] which, under the assumption $h_G^{\mathrm{Rok}}(X, \mu) < \infty$, constructed a finite generating partition without any control over its cardinality or distribution. The major difficulty which the present work overcomes is that all prior proofs of Krieger's theorem relied critically upon the classical Rokhlin lemma and Shannon–McMillan–Breiman theorem, and these tools do not exist for actions of general countable groups.

We remark that in order for a partition α to exist as described in Theorem I.3, it is necessary that $h_G^{\mathrm{Rok}}(X,\mu) \leq \mathrm{H}(\bar{p})$. So the above theorem is optimal since in general there are actions where the infimum $h_G^{\mathrm{Rok}}(X,\mu)$ is not achieved, such as free ergodic actions which are not isomorphic to any Bernoulli shift (see Corollary I.10 below).

If $h_G^{\mathrm{Rok}}(X,\mu) < \log(k)$ then using $\bar{p} = (p_0, \ldots, p_{k-1})$ where each $p_i = 1/k$ we obtain the following:

Corollary I.4. *Let G be a countably infinite group acting ergodically, but not necessarily freely, by measure-preserving bijections on a non-atomic standard probability space (X,μ). If $h_G^{\mathrm{Rok}}(X,\mu) < \log(k)$, then there is a generating partition α with $|\alpha| = k$.*

We mention that Corollary I.4 is the first non-free action version of Krieger's finite generator theorem. Furthermore, we believe that Corollary I.4 (together with the Rokhlin generator theorem for amenable groups [48]) is the first explicit proof of Krieger's finite generator theorem for free ergodic actions of countable amenable groups. In fact, we obtain the following strong form of Denker's theorem for amenable groups:

Corollary I.5. *Let G be a countably infinite amenable group and let $G \curvearrowright (X,\mu)$ be a free ergodic p.m.p. action. If $\bar{p} = (p_i)$ is any finite or countable probability vector with $h_G(X,\mu) < \mathrm{H}(\bar{p})$ then there exists a generating partition $\alpha = \{A_i : 0 \leq i < |\bar{p}|\}$ with $\mu(A_i) = p_i$ for every $0 \leq i < |\bar{p}|$.*

We point out that Theorem I.3 shows that a positive answer to Question I.1.(1) implies a positive answer to I.1.(2).

Rather than proving Theorem I.3 directly, we instead prove a stronger but more technical result which is a generalization of the "relative" Krieger finite generator theorem. The relative version of Krieger's theorem for $\mathbb{Z}$ actions was first proven by Kifer and Weiss [32] in 2002. It states that if $\mathbb{Z} \curvearrowright (X, \mu)$ is a free ergodic p.m.p. action, $\mathcal{F}$ is a $\mathbb{Z}$-invariant sub-σ-algebra, and the relative entropy satisfies $h_{\mathbb{Z}}(X, \mu | \mathcal{F}) < \log(k)$, then there is a Borel partition α consisting of k sets such that $\sigma\text{-alg}_G(\alpha) \vee \mathcal{F}$ is the entire Borel σ-algebra $\mathcal{B}(X)$. This result was later extended by Danilenko and Park [13] to free ergodic actions of amenable groups under the assumption that $\mathcal{F}$ induces a class-bijective factor.

For a p.m.p. ergodic action $G \curvearrowright (X, \mu)$ and a G-invariant sub-σ-algebra $\mathcal{F}$, we define the *relative Rokhlin entropy* $h_G^{\mathrm{Rok}}(X, \mu | \mathcal{F})$ to be

$$\inf \Big\{ \mathrm{H}(\alpha | \mathcal{F}) \, : \, \alpha \text{ is a countable Borel partition and } \sigma\text{-alg}_G(\alpha) \vee \mathcal{F} = \mathcal{B}(X) \Big\}.$$

We refer the reader to Chapter II for the definition of the conditional Shannon entropy $\mathrm{H}(\alpha | \mathcal{F})$, but we remark that when $\mathcal{F} = \{X, \varnothing\}$ we have $\mathrm{H}(\alpha | \mathcal{F}) = \mathrm{H}(\alpha)$. We observe in Proposition X.1 that for free ergodic actions of amenable groups the relative Rokhlin entropy coincides with relative Kolmogorov–Sinai entropy. Similar to the Rudolph–Weiss theorem [43], we observe in Proposition III.4 that $h_G^{\mathrm{Rok}}(X, \mu | \mathcal{F})$ is invariant under orbit equivalences for which the orbit-change cocycle is $\mathcal{F}$-measurable.

Before stating the stronger version of our main theorem, we introduce some additional terminology. A *pre-partition* of X is a countable collection of pairwise-disjoint subsets of X. We say that another pre-partition β extends α, written $\beta \sqsupseteq \alpha$, if there is an injection $\iota : \alpha \to \beta$ with $A \subseteq \iota(A)$ for every $A \in \alpha$. Equivalently, $\beta \sqsupseteq \alpha$ if and only if the restriction of β to $\cup \alpha$ coincides with α.

For a Borel pre-partition α, we define the *reduced σ-algebra* $\sigma\text{-alg}_G^{\mathrm{red}}(\alpha)$ to be the collection of Borel sets $R \subseteq X$ such that there is a conull $X' \subseteq X$ satisfying:

$$\text{for every } r \in R \cap X' \text{ and } x \in X' \setminus R \text{ there is } g \in G \text{ with } g \cdot r, g \cdot x \in \cup \alpha$$

and with $g \cdot r$ and $g \cdot x$ lying in distinct classes of α.

It is a basic exercise to verify that $\sigma\text{-alg}^{\text{red}}_G(\alpha)$ is indeed a σ-algebra.

The definition of reduced σ-algebra may seem a bit odd at first, but comes about naturally from our work here and significantly simplifies the proof of Theorem I.16 below. A key property of this definition is that if β is any partition extending α then one automatically has $\sigma\text{-alg}_G(\beta) \supseteq \sigma\text{-alg}^{\text{red}}_G(\alpha)$ (Lemma II.5). Another important property is that if $G \curvearrowright (Y, \nu)$ is a factor of (X, μ) via $\phi : (X, \mu) \to (Y, \nu)$, then for any pre-partition α of Y we have $\sigma\text{-alg}^{\text{red}}_G(\phi^{-1}(\alpha)) = \phi^{-1}(\sigma\text{-alg}^{\text{red}}_G(\alpha))$ (Lemma II.6). These properties can be quite useful for specialized constructions. For example, one could imagine constructing two pre-partitions α^1 and α^2 which achieve different goals. If $\cup\alpha^1$ is disjoint from $\cup\alpha^2$, then one can choose a common extension partition α and automatically have $\sigma\text{-alg}_G(\alpha) \supseteq \sigma\text{-alg}^{\text{red}}_G(\alpha^1) \vee \sigma\text{-alg}^{\text{red}}_G(\alpha^2)$.

Theorem I.6. *Let G be a countably infinite group acting ergodically, but not necessarily freely, by measure-preserving bijections on a non-atomic standard probability space (X, μ). Let $\mathcal{F}$ be a G-invariant sub-σ-algebra of X. If $0 < r \leq 1$ and $\bar{p} = (p_i)$ is any finite or countable probability vector with $h^{\text{Rok}}_G(X, \mu | \mathcal{F}) < r \cdot \mathrm{H}(\bar{p})$, then there is a Borel pre-partition $\alpha = \{A_i : 0 \leq i < |\bar{p}|\}$ with $\mu(\cup\alpha) = r$, $\mu(A_i) = r \cdot p_i$ for every $0 \leq i < |\bar{p}|$, and $\sigma\text{-alg}^{red}_G(\alpha) \vee \mathcal{F} = \mathcal{B}(X)$.*

The above result is new even in the case $G = \mathbb{Z}$ and $\mathcal{F} = \{X, \varnothing\}$. We mention that the parameter r is needed for some of our later results. With $r = 1$, this result strengthens the prior versions of the relative Krieger finite generator theorem, and with $\mathcal{F} = \{X, \varnothing\}$ it implies Theorem I.3. We point out that we do not assume any properties of $\mathcal{F}$, and in particular we do not require that $\mathcal{F}$ induce a class-bijective factor.

Observe that by using $r = 1$, Theorem I.6 implies that we may use $\mathrm{H}(\alpha)$ in place of $\mathrm{H}(\alpha|\mathcal{F})$ in the definition of $h_G^{\mathrm{Rok}}(X, \mu|\mathcal{F})$. From this observation, we deduce the following sub-additive identity.

Corollary I.7. *Let G be a countably infinite group acting ergodically, but not necessarily freely, by measure-preserving bijections on a non-atomic standard probability space (X, μ). If $G \curvearrowright (Y, \nu)$ is a factor of $G \curvearrowright (X, \mu)$ and $\mathcal{F}$ is the sub-σ-algebra of X associated to Y then*

$$h_G^{\mathrm{Rok}}(X, \mu) \leq h_G^{\mathrm{Rok}}(Y, \nu) + h_G^{\mathrm{Rok}}(X, \mu|\mathcal{F}).$$

The inequality above can be strict, for example when $h_G^{\mathrm{Rok}}(X, \mu) < h_G^{\mathrm{Rok}}(Y, \nu)$. A strict inequality is common for actions of non-amenable groups [47].

1.3 Applications

We use Theorem I.6 to study the Rokhlin entropy of Bernoulli shifts and investigate Question I.2. Recall that for a standard probability space (L, λ) the *Bernoulli shift* over G with *base space* (L, λ) is simply the product space (L^G, λ^G) equipped with the natural left-shift action of G:

$$\text{for } g, h \in G \text{ and } x \in L^G \quad (g \cdot x)(h) = x(g^{-1}h).$$

The *Shannon entropy* of the base space is

$$\mathrm{H}(L, \lambda) = \sum_{\ell \in L} -\lambda(\ell) \cdot \log \lambda(\ell)$$

if λ has countable support, and $\mathrm{H}(L, \lambda) = \infty$ otherwise. Every Bernoulli shift (L^G, λ^G) comes with the canonical, possibly uncountable, generating partition $\mathscr{L} = \{R_\ell : \ell \in L\}$, where

$$R_\ell = \{x \in L^G : x(1_G) = \ell\}.$$

Note that if $\mathrm{H}(L,\lambda) < \infty$ then $\mathscr{L}$ is countable and $\mathrm{H}(\mathscr{L}) = \mathrm{H}(L,\lambda)$. Thus one always has $h_G^{\mathrm{Rok}}(L^G,\lambda^G) \leq \mathrm{H}(L,\lambda)$.

A fundamental open problem in ergodic theory is to determine, for every countably infinite group G, whether $(2^G, u_2^G)$ can be isomorphic to $(3^G, u_3^G)$. Here we write n for $\{0,\ldots,n-1\}$ and u_n for the normalized counting measure on $\{0,\ldots,n-1\}$. Note that $\mathrm{H}(n,u_n) = \log(n)$. For amenable groups G, the Bernoulli shift (L^G,λ^G) has Kolmogorov–Sinai entropy $\mathrm{H}(L,\lambda)$, and thus $(2^G, u_2^G)$ and $(3^G, u_3^G)$ are non-isomorphic. In 2010, groundbreaking work of Bowen [6], together with improvements by Kerr and Li [29], created a notion of sofic entropy for p.m.p. actions of sofic groups. For sofic G, the Bernoulli shift (L^G,λ^G) has sofic entropy $\mathrm{H}(L,\lambda)$ [6, 31]. Thus $(2^G, u_2^G)$ and $(3^G, u_3^G)$ are non-isomorphic for sofic G. Based on these results, it seems that the following statement may be true of all countably infinite groups G:

INV : $\mathrm{H}(L,\lambda)$ is an isomorphism invariant for (L^G,λ^G).

Remark I.8. Another important question is whether $\mathrm{H}(L,\lambda) = \mathrm{H}(K,\kappa)$ implies that (L^G,λ^G) is isomorphic to (K^G,κ^G). In 1970, Ornstein famously answered this question positively for $G = \mathbb{Z}$, thus completely classifying Bernoulli shifts over $\mathbb{Z}$ up to isomorphism [36, 37]. This result was extended to amenable groups by Ornstein and Weiss in 1987 [39]. Work of Stepin shows that this property is retained under passage to supergroups [49], so the isomorphism result extends to all groups which contain an infinite amenable subgroup. In 2012, Bowen proved that for every countably infinite group G, if $\mathrm{H}(L,\lambda) = \mathrm{H}(K,\kappa)$ and the supports of λ and κ each have cardinality at least 3, then (L^G,λ^G) is isomorphic to (K^G,κ^G) [8]. Thus, this question is nearly resolved with only the case of a two atom base space incomplete.

We previously noted that one always has $h_G^{\mathrm{Rok}}(L^G,\lambda^G) \leq \mathrm{H}(L,\lambda)$. When G is sofic, Rokhlin entropy is bounded below by sofic entropy and thus $h^{\mathrm{Rok}}_G(L^G, \lambda^G) = \mathrm{H}(L, \lambda)$ whenever G is sofic. Since the definition of Rokhlin entropy does not require the acting group to be sofic, the statement

$$\mathbf{RBS} : h^{\mathrm{Rok}}_G(L^G, \lambda^G) = \mathrm{H}(L, \lambda) \text{ for every standard probability space } (L, \lambda).$$

(acronym for Rokhlin entropy of Bernoulli Shifts) may be true for all countably infinite groups G. Notice that $\mathbf{RBS} \Rightarrow \mathbf{INV}$.

We investigate **RBS** and along the way we further develop the theory of Rokhlin entropy. The canonical generating partition $\mathscr{L}$ of (L^G, λ^G) has the property that its translates are mutually independent. Our first result uses the joint distributions of translates of a generating partition in order to bound Rokhlin entropy.

Theorem I.9. *Let G be a countably infinite group, let $G \curvearrowright (X, \mu)$ be a free p.m.p. ergodic action, and let α be a countable generating partition. If $T \subseteq G$ is finite, $\epsilon > 0$, and $\frac{1}{|T|} \cdot \mathrm{H}(\alpha^T) < \mathrm{H}(\alpha) - \epsilon$, then $h^{\mathrm{Rok}}_G(X, \mu) < \mathrm{H}(\alpha) - \epsilon/(16|T|^3)$.*

Since the equality $\mathrm{H}(\alpha^T) = |T| \cdot \mathrm{H}(\alpha)$ implies that the T-translates of α are mutually independent when $\mathrm{H}(\alpha) < \infty$, we obtain the following.

Corollary I.10. *Let G be a countably infinite group acting freely and ergodically on a standard probability space (X, μ) by measure-preserving bijections. If α is a countable generating partition and*

$$h^{\mathrm{Rok}}_G(X, \mu) = \mathrm{H}(\alpha) < \infty,$$

then (X, μ) is isomorphic to a Bernoulli shift.

As the sofic entropy of an ergodic action is always bounded above by Rokhlin entropy [6], we have the following immediate corollary.

Corollary I.11. *Let G be a sofic group with sofic approximation Σ, and let G act freely and ergodically on a standard probability space (X, μ) by measure-preserving* *bijections. If α is a countable generating partition and the sofic entropy $h_G^\Sigma(X, \mu)$ satisfies $h_G^\Sigma(X, \mu) = \mathrm{H}(\alpha) < \infty$, then (X, μ) is isomorphic to a Bernoulli shift.*

From Theorem I.9 we derive a few properties which would follow if **RBS** were found to be true. Recall that an action $G \curvearrowright (X, \mu)$ of an amenable group G is said to have *completely positive entropy* if every factor $G \curvearrowright (Y, \nu)$ of (X, μ), with Y not essentially a single point, has positive Kolmogorov–Sinai entropy. For $G = \mathbb{Z}$, these actions are also called Kolmogorov or K-automorphisms. The standard example of completely positive entropy actions are Bernoulli shifts (see [43]). In fact, for amenable groups factors of Bernoulli shifts are Bernoulli [39], but it is unknown if this holds for any non-amenable group. Recently, it was proven by Kerr that Bernoulli shifts over sofic groups have completely positive sofic entropy [28]. Along these lines, we obtain the following corollary of Theorem I.9.

Corollary I.12. *Let G be a countably infinite group. Assume that $h_G^{\mathrm{Rok}}(L^G, \lambda^G) = \mathrm{H}(L, \lambda)$ for all standard probability spaces (L, λ). Then every Bernoulli shift over G has completely positive Rokhlin entropy.*

Our next corollary relates to two well-known open conjectures from outside ergodic theory. The first is *Kaplansky's direct finiteness conjecture*, which states that for every countable group G and every field K, if a and b are elements of the group ring $K[G]$ and satisfy $ab = 1$ then $ba = 1$. Kaplansky proved this for $K = \mathbb{C}$ in 1972 [23] (see also a shorter proof by Burger and Valette [10]). For general fields K, this conjecture was proven for abelian groups by Ara, O'Meara, and Perera in 2002 [2], and then proven for sofic groups by Elek and Szabó in 2004 [16]. This conjecture has also been verified for some groups whose soficity is currently unknown [52, 3].

The second conjecture is *Gottschalk's surjunctivity conjecture*, which states that if G is a countable group, $n \in \mathbb{N}$, and $\phi : n^G \to n^G$ is a continuous G-equivariant injection, then ϕ is surjective. This conjecture has a simple topological proof when G is residually finite (this is due to Lawton, see [20] or [54]), and can be proven for amenable groups using topological entropy. Gromov proved the conjecture for sofic groups, and in fact he defined the class of sofic groups for this purpose [22, 54]. Later, after the discovery of sofic entropy, a topological entropy proof was given for sofic groups [29]. We point out that it is known that Gottschalk's surjunctivity conjecture implies Kaplansky's direct finiteness conjecture [11, Section I.5].

From Corollary I.10 we deduce the following.

Corollary I.13. *Let G be a countably infinite group. Assume that $h_G^{\mathrm{Rok}}(L^G, \lambda^G) = \mathrm{H}(L, \lambda)$ for all standard probability spaces (L, λ). Then G satisfies Gottschalk's surjunctivity conjecture and Kaplansky's direct finiteness conjecture.*

If we define the statements

CPE : Every Bernoulli shift over G has completely positive Rokhlin entropy.

GOT : G satisfies Gottschalk's surjunctivity conjecture.

KAP : G satisfies Kaplansky's direct finiteness conjecture.

then from earlier comments and Corollaries I.12 and I.13 we deduce that for every countably infinite group G

$$\mathbf{RBS} \;\Rightarrow\; \mathbf{INV} + \mathbf{CPE} + \mathbf{GOT} + \mathbf{KAP}.$$

We now turn our attention to the validity of **RBS**. A priori, there is nothing obvious one can say about $h_G^{\mathrm{Rok}}(L^G, \lambda^G)$ except that

$$h_G^{\mathrm{Rok}}((L \times K)^G, (\lambda \times \kappa)^G) \leq h_G^{\mathrm{Rok}}(L^G, \lambda^G) + h_G^{\mathrm{Rok}}(K^G, \kappa^G) \leq \mathrm{H}(L, \lambda) + \mathrm{H}(K, \kappa).$$

Indeed, we do not know if Rokhlin entropy is additive under direct products, even for Bernoulli shifts.

For a countably infinite group G, define

$$h^{\mathrm{Rok}}_{sup}(G) = \sup_{G \curvearrowright (X,\mu)} h^{\mathrm{Rok}}_G(X, \mu),$$

where the supremum is taken over all free ergodic p.m.p. actions $G \curvearrowright (X, \mu)$ with $h^{\mathrm{Rok}}_G(X, \mu) < \infty$. For non-sofic groups G, we do not know if either of the following two statements are true.

POS : There is a free ergodic p.m.p. action $G \curvearrowright (X, \mu)$ with $h^{\mathrm{Rok}}_G(X, \mu) > 0$.

INF : $h^{\mathrm{Rok}}_{sup}(G) = \infty$.

In order to study **RBS**, we first use Theorem I.6 in order to develop the following analog of the classical Kolmogorov–Sinai theorem from entropy theory. Recall that if G is amenable then the Kolmogorov–Sinai theorem states that the Kolmogorov–Sinai entropy $h_G(X, \mu)$ of $G \curvearrowright (X, \mu)$ satisfies

$$h_G(X, \mu) = \sup_{\alpha} \inf_{\substack{T \subseteq G \\ T \text{ finite}}} \frac{1}{|T|} \cdot \mathrm{H}(\alpha^T),$$

where the supremum is over all countable partitions α with $\mathrm{H}(\alpha) < \infty$.

Theorem I.14. *Let G be a countable group acting ergodically, but not necessarily freely, by measure-preserving bijections on a standard probability space (X, μ). Let $(\alpha_n)_{n \in \mathbb{N}}$ be an increasing sequence of partitions with $\mathrm{H}(\alpha_n) < \infty$ and $\mathcal{B}(X) = \bigvee_{n \in \mathbb{N}} \sigma\text{-}\mathrm{alg}_G(\alpha_n)$. If*

$$\inf_{n \in \mathbb{N}} \lim_{\epsilon \to 0} \sup_{m \in \mathbb{N}} \inf_{k \in \mathbb{N}} \inf_{\substack{T \subseteq G \\ T \text{ finite}}} \inf \left\{ \mathrm{H}(\beta \mid \alpha_n^T) \, : \, \beta \leq \alpha_k^T, \ \mathrm{H}(\alpha_m \mid \beta^T) < \epsilon \right\}$$

is positive then $h^{\mathrm{Rok}}_G(X, \mu) = \infty$. On the other hand, if the expression above is equal *to* 0 *then*

$$h_G^{\mathrm{Rok}}(X,\mu) = \lim_{\epsilon\to 0} \sup_{m\in\mathbb{N}} \inf_{k\in\mathbb{N}} \inf_{\substack{T\subseteq G \\ T \text{ finite}}} \inf \Big\{ \mathrm{H}(\beta) \ : \ \beta \leq \alpha_k^T, \ \mathrm{H}(\alpha_m \mid \beta^T) < \epsilon \Big\}.$$

We do not know if requiring the first expression in Theorem I.14 to be 0 is superfluous. Although the connection may not be obvious, this is closely related to whether **POS** implies **INF** (see the discussion following Corollary XV.7).

The main utility of Theorem I.14 is that it reveals new properties of Rokhlin entropy (in addition to the corollary below, see also Corollaries XIII.4, XIII.5, XIII.7, XIII.8, and XIII.9). This theorem and its corollaries are important ingredients to our main theorems.

Corollary I.15. *Let G be a countable group, let L be a finite set, and let L^G have the product topology. Then the map taking invariant ergodic Borel probability measures μ to $h_G^{\mathrm{Rok}}(L^G,\mu)$ is upper-semicontinuous in the weak*-topology.*

We investigate **RBS** by an approximation argument via Corollary I.15. The required ingredient is the construction of generating partitions α which are almost Bernoulli in the sense that $\mathrm{H}(\alpha^T)/|T| > \mathrm{H}(\alpha) - \epsilon$ for some large but finite $T \subseteq G$ and some small $\epsilon > 0$. By well known properties of Shannon entropy [15, Fact 3.1.3], this condition is equivalent to saying that the T-translates of α are close to being mutually independent. This theorem may be viewed as a generalization of a similar result obtained by Grillenberger and Krengel for $G = \mathbb{Z}$ [21].

Theorem I.16. *Let G be a countably infinite group acting freely and ergodically on a standard probability space (X,μ) by measure-preserving bijections. If $\bar{p} = (p_i)$ is any finite or countable probability vector with $h_G^{\mathrm{Rok}}(X,\mu) < \mathrm{H}(\bar{p}) < \infty$, then for every finite $T \subseteq G$ and $\epsilon > 0$ there is a generating partition $\alpha = \{A_i \ : \ 0 \leq i < |\bar{p}|\}$ with*

$\mu(A_i) = p_i$ for every $0 \leq i < |\bar{p}|$ and

$$\frac{1}{|T|} \cdot \mathrm{H}(\alpha^T) > \mathrm{H}(\alpha) - \epsilon.$$

The proof of Theorem I.3, upon which the above result is based, takes place almost exclusively within the pseudo-group of the induced orbit equivalence relation. It is therefore a bit unexpected that we are able to control the interaction among the T-translates of α in the above theorem.

The above theorem strengthens the result of Abért and Weiss that all free actions weakly contain a Bernoulli shift [1]. Specifically, assuming only that $\mathrm{H}(\bar{p}) > 0$, they proved the existence of an α which is not necessarily generating but otherwise satisfies the conditions stated in Theorem I.16.

Theorem I.16 allows us to investigate **RBS** for $\mathrm{H}(L, \lambda) < \infty$.

Theorem I.17. *Let G be a countably infinite group and let (L, λ) be a standard probability space with $\mathrm{H}(L, \lambda) < \infty$. Then*

$$h_G^{\mathrm{Rok}}(L^G, \lambda^G) = \min\Big(\mathrm{H}(L, \lambda),\ h_{sup}^{\mathrm{Rok}}(G)\Big).$$

Note that when $h_G^{\mathrm{Rok}}(L^G, \lambda^G) < \mathrm{H}(L, \lambda)$, the supremum $h_{sup}^{\mathrm{Rok}}(G)$ is achieved by (L^G, λ^G). We point out that the above theorem places a significant restriction on the nature of the map $\mathrm{H}(L, \lambda) \mapsto h_G^{\mathrm{Rok}}(L^G, \lambda^G)$. Prior to obtaining this theorem, there is no obvious reason why this map should be monotone or even piece-wise linear.

Next we consider the case $\mathrm{H}(L, \lambda) = \infty$. In this case we obtain a result stronger than Theorem I.17. This is surprising from a historical perspective, since when Kolmogorov defined entropy in 1958 he could only handle Bernoulli shifts with a finite Shannon entropy base [33, 34]. It was not until the improvements of Sinai that infinite Shannon entropy bases could be considered [44]. Similarly, when Bowen defined sofic entropy he studied Bernoulli shifts with both finite and infinite Shannon entropy bases [6], but he was only fully successful in the finite case. The infinite case was resolved through improvements by Kerr and Li [29, 31, 27].

Theorem I.18. *Let G be a countably infinite group and let (L, λ) be a standard probability space with $\mathrm{H}(L, \lambda) = \infty$. Then $h_G^{\mathrm{Rok}}(L^G, \lambda^G) = \infty$ if and only if there exists a free ergodic p.m.p. action $G \curvearrowright (X, \mu)$ with $h_G^{\mathrm{Rok}}(X, \mu) > 0$.*

Thus, if $\mathrm{H}(L, \lambda) = \infty$ then $h_G^{\mathrm{Rok}}(L^G, \lambda^G)$ is either 0 or infinity.

It follows from Theorems I.17 and I.18 that for every countably infinite group G

$$\mathbf{INF} \Rightarrow \mathbf{RBS}.$$

Theorem I.19. *Let P be a countable group containing arbitrarily large finite subgroups. If G is any countably infinite group with $h_{sup}^{\mathrm{Rok}}(G) < \infty$ then $h_{sup}^{\mathrm{Rok}}(P \times G) = 0$.*

Thus $(\forall G \ \mathbf{POS}) \Rightarrow (\forall G \ \mathbf{INF})$. Putting all of our results together, we obtain the following.

Corollary I.20. *Assume that every countably infinite group G admits a free ergodic p.m.p. action with $h_G^{\mathrm{Rok}}(X, \mu) > 0$. Then:*

(i) $h_G^{\mathrm{Rok}}(L^G, \lambda^G) = \mathrm{H}(L, \lambda)$ for every countably infinite group G and every probability space (L, λ);

(ii) Every Bernoulli shift over any countably infinite group has completely positive Rokhlin entropy;

(iii) Gottschalk's surjunctivity conjecture is true;

(iv) Kaplansky's direct finiteness conjecture is true.

This corollary indicates that the validity of ($\forall G$ **POS**) should be considered an important open problem.

Finally, for convenience to the reader we summarize the implications we uncovered in the two lines below:

$$\mathbf{INF} \Rightarrow \mathbf{RBS} \Rightarrow \mathbf{INV} + \mathbf{CPE} + \mathbf{GOT} + \mathbf{KAP}$$

$$(\forall G\ \mathbf{POS}) \Rightarrow (\forall G\ \mathbf{INF}).$$

1.4 Outline

The proof of Theorem I.6 is entirely self-contained and only uses the definition of ergodicity, standard properties of Shannon entropy, and Stirling's formula. The proof generally ignores the action of the group but instead works almost exclusively within the pseudo-group of the induced orbit-equivalence relation. We review basic properties of the pseudo-group in Chapter III. The important advantage of working within the pseudo-group is that we are able to obtain a suitable replacement to both the Rokhlin lemma and the Shannon–McMillan–Breiman theorem. We present this replacement in Chapter IV. A significant difficulty of working within the pseudo-group is that the notion of "generating" partition is lost. In a spirit somewhat similar to work of Rudolph–Weiss [43], we must maintain careful control over sub-σ-algebras and the measurability properties of cocycles which relate elements of the pseudo-group to the action of G. This is the most challenging part of the proof, and it is essentially the only time when we must use the original group action. The coding machinery needed for this task is presented in Chapters V and VI. One final main ingredient is a procedure for replacing countably infinite partitions with finite ones. This procedure originates from prior work of the author in [46] and is presented in Chapter VII. In Chapter VIII we review a few well known counting lemmas related to Shannon entropy. Then in Chapter IX we collect our tools together and mimic the classical proof of Krieger's finite generator theorem and thus establish Theorem I.6.

We remark that if one is only interested in obtaining a finite generating partition, then only Chapters II, III, and VII are needed. Indeed, the latter two chapters essentially recreate the proof of this fact by the author in [46]. The novelty of Theorem I.6 is its precise control over the cardinality and distribution of the generating partition, and the new ideas needed for this stronger result are the content of Chapters IV, V, and VI.

In Chapter X we show that relative Rokhlin entropy and relative Kolmogorov–Sinai entropy coincide. We review the Rokhlin metric on the space of partitions and some of its basic properties in Chapter XI. Then in Chapter XII we study the joint distributions among translates of a given generating partition and prove Theorem I.9. This chapter also contains the proofs of Corollaries I.12 and I.13. Next we study computability aspects of Rokhlin entropy and present the proof of Theorem I.14 in Chapter XIII. Chapters XII and XIII not only develop important properties of Rokhlin entropy, but also serve as vital steps towards the study of the Rokhlin entropy of Bernoulli shifts. In Chapter XIV we construct generating partitions which are approximately Bernoulli and establish Theorem I.16. We are then able to study the Rokhlin entropy of Bernoulli shifts in Chapter XV and prove Theorems I.17, I.18, and I.19.

CHAPTER II

Preliminaries

Let (X, μ) be a standard probability space. For $\mathcal{C} \subseteq \mathcal{B}(X)$, we let $\sigma\text{-alg}(\mathcal{C})$ denote the smallest sub-σ-algebra containing $\mathcal{C} \cup \{X\}$ and the μ-null sets (not to be confused with the notation $\sigma\text{-alg}_G(\mathcal{C})$ from the introduction). For a collection of partitions α_i, we let $\bigvee_{i \in I} \alpha_i$ denote the coarsest partition finer than every α_i. Note that $\bigvee_{i \in I} \alpha_i$ may be uncountable. Similarly, for a collection of sub-σ-algebras $\mathcal{F}_i$, we let $\bigvee_{i \in I} \mathcal{F}_i$ denote the smallest σ-algebra containing every $\mathcal{F}_i$.

Every probability space (X, μ) which we consider will be assumed to be standard. In particular, X will be a standard Borel space. A well-known property of standard Borel spaces is that they are countably generated [25, Prop. 12.1], meaning there is a sequence $B_n \subseteq X$ of Borel sets such that $\mathcal{B}(X)$ is the smallest σ-algebra containing all of the sets B_n. This implies that there is an increasing sequence α_n of finite Borel partitions of X such that $\mathcal{B}(X) = \bigvee_{n \in \mathbb{N}} \sigma\text{-alg}(\alpha_n)$.

Throughout this paper, whenever working with a probability space (X, μ) we will generally ignore sets of measure zero. In particular, we write $A = B$ for $A, B \subseteq X$ if their symmetric difference is null. Also, by a partition of X we will mean a collection of pairwise-disjoint Borel sets whose union is conull. In particular, we allow partitions to contain the empty set. Similarly, we will use the term *probability vector* more freely than described in the introduction. A probability vector $\bar{p} = (p_i)$ will be any finite or countable ordered tuple of non-negative real numbers which sum to 1 (so some terms p_i may be 0). We say that another probability vector $\bar{q}$ is *coarser* than $\bar{p}$ if there is a partition $\mathcal{Q} = \{Q_j : 0 \leq j < |\bar{q}|\}$ of the integers $\{0 \leq i < |\bar{p}|\}$ such that for every $0 \leq j < |\bar{q}|$

$$q_j = \sum_{i \in Q_j} p_i.$$

For a countable ordered partition $\alpha = \{A_i : 0 \leq i < |\alpha|\}$ we let $\mathrm{dist}(\alpha)$ denote the probability vector $\bar{p}$ satisfying $p_i = \mu(A_i)$. For two partitions α and β, we say β is *coarser* than α, or α is *finer* than β, written $\beta \leq \alpha$, if every $B \in \beta$ is the union of classes of α. We let $\mathscr{P}_{\mathrm{H}}$ denote the set of countable Borel partitions α with $\mathrm{H}(\alpha) < \infty$. The space $\mathscr{P}_{\mathrm{H}}$ is a complete separable metric space [15, Fact 1.7.15] under the *Rokhlin metric* d_μ^{Rok} defined by

$$d_\mu^{\mathrm{Rok}}(\alpha, \beta) = \mathrm{H}(\alpha \mid \beta) + \mathrm{H}(\beta \mid \alpha).$$

At times, we will consider the space of all Borel probability measures on X. Recall that the space of Borel probability measures on X has a natural standard Borel structure which is generated by the maps $\mu \mapsto \mu(A)$ for $A \subseteq X$ Borel [25, Theorem 17.24]. If X is furthermore a compact space, then we equip the space of Borel probability measures on X with the weak*-topology. This topology is defined to be the weakest topology such that for every continuous function $f : X \to \mathbb{R}$ the map $\mu \mapsto \int f \, d\mu$ is continuous. For a standard Borel space X and a Borel action $G \curvearrowright X$, we write $\mathscr{E}_G(X)$ for the collection of ergodic invariant Borel probability measures on X.

A probability space (Y, ν) is a *factor* of (X, μ) if there exists a measure-preserving map $\pi : (X, \mu) \to (Y, \nu)$. Every factor $\pi : (X, \mu) \to (Y, \nu)$ is uniquely associated (mod μ-null sets) to a sub-σ-algebra $\mathcal{F}$ of X, and conversely every sub-σ-algebra $\mathcal{F}$

of (X, μ) is uniquely associated (up to isomorphism) to a factor $\pi : (X, \mu) \to (Y, \nu)$ [19, Theorem 2.15]. Since the factor Y is always standard Borel and thus countably generated, for any sub-σ-algebra $\mathcal{F}$ of X there is an increasing sequence of finite partitions γ_n with $\mathcal{F} = \bigvee_{n \in \mathbb{N}} \sigma\text{-alg}(\gamma_n)$ mod μ-null sets.

If G acts on (X, μ) and on (Y, ν), then we say that $G \curvearrowright (Y, \nu)$ is a factor of (X, μ) if there exists a measure-preserving G-equivariant map $\pi : (X, \mu) \to (Y, \nu)$. Under the correspondence described in the previous paragraph, factors $G \curvearrowright (Y, \nu)$ of (X, μ) are in one-to-one correspondence with G-invariant sub-σ-algebras $\mathcal{F} \subseteq \mathcal{B}(X)$. We will make frequent use of the following theorem.

Theorem II.1 (Seward–Tucker-Drob [48])**.** *Let G be a countably infinite group and let $G \curvearrowright (X, \mu)$ be a free p.m.p. ergodic action. Then for every $\epsilon > 0$ there is a factor $G \curvearrowright (Y, \nu)$ of (X, μ) such that $h_G^{\mathrm{Rok}}(Y, \nu) < \epsilon$ and G acts freely on Y.*

If $\pi : (X, \mu) \to (Y, \nu)$ is a factor map, then there is an essentially unique Borel map associating each $y \in Y$ to a Borel probability measure μ_y on X such that $\mu = \int \mu_y \, d\nu(y)$ and $\mu_y(\pi^{-1}(y)) = 1$ [19, Theorem A.7]. We call this the *disintegration* of μ over ν.

Let (X, μ) be a probability space, and let $\mathcal{F}$ be a sub-σ-algebra. Let $\pi : (X, \mu) \to (Y, \nu)$ be the associated factor, and let $\mu = \int \mu_y \, d\nu(y)$ be the disintegration of μ over ν. For a countable Borel partition α of X, the *conditional Shannon entropy* of α relative to $\mathcal{F}$ is

$$\mathrm{H}(\alpha \,|\, \mathcal{F}) = \int_Y \sum_{A \in \alpha} -\mu_y(A) \cdot \log \mu_y(A) \; d\nu(y) = \int_Y \mathrm{H}_{\mu_y}(\alpha) \; d\nu(y).$$

When necessary, we will write $\mathrm{H}_\mu(\alpha \,|\, \mathcal{F})$ to emphasize the measure. If $\mathcal{F} = \{X, \varnothing\}$ is the trivial σ-algebra then $\mathrm{H}(\alpha \,|\, \mathcal{F}) = \mathrm{H}(\alpha)$. For a countable partition β of X we set

$\mathrm{H}(\alpha \mid \beta) = \mathrm{H}(\alpha \mid \sigma\text{-alg}(\beta))$. For $B \subseteq X$ we write

$$\mathrm{H}_B(\alpha \mid \mathcal{F}) = \mathrm{H}_{\mu_B}(\alpha \mid \mathcal{F}),$$

where μ_B is the normalized restriction of μ to B defined by $\mu_B(A) = \mu(A \cap B)/\mu(B)$. Since for $B \in \mathcal{F}$ we have $\mu_B = \int \mu_y \, d\nu_{\pi(B)}(y)$, it follows that if $\beta \subseteq \mathcal{F}$ is a countable partition of X then

$$\mathrm{H}(\alpha \mid \mathcal{F}) = \sum_{B \in \beta} \mu(B) \cdot \mathrm{H}_B(\alpha \mid \mathcal{F}).$$

In particular,

$$\mathrm{H}(\alpha \mid \beta) = \sum_{B \in \beta} \mu(B) \cdot \mathrm{H}_B(\alpha).$$

We will need the following standard properties of Shannon entropy (proofs can be found in [15]):

Lemma II.2. *Let (X, μ) be a standard probability space, let α and β be countable Borel partitions of X, and let $\mathcal{F}$ and Σ be sub-σ-algebras. Then*

(i) $\mathrm{H}(\alpha \mid \mathcal{F}) = 0$ *if and only if* $\alpha \subseteq \mathcal{F}$ *mod null sets;*

(ii) $\mathrm{H}(\alpha \mid \mathcal{F}) \leq \log |\alpha|$*;*

(iii) *if* $\alpha \geq \beta$ *then* $\mathrm{H}(\alpha \mid \mathcal{F}) \geq \mathrm{H}(\beta \mid \mathcal{F})$*;*

(iv) *if* $\Sigma \subseteq \mathcal{F}$ *then* $\mathrm{H}(\alpha \mid \Sigma) \geq \mathrm{H}(\alpha \mid \mathcal{F})$*;*

(v) $\mathrm{H}(\alpha \vee \beta \mid \mathcal{F}) = \mathrm{H}(\beta \mid \mathcal{F}) + \mathrm{H}(\alpha \mid \sigma\text{-alg}(\beta) \vee \mathcal{F})$*;*

(vi) *if* $\mathrm{H}(\alpha), \mathrm{H}(\beta) < \infty$ *then* $\mathrm{H}(\alpha \vee \beta) = \mathrm{H}(\alpha) + \mathrm{H}(\beta)$ *if and only if* α *and* β *are independent;*

(vii) *if* $\alpha = \bigvee_{n \in \mathbb{N}} \alpha_n$ *is countable, then* $\mathrm{H}(\alpha \mid \mathcal{F}) = \lim_{k \to \infty} \mathrm{H}(\bigvee_{0 \leq n \leq k} \alpha_k \mid \mathcal{F})$*;*

(viii) *if* $\mathrm{H}(\alpha) < \infty$ *then* $\mathrm{H}(\alpha \mid \bigvee_{n \in \mathbb{N}} \mathcal{F}_n) = \lim_{k \to \infty} \mathrm{H}(\alpha \mid \bigvee_{0 \leq n \leq k} \mathcal{F}_n)$.

We will also need the following basic fact.

Lemma II.3. *Let (X, μ) be a probability space and let (α_n) be a sequence of countable partitions of X. If $\sum_{n \in \mathbb{N}} \mathrm{H}(\alpha_n) < \infty$ then $\beta = \bigvee_{n \in \mathbb{N}} \alpha_n$ is essentially countable.*

Proof. If for each n there is a coarsening $\zeta_n = \{Z_n, X \setminus Z_n\}$ of α_n such that the sequence of measures $\mu(Z_n)$ has an accumulation point in $(0, 1)$, then $\infty = \sum \mathrm{H}(\zeta_n) \leq \sum \mathrm{H}(\alpha_n)$, a contradiction. Let C_n be the piece of α_n of largest measure, and set $\xi_n = \{C_n, X \setminus C_n\}$. We must have $\mu(C_n)$ tends to 1 as otherwise there would exist partitions ζ_n as described above. We have $\sum \mathrm{H}(\xi_n) \leq \sum \mathrm{H}(\alpha_n) < \infty$. Since $x < \mathrm{H}(x, 1-x)$ for all x sufficiently close to 0, we deduce that $\sum \mu(X \setminus C_n) < \infty$. Now the Borel–Cantelli lemma states that almost-every $x \in X$ lies in only finitely many of the sets $X \setminus C_n$. So almost-every $x \in X$ lies in $C_n \in \alpha_n$ for all sufficiently large n. Let X_n be the set of x with $x \notin C_n$ but $x \in C_m$ for all $m > n$. Then the X_n's are pairwise disjoint, have conull union, and β is countable when restricted to any X_n. $\square$

We note a few lemmas related to σ-algebras and reduced σ-algebras which we will need.

Lemma II.4. *Let (X, μ) be a probability space and let $\mathcal{C}$ be a countable algebra of Borel sets. Then $A \in \sigma\text{-alg}(\mathcal{C})$ if and only if A is Borel and there is a conull $X' \subseteq X$ such that for every $a \in A \cap X'$ and $x \in X' \setminus A$ there is a set $C \in \mathcal{C}$ which separates a and x.*

Proof. Let Σ be the collection of sets A satisfying the condition described in the statement of the lemma. Then Σ contains $\mathcal{C} \cup \{X\}$ and the null sets, and it is easy to see that Σ is a σ-algebra. Thus $\sigma\text{-alg}(\mathcal{C}) \subseteq \Sigma$.

Enumerate $\mathcal{C}$ as $C_1, C_2, \cdots$ and define $\pi : X \to \{0, 1\}^{\mathbb{N}}$ by the rule $\pi(x)(n) = 1$ if and only if $x \in C_n$. Note that π is $\sigma\text{-alg}(\mathcal{C})$-measurable. Now fix a set $A \in \Sigma$. Then there is a conull $X' \subseteq X$ so that for all $a \in A \cap X'$ and $x \in X' \setminus A$ we have $\pi(a) \neq \pi(x)$. Consider the set $\pi(A \cap X')$. Note that $X' \cap \pi^{-1}(\pi(A \cap X')) = A \cap X'$. A priori, we do not know if $\pi(A \cap X')$ is Borel. However, since Borel probability measures are regular [25, Theorem 17.10], there is an F_σ-set $E \subseteq \pi(A \cap X')$ and a G_δ-set $F \supseteq \pi(A \cap X')$ with $\pi_*(\mu)(F \setminus E) = 0$. Then we have $X' \cap \pi^{-1}(E) \subseteq A \cap X' \subseteq \pi^{-1}(F)$, $\pi^{-1}(E), \pi^{-1}(F) \in \sigma\text{-alg}(\mathcal{C})$, and $\mu(\pi^{-1}(F) \setminus \pi^{-1}(E)) = 0$. Since A is Borel and differs from an element of $\sigma\text{-alg}(\mathcal{C})$ by a null set, we must have $A \in \sigma\text{-alg}(\mathcal{C})$. $\square$

Lemma II.5. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. action, and let α be a pre-partition. If β is a countable pre-partition and $\beta \sqsupseteq \alpha$ then $\sigma\text{-alg}_G^{red}(\beta) \supseteq \sigma\text{-alg}_G^{red}(\alpha)$. In particular, if β is a countable partition and $\beta \sqsupseteq \alpha$ then $\sigma\text{-alg}_G(\beta) \supseteq \sigma\text{-alg}_G^{red}(\alpha)$.*

Proof. Fix $R \in \sigma\text{-alg}_G^{\mathrm{red}}(\alpha)$. By definition of $\sigma\text{-alg}_G^{\mathrm{red}}(\alpha)$, there is a conull $X' \subseteq X$ such that for all $r \in R \cap X'$ and $x \in X' \setminus R$ there is $g_0 \in G$ with $g_0 \cdot r, g_0 \cdot x \in \cup\alpha$ and such that α separates $g_0 \cdot r$ and $g_0 \cdot x$. Since the restriction of β to $\cup\alpha$ is equal to α, we also have that β separates $g_0 \cdot r$ and $g_0 \cdot x$. We conclude that $R \in \sigma\text{-alg}_G^{\mathrm{red}}(\beta)$. If β is in fact a partition, then $\sigma\text{-alg}_G^{\mathrm{red}}(\beta) = \sigma\text{-alg}_G(\beta)$ be Lemma II.4. $\square$

Lemma II.6. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. action and let $G \curvearrowright (Y, \nu)$ be a factor of (X, μ) under the map $\pi : (X, \mu) \to (Y, \nu)$. If α is a countable pre-partition of Y then $\sigma\text{-alg}_G^{red}(\pi^{-1}(\alpha)) = \pi^{-1}(\sigma\text{-alg}_G^{red}(\alpha))$.*

Proof. It is a routine exercise to check $\pi^{-1}(\sigma\text{-alg}_G^{\mathrm{red}}(\alpha)) \subseteq \sigma\text{-alg}_G^{\mathrm{red}}(\pi^{-1}(\alpha))$. So fix $R \in \sigma\text{-alg}_G^{\mathrm{red}}(\pi^{-1}(\alpha))$. If there is a Borel $R' \subseteq Y$ with $R = \pi^{-1}(R')$, then again it follows easily from the definitions that $R' \in \sigma\text{-alg}_G^{\mathrm{red}}(\alpha)$ and thus $R \in \pi^{-1}(\sigma\text{-alg}_G^{\mathrm{red}}(\alpha))$. However, by Lemma II.4 we necessarily have $R \in \sigma\text{-alg}_G(\pi^{-1}(\alpha)) = \pi^{-1}(\sigma\text{-alg}_G(\alpha))$. Thus there is $R' \subseteq Y$ with $R = \pi^{-1}(R')$. $\square$

If $G \curvearrowright (X, \mu)$ is a p.m.p. ergodic action and $\mathcal{F}$ is a G-invariant sub-σ-algebra, then the *relative Rokhlin entropy* $h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F})$ is

$$\inf \Big\{ \mathrm{H}(\alpha \mid \mathcal{F}) \, : \, \alpha \text{ is a countable Borel partition and } \sigma\text{-}\mathrm{alg}_G(\alpha) \vee \mathcal{F} = \mathcal{B}(X) \Big\}.$$

For a collection $\mathcal{C}$ of Borel sets we define the *outer Rokhlin entropy* as

$$h_{G,X}^{\mathrm{Rok}}(\mathcal{C}|\mathcal{F}) = \inf \Big\{ \mathrm{H}(\alpha|\mathcal{F}) : \alpha \text{ is a countable Borel partition and } \mathcal{C} \subseteq \sigma\text{-}\mathrm{alg}_G(\alpha) \vee \mathcal{F} \Big\}.$$

When $\mathcal{F} = \{X, \varnothing\}$ we simply write $h_{G,X}^{\mathrm{Rok}}(\mathcal{C})$ for $h_{G,X}^{\mathrm{Rok}}(\mathcal{C}|\mathcal{F})$. If $G \curvearrowright (Y, \nu)$ is a factor of (X, μ), then we define $h_{G,X}^{\mathrm{Rok}}(Y) = h_{G,X}^{\mathrm{Rok}}(\Sigma)$, where Σ is the G-invariant sub-σ-algebra of X associated to Y.

CHAPTER III

The pseudo-group of an ergodic action

For a p.m.p. action $G \curvearrowright (X, \mu)$ we let E_G^X denote the induced orbit equivalence relation:

$$E_G^X = \{(x, y) : \exists g \in G, \quad g \cdot x = y\}.$$

The *pseudo-group* of E_G^X, denoted $[[E_G^X]]$, is the set of all Borel bijections $\theta : \mathrm{dom}(\theta) \to \mathrm{rng}(\theta)$ where $\mathrm{dom}(\theta), \mathrm{rng}(\theta) \subseteq X$ are Borel and $\theta(x) \in G \cdot x$ for every $x \in \mathrm{dom}(\theta)$. The *full group* of E_G^X, denoted $[E_G^X]$, is the set of all $\theta \in [[E_G^X]]$ with $\mathrm{dom}(\theta) = \mathrm{rng}(\theta) = X$ (i.e. conull in X).

For every $\theta \in [[E_G^X]]$ there is a Borel partition $\{Z_g^\theta : g \in G\}$ of $\mathrm{dom}(\theta)$ such that $\theta(x) = g \cdot x$ for every $x \in Z_g^\theta$. Thus, an important fact which we will use repeatedly is that every $\theta \in [[E_G^X]]$ is measure-preserving. We mention that the sets Z_g^θ are in general not uniquely determined from θ since the action of G might not be free. It will be necessary to keep record of such decompositions $\{Z_g^\theta\}$ for $\theta \in [[E_G^X]]$. The precise notion we need is the following.

Definition III.1. Let $G \curvearrowright (X, \mu)$ be a p.m.p. action, let $\theta \in [[E_G^X]]$, and let $\mathcal{F}$ be a G-invariant sub-σ-algebra. We say that θ is *$\mathcal{F}$-expressible* if $\mathrm{dom}(\theta), \mathrm{rng}(\theta) \in \mathcal{F}$ and there is a $\mathcal{F}$-measurable partition $\{Z_g^\theta : g \in G\}$ of $\mathrm{dom}(\theta)$ such that $\theta(x) = g \cdot x$ for every $x \in Z_g^\theta$ and all $g \in G$.

We observe two simple facts on the notion of expressibility.

Lemma III.2. *Let $G \curvearrowright (X,\mu)$ be a p.m.p. action and let $\mathcal{F}$ be a G-invariant sub-σ-algebra. If $\theta \in [[E_G^X]]$ is $\mathcal{F}$-expressible and $A \subseteq X$, then $\theta(A) = \theta(A \cap \mathrm{dom}(\theta))$ is $\sigma\text{-alg}_G(\{A\}) \vee \mathcal{F}$-measurable. In particular, if $A \in \mathcal{F}$ then $\theta(A) \in \mathcal{F}$.*

Proof. Fix a $\mathcal{F}$-measurable partition $\{Z_g^\theta : g \in G\}$ of $\mathrm{dom}(\theta)$ such that $\theta(x) = g \cdot x$ for all $x \in Z_g^\theta$. Then

$$\theta(A) = \bigcup_{g \in G} g \cdot (A \cap Z_g^\theta) \in \sigma\text{-alg}_G(\{A\}) \vee \mathcal{F}. \qquad \square$$

Lemma III.3. *Let $G \curvearrowright (X,\mu)$ be a p.m.p. action and let $\mathcal{F}$ be a G-invariant sub-σ-algebra. If $\theta, \phi \in [[E_G^X]]$ are $\mathcal{F}$-expressible then so are θ^{-1} and $\theta \circ \phi$.*

Proof. Fix $\mathcal{F}$-measurable partitions $\{Z_g^\theta : g \in G\}$ and $\{Z_g^\phi : g \in G\}$ of $\mathrm{dom}(\theta)$ and $\mathrm{dom}(\phi)$, respectively, satisfying $\theta(x) = g \cdot x$ for all $x \in Z_g^\theta$ and $\phi(x) = g \cdot x$ for all $x \in Z_g^\phi$. Define for $g \in G$

$$Z_g^{\theta^{-1}} = g^{-1} \cdot Z_{g^{-1}}^\theta.$$

Then each $Z_g^{\theta^{-1}}$ is $\mathcal{F}$-measurable since $\mathcal{F}$ is G-invariant. It is easily checked that $\{Z_g^{\theta^{-1}} : g \in G\}$ partitions $\mathrm{rng}(\theta)$ and satisfies $\theta^{-1}(x) = g \cdot x$ for all $x \in Z_g^{\theta^{-1}}$. Thus θ^{-1} is $\mathcal{F}$-expressible.

Observe that by the previous lemma, $\phi^{-1}(Z_g^\theta) \in \mathcal{F}$ for every $g \in G$ since ϕ^{-1} is $\mathcal{F}$-expressible. Notice that the sets $Z_g^\phi \cap \phi^{-1}(Z_h^\theta)$ partition $\mathrm{dom}(\theta \circ \phi)$. Define for $g \in G$

$$Z_g^{\theta \circ \phi} = \bigcup_{h \in G} \Big(Z_{h^{-1}g}^\phi \cap \phi^{-1}(Z_h^\theta) \Big).$$

These sets are $\mathcal{F}$-measurable and pairwise-disjoint and we have $\theta \circ \phi(x) = g \cdot x$ for all $x \in Z_g^{\theta \circ \phi}$. $\square$

With the aid of Lemma III.2, we observe a basic property of relative Rokhlin entropy. The proposition below resembles a theorem of Rudolph and Weiss from classical entropy theory [43]. Note that if G and Γ act on (X, μ) with the same orbits then $E_G^X = E_\Gamma^X$ and $[[E_G^X]] = [[E_\Gamma^X]]$. In this situation, we say that $\theta \in [[E_G^X]]$ is $(G, \mathcal{C})$-expressible if it is $\mathcal{C}$-expressible with respect to the G-action $G \curvearrowright (X, \mu)$.

Proposition III.4. *Let G and Γ be countable groups, and let $G \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (X, \mu)$ be p.m.p. ergodic actions having the same orbits. Suppose that $\mathcal{F}$ is a G and Γ invariant sub-σ-algebra such that the transformation associated to each $g \in G$ is $(\Gamma, \mathcal{F})$-expressible and similarly the transformation associated to each $\gamma \in \Gamma$ is $(G, \mathcal{F})$-expressible. Then*

$$h_G^{\mathrm{Rok}}(X, \mu | \mathcal{F}) = h_\Gamma^{\mathrm{Rok}}(X, \mu | \mathcal{F}).$$

Proof. It suffices to show that for every countable partition α, $\sigma\text{-alg}_G(\alpha) \vee \mathcal{F} = \sigma\text{-alg}_\Gamma(\alpha) \vee \mathcal{F}$. Indeed, since the transformation associated to each $g \in G$ is $(\Gamma, \mathcal{F})$-expressible and $\alpha \subseteq \sigma\text{-alg}_\Gamma(\alpha) \vee \mathcal{F}$, it follows from Lemma III.2 that the σ-algebra $\sigma\text{-alg}_\Gamma(\alpha) \vee \mathcal{F}$ is G-invariant. Therefore $\sigma\text{-alg}_G(\alpha) \vee \mathcal{F} \subseteq \sigma\text{-alg}_\Gamma(\alpha) \vee \mathcal{F}$. With the same argument we obtain the reverse containment. $\square$

The lemma below and the corollaries which follow it provide us with all elements of the pseudo-group $[[E_G^X]]$ which will be needed in forthcoming chapters.

Lemma III.5. *Let $G \curvearrowright (X, \mu)$ be an ergodic p.m.p. action. Let $A, B \subseteq X$ be Borel sets with $0 < \mu(A) \leq \mu(B)$. Then there exists a $\sigma\text{-alg}_G(\{A, B\})$-expressible function $\theta \in [[E_G^X]]$ with $\mathrm{dom}(\theta) = A$ and $\mathrm{rng}(\theta) \subseteq B$.*

Proof. Let $g_0, g_1, \ldots$ be an enumeration of G. Set $Z_{g_0}^\theta = A \cap g_0^{-1} \cdot B$ and inductively define

$$Z_{g_n}^\theta = \left(A \setminus \left(\bigcup_{i=0}^{n-1} Z_{g_i}^\theta\right)\right) \bigcap g_n^{-1} \cdot \left(B \setminus \left(\bigcup_{i=0}^{n-1} g_i \cdot Z_{g_i}^\theta\right)\right).$$

Define $\theta : \bigcup_{n\in\mathbb{N}} Z^\theta_{g_n} \to B$ by setting $\theta(x) = g_n \cdot x$ for $x \in Z^\theta_{g_n}$. Clearly θ is $\sigma\text{-alg}_G(\{A, B\})$-expressible.

Set $C = A \setminus \mathrm{dom}(\theta)$. Towards a contradiction, suppose that $\mu(C) > 0$. Then we have

$$\mu(\mathrm{rng}(\theta)) = \mu(\mathrm{dom}(\theta)) < \mu(A) \leq \mu(B).$$

So $\mu(B \setminus \mathrm{rng}(\theta)) > 0$ and by ergodicity there is $n \in \mathbb{N}$ with

$$\mu\Big(C \cap g_n^{-1} \cdot (B \setminus \mathrm{rng}(\theta))\Big) > 0.$$

However, this implies that $\mu(C \cap Z^\theta_{g_n}) > 0$, a contradiction. We conclude that, up to a null set, $\mathrm{dom}(\theta) = A$. □

Corollary III.6. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action. If $C \subseteq B \subseteq X$ and $\mu(C) = \frac{1}{n} \cdot \mu(B)$ with $n \in \mathbb{N}$, then there is a $\sigma\text{-alg}_G(\{C, B\})$-measurable partition ξ of B into n pieces with each piece having measure $\frac{1}{n} \cdot \mu(B)$ and with $C \in \xi$.*

Proof. Set $C_1 = C$. Once $\sigma\text{-alg}_G(\{C, B\})$-measurable subsets $C_1, \ldots, C_{k-1}$ of B, each of measure $\frac{1}{n} \cdot \mu(B)$, have been defined, we apply Lemma III.5 to get a $\sigma\text{-alg}_G(\{C, B\})$-expressible function $\theta \in [[E^X_G]]$ with $\mathrm{dom}(\theta) = C$ and

$$\mathrm{rng}(\theta) \subseteq B \setminus (C_1 \cup \cdots \cup C_{k-1}).$$

We set $C_k = \theta(C)$. We note that $\mu(C_k) = \frac{1}{n} \cdot \mu(B)$ and $C_k \in \sigma\text{-alg}_G(\{C, B\})$ by Lemma III.2. Finally, set $\xi = \{C_1, \ldots, C_n\}$. □

In the corollary below we write $\mathrm{id}_A \in [[E^X_G]]$ for the identity function on A for $A \subseteq X$.

Corollary III.7. *Let $G \curvearrowright (X, \mu)$ be an ergodic p.m.p. action. If $\xi = \{C_1, \ldots, C_n\}$ is a collection of pairwise disjoint Borel sets of equal measure, then there is a $\sigma\text{-alg}_G(\xi)$-*

expressible function $\theta \in [[E_G^X]]$ *which cyclically permutes the members of* ξ*, meaning that* $\mathrm{dom}(\theta) = \mathrm{rng}(\theta) = \cup\xi$*,* $\theta(C_k) = C_{k+1}$ *for* $1 \leq k < n$*,* $\theta(C_n) = C_1$*, and* $\theta^n = \mathrm{id}_{\cup\xi}$*.*

Proof. By Lemma III.5, for each $2 \leq k \leq n$ there is a σ-$\mathrm{alg}_G(\xi)$-expressible function $\phi_k \in [[E_G^X]]$ with $\mathrm{dom}(\phi_k) = C_1$ and $\mathrm{rng}(\phi_k) = C_k$. We define $\theta : \cup\xi \to \cup\xi$ by

$$\theta(x) = \begin{cases} \phi_2(x) & \text{if } x \in C_1 \\ \phi_{k+1} \circ \phi_k^{-1}(x) & \text{if } x \in C_k \text{ and } 1 < k < n \\ \phi_n^{-1}(x) & \text{if } x \in C_n. \end{cases}$$

Then θ cyclically permutes the members of ξ and has order n. Finally, each restriction $\theta \restriction C_k$ is σ-$\mathrm{alg}_G(\xi)$-expressible by Lemma III.3 and thus θ is σ-$\mathrm{alg}_G(\xi)$-expressible. $\square$

CHAPTER IV

Finite subequivalence relations

In this chapter we construct finite subequivalence relations which will be used to replace the traditional role of the Rokhlin lemma and the Shannon–McMillan–Breiman theorem. We begin with a technical lemma.

Lemma IV.1. *Let $\bar{a} = (a_1, a_2, \ldots, a_p)$ be a probability vector and let $\epsilon > 0$. Then there exists $n \in \mathbb{N}$, probability vectors $\bar{r}^j = (r_1^j, r_2^j, \ldots, r_p^j)$ having rational entries with denominator n, and a probability vector $\bar{c} = (c_1, c_2, \ldots, c_p)$ such that $|a_i - r_i^j| < \epsilon$ for all i, j and $\bar{a} = \sum_{j=1}^{p} c_j \cdot \bar{r}^j$.*

Proof. Without loss of generality, we may suppose that $a_p > 0$. Fix $n \in \mathbb{N}$ with $n > (p-1)/\epsilon$ and $n > 2(p-1)/a_p$. For $i < p$ let $k_i \in \mathbb{N}$ satisfy $k_i/n \leq a_i < (k_i+1)/n$ and let $\lambda_i \in (0, 1]$ be such that

$$a_i = \lambda_i \cdot \frac{k_i}{n} + (1 - \lambda_i) \cdot \frac{k_i + 1}{n.}$$

Set $\lambda_0 = 0$ and $\lambda_p = 1$. By reordering a_1 through a_{p-1} if necessary, we may suppose that

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{p-1} \leq \lambda_p = 1.$$

For $1 \leq j \leq p$ set $c_j = \lambda_j - \lambda_{j-1}$. Then $\bar{c} = (c_1, \ldots, c_p)$ is a probability vector. Since

$\sum_{j=1}^{i} c_j = \lambda_i$ and $\sum_{j=i+1}^{p} c_j = 1 - \lambda_i$, we deduce that

$$\forall i < p \qquad a_i = \sum_{j=1}^{i} c_j \cdot \frac{k_i}{n} + \sum_{j=i+1}^{p} c_j \cdot \frac{k_i+1}{n}. \tag{4.1}$$

Now define $\bar{r}^j = (r_1^j, r_2^j, \ldots, r_p^j)$ by:

$$r_i^j = \begin{cases} \frac{k_i}{n} & \text{if } j \leq i \neq p \\ \frac{k_i+1}{n} & \text{if } j > i \\ 1 - \sum_{t=1}^{p-1} r_t^j & \text{if } i = p. \end{cases}$$

Clearly $\bar{r}^j$ has rational entries with denominator n. Furthermore $|r_i^j - a_i| \leq 1/n < \epsilon$ for $i < p$ and

$$\left|r_p^j - a_p\right| = \left|1 - \sum_{t=1}^{p-1} r_t^j - a_p\right| = \left|\sum_{t=1}^{p-1} \left(a_t - r_t^j\right)\right| \leq \frac{p-1}{n} < \epsilon.$$

From the expression above we also deduce that $r_p^j > 0$ so that $\bar{r}^j$ is indeed a probability vector. It follows from (4.1) that $a_i = \sum_{j=1}^{p} c_j \cdot r_i^j$ for all $i < p$, and since $(a_1, \ldots, a_p)$ and $\sum_{j=1}^{p} c_j \cdot \bar{r}^j$ are both length-p probability vectors whose first $(p-1)$-many coordinates agree, we must have $a_p = \sum_{j=1}^{p} c_j \cdot r_p^j$. $\square$

For an equivalence relation E on X and $x \in X$, we write $[x]_E$ for the E-class of x. Recall that a set $T \subseteq X$ is a *transversal* for E if $|T \cap [x]_E| = 1$ for almost-every $x \in X$. We will work with equivalence relations which are generated by an element of the pseudo-group in the following sense.

Definition IV.2. Let $G \curvearrowright (X, \mu)$ be a p.m.p. action, let $B \subseteq X$ be a Borel set of positive measure, and let E be an equivalence relation on B with $E \subseteq E_G^X \cap B \times B$. We say that E is *generated by* $\theta \in [[E_G^X]]$ if $\mathrm{dom}(\theta) = \mathrm{rng}(\theta) = B$ and $[x]_E = \{\theta^i(x) : i \in \mathbb{Z}\}$ for almost-all $x \in B$. In this case, we write $E = E_\theta$.

Lemma IV.3. *Let $G \curvearrowright (X,\mu)$ be an ergodic p.m.p. action, let $B \subseteq X$ have positive measure, let α be a finite partition of X, and let $\epsilon > 0$. Then there is an equivalence relation E on B with $E \subseteq E_G^X \cap B \times B$ and $n \in \mathbb{N}$ so that for μ-almost-every $x \in B$, the E-class of x has cardinality n and*

$$\forall A \in \alpha \qquad \frac{\mu(A \cap B)}{\mu(B)} - \epsilon < \frac{|A \cap [x]_E|}{|[x]_E|} < \frac{\mu(A \cap B)}{\mu(B)} + \epsilon.$$

Moreover, E admits a $\sigma\text{-alg}_G(\alpha \cup \{B\})$-measurable transversal and is generated by a $\sigma\text{-alg}_G(\alpha \cup \{B\})$-expressible function $\theta : B \to B$ in $[[E_G^X]]$ which satisfies $\theta^n = \mathrm{id}_B$.

Proof. Let $\pi : (X,\mu) \to (Y,\nu)$ be the factor map associated to the G-invariant sub-σ-algebra generated by $\alpha \cup \{B\}$. Enumerate α as $\alpha = \{A_1, A_2, \ldots, A_p\}$. Set $B' = \pi(B)$ and $\alpha' = \{A'_i : 1 \leq i \leq p\}$ where $A'_i = \pi(A_i)$. Note that α' is a partition of (Y,ν) and that $\nu(A'_i \cap B') = \mu(A_i \cap B)$.

First, let's suppose that (Y,ν) is non-atomic. By Lemma IV.1 there are $n \in \mathbb{N}$, probability vectors $\bar{r}^j$ having rational entries with denominator n, and a probability vector $\bar{c}$ such that

$$\left| \frac{\nu(A'_i \cap B')}{\nu(B')} - r_i^j \right| < \epsilon$$

for all i, j and

$$\left(\frac{\nu(A'_1 \cap B')}{\nu(B')}, \; \frac{\nu(A'_2 \cap B')}{\nu(B')}, \ldots, \; \frac{\nu(A'_p \cap B')}{\nu(B')} \right) = \sum_{j=1}^{p} c_j \cdot (r_1^j, r_2^j, \ldots, r_p^j).$$

Since (Y,ν) is non-atomic and since α' is a partition, we can partition B' into sets $\{Z'_1, Z'_2, \ldots, Z'_p\}$ such that for every j, $\nu(Z'_j)/\nu(B') = c_j$ and

$$\left(\frac{\nu(Z'_j \cap A'_1)}{\nu(B')}, \; \frac{\nu(Z'_j \cap A'_2)}{\nu(B')}, \ldots, \; \frac{\nu(Z'_j \cap A'_p)}{\nu(B')} \right) = c_j \cdot (r_1^j, r_2^j, \ldots, r_p^j).$$

It follows that $\nu(Z'_j \cap A'_i)/\nu(Z'_j) = r_i^j$ is rational with denominator n for all i, j. This implies that there is a partition ξ'_j of Z'_j into n pieces each of measure $\frac{1}{n} \cdot \nu(Z'_j)$ such that for every i, $Z'_j \cap A'_i$ is the union of $n \cdot r_i^j$ many classes of ξ'_j.

Set $Z_j = \pi^{-1}(Z'_j)$ and $\xi_j = \pi^{-1}(\xi'_j)$. Then Z_j and the classes of ξ_j all automatically lie in $\sigma\text{-alg}_G(\alpha \cup \{B\})$. For each j, apply Corollary III.7 to get a $\sigma\text{-alg}_G(\alpha \cup \{B\})$-expressible function $\phi_j \in [[E^X_G]]$ which cyclically permutes the classes of ξ_j. So in particular $\mathrm{dom}(\phi_j) = \mathrm{rng}(\phi_j) = Z_j$, and $\phi_j^n = \mathrm{id}_{Z_j}$. Set $\theta = \phi_1 \cup \phi_2 \cup \cdots \cup \phi_p$ and set $E = E_\theta$. Then for μ-almost-every $x \in B$, the E-class of x has cardinality n and if $x \in Z_j$ then

$$\forall i \qquad \left| \frac{\mu(A_i \cap B)}{\mu(B)} - \frac{|A_i \cap [x]_E|}{|[x]_E|} \right| = \left| \frac{\mu(A_i \cap B)}{\mu(B)} - r_i^j \right| < \epsilon.$$

Finally, if we fix some $C_j \in \xi_j$ for each j, then $\bigcup_{1 \leq j \leq p} C_j \in \sigma\text{-alg}_G(\alpha \cup \{B\})$ is a transversal for E.

In the case that (Y, ν) has an atom, we deduce by ergodicity that, modulo a null set, Y is finite. Say $|Y| = m$ and each point in Y has measure $\frac{1}{m}$. Set $n = |B'|$. Clearly there are integers $k_i \in \mathbb{N}$, with $\sum_{i=1}^p k_i = n$ and

$$\frac{\mu(A_i \cap B)}{\mu(B)} = \frac{\nu(A'_i \cap B')}{\nu(B')} = \frac{k_i/m}{n/m} = \frac{k_i}{n}.$$

Let ξ' be the partition of B' into points, and pull back ξ' to a partition ξ of B. Now apply Corollary III.7 and follow the argument from the non-atomic case. $\square$

Corollary IV.4. *Let $G \curvearrowright (X, \mu)$ be an ergodic p.m.p. action, let $B \subseteq X$ have positive measure, let $\epsilon > 0$, and let $F = \{f : B \to \mathbb{R}\}$ be a finite collection of finite valued Borel functions. Then there is an equivalence relation E on B with $E \subseteq E^X_G \cap B \times B$ and $n \in \mathbb{N}$ so that for μ-almost-every $x \in B$, the E-class of x has cardinality n and*

$$\forall f \in F \qquad \frac{1}{\mu(B)} \cdot \int_B f \; d\mu - \epsilon < \frac{1}{|[x]_E|} \cdot \sum_{y \in [x]_E} f(y) < \frac{1}{\mu(B)} \cdot \int_B f \; d\mu + \epsilon.$$

Moreover, if each $f \in F$ is $\mathcal{F}$-measurable then E admits a $\sigma\text{-alg}_G(\mathcal{F} \cup \{B\})$-measurable transversal and is generated by a $\sigma\text{-alg}_G(\mathcal{F} \cup \{B\})$-expressible function $\theta : B \to B$ in $[[E^X_G]]$ which satisfies $\theta^n = \mathrm{id}_B$.

Proof. Define a partition α of B so that $x, y \in B$ lie in the same piece of α if and only if $f(x) = f(y)$ for all $f \in F$. Then α is a finite partition. Now the desired equivalence relation E is obtained from Lemma IV.3. $\square$

The conclusions of the previous lemma and corollary are not too surprising since you are allowed to "see" the sets which you wish to mix, i.e. you are allowed to use $\sigma\text{-alg}_G(\alpha \cup \{B\})$. The following proposition however is unexpected. It roughly says that you can achieve the same conclusion even if you are restricted to only seeing a very small sub-σ-algebra. We will use this in the same fashion one typically uses the Rokhlin lemma and the Shannon–McMillan–Breiman theorem, although technically the proposition below bears more similarity with the Rokhlin lemma and the ergodic theorem.

Let us say a few words on the Rokhlin lemma to highlight the similarity. For a free p.m.p. action $\mathbb{Z} \curvearrowright (X, \mu)$, $n \in \mathbb{N}$, and $\epsilon > 0$, the Rokhlin lemma provides a Borel set $S \subseteq X$ such that the sets $i \cdot S$, $0 \leq i \leq n-1$, are pairwise disjoint and union to a set having measure at least $1-\epsilon$. The set S naturally produces a subequivalence relation E defined as follows. For $x \in X$ set $x_S = (-i) \cdot x$ where $(-i) \cdot x \in S$ and $(-j) \cdot x \notin S$ for all $0 \leq j < i$. We set $x \; E \; y$ if and only if $x_S = y_S$. Clearly every E class has cardinality at least n, and a large measure of E-classes have cardinality precisely n. A key fact which is frequently used in classical results such as Krieger's theorem is that the equivalence relation E is easily described. Specifically, S is small since $\mu(S) \leq 1/n$, and so E can be defined by using the small sub-σ-algebra $\sigma\text{-alg}_{\mathbb{Z}}(\{S\})$.

Proposition IV.5. *Let $G \curvearrowright (X, \mu)$ be an ergodic p.m.p. action with (X, μ) non-atomic, let α be a finite collection of Borel subsets of X, let $\epsilon > 0$, and let $N \in \mathbb{N}$. Then there are $n \geq N$, Borel sets $S_1, S_2 \subseteq X$ with $\mu(S_1) + \mu(S_2) < \epsilon$, and a $\sigma\text{-alg}_G(\{S_1, S_2\})$-expressible $\theta \in [E_G^X]$ such that E_θ admits a $\sigma\text{-alg}_G(\{S_1, S_2\})$-*

measurable transversal, and for almost-every $x \in X$ *we have* $|[x]_{E_\theta}| = n$ *and*

$$\forall A \in \alpha \qquad \mu(A) - \epsilon < \frac{|A \cap [x]_{E_\theta}|}{|[x]_{E_\theta}|} < \mu(A) + \epsilon.$$

Proof. Pick $m > \max(4/\epsilon,\ N)$ with $m \in \mathbb{N}$ and

$$|\alpha| \cdot \log_2(m+1) < \frac{\epsilon}{4} \cdot m.$$

Let $S_1 \subseteq X$ be any Borel set with $\mu(S_1) = \frac{1}{m} < \frac{\epsilon}{4}$. Apply Corollaries III.6 and III.7 to obtain a $\sigma\text{-alg}_G(\{S_1\})$-expressible function $h \in [E_G^X]$ such that $\mathrm{dom}(h) = \mathrm{rng}(h) = X$, $h^m = \mathrm{id}_X$, and such that $\{h^i(S_1) : 0 \leq i < m\}$ is a partition of X. The induced Borel equivalence relation E_h is finite, in fact almost-every E_h-class has cardinality m, and it has S_1 as a transversal. We imagine the classes of E_h as extending horizontally to the right, and we visualize S_1 as a vertical column.

We consider the distribution of $\alpha \restriction [s]_{E_h}$ for each $s \in S_1$. For $A \in \alpha$ define $d_A : S_1 \to \mathbb{R}$ by

$$d_A(s) = \frac{|A \cap [s]_{E_h}|}{|[s]_{E_h}|} = \frac{1}{m} \cdot \Big|A \cap [s]_{E_h}\Big|.$$

Note that for each $A \in \alpha$

$$\int_{S_1} d_A \ d\mu = \frac{1}{m} \cdot \mu(A) = \mu(S_1) \cdot \mu(A).$$

By Corollary IV.4 there is $k \in \mathbb{N}$ and an equivalence relation $E_v \subseteq E_G^X \cap S_1 \times S_1$ on S_1 such that for almost every $s \in S_1$, the E_v-class of s has cardinality k and

$$\forall A \in \alpha \qquad \mu(A) - \epsilon < \frac{1}{|[s]_{E_v}|} \cdot \sum_{s' \in [s]_{E_v}} d_A(s') < \mu(A) + \epsilon.$$

Moreover, if we let $\mathcal{F}$ denote the G-invariant sub-σ-algebra generated by the functions d_A, $A \in \alpha$, then E_v admits a $\sigma\text{-alg}_G(\mathcal{F} \cup \{S_1\})$-measurable transversal T and is generated by a $\sigma\text{-alg}_G(\mathcal{F} \cup \{S_1\})$-expressible function $v \in [[E_G^X]]$ which satisfies $\mathrm{dom}(v) = \mathrm{rng}(v) = S_1$ and $v^k = \mathrm{id}_{S_1}$.

Let $E = E_v \vee E_h$ be the equivalence relation generated by E_v and E_h. Then $T \subseteq S_1$ is a transversal for E, and for every $s \in T$

$$\Big|[s]_E\Big| = \sum_{s' \in [s]_{E_v}} \Big|[s']_{E_h}\Big| = k \cdot m.$$

Setting $n = k \cdot m \geq N$, we have that almost every E-class has cardinality n. Also, for every $A \in \alpha$ and $s \in T$ we have

$$\frac{|A \cap [s]_E|}{|[s]_E|} = \frac{1}{k \cdot m} \cdot \sum_{s' \in [s]_{E_v}} \Big|A \cap [s']_{E_h}\Big| = \frac{1}{|[s]_{E_v}|} \cdot \sum_{s' \in [s]_{E_v}} d_A(s').$$

It follows that for μ-almost-every $x \in X$

$$\forall A \in \alpha \qquad \mu(A) - \epsilon < \frac{|A \cap [x]_E|}{|[x]_E|} < \mu(A) + \epsilon.$$

Now consider the partition $\xi = \{T_{i,j} \,:\, 0 \leq i < k,\ 0 \leq j < m\}$ of X where

$$T_{i,j} = h^j \circ v^i(T).$$

Note that $T_{i,j} \in \sigma\text{-alg}_G(\mathcal{F} \cup \{S_1\})$ by Lemmas III.2 and III.3. We will define a function $\theta \in [E_G^X]$ which generates E by defining θ on each piece of ξ. We define

$$\theta \upharpoonright T_{i,j} = \begin{cases} h \upharpoonright T_{i,j} & \text{if } j + 1 < m \\ \\ v \circ h \upharpoonright T_{i,j} & \text{if } j + 1 = m. \end{cases}$$

In regard to the second case above, one should observe that $h(T_{i,m-1}) = T_{i,0}$ since $h^m = \mathrm{id}_X$. Since v satisfies $v^k = \mathrm{id}_{S_1}$ and $n = k \cdot m$, we see that θ satisfies $\theta^n = \mathrm{id}_X$. We also have $E = E_\theta$. Finally, θ is $\sigma\text{-alg}_G(\mathcal{F} \cup \{S_1\})$-expressible since each restriction $\theta \upharpoonright T_{i,j}$ is $\sigma\text{-alg}_G(\mathcal{F} \cup \{S_1\})$-expressible by Lemma III.3.

To complete the proof, we must find a Borel set $S_2 \subseteq X$ with $\mu(S_2) < \frac{3}{4} \cdot \epsilon < \epsilon - \mu(S_1)$ such that $\mathcal{F} \subseteq \sigma\text{-alg}_G(\{S_1, S_2\})$. Notice that $|\mathrm{rng}(d_A)| \leq m + 1$ for every $A \in \alpha$ and therefore the product map

$$d_\alpha = \prod_{A \in \alpha} d_A : S_1 \to \Big\{0, \frac{1}{m}, \frac{2}{m}, \ldots, 1\Big\}^\alpha$$

has an image of cardinality at most $(m+1)^\alpha$. Set $\ell = \lceil(\epsilon/4)\cdot m\rceil$ (i.e. the least integer greater than or equal to $(\epsilon/4)\cdot m$). Since $(\epsilon/4)\cdot m > 1$ we have that $\ell < (\epsilon/2)\cdot m$. By our choice of m we have

$$(m+1)^{|\alpha|} < 2^{(\epsilon/4)\cdot m} \leq 2^\ell.$$

Therefore there is an injection

$$r : \{0, 1/m, \ldots, 1\}^\alpha \to \{0,1\}^\ell.$$

Now we will define S_2 so that, for every $s \in S_1$, the integers $\{1 \leq i \leq \ell : h^i(s) \in S_2\}$ will encode the value $r \circ d_\alpha(s)$. Specifically, we define

$$S_2 = \{h^i(s) \,:\, 1 \leq i \leq \ell,\ s \in S_1,\ r(d_\alpha(s))(i) = 1\}.$$

We have that $S_2 \subseteq \bigcup_{1\leq i\leq \ell} h^i(S_1)$ and therefore

$$\mu(S_2) \leq \ell \cdot \mu(S_1) < \left(\frac{\epsilon}{2}\cdot m\right)\cdot\frac{1}{m} = \frac{\epsilon}{2}$$

as required. Finally, we check that $\mathcal{F} \subseteq \sigma\text{-alg}_G(\{S_1, S_2\})$. Fix $p \in \{0, 1/m, \ldots, 1\}^\alpha$. Set

$$I_p^0 = \{1 \leq i \leq \ell \,:\, r(p)(i) = 0\} \quad \text{and} \quad I_p^1 = \{1 \leq i \leq \ell \,:\, r(p)(i) = 1\}.$$

Then for $s \in S_1$ we have

$$\begin{aligned}
d_\alpha(s) = p &\iff r(d_\alpha(s)) = r(p)\\
&\iff (\forall i \in I_p^0)\ \ h^i(s) \notin S_2 \quad \text{and} \quad (\forall i \in I_p^1)\ \ h^i(s) \in S_2\\
&\iff s \in S_1 \cap \left(\bigcap_{i\in I_p^0} h^{-i}(X \setminus S_2)\right) \cap \left(\bigcap_{i\in I_p^1} h^{-i}(S_2)\right).
\end{aligned}$$

So $d_\alpha^{-1}(p) \in \sigma\text{-alg}_G(\{S_1, S_2\})$ by Lemmas III.2 and III.3. Thus $\mathcal{F} \subseteq \sigma\text{-alg}_G(\{S_1, S_2\})$.

□

CHAPTER V

Construction of a non-trivial reduced σ-algebra

This chapter is devoted to building a pre-partition β with $\sigma\text{-alg}_G^{\mathrm{red}}(\beta) \neq \{X, \varnothing\}$. We will in fact build β with some additional properties which will be needed in the next chapter.

The pre-partition β will consist of two (disjoint) sets B_0, B_1. On an intuitive level, it is likely helpful to imagine points in B_0 as "labeled with 0", points in B_1 as "labeled with 1", and points in $X \setminus (B_0 \cup B_1)$ as "unlabeled." The purpose of this chapter is to build B_0, B_1 and a set R, $0 < \mu(R) < 1$, with $R \in \sigma\text{-alg}_G^{\mathrm{red}}(\beta)$. Intuitively, this means that for every point $x \in X$ and every $\{0,1\}$-labeling of the orbit of x which extends the $\{0,1\}$-labeling coming from $\{B_0, B_1\}$, one can determine from this labeling whether or not $x \in R$. In approaching this coding problem we are guided by previous works of the author. Specifically, we draw upon the notions of and constructions for "locally recognizable functions" and "membership tests" developed in [17] and [18]. Those constructions were done in a purely combinatorial framework. Some of these constructions were generalized to the Borel setting in [48] under the name "recognizable sets," and this influences our methods here as well. However, our constraints and goals are different in the present work, and the constructions in this chapter and the next differ greatly from those in [17, 18, 48].

A naive but suggestive idea for building R, B_0, B_1 is to fix a finite window $W \subseteq G$, label all points in $W \cdot R$ with 1 (i.e. set $B_1 = W \cdot R$) and try to arrange B_0 so that for every $x \notin R$ there is a point in $W \cdot x$ labeled 0. This naive approach is the right idea but does not quite work. For example, this will fail if W has too much symmetry, such as if W is a finite subgroup. If W is a finite subgroup then this construction might not distinguish R from $W \cdot R$. In the case of free actions it is not hard to choose W in a more intelligent way and get this argument to work (see the construction of locally recognizable functions in [18]). However, for non-free actions it is not easy to make this argument work, but we do so in this chapter. One indication of the difficulty for non-free actions is that there may be points x for which $W \cdot x = \{x\}$. To overcome the difficulties of non-free actions we will construct group elements c and $Q = \{q_1, \ldots, q_6\}$. We will arrange the construction so that for $r \in R$ the labels of the points $q_i \cdot r$ ("query points") will contain useful information (q_1 and q_2 will be used in this chapter, while $q_3, \ldots, q_6$ will be used in the next). The point $c \cdot r$ will be one final checkpoint for verifying that $r \in R$.

We remark that this is the only chapter where we truly work with the original action of G rather than the pseudo-group. We thus believe that allowing for non-free actions does not significantly impact the length or the complexity of the proof of the main theorem.

Lemma V.1. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. action. If $Y \subseteq X$ is Borel and $F \subseteq G$ is finite, then there exists a Borel set $D \subseteq Y$ such that $Y \subseteq F^{-1}F \cdot D$ and $F \cdot d \cap F \cdot d' = \varnothing$ for all $d \neq d' \in D$. In particular, if $\mu(Y) > 0$ then $\mu(D) > 0$.*

This is a special case of a more general result due to Kechris–Solecki–Todorcevic [26, Prop. 4.2 and Prop. 4.5]. As a convenience to the reader, we include a proof below.

Proof. Since X is a standard Borel space, there is a sequence B_n of Borel sets which separates points, meaning that for all $x \neq y \in X$ there is n with B_n containing one, but not both, of x and y [25, Prop. 12.1]. For $1 \leq k \leq |F^{-1}F|$, set $Y_k = \{y \in Y : |F^{-1}F \cdot y| = k\}$. Let $\mathcal{C}$ be the G-invariant algebra generated by the sets $\{B_n : n \in \mathbb{N}\} \cup \{Y_k : 1 \leq k \leq |F^{-1}F|\}$. Then $\mathcal{C}$ is countable. Let C_n, $n \in \mathbb{N}$, enumerate the elements of $\mathcal{C}$ satisfying $F \cdot x \cap F \cdot x' = \varnothing$ for all $x \neq x' \in C_n$. Inductively define $D_1 = Y \cap C_1$ and

$$D_{i+1} = D_i \cup \Big((Y \cap C_{i+1}) \setminus F^{-1}F \cdot D_i \Big).$$

Set $D = \bigcup_{i \in \mathbb{N}} D_i \subseteq Y$.

Consider $y \in Y$. Say $y \in Y_k$ and suppose that $F^{-1}F \cdot y$ consists of the distinct points $f_1 \cdot y, \ldots, f_k \cdot y$. Since $\mathcal{C}$ separates points and is an algebra, there are pairwise disjoint sets $A_1, \ldots, A_k \in \mathcal{C}$ with $f_i \cdot y \in A_i$ for each $1 \leq i \leq k$. Set $A_y = Y_k \cap \bigcap_{i=1}^{k} f_i^{-1} \cdot A_i \in \mathcal{C}$. Then for $y' \in A_y$ we have that $F^{-1}F \cdot y'$ has cardinality k and consists of the points $f_i \cdot y' \in A_i$ for $1 \leq i \leq k$. It follows that $F \cdot y' \cap F \cdot y'' = \varnothing$ for all $y' \neq y'' \in A_y$. Thus $A_y = C_n$ for some n. Clearly $y \in A_y = C_n$. It follows that either $y \in D_n$ or else $y \in F^{-1}F \cdot D_{n-1}$. In either case, $y \in F^{-1}F \cdot D$. We conclude that $Y \subseteq F^{-1}F \cdot D$.

Finally, fix $d \neq d' \in D$. Let i and j be least with $d \in C_i$ and $d' \in C_j$. If $i = j$ then $F \cdot d \cap F \cdot d' = \varnothing$ and we are done. So without loss of generality, suppose that $i > j$. Since $d' \in C_j \cap D$ we must have $d' \in D_j$. As $d \in D \cap C_i$ and $i > j$ is minimal, we have $d \in D_i \setminus D_{i-1}$ and therefore $d \notin F^{-1}F \cdot d'$. Thus $Fd \cap Fd' = \varnothing$. $\square$

Lemma V.2. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action with (X, μ) non-atomic, and let $\delta > 0$. Then there exists a finite symmetric set $W \subseteq G$ with $1_G \in W$ and a Borel set $D \subseteq X$ such that $0 < \mu(D) < \delta$ and $W \cdot x \cap D \neq \varnothing$ for all $x \in X$.*

Proof. Since $G \curvearrowright (X, \mu)$ is ergodic and (X, μ) is non-atomic, it must be that almost-every orbit is infinite. So there is a finite set $F \subseteq G$ such that

$$A = \{x \in X \,:\, |F \cdot x| > 2/\delta\}$$

satisfies $\mu(A) > 1 - \delta/2$. Fix such a set F. Apply Lemma V.1 to obtain a Borel set $D_A \subseteq A$ of positive measure with $A \subseteq F^{-1}F \cdot D_A$ and $F \cdot d \cap F \cdot d' = \varnothing$ for all $d \neq d' \in D_A$. Then

$$\frac{2}{\delta} \cdot \mu(D_A) \leq \int_{D_A} |F \cdot x| \, d\mu(x) \leq 1$$

and $\mu(D_A) \leq \delta/2$. Again apply Lemma V.1 to obtain a Borel set $D_0 \subseteq (X \setminus F^{-1}F \cdot D_A)$ with $(X \setminus F^{-1}F \cdot D_A) \subseteq F^{-1}F \cdot D_0$ and $F \cdot d \cap F \cdot d' = \varnothing$ for all $d \neq d' \in D_0$. Set $D = D_0 \cup D_A$. Then $F \cdot d \cap F \cdot d' = \varnothing$ for all $d \neq d' \in D$ and $F^{-1}F \cdot D = X$. Also,

$$0 < \mu(D) \leq \mu(D_A) + \mu(X \setminus F^{-1}F \cdot D_A) \leq \mu(D_A) + \mu(X \setminus A) < \delta/2 + \delta/2 = \delta.$$

We set $W = F^{-1}F$. Then W is symmetric and $1_G \in W$. Finally, since $X = F^{-1}F \cdot D = W \cdot D$, we obtain $W \cdot x \cap D \neq \varnothing$ for all $x \in X$. □

Lemma V.3 (B.H. Neumann, [40])**.** *Let G be a group, and let H_i, $1 \leq i \leq n$, be subgroups of G. Suppose there are group elements $g_i \in G$ so that*

$$G = \bigcup_{i=1}^{n} g_i \cdot H_i.$$

Then there is i such that $|G : H_i| < \infty$.

As a convenience to the reader, we include a proof below.

Proof. The lemma is immediate if all of the H_i are equal to a fixed subgroup H. Now inductively assume that the lemma is true for every n whenever there are less than c distinct groups among $H_1, \ldots, H_n$. Let $n \geq 1$ and let $H_1, \ldots, H_n$ be a sequence of c distinct subgroups, and let $g_1, \ldots, g_n \in G$ be such that $G = \bigcup g_i \cdot H_i$. Set $H = H_n$. By reordering the H_i's if necessary, we may suppose that there is $m \leq n$ with $H_i = H$

if and only if $i \geq m$. If $G = \bigcup_{i=m}^{n} g_i \cdot H$ then $H = H_n$ has finite index in G and we are done. Otherwise, there is $a \in G$ with $a \cdot H$ disjoint from each of $g_i \cdot H$ for $i \geq m$. Then we must have

$$a \cdot H \subseteq \bigcup_{i=1}^{m-1} g_i \cdot H_i \qquad \text{and hence} \qquad H \subseteq \bigcup_{i=1}^{m-1} a^{-1} g_i \cdot H_i.$$

So we obtain

$$G = \bigcup_{i=1}^{m-1} g_i \cdot H_i \cup \bigcup_{j=m}^{n} \bigcup_{i=1}^{m-1} g_j a^{-1} g_i \cdot H_i.$$

Since there are now $c - 1$ distinct subgroups appearing on the right-hand side, we conclude from the inductive hypothesis that there is $i \leq m-1$ with $|G : H_i| < \infty$. $\square$

Corollary V.4. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action with (X, μ) non-atomic. Let $R \subseteq X$ have positive measure and let $W, T \subseteq G$ be finite. Then there are a Borel set $R' \subseteq R$ with $\mu(R') > 0$ and $c \in G$ such that $cW \cdot R' \cap T \cdot R' = \varnothing$.*

Proof. Our assumptions imply that almost-every orbit is infinite. So for μ-almost-every $r \in R$ the stability group $\mathrm{Stab}(r) = \{g \in G : g \cdot r = r\}$ has infinite index in G and thus by Lemma V.3

$$T \cdot \mathrm{Stab}(r) \cdot W^{-1} = \bigcup_{t \in T} \bigcup_{w \in W} tw^{-1} \cdot (w\mathrm{Stab}(r)w^{-1}) \neq G.$$

As G is countable, there is $c \in G$ and a non-null Borel set $R_0 \subseteq R$ with

$$c \notin T \cdot \mathrm{Stab}(r) \cdot W^{-1}$$

for all $r \in R_0$. It follows that $cW \cdot r \cap T \cdot r = \varnothing$ for all $r \in R_0$. Now apply Lemma V.1 to get positive measure Borel set $R' \subseteq R_0$ with $(cW \cup T) \cdot r \cap (cW \cup T) \cdot r' = \varnothing$ for all $r \neq r' \in R'$. $\square$

Lemma V.5. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action with (X, μ) non-atomic. Let $R, Y \subseteq X$ be positive measure Borel sets and let $T \subseteq G$ be finite. Then there* *are $q \in G$ and a Borel set $R' \subseteq R$ of positive measure such that $q \cdot R' \subseteq Y$ and $q \cdot R' \cap T \cdot R' = \varnothing$.*

Proof. Let $R_0 \subseteq R$ be a Borel set with $\mu(R_0) > 0$ and $\mu(Y \setminus T \cdot R_0) > 0$. By ergodicity, there is $q \in G$ such that $q \cdot R_0 \cap (Y \setminus T \cdot R_0)$ has positive measure. Set

$$R' = q^{-1} \cdot \Big(q \cdot R_0 \cap (Y \setminus T \cdot R_0) \Big). \qquad \square$$

The following lemma is rather technical to state, but its proof is short. This lemma will play an important role in the proposition which follows.

Lemma V.6. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action with (X, μ) non-atomic. Let $Y \subseteq X$ be a Borel set of positive measure, let $W \subseteq G$ be finite and symmetric with $1_G \in W$, and let $m \in \mathbb{N}$. Then there exist $n \in \mathbb{N}$, $F \cup Q \cup \{c\} \subseteq G$, and a Borel set $R \subseteq X$ with $Q = \{q_1, \ldots, q_6\}$, $\mu(R) = \frac{1}{n}$, $n > m \cdot |F|$, and satisfying the following:*

(i) $Q \cdot R \subseteq Y$;

(ii) $|(\{c\} \cup Q) \cdot r \setminus W \cdot r| = 7$ *for all* $r \in R$;

(iii) $(W \cup \{c\} \cup Q)^2 \subseteq F$;

(iv) $F \cdot r \cap F \cdot r' = \varnothing$ *for all* $r \neq r' \in R$;

(v) $Wq \cdot R \cap (W \cup \{c\} \cup Q) \cdot R \subseteq q \cdot R$ *for every* $q \in Q$;

(vi) $Qc \cdot R \cap (W \cup \{c\} \cup Q) \cdot R \subseteq c \cdot R$;

(vii) $cW \cdot R \cap (W \cup \{c\} \cup Q) \cdot R \subseteq c \cdot R$;

(viii) for all $r \in R$, *either* $q_1 c \cdot r \neq c \cdot r$ *or* $q_2 c \cdot r = c \cdot r$.

Proof. Set $R_0 = X$. By induction on $1 \leq i \leq 6$ we choose $q_i \in G$ and a Borel set $R_i \subseteq R_{i-1}$ such that $\mu(R_i) > 0$, $q_i \cdot R_i \subseteq Y$, and

$$q_i \cdot R_i \cap W(W \cup \{q_j \,:\, j < i\}) \cdot R_i = \varnothing.$$

Both the base case and the inductive steps are taken care of by Lemma V.5. Set $Q = \{q_1, q_2, \ldots, q_6\}$. Then $q_i \cdot R_6 \subseteq q_i \cdot R_i \subseteq Y$ and $|Q \cdot r \setminus W \cdot r| = 6$ for all $r \in R_6$. Now apply Corollary V.4 to obtain $c \in G$ and a Borel set $R_c \subseteq R_6$ with $\mu(R_c) > 0$ and

$$cW \cdot R_c \cap (\{1_G\} \cup Q^{-1})(W \cup Q \cup WQ) \cdot R_c = \varnothing.$$

Set $F = (W \cup \{c\} \cup Q)^2$ so that (iii) is satisfied.

If there is $q \in Q$ with $qc \cdot r = c \cdot r$ for all $r \in R_c$, then set $R' = R_c$ and re-index the elements of Q so that $q_2 = q$. Otherwise, we may re-index Q and find a Borel set $R' \subseteq R_c$ of positive measure with $q_1 c \cdot r \neq c \cdot r$ for all $r \in R'$. Now apply Lemma V.1 to obtain a positive measure Borel set $R \subseteq R'$ with $F \cdot r \cap F \cdot r' = \varnothing$ for all $r \neq r' \in R$. By shrinking R if necessary, we may suppose that $\mu(R) = \frac{1}{n} < \frac{1}{m \cdot |F|}$ for some $n > m \cdot |F|$. Then (iv) is immediately satisfied, (viii) is satisfied since $R \subseteq R'$, and (i) is satisfied since $R \subseteq R_6$. Clause (ii) also holds since $c \cdot r \in cW \cdot r$ is disjoint from $(W \cup Q) \cdot r$ for every $r \in R$.

Recall that $W = W^{-1}$ and $1_G \in W$. Fix $1 \leq i \leq 6$. By the definition of q_i we have $Wq_i \cdot R \cap W \cdot R = \varnothing$, and if $j \neq i$ then $Wq_i \cdot R \cap q_j \cdot R = \varnothing$. Also, the definition of c implies that $Wq_i \cdot R \cap c \cdot R = \varnothing$. Therefore

$$Wq_i \cdot R \cap (W \cup \{c\} \cup Q) \cdot R \subseteq q_i \cdot R,$$

establishing (v). By definition of c we have $Qc \cdot R \cap (W \cup Q) \cdot R = \varnothing$. So (vi) follows. Similarly, $cW \cdot R \cap (W \cup Q) \cdot R = \varnothing$ and (vii) follows. □

Proposition V.7. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action with (X, μ) non-atomic. Let $W \subseteq G$ be finite and symmetric with $1_G \in W$, and let $D \subseteq X$ be a Borel set with $W \cdot x \cap D \neq \varnothing$ for all $x \in X$. Assume that the set $Y = \{x \in X : |W \cdot x| \geq 2\}$*

has positive measure, and let $F \cup Q \cup \{c\} \subseteq G$ *and* $R \subseteq X$ *be as in Lemma V.6. If* $\beta = \{B_0, B_1\}$ *is a pre-partition with:*

(1) $(W \cup \{c, q_1\}) \cdot R \subseteq B_1$*;*

(2) $B_1 \cap F \cdot R \subseteq (W \cup \{c\} \cup Q) \cdot R$*; and*

(3) $(D \setminus F \cdot R) \cup (F \cdot R \setminus B_1) \cup q_2 \cdot R \subseteq B_0$*,*

then $R \in \sigma\text{-alg}_G^{red}(\beta)$*.*

Proof. We will make use of clauses (i) through (viii) of Lemma V.6. We first make three claims.

Claim: If $q \in Q$ and $r \in R$ then $W \cdot (q \cdot r) \not\subseteq B_1$.

Fix $q \in Q$ and $r \in R$. By (i) $q \cdot r \in Y$ and hence there is $w \in W$ with $wq \cdot r \neq q \cdot r$. It follows $wq \cdot r \notin q \cdot R$ by (iii) and (iv). Thus (v) and (2) imply

$$wq \cdot r \notin (W \cup \{c\} \cup Q) \cdot R \supseteq B_1 \cap F \cdot R.$$

Since $wq \cdot r \in F \cdot R$, we deduce that $wq \cdot r \notin B_1$, establishing the claim.

Claim: If $r \in R$ then either $q_1 \cdot (c \cdot r) \notin B_1$ or $q_2 \cdot (c \cdot r) \notin B_0$.

Fix $r \in R$. By (viii) we have that either $q_1 c \cdot r \neq c \cdot r$ or $q_2 c \cdot r = c \cdot r$. In the latter case, (1) gives $q_2 c \cdot r = c \cdot r \in B_1$, and since $B_1 \cap B_0 = \varnothing$, we conclude that $q_2 c \cdot r \notin B_0$. So we may assume $q_1 c \cdot r \neq c \cdot r$. Then $q_1 c \cdot r \notin c \cdot R$ by (iii) and (iv). Hence (vi) and (2) give

$$q_1 c \cdot r \notin (W \cup \{c\} \cup Q) \cdot R \supseteq B_1 \cap F \cdot R.$$

Since $q_1 c \cdot r \in F \cdot R$, we obtain $q_1 c \cdot r \notin B_1$ and we are done.

Claim: $R \in \sigma\text{-alg}_G^{\mathrm{red}}(\beta)$.

If $r \in R$ then it is immediate from (1) and (3) that $(W \cup \{c, q_1\}) \cdot r \subseteq B_1$ and $q_2 \cdot r \in B_0$. So it suffices to show that if $x \notin R$ then either $(W \cup \{c, q_1\}) \cdot x \cap B_0 \neq \varnothing$

or $q_2 \cdot x \in B_1$. Let $w \in W$ be such that $w \cdot x \in D$. If $w \cdot x \in B_0$ then we are done. So suppose that $w \cdot x \notin B_0$. Since $w \cdot x \in D \setminus B_0$, it follows from (3) that $w \cdot x \in F \cdot R$. By (3) we also have $F \cdot R \subseteq B_0 \cup B_1$, and since $w \cdot x \notin B_0$ we must have $w \cdot x \in F \cdot R \cap B_1$. So by (2) $w \cdot x \in (W \cup \{c\} \cup Q) \cdot R$. If $x \in B_0$ then we are done since $1_G \in W$. So suppose that $x \notin B_0$. Since W is symmetric, $x \in W \cdot w \cdot x$ and hence $x \in F \cdot R$ by (iii). Again, by (3) we have $x \in F \cdot R \subseteq B_0 \cup B_1$, so $x \notin B_0$ implies $x \in B_1 \cap F \cdot R$. Applying (2), we obtain $x \in (W \cup \{c\} \cup Q) \cdot R$. From the previous two claims we see that we are done if $x \in (\{c\} \cup Q) \cdot R$. So suppose that $x \in W \cdot R$. Since $x \notin R$, it follows from (vii) that $c \cdot x \notin (W \cup \{c\} \cup Q) \cdot R$. By (iii) we have $c \cdot x \in F \cdot R$, so by applying (2) we find that $c \cdot x \notin B_1$. Again, (3) gives $c \cdot x \in F \cdot R \subseteq B_0 \cup B_1$, so we must have $c \cdot x \in B_0$. This completes the proof. $\square$

CHAPTER VI

Coding small sets

In this chapter we develop a method for perturbing a given pre-partition α to obtain a pre-partition α' with the property that $\sigma\text{-alg}_G^{\text{red}}(\alpha')$ contains pre-specified small sets. The construction in this chapter is intended to complement Proposition IV.5. The construction we present involves a delicate coding procedure which is inspired by techniques in [17], [18], and [48].

Proposition VI.1. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action with (X, μ) non-atomic and let $0 < \kappa < 1$. Then there are $0 < \epsilon < \kappa$ and a Borel set $M \subseteq X$ with $\mu(M) = \kappa$ with the following property: if $S_1, S_2 \subseteq X$ satisfy $\mu(S_1) + \mu(S_2) < \epsilon$, then there is a two-piece partition $\beta = \{B_0, B_1\}$ of M such that $S_1, S_2 \in \sigma\text{-alg}_G^{red}(\beta)$.*

Proof. By Lemma V.2, there is a finite symmetric set $W \subseteq G$ with $1_G \in W$ and a Borel set $D \subseteq X$ with $\mu(D) < \kappa/2$ such that $W \cdot x \cap D \neq \varnothing$ for all $x \in X$. Note that if $|W \cdot x| = 1$ then $x \in D$. Thus the set $Y = \{x \in X : |W \cdot x| \geq 2\}$ has positive measure. Apply Lemma V.6 to obtain $F \cup \{c\} \cup Q \subseteq G$ with $Q = \{q_1, \ldots, q_6\}$ and $R \subseteq X$ with $\mu(R) = \frac{1}{n}$, where $n > 2|F|/\kappa$. Fix $k \in \mathbb{N}$ with

$$\log_2(2nk) < k - 1$$

and let Z_1 and Z_2 be disjoint Borel subsets of R with $\mu(Z_1) = \mu(Z_2) = \frac{1}{2nk}$. Set

$Z = Z_1 \cup Z_2$ and note that $\mu(Z) = \frac{1}{nk} = \frac{1}{k} \cdot \mu(R)$. Fix $\epsilon > 0$ with $\epsilon < \frac{1}{6nk} < \kappa$. Let $M \subseteq X$ be any Borel set with $D \cup F \cdot R \subseteq M$ and $\mu(M) = \kappa$.

Apply Corollaries III.6 and III.7 to obtain a $\sigma\text{-alg}_G(\{Z, R\})$-expressible function $\rho \in [[E_G^X]]$ such that $\mathrm{dom}(\rho) = \mathrm{rng}(\rho) = R$, $\rho^k = \mathrm{id}_R$, and such that $\{\rho^i(Z) : 0 \leq i < k\}$ is a partition of R. For each $j = 1, 2$, again apply these corollaries to obtain a $\sigma\text{-alg}_G(\{Z_j\})$-expressible function $\psi_j \in [E_G^X]$ such that $\mathrm{dom}(\psi_j) = \mathrm{rng}(\psi_j) = X$, $\psi_j^{2nk} = \mathrm{id}_X$, and such that $\{\psi_j^i(Z_j) : 0 \leq i < 2nk\}$ is a partition of X. We mention that there are no assumed relationships between ψ_1, ψ_2, and ρ.

Let $S_1, S_2 \subseteq X$ be Borel sets with $\mu(S_1) + \mu(S_2) < \epsilon$. Our intention will be to encode how the sets $\psi_1^i(Z_1)$ meet S_1 and similarly how the sets $\psi_2^i(Z_2)$ meet S_2. For $1 \leq m \leq 2nk$ and $j = 1, 2$, let Z_j^m be the set of $z \in Z_j$ such that

$$|\{0 \leq i < 2nk : \psi_j^i(z) \in S_j\}| \geq m.$$

Then $Z_j^1 \supseteq Z_j^2 \supseteq \cdots \supseteq Z_j^{2nk}$ and

$$\sum_{m=1}^{2nk} \mu(Z_1^m \cup Z_2^m) = \mu(S_1) + \mu(S_2) < \epsilon.$$

Setting $Z_j^* = Z_j \setminus Z_j^1$, we have

$$\mu(Z_j^*) = \mu(Z_j) - \mu(Z_j^1) > \frac{1}{2nk} - \epsilon > \frac{1}{3nk} > 2\epsilon.$$

In particular,

$$\mu(Z_1^* \cup Z_2^*) - \sum_{m=1}^{2nk} \mu(Z_1^m \cup Z_2^m) > 4\epsilon - \epsilon = 3\epsilon. \tag{6.1}$$

Set $Z^m = Z_1^m \cup Z_2^m$ and $Z^* = Z_1^* \cup Z_2^*$.

For each $1 \leq m \leq 2nk$ we wish to build a function $\theta_m \in [[E_G^X]]$ which is expressible with respect to $\sigma\text{-alg}_G(\{Z^*, Z^1, \ldots, Z^m\})$ and satisfies $\mathrm{dom}(\theta_m) = Z^m$ and

$$\mathrm{rng}(\theta_m) \subseteq Z^* \setminus \bigcup_{k=1}^{m-1} \theta_k(Z^k).$$

We construct these functions inductively. When $m = 1$, we have $\mu(Z^1) < \epsilon < \mu(Z^*)$ and thus θ_1 is obtained immediately from Lemma III.5. Now suppose that θ_1 through θ_{m-1} have been defined. Then

$$Z^* \setminus \bigcup_{k=1}^{m-1} \theta_k(Z^k)$$

lies in $\sigma\text{-alg}_G(\{Z^*, Z^1, \ldots, Z^{m-1}\})$ by Lemma III.2. By (6.1) we have

$$\mu(Z^m) < \epsilon < \mu(Z^*) - \sum_{k=1}^{m-1} \mu(Z^k) = \mu\left(Z^* \setminus \bigcup_{k=1}^{m-1} \theta_k(Z^k)\right).$$

Therefore we may apply Lemma III.5 to obtain θ_m. This completes the construction.

Define $f : \bigcup_{m=1}^{2nk} \mathrm{rng}(\theta_m) \to \{0, 1, \ldots, 2nk-1\}$ by setting $f(\theta_m(z)) = \ell$ for $z \in Z_j^m$ if and only if $\psi_j^\ell(z) \in S_j$, and

$$|\{0 \leq i \leq \ell \,:\, \psi_j^i(z) \in S_j\}| = m.$$

For $i, t \in \mathbb{N}$ we let $\mathbb{B}_i(t) \in \{0,1\}$ denote the i^{th} digit in the binary expansion of t (so $\mathbb{B}_i(t) = 0$ for all $i > \log_2(t) + 1$). Now define a Borel set $B_1 \subseteq X$ by the rule

$$x \in B_1 \iff \begin{cases} x \in W \cdot R & \text{or} \\ x \in c \cdot R & \text{or} \\ x \in q_1 \cdot R & \text{or} \\ x \in q_3 \cdot Z & \text{or} \\ x \in q_4 \cdot Z_1 & \text{or} \\ x \in q_5 \cdot Z^1 & \text{or} \\ x \in q_6 \cdot \theta_m(Z^{m+1}) & \text{for some } 1 \leq m < 2nk, \text{ or} \\ x = q_6 \cdot \rho^i(z) & \text{where } 1 \leq i < k,\ z \in \mathrm{dom}(f), \\ & \qquad\qquad \text{and } \mathbb{B}_i(f(z)) = 1. \end{cases}$$

It is important to note that $B_1 \subseteq (W \cup \{c\} \cup Q) \cdot R$. In particular, $B_1 \subseteq F \cdot R$ by Lemma V.6.(iii). We also define the Borel set

$$B_0 = M \setminus B_1 = (M \setminus (D \cup F \cdot R)) \cup (D \setminus F \cdot R) \cup (F \cdot R \setminus B_1).$$

Note that clauses (iii) and (iv) of Lemma V.6 imply that for every $r \neq r' \in R$

$$(W \cup \{c\} \cup Q) \cdot r \cap (W \cup \{c\} \cup Q) \cdot r' = \varnothing.$$

Thus from clause (ii) of Lemma V.6 we obtain the following one-way implications

$$x \in B_0 \Longleftarrow \begin{cases} x \in q_2 \cdot R & \text{or} \\ x \in q_3 \cdot (R \setminus Z) & \text{or} \\ x \in q_4 \cdot (R \setminus Z_1) & \text{or} \\ x \in q_5 \cdot (R \setminus Z^1) & \text{or} \\ x \in q_6 \cdot \bigcap_{m=1}^{2nk-1} (Z \setminus \theta_m(Z^{m+1})) & \text{or} \\ x = q_6 \cdot \rho^i(z) & \text{where } 1 \leq i < k,\ z \in \mathrm{dom}(f), \\ & \qquad \text{and } \mathbb{B}_i(f(z)) = 0. \end{cases}$$

In particular, $q_2 \cdot R \subseteq B_0$. We therefore see that $\beta = \{B_0, B_1\}$ satisfies the assumptions of Proposition V.7.

We will now check that $S_1, S_2 \in \sigma\text{-alg}_G^{\mathrm{red}}(\beta)$. By Proposition V.7 we have $R \in \sigma\text{-alg}_G^{\mathrm{red}}(\beta)$. By G-invariance of $\sigma\text{-alg}_G^{\mathrm{red}}(\beta)$, we have $q_i \cdot R \in \sigma\text{-alg}_G^{\mathrm{red}}(\beta)$ for $1 \leq i \leq 6$. It immediately follows from the definition of $\sigma\text{-alg}_G^{\mathrm{red}}(\beta)$ that $B_0 \cap q_i \cdot R$ and $B_1 \cap q_i \cdot R$ lie $\sigma\text{-alg}_G^{\mathrm{red}}(\beta)$. Defining the partition

$$\gamma = \Big\{R, X \setminus (R \cup Q \cdot R)\Big\} \cup \Big\{B_0 \cap q_i \cdot R : 1 \leq i \leq 6\Big\} \cup \Big\{B_1 \cap q_i \cdot R : 1 \leq i \leq 6\Big\},$$

we have $\sigma\text{-alg}_G(\gamma) \subseteq \sigma\text{-alg}_G^{\mathrm{red}}(\beta)$. It suffices to show that $S_1, S_2 \in \sigma\text{-alg}_G(\gamma)$.

We have $x \in Z$ if and only if $q_3 \cdot x \in B_1 \cap q_3 \cdot R \in \gamma$. Since $R \in \gamma$, we have that both Z and $R \setminus Z$ lie in $\sigma\text{-alg}_G(\gamma)$. Similarly, $x \in Z_1$ if and only if $q_4 \cdot x \in B_1 \cap q_4 \cdot R$. We conclude that $Z_1, Z_2, Z, R \in \sigma\text{-alg}_G(\gamma)$. It follows that ρ, ψ_1, and ψ_2 are $\sigma\text{-alg}_G(\gamma)$-expressible.

We prove by induction on $1 \leq m \leq 2nk$ that $Z^m, Z_1^m, Z_2^m \in \sigma\text{-alg}_G(\gamma)$ and that θ_m is $\sigma\text{-alg}_G(\gamma)$-expressible. Since $x \in Z^1$ if and only if $q_5 \cdot x \in B_1 \cap q_5 \cdot R$, we have $Z^1 \in \sigma\text{-alg}_G(\gamma)$. Also $Z_1^1 = Z^1 \cap Z_1$ and $Z_2^1 = Z^1 \cap Z_2$ are in $\sigma\text{-alg}_G(\gamma)$. So $Z^* = Z \setminus Z^1$, $Z_1^* = Z_1 \setminus Z_1^1$, and $Z_2^* = Z_2 \setminus Z_2^1$ are in $\sigma\text{-alg}_G(\gamma)$ as well. It follows that θ_1 is $\sigma\text{-alg}_G(\gamma)$-expressible. Now inductively suppose that $Z^i \in \sigma\text{-alg}_G(\gamma)$ and that θ_i is $\sigma\text{-alg}_G(\gamma)$-expressible for all $1 \leq i \leq m$. Then $z \in Z^{m+1}$ if and only if $z \in Z^m$ and $q_6 \cdot \theta_m(z) \in B_1 \cap q_6 \cdot R$. In other words,

$$Z^{m+1} = \theta_m^{-1}\Big(q_6^{-1} \cdot (B_1 \cap q_6 \cdot R)\Big).$$

Thus $Z^{m+1} \in \sigma\text{-alg}_G(\gamma)$ by Lemmas III.2 and III.3. Similarly, $Z_1^{m+1} = Z^{m+1} \cap Z_1$ and $Z_2^{m+1} = Z^{m+1} \cap Z_2$ are in $\sigma\text{-alg}_G(\gamma)$. Finally, θ_{m+1} is expressible with respect to $\sigma\text{-alg}_G(\{Z^*, Z^1, \ldots, Z^{m+1}\}) \subseteq \sigma\text{-alg}_G(\gamma)$. This completes the inductive argument.

Now to complete the proof we show that $S_1, S_2 \in \sigma\text{-alg}_G(\gamma)$. We first argue that f is $\sigma\text{-alg}_G(\gamma)$-measurable. It follows from the previous paragraph and Lemma III.2 that $\mathrm{dom}(f) \in \sigma\text{-alg}_G(\gamma)$. Observe that the numbers $\ell \in \mathrm{rng}(f)$ are distinguished by their first $(k-1)$-binary digits $\mathbb{B}_i(\ell)$, $1 \leq i < k$, since by construction $\log_2(2nk) < k - 1$. So for $0 \leq \ell < 2nk$, if we set $I_0 = \{1 \leq i < k : \mathbb{B}_i(\ell) = 0\}$ and $I_1 = \{1 \leq i < k\} \setminus I_0$ then we have

$$f^{-1}(\ell) = \mathrm{dom}(f) \cap \bigcap_{i \in I_0} \rho^{-i}\Big(q_6^{-1} \cdot (B_0 \cap q_6 \cdot R)\Big) \cap \bigcap_{i \in I_1} \rho^{-i}\Big(q_6^{-1} \cdot (B_1 \cap q_6 \cdot R)\Big).$$

Thus $f^{-1}(\ell) \in \sigma\text{-alg}_G(\gamma)$ by Lemmas III.2 and III.3. Now suppose that $x \in S_j$. Then there is $z \in Z_j$ and $0 \leq \ell < 2nk$ with $x = \psi_j^\ell(z)$. It follows that $z \in Z_j^m$ where

$$m = |\{0 \leq i \leq \ell \,:\, \psi_j^i(z) \in S_j\}|.$$

Furthermore, $\ell = f(\theta_m(z))$. Conversely, if there is $1 \leq m \leq 2nk$, $z \in Z_j^m$, and $0 \leq \ell < 2nk$ with $x = \psi_j^\ell(z)$ and $f(\theta_m(z)) = \ell$, then $x \in S_j$. Therefore

$$S_j = \bigcup_{\ell=0}^{2nk-1} \bigcup_{m=1}^{2nk} \psi_j^\ell\Big(Z_j \cap \theta_m^{-1}(f^{-1}(\ell))\Big) \in \sigma\text{-alg}_G(\gamma) \subseteq \sigma\text{-alg}_G^{\mathrm{red}}(\beta). \qquad \square$$

We call a probability vector $\bar{p} = (p_i)$ *non-trivial* if there are $i \neq j$ with $p_i, p_j > 0$.

Corollary VI.2. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action with (X, μ) non-atomic, let $\bar{p}$ be a non-trivial probability vector, let $0 < r \leq 1$, and let $\delta > 0$. Then there are $0 < \epsilon < r\delta$, and a Borel set $M \subseteq X$ with $\mu(M) = r\delta$ with the following property: if $S_1, S_2 \subseteq X$ satisfy $\mu(S_1) + \mu(S_2) < \epsilon$ and $\alpha = \{A_i \,:\, 0 \leq i < |\bar{p}|\}$ is a pre-partition with $\cup\alpha \subseteq X \setminus M$ and*

$$\mu(A_i) < \min\Big((rp_i + r\epsilon) \cdot \mu(X \setminus M), \quad rp_i\Big)$$

for all $0 \leq i < |\bar{p}|$, then there is a pre-partition $\alpha' = \{A'_i \,:\, 0 \leq i < |\bar{p}|\}$ with $A_i \subseteq A'_i$ and $\mu(A'_i) = rp_i$ for every i and $S_1, S_2 \in \sigma\text{-alg}_G^{red}(\alpha')$.

Proof. Without loss of generality, we may suppose that $p_0, p_1 > 0$. Pick $0 < \kappa < r\delta$ so that for $i = 0, 1$

$$(rp_i + r\kappa) \cdot (1 - r\delta) < rp_i - \kappa.$$

Apply Proposition VI.1 to get $0 < \epsilon < \kappa$ and $M' \subseteq X$ with $\mu(M') = \kappa$. Fix any set $M \supseteq M'$ with $\mu(M) = r\delta$.

Now let $S_1, S_2 \subseteq X$ with $\mu(S_1) + \mu(S_2) < \epsilon$ and let $\alpha = \{A_i \,:\, 0 \leq i < |\bar{p}|\}$ be a pre-partition with $\cup\alpha \subseteq X \setminus M$ and

$$\mu(A_i) < \min\Big((rp_i + r\epsilon) \cdot \mu(X \setminus M), \quad rp_i\Big)$$

for all i. Then by Proposition VI.1 there is a partition $\beta = \{B_0, B_1\}$ of M' such that $S_1, S_2 \in \sigma\text{-alg}^{\mathrm{red}}_G(\beta)$. For $i = 0, 1$, our choice of κ gives

$$\mu(A_i) < \min\Big((rp_i + r\epsilon) \cdot (1 - r\delta), rp_i\Big) < (rp_i + r\kappa) \cdot (1 - r\delta) < rp_i - \kappa < rp_i - \mu(B_i).$$

Set $C_0 = A_0 \cup B_0$, $C_1 = A_1 \cup B_1$, and $C_i = A_i$ for $2 \le i < |\bar{p}|$. Then $\{C_i : 0 \le i < |\bar{p}|\}$ is a collection of pairwise disjoint Borel subsets of X with $\mu(C_i) < rp_i$ for every $0 \le i < |\bar{p}|$. Since (X, μ) is non-atomic, there exists a pre-partition $\alpha' = \{A'_i : 0 \le i < |\bar{p}|\}$ with $A_i \subseteq C_i \subseteq A'_i$ and $\mu(A'_i) = rp_i$ for every i. By construction α' extends β and hence $S_1, S_2 \in \sigma\text{-alg}^{\mathrm{red}}_G(\beta) \subseteq \sigma\text{-alg}^{\mathrm{red}}_G(\alpha')$ by Lemma II.5. $\square$

CHAPTER VII

Countably infinite partitions

In this chapter, we show how to replace countably infinite partitions by finite ones. This will allow us to carry-out counting arguments in proving the main theorem. Our work in this section improves upon methods used by the author in [46].

For a finite set S we let $S^{<\omega}$ denote the set of all finite words with letters in S (the ω in the superscript denotes the first infinite ordinal). For $z \in S^{<\omega}$ we let $|z|$ denote the length of the word z. The lemma below is a strengthened version of a similar lemma due to Krieger [35].

Lemma VII.1. *Let (X, μ) be a probability space, let $\mathcal{F}$ be a sub-σ-algebra, let (Y, ν) be the associated factor of (X, μ), and let $\mu = \int_Y \mu_y \; d\nu(y)$ be the corresponding decomposition of μ. If ξ is a countable Borel partition of X with $\mathrm{H}(\xi|\mathcal{F}) < \infty$, then there is a Borel function $L : Y \times \xi \to \{0, 1, 2\}^{<\omega}$ such that ν-almost-every restriction $L(y, \cdot) : \xi \to \{0, 1, 2\}^{<\omega}$ is injective and*

$$\int_Y \sum_{C \in \xi} |L(y, C)| \cdot \mu_y(C) \; d\nu(y) < \infty.$$

Proof. If ξ is finite then we can simply fix an injection $L : \xi \to \{0, 1, 2\}^k$ for some $k \in \mathbb{N}$. So suppose that ξ is infinite. Say $\xi = \{C_1, C_2, \ldots\}$. Let $\sigma : Y \to \mathrm{Sym}(\mathbb{N})$ be the unique map satisfying for all $n \in \mathbb{N}$: either $\mu_y(C_{\sigma(y)(n+1)}) < \mu_y(C_{\sigma(y)(n)})$ or else

$\mu_y(C_{\sigma(y)(n+1)}) = \mu_y(C_{\sigma(y)(n)})$ and $\sigma(y)(n+1) > \sigma(y)(n)$. Since each map $y \mapsto \mu_y(C_k)$ is Borel (see Chapter II), we see that σ is Borel.

For each n let $t(n) \in \{0,1,2\}^{<\omega}$ be the ternary expansion of n. Note that $|t(n)| \leq \log_3(n) + 1$. For $y \in Y$ and $C_k \in \xi$ define $L(y, C_k) = t(\sigma(y)^{-1}(k))$. Then L is a Borel function and it can be equivalently expressed as

$$L(y, C_{\sigma(y)(n)}) = t(n).$$

If $|t(n)| = |L(y, C_{\sigma(y)(n)})| > -\log \mu_y(C_{\sigma(y)(n)})$ then for all $k \leq n$

$$\mu_y(C_{\sigma(y)(k)}) \geq \mu_y(C_{\sigma(y)(n)}) > e^{-|t(n)|} \geq \frac{1}{e} \cdot e^{-\log_3(n)} = \frac{1}{e} \cdot n^{-\log_3(e)}.$$

Thus

$$\frac{1}{e} \cdot n^{1-\log_3(e)} = n \cdot \frac{1}{e} \cdot n^{-\log_3(e)} < \sum_{k=1}^{n} \mu_y(C_{\sigma(y)(k)}) \leq 1,$$

and hence $n \leq \exp(1/(1 - \log_3(e)))$. Letting m be the least integer greater than $\exp(1/(1 - \log_3(e)))$, we have that $|L(y, C_{\sigma(y)(n)})| \leq -\log \mu_y(C_{\sigma(y)(n)})$ for all $y \in Y$ and all $n > m$. Therefore

$$\begin{aligned}
\sum_{n \in \mathbb{N}} |L(y, C_{\sigma(y)(n)})| \cdot \mu_y(C_{\sigma(y)(n)}) &\leq m \cdot |t(m)| + \sum_{n > m} |L(y, C_{\sigma(y)(n)})| \cdot \mu_y(C_{\sigma(y)(n)}) \\
&\leq m \cdot |t(m)| + \sum_{n \in \mathbb{N}} -\mu_y(C_n) \log \mu_y(C_n) \\
&= m \cdot |t(m)| + \mathrm{H}_{\mu_y}(\xi).
\end{aligned}$$

Integrating both sides over Y and using $\int_Y \mathrm{H}_{\mu_y}(\xi)\ d\nu(y) = \mathrm{H}(\xi|\mathcal{F}) < \infty$ completes the proof. $\square$

Proposition VII.2. *Let $G \curvearrowright (X, \mu)$ be an ergodic p.m.p. action, let $\mathcal{F}$ be a G-invariant sub-σ-algebra, and let ξ be a countable Borel partition with $\mathrm{H}(\xi|\mathcal{F}) < \infty$. Then for every $\epsilon > 0$ there is a finite Borel partition α with $\sigma\text{-alg}_G(\alpha) \vee \mathcal{F} = \sigma\text{-alg}_G(\xi) \vee \mathcal{F}$ and $\mathrm{H}(\alpha|\mathcal{F}) < \mathrm{H}(\xi|\mathcal{F}) + \epsilon$.*

Proof. Let $\pi : (X, \mu) \to (Y, \nu)$ be the factor map associated to $\mathcal{F}$, and let $\mu = \int \mu_y \, d\nu(y)$ be the corresponding decomposition of μ. Apply Lemma VII.1 to obtain a Borel function $L : Y \times \xi \to \{0,1,2\}^{<\omega}$ such that ν-almost-every restriction $L(y, \cdot) : \xi \to \{0,1,2\}^{<\omega}$ is injective and

$$\int_Y \sum_{C \in \xi} |L(y, C)| \cdot \mu_y(C) \; d\nu(y) < \infty.$$

We define $\ell : X \to \{0,1,2\}^{<\omega}$ by

$$\ell(x) = L(\pi(x), C)$$

for $x \in C \in \xi$. Observe that ℓ is σ-alg$(\xi) \vee \mathcal{F}$-measurable and

$$\int_X |\ell(x)| \; d\mu(x) = \int_Y \int_X |\ell(x)| \; d\mu_y(x) \; d\nu(y) = \int_Y \sum_{C \in \xi} |L(y, C)| \cdot \mu_y(C) \; d\nu(y) < \infty.$$

For $n \in \mathbb{N}$ let $\mathcal{P}_n = \{P_n, X \setminus P_n\}$ where

$$P_n = \{x \in X \; : \; |\ell(x)| \geq n\}.$$

Then the P_n's are decreasing and have empty intersection. Refine $\mathcal{P}_n$ to $\beta_n = \{X \setminus P_n, B_n^0, B_n^1, B_n^2\}$ where for $i \in \{0,1,2\}$

$$B_n^i = \{x \in P_n \; : \; \ell(x)(n) = i\}.$$

For $n \in \mathbb{N}$ define

$$\gamma_n = \bigvee_{k \leq n} \beta_k.$$

Since each restriction $L(y, \cdot) : \xi \to \{0,1,2\}^{<\omega}$ is injective we have that

$$\xi \subseteq \mathcal{F} \vee \bigvee_{n \in \mathbb{N}} \sigma\text{-alg}(\gamma_n). \tag{7.1}$$

Fix $0 < \delta < \min(1/4, \epsilon/2)$ with

$$-\delta \cdot \log(\delta) - (1 - \delta) \cdot \log(1 - \delta) + \delta \cdot \log(7) < \epsilon.$$

Since

$$\sum_{n\in\mathbb{N}} \mu(P_n) = \int_X |\ell(x)|\, d\mu(x) < \infty$$

we may fix $N \in \mathbb{N}$ so that $\sum_{n=N}^{\infty} \mu(P_n) < \delta$. Observe that in particular $\mu(P_N) < \delta$ and thus

$$\mu(P_N) + \sum_{n=N}^{\infty} \mu(P_n) < 2\delta < 1/2.$$

For $n \geq N$ we seek to build $\sigma\text{-alg}_G(\mathcal{P}_n \vee \gamma_{n-1})$-expressible functions $\theta_n \in [[E_G^X]]$ with $\mathrm{dom}(\theta_n) = P_n$ and

$$\mathrm{rng}(\theta_n) \subseteq X \setminus \left(P_N \cup \bigcup_{k=N}^{n-1} \theta_k(P_k) \right).$$

We build the θ_n's by induction on $n \geq N$. To begin we note that $\mu(P_N) < \mu(X \setminus P_N)$ and we apply Lemma III.5 to obtain θ_N. Now assume that $\theta_N, \ldots, \theta_{n-1}$ have been defined and posses the properties stated above. Then since γ_{n-1} refines $\mathcal{P}_k \vee \gamma_{k-1}$ for every $k < n$, we obtain from Lemma III.2

$$P_N \cup \bigcup_{k=N}^{n-1} \theta_k(P_k) \in \sigma\text{-alg}_G(\gamma_{n-1}).$$

Also, by our choice of N we have that

$$\begin{aligned} \mu(P_n) \leq \mu(P_N) < \frac{1}{2} < 1 - 2\delta < 1 - \mu(P_N) - \sum_{k=N}^{n-1} \mu(P_k) \\ = \mu\left(X \setminus \left(P_N \cup \textstyle\bigcup_{k=N}^{n-1} \theta_k(P_k) \right) \right). \end{aligned}$$

Therefore we may apply Lemma III.5 to obtain θ_n. This defines the functions θ_n, $n \geq N$.

Define the partition $\beta = \{X \setminus P, B^0, B^1, B^2\}$ of X by

$$P = \bigcup_{n \geq N} \theta_n(P_n);$$

$$B^i = \bigcup_{n \geq N} \theta_n(B_n^i).$$

Note that the above expressions do indeed define a partition of X since the images of the θ_n's are pairwise disjoint. Also define $\mathcal{Q} = \{Q, X \setminus Q\}$ where

$$Q = \bigcup_{n \geq N} \theta_n(P_{n+1}).$$

Note that Q is contained in P and so β might not refine $\mathcal{Q}$. Set $\alpha = \gamma_N \vee \beta \vee \mathcal{Q}$. Then α is finite. Using Lemma II.2 and the facts that $X \setminus P \in \beta \vee \mathcal{Q}$, $\mu(P) < \delta$, and $\mathrm{H}_{\mu_y}(\gamma_N) \leq \mathrm{H}_{\mu_y}(\xi)$ for ν-almost-every $y \in Y$ (since ξ μ_y-almost-everywhere refines γ_N), we obtain

$$\begin{aligned}
\mathrm{H}(\alpha|\mathcal{F}) &\leq \mathrm{H}(\gamma_N|\mathcal{F}) + \mathrm{H}(\beta \vee \mathcal{Q}) \\
&= \mathrm{H}(\gamma_N|\mathcal{F}) + \mathrm{H}(\{P, X \setminus P\}) + \mathrm{H}(\beta \vee \mathcal{Q}|\{P, X \setminus P\}) \\
&\leq \mathrm{H}(\gamma_N|\mathcal{F}) - \mu(P) \cdot \log \mu(P) - \mu(X \setminus P) \log \mu(X \setminus P) + \mu(P) \cdot \log(7) \\
&< \mathrm{H}(\gamma_N|\mathcal{F}) + \epsilon \\
&= \int_Y \mathrm{H}_{\mu_y}(\gamma_N) \, d\nu(y) + \epsilon \\
&\leq \int_Y \mathrm{H}_{\mu_y}(\xi) \, d\nu(y) + \epsilon \\
&= \mathrm{H}(\xi|\mathcal{F}) + \epsilon.
\end{aligned}$$

Thus it only remains to check that $\sigma\text{-alg}_G(\alpha) \vee \mathcal{F} = \sigma\text{-alg}_G(\xi) \vee \mathcal{F}$.

First notice that the function ℓ and all of the partitions γ_n and $\mathcal{P}_n$ are $\sigma\text{-alg}_G(\xi) \vee \mathcal{F}$-measurable and therefore each θ_k is $\sigma\text{-alg}_G(\xi) \vee \mathcal{F}$-expressible. It follows from Lemma III.2 that β, $\mathcal{Q}$, and α are $\sigma\text{-alg}_G(\xi) \vee \mathcal{F}$-measurable. Thus $\sigma\text{-alg}_G(\alpha) \vee \mathcal{F} \subseteq \sigma\text{-alg}_G(\xi) \vee \mathcal{F}$. Now we consider the reverse inclusion. By induction and by (7.1) it suffices to assume that $\gamma_k \subseteq \sigma\text{-alg}_G(\alpha)$ and prove that $\gamma_{k+1} \subseteq \sigma\text{-alg}_G(\alpha)$ as well. This is immediate when $k \leq N$. So assume that $k \geq N$ and that $\gamma_k \subseteq \sigma\text{-alg}_G(\alpha)$. Since θ_k is expressible with respect to $\sigma\text{-alg}_G(\gamma_k) \subseteq \sigma\text{-alg}_G(\alpha)$, we have that

$$P_{k+1} = \theta_k^{-1}(Q) \in \sigma\text{-alg}_G(\alpha)$$

by Lemmas III.2 and III.3. Therefore $\mathcal{P}_{k+1} \subseteq \sigma\text{-alg}_G(\alpha)$. Now since θ_{k+1} is expressible with respect to $\sigma\text{-alg}_G(\mathcal{P}_{k+1} \vee \gamma_k) \subseteq \sigma\text{-alg}_G(\alpha)$ we have that for $i \in \{0, 1, 2\}$

$$B^i_{k+1} = \theta^{-1}_{k+1}(B^i) \in \sigma\text{-alg}_G(\alpha)$$

by Lemmas III.2 and III.3. Thus $\beta_{k+1} \subseteq \sigma\text{-alg}_G(\alpha)$ and we conclude that $\gamma_{k+1} \subseteq \sigma\text{-alg}_G(\alpha)$. This completes the proof. $\square$

CHAPTER VIII

Distributions on finite sets

For a finite probability vector $\bar{q}$, $\epsilon > 0$, and $n \in \mathbb{N}$, we let $L^n_{\bar{q},\epsilon}$ be the set of functions $\ell : \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, |\bar{q}|-1\}$ which approximate the distribution of $\bar{q}$, meaning

$$\forall 0 \leq t < |\bar{q}| \qquad \left| \frac{|\ell^{-1}(t)|}{n} - q_t \right| \leq \epsilon.$$

Similarly, if (X, μ) is a probability space and ξ is a finite partition of X, then we let $L^n_{\xi,\epsilon}$ be the set of functions $\ell : \{0, 1, \ldots, n-1\} \to \xi$ such that

$$\forall C \in \xi \qquad \left| \frac{|\ell^{-1}(C)|}{n} - \mu(C) \right| \leq \epsilon.$$

We define a metric d on the set $L^n_{\bar{q},\infty}$ by

$$d(\ell, \ell') = \frac{1}{n} \cdot \Big| \{0 \leq i < n \, : \, \ell(i) \neq \ell'(i)\} \Big|.$$

If ξ and β are finite partitions of (X, μ) and ξ is finer than β, then we define the *coarsening map* $\pi_\beta : \xi \to \beta$ to be the unique map satisfying $C \subseteq \pi_\beta(C)$ for all $C \in \xi$. By applying π_β coordinate-wise, we obtain a map $\pi_\beta : L^n_{\xi,\infty} \to L^n_{\beta,\infty}$.

This chapter consists of some simple counting lemmas related to the sets $L^n_{\bar{q},\epsilon}$ and $L^n_{\xi,\epsilon}$.

Lemma VIII.1. *Let (X, μ) be a probability space, let ξ and β be finite partitions of X, and let $\delta > 0$. Suppose that ξ refines β and let $\pi_\beta : \xi \to \beta$ be the coarsening map.*

Then there is $\epsilon_0 > 0$ *and* $n_0 \in \mathbb{N}$ *so that for all* $0 < \epsilon \leq \epsilon_0$, $n \geq n_0/\epsilon$, *and every* $b \in L^n_{\beta,\epsilon}$

$$\exp\Big(n \cdot \mathrm{H}(\xi|\beta) - n \cdot \delta\Big) \leq \Big|\Big\{c \in L^n_{\xi,\epsilon} \,:\, \pi_\beta(c) = b\Big\}\Big| \leq \exp\Big(n \cdot \mathrm{H}(\xi|\beta) + n \cdot \delta\Big).$$

Proof. Without loss of generality, we may assume that $\mu(C) > 0$ for all $C \in \xi$. For a ξ-indexed probability vector $\bar{q} = (q_C)_{C\in\xi}$ we let $\pi_\beta(\bar{q}) = (q_B)_{B\in\beta}$ denote the coarsening of $\bar{q}$ induced by π_β, specifically $q_B = \sum_{C\subseteq B} q_C$. Note that if $q_C = \mu(C)$ for every $C \in \xi$ then

$$\mathrm{H}(\xi|\beta) = \sum_{B\in\beta}\sum_{C\subseteq B} -q_C \cdot \log(q_C/q_B).$$

Choose $\epsilon_0 > 0$ so that whenever $\bar{q}$ is a ξ-indexed probability vector with $|q_C - \mu(C)| < \epsilon_0$ for all $C \in \xi$ we have

$$\left|\mathrm{H}(\xi|\beta) - \sum_{B\in\beta}\sum_{C\subseteq B} -q_C \cdot \log(q_C/q_B)\right| < \delta/3.$$

By shrinking ϵ_0 further, we may assume that $\mu(C) > 2\epsilon_0$ for all $C \in \xi$.

Recall that Stirling's formula states that $n!$ is asymptotic to $\sqrt{2\pi n} \cdot n^n \cdot e^{-n}$. Therefore $\frac{1}{n}\log(n!) - \log(n) + 1$ converges to 0. Let n_0 be such that both

$$\left|\frac{1}{n} \cdot \log(n!) - \log(n) + 1\right| < \frac{\delta}{3(|\xi| + |\beta|)}$$

and $(3\epsilon_0 \cdot n)^{|\xi|} < \exp(n \cdot \delta/3)$ for all $n \geq n_0$.

Fix $0 < \epsilon \leq \epsilon_0$, $n \geq n_0/\epsilon$ and $b \in L^n_{\beta,\epsilon}$. Let Q be the set of ξ-indexed probability vectors $\bar{q} = (q_C)_{C\in\xi}$ such that $|q_C - \mu(C)| \leq \epsilon$, $n \cdot q_C \in \mathbb{N}$ for all $C \in \xi$, and $n \cdot q_B = |b^{-1}(B)|$ for all $B \in \beta$. For $\bar{q} \in Q$, basic combinatorics gives

$$\Big|\{\ell \in L^n_{\bar{q},0} \,:\, \pi_\beta(\ell) = b\}\Big| = \prod_{B\in\beta} \frac{(n \cdot q_B)!}{\prod_{C\subseteq B}(n \cdot q_C)!}.$$

Since $q_C \cdot n \geq \epsilon \cdot n \geq n_0$ for all $C \in \xi$, there is κ with $|\kappa| < \delta/3$ such that

$$\begin{aligned}
&\frac{1}{n} \cdot \log |\{\ell \in L^n_{\bar{q},0} \,:\, \pi_\beta(\ell) = b\}| \\
&\qquad = \frac{1}{n} \cdot \sum_{B \in \beta} \log((n \cdot q_B)!) - \frac{1}{n} \cdot \sum_{C \in \xi} \log((n \cdot q_C)!) \\
&\qquad = \sum_{B \in \beta} q_B \cdot (\log(n \cdot q_B) - 1) - \sum_{C \in \xi} q_C \cdot (\log(n \cdot q_C) - 1) + \kappa \\
&\qquad = \sum_{B \in \beta} q_B \cdot \log(q_B) - \sum_{C \in \xi} q_C \cdot \log(q_C) + \kappa \\
&\qquad = \sum_{B \in \beta} \sum_{C \subseteq B} -q_C \cdot \log(q_C/q_B) + \kappa
\end{aligned}$$

So our choice of ϵ_0 gives

$$\exp\Big(n \cdot \mathrm{H}(\xi|\beta) - n \cdot 2\delta/3\Big) < \Big|\{\ell \in L^n_{\bar{q},0} \,:\, \pi_\beta(\ell) = b\}\Big| < \exp\Big(n \cdot \mathrm{H}(\xi|\beta) + 2\delta/3\Big).$$

Finally,

$$\{c \in L^n_{\xi,\epsilon} \,:\, \pi_\beta(c) = b\} = \bigcup_{\bar{q} \in Q} \{\ell \in L^n_{\bar{q},0} \,:\, \pi_\beta(\ell) = b\},$$

and since $|Q| \leq (3\epsilon \cdot n)^{|\xi|} \leq \exp(n \cdot \delta/3)$, the claim follows. $\square$

By taking β to be the trivial partition in the previous lemma, we obtain the following.

Corollary VIII.2. *Let $\bar{q}$ be a finite probability vector and let $\delta > 0$. Then there is $\epsilon_0 > 0$ and $n_0 \in \mathbb{N}$ so that for all $0 < \epsilon \leq \epsilon_0$ and all $n \geq n_0/\epsilon$*

$$\exp\Big(n \cdot \mathrm{H}(\bar{q}) - n \cdot \delta\Big) \leq \Big|L^n_{\bar{q},\epsilon}\Big| \leq \exp\Big(n \cdot \mathrm{H}(\bar{q}) + n \cdot \delta\Big).$$

Corollary VIII.3. *Fix $0 < \kappa < 1$. Then for sufficiently large n we have*

$$\binom{n}{\lfloor \kappa \cdot n \rfloor} \leq \exp\Big(n \cdot 2 \cdot \mathrm{H}(\kappa, 1 - \kappa)\Big),$$

where $\lfloor \kappa \cdot n \rfloor$ is the greatest integer less than or equal to $\kappa \cdot n$.

Proof. Set $\bar{q} = (1-\kappa, \kappa)$. By definition $\binom{n}{\lfloor \kappa \cdot n \rfloor}$ is the number of subsets of $\{0, \ldots, n-1\}$ having cardinality $\lfloor \kappa \cdot n \rfloor$. Such subsets naturally correspond, via their characteristic functions, to elements of $L^n_{\bar{q},\epsilon}$ when $n > 1/\epsilon$. Thus when $n > 1/\epsilon$ we have $\binom{n}{\lfloor \kappa \cdot n \rfloor} \leq |L^n_{\bar{q},\epsilon}|$. Now apply Corollary VIII.2 with $\delta < \mathrm{H}(\kappa, 1-\kappa)$ to obtain ϵ with

$$\binom{n}{\lfloor \kappa \cdot n \rfloor} \leq |L^n_{\bar{q},\epsilon}| \leq \exp\Big(n \cdot 2 \cdot \mathrm{H}(\kappa, 1-\kappa)\Big)$$

for all $n > 1/\epsilon$. □

Corollary VIII.4. *Let (X, μ) be a probability space, let ξ and β be finite partitions of X with ξ finer than β, let $\bar{q}$ be a finite probability vector, and let $0 < r \leq 1$. Assume that $\mathrm{H}(\xi|\beta) < r \cdot \mathrm{H}(\bar{q})$. Then there are $\delta > 0$, $\epsilon_0 > 0$, and $n_0 \in \mathbb{N}$ such that for all $0 < \epsilon \leq \epsilon_0$ and all $n \geq n_0/\epsilon$, there are injections*

$$f_b : \{c \in L^n_{\xi,\epsilon} \,:\, \pi_\beta(c) = b\} \to L^{\lfloor r \cdot n \rfloor}_{\bar{q},\epsilon}$$

for every $b \in L^n_{\beta,\epsilon}$ such that $d(f_b(c), f_{b'}(c')) > 20\delta|\bar{q}|$ whenever $f_b(c) \neq f_{b'}(c')$.

Proof. Fix $\delta > 0$ such that $20\delta|\bar{q}| < 1/2$, and

$$\mathrm{H}(\xi|\beta) < r \cdot \mathrm{H}(\bar{q}) - \delta - r\delta - 2r \cdot \mathrm{H}(20\delta|\bar{q}|, 1 - 20\delta|\bar{q}|) - 20\delta|\bar{q}|r \cdot \log|\bar{q}|.$$

Fix $m \in \mathbb{N}$ with $rm \cdot (\delta/2) > \mathrm{H}(\bar{q})$. By Lemma VIII.1 and Corollaries VIII.2 and VIII.3 there are $\epsilon_0 > 0$ and $n_0 \geq m$ such that for all $0 < \epsilon \leq \epsilon_0$, $n \geq n_0/\epsilon$, and all $b \in L^n_{\beta,\epsilon}$

$$\begin{aligned}
\Big|\Big\{c \in L^n_{\xi,\epsilon} \,:\, \pi_\beta(c) = b\Big\}\Big| &\leq \exp\Big(n \cdot \mathrm{H}(\xi|\beta) + n \cdot \delta\Big), \\
\Big|L^{\lfloor r \cdot n \rfloor}_{\bar{q},\epsilon}\Big| &\geq \exp\Big(\lfloor rn \rfloor \cdot \mathrm{H}(\bar{q}) - \lfloor rn \rfloor \cdot \delta/2\Big) \\
&\geq \exp\Big(rn \cdot \mathrm{H}(\bar{q}) - rn \cdot \delta\Big), \\
\text{and} \qquad \binom{\lfloor r \cdot n \rfloor}{\lfloor 20\delta|\bar{q}|r \cdot n \rfloor} &\leq \exp\Big(rn \cdot 2 \cdot \mathrm{H}(20\delta|\bar{q}|, 1 - 20\delta|\bar{q}|)\Big).
\end{aligned}$$

Then by our choice of δ we have that for all $\epsilon \leq \epsilon_0$, $n \geq n_0/\epsilon$, and all $b \in L^n_{\beta,\epsilon}$

$$\left|\left\{c \in L^n_{\xi,\epsilon} \,:\, \pi_\beta(c) = b\right\}\right| \leq \exp\Big(n \cdot \mathrm{H}(\xi|\beta) + n \cdot \delta\Big)$$

$$< \exp\Big(nr \cdot \mathrm{H}(\bar{q}) - n \cdot r\delta - n \cdot 2r \cdot \mathrm{H}(20\delta|\bar{q}|, 1 - 20\delta|\bar{q}|) - n \cdot 20\delta|\bar{q}|r \cdot \log|\bar{q}|\Big)$$

$$\leq \left|L^{\lfloor r\cdot n\rfloor}_{\bar{q},\epsilon}\right| \cdot \binom{\lfloor r \cdot n\rfloor}{\lfloor 20\delta|\bar{q}|r \cdot n\rfloor}^{-1} \cdot |\bar{q}|^{-n20\delta|\bar{q}|r}. \tag{8.1}$$

Now fix $0 < \epsilon \leq \epsilon_0$ and $n \geq n_0/\epsilon$. For $V \subseteq L^{\lfloor r\cdot n\rfloor}_{\bar{q},\infty}$ let

$$B_d(V;\rho) = \{\ell \in L^{\lfloor r\cdot n\rfloor}_{\bar{q},\infty} \,:\, \exists v \in V \quad d(\ell, v) \leq \rho\}.$$

Basic combinatorics implies that for $\rho < 1/2$

$$\Big|B_d(V;\rho)\Big| \leq |V| \cdot \binom{\lfloor r \cdot n\rfloor}{\lfloor \rho \cdot \lfloor r \cdot n\rfloor\rfloor} \cdot |\bar{q}|^{\rho r\cdot n} \leq |V| \cdot \binom{\lfloor r \cdot n\rfloor}{\lfloor \rho r \cdot n\rfloor} \cdot |\bar{q}|^{\rho r\cdot n}.$$

Let $K \subseteq L^{\lfloor r\cdot n\rfloor}_{\bar{q},\epsilon}$ be maximal with the property that $d(k,k') > 20\delta|\bar{q}|$ for all $k \neq k' \in K$. Then by maximality of K we have $L^{\lfloor r\cdot n\rfloor}_{\bar{q},\epsilon} \subseteq B_d(K; 20\delta|\bar{q}|)$. Therefore

$$\left|L^{\lfloor r\cdot n\rfloor}_{\bar{q},\epsilon}\right| \leq \Big|B_d(K; 20\delta|\bar{q}|)\Big| \leq |K| \cdot \binom{\lfloor r \cdot n\rfloor}{\lfloor 20\delta|\bar{q}|r \cdot n\rfloor} \cdot |\bar{q}|^{20\delta|\bar{q}|r\cdot n}.$$

So $|\{c \in L^n_{\xi,\epsilon} \,:\, \pi_\beta(c) = b\}| < |K|$ for every $b \in L^n_{\beta,\epsilon}$ by (8.1). Thus we may choose injections $f_b : \{c \in L^n_{\xi,\epsilon} \,:\, \pi_\beta(c) = b\} \to K \subseteq L^{\lfloor r\cdot n\rfloor}_{\bar{q},\epsilon}$ for every $b \in L^n_{\beta,\epsilon}$. $\square$

Lemma VIII.5. *Let $\bar{q}$ be a finite probability vector, let $\epsilon, \delta > 0$, and let $n \in \mathbb{N}$. Assume that $\epsilon < \delta < 1$, and $\delta \cdot n > 1$. If $\ell \in L^n_{\bar{q},\epsilon}$ then there is $J \subseteq \{0, 1, \ldots, n-1\}$ such that $|J| < 3\delta|\bar{q}| \cdot n$ and*

$$\forall 0 \leq t < |\bar{q}| \qquad \frac{1}{n} \cdot \Big|\ell^{-1}(t) \setminus J\Big| < \min\Big((q_t + \epsilon)(1 - \delta), \quad q_t\Big).$$

Proof. For $0 \le t < |\bar{q}|$ we have

$$\begin{aligned}
\left|\ell^{-1}(t)\right| &- \min\Big((n \cdot q_t + n \cdot \epsilon)(1-\delta), \quad n \cdot q_t\Big) \\
&\le n \cdot q_t + n \cdot \epsilon - \min\Big((n \cdot q_t + n \cdot \epsilon)(1-\delta), \quad n \cdot q_t\Big) \\
&\le \max\Big(\delta \cdot (n \cdot q_t + n \cdot \epsilon),\ n \cdot \epsilon\Big) \\
&\le 2\delta \cdot n.
\end{aligned}$$

Therefore we may pick $J_t \subseteq \ell^{-1}(t)$ with

$$\left|\ell^{-1}(t)\right| - \min\Big((n \cdot q_t + n \cdot \epsilon)(1-\delta), \quad n \cdot q_t\Big) < \left|J_t\right| \le 2\delta \cdot n + 1 < 3\delta \cdot n.$$

Finally, we set $J = \bigcup_{t=0}^{|\bar{q}|-1} J_t$. ☐

CHAPTER IX

Krieger's finite generator theorem

Let (X, μ) be a probability space. If $\xi = \{C_t : 0 \leq t < |\xi|\}$ is an ordered partition of X, then we let $\mathrm{dist}(\xi)$ be the probability vector with $\mathrm{dist}(\xi)(t) = \mu(C_t)$ for $0 \leq t < |\xi|$. By associating ξ to the probability vector $\mathrm{dist}(\xi)$ in this manner, we also identify the two sets $L^n_{\xi,\epsilon}$ and $L^n_{\mathrm{dist}(\xi),\epsilon}$. We will also find it helpful to write L^n for the set of all functions $\ell : \{0, 1, \ldots, n-1\} \to \mathbb{N} \cup \{0\}$. For $k \leq n$, $\ell \in L^n$, and $\ell' \in L^k$ we define

$$d(\ell, \ell') = d(\ell', \ell) = \frac{1}{k} \cdot \Big|\{0 \leq i < k \,:\, \ell(i) \neq \ell'(i)\}\Big|.$$

When $n = k$, $d(\cdot, \cdot)$ coincides with the metric defined at the start of Chapter VIII.

Let $G \curvearrowright (X, \mu)$ be a p.m.p. action, and let ξ be a partition of X. If $n \in \mathbb{N}$ and $\theta \in [E^X_G]$ has the property that almost-every E_θ class has cardinality n, then we can associate to each $x \in X$ its (ξ, θ)-name $\mathcal{N}^\theta_\xi(x) \in L^n_{\xi,\infty}$ defined by setting $\mathcal{N}^\theta_\xi(x)(i) = C$ if $\theta^i(x) \in C \in \xi$. If furthermore ξ is an ordered partition then we may view $\mathcal{N}^\theta_\xi(x)$ as an element of $L^n_{\mathrm{dist}(\xi),\infty} \subseteq L^n$.

We now present the main theorem. As a corollary we will obtain Theorem I.6 from the introduction.

Theorem IX.1. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action with (X, μ) non-atomic, and let $\mathcal{F}$ be a G-invariant sub-σ-algebra. If ξ is a countable Borel partition of X,*

$0 < r \leq 1$, *and* $\bar{p}$ *is a probability vector with* $\mathrm{H}(\xi|\mathcal{F}) < r \cdot \mathrm{H}(\bar{p})$, *then there is a Borel pre-partition* $\alpha = \{A_i : 0 \leq i < |\bar{p}|\}$ *with* $\mu(A_i) = rp_i$ *for every* i *and* $\sigma\text{-alg}_G(\xi) \vee \mathcal{F} \subseteq \sigma\text{-alg}_G^{red}(\alpha) \vee \mathcal{F}$.

Proof. Apply Proposition VII.2 to obtain a finite Borel partition ξ' with $\sigma\text{-alg}_G(\xi') \vee \mathcal{F} = \sigma\text{-alg}_G(\xi) \vee \mathcal{F}$ and $\mathrm{H}(\xi'|\mathcal{F}) < r \cdot \mathrm{H}(\bar{p})$. Since ξ' is finite, by Lemma II.2 we have that $\mathrm{H}(\xi'|\mathcal{F})$ is equal to the infimum of $\mathrm{H}(\xi'|\beta)$ over finite $\mathcal{F}$-measurable partitions β of X. So fix a finite $\mathcal{F}$-measurable partition β with $\mathrm{H}(\xi'|\beta) < r \cdot \mathrm{H}(\bar{p})$. Since $\mathrm{H}(\xi' \vee \beta|\beta) = \mathrm{H}(\xi'|\beta)$ and $\sigma\text{-alg}_G(\xi' \vee \beta) \vee \mathcal{F} = \sigma\text{-alg}_G(\xi') \vee \mathcal{F}$, we may replace ξ' with $\xi' \vee \beta$ if necessary and assume that ξ' refines β. Let $\pi_\beta : \xi' \to \beta$ be the coarsening map. Finally, by Lemma II.2 we may let $\bar{q}$ be a finite probability vector which coarsens $\bar{p}$ and satisfies $\mathrm{H}(\xi'|\beta) < r \cdot \mathrm{H}(\bar{q}) \leq r \cdot \mathrm{H}(\bar{p})$.

Let $0 < \delta < 1$, $\epsilon_0 > 0$, and $n_0 \in \mathbb{N}$ be as given by Corollary VIII.4. Let $0 < \epsilon < r\delta$, and $M \subseteq X$ with $\mu(M) = r \cdot \delta$ be as given by Corollary VI.2. Note that replacing ϵ by a smaller quantity will not interfere with applying Corollary VI.2, so we may assume that $\epsilon \leq \epsilon_0$. We may also increase n_0 if necessary so that $n_0 \cdot r\delta > 1$ and $\lfloor r \cdot n_0 \rfloor > r \cdot n_0/2$. By Proposition IV.5 there are $n \geq n_0/\epsilon$, Borel sets $S_1, S_2 \subseteq X$ with $\mu(S_1) + \mu(S_2) < \epsilon$, and a $\sigma\text{-alg}_G(\{S_1, S_2\})$-expressible $\theta \in [E_G^X]$ such that E_θ admits a $\sigma\text{-alg}_G(\{S_1, S_2\})$-measurable transversal Y and such that for μ-almost-every $x \in X$, the E_θ class of x has cardinality n,

$$\begin{aligned} &\forall C \in \xi' \cup \beta &\qquad& \mu(C) - \epsilon < \frac{|C \cap [x]_{E_\theta}|}{|[x]_{E_\theta}|} < \mu(C) + \epsilon, \\ &\text{and} && \frac{|M \cap [x]_{E_\theta}|}{|[x]_{E_\theta}|} < \mu(M) + r \cdot \delta = 2r \cdot \delta. \end{aligned}$$

So we have that $\mathcal{N}_{\xi'}^\theta(y) \in L_{\xi',\epsilon}^n$ and $\mathcal{N}_\beta^\theta(y) \in L_{\beta,\epsilon}^n$ for almost-every $y \in Y$. Set $k = \lfloor r \cdot n \rfloor$ and let $f_b : \{c \in L_{\xi',\epsilon}^n : \pi_\beta(c) = b\} \to L_{\bar{q},\epsilon}^k$ be the injections provided by Corollary VIII.4 for $b \in L_{\beta,\epsilon}^n$. For $y \in Y$ set $b_y = \mathcal{N}_\beta^\theta(y)$, $c_y = \mathcal{N}_{\xi'}^\theta(y)$, and $\tilde{a}_y = f_{b_y}(c_y) \in L_{\bar{q},\epsilon}^k$.

Also define

$$M_y = \{0 \leq i < n \,:\, \theta^i(y) \in M\}.$$

Then $|M_y| < 2\delta r \cdot n$ for μ-almost-every $y \in Y$. Since $\tilde{a}_y \in L^k_{\bar{q},\epsilon}$, Lemma VIII.5 provides a set $J_y \subseteq \{0, 1, \ldots, k-1\}$ with $|J_y| < 3r\delta|\bar{q}| \cdot n$ such that for all $0 \leq t < |\bar{q}|$

$$\begin{aligned} \frac{1}{k} \cdot \left|\tilde{a}_y^{-1}(t) \setminus (M_y \cup J_y)\right| &< \min\Big((q_t + \epsilon)(1 - r\delta), \quad q_t\Big) \\ &= \min\Big((q_t + \epsilon)\mu(X \setminus M), \quad q_t\Big). \end{aligned} \tag{9.1}$$

Clearly we can arrange the map $y \mapsto J_y$ to be Borel. We then let J be the Borel set

$$J = \{\theta^j(y) \,:\, y \in Y,\ j \in J_y\}.$$

Define a pre-partition $\mathcal{Q} = \{Q_t \,:\, 0 \leq t < |\bar{q}|\}$ by setting

$$Q_t = \{\theta^i(y) \,:\, y \in Y,\ 0 \leq i < k,\ i \notin M_y \cup J_y, \text{ and } \tilde{a}_y(i) = t\}.$$

Observe that $\mu(Y) = 1/n$ since Y is a transversal for E_θ. By (9.1) we have that for every $0 \leq t < |\bar{q}|$

$$\begin{aligned} \mu(Q_t) = \int_Y |\tilde{a}_y^{-1}(t) \setminus (M_y \cup J_y)|\ d\mu(y) &< \frac{k}{n} \cdot \min\Big((q_t + \epsilon)\mu(X \setminus M),\ q_t\Big) \\ &< \min\Big((r \cdot q_t + r \cdot \epsilon)\mu(X \setminus M),\ r \cdot q_t\Big) \end{aligned}$$

Now apply Corollary VI.2 to get a pre-partition $\alpha' = \{A'_t \,:\, 0 \leq t < |\bar{q}|\}$ of X with $Q_t \subseteq A'_t$ and $\mu(A'_t) = r \cdot q_t$ for every t, and with $S_1, S_2 \in \sigma\text{-alg}^{\text{red}}_G(\alpha')$. We have that θ is expressible and Y is measurable with respect to $\sigma\text{-alg}_G(\{S_1, S_2\}) \subseteq \sigma\text{-alg}^{\text{red}}_G(\alpha')$. By Lemma III.3 it follows that θ^i is $\sigma\text{-alg}^{\text{red}}_G(\alpha')$-expressible for all $i \in \mathbb{Z}$.

We claim that the map $y \in Y \mapsto \tilde{a}_y$ is $\sigma\text{-alg}^{\text{red}}_G(\alpha')$-measurable. We check this via the definition of a reduced σ-algebra. Fix $y \in Y$ and $x \in X$ with either $x \notin Y$ or $\tilde{a}_x \neq \tilde{a}_y$. If $x \notin Y$ then we are done since $Y \in \sigma\text{-alg}^{\text{red}}_G(\alpha')$. So suppose that $x \in Y$

and $\tilde{a}_x \neq \tilde{a}_y$. Then $d(\tilde{a}_y, \tilde{a}_x) > 20\delta|\bar{q}|$. Set $I = \{0 \leq i < k : \tilde{a}_y(i) \neq \tilde{a}_x(i)\}$ and note $|I| > 20\delta|\bar{q}| \cdot k$. Since

$$\left| M_y \cup J_y \cup M_x \cup J_x \right| < 10r\delta|\bar{q}| \cdot n < 20\delta|\bar{q}| \cdot k < |I|,$$

we may fix $i \in I \setminus (M_y \cup J_y \cup M_x \cup J_x)$. Since θ^i is $\sigma\text{-alg}_G^{\text{red}}(\alpha')$-expressible, there is a $\sigma\text{-alg}_G^{\text{red}}(\alpha')$-measurable partition $\{Z_g : g \in G\}$ of X such that $\theta^i(z) = g \cdot z$ for all $g \in G$ and $z \in Z_g$. If y and x are separated by the partition $\{Z_g : g \in G\}$ then, since this partition is $\sigma\text{-alg}_G^{\text{red}}(\alpha')$-measurable, there must be $h \in G$ with both $h \cdot y$ and $h \cdot x$ lying in $\cup\alpha'$ and separated by α'. We are done in this case. So assume there is $g \in G$ with $y, x \in Z_g$. Then $g \cdot y = \theta^i(y)$ lies in $Q_t \subseteq A'_t$ where $t = \tilde{a}_y(i)$ and similarly $g \cdot x = \theta^i(x)$ lies in $Q_s \subseteq A'_s$ where $s = \tilde{a}_x(i)$. As $t \neq s$ we have that $g \cdot y$ and $g \cdot x$ lie in $\cup\alpha'$ and are separated by α'. This proves the claim.

We observe that the map $y \in Y \mapsto b_y$ is $\sigma\text{-alg}_G^{\text{red}}(\alpha') \vee \mathcal{F}$-measurable since the value of b_y is entirely determined by the location of y in the partition $\bigvee_{i=0}^{n-1} \theta^{-i}(\beta) \restriction Y$ of Y. This partition is $\sigma\text{-alg}_G^{\text{red}}(\alpha') \vee \mathcal{F}$-measurable by Lemmas III.2 and III.3. So the map $y \in Y \mapsto (b_y, \tilde{a}_y)$ is $\sigma\text{-alg}_G^{\text{red}}(\alpha') \vee \mathcal{F}$-measurable. Since $c_y = f_{b_y}^{-1}(\tilde{a}_y)$, it follows that the map $y \in Y \mapsto c_y$ is $\sigma\text{-alg}_G^{\text{red}}(\alpha') \vee \mathcal{F}$-measurable as well. For $C_t \in \xi'$ we have

$$\begin{aligned} C_t &= \{\theta^i(y) : y \in Y,\ 0 \leq i < n,\ \text{and } c_y(i) = t\} \\ &= \bigcup_{i=0}^{n-1} \theta^i\Big(\{y \in Y : c_y(i) = t\}\Big). \end{aligned}$$

Therefore $\xi' \subseteq \sigma\text{-alg}_G^{\text{red}}(\alpha') \vee \mathcal{F}$ by Lemmas III.2 and III.3. We conclude that

$$\sigma\text{-alg}_G(\xi) \vee \mathcal{F} = \sigma\text{-alg}_G(\xi') \vee \mathcal{F} \subseteq \sigma\text{-alg}_G^{\text{red}}(\alpha') \vee \mathcal{F}.$$

Finally, since (X, μ) is non-atomic, $\mu(A'_t) = r \cdot q_t$, and $\bar{q}$ is a coarsening of $\bar{p}$, there is a refinement α of α' with $\mu(A_t) = r \cdot p_t$ for all $0 \leq t < |\bar{p}|$. Clearly we still have $\sigma\text{-alg}_G(\xi) \vee \mathcal{F} \subseteq \sigma\text{-alg}_G^{\text{red}}(\alpha) \vee \mathcal{F}$. $\square$

Now Theorem I.6 follows quickly.

Proof of Theorem I.6. By assumption $h^{\mathrm{Rok}}_G(X,\mu|\mathcal{F}) < r \cdot \mathrm{H}(\bar{p})$. Thus there exists a partition ξ satisfying $\mathrm{H}(\xi|\mathcal{F}) < r \cdot \mathrm{H}(\bar{p})$ and $\sigma\text{-alg}_G(\xi) \vee \mathcal{F} = \mathcal{B}(X)$. By applying Theorem IX.1 we obtain a pre-partition $\alpha = \{A_i : 0 \leq i < |\bar{p}|\}$ with $\mu(A_i) = rp_i$ for every $0 \leq i < |\bar{p}|$ and $\mathcal{B}(X) = \sigma\text{-alg}_G(\xi) \vee \mathcal{F} \subseteq \sigma\text{-alg}^{\mathrm{red}}_G(\alpha) \vee \mathcal{F}$. $\square$

By letting $\mathcal{F} = \{X, \varnothing\}$ be the trivial σ-algebra, we obtain the following.

Corollary IX.2. *Let $G \curvearrowright (X,\mu)$ be a p.m.p. ergodic action with (X,μ) non-atomic. If ξ is a countable Borel partition of X, $0 < r \leq 1$, and $\bar{p}$ is a probability vector with $\mathrm{H}(\xi) < r \cdot \mathrm{H}(\bar{p})$, then there is a Borel pre-partition $\alpha = \{A_i : 0 \leq i < |\bar{p}|\}$ with $\mu(A_i) = r \cdot p_i$ for every $0 \leq i < |\bar{p}|$ and $\xi \subseteq \sigma\text{-alg}^{red}_G(\alpha)$.*

Just as Theorem I.6 follows from Theorem IX.1, we see that Theorem I.3 follows from Corollary IX.2. We mention that in the above corollary, $\sigma\text{-alg}_G(\xi)$ could correspond to a purely atomic factor $G \curvearrowright (Y,\nu)$ of $G \curvearrowright (X,\mu)$. In this case Theorem I.3 would not be applicable, and so Corollary IX.2 offers a bit more generality.

Corollary IX.3. *Let $G \curvearrowright (X,\mu)$ be a p.m.p. ergodic action with (X,μ) non-atomic, and let $\mathcal{F}$ be a G-invariant sub-σ-algebra. If $G \curvearrowright (Y,\nu)$ is a factor of $G \curvearrowright (X,\mu)$ and Σ is the sub-σ-algebra of X associated to Y then*

$$h^{\mathrm{Rok}}_G(X,\mu|\mathcal{F}) \leq h^{\mathrm{Rok}}_G(Y,\nu) + h^{\mathrm{Rok}}_G(X,\mu|\mathcal{F} \vee \Sigma).$$

Proof. This is immediate if either $h^{\mathrm{Rok}}_G(Y,\nu)$ or $h^{\mathrm{Rok}}_G(X,\mu|\mathcal{F}\vee\Sigma)$ is infinite, so suppose that both are finite. Fix $\epsilon > 0$ and fix a generating partition β' for $G \curvearrowright (Y,\nu)$ with $\mathrm{H}(\beta') < h^{\mathrm{Rok}}_G(Y,\nu) + \epsilon/2$. Pull back β' to a partition β of X. Then $\mathrm{H}(\beta) = \mathrm{H}(\beta')$ and $\sigma\text{-alg}_G(\beta) = \Sigma$. By definition of $h^{\mathrm{Rok}}_G(X,\mu|\mathcal{F} \vee \Sigma)$, there is a partition γ' of X with

$$\mathrm{H}(\gamma'|\mathcal{F} \vee \Sigma) < h^{\mathrm{Rok}}_G(X,\mu|\mathcal{F} \vee \Sigma) + \epsilon/2$$

and $\sigma\text{-alg}_G(\gamma') \vee \mathcal{F} \vee \Sigma = \mathcal{B}(X)$. Apply Theorem IX.1 to get a partition γ of X with

$$\mathrm{H}(\gamma) < h_G^{\mathrm{Rok}}(X, \mu|\mathcal{F} \vee \Sigma) + \epsilon/2$$

and $\sigma\text{-alg}_G(\gamma) \vee \mathcal{F} \vee \Sigma = \mathcal{B}(X)$. Then

$$\mathcal{B}(X) = \sigma\text{-alg}_G(\gamma) \vee \mathcal{F} \vee \Sigma = \sigma\text{-alg}_G(\gamma \vee \beta) \vee \mathcal{F},$$

and hence

$$h_G^{\mathrm{Rok}}(X, \mu|\mathcal{F}) \leq \mathrm{H}(\beta \vee \gamma|\mathcal{F}) \leq \mathrm{H}(\beta) + \mathrm{H}(\gamma) < h_G^{\mathrm{Rok}}(Y, \nu) + h_G^{\mathrm{Rok}}(X, \mu|\mathcal{F} \vee \Sigma). \quad \square$$

Essentially the same proof yields the following.

Corollary IX.4. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action with (X, μ) non-atomic, and let $\mathcal{F}$ be a G-invariant sub-σ-algebra. If α is a partition and $\mathcal{C} \subseteq \mathcal{B}(X)$ then*

$$h_{G,X}^{\mathrm{Rok}}(\alpha \mid \mathcal{F}) \leq h_{G,X}^{\mathrm{Rok}}(\mathcal{C} \mid \mathcal{F}) + h_{G,X}^{\mathrm{Rok}}(\alpha \mid \mathcal{F} \vee \sigma\text{-alg}_G(\mathcal{C})),$$

$$\textit{and} \quad h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F}) \leq h_{G,X}^{\mathrm{Rok}}(\mathcal{C} \mid \mathcal{F}) + h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F} \vee \sigma\text{-alg}_G(\mathcal{C})).$$

CHAPTER X

Relative Rokhlin entropy and amenable groups

We verify that for free ergodic actions of amenable groups, relative Rokhlin entropy and relative Kolmogorov–Sinai entropy agree. This result was previously established in the non-relative case by the author and Tucker-Drob [48].

We first recall the definition of relative Kolmogorov–Sinai entropy. Let G be a countably infinite amenable group, and let $G \curvearrowright (X, \mu)$ be a free p.m.p. action. For a partition α and a finite set $T \subseteq G$, we write α^T for the join $\bigvee_{t \in T} t \cdot \alpha$, where $t \cdot \alpha = \{t \cdot A : A \in \alpha\}$. Given a G-invariant sub-σ-algebra $\mathcal{F}$, the relative Kolmogorov–Sinai entropy is defined as

$$h_G(X, \mu | \mathcal{F}) = \sup_{\alpha} \inf_{T \subseteq G} \frac{1}{|T|} \cdot \mathrm{H}(\alpha^T | \mathcal{F}),$$

where α ranges over all finite partitions and T ranges over finite subsets of G [13]. Equivalently, one can replace the infimum with a limit over a Følner sequence (T_n) [39]. Recall that a sequence $T_n \subseteq G$ of finite sets is a *Følner sequence* if

$$\lim_{n \to \infty} \frac{|\partial_K(T_n)|}{|T_n|} = 0$$

for every finite $K \subseteq G$, where $\partial_K(T) = \{t \in T : tK \not\subseteq T\}$. We also write $\mathcal{I}_K(T)$ for $T \setminus \partial_K(T)$.

Proposition X.1. *Let G be a countably infinite amenable group, let $G \curvearrowright (X, \mu)$ be a free ergodic action, and let $\mathcal{F}$ be a G-invariant sub-σ-algebra. Then the relative Kolmogorov–Sinai entropy and relative Rokhlin entropy coincide:*

$$h_G(X, \mu|\mathcal{F}) = h_G^{\mathrm{Rok}}(X, \mu|\mathcal{F}).$$

Proof. We first show that $h_G(X, \mu|\mathcal{F}) \leq h_G^{\mathrm{Rok}}(X, \mu|\mathcal{F})$. If $h_G^{\mathrm{Rok}}(X, \mu|\mathcal{F}) = \infty$ then there is nothing to show. So suppose that $h_G^{\mathrm{Rok}}(X, \mu|\mathcal{F}) < \infty$ and fix $\epsilon > 0$. Let α be a countable partition with $\sigma\text{-alg}_G(\alpha) \vee \mathcal{F} = \mathcal{B}(X)$ and $\mathrm{H}(\alpha|\mathcal{F}) < h_G^{\mathrm{Rok}}(X, \mu|\mathcal{F}) + \epsilon$. Let β be any finite partition of X and let (T_n) be a Følner sequence. Then by Lemma II.2

$$0 = \mathrm{H}(\beta|\sigma\text{-alg}_G(\alpha) \vee \mathcal{F}) = \inf_{K \subseteq G} \mathrm{H}(\beta|\alpha^K \vee \mathcal{F}),$$

where K ranges over finite subsets of G. Fix $K \subseteq G$ so that $\mathrm{H}(\beta|\alpha^K \vee \mathcal{F}) < \epsilon$. Note that $\mathrm{H}(t \cdot \beta|\alpha^{tK} \vee \mathcal{F}) < \epsilon$ for all $t \in G$. Therefore

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{|T_n|} \cdot \mathrm{H}(\beta^{T_n}|\mathcal{F}) \\
&\qquad \leq \lim_{n\to\infty} \frac{1}{|T_n|} \cdot \mathrm{H}(\alpha^{T_n} \vee \beta^{T_n}|\mathcal{F}) \\
&\qquad = \lim_{n\to\infty} \frac{1}{|T_n|} \cdot \mathrm{H}(\alpha^{T_n}|\mathcal{F}) + \frac{1}{|T_n|} \cdot \mathrm{H}(\beta^{T_n}|\alpha^{T_n} \vee \mathcal{F}) \\
&\qquad \leq \lim_{n\to\infty} \frac{1}{|T_n|} \cdot \sum_{t \in T_n} \Big(\mathrm{H}(t \cdot \alpha|\mathcal{F}) + \mathrm{H}(t \cdot \beta|\alpha^{T_n} \vee \mathcal{F})\Big) \\
&\qquad < \lim_{n\to\infty} h_G^{\mathrm{Rok}}(X, \mu|\mathcal{F}) + \epsilon + \frac{|\mathcal{I}_K(T_n)|}{|T_n|} \cdot \epsilon + \frac{|\partial_K(T_n)|}{|T_n|} \cdot \mathrm{H}(\beta) \\
&\qquad = h_G^{\mathrm{Rok}}(X, \mu|\mathcal{F}) + 2\epsilon.
\end{aligned}$$

Now let ϵ tend to 0 and then take the supremum over all β.

Now we argue that $h_G^{\mathrm{Rok}}(X, \mu|\mathcal{F}) \leq h_G(X, \mu|\mathcal{F})$. Since the action of G is free, a theorem of Seward and Tucker-Drob [48] states that there is a factor action $G \curvearrowright (Z, \eta)$ of (X, μ) such that the action of G on Z is free and $h_G^{\mathrm{Rok}}(Z, \eta) < \epsilon$. Let

Σ be the G-invariant sub-σ-algebra of X associated to Z, and let $G \curvearrowright (Y, \nu)$ be the factor of (X, μ) associated to $\mathcal{F} \vee \Sigma$. Then G acts freely on (Y, ν) since (Y, ν) factors onto (Z, η). By the Ornstein–Weiss theorem [38], all free ergodic actions of countably infinite amenable groups are orbit equivalent. In particular, there is a free ergodic p.m.p. action $\mathbb{Z} \curvearrowright (Y, \nu)$ which has the same orbits as $G \curvearrowright (Y, \nu)$ and has 0 Kolmogorov–Sinai entropy, $h_{\mathbb{Z}}(Y, \nu) = 0$. By the Rokhlin generator theorem [41], we have $h^{\mathrm{Rok}}_{\mathbb{Z}}(Y, \nu) = 0$ as well.

Let's say $\mathbb{Z} = \langle t \rangle$. Define $c : Y \to G$ by

$$c(y) = g \Longleftrightarrow t \cdot y = g \cdot y.$$

Let $f : (X, \mu) \to (Y, \nu)$ be the factor map, and let $\mathbb{Z}$ act on (X, μ) by setting

$$t \cdot x = c(f(x)) \cdot x.$$

Then $\mathcal{F} \vee \Sigma$ and the actions of G and $\mathbb{Z}$ on (X, μ) satisfy the assumptions of Proposition III.4. Equivalently, in the terminology of Rudolph–Weiss [43], the orbit-change cocycles between the actions of G and $\mathbb{Z}$ on X are $\mathcal{F} \vee \Sigma$-measurable. Thus $h_G(X, \mu | \mathcal{F} \vee \Sigma) = h_{\mathbb{Z}}(X, \mu | \mathcal{F} \vee \Sigma)$ by [43, Theorem 2.6]. Also, since $h^{\mathrm{Rok}}_{\mathbb{Z}}(X, \mu | \mathcal{F} \vee \Sigma) \leq h^{\mathrm{Rok}}_{\mathbb{Z}}(X, \mu)$ and $h^{\mathrm{Rok}}_{\mathbb{Z}}(Y, \nu) = 0$, it follows from Corollary IX.3 that

$$h^{\mathrm{Rok}}_{\mathbb{Z}}(X, \mu | \mathcal{F} \vee \Sigma) = h^{\mathrm{Rok}}_{\mathbb{Z}}(X, \mu). \tag{10.1}$$

We have

$$\begin{aligned}
h_G(X, \mu | \mathcal{F} \vee \Sigma) &= h_{\mathbb{Z}}(X, \mu | \mathcal{F} \vee \Sigma) && \text{by the Rudolph–Weiss theorem [43]} \\
&= h_{\mathbb{Z}}(X, \mu) - h_{\mathbb{Z}}(Y, \nu) && \text{by the Abramov–Rokhlin theorem [4]} \\
&= h_{\mathbb{Z}}(X, \mu) && \text{since } h_{\mathbb{Z}}(Y, \nu) = 0 \\
&= h^{\mathrm{Rok}}_{\mathbb{Z}}(X, \mu) && \text{by the Rokhlin generator theorem [41]} \\
&= h^{\mathrm{Rok}}_{\mathbb{Z}}(X, \mu | \mathcal{F} \vee \Sigma) && \text{by Equation 10.1} \\
&= h^{\mathrm{Rok}}_{G}(X, \mu | \mathcal{F} \vee \Sigma) && \text{by Proposition III.4}
\end{aligned}$$

So $h_G(X,\mu|\mathcal{F}\vee\Sigma) = h_G^{\mathrm{Rok}}(X,\mu|\mathcal{F}\vee\Sigma)$. Also, it is immediate from the definitions that $h_G(X,\mu|\mathcal{F}\vee\Sigma) \leq h_G(X,\mu|\mathcal{F})$. Finally, by Corollary IX.3 we have

$$h_G^{\mathrm{Rok}}(X,\mu|\mathcal{F}) \leq h_G^{\mathrm{Rok}}(Z,\eta)+h_G^{\mathrm{Rok}}(X,\mu|\mathcal{F}\vee\Sigma) < \epsilon+h_G(X,\mu|\mathcal{F}\vee\Sigma) \leq \epsilon+h_G(X,\mu|\mathcal{F}).$$

Now let ϵ tend to 0. $\square$

CHAPTER XI

Metrics on the space of partitions

Let (X, μ) be a probability space. Recall that the *measure algebra* of (X, μ) is the algebra of equivalence classes of Borel sets mod null sets together with the metric $d_\mu(A, B) = \mu(A \triangle B)$. There is a closely related metric d_μ on the space of all countable Borel partitions $\mathscr{P}$ defined by

$$d_\mu(\alpha, \beta) = \inf \Big\{ \mu(Y) \,:\, Y \subseteq X \text{ and } \alpha \restriction (X \setminus Y) = \beta \restriction (X \setminus Y) \Big\}.$$

We will tend to work more frequently with the space $\mathscr{P}_{\mathrm{H}}$ of countable Borel partitions α satisfying $\mathrm{H}(\alpha) < \infty$. In addition to the metric d_μ, this space also has the *Rokhlin metric* d_μ^{Rok} defined by

$$d_\mu^{\mathrm{Rok}}(\alpha, \beta) = \mathrm{H}(\alpha \mid \beta) + \mathrm{H}(\beta \mid \alpha).$$

In this chapter we collect some known properties of these metric spaces for which there is no good reference in the existing literature.

Lemma XI.1. *Let G be a countable group, let $G \curvearrowright (X, \mu)$ be a p.m.p. action, and let $\alpha, \beta, \xi \in \mathscr{P}_{\mathrm{H}}$. Then:*

(i) $d_\mu^{\mathrm{Rok}}(\beta^T, \xi^T) \leq |T| \cdot d_\mu^{\mathrm{Rok}}(\beta, \xi)$ for every finite $T \subseteq G$;

(ii) $d_\mu^{\mathrm{Rok}}(\alpha \vee \beta, \alpha \vee \xi) \leq d_\mu^{\mathrm{Rok}}(\beta, \xi)$;

(iii) $|\mathrm{H}(\beta) - \mathrm{H}(\xi)| \leq d_\mu^{\mathrm{Rok}}(\beta, \xi)$*;*

(iv) $|\mathrm{H}(\beta \mid \alpha) - \mathrm{H}(\xi \mid \alpha)| \leq d_\mu^{\mathrm{Rok}}(\beta, \xi)$*;*

(v) $|\mathrm{H}(\alpha \mid \beta) - \mathrm{H}(\alpha \mid \xi)| \leq 2 \cdot d_\mu^{\mathrm{Rok}}(\beta, \xi)$*.*

Proof. We have

$$\mathrm{H}(\beta^T \mid \xi^T) \leq \sum_{t \in T} \mathrm{H}(t \cdot \beta \mid \xi^T) \leq \sum_{t \in T} \mathrm{H}(t \cdot \beta \mid t \cdot \xi) = |T| \cdot \mathrm{H}(\beta \mid \xi),$$

where the final equality holds since G acts measure-preservingly. This establishes (i). Item (ii) is immediate since $\mathrm{H}(\alpha \vee \beta \mid \alpha \vee \xi) = \mathrm{H}(\beta \mid \alpha \vee \xi) \leq \mathrm{H}(\beta \mid \xi)$. For (iii), we may assume that $\mathrm{H}(\beta) \geq \mathrm{H}(\xi)$. Then we have

$$\mathrm{H}(\beta) - \mathrm{H}(\xi) \leq \mathrm{H}(\beta \vee \xi) - \mathrm{H}(\xi) = \mathrm{H}(\beta \mid \xi) \leq d_\mu^{\mathrm{Rok}}(\beta, \xi).$$

Items (iv) and (v) follow from (ii) and (iii) by using the identities $\mathrm{H}(\beta \mid \alpha) = \mathrm{H}(\alpha \vee \beta) - \mathrm{H}(\alpha)$ and $\mathrm{H}(\alpha \mid \beta) = \mathrm{H}(\alpha \vee \beta) - \mathrm{H}(\beta)$. □

In the next lemma we will use the well-known property [15, Fact 1.7.7] that for every $n \in \mathbb{N}$, the restrictions of d_μ and d_μ^{Rok} to the space of n-piece partitions are uniformly equivalent. Moreover, d_μ is always uniformly dominated by d_μ^{Rok}, meaning that for every $\epsilon > 0$ there is $\delta > 0$ such that if $\alpha, \beta \in \mathscr{P}_{\mathrm{H}}$ and $d_\mu^{\mathrm{Rok}}(\alpha, \beta) < \delta$ then $d_\mu(\alpha, \beta) < \epsilon$.

Lemma XI.2. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. action. Let $T \subseteq G$ be finite, let $\alpha \in \mathscr{P}_{\mathrm{H}}$, and let β be a coarsening of α^T. For every $\epsilon > 0$ there is $\delta > 0$ so that if $\alpha' \in \mathscr{P}_{\mathrm{H}}$ and $d_\mu^{\mathrm{Rok}}(\alpha', \alpha) < \delta$, then there is a coarsening β' of α'^T with $d_\mu^{\mathrm{Rok}}(\beta', \beta) < \epsilon$.*

Proof. By Lemma II.2, there is a finite partition β_0 coarser than β with $d_\mu^{\mathrm{Rok}}(\beta_0, \beta) < \epsilon/2$. Set $n = |\beta_0|$ and let $\kappa > 0$ be such that $d_\mu^{\mathrm{Rok}}(\zeta, \zeta') < \epsilon/2$ whenever ζ and ζ' are n-piece partitions with $d_\mu(\zeta, \zeta') < \kappa$. Let $\delta > 0$ be such that $d_\mu(\xi, \xi') < \kappa/|T|$ whenever

$\xi, \xi' \in \mathscr{P}_{\mathrm{H}}$ satisfy $d_\mu^{\mathrm{Rok}}(\xi, \xi') < \delta$. Now let $\alpha' \in \mathscr{P}_{\mathrm{H}}$ with $d_\mu^{\mathrm{Rok}}(\alpha', \alpha) < \delta$. Then $d_\mu(\alpha', \alpha) < \kappa/|T|$ and hence $d_\mu(\alpha'^T, \alpha^T) < \kappa$. This means there is a set $Y \subseteq X$ with $\mu(Y) < \kappa$ and $\alpha'^T \restriction (X \setminus Y) = \alpha^T \restriction (X \setminus Y)$. Thus there is a n-piece coarsening β' of α'^T with $\beta' \restriction (X \setminus Y) = \beta_0 \restriction (X \setminus Y)$. So $d_\mu(\beta', \beta_0) < \kappa$ and hence $d_\mu^{\mathrm{Rok}}(\beta', \beta_0) < \epsilon/2$. We conclude that $d_\mu^{\mathrm{Rok}}(\beta', \beta) < \epsilon$. □

Lemma XI.3. *Let (X, μ) be a probability space, and let $\mathcal{A}$ be an algebra of Borel sets which is d_μ-dense in a sub-σ-algebra $\mathcal{F}$. If $\beta \in \mathscr{P}_{\mathrm{H}}$, $\beta \subseteq \mathcal{F}$, and $\epsilon > 0$ then there is a partition $\beta' \subseteq \mathcal{A}$ with $d_\mu^{\mathrm{Rok}}(\beta', \beta) < \epsilon$.*

Proof. By Lemma II.2 there is a finite partition β_0 coarser than β with $d_\mu^{\mathrm{Rok}}(\beta_0, \beta) < \epsilon/2$. Set $n = |\beta_0|$ and let $\delta > 0$ be such that $d_\mu^{\mathrm{Rok}}(\zeta, \zeta') < \epsilon/2$ whenever ζ and ζ' are n-piece partitions with $d_\mu(\zeta, \zeta') < \delta$. Since $\mathcal{A}$ is dense in $\mathcal{F}$ there is a n-piece partition $\beta' \subseteq \mathcal{A}$ with $d_\mu(\beta', \beta_0) < \delta$. Then $d_\mu^{\mathrm{Rok}}(\beta', \beta_0) < \epsilon/2$ and $d_\mu^{\mathrm{Rok}}(\beta', \beta) < \epsilon$. □

Corollary XI.4. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. action, let $\mathcal{F}$ be a sub-σ-algebra, and let α be a partition with $\mathcal{F} \subseteq \sigma\text{-alg}_G(\alpha)$. If $\beta \in \mathscr{P}_{\mathrm{H}}$, $\beta \subseteq \mathcal{F}$, and $\epsilon > 0$, then there exists a finite $T \subseteq G$ and a coarsening β' of α^T with $d_\mu^{\mathrm{Rok}}(\beta', \beta) < \epsilon$.*

Proof. The σ-algebra generated by the sets $g \cdot A$, $g \in G$, $A \in \alpha$, contains $\mathcal{F}$. Therefore the algebra generated by these sets is dense in $\mathcal{F}$. □

The same proof also provides the following.

Corollary XI.5. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. action, let $\mathcal{F}$ be a sub-σ-algebra, and let (α_n) be an increasing sequence of partitions with $\mathcal{F} \subseteq \bigvee_{n \in \mathbb{N}} \sigma\text{-alg}_G(\alpha_n)$. If $\beta \in \mathscr{P}_{\mathrm{H}}$, $\beta \subseteq \mathcal{F}$, and $\epsilon > 0$, then there exist $k \in \mathbb{N}$, a finite $T \subseteq G$, and a coarsening β' of α_k^T with $d_\mu^{\mathrm{Rok}}(\beta', \beta) < \epsilon$.*

CHAPTER XII

Translations and independence

In this chapter we show that if the Rokhlin entropy of a free ergodic action is realized by a generating partition, then the action is isomorphic to a Bernoulli shift.

Lemma XII.1. *Let G be a countably infinite group, let $G \curvearrowright (X, \mu)$ be a free p.m.p. action, and let $T \subseteq G$ be finite. Then there is a Borel partition ξ of X such that for every $C \in \xi$ we have $\mu(C) \geq \frac{1}{4} \cdot |T|^{-4}$ and $t \cdot C \cap s \cdot C = \varnothing$ for all $t \neq s \in T$.*

Proof. If $|T| = 1$ then by setting $\xi = \{X\}$ we are done. So assume $|T| \geq 2$. Since the action is free, the condition $t \cdot C \cap s \cdot C = \varnothing$ for all $t \neq s \in T$ is equivalent to the condition $T \cdot c \cap T \cdot c' = \varnothing$ for all $c \neq c' \in C$. By repeatedly applying Lemma V.1 we can inductively construct disjoint sets $C_1, C_2, \ldots$ such that for every i

$$X \setminus (C_1 \cup C_2 \cup \cdots \cup C_{i-1}) \subseteq T^{-1}T \cdot C_i$$

and $T \cdot c \cap T \cdot c' = \varnothing$ for all $c \neq c' \in C_i$. We claim that there is $n \leq |T^{-1}T| + 1$ such that $X = C_1 \cup \cdots \cup C_n$. If not, then there is $x \in X \setminus (C_1 \cup \cdots \cup C_{|T^{-1}T|+1})$. Then $x \in T^{-1}T \cdot C_i$ for every i and hence $T^{-1}T \cdot x$ meets every C_i, $1 \leq i \leq |T^{-1}T| + 1$. This contradicts the C_i's being disjoint.

Set $\xi = \{C_i : 1 \leq i \leq n\}$. If $\mu(C_i) < \frac{1}{4} \cdot |T|^{-4}$ for some i, then since ξ is a partition of X with $|\xi| \leq 2|T|^2$, there must be some j with $\mu(C_j) > \frac{1}{2}|T|^{-2}$. So

$$\mu\Big(C_j \setminus T^{-1}T \cdot C_i\Big) \geq \frac{1}{2|T|^2} - \frac{|T|^2}{4|T|^4} = \frac{1}{4|T|^2} > 2 \cdot \frac{1}{4|T|^4}.$$

Thus by removing from C_j a subset $B \subseteq C_j \setminus T^{-1}T \cdot C_i$ having measure $\mu(B) = \frac{1}{4} \cdot |T|^{-4}$ and by enlarging C_i to contain B, we will have reduced the number of sets in ξ having measure less than $\frac{1}{4} \cdot |T|^{-4}$. This process can be repeated until every set in ξ has measure at least $\frac{1}{4} \cdot |T|^{-4}$. □

Let $G \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action, and let α be a generating partition with $\mathrm{H}(\alpha) < \infty$. If (X, μ) is not isomorphic to a Bernoulli shift, then the G-translates of α cannot be mutually independent. Thus, there is a finite set $T \subseteq G$ with $\mathrm{H}(\alpha^T) < |T| \cdot \mathrm{H}(\alpha)$. So it suffices to show that $\mathrm{H}(\alpha^T) < |T| \cdot \mathrm{H}(\alpha)$ implies $h_G^{\mathrm{Rok}}(X, \mu) < \mathrm{H}(\alpha)$. It is interesting to note that when G is amenable and the action on (X, μ) is free and ergodic, the Rokhlin entropy coincides with Kolmogorov–Sinai entropy and therefore $h_G^{\mathrm{Rok}}(X, \mu)$ is equal to the infimum of $\mathrm{H}(\alpha^T)/|T|$ for finite $T \subseteq G$. While this equality is known to fail for non-amenable groups, it is unknown if an inequality holds.

Question XII.2. Let G be a countably infinite group, let $G \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action, and let α be a generating partition with $\mathrm{H}(\alpha) < \infty$. Is it true that

$$h_G^{\mathrm{Rok}}(X, \mu) \leq \inf_{\substack{T \subseteq G \\ T \text{ finite}}} \frac{1}{|T|} \cdot \mathrm{H}(\alpha^T)?$$

What if the right-hand side is 0?

We remark that the f-invariant, an isomorphism invariant for actions of finite rank free groups introduced by Bowen [5], does satisfy the inequality appearing in Question XII.2 [45].

The theorem below is an attempt at answering Question XII.2. Recall the notion of outer Rokhlin entropy $h_{G,X}^{\mathrm{Rok}}(\mathcal{C} \mid \mathcal{F})$ defined in Chapter II.

Theorem XII.3. *Let G be a countably infinite group, let $G \curvearrowright (X, \mu)$ be a free p.m.p. ergodic action, and let $\mathcal{F}$ be a G-invariant sub-σ-algebra. If α is a countable partition, $T \subseteq G$ is finite, $\epsilon > 0$, and $\frac{1}{|T|} \cdot \mathrm{H}(\alpha^T \,|\, \mathcal{F}) < \mathrm{H}(\alpha \,|\, \mathcal{F}) - \epsilon$, then $h^{\mathrm{Rok}}_{G,X}(\alpha \,|\, \mathcal{F}) < \mathrm{H}(\alpha \mid \mathcal{F}) - \epsilon/(16|T|^3)$.*

Proof. By invariance of μ and $\mathcal{F}$, $\mathrm{H}(\alpha^{sT} \mid \mathcal{F}) = \mathrm{H}(\alpha^T \mid \mathcal{F})$ for all $s \in G$. So by replacing T with a translate sT we may assume that $1_G \in T$. By Theorem II.1, there is a factor $G \curvearrowright (Z, \eta)$ of (X, μ) such that the action of G on Z is free and $h^{\mathrm{Rok}}_G(Z, \eta) < \epsilon/(16 \cdot |T|^3)$. Let Σ be the G-invariant sub-σ-algebra of X associated to Z. If $\mathrm{H}(\alpha \mid \mathcal{F} \vee \Sigma) \leq \mathrm{H}(\alpha \mid \mathcal{F}) - \epsilon/2$, then by Corollary IX.4

$$\begin{aligned} h^{\mathrm{Rok}}_{G,X}(\alpha \mid \mathcal{F}) &\leq h^{\mathrm{Rok}}_{G,X}(\Sigma \mid \mathcal{F}) + h^{\mathrm{Rok}}_{G,X}(\alpha \mid \mathcal{F} \vee \Sigma) \\ &\leq h^{\mathrm{Rok}}_G(Z, \eta) + \mathrm{H}(\alpha \mid \mathcal{F} \vee \Sigma) \\ &< \frac{\epsilon}{16 \cdot |T|^3} + \mathrm{H}(\alpha \mid \mathcal{F}) - \frac{\epsilon}{2} \\ &< \mathrm{H}(\alpha \mid \mathcal{F}) - \frac{\epsilon}{16|T|^3}, \end{aligned}$$

and thus we are done. So assume $\mathrm{H}(\alpha \mid \Sigma \vee \mathcal{F}) > \mathrm{H}(\alpha \mid \mathcal{F}) - \epsilon/2$. Note that

$$\frac{1}{|T|} \cdot \mathrm{H}(\alpha^T \mid \mathcal{F} \vee \Sigma) \leq \frac{1}{|T|} \cdot \mathrm{H}(\alpha^T \mid \mathcal{F}) < \mathrm{H}(\alpha \mid \mathcal{F}) - \epsilon < \mathrm{H}(\alpha \mid \mathcal{F} \vee \Sigma) - \epsilon/2.$$

By definition the action $G \curvearrowright (Z, \eta)$ is free. So we can apply Lemma XII.1 to obtain a partition $\xi \subseteq \Sigma$ of X such that for every $C \in \xi$ we have $t^{-1} \cdot C \cap s^{-1} \cdot C = \varnothing$ for all $t \neq s \in T$ and $\mu(C) \geq \frac{1}{4} \cdot |T|^{-4}$.

Let $\pi : (X, \mu) \to (Y, \nu)$ be the factor associated to $\mathcal{F} \vee \Sigma$, and let $\mu = \int \mu_y \, d\nu(y)$

be the disintegration of μ over ν. We have

$$\begin{aligned}
&\sum_{C\in\xi}\int_{\pi(C)}\left(\sum_{t\in T}\mathrm{H}_{\mu_y}(t\cdot\alpha)-\mathrm{H}_{\mu_y}(\alpha^T)\right)\,d\nu(y)\\
&\quad=\int_Y\left(\sum_{t\in T}\mathrm{H}_{\mu_y}(t\cdot\alpha)-\mathrm{H}_{\mu_y}(\alpha^T)\right)\,d\nu(y)\\
&\quad=\sum_{t\in T}\mathrm{H}(t\cdot\alpha\mid\mathcal{F}\vee\Sigma)-\mathrm{H}(\alpha^T\mid\mathcal{F}\vee\Sigma)\\
&\quad=|T|\cdot\mathrm{H}(\alpha\mid\mathcal{F}\vee\Sigma)-\mathrm{H}(\alpha^T\mid\mathcal{F}\vee\Sigma)\\
&\quad>|T|\cdot\frac{\epsilon}{2}.
\end{aligned}$$

So we can fix $D\in\xi$ with

$$\int_{\pi(D)}\left(\sum_{t\in T}\mathrm{H}_{\mu_y}(t\cdot\alpha)-\mathrm{H}_{\mu_y}(\alpha^T)\right)\,d\nu(y)>|T|\cdot\frac{\epsilon}{2}\cdot\mu(D).$$

Set $R=T^{-1}\cdot D$ and observe that $\mu(R)=|T|\cdot\mu(D)$. Note that for almost-every $y\in Y$ and all $g\in G$ we have $\mu_y(E)=\mu_{g\cdot y}(g\cdot E)$ for Borel $E\subseteq X$ and hence also $\mathrm{H}_{\mu_y}(\alpha)=\mathrm{H}_{\mu_{g\cdot y}}(g\cdot\alpha)$. Thus

$$\begin{aligned}
&\mathrm{H}_R(\alpha\mid\mathcal{F}\vee\Sigma)-\frac{1}{|T|}\cdot\mathrm{H}_D(\alpha^T\mid\mathcal{F}\vee\Sigma)\\
&\quad=\frac{1}{\mu(R)}\cdot\int_{T^{-1}\cdot\pi(D)}\mathrm{H}_{\mu_y}(\alpha)\,d\nu(y)-\frac{1}{|T|\cdot\mu(D)}\cdot\int_{\pi(D)}\mathrm{H}_{\mu_y}(\alpha^T)\,d\nu(y)\\
&\quad=\frac{1}{|T|\cdot\mu(D)}\cdot\sum_{t\in T}\int_{t^{-1}\cdot\pi(D)}\mathrm{H}_{\mu_y}(\alpha)\,d\nu(y)-\frac{1}{|T|\cdot\mu(D)}\cdot\int_{\pi(D)}\mathrm{H}_{\mu_y}(\alpha^T)\,d\nu(y)\\
&\quad=\frac{1}{|T|\cdot\mu(D)}\cdot\int_{\pi(D)}\left(\sum_{t\in T}\mathrm{H}_{\mu_y}(t\cdot\alpha)-\mathrm{H}_{\mu_y}(\alpha^T)\right)\,d\nu(y)\\
&\quad>\frac{\epsilon}{2}.
\end{aligned}$$

Define a new partition

$$\beta=\Big(\alpha\restriction(X\setminus R)\Big)\cup\Big\{R\setminus D\Big\}\cup\Big(\alpha^T\restriction D\Big).$$

Observe that $D \subseteq R$ since $1_G \in T$. Let γ be the partition of X consisting of the sets $t^{-1} \cdot D$, $t \in T$, and $X \setminus R$. Then $\gamma \subseteq \Sigma$ and α is coarser than

$$\alpha \vee \gamma = \Big(\alpha \upharpoonright (X \setminus R)\Big) \cup \bigcup_{t \in T} \Big(\alpha \upharpoonright t^{-1} \cdot D\Big).$$

Since $\alpha \upharpoonright (X \setminus R) \subseteq \beta$ and for each $t \in T$ the partition $t \cdot (\alpha \upharpoonright t^{-1} \cdot D) = (t \cdot \alpha \upharpoonright D)$ of D is coarser than $\alpha^T \upharpoonright D$, we see that

$$\alpha \leq \alpha \vee \gamma \subseteq \sigma\text{-alg}_G(\beta) \vee \Sigma.$$

Therefore $h^{\mathrm{Rok}}_{G,X}(\alpha \,|\, \mathcal{F} \vee \Sigma) \leq \mathrm{H}(\beta \,|\, \mathcal{F} \vee \Sigma)$.

Since $R, D \in \Sigma$ and $\mu(R) = |T| \cdot \mu(D) \geq \frac{1}{4} \cdot |T|^{-3}$ we have

$$\begin{aligned}
\mathrm{H}(\beta \,|\, \mathcal{F} \vee \Sigma) &= \mu(X \setminus R) \cdot \mathrm{H}_{X \setminus R}(\alpha \,|\, \mathcal{F} \vee \Sigma) + \mu(D) \cdot \mathrm{H}_D(\alpha^T \,|\, \mathcal{F} \vee \Sigma) \\
&= \mu(X \setminus R) \cdot \mathrm{H}_{X \setminus R}(\alpha \,|\, \mathcal{F} \vee \Sigma) + \mu(R) \cdot \frac{1}{|T|} \cdot \mathrm{H}_D(\alpha^T \,|\, \mathcal{F} \vee \Sigma) \\
&< \mu(X \setminus R) \cdot \mathrm{H}_{X \setminus R}(\alpha \,|\, \mathcal{F} \vee \Sigma) + \mu(R) \cdot \mathrm{H}_R(\alpha \,|\, \mathcal{F} \vee \Sigma) - \mu(R) \cdot \frac{\epsilon}{2} \\
&= \mathrm{H}(\alpha \,|\, \mathcal{F} \vee \Sigma) - \mu(R) \cdot \frac{\epsilon}{2} \\
&\leq \mathrm{H}(\alpha \,|\, \mathcal{F} \vee \Sigma) - \frac{\epsilon}{8|T|^3}
\end{aligned}$$

Therefore

$$\begin{aligned}
h^{\mathrm{Rok}}_{G,X}(\alpha \,|\, \mathcal{F} \vee \Sigma) + h^{\mathrm{Rok}}_G(Z, \eta) &\leq \mathrm{H}(\beta \,|\, \mathcal{F} \vee \Sigma) + h^{\mathrm{Rok}}_G(Z, \eta) \\
&< \mathrm{H}(\alpha \,|\, \mathcal{F} \vee \Sigma) - \frac{\epsilon}{8|T|^3} + \frac{\epsilon}{16 \cdot |T|^3} \\
&\leq \mathrm{H}(\alpha \,|\, \mathcal{F}) - \frac{\epsilon}{16|T|^3}.
\end{aligned}$$

Thus we are done by Corollary IX.4. □

We will also need the following variant of Theorem XII.3 where we replace both instances of $\mathrm{H}(\alpha \,|\, \mathcal{F})$ with $\mathrm{H}(\alpha)$.

Corollary XII.4. *Let G be a countably infinite group, let $G \curvearrowright (X, \mu)$ be a free p.m.p. ergodic action, and let $\mathcal{F}$ be a G-invariant sub-σ-algebra. If α is a countable partition, $T \subseteq G$ is finite, $\epsilon > 0$, and $\frac{1}{|T|} \cdot \mathrm{H}(\alpha^T \mid \mathcal{F}) < \mathrm{H}(\alpha) - \epsilon$, then $h^{\mathrm{Rok}}_{G,X}(\alpha \mid \mathcal{F}) < \mathrm{H}(\alpha) - \epsilon/(32|T|^3)$.*

Proof. If $\mathrm{H}(\alpha \mid \mathcal{F}) < \mathrm{H}(\alpha) - \epsilon/2$ then clearly

$$h^{\mathrm{Rok}}_{G,X}(\alpha \mid \mathcal{F}) \leq \mathrm{H}(\alpha \mid \mathcal{F}) < \mathrm{H}(\alpha) - \frac{\epsilon}{32|T|^3}.$$

So suppose that $\mathrm{H}(\alpha \mid \mathcal{F}) \geq \mathrm{H}(\alpha) - \epsilon/2$. Then

$$\mathrm{H}(\alpha^T \mid \mathcal{F}) < |T| \cdot \mathrm{H}(\alpha) - |T| \cdot \epsilon \leq |T| \cdot \mathrm{H}(\alpha \mid \mathcal{F}) - |T| \cdot \epsilon/2.$$

In this case we can apply Theorem XII.3. □

We recall the simple fact that a free ergodic p.m.p. action $G \curvearrowright (X, \mu)$ is isomorphic to a Bernoulli shift if and only if there is a generating partition whose G-translates are mutually independent.

Corollary XII.5. *Let G be a countably infinite group and let $G \curvearrowright (X, \mu)$ be a free p.m.p. ergodic action. If α is a generating partition with $\mathrm{H}(\alpha) = h^{\mathrm{Rok}}_G(X, \mu) < \infty$ then $G \curvearrowright (X, \mu)$ is isomorphic to a Bernoulli shift.*

Proof. Since $h^{\mathrm{Rok}}_G(X, \mu) = \mathrm{H}(\alpha)$, Theorem XII.3 implies that $\mathrm{H}(\alpha^T) = |T| \cdot \mathrm{H}(\alpha)$ for every finite $T \subseteq G$. Since $\mathrm{H}(\alpha) < \infty$, this implies that the G-translates of α are mutually independent. As α is a generating partition, it follows that $G \curvearrowright (X, \mu)$ is isomorphic to a Bernoulli shift. □

As a quick corollary of Theorem XII.3, we obtain a relationship between the Rokhlin entropy values of Bernoulli shifts and Gottschalk's surjunctivity conjecture.

Corollary XII.6. *Let G be a countably infinite group. Assume that $h^{\mathrm{Rok}}_G(k^G, u_k^G) = \log(k)$ for every $k \in \mathbb{N}$. Then G satisfies Gottschalk's surjunctivity conjecture and Kaplansky's direct finiteness conjecture.*

Proof. We verify Gottschalk's surjunctivity conjecture as Kaplansky's direct finiteness conjecture will then hold automatically [11, Section I.5]. Let $k \geq 2$ and let $\phi : k^G \to k^G$ be a continuous G-equivariant injection. Set $(Y, \nu) = (\phi(k^G), \phi_*(u_k^G))$ where $\nu = \phi_*(u_k^G)$ is the push-forward measure. Let $\mathscr{L} = \{R_i : 0 \leq i < k\}$ denote the canonical generating partition for k^G, where

$$R_i = \{x \in k^G : x(1_G) = i\}.$$

Note that $\mathscr{L} \restriction Y$ is generating for Y. Since ϕ is injective, it is an isomorphism between (k^G, u_k^G) and (Y, ν). Therefore

$$\log(k) = h^{\mathrm{Rok}}_G(k^G, u_k^G) = h^{\mathrm{Rok}}_G(Y, \nu) \leq \mathrm{H}_\nu(\mathscr{L}) \leq \log|\mathscr{L}| = \log(k).$$

So $h^{\mathrm{Rok}}_G(Y, \nu) = \mathrm{H}_\nu(\mathscr{L}) = \log(k)$. In particular, $\mathrm{H}_\nu(\mathscr{L}^T) = |T| \cdot \mathrm{H}_\nu(\mathscr{L})$ for all finite $T \subseteq G$ by Theorem XII.3.

Towards a contradiction, suppose that ϕ is not surjective. Then its image is a proper closed subset of k^G and hence there is some finite $T \subseteq G$ and $w \in k^{T^{-1}}$ such that $y \restriction T^{-1} \neq w$ for all $y \in Y$. This implies that $|\mathscr{L}^T \restriction Y| \leq k^{|T|} - 1$. So

$$\mathrm{H}_\nu(\mathscr{L}^T) \leq \log|\mathscr{L}^T \restriction Y| \leq \log(k^{|T|} - 1) < |T| \cdot \log(k) = |T| \cdot \mathrm{H}_\nu(\mathscr{L}),$$

a contradiction. □

Finally, we use Theorem XII.3 to investigate the completely positive outer Rokhlin entropy property of Bernoulli shifts. We say that an ergodic action $G \curvearrowright (X, \mu)$ has *completely positive outer Rokhlin entropy* if every factor $G \curvearrowright (Y, \nu)$ which is non-trivial (i.e. Y is not a single point) satisfies $h^{\mathrm{Rok}}_{G,X}(Y) > 0$.

Corollary XII.7. *Let G be a countably infinite group. Assume that $h^{\mathrm{Rok}}_G(L^G, \lambda^G) = \mathrm{H}(L, \lambda)$ for every probability space (L, λ). Then every Bernoulli shift over G has completely positive outer Rokhlin entropy.*

Proof. Let (L, λ) be a probability space, and let $G \curvearrowright (Y, \nu)$ be a non-trivial factor of (L^G, λ^G). Let $\mathcal{F}$ be the G-invariant sub-σ-algebra of L^G associated to (Y, ν).

First let us outline the idea of the proof in the case that $\mathrm{H}(L, \lambda) < \infty$. Let $\mathscr{L}$ be the canonical partition of L^G. If $\mathcal{P}$ is any non-trivial partition contained in $\mathcal{F}$ then since $\mathscr{L}$ is a generating partition there must be a finite $T \subseteq G$ and $\beta \leq \mathscr{L}^T$ with $d^{\mathrm{Rok}}_{\lambda^G}(\beta, \mathcal{P})$ very small. It follows that

$$\mathrm{H}(\mathscr{L}^T \mid \beta) = \mathrm{H}(\mathscr{L}^T) - \mathrm{H}(\beta) = |T| \cdot \mathrm{H}(\mathscr{L}) - \mathrm{H}(\beta)$$

is very close to $\mathrm{H}(\mathscr{L}^T \mid \mathcal{P}) \geq \mathrm{H}(\mathscr{L}^T \mid \mathcal{F})$. Therefore $\mathrm{H}(\mathscr{L}^T \mid \mathcal{F}) < |T| \cdot \mathrm{H}(\mathscr{L})$ and thus $h^{\mathrm{Rok}}_{G,L^G}(\mathscr{L} \mid \mathcal{F}) < \mathrm{H}(\mathscr{L})$ by Corollary XII.4. If $h^{\mathrm{Rok}}_{G,L^G}(Y, \nu) = 0$ then by applying Corollary IX.4 we obtain

$$h^{\mathrm{Rok}}_G(L^G, \lambda^G) \leq h^{\mathrm{Rok}}_{G,L^G}(Y, \nu) + h^{\mathrm{Rok}}_{G,L^G}(\mathscr{L} \mid \mathcal{F}) = h^{\mathrm{Rok}}_{G,L^G}(\mathscr{L} \mid \mathcal{F}) < \mathrm{H}(\mathscr{L}) = \mathrm{H}(L, \lambda),$$

a contradiction.

Note that in the argument above we only needed that $h^{\mathrm{Rok}}_G(L^G, \lambda^G) = \mathrm{H}(L, \lambda)$ for this fixed choice of (L, λ). Below we discuss the general case where $\mathrm{H}(L, \lambda)$ need not be finite. In this case the argument is more technical and requires that $h^{\mathrm{Rok}}_G(L^G, \lambda^G) = \mathrm{H}(L, \lambda)$ for all probability spaces (L, λ).

Fix an increasing sequence of finite partitions $\mathscr{L}_n$ of L with $\mathcal{B}(L) = \bigvee_{n \in \mathbb{N}} \sigma\text{-alg}(\mathscr{L}_n)$, and let (L_k, λ_k) denote the factor of (L, λ) associated to $\mathscr{L}_k$. Let $\mathscr{L} = \{R_\ell : \ell \in L\}$ be the canonical partition of L^G, where $R_\ell = \{x \in L^G : x(1_G) = \ell\}$. We identify each of the partitions $\mathscr{L}_k$ as coarsenings of $\mathscr{L} \subseteq \mathcal{B}(L^G)$. Note that (L_k^G, λ_k^G) is the factor of (L^G, λ^G) associated to $\sigma\text{-alg}_G(\mathscr{L}_k)$. When working with L^G_k, for $m \leq k$ we view $\mathscr{L}_m$ as a partition of L^G_k in the natural way.

Fix a non-trivial finite partition $\mathcal{P} \subseteq \mathcal{F}$, and fix $\epsilon > 0$ with $8\epsilon < \mathrm{H}(\mathcal{P})$. By Corollary XI.5, there is $m \in \mathbb{N}$, finite $T \subseteq G$, and $\beta \leq \mathscr{L}_m^T$ with $d^{\mathrm{Rok}}_{\lambda^G}(\beta, \mathcal{P}) < \epsilon$. Now fix $\delta > 0$ with

$$\delta < \frac{\epsilon}{128|T|^4}.$$

Fix a partition $\mathcal{Q}$ with $\mathcal{F} \subseteq \sigma\text{-alg}_G(\mathcal{Q})$ and $\mathrm{H}(\mathcal{Q}) < h^{\mathrm{Rok}}_{G,L^G}(Y) + \delta$. By Corollary XI.4, there is a finite $U \subseteq G$ and $\mathcal{P}' \leq \mathcal{Q}^U$ with $d^{\mathrm{Rok}}_{\lambda^G}(\mathcal{P}', \mathcal{P}) < \epsilon$. Now by Lemma XI.2 and Corollary XI.5 there is $k \geq m$, $\gamma \subseteq \sigma\text{-alg}_G(\mathscr{L}_k)$ with $d^{\mathrm{Rok}}_{\lambda^G}(\gamma, \mathcal{Q}) < \delta$, and $\beta' \leq \gamma^U$ with $d^{\mathrm{Rok}}_{\lambda^G}(\beta', \mathcal{P}') < \epsilon$. Note that

$$\mathrm{H}(\gamma) \leq \mathrm{H}(\mathcal{Q}) + d^{\mathrm{Rok}}_{\lambda^G}(\gamma, \mathcal{Q}) < h^{\mathrm{Rok}}_{G,L^G}(Y) + 2\delta.$$

Since $\beta' \subseteq \sigma\text{-alg}_G(\gamma)$, $\beta \leq \mathscr{L}_m^T$, and

$$d^{\mathrm{Rok}}_{\lambda^G}(\beta', \beta) \leq d^{\mathrm{Rok}}_{\lambda^G}(\beta', \mathcal{P}') + d^{\mathrm{Rok}}_{\lambda^G}(\mathcal{P}', \mathcal{P}) + d^{\mathrm{Rok}}_{\lambda^G}(\mathcal{P}, \beta) < 3\epsilon,$$

it follows from Lemma XI.1.(v) that

$$\begin{aligned}
\mathrm{H}(\mathscr{L}_m^T \mid \sigma\text{-alg}_G(\gamma)) &\leq \mathrm{H}(\mathscr{L}_m^T \mid \beta') \\
&< \mathrm{H}(\mathscr{L}_m^T \mid \beta) + 6\epsilon \\
&= \mathrm{H}(\mathscr{L}_m^T) - \mathrm{H}(\beta) + 6\epsilon \\
&< \mathrm{H}(\mathscr{L}_m^T) - \mathrm{H}(\mathcal{P}) + 7\epsilon \\
&< \mathrm{H}(\mathscr{L}_m^T) - \epsilon \\
&= |T| \cdot \mathrm{H}(\mathscr{L}_m) - \epsilon.
\end{aligned}$$

Since $\gamma \cup \mathscr{L}_m \subseteq \sigma\text{-alg}_G(\mathscr{L}_k)$, we may work inside (L^G_k, λ^G_k) and apply Corollary XII.4 to get

$$h^{\mathrm{Rok}}_{G,L^G_k}(\mathscr{L}_m \mid \sigma\text{-alg}_G(\gamma)) < \mathrm{H}(\mathscr{L}_m) - \frac{\epsilon}{32|T|^4}. \tag{12.1}$$

Now two applications of Corollary IX.4 and (12.1) give

$$
\begin{aligned}
h_G^{\mathrm{Rok}}(L_k^G, \lambda_k^G) &\le h_{G,L_k^G}^{\mathrm{Rok}}(\gamma) + h_G^{\mathrm{Rok}}(L_k^G, \lambda_k^G \mid \sigma\text{-}\mathrm{alg}_G(\gamma)) \\
&\le \mathrm{H}(\gamma) + h_{G,L_k^G}^{\mathrm{Rok}}(\mathscr{L}_m \mid \sigma\text{-}\mathrm{alg}_G(\gamma)) + h_G^{\mathrm{Rok}}(L_k^G, \lambda_k^G \mid \sigma\text{-}\mathrm{alg}_G(\mathscr{L}_m \vee \gamma)) \\
&< h_{G,L^G}^{\mathrm{Rok}}(Y) + 2\delta + \mathrm{H}(\mathscr{L}_m) - \frac{\epsilon}{32|T|^4} + h_G^{\mathrm{Rok}}(L_k^G, \lambda_k^G \mid \sigma\text{-}\mathrm{alg}_G(\mathscr{L}_m)). \qquad (12.2)
\end{aligned}
$$

By assumption $h_G^{\mathrm{Rok}}(L_k^G, \lambda_k^G) = \mathrm{H}(L_k, \lambda_k) = \mathrm{H}(\mathscr{L}_k)$. So by Corollary IX.3 we have

$$
\begin{aligned}
h_G^{\mathrm{Rok}}(L_k^G, \lambda_k^G) &\le \mathrm{H}(\mathscr{L}_m) + h_G^{\mathrm{Rok}}(L_k^G, \lambda_k^G \mid \sigma\text{-}\mathrm{alg}_G(\mathscr{L}_m)) \\
&\le \mathrm{H}(\mathscr{L}_m) + \mathrm{H}(\mathscr{L}_k \mid \mathscr{L}_m) \\
&= \mathrm{H}(\mathscr{L}_k) \\
&= h_G^{\mathrm{Rok}}(L_k^G, \lambda_k^G),
\end{aligned}
$$

implying that $\mathrm{H}(\mathscr{L}_m) + h_G^{\mathrm{Rok}}(L_k^G, \lambda_k^G \mid \sigma\text{-}\mathrm{alg}_G(\mathscr{L}_m)) = h_G^{\mathrm{Rok}}(L_k^G, \lambda_k^G)$. Plugging this into (12.2) we obtain

$$
h_{G,L^G}^{\mathrm{Rok}}(Y) > \frac{\epsilon}{32|T|^4} - 2\delta > \frac{\epsilon}{64|T|^4} > 0. \qquad \square
$$

CHAPTER XIII

Kolmogorov and Kolmogorov–Sinai theorems

In this chapter we study the computational properties of $h_G^{\mathrm{Rok}}(X, \mu)$ for an ergodic p.m.p. action $G \curvearrowright (X, \mu)$. It will be advantageous to allow (X, μ) to be either atomless or purely atomic, and therefore we will need the following simple observation.

Lemma XIII.1. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action, and let $\mathcal{F}$ be a G-invariant sub-σ-algebra. If (X, μ) has an atom and $\mathcal{F} \neq \mathcal{B}(X)$ then $h_G^{\mathrm{Rok}}(X, \mu \,|\, \mathcal{F})$ is the minimum of $\mathrm{H}(\beta \,|\, \mathcal{F})$ over all Borel partitions β with $\mathrm{H}(\beta \,|\, \mathcal{F}) > 0$.*

Proof. By ergodicity, X is finite after removing a null set. Say $|X| = n$ with each point having measure $1/n$. Then $\mathcal{F}$ is a finite σ-algebra and is therefore generated by a finite G-invariant partition ζ of X. Each $Z \in \zeta$ has the same cardinality, say $|Z| = k$ for all $Z \in \zeta$. So $\mu(Z) = k/n$ for every $Z \in \zeta$. Our assumption $\mathcal{B}(X) \neq \mathcal{F}$ implies that $k > 1$. Let $\alpha = \{A_0, A_1\}$ be a two-piece partition with A_0 consisting of a single point. Then α is generating and in particular $\sigma\text{-alg}_G(\alpha) \vee \mathcal{F} = \mathcal{B}(X)$. If β is any Borel partition of X with $\mathrm{H}(\beta \,|\, \mathcal{F}) > 0$, then it admits a two-piece coarsening $\xi = \{C, X \setminus C\}$ with $\mathrm{H}(\xi \,|\, \mathcal{F}) > 0$. Pick any $Z' \in \zeta$ with $\xi \upharpoonright Z'$ non-trivial and set $m = |C \cap Z'|$. Then $1 \leq m \leq k - 1$ and we have

$$\mathrm{H}(\beta \,|\, \mathcal{F}) \geq \mathrm{H}(\xi \,|\, \mathcal{F}) \geq \frac{k}{n} \cdot \mathrm{H}\left(\frac{m}{k}, 1 - \frac{m}{k}\right) \geq \frac{k}{n} \cdot \mathrm{H}\left(\frac{1}{k}, 1 - \frac{1}{k}\right) = \mathrm{H}(\alpha \,|\, \mathcal{F}). \qquad \square$$

Recall that a real-valued function f on a topological space X is called *upper-semicontinuous* if for every $x \in X$ and $\epsilon > 0$ there is an open set U containing x with $f(y) < f(x) + \epsilon$ for all $y \in U$. When X is first countable, this is equivalent to saying that $f(x) \geq \limsup f(x_n)$ whenever (x_n) is a sequence converging to x. We observe a simple property.

Lemma XIII.2. *Let X be a topological space, let $f_\epsilon : X \to [0, \infty)$, $\epsilon > 0$, be a family of upper-semicontinuous functions and set $g = \lim_{\epsilon \to 0} f_\epsilon$. Assume that $f_\delta(x) \geq f_\epsilon(x)$ for $\delta < \epsilon$ and that $f_\epsilon(x) \geq g(x) - \epsilon$. Then $g : X \to \mathbb{R}$ is upper-semicontinuous.*

Proof. Fix $x \in X$ and $\epsilon > 0$. Since $f_{\epsilon/2}$ is upper-semicontinuous, there is an open neighborhood U of x with $f_{\epsilon/2}(y) < f_{\epsilon/2}(x) + \epsilon/2$ for all $y \in U$. Then for $y \in U$ we have $g(y) \leq f_{\epsilon/2}(y) + \epsilon/2 \leq f_{\epsilon/2}(x) + \epsilon \leq g(x) + \epsilon$. $\square$

We now present the analogue of the Kolmogorov–Sinai theorem [44]. We remind the reader that the partitions α_n and γ_n mentioned below always exist (see Chapter II). The theorem below is a relative version of Theorem I.14 stated in the introduction. In particular, Theorem I.14 follows immediately from the theorem below by taking $\mathcal{F} = \{X, \varnothing\}$.

Theorem XIII.3. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action, and let $\mathcal{F}$ be a G-invariant sub-σ-algebra. Let $(\alpha_n)_{n \in \mathbb{N}}$ and $(\gamma_n)_{n \in \mathbb{N}}$ be increasing sequences of partitions satisfying $\mathrm{H}(\alpha_n), \mathrm{H}(\gamma_n) < \infty$, $\mathcal{B}(X) = \bigvee_{n \in \mathbb{N}} \sigma\text{-alg}_G(\alpha_n \vee \gamma_n)$, and $\mathcal{F} = \bigvee_{n \in \mathbb{N}} \sigma\text{-alg}_G(\gamma_n)$. If*

$$(13.1)\quad \inf_{n \in \mathbb{N}} \lim_{\epsilon \to 0} \sup_{m \in \mathbb{N}} \inf_{k \in \mathbb{N}} \inf_{\substack{T \subseteq G \\ T \text{ finite}}} \inf \left\{ \mathrm{H}(\beta \,|\, \alpha_n^T \vee \gamma_k^T) \,:\, \beta \leq \alpha_k^T \vee \gamma_k^T,\ \mathrm{H}(\alpha_m \,|\, \beta^T \vee \gamma_k^T) < \epsilon \right\}$$

is positive then $h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F}) = \infty$. On the other hand, if the expression above is *equal to* 0 *then* $h_G^{\mathrm{Rok}}(X,\mu \mid \mathcal{F})$ *is equal to*

$$\lim_{\epsilon\to 0}\sup_{m\in\mathbb{N}}\inf_{k\in\mathbb{N}}\inf_{\substack{T\subseteq G\\ T \text{ finite}}}\inf\Big\{\mathrm{H}(\beta \mid \gamma_k^T) \,:\, \beta \leq \alpha_k^T \vee \gamma_k^T,\ \mathrm{H}(\alpha_m \mid \beta^T \vee \gamma_k^T) < \epsilon\Big\}. \tag{13.2}$$

Note that one can equivalently use limits for n, m, and k in the above formulas. In particular, the expressions above are only of interest when $n << m << k$.

Proof. If $\mathcal{F} = \mathcal{B}(X)$ then $h_G^{\mathrm{Rok}}(X,\mu \mid \mathcal{F})$ and all expressions above are 0. So we assume that $\mathcal{F}$ is a proper sub-σ-algebra. First suppose that $h_G^{\mathrm{Rok}}(X,\mu \,|\, \mathcal{F}) < \infty$. Fix a countable partition ξ with $\mathrm{H}(\xi \mid \mathcal{F}) < \infty$ and $\sigma\text{-alg}_G(\xi) \vee \mathcal{F} = \mathcal{B}(X)$. Fix $\delta > 0$. If (X,μ) has an atom then X is essentially finite and $\mathrm{H}(\xi) < \infty$. In this case set $\xi' = \xi$. Otherwise, if (X,μ) is non-atomic then we can apply Theorem I.6 to get a partition ξ' with $\mathrm{H}(\xi') < \infty$, $\mathrm{H}(\xi' \mid \mathcal{F}) < \mathrm{H}(\xi \mid \mathcal{F}) + \delta/2$ and $\sigma\text{-alg}_G(\xi') \vee \mathcal{F} = \mathcal{B}(X)$. Since $\mathrm{H}(\xi') < \infty$, we can fix $n \in \mathbb{N}$ with

$$\mathrm{H}(\xi' \mid \sigma\text{-alg}_G(\alpha_n \vee \gamma_n)) < \delta/2 \quad \text{and} \quad \mathrm{H}(\xi' \mid \sigma\text{-alg}_G(\gamma_n)) < \mathrm{H}(\xi \mid \mathcal{F}) + \delta/2.$$

Fix $m \in \mathbb{N}$ and $0 < \epsilon < \delta$. Let $k_0 \in \mathbb{N}$ and $T_0 \subseteq G$ be finite with:

$$\begin{aligned} \mathrm{H}(\xi' \mid \alpha_n^{T_0} \vee \gamma_n^{T_0}) &< \delta/2,\\ \mathrm{H}(\xi' \mid \gamma_n^{T_0}) &< \mathrm{H}(\xi \mid \mathcal{F}) + \delta/2,\\ \text{and} \quad \mathrm{H}(\alpha_m \mid \xi'^{T_0} \vee \gamma_{k_0}^{T_0}) &< \epsilon/2. \end{aligned}$$

Apply Corollary XI.5 to get $k \geq \max(k_0, n)$, a finite $T \subseteq G$ with $T_0 \subseteq T$, and a partition $\beta \leq \alpha_k^T \vee \gamma_k^T$ with $d_\mu^{\mathrm{Rok}}(\beta, \xi') < \epsilon/(4|T_0|)$. Then

$$\mathrm{H}(\alpha_m \mid \beta^T \vee \gamma_k^T) \leq \mathrm{H}(\alpha_m \mid \beta^{T_0} \vee \gamma_{k_0}^{T_0}) \leq \mathrm{H}(\alpha_m \mid \xi'^{T_0} \vee \gamma_{k_0}^{T_0}) + 2|T_0| \cdot d_\mu^{\mathrm{Rok}}(\beta, \xi') < \epsilon.$$

Furthermore,

$$\mathrm{H}(\beta \mid \alpha_n^T \vee \gamma_k^T) \leq \mathrm{H}(\beta \mid \alpha_n^{T_0} \vee \gamma_n^{T_0}) \leq \mathrm{H}(\xi' \mid \alpha_n^{T_0} \vee \gamma_n^{T_0}) + d_\mu^{\mathrm{Rok}}(\beta, \xi') < \delta$$

and

$$\mathrm{H}(\beta \mid \gamma_k^T) \le \mathrm{H}(\beta \mid \gamma_n^{T_0}) \le \mathrm{H}(\xi' \mid \gamma_n^{T_0}) + d_\mu^{\mathrm{Rok}}(\beta, \xi') < \mathrm{H}(\xi \mid \mathcal{F}) + \delta.$$

Thus since m and ϵ do not depend on ξ or δ we deduce that (13.1) is less than or equal to δ and (13.2) is less than or equal to $\mathrm{H}(\xi \mid \mathcal{F}) + \delta$. Since ξ and δ were arbitrary, (13.1) must be 0 and (13.2) must be at most $h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F})$. Note that (13.2) must always be bounded above by $h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F})$ since this trivially holds when $h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F}) = \infty$.

Now suppose that (13.1) is 0. We will show that $h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F})$ is less than or equal to (13.2). Denote the value of (13.2) by h'. This is automatic if h' is infinite, so we assume that it is finite.

First assume that (X, μ) has an atom. Fix m sufficiently large so that $\mathrm{H}(\alpha_m \mid \mathcal{F}) > 0$. Such an m exists since we are assuming that $\mathcal{F}$ is properly contained in $\mathcal{B}(X)$. Now let $\epsilon < \mathrm{H}(\alpha_m \mid \mathcal{F})$. If β is a partition and $\mathrm{H}(\alpha_m \mid \beta^T \vee \gamma_k^T) < \epsilon$ then $\beta^T \not\subseteq \mathcal{F}$. Since $\mathcal{F}$ is G-invariant, $\beta \not\subseteq \mathcal{F}$ and hence $\mathrm{H}(\beta \mid \mathcal{F}) > 0$ by Lemma II.2. Therefore it follows from Lemma XIII.1 that $h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F})$ is less than or equal to h'.

Now assume that (X, μ) is non-atomic. Fix $\delta > 0$. Since (13.1) is 0, for each $i \ge 1$ we can pick $n(i)$ with

$$\limsup_{\epsilon \to 0} \sup_{m \in \mathbb{N}} \inf_{k \in \mathbb{N}} \inf_{\substack{T \subseteq G \\ T \text{ finite}}} \inf \left\{ \mathrm{H}(\beta \mid \alpha_{n(i)}^T \vee \gamma_k^T) \, : \, \beta \le \alpha_k^T \vee \gamma_k^T, \ \mathrm{H}(\alpha_m \mid \beta^T \vee \gamma_k^T) < \epsilon \right\} < \frac{\delta}{2^i}.$$

Next, for $i \ge 1$ we consider $\epsilon = \delta / 2^i$ and $m = n(i+1)$ in the above expression in order to obtain a partition β_i of X with

$$\mathrm{H}\Big(\beta_i \Big| \sigma\text{-alg}_G(\alpha_{n(i)}) \vee \mathcal{F}\Big) < \frac{\delta}{2^i} \quad \text{and} \quad \mathrm{H}\Big(\alpha_{n(i+1)} \Big| \sigma\text{-alg}_G(\beta_i) \vee \mathcal{F}\Big) < \frac{\delta}{2^i}.$$

By Theorem I.6, there are partitions ξ_i with $\mathrm{H}(\xi_i) < \delta/2^i$ and $\alpha_{n(i+1)} \subseteq \sigma\text{-alg}_G(\beta_i \vee \xi_i) \vee \mathcal{F}$. Apply Theorem I.6 again to obtain partitions β_i' with $\mathrm{H}(\beta_i') < \delta/2^i$ and $\beta_i \subseteq \sigma\text{-alg}_G(\beta_i' \vee \alpha_{n(i)}) \vee \mathcal{F}$. Observe that

$$\alpha_{n(i+1)} \subseteq \sigma\text{-alg}_G(\beta_i \vee \xi_i) \vee \mathcal{F} \subseteq \sigma\text{-alg}_G(\alpha_{n(i)} \vee \beta_i' \vee \xi_i) \vee \mathcal{F}.$$

Now, by considering (13.2) with $\epsilon < \delta$ and $m = n(1)$ we obtain a partition ζ with $\mathrm{H}(\zeta \mid \mathcal{F}) < h' + \delta$ and $\mathrm{H}(\alpha_{n(1)} \mid \sigma\text{-alg}_G(\zeta) \vee \mathcal{F}) < \delta$. Apply Theorem I.6 to obtain a partition ζ' with $\mathrm{H}(\zeta') < \delta$ and $\alpha_{n(1)} \subseteq \sigma\text{-alg}_G(\zeta \vee \zeta') \vee \mathcal{F}$. Then by induction we have that for all i

$$\alpha_{n(i)} \subseteq \sigma\text{-alg}_G\Big(\zeta \vee \zeta' \vee \beta'_1 \vee \xi_1 \vee \cdots \vee \beta'_{i-1} \vee \xi_{i-1}\Big) \vee \mathcal{F}. \tag{13.3}$$

Since

$$\sum_{i=1}^{\infty} \Big(\mathrm{H}(\beta'_i) + \mathrm{H}(\xi_i)\Big) < \sum_{i=1}^{\infty} 2 \cdot \frac{\delta}{2^i} = 2\delta$$

is finite, the partition $\chi = \bigvee_{i \geq 1} \beta'_i \vee \xi_i$ is essentially countable and satisfies $\mathrm{H}(\chi) < 2\delta$ (see Lemmas II.2 and II.3). From (13.3) we see that $\mathcal{B}(X) = \sigma\text{-alg}_G(\zeta \vee \zeta' \vee \chi) \vee \mathcal{F}$ and hence

$$h^{\mathrm{Rok}}_G(X, \mu \mid \mathcal{F}) \leq \mathrm{H}(\zeta \vee \zeta' \vee \chi \mid \mathcal{F}) \leq \mathrm{H}(\zeta \mid \mathcal{F}) + \mathrm{H}(\zeta') + \mathrm{H}(\chi) < h' + 4\delta. \qquad \square$$

Recall that for a standard Borel space X and a Borel action $G \curvearrowright X$, we write $\mathscr{E}_G(X)$ for the collection of ergodic invariant Borel probability measures on X.

Corollary XIII.4. *Let G be a countable group, let X be a standard Borel space, let $G \curvearrowright X$ be a Borel action, and let $\mathcal{F}$ be a G-invariant sub-σ-algebra. Suppose there is a countable collection of Borel sets $\mathcal{C}$ such that $\mathcal{F}$ is the smallest G-invariant σ-algebra containing $\mathcal{C}$. Then the map $\mu \in \mathscr{E}_G(X) \mapsto h^{\mathrm{Rok}}_G(X, \mu \mid \mathcal{F})$ is Borel.*

Proof. Since X is a standard Borel space, there is a countable collection of Borel sets $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ such that $\mathcal{B}(X)$ is the smallest σ-algebra containing $\mathcal{A}$. In particular, there is an increasing sequence (α_n) of finite Borel partitions of X which mutually generate $\mathcal{B}(X)$. Similarly, our assumptions imply that there is an increasing sequence (γ_n) of finite Borel partitions such that $\mathcal{F}$ is the smallest G-invariant σ-algebra containing all of the γ_n's. The space $\mathscr{E}_G(X)$ of invariant ergodic

Borel probability measures μ on X has a natural standard Borel structure which is generated by the maps $\mu \mapsto \mu(A)$ for $A \subseteq X$ Borel [25, Theorem 17.24]. In particular, for finite $T \subseteq G$ and for finite Borel partitions β the maps $\mu \mapsto \mathrm{H}_\mu(\beta \,|\, \gamma_k^T)$, $\mu \mapsto \mathrm{H}_\mu(\beta \,|\, \alpha_n^T \vee \gamma_k^T)$, and $\mu \mapsto \mathrm{H}_\mu(\alpha_m \,|\, \beta^T \vee \gamma_k^T)$ are Borel. So the claim follows from Theorem XIII.3. □

From Theorem XIII.3 we derive the following analogue of the Kolmogorov theorem from entropy theory [33, 34]. Recall that the classical Kolmogorov theorem states that if G is amenable, $G \curvearrowright (X, \mu)$ is an ergodic p.m.p. action, and α is a generating partition with $\mathrm{H}(\alpha) < \infty$, then the Kolmogorov–Sinai entropy $h_G(X, \mu)$ satisfies

$$h_G(X, \mu) = \inf_{\substack{T \subseteq G \\ T \text{ finite}}} \frac{1}{|T|} \cdot \mathrm{H}(\alpha^T).$$

Corollary XIII.5. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action. Let $\mathcal{F}$ be a G-invariant sub-σ-algebra and let $(\gamma_n)_{n \in \mathbb{N}}$ be an increasing sequence of partitions with $\mathrm{H}(\gamma_n) < \infty$ and $\mathcal{F} = \bigvee_{n \in \mathbb{N}} \sigma\text{-alg}_G(\gamma_n)$. If α is a partition with $\mathrm{H}(\alpha) < \infty$ and $\sigma\text{-alg}_G(\alpha) \vee \mathcal{F} = \mathcal{B}(X)$ then*

$$h_G^{\mathrm{Rok}}(X, \mu \,|\, \mathcal{F}) = \lim_{\epsilon \to 0} \inf_{k \in \mathbb{N}} \inf_{\substack{T \subseteq G \\ T \textit{ finite}}} \inf \Big\{ \mathrm{H}(\beta \,|\, \gamma_k^T) \,:\, \beta \leq \alpha^T \vee \gamma_k^T \text{ and } \mathrm{H}(\alpha \,|\, \beta^T \vee \gamma_k^T) < \epsilon \Big\}.$$

Proof. We have that $h_G^{\mathrm{Rok}}(X, \mu | \mathcal{F}) \leq \mathrm{H}(\alpha) < \infty$. So, setting $\alpha_n = \alpha$ for all $n \in \mathbb{N}$, we know by Theorem XIII.3 that $h_G^{\mathrm{Rok}}(X, \mu \,|\, \mathcal{F})$ is given by (13.2). Since each $\alpha_n = \alpha$, this is identical to the formula above. □

Next, we make a simple observation.

Lemma XIII.6. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action. Let $\mathcal{F}$ be a G-invariant sub-σ-algebra and let $(\gamma_n)_{n \in \mathbb{N}}$ be an increasing sequence of partitions with $\mathrm{H}(\gamma_n) < \infty$ and $\mathcal{F} = \bigvee_{n \in \mathbb{N}} \sigma\text{-alg}_G(\gamma_n)$. If α is a partition with $\mathrm{H}(\alpha) < \infty$ and $\sigma\text{-alg}_G(\alpha) \vee \mathcal{F} =$*

$\mathcal{B}(X)$ *then for every* $\epsilon > 0$

$$\inf_{k\in\mathbb{N}} \inf_{\substack{T\subseteq G\\ T \text{ finite}}} \inf \Big\{ \mathrm{H}(\beta \,|\, \gamma_k^T) \,:\, \beta \leq \alpha^T \vee \gamma_k^T \text{ and } \mathrm{H}(\alpha \,|\, \beta^T \vee \gamma_k^T) < \epsilon \Big\} \geq h_G^{\mathrm{Rok}}(X, \mu \,|\, \mathcal{F}) - \epsilon.$$

Proof. Fix $\epsilon > 0$. First suppose that (X, μ) has an atom. Then by ergodicity X is finite. Fix $k \in \mathbb{N}$, $T \subseteq G$, and $\beta \leq \alpha^T \vee \gamma_k^T$ with $\mathrm{H}(\alpha \,|\, \beta^T \vee \gamma_k^T) < \epsilon$. If $\mathrm{H}(\beta \,|\, \mathcal{F}) > 0$ then

$$\mathrm{H}(\beta \,|\, \gamma_k^T) \geq \mathrm{H}(\beta \,|\, \mathcal{F}) \geq h_G^{\mathrm{Rok}}(X, \mu \,|\, \mathcal{F})$$

by Lemma XIII.1 and we are done. On the other hand, if $\mathrm{H}(\beta \,|\, \mathcal{F}) = 0$ then $\beta \subseteq \mathcal{F}$ by Lemma II.2 and thus

$$h_G^{\mathrm{Rok}}(X, \mu \,|\, \mathcal{F}) \leq \mathrm{H}(\alpha \,|\, \mathcal{F}) \leq \mathrm{H}(\alpha \,|\, \beta^T \vee \gamma_k^T) < \epsilon.$$

It follows that $\mathrm{H}(\beta \,|\, \gamma_k^T) \geq 0 > h_G^{\mathrm{Rok}}(X, \mu \,|\, \mathcal{F}) - \epsilon$.

Now suppose that (X, μ) is non-atomic. If β is a partition with $\mathrm{H}(\alpha \,|\, \beta^T \vee \gamma_k^T) < \epsilon$, then by applying Theorem I.6 we can obtain a partition ξ with $\mathrm{H}(\xi) < \epsilon$ and $\alpha \subseteq \sigma\text{-alg}_G(\beta \vee \xi) \vee \mathcal{F}$. Then $\mathcal{B}(X) = \sigma\text{-alg}_G(\alpha) \vee \mathcal{F} = \sigma\text{-alg}_G(\beta \vee \xi) \vee \mathcal{F}$ so that

$$h_G^{\mathrm{Rok}}(X, \mu \,|\, \mathcal{F}) \leq \mathrm{H}(\beta \vee \xi \,|\, \mathcal{F}) \leq \mathrm{H}(\beta \,|\, \mathcal{F}) + \mathrm{H}(\xi) < \mathrm{H}(\beta \,|\, \gamma_k^T) + \epsilon.$$

It follows that $\mathrm{H}(\beta \,|\, \gamma_k^T) > h_G^{\mathrm{Rok}}(X, \mu \,|\, \mathcal{F}) - \epsilon$ as required. □

For a p.m.p. action $G \curvearrowright (X, \mu)$ and a partition α of X, the G-invariant σ-algebra $\sigma\text{-alg}_G(\alpha)$ is associated to a factor $G \curvearrowright (Y, \nu)$ of (X, μ). From Corollary XIII.5 we obtain the following dependence of $h_G^{\mathrm{Rok}}(Y, \nu)$ on α. Recall from Chapter II that $\mathscr{P}_{\mathrm{H}}$ is the space of all countable Borel partitions α with $\mathrm{H}(\alpha) < \infty$.

Corollary XIII.7. *Let* $G \curvearrowright (X, \mu)$ *be a p.m.p. ergodic action and let* $\mathcal{F}$ *be a* G*-invariant sub-*σ*-algebra. For* $\alpha \in \mathscr{P}_{\mathrm{H}}$*, let* $G \curvearrowright (Y_\alpha, \nu_\alpha)$ *be the factor of* (X, μ) *associated to* $\sigma\text{-alg}_G(\alpha) \vee \mathcal{F}$*, and let* $\mathcal{F}_\alpha$ *be the image of* $\mathcal{F}$ *in* Y_α*. Then the map*

$$\alpha \in \mathscr{P}_{\mathrm{H}} \mapsto h_G^{\mathrm{Rok}}(Y_\alpha, \nu_\alpha \,|\, \mathcal{F}_\alpha)$$

is upper-semicontinuous in the metric d_μ^{Rok}.

Proof. Fix an increasing sequence $(\gamma_n)_{n \in \mathbb{N}}$ of finite partitions of X satisfying $\mathcal{F} = \bigvee_{n \in \mathbb{N}} \sigma\text{-alg}_G(\gamma_n)$. Such a sequence always exists; see Chapter II. Set

$$f_\epsilon(\alpha) = \inf_{k \in \mathbb{N}} \inf_{\substack{T \subseteq G \\ T \text{ finite}}} \inf \left\{ \mathrm{H}(\beta \mid \gamma_k^T) \, : \, \beta \leq \alpha^T \vee \gamma_k^T \text{ and } \mathrm{H}(\alpha \mid \beta^T \vee \gamma_k^T) < \epsilon \right\}$$

and set $g(\alpha) = \lim_{\epsilon \to 0} f_\epsilon(\alpha)$. Using the natural one-to-one measure-preserving correspondence between the σ-algebras $\mathcal{B}(Y_\alpha)$ and $\sigma\text{-alg}_G(\alpha) \vee \mathcal{F}$, we see by Corollary XIII.5 that $g(\alpha) = h_G^{\mathrm{Rok}}(Y_\alpha, \nu_\alpha \mid \mathcal{F}_\alpha)$. Each function f_ϵ is upper-semicontinuous in d_μ^{Rok} by Lemmas XI.1 and XI.2, and $f_\epsilon(\alpha) \geq g(\alpha) - \epsilon$ by Lemma XIII.6. Therefore $g(\alpha)$ is upper-semicontinuous by Lemma XIII.2. $\square$

In fact, with the same proof we obtain the following.

Corollary XIII.8. *Let* $G \curvearrowright (X, \mu)$ *be a p.m.p. ergodic action. For* $\alpha, \gamma \in \mathscr{P}_{\mathrm{H}}$*, let* $G \curvearrowright (Y_{(\alpha,\gamma)}, \nu_{(\alpha,\gamma)})$ *be the factor of* (X, μ) *associated to* $\sigma\text{-alg}_G(\alpha \vee \gamma)$*, and let* γ' *be the image of* γ *in* $Y_{(\alpha,\gamma)}$*. Then the map*

$$(\alpha, \gamma) \in \mathscr{P}_{\mathrm{H}} \times \mathscr{P}_{\mathrm{H}} \mapsto h_G^{\mathrm{Rok}}(Y_{(\alpha,\gamma)}, \nu_{(\alpha,\gamma)} | \sigma\text{-alg}_G(\gamma'))$$

is upper-semicontinuous in the metric $d_\mu^{\mathrm{Rok}} \times d_\mu^{\mathrm{Rok}}$.

The upper-semicontinuity property provides the following alternative method for computing Rokhlin entropy.

Corollary XIII.9. *Let* $G \curvearrowright (X, \mu)$ *be a p.m.p. ergodic action, let* $\mathcal{F}$ *be a* G*-invariant sub-*σ*-algebra, and let* α *be a partition with* $\mathrm{H}(\alpha) < \infty$ *and* $\sigma\text{-alg}_G(\alpha) \vee \mathcal{F} = \mathcal{B}(X)$*. Fix an increasing sequence of partitions* $\alpha_n \leq \alpha$ *with* $\alpha = \bigvee_{n \in \mathbb{N}} \alpha_n$*, and for each* n *let* $G \curvearrowright (Y_n, \nu_n)$ *be the factor of* (X, μ) *associated to* $\sigma\text{-alg}_G(\alpha_n) \vee \mathcal{F}$*. Also let* $\mathcal{F}_n$ *be the image of* $\mathcal{F}$ *in* Y_n*. Then* $h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F}) = \lim_{n \to \infty} h_G^{\mathrm{Rok}}(Y_n, \nu_n \mid \mathcal{F}_n)$*.*

Proof. If (X, μ) has an atom then X is essentially finite and so is α. Thus the claim is trivial in this case since $\alpha_n = \alpha$, $Y_n = X$, and $\mathcal{F}_n = \mathcal{F}$ for all sufficiently large n. Now suppose that (X, μ) is non-atomic. Observe that $d_\mu^{\mathrm{Rok}}(\alpha_n, \alpha) = \mathrm{H}(\alpha \,|\, \alpha_n)$ tends to 0 by Lemma II.2. Fix $\epsilon > 0$. By Corollary XIII.7 there is $\delta > 0$ so that if β is any partition with $d_\mu^{\mathrm{Rok}}(\beta, \alpha) < \delta$ then $h_G^{\mathrm{Rok}}(Y_\beta, \nu_\beta \mid \mathcal{F}_\beta) < h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F}) + \epsilon$, where (Y_β, ν_β) is the factor associated to $\sigma\text{-alg}_G(\beta) \vee \mathcal{F}$ and $\mathcal{F}_\beta$ is the image of $\mathcal{F}$. Let n be sufficiently large so that $d_\mu^{\mathrm{Rok}}(\alpha_n, \alpha) < \min(\delta, \epsilon/2)$. Then $h_G^{\mathrm{Rok}}(Y_n, \nu_n \mid \mathcal{F}_n) < h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F}) + \epsilon$. For the other inequality, fix a partition ξ_n of Y_n with $\mathrm{H}(\xi_n \mid \mathcal{F}_n) < h_G^{\mathrm{Rok}}(Y_n, \nu_n \mid \mathcal{F}_n) + \epsilon/2$ and $\sigma\text{-alg}_G(\xi_n) \vee \mathcal{F}_n = \mathcal{B}(Y_n)$. Pull back ξ_n to a partition ξ of X. Then

$$\mathrm{H}(\xi \mid \mathcal{F}) = \mathrm{H}(\xi_n \mid \mathcal{F}_n) < h_G^{\mathrm{Rok}}(Y_n, \nu_n \mid \mathcal{F}_n) + \epsilon/2$$

and $\sigma\text{-alg}_G(\xi) \vee \mathcal{F} = \sigma\text{-alg}_G(\alpha_n) \vee \mathcal{F}$. We have $\mathrm{H}(\alpha \,|\, \sigma\text{-alg}_G(\xi) \vee \mathcal{F}) \leq \mathrm{H}(\alpha \,|\, \alpha_n) < \epsilon/2$, so by Theorem I.6 there is a partition ζ with $\mathrm{H}(\zeta) < \epsilon/2$ and $\alpha \subseteq \sigma\text{-alg}_G(\zeta \vee \xi) \vee \mathcal{F}$. Thus $\sigma\text{-alg}_G(\zeta \vee \xi) \vee \mathcal{F} = \mathcal{B}(X)$ and hence

$$h_G^{\mathrm{Rok}}(X, \mu \,|\, \mathcal{F}) \leq \mathrm{H}(\zeta \vee \xi \,|\, \mathcal{F}) \leq \mathrm{H}(\xi \,|\, \mathcal{F}) + \mathrm{H}(\zeta) < h_G^{\mathrm{Rok}}(Y_n, \nu_n \,|\, \mathcal{F}_n) + \epsilon. \qquad \square$$

Finally, we consider the upper-semicontinuity of Rokhlin entropy as a function of the ergodic probability measure.

Corollary XIII.10. *Let G be a countable group, let L be a finite set, and let L^G have the product topology. Let $\mathcal{C}$ be a countable collection of clopen sets, and let $\mathcal{F}$ be the smallest G-invariant σ-algebra containing $\mathcal{C}$. Then the map $\mu \in \mathscr{E}_G(L^G) \mapsto h_G^{\mathrm{Rok}}(L^G, \mu \,|\, \mathcal{F})$ is upper-semicontinuous in the weak*-topology.*

Proof. Let $\mathscr{L} = \{R_\ell : \ell \in L\}$ be the canonical generating partition for L^G, where $R_\ell = \{x \in L^G : x(1_G) = \ell\}$. Choose an increasing sequence of finite partitions γ_k contained in the algebra generated by $\mathcal{C}$ with $\mathcal{F} = \bigvee_{k \in \mathbb{N}} \sigma\text{-alg}_G(\gamma_k)$. Then any set D

in $\mathscr{L}^T$, γ_k^T, or any $\beta \leq \mathscr{L}^T$ is clopen and hence the map $\mu \mapsto \mu(D)$ is continuous. Similarly, the maps $\mu \mapsto \mathrm{H}_\mu(\beta|\gamma_k^T)$ and $\mu \mapsto \mathrm{H}_\mu(\mathscr{L}|\beta^T \vee \gamma_k^T)$ are continuous. Therefore each function

$$f_\epsilon(\mu) = \inf_{k \in \mathbb{N}} \inf_{\substack{T \subseteq G \\ T \text{ finite}}} \inf \left\{ \mathrm{H}_\mu(\beta \mid \gamma_k^T) \,:\, \beta \leq \mathscr{L}^T \text{ and } \mathrm{H}_\mu(\mathscr{L} \mid \beta^T \vee \gamma_k^T) < \epsilon \right\}$$

is upper-semicontinuous. Setting $g(\mu) = \lim_{\epsilon \to 0} f_\epsilon(\mu)$, Corollary XIII.5 implies that $g(\mu) = h_G^{\mathrm{Rok}}(L^G, \mu \mid \mathcal{F})$. By Lemmas XIII.6 and XIII.2 we have that $g(\mu)$ is upper-semicontinuous. $\square$

CHAPTER XIV

Approximately Bernoulli partitions

In this chapter we will show how to construct generating partitions which are approximately Bernoulli. This will allow us to use Corollary XIII.10 in order to study the Rokhlin entropy values of Bernoulli shifts. We begin with a few lemmas.

Lemma XIV.1. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action, let $\mathcal{F}$ be a G-invariant sub-σ-algebra, and let $B \in \mathcal{F}$ with $\mu(B) > 0$. Then there is a finite collection $\Phi \subseteq [[E_G^X]]$ of $\mathcal{F}$-expressible functions such that $\{\mathrm{dom}(\phi) : \phi \in \Phi\}$ partitions X and $\mathrm{rng}(\phi) \subseteq B$ for every $\phi \in \Phi$.*

Proof. We claim that there is a finite partition $\gamma \subseteq \mathcal{F}$ with $\mu(C) \leq \mu(B)$ for every $C \in \gamma$. If the factor $G \curvearrowright (Y, \nu)$ of (X, μ) associated to $\mathcal{F}$ is purely atomic then we can simply let γ be the pre-image of the partition of Y into points. On the other hand, if (Y, ν) is non-atomic then we can find such a partition in Y and let γ be its pre-image. Now by Lemma III.5, for every $C \in \gamma$ there is an $\mathcal{F}$-expressible $\phi_C \in [[E_G^X]]$ with $\mathrm{dom}(\phi_C) = C$ and $\mathrm{rng}(\phi_C) \subseteq B$. Then $\Phi = \{\phi_C : C \in \gamma\}$ has the desired properties. $\square$

Lemma XIV.2. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action with (X, μ) non-atomic, let $\mathcal{F}$ be a G-invariant sub-σ-algebra, and let $B \in \mathcal{F}$. If ξ is a countable partition of*

X and $\bar{p} = (p_i)$ is a probability vector with

$$\mathrm{H}(\xi \mid \mathcal{F}) < \mu(B) \cdot \mathrm{H}(\bar{p}),$$

then there is a partition $\alpha = \{A_i : 0 \leq i < |\bar{p}|\}$ of B with $\mu(A_i) = p_i \cdot \mu(B)$ for every $0 \leq i < |\bar{p}|$ and with $\xi \subseteq \sigma\text{-alg}_G(\alpha') \vee \mathcal{F}$ for every partition α' of X extending α.

Proof. Let $\Phi \subseteq [[E_G^X]]$ be as given by Lemma XIV.1. For $\phi \in \Phi$, define a partition ξ_ϕ of X by

$$\xi_\phi = \Big\{X \setminus \mathrm{rng}(\phi)\Big\} \cup \phi\Big(\xi \restriction \mathrm{dom}(\phi)\Big),$$

and set $\zeta = \bigvee_{\phi \in \Phi} \xi_\phi$. Note that ζ is countable since Φ is finite. Also observe that

$$\mu(\mathrm{rng}(\phi)) \cdot \mathrm{H}_{\mathrm{rng}(\phi)}(\xi_\phi \mid \mathcal{F}) = \mu(\mathrm{dom}(\phi)) \cdot \mathrm{H}_{\mathrm{dom}(\phi)}(\xi \mid \mathcal{F}) \tag{14.1}$$

since ϕ is a $\mathcal{B}(X)$ and $\mathcal{F}$ measure-preserving bijection from $\mathrm{dom}(\phi)$ to $\mathrm{rng}(\phi)$ by Lemma III.2.

We claim that $\xi \subseteq \sigma\text{-alg}_G(\zeta) \vee \mathcal{F}$. Consider $C \in \xi$ and $\phi \in \Phi$. Since ϕ is $\mathcal{F}$-expressible, we have $\mathrm{rng}(\phi) \in \mathcal{F}$. Thus $\xi_\phi \restriction \mathrm{rng}(\phi) \subseteq \sigma\text{-alg}_G(\zeta) \vee \mathcal{F}$. It follows from Lemmas III.2 and III.3 that

$$\phi^{-1}(\xi_\phi \restriction \mathrm{rng}(\phi)) \subseteq \sigma\text{-alg}_G(\zeta) \vee \mathcal{F}.$$

Since $C \cap \mathrm{dom}(\phi)$ is an element of the set on the left, and since C is the union of $C \cap \mathrm{dom}(\phi)$ for $\phi \in \Phi$, we conclude that $\xi \subseteq \sigma\text{-alg}_G(\zeta) \vee \mathcal{F}$.

For $g \in G$ define $\gamma_g \in [[E_G^X]]$ with $\mathrm{dom}(\gamma_g) = \mathrm{rng}(\gamma_g) = B$ by the rule

$$\gamma_g(x) = y \iff y = g^i \cdot x \text{ where } i > 0 \text{ is least with } g^i \cdot x \in B.$$

By the Poincaré recurrence theorem, the domain and range of γ_g are indeed conull in B. Note that γ_g is $\mathcal{F}$-expressible since $B \in \mathcal{F}$. Let Γ be the group of transformations of B generated by $\{\gamma_g : g \in G\}$. Then every $\gamma \in \Gamma$ is $\mathcal{F}$ expressible by Lemma III.3. Let μ_B denote the normalized restriction of μ to B, so that $\mu_B(A) = \mu(A \cap B)/\mu(B)$. Since μ is ergodic, it is not difficult to check that the action of Γ on (B, μ_B) is ergodic. Similarly, since μ is non-atomic μ_B is non-atomic as well. Using (14.1) and the fact that $\mathrm{dom}(\phi), \mathrm{rng}(\phi) \in \mathcal{F}$, we have

$$\begin{aligned}
\mu(B) \cdot \mathrm{H}_{\mu_B}(\zeta \mid \mathcal{F}) &= \mu(B) \cdot \mathrm{H}_B(\zeta \mid \mathcal{F}) \\
&\leq \sum_{\phi \in \Phi} \mu(B) \cdot \mathrm{H}_B(\xi_\phi \mid \mathcal{F}) \\
&= \sum_{\phi \in \Phi} \mu(\mathrm{rng}(\phi)) \cdot \mathrm{H}_{\mathrm{rng}(\phi)}(\xi_\phi \mid \mathcal{F}) \\
&= \sum_{\phi \in \Phi} \mu(\mathrm{dom}(\phi)) \cdot \mathrm{H}_{\mathrm{dom}(\phi)}(\xi \mid \mathcal{F}) \\
&= \mathrm{H}(\xi \mid \mathcal{F}) \\
&< \mu(B) \cdot \mathrm{H}(\bar{p}).
\end{aligned}$$

So by Theorem I.6 there is a partition $\alpha = \{A_i : 0 \leq i < |\bar{p}|\}$ of B with $\mu_B(A_i) = p_i$ for every $0 \leq i < |\bar{p}|$ and with $\zeta \upharpoonright B \subseteq \sigma\text{-alg}_\Gamma(\alpha) \vee \mathcal{F}$. Since $\zeta \upharpoonright (X \setminus B)$ is trivial and $X \setminus B \in \mathcal{F}$, it follows that $\zeta \subseteq \sigma\text{-alg}_\Gamma(\alpha) \vee \mathcal{F}$.

Since $A_i \subseteq B$ and $\mu_B(A_i) = p_i$, it follows that $\mu(A_i) = p_i \cdot \mu(B)$. Now let α' be a partition of X extending α. Since Γ is $\mathcal{F}$-expressible, it follows from Lemma III.2 that $\sigma\text{-alg}_G(\alpha') \vee \mathcal{F}$ is Γ-invariant. Since also $B \in \mathcal{F}$ and $\alpha = \alpha' \upharpoonright B$, we have $\sigma\text{-alg}_\Gamma(\alpha) \vee \mathcal{F} \subseteq \sigma\text{-alg}_G(\alpha') \vee \mathcal{F}$. Therefore $\zeta \subseteq \sigma\text{-alg}_G(\alpha') \vee \mathcal{F}$ and hence

$$\xi \subseteq \sigma\text{-alg}_G(\zeta) \vee \mathcal{F} \subseteq \sigma\text{-alg}_G(\alpha') \vee \mathcal{F}. \qquad \square$$

The following lemma is, in some ways, a strengthening of Theorem I.6.

Lemma XIV.3. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. ergodic action with (X, μ) non-atomic, let $\mathcal{F}$ be a G-invariant sub-σ-algebra, and let ξ be a countable Borel partition of X. If*

$\beta \subseteq \mathcal{F}$ is a collection of pairwise disjoint Borel sets and $\{\bar{p}^B : B \in \beta\}$ is a collection of probability vectors with

$$\mathrm{H}(\xi \mid \mathcal{F}) < \sum_{B \in \beta} \mu(B) \cdot \mathrm{H}(\bar{p}^B),$$

then there is a partition $\alpha = \{A_i : 0 \leq i < |\alpha|\}$ of $\cup\beta$ with $\mu(A_i \cap B) = p_i^B \cdot \mu(B)$ for every $B \in \beta$ and $0 \leq i < |\alpha|$ and with $\xi \subseteq \sigma\text{-alg}_G(\alpha') \vee \mathcal{F}$ for every partition α' of X extending α.

Proof. Fix $\epsilon > 0$ with

$$\mathrm{H}(\xi \mid \mathcal{F}) < \sum_{B \in \beta} \mu(B) \cdot \mathrm{H}(\bar{p}^B) - \epsilon \cdot \mu(\cup\beta).$$

For each $B \in \beta$, fix any probability vector $\bar{q}^B$ satisfying

$$\mu(B) \cdot \mathrm{H}(\bar{p}^B) - \epsilon \cdot \mu(B) < \mathrm{H}(\bar{q}^B) < \mu(B) \cdot \mathrm{H}(\bar{p}^B).$$

Let $\bar{r}$ be the probability vector which represents the independent join of the $\bar{q}^B$'s. Specifically, $\bar{r} = (r_\pi)_{\pi \in \mathbb{N}^\beta}$ where

$$r_\pi = \prod_{B \in \beta} q^B_{\pi(B)}.$$

Then

$$\mathrm{H}(\bar{r}) = \sum_{B \in \beta} \mathrm{H}(\bar{q}^B) > \sum_{B \in \beta} \mu(B) \cdot \mathrm{H}(\bar{p}^B) - \epsilon \cdot \mu(\cup\beta) > \mathrm{H}(\xi \mid \mathcal{F}).$$

So by Theorem I.6 there is a partition $\gamma = \{C_\pi : \pi \in \mathbb{N}^\beta\}$ with $\xi \subseteq \sigma\text{-alg}_G(\gamma) \vee \mathcal{F}$ and with $\mu(C_\pi) = r_\pi$ for every $\pi \in \mathbb{N}^\beta$.

For each $B \in \beta$, let γ^B be the coarsening of γ associated to $\bar{q}^B$. Specifically, $\gamma^B = \{C_i^B : 0 \leq i < |\bar{q}^B|\}$ where

$$C_i^B = \bigcup_{\substack{\pi \in \mathbb{N}^\beta \\ \pi(B) = i}} C_\pi.$$

Note that $\gamma = \bigvee_{B \in \beta} \gamma^B$. Also note that $\mu(C_i^B) = q_i^B$ and $\mathrm{H}(\gamma^B) = \mathrm{H}(\bar{q}^B) < \mu(B) \cdot \mathrm{H}(\bar{p}^B)$. For each $B \in \beta$ we apply Lemma XIV.2 to γ^B in order to obtain a partition $\alpha^B = \{A_i^B : 0 \leq i < |\bar{p}^B|\}$ of B with $\mu(A_i^B) = \mu(B) \cdot p_i^B$ and $\gamma^B \subseteq \sigma\text{-alg}_G(\zeta) \vee \mathcal{F}$ for every partition ζ of X extending α^B. Now define $\alpha = \{A_i : 0 \leq i < |\alpha|\}$ where $A_i = \bigcup_{B \in \beta} A_i^B$. Then for $B \in \beta$ and $0 \leq i < |\alpha|$ we have $\mu(A_i \cap B) = \mu(A_i^B) = p_i^B \cdot \mu(B)$. Furthermore, if α' is a partition of X which extends α, then α' extends every α^B and hence $\gamma^B \subseteq \sigma\text{-alg}_G(\alpha') \vee \mathcal{F}$. It follows that

$$\xi \subseteq \sigma\text{-alg}_G(\gamma) \vee \mathcal{F} \subseteq \sigma\text{-alg}_G(\alpha') \vee \mathcal{F}. \qquad \square$$

We will need the result of Abért and Weiss that all free actions weakly contain Bernoulli shifts [1]. The following is a slightly modified statement of their result, obtained by invoking [1, Lemma 5] and performing a perturbation.

Theorem XIV.4 (Abért–Weiss [1])**.** *Let $G \curvearrowright (X, \mu)$ be a p.m.p. free action, and let $\bar{p} = (p_i)$ be a finite probability vector. If $T \subseteq G$ is finite and $\epsilon > 0$, then there is a partition $\gamma = \{C_i : 0 \leq i < |\bar{p}|\}$ of X such that $\mu(C_i) = p_i$ for every $0 \leq i < |\bar{p}|$ and $\mathrm{H}(\gamma^T)/|T| > \mathrm{H}(\gamma) - \epsilon$.*

We are almost ready to construct approximately Bernoulli generating partitions. For this construction we will find it more convenient to use Borel partitions of $([0,1], \lambda)$, where λ is Lebesgue measure, in place of probability vectors. We first make a simple observation.

Lemma XIV.5. *If $\mathcal{Q} \leq \mathcal{P}$ are finite partitions of $([0,1], \lambda)$ and $0 < r < \mathrm{H}(\mathcal{P} \mid \mathcal{Q})$, then there is a finite partition $\mathcal{R}$ such that $\mathcal{Q} \leq \mathcal{R}$ and $\mathrm{H}(\mathcal{P} \mid \mathcal{R}) = r$.*

Proof. Fix a d_λ^{Rok}-continuous 1-parameter family of finite partitions $\mathcal{Q}_t$, $0 \leq t \leq 1$, such that $\mathcal{Q}_0 = \mathcal{Q}$, $\mathcal{Q}_1 = \mathcal{P}$, and $\mathcal{Q} \leq \mathcal{Q}_t$ for all t. The function $t \mapsto \mathrm{H}(\mathcal{P} \mid \mathcal{Q}_t)$ is continuous, $\mathrm{H}(\mathcal{P} \mid \mathcal{Q}_0) = \mathrm{H}(\mathcal{P} \mid \mathcal{Q}) > r$, and $\mathrm{H}(\mathcal{P} \mid \mathcal{Q}_1) = \mathrm{H}(\mathcal{P} \mid \mathcal{P}) = 0$. Therefore there is $t \in (0, 1)$ with $\mathrm{H}(\mathcal{P} \mid \mathcal{Q}_t) = r$. Set $\mathcal{R} = \mathcal{Q}_t$. $\square$

For countable partitions α and β of (X, μ) recall from Chapter XI the metric

$$d_\mu(\alpha, \beta) = \inf \Big\{ \mu(Y) \, : \, Y \subseteq X \text{ and } \alpha \restriction (X \setminus Y) = \beta \restriction (X \setminus Y) \Big\}.$$

For every $n \in \mathbb{N}$ the restrictions of d_μ and d_μ^{Rok} to the space of n-piece partitions are uniformly equivalent [15, Fact 1.7.7]. We will temporarily need to use this metric in the proof of the next theorem.

Recall that for a countable ordered partition $\alpha = \{A_i : 0 \leq i < |\alpha|\}$ we let $\mathrm{dist}(\alpha)$ denote the probability vector having i^{th} term $\mu(A_i)$. For $B \subseteq X$ we also write $\mathrm{dist}_B(\alpha)$ for the probability vector having i^{th} term $\mu(A_i \cap B)/\mu(B)$.

Theorem XIV.6. *Let G be a countably infinite group and let $G \curvearrowright (X, \mu)$ be a free p.m.p. ergodic action. Let $\mathcal{P}$ and $\mathcal{Q}$ be ordered countable partitions of $([0, 1], \lambda)$ with $\mathcal{Q} \leq \mathcal{P}$ and $\mathrm{H}(\mathcal{P}) < \infty$. If $h_G^{\mathrm{Rok}}(X, \mu) < \mathrm{H}(\mathcal{P} \mid \mathcal{Q})$, then for every finite $T \subseteq G$ and $\epsilon > 0$ there is an ordered generating partition α with $\mathrm{dist}(\alpha) = \mathrm{dist}(\mathcal{P})$,*

$$\frac{1}{|T|} \cdot \mathrm{H}(\alpha^T) > \mathrm{H}(\alpha) - \epsilon,$$

and $h_{G,X}^{\mathrm{Rok}}(\beta) < \epsilon$, where β is the coarsening of α corresponding to $\mathcal{Q} \leq \mathcal{P}$.

We point out that we do not prove any relative Rokhlin entropy version of this theorem. We believe that a relative version should be true, but its proof would require modifying the Abért–Weiss argument.

Proof. First assume that $\mathcal{P}$ is finite. Apply Lemma XIV.5 to obtain a finite partition $\mathcal{R}$ of $[0, 1]$ which is finer than $\mathcal{Q}$ and satisfies

$$h_G^{\mathrm{Rok}}(X, \mu) < \mathrm{H}(\mathcal{P} \mid \mathcal{R}) < h_G^{\mathrm{Rok}}(X, \mu) + \frac{\epsilon}{256 \cdot |T|^3}.$$

Without loss of generality, we may assume that $\lambda(R) > 0$ for every $R \in \mathcal{R}$. Set $s = \min_{R \in \mathcal{R}} \lambda(R)$. Since d_μ and d_μ^{Rok} are uniformly equivalent on the space of partitions of X having at most $|\mathcal{P}|$ pieces, there is

$$0 < \kappa < \frac{\epsilon}{256 \cdot |T|^3 \cdot \mathrm{H}(\mathcal{P})}$$

satisfying

$$h_G^{\mathrm{Rok}}(X, \mu) < (1 - \kappa) \cdot \mathrm{H}(\mathcal{P} \mid \mathcal{R})$$

such that $d_\mu^{\mathrm{Rok}}(\xi, \xi') < \epsilon/8$ whenever ξ and ξ' are partitions of X with at most $|\mathcal{P}|$ pieces and with $d_\mu(\xi, \xi') \leq \kappa$.

By Theorem II.1, there is a factor $G \curvearrowright (Y, \nu)$ of (X, μ) such that

$$h_G^{\mathrm{Rok}}(Y, \nu) < s\kappa \cdot \mathrm{H}(\mathcal{P}) < \frac{\epsilon}{256 \cdot |T|^3}$$

and G acts freely on (Y, ν). Let $\mathcal{F}$ be the sub-σ-algebra of X associated to (Y, ν). Note that by Corollary IX.3

$$h_G^{\mathrm{Rok}}(X, \mu) \leq h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F}) + h_G^{\mathrm{Rok}}(Y, \nu) < h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F}) + \frac{\epsilon}{256 \cdot |T|^3}.$$

Therefore

$$\mathrm{H}(\mathcal{P} \mid \mathcal{R}) < h_G^{\mathrm{Rok}}(X, \mu) + \frac{\epsilon}{256 \cdot |T|^3} < h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F}) + \frac{\epsilon}{128 \cdot |T|^3}. \tag{14.2}$$

Since G acts freely on (Y, ν), the Abért–Weiss theorem implies that there is an ordered partition $\gamma = \{C_k : 0 \leq k < |\mathcal{R}|\} \subseteq \mathcal{F}$ with $\mathrm{dist}(\gamma) = \mathrm{dist}(\mathcal{R})$ and

$$\frac{1}{|T|} \cdot \mathrm{H}(\gamma^T) > \mathrm{H}(\gamma) - \frac{\epsilon}{2}. \tag{14.3}$$

By construction $h_G^{\mathrm{Rok}}(Y, \nu) < s\kappa \cdot \mathrm{H}(\mathcal{P})$. So by applying Theorem I.6 to (Y, ν) (and invoking Lemma II.6) we obtain a set $Z_0 \in \mathcal{F}$ with $\mu(Z_0) = s\kappa$ and a partition $\alpha^0 = \{A_i^0 : 0 \leq i < |\mathcal{P}|\} \subseteq \mathcal{F}$ of Z_0 with $\mathcal{F} = \sigma\text{-}\mathrm{alg}_G^{\mathrm{red}}(\alpha^0)$ and

$$\mu(A_i^0) = s\kappa \cdot \lambda(P_i) = \mu(Z_0) \cdot \lambda(P_i) \tag{14.4}$$

for every $0 \le i < |\mathcal{P}|$. Note that

$$\mu(Z_0 \cap C_k) \le \mu(Z_0) = s\kappa \le \kappa \cdot \lambda(R_k) = \kappa \cdot \mu(C_k)$$

for all $0 \le k < |\mathcal{R}|$ since $\mathrm{dist}(\gamma) = \mathrm{dist}(\mathcal{R})$. Since (Y, ν) is non-atomic and $\{Z_0\} \cup \gamma \subseteq \mathcal{F}$, it follows from the above inequality that there exists $Z_1 \in \mathcal{F}$ such that $Z_1 \cap Z_0 = \varnothing$, $\mu(Z_1) = 1 - \kappa$, and $\mu(Z_1 \cap C) = (1 - \kappa) \cdot \mu(C)$ for every $C \in \gamma$.

Consider the collection $\gamma \restriction Z_1$ of pairwise disjoint sets. For each $C_k \cap Z_1 \in \gamma \restriction Z_1$ define the probability vector $\bar{p}^{C_k \cap Z_1} = \mathrm{dist}_{R_k}(\mathcal{P})$. We have

$$\begin{aligned} h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F}) &\le h_G^{\mathrm{Rok}}(X, \mu) \\ &< (1 - \kappa) \cdot \mathrm{H}(\mathcal{P} \mid \mathcal{R}) \\ &= \sum_{0 \le k < |\mathcal{R}|} (1 - \kappa)\lambda(R_k) \cdot \mathrm{H}_{R_k}(\mathcal{P}) \\ &= \sum_{0 \le k < |\mathcal{R}|} \mu(C_k \cap Z_1) \cdot \mathrm{H}(\bar{p}^{C_k \cap Z_1}). \end{aligned}$$

So by Lemma XIV.3, there is a partition $\alpha^1 = \{A_i^1 : 0 \le i < |\mathcal{P}|\}$ of Z_1 with

$$\mu(A_i^1 \cap C_k \cap Z_1) = \frac{\lambda(R_k \cap P_i)}{\lambda(R_k)} \cdot \mu(C_k \cap Z_1) = (1 - \kappa) \cdot \lambda(R_k \cap P_i) \tag{14.5}$$

for every i and k and with $\sigma\text{-alg}_G(\alpha') \vee \mathcal{F} = \mathcal{B}(X)$ for all partitions α' extending α^1. Note that

$$\mu(A_i^1) = (1 - \kappa) \cdot \lambda(P_i) = \mu(Z_1) \cdot \lambda(P_i) \tag{14.6}$$

for every i.

Set $Z_2 = X \setminus (Z_0 \cup Z_1)$. Pick any partition $\alpha^2 = \{A_i^2 : 0 \le i < |\mathcal{P}|\}$ of Z_2 with

$$\mu(A_i^2) = \lambda(P_i) \cdot \mu(Z_2) \tag{14.7}$$

for every i. Set $\alpha = \{A_i : 0 \le i < |\mathcal{P}|\}$ where $A_i = A_i^0 \cup A_i^1 \cup A_i^2$. Then $\mu(A_i) = \lambda(P_i)$ for every i by (14.4), (14.6), and (14.7). Additionally, α extends α^0 and thus

$\mathcal{F} \subseteq \sigma\text{-alg}_G(\alpha)$ by Lemma II.5. Similarly, α extends α^1 so

$$\mathcal{B}(X) = \sigma\text{-alg}_G(\alpha) \vee \mathcal{F} = \sigma\text{-alg}_G(\alpha).$$

Thus α is generating.

By (14.5), the partition $\alpha \vee \gamma$ almost has the same distribution as $\mathcal{P} \vee \mathcal{R}$. We next perturb α so that the joint distribution with γ will be precisely the distribution of $\mathcal{P} \vee \mathcal{R}$. Using (14.5), we may pick a partition $\alpha^* = \{A_i^* : 0 \leq i < |\mathcal{P}|\}$ extending α^1 and satisfying $\mu(A_i^* \cap C_k) = \lambda(P_i \cap R_k)$ for all $0 \leq i < |\mathcal{P}|$ and $0 \leq k < |\mathcal{R}|$. Then $\mathrm{dist}(\alpha) = \mathrm{dist}(\alpha^*) = \mathrm{dist}(\mathcal{P})$ and $d_\mu(\alpha, \alpha^*) \leq \mu(Z_0 \cup Z_2) = \kappa$. It follows from the definition of κ that $d_\mu^{\mathrm{Rok}}(\alpha, \alpha^*) < \epsilon/8$ and thus by (14.2)

$$\begin{aligned} \mathrm{H}(\alpha \mid \gamma) &< \mathrm{H}(\alpha^* \mid \gamma) + \epsilon/8 \\ &= \mathrm{H}(\mathcal{P} \mid \mathcal{R}) + \epsilon/8 \\ &< h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F}) + \epsilon/4 \\ &\leq \mathrm{H}(\alpha \mid \mathcal{F}) + \epsilon/4. \end{aligned} \tag{14.8}$$

Let β and β^* be the coarsenings of α and α^*, respectively, corresponding to the coarsening $\mathcal{Q}$ of $\mathcal{P}$. Since $\mu(A_i^* \cap C_k) = \lambda(P_i \cap R_k)$ for all i and k, there is an isomorphism $(X, \mu) \to ([0,1], \lambda)$ of measure spaces which identifies α^* with $\mathcal{P}$ and γ with $\mathcal{R}$. Since $\mathcal{Q}$ is coarser than $\mathcal{R}$, it follows that β^* is coarser than γ. So $\beta^* \subseteq \mathcal{F}$ and hence $h_{G,X}^{\mathrm{Rok}}(\beta^*) \leq h_G^{\mathrm{Rok}}(Y, \nu) < \epsilon/8$. Additionally, $d_\mu(\alpha, \alpha^*) \leq \kappa$ implies $d_\mu(\beta, \beta^*) \leq \kappa$ and thus $d_\mu^{\mathrm{Rok}}(\beta, \beta^*) < \epsilon/8$. It follows that $\mathrm{H}(\beta \mid \beta^*) < \epsilon/8$ and hence $h_{G,X}^{\mathrm{Rok}}(\beta) < \epsilon/4 < \epsilon$ as required.

Finally, we check that $\mathrm{H}(\alpha^T)/|T| > \mathrm{H}(\alpha) - \epsilon$. Using (14.2) and the fact that

$Z_0, Z_1, Z_2 \in \mathcal{F}$, we have

$$\begin{aligned}
\mathrm{H}(\alpha \mid \mathcal{F}) &= \mu(Z_0 \cup Z_2) \cdot \mathrm{H}_{Z_0 \cup Z_2}(\alpha \mid \mathcal{F}) + \mu(Z_1) \cdot \mathrm{H}_{Z_1}(\alpha \mid \mathcal{F}) \\
&\leq \mu(Z_0 \cup Z_2) \cdot \mathrm{H}_{Z_0 \cup Z_2}(\alpha) + \mathrm{H}_{Z_1}(\alpha \mid \gamma) \\
&= \kappa \cdot \mathrm{H}(\mathcal{P}) + \mathrm{H}(\mathcal{P} \mid \mathcal{R}) \\
&< \frac{\epsilon}{256 \cdot |T|^3} + h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F}) + \frac{\epsilon}{128 \cdot |T|^3} \\
&< h_G^{\mathrm{Rok}}(X, \mu \mid \mathcal{F}) + \frac{\epsilon}{64 \cdot |T|^3}
\end{aligned}$$

Applying Theorem XII.3, we conclude that

$$\frac{1}{|T|} \cdot \mathrm{H}(\alpha^T \mid \gamma^T) \geq \frac{1}{|T|} \cdot \mathrm{H}(\alpha^T \mid \mathcal{F}) \geq \mathrm{H}(\alpha \mid \mathcal{F}) - \frac{\epsilon}{4}.$$

From the above inequality and (14.8) we obtain

$$\frac{1}{|T|} \cdot \mathrm{H}(\alpha^T \mid \gamma^T) > \mathrm{H}(\alpha \mid \gamma) - \frac{\epsilon}{2}. \tag{14.9}$$

Also, we observe that

$$\mathrm{H}(\gamma^T \mid \alpha^T) \leq \sum_{t \in T} \mathrm{H}(t \cdot \gamma \mid \alpha^T) \leq \sum_{t \in T} \mathrm{H}(t \cdot \gamma \mid t \cdot \alpha) = |T| \cdot \mathrm{H}(\gamma \mid \alpha). \tag{14.10}$$

Therefore, using (14.3), (14.9), and (14.10), we have

$$\begin{aligned}
\frac{1}{|T|} \cdot \mathrm{H}(\alpha^T) &= \frac{1}{|T|} \cdot \mathrm{H}(\alpha^T \vee \gamma^T) - \frac{1}{|T|} \cdot \mathrm{H}(\gamma^T \mid \alpha^T) \\
&= \frac{1}{|T|} \cdot \mathrm{H}(\gamma^T) + \frac{1}{|T|} \cdot \mathrm{H}(\alpha^T \mid \gamma^T) - \frac{1}{|T|} \cdot \mathrm{H}(\gamma^T \mid \alpha^T) \\
&> \mathrm{H}(\gamma) - \epsilon/2 + \mathrm{H}(\alpha \mid \gamma) - \epsilon/2 - \mathrm{H}(\gamma \mid \alpha) \\
&= \mathrm{H}(\alpha \vee \gamma) - \epsilon - \mathrm{H}(\gamma \mid \alpha) \\
&= \mathrm{H}(\alpha) - \epsilon.
\end{aligned}$$

To complete the proof, we consider the case where $\mathcal{P}$ is countably infinite. By Lemma II.2, there is a finite $\mathcal{Q}_0 \leq \mathcal{Q}$ so that $\mathrm{H}(\mathcal{Q} \mid \mathcal{Q}_0) < \epsilon/2$. Note that $h_G^{\mathrm{Rok}}(X, \mu) <$

$\mathrm{H}(\mathcal{P}\,|\,\mathcal{Q}) \leq \mathrm{H}(\mathcal{P}\,|\,\mathcal{Q}_0)$. Now choose a finite $\mathcal{P}_0 \leq \mathcal{P}$ such that $\mathcal{Q}_0 \leq \mathcal{P}_0$, $\mathrm{H}(\mathcal{P}|\mathcal{P}_0) < \epsilon/2$, and $h^{\mathrm{Rok}}_G(X,\mu) < \mathrm{H}(\mathcal{P}_0\,|\,\mathcal{Q}_0)$. Apply the above argument to get a generating partition α_0 with $\mathrm{dist}(\alpha_0) = \mathrm{dist}(\mathcal{P}_0)$, $\mathrm{H}(\alpha_0^T)/|T| > \mathrm{H}(\alpha_0) - \epsilon/2$, and $h^{\mathrm{Rok}}_{G,X}(\beta_0) < \epsilon/2$, where β_0 is the coarsening of α_0 corresponding to $\mathcal{Q}_0$. Since (X,μ) is non-atomic, we may choose $\alpha \geq \alpha_0$ with $\mathrm{dist}(\alpha) = \mathcal{P}$. Clearly α is still generating. Since $\mathrm{H}(\alpha \mid \alpha_0) = \mathrm{H}(\mathcal{P} \mid \mathcal{P}_0) < \epsilon/2$, we have

$$\frac{1}{|T|} \cdot \mathrm{H}(\alpha^T) \geq \frac{1}{|T|} \cdot \mathrm{H}(\alpha_0^T) > \mathrm{H}(\alpha_0) - \epsilon/2 > \mathrm{H}(\alpha) - \epsilon.$$

Finally, if β is the coarsening of α corresponding to $\mathcal{Q}$ then $\mathrm{H}(\beta|\beta_0) = \mathrm{H}(\mathcal{Q}|\mathcal{Q}_0) < \epsilon/2$ and hence $h^{\mathrm{Rok}}_{G,X}(\beta) < h^{\mathrm{Rok}}_{G,X}(\beta_0) + \epsilon/2 < \epsilon$. $\square$

CHAPTER XV

Rokhlin entropy of Bernoulli shifts

In order to investigate the Rokhlin entropy values of Bernoulli shifts, we first restate Theorem XIV.6 in terms of isomorphisms.

Corollary XV.1. *Let G be a countably infinite group and let $G \curvearrowright (X, \mu)$ be a free p.m.p. ergodic action. Let (L, λ) be a probability space with L finite. Let $\mathscr{L}$ be the canonical partition of L^G, and let $\mathscr{K}$ be a partition coarser than $\mathscr{L}$. If $h^{\mathrm{Rok}}_G(X, \mu) < \mathrm{H}(\mathscr{L} | \mathscr{K})$, then for every open neighborhood $U \subseteq \mathscr{E}_G(L^G)$ of λ^G and every $\epsilon > 0$, there is a G-equivariant isomorphism $\phi : (X, \mu) \to (L^G, \nu)$ with $\nu \in U$ and $h^{\mathrm{Rok}}_{G,(L^G,\nu)}(\mathscr{K}) < \epsilon$.*

Proof. By definition, $\mathscr{L} = \{R_\ell : \ell \in L\}$ where

$$R_\ell = \{y \in L^G : y(1_G) = \ell\}.$$

Since U is open, there are continuous functions $f_1, \ldots, f_n$ on L^G and $\kappa_1 > 0$ such that for all $\nu \in \mathscr{E}_G(L^G)$

$$\left| \int f_i \; d\lambda^G - \int f_i \; d\nu \right| < \kappa_1 \text{ for all } 1 \leq i \leq n \Longrightarrow \nu \in U.$$

Since L^G is compact, each f_i is uniformly continuous and therefore there is a finite $T \subseteq G$ and continuous $\mathscr{L}^T$-measurable functions f'_i such that $\|f_i - f'_i\| < \kappa_1/2$ for each $1 \le i \le n$, where $\|\cdot\|$ denotes the sup-norm. Therefore there is $\kappa_2 > 0$ such that for all $\nu \in \mathscr{E}_G(L^G)$

$$\left|\lambda^G(D) - \nu(D)\right| < \kappa_2 \text{ for all } D \in \mathscr{L}^T \Longrightarrow \nu \in U.$$

Enumerate T as $t_1, \ldots, t_m$ and set $T_i = \{t_1, \ldots, t_{i-1}\}$. If $D = \bigcap_{i=1}^m t_i \cdot R_i \in \mathscr{L}^T$, then setting $D_j = \bigcap_{i=1}^{j-1} t_i \cdot R_i \in \mathscr{L}^{T_j}$ we have $D_{i+1} = D_i \cap t_i \cdot R_i$ and hence

$$\nu(D) = \prod_{i=1}^m \frac{\nu(D_i \cap t_i \cdot R_i)}{\nu(D_i)}.$$

Since m-fold multiplication of elements of $[0,1]$ is uniformly continuous, there is $\kappa_3 > 0$ such that the condition

$$\left|\lambda^G(R) - \frac{\nu(D \cap t_i \cdot R)}{\nu(D)}\right| < \kappa_3 \text{ for all } 1 \le i \le m,\ D \in \mathscr{L}^{T_i},\text{ and } R \in \mathscr{L}$$

implies $\nu \in U$. Above we have used the fact that $\lambda^G(D \cap t_i \cdot R)/\lambda^G(D) = \lambda^G(R)$ for $1 \le i \le m$, $D \in \mathscr{L}^{T_i}$, and $R \in \mathscr{L}$. Finally, by standard properties of Shannon entropy [15, Fact 3.1.3], there is $\kappa_4 > 0$ such that the condition

$$|\lambda^G(R) - \nu(R)| < \kappa_4 \text{ and } \mathrm{H}_\nu(t_i \cdot \mathscr{L} \,|\, \mathscr{L}^{T_i}) > \mathrm{H}_\nu(\mathscr{L}) - \kappa_4 \text{ for all } R \in \mathscr{L} \text{ and } 1 \le i \le m$$

implies $\nu \in U$.

Now apply Theorem XIV.6 to obtain a generating partition $\alpha = \{A_\ell : \ell \in L\}$ of X satisfying $\mu(A_\ell) = \lambda^G(R_\ell)$ for every $\ell \in L$, $\mathrm{H}(\alpha^T) > |T| \cdot \mathrm{H}(\alpha) - \kappa_4$, and $h^{\mathrm{Rok}}_{G,X}(\beta) < \epsilon$, where β is the coarsening of α corresponding to $\mathscr{K}$. Since α is generating and its classes are indexed by L, it induces a G-equivariant isomorphism $\phi : (X, \mu) \to (L^G, \nu)$ which identifies α with $\mathscr{L}$ and β with $\mathscr{K}$. We immediately have $\nu(R_\ell) = \mu(A_\ell) = \lambda^G(R_\ell)$ for every $\ell \in L$ and $h^{\mathrm{Rok}}_{G,(L^G,\nu)}(\mathscr{K}) = h^{\mathrm{Rok}}_{G,X}(\beta) < \epsilon$. Also,

$$\sum_{i=1}^m \Big(\mathrm{H}(\alpha) - \mathrm{H}(t_i \cdot \alpha \mid \alpha^{T_i})\Big) = |T| \cdot \mathrm{H}(\alpha) - \mathrm{H}(\alpha^T) < \kappa_4.$$

Since each summand on the left is non-negative, we deduce that

$$\mathrm{H}_\nu(t_i \cdot \mathscr{L} \mid \mathscr{L}^{T_i}) = \mathrm{H}(t_i \cdot \alpha \mid \alpha^{T_i}) > \mathrm{H}(\alpha) - \kappa_4 = \mathrm{H}_\nu(\mathscr{L}) - \kappa_4$$

for every $1 \leq i \leq m$. We conclude that $\nu \in U$. □

Fix a countably infinite group G. Recall from the introduction the quantity

$$h^{\mathrm{Rok}}_{sup}(G) = \sup_{G \curvearrowright (X,\mu)} h^{\mathrm{Rok}}_G(X,\mu),$$

where the supremum is taken over all free ergodic p.m.p. actions $G \curvearrowright (X,\mu)$ with $h^{\mathrm{Rok}}_G(X,\mu) < \infty$. If there is a free ergodic p.m.p. action $G \curvearrowright (X,\mu)$ with $h^{\mathrm{Rok}}_G(X,\mu) = \infty$, we do not know if it necessarily follows that $h^{\mathrm{Rok}}_{sup}(G) = \infty$. In particular, we do not know if $G \curvearrowright (X,\mu)$ must factor onto free actions having large but finite Rokhlin entropy values. However, we have the following.

Lemma XV.2. *Let G be a countably infinite group and let $G \curvearrowright (X,\mu)$ be a free p.m.p. ergodic action. If $h^{\mathrm{Rok}}_G(X,\mu) < \infty$ then for every $0 \leq t \leq h^{\mathrm{Rok}}_G(X,\mu)$ and $\delta > 0$ there is a factor $G \curvearrowright (Y,\nu)$ of (X,μ) such that G acts freely on Y and $h^{\mathrm{Rok}}_G(Y,\nu) \in (t-\delta, t+\delta)$.*

Proof. Let $\bar{p}$ be a probability vector with $\mathrm{H}(\bar{p}) = t$, and let $\bar{q}$ be a probability vector with $h^{\mathrm{Rok}}_G(X,\mu) - t < \mathrm{H}(\bar{q}) < h^{\mathrm{Rok}}_G(X,\mu) - t + \delta$. Let $\bar{r}$ be the probability vector which represents the independent join of $\bar{p}$ and $\bar{q}$. Specifically, $\bar{r} = (r_{i,j})$ where $r_{i,j} = p_i \cdot q_j$. We have $\mathrm{H}(\bar{r}) = \mathrm{H}(\bar{p}) + \mathrm{H}(\bar{q})$ so $h^{\mathrm{Rok}}_G(X,\mu) < \mathrm{H}(\bar{r})$. By Theorem I.3 there is a generating partition $\gamma = \{C_{i,j}\}$ with $\mu(C_{i,j}) = r_{i,j}$. Let $\alpha = \{A_i \,:\, 0 \leq i < |\bar{p}|\}$ be the coarsening of γ associated to $\bar{p}$, meaning

$$A_i = \cup\{C_{i,j} \,:\, 0 \leq j < |\bar{q}|\}.$$

Similarly define $\beta = \{B_j \,:\, 0 \leq j < |\bar{q}|\}$ by

$$B_j = \cup\{C_{i,j} \,:\, 0 \leq i < |\bar{p}|\}.$$

Then $\mathrm{dist}(\alpha) = \bar{p}$, $\mathrm{dist}(\beta) = \bar{q}$, and $\alpha \vee \beta = \gamma$.

By Theorem II.1, there is a factor $G \curvearrowright (Z, \eta)$ of (X, μ) with $h_G^{\mathrm{Rok}}(Z, \eta) < \delta$. Let ζ' be a generating partition for Z with $\mathrm{H}(\zeta') < \delta$, and let ζ be the pre-image of ζ' in X. Let $G \curvearrowright (Y, \nu)$ be the factor of (X, μ) associated to $\sigma\text{-alg}_G(\alpha \vee \zeta)$. Clearly $\alpha \vee \zeta$ pushes forward to a generating partition $\alpha' \vee \zeta''$ of Y with $\mathrm{H}(\alpha') = \mathrm{H}(\bar{p})$ and $\mathrm{H}(\zeta'') < \delta$. So $h_G^{\mathrm{Rok}}(Y, \nu) \leq \mathrm{H}(\alpha' \vee \zeta'') < t + \delta$. By Corollary IX.3 we also have

$$h_G^{\mathrm{Rok}}(Y, \nu) \geq h_G^{\mathrm{Rok}}(X, \mu) - h_G^{\mathrm{Rok}}(X, \mu \,|\, \sigma\text{-alg}_G(\alpha \vee \zeta)) \geq h_G^{\mathrm{Rok}}(X, \mu) - \mathrm{H}(\beta) > t - \delta.$$

Finally, $G \curvearrowright (Y, \nu)$ must be a free action since it factors onto (Z, η). $\square$

We now focus our attention on the Rokhlin entropy values of Bernoulli shifts. Let (L, λ) be a probability space and let $\mathscr{L}$ be the canonical partition of L^G. If $\mathscr{K}$ is a partition coarser than $\mathscr{L}$, then the translates of $\mathscr{K}$ are mutually independent and the factor associated to $\sigma\text{-alg}_G(\mathscr{K})$ is a Bernoulli shift $G \curvearrowright (K^G, \kappa^G)$. In order to emphasize the fact that $\sigma\text{-alg}_G(\mathscr{K})$ corresponds to a Bernoulli factor of (L^G, λ^G), for the remainder of this chapter we will write $\mathscr{K}^G$ for $\sigma\text{-alg}_G(\mathscr{K})$.

Proposition XV.3. *Let G be a countably infinite group and let (L, λ) be a probability space with L finite. Let $\mathscr{L}$ be the canonical partition of L^G and let $\mathscr{K}$ be a partition coarser than $\mathscr{L}$. Then*

$$h_G^{\mathrm{Rok}}\big(L^G, \lambda^G \,|\, \mathscr{K}^G\big) = \min\Big(\mathrm{H}(\mathscr{L} \,|\, \mathscr{K}), \quad h_{sup}^{\mathrm{Rok}}(G)\Big).$$

Proof. We immediately have $h_G^{\mathrm{Rok}}(L^G, \lambda^G \,|\, \mathscr{K}^G) \leq \mathrm{H}(\mathscr{L} \,|\, \mathscr{K})$ since $\mathscr{L}$ is a generating partition. We will show that there does not exist any free p.m.p. ergodic action $G \curvearrowright (X, \mu)$ with

$$h_G^{\mathrm{Rok}}(L^G, \lambda^G \,|\, \mathscr{K}^G) < h_G^{\mathrm{Rok}}(X, \mu) < \mathrm{H}(\mathscr{L} \,|\, \mathscr{K}).$$

From Lemma XV.2 it will follow that either $h_G^{\mathrm{Rok}}(L^G, \lambda^G \mid \mathscr{K}^G) = \mathrm{H}(\mathscr{L} \mid \mathscr{K})$ or else $h_G^{\mathrm{Rok}}(L^G, \lambda^G \mid \mathscr{K}^G) \geq h_G^{\mathrm{Rok}}(X, \mu)$ for every free p.m.p. ergodic action $G \curvearrowright (X, \mu)$ with $h_G^{\mathrm{Rok}}(X, \mu) < \infty$.

Towards a contradiction, suppose that $G \curvearrowright (X, \mu)$ is a free p.m.p. ergodic action with $h_G^{\mathrm{Rok}}(L^G, \lambda^G \mid \mathscr{K}^G) < h_G^{\mathrm{Rok}}(X, \mu) < \mathrm{H}(\mathscr{L} \mid \mathscr{K})$. Fix $\epsilon > 0$ with

$$h_G^{\mathrm{Rok}}(L^G, \lambda^G \mid \mathscr{K}^G) + \epsilon < h_G^{\mathrm{Rok}}(X, \mu).$$

By Corollary XIII.10, there is an open neighborhood $U \subseteq \mathscr{E}_G(L^G)$ of λ^G such that $h_G^{\mathrm{Rok}}(L^G, \nu \mid \mathscr{K}^G) < h_G^{\mathrm{Rok}}(L^G, \lambda^G \mid \mathscr{K}^G) + \epsilon/2$ for all $\nu \in U$. By Corollary XV.1, there is a G-equivariant isomorphism $\phi : (X, \mu) \to (L^G, \nu)$ with $\nu \in U$ and $h_{G,(L^G,\nu)}^{\mathrm{Rok}}(\mathscr{K}) < \epsilon/2$. Then by Corollary IX.4

$$\begin{aligned} h_G^{\mathrm{Rok}}(X, \mu) &= h_G^{\mathrm{Rok}}(L^G, \nu) \\ &\leq h_{G,(L^G,\nu)}^{\mathrm{Rok}}(\mathscr{K}) + h_G^{\mathrm{Rok}}(L^G, \nu \mid \mathscr{K}^G) \\ &< h_G^{\mathrm{Rok}}(L^G, \lambda^G \mid \mathscr{K}^G) + \epsilon \\ &< h_G^{\mathrm{Rok}}(X, \mu), \end{aligned}$$

a contradiction. □

Theorem XV.4. *Let G be a countably infinite group and let (L, λ) be a probability space with $\mathrm{H}(L, \lambda) < \infty$. Then*

$$h_G^{\mathrm{Rok}}(L^G, \lambda^G) = \min\Big(\mathrm{H}(L, \lambda), \quad h_{sup}^{\mathrm{Rok}}(G)\Big).$$

Proof. Let $\mathscr{L} = \{R_\ell : \ell \in L\}$ be the canonical partition of L^G where

$$R_\ell = \{y \in L^G : y(1_G) = \ell\}.$$

Let $\mathscr{L}_n$ be an increasing sequence of finite partitions which are coarser than $\mathscr{L}$ and satisfy $\mathscr{L} = \bigvee_{n \in \mathbb{N}} \mathscr{L}_n$. The algebra generated by $\mathscr{L}_n$ corresponds to a factor (L_n, λ_n)

of (L, λ), and the factor of (L^G, λ^G) corresponding to $\mathscr{L}_n^G$ is (L_n^G, λ_n^G). By Corollary XIII.9 $h_G^{\mathrm{Rok}}(L^G, \lambda^G) = \lim_{n\to\infty} h_G^{\mathrm{Rok}}(L_n^G, \lambda_n^G)$. The claim now follows by applying Proposition XV.3 to each (L_n^G, λ_n^G) and using the fact that $\mathrm{H}(L_n, \lambda_n) = \mathrm{H}(\mathscr{L}_n)$ converges to $\mathrm{H}(\mathscr{L}) = \mathrm{H}(L, \lambda)$. □

We next handle the case where $\mathrm{H}(L, \lambda) = \infty$, but first we need a lemma.

Lemma XV.5. *Let (L, λ) be a probability space with $\mathrm{H}(L, \lambda) = \infty$, and let $c > 0$. Then there exists a sequence of finite partitions $(\mathscr{L}_n)_{n\in\mathbb{N}}$ with $\bigvee_{n\in\mathbb{N}} \sigma\text{-alg}(\mathscr{L}_n) = \mathcal{B}(L)$ and*

$$\mathrm{H}\Big(\mathscr{L}_m \Big| \bigvee_{n\neq m} \mathscr{L}_n\Big) > c$$

for all $m \in \mathbb{N}$.

Proof. First suppose that L is essentially countable. For $\ell \in L$ we will write $\lambda(\ell)$ for $\lambda(\{\ell\})$. Since

$$\sum_{\ell\in L} -\lambda(\ell)\cdot\log\lambda(\ell) = \mathrm{H}(L, \lambda) = \infty,$$

we can partition L into finite sets I_n with

$$\sum_{\ell\in I_n} -\lambda(\ell)\cdot\log\lambda(\ell) > c + \log(2)$$

for all n. Define

$$\mathscr{L}_n = \{L \setminus I_n\} \cup \Big\{\{\ell\} : \ell \in I_n\Big\}.$$

Note that $\mathrm{H}(\mathscr{L}_n) > c + \log(2)$. Clearly $\mathscr{L}_n$ is finite and $\bigvee_{n\in\mathbb{N}} \sigma\text{-alg}(\mathscr{L}_n) = \mathcal{B}(L)$. Additionally, we have $I_n \in \bigvee_{k\neq n} \sigma\text{-alg}(\mathscr{L}_k)$ since $L \setminus I_n$ is the union of all singleton sets contained in $\bigvee_{k \neq n} \sigma\text{-alg}(\mathscr{L}_k)$. Therefore

$$\begin{aligned}
\mathrm{H}\Big(\mathscr{L}_n \mid \textstyle\bigvee_{k \neq n} \mathscr{L}_k\Big) &= \mathrm{H}(\mathscr{L}_n \mid \{I_n, L \setminus I_n\}) \\
&= \mathrm{H}(\mathscr{L}_n) - \mathrm{H}(\{I_n, L \setminus I_n\}) \\
&> \mathrm{H}(\mathscr{L}_n) - \log(2) \\
&> c.
\end{aligned}$$

Now suppose that (L, λ) is not essentially countable. Then L decomposes into a non-atomic part $B \subseteq L$ and a purely atomic part $A \subseteq L$ with $\{B, A\}$ a partition of L and $\lambda(B) > 0$. Fix any increasing sequence α_n of finite partitions of A with $\mathcal{B}(L) \restriction A = \bigvee_{n \in \mathbb{N}} \sigma\text{-alg}(\alpha_n) \restriction A$. Choose a probability vector $\bar{p}$ with $\mu(B) \cdot \mathrm{H}(\bar{p}) > c$, and let λ_B be the normalized restriction of λ to B. Since B has no atoms, we can find a sequence of λ_B-independent ordered partitions β_n of B with $\mathrm{dist}_{\lambda_B}(\beta_n) = \bar{p}$ for every n and with $\mathcal{B}(L) \restriction B = \bigvee_{n \in \mathbb{N}} \sigma\text{-alg}(\beta_n) \restriction B$. Now set $\mathscr{L}_n = \beta_n \cup \alpha_n$. Then $\mathscr{L}_n$ is finite and $\mathcal{B}(L) = \bigvee_{n \in \mathbb{N}} \sigma\text{-alg}(\mathscr{L}_n)$. Finally, since $\{B, A\}$ is coarser than every $\mathscr{L}_n$ we have

$$\begin{aligned}
\mathrm{H}\Big(\mathscr{L}_m \mid \textstyle\bigvee_{n \neq m} \mathscr{L}_n\Big) &\geq \lambda(B) \cdot \mathrm{H}_B\Big(\mathscr{L}_m \mid \textstyle\bigvee_{n \neq m} \mathscr{L}_n\Big) \\
&= \lambda(B) \cdot \mathrm{H}_B\Big(\beta_m \mid \textstyle\bigvee_{n \neq m} \beta_n\Big) \\
&= \lambda(B) \cdot \mathrm{H}(\bar{p}) \\
&> c.
\end{aligned}$$

□

Theorem XV.6. *Let G be a countably infinite group, and let (L, λ) be a probability space with $\mathrm{H}(L, \lambda) = \infty$. Then $h_G^{\mathrm{Rok}}(L^G, \lambda^G) = \infty$ if and only if there is a free ergodic p.m.p. action $G \curvearrowright (X, \mu)$ with $h_G^{\mathrm{Rok}}(X, \mu) > 0$.*

Proof. One implication is immediate: if $h_G^{\mathrm{Rok}}(L^G, \lambda^G) = \infty$ then $h_G^{\mathrm{Rok}}(X, \mu) > 0$ with $(X, \mu) = (L^G, \lambda^G)$. So suppose that $G \curvearrowright (X, \mu)$ is a free p.m.p. ergodic action with $h_G^{\mathrm{Rok}}(X, \mu) > 0$. Let (α_n) be an increasing sequence of finite partitions of X with $\mathcal{B}(X) = \bigvee_{n \in \mathbb{N}} \sigma\text{-alg}_G(\alpha_n)$. Using Theorem II.1, we may choose α_1 so that G acts freely on the factor (Z, η) of (X, μ) associated to $\sigma\text{-alg}_G(\alpha_1)$. From Theorem XIII.3 we have that at least one of the following two quantities is positive:

$$\inf_{n \in \mathbb{N}} \lim_{\epsilon \to 0} \sup_{m \in \mathbb{N}} \inf_{k \in \mathbb{N}} \inf_{\substack{T \subseteq G \\ T \text{ finite}}} \inf \left\{ \mathrm{H}(\beta \mid \alpha_n^T) \,:\, \beta \leq \alpha_k^T,\ \mathrm{H}(\alpha_m \mid \beta^T) < \epsilon \right\}$$

$$\lim_{\epsilon \to 0} \sup_{m \in \mathbb{N}} \inf_{k \in \mathbb{N}} \inf_{\substack{T \subseteq G \\ T \text{ finite}}} \inf \left\{ \mathrm{H}(\beta) \,:\, \beta \leq \alpha_k^T,\ \mathrm{H}(\alpha_m \mid \beta^T) < \epsilon \right\}.$$

Since the first expression is less than or equal to the second, the second expression must be positive. Fix ϵ_0 and $m \in \mathbb{N}$ with

$$\inf_{k \in \mathbb{N}} \inf_{\substack{T \subseteq G \\ T \text{ finite}}} \inf \left\{ \mathrm{H}(\beta) \,:\, \beta \leq \alpha_k^T,\ \mathrm{H}(\alpha_m \mid \beta^T) < \epsilon_0 \right\} > 0.$$

Since the above expression increases in value as ϵ_0 decreases, we see that

$$\lim_{\epsilon \to 0} \inf_{\substack{T \subseteq G \\ T \text{ finite}}} \inf \left\{ \mathrm{H}(\beta) \,:\, \beta \leq \alpha_m^T,\ \mathrm{H}(\alpha_m \mid \beta^T) < \epsilon \right\} > 0. \tag{15.1}$$

Let $G \curvearrowright (Y, \nu)$ be the factor of (X, μ) associated to $\sigma\text{-alg}_G(\alpha_m)$. From Corollary XIII.5 and (15.1) we obtain $h_G^{\mathrm{Rok}}(Y, \nu) > 0$. Additionally, (Y, ν) factors onto (Z, η) since α_m refines α_1. So G acts freely on Y and $0 < h_G^{\mathrm{Rok}}(Y, \nu) \leq \mathrm{H}(\alpha_m) < \infty$. Set $c = h_G^{\mathrm{Rok}}(Y, \nu)$.

Apply Lemma XV.5 to get a sequence $\mathscr{L}_n$ of finite non-trivial partitions of L with $\mathcal{B}(L) = \bigvee_{n \in \mathbb{N}} \sigma\text{-alg}(\mathscr{L}_n)$ and $\mathrm{H}(\mathscr{L}_m \mid \bigvee_{n \neq m} \mathscr{L}_n) \geq c$ for all m. For $m \leq k$ set

$$\mathscr{L}_{[0,k]} = \bigvee_{0 \leq i \leq k} \mathscr{L}_i \quad \text{and} \quad \mathscr{L}_{[0,k],m} = \bigvee_{0 \leq i \neq m \leq k} \mathscr{L}_i.$$

Note that for $k \geq m$ we have $\mathrm{H}(\mathscr{L}_{[0,k]} \mid \mathscr{L}_{[0,k],m}) \geq c$ by construction. We let $(L_{[0,k]}, \lambda_{[0,k]})$ denote the factor of (L, λ) associated to $\mathscr{L}_{[0,k]}$. Let $\mathscr{L} = \{R_\ell : \ell \in L\}$

be the canonical (possibly uncountable) partition of L^G defined by

$$R_\ell = \{w \in L^G : w(1_G) = \ell\}.$$

Note that $\mathcal{B}(L^G) = \mathscr{L}^G$. We identify each of the partitions $\mathscr{L}_m$, $\mathscr{L}_{[0,k]}$, and $\mathscr{L}_{[0,k],m}$ as coarsenings of $\mathscr{L} \subseteq \mathcal{B}(L^G)$. Note that $(L^G_{[0,k]}, \lambda^G_{[0,k]})$ is the factor of (L^G, λ^G) associated to $\mathscr{L}^G_{[0,k]}$. As each $\mathscr{L}_n$ is non-trivial, the space $(L_{[0,k]}, \lambda_{[0,k]})$ is not essentially a single point and hence $\lambda^G_{[0,k]}$ is non-atomic.

The partitions $\mathscr{L}_{[0,k]}$ are increasing with k and $\mathscr{L}^G = \bigvee_{k\in\mathbb{N}} \mathscr{L}^G_{[0,k]}$. By Theorem XIII.3, it suffices to show that

$$\inf_{n\in\mathbb{N}} \lim_{\epsilon\to 0} \sup_{m\in\mathbb{N}} \inf_{k\in\mathbb{N}} \inf_{\substack{T\subseteq G\\ T \text{ finite}}} \inf \left\{ \mathrm{H}(\beta \mid \mathscr{L}^T_{[0,n]}) \ : \ \beta \leq \mathscr{L}^T_{[0,k]}, \ \mathrm{H}(\mathscr{L}_{[0,m]} \mid \beta^T) < \epsilon \right\} \tag{15.2}$$

is positive. Note that above one can change $\inf_{k\in\mathbb{N}}$ to $\lim_{k\to\infty}$ without changing the value of the expression. So it suffices to fix $n < m \leq k$ and $0 < \epsilon < c/2$ and show that the remaining portion of (15.2) is uniformly bounded away from 0. Suppose that $\beta \subseteq \mathscr{L}^G_{[0,k]}$ and $\mathrm{H}(\mathscr{L}_{[0,m]} \mid \sigma\text{-alg}_G(\beta)) < c/2$. Since $\mathscr{L}_{[0,m]} \leq \mathscr{L}_{[0,k]}$ and $\lambda^G_{[0,k]}$ is non-atomic, by Theorem I.6 there is a partition $\gamma \subseteq \mathscr{L}^G_{[0,k]}$ with $\mathrm{H}(\gamma) < c/2$ and $\mathscr{L}_{[0,m]} \subseteq \sigma\text{-alg}_G(\beta \vee \gamma)$. Then

$$\mathscr{L}_{[0,k]} \subseteq \sigma\text{-alg}_G(\beta \vee \gamma) \vee \mathscr{L}^G_{[0,k],m}$$

and

$$\mathrm{H}(\beta \vee \gamma \mid \mathscr{L}^G_{[0,k],m}) \leq \mathrm{H}(\beta \mid \mathscr{L}^G_{[0,n]}) + \mathrm{H}(\gamma) < \mathrm{H}(\beta \mid \mathscr{L}^G_{[0,n]}) + c/2.$$

Therefore

$$h^{\mathrm{Rok}}_G\left(L^G_{[0,k]}, \lambda^G_{[0,k]} \middle| \mathscr{L}^G_{[0,k],m}\right) \leq \mathrm{H}(\beta \mid \mathscr{L}^G_{[0,n]}) + c/2.$$

Applying Proposition XV.3 with $\mathscr{K} = \mathscr{L}_{[0,k],m}$ we obtain

$$\begin{aligned} c &= \min\Big(\mathrm{H}(\mathscr{L}_{[0,k]} \mid \mathscr{L}_{[0,k],m}),\ h_G^{\mathrm{Rok}}(Y,\nu)\Big) \\ &\leq h_G^{\mathrm{Rok}}\Big(L_{[0,k]}^G, \lambda_{[0,k]}^G \Big| \mathscr{L}_{[0,k],m}^G\Big) \\ &< \mathrm{H}(\beta \mid \mathscr{L}_{[0,n]}^G) + c/2. \end{aligned}$$

So $\mathrm{H}(\beta \mid \mathscr{L}_{[0,n]}^G) > c/2$ and hence (15.2) is at least $c/2 > 0$. We conclude that $h_G^{\mathrm{Rok}}(L^G, \lambda^G) = \infty$. □

Corollary XV.7. *Let G be a countably infinite group. The following are equivalent:*

(i) $h_{sup}^{\mathrm{Rok}}(G) > 0$*;*

(ii) there is a free ergodic p.m.p. action with $0 < h_G^{\mathrm{Rok}}(X,\mu) < \infty$*;*

(iii) there is a free ergodic p.m.p. action with $h_G^{\mathrm{Rok}}(X,\mu) = \infty$*.*

Proof. The equivalence of (i) and (ii) is by definition. Theorem XV.6 shows that (ii) implies (iii), and the implication (iii) implies (ii) was deduced in the first paragraph of the proof of Theorem XV.6. □

We mention that if in Theorem XIII.3 the second expression always coincides with Rokhlin entropy, then from a free ergodic action $G \curvearrowright (Y,\nu)$ with $h_G^{\mathrm{Rok}}(Y,\nu) = \infty$ one could use the argument in the first paragraph of the proof of Theorem XV.6 to show that (Y,ν) has free factors with arbitrarily large but finite Rokhlin entropy values. From Corollary XV.7 it would then follow that $h_{sup}^{\mathrm{Rok}}(G) > 0$ implies $h_{sup}^{\mathrm{Rok}}(G) = \infty$.

Theorem XV.8. *Let P be a countable group containing arbitrarily large finite subgroups. If G is any countably infinite group with $h_{sup}^{\mathrm{Rok}}(G) < \infty$ then $h_{sup}^{\mathrm{Rok}}(P \times G) = 0$.*

Proof. Set $\Gamma = P \times G$. Let (L,λ) be a probability space with L finite and $\mathrm{H}(L,\lambda) > 0$, and consider the Bernoulli shift $(L^\Gamma, \lambda^\Gamma)$. By Theorem XV.4 it suffices to show that $h_\Gamma^{\mathrm{Rok}}(L^\Gamma, \lambda^\Gamma) = 0$.

Fix $\epsilon > 0$, fix $k \in \mathbb{N}$ with $h^{\mathrm{Rok}}_{sup}(G) < \log(k)$, and fix a finite subgroup $T \leq P$ with $\log(k)/|T| < \epsilon$. Let $\mathscr{L} = \{R_\ell : \ell \in L\}$ be the canonical partition of L^Γ, where

$$R_\ell = \{x \in L^\Gamma : x(1_\Gamma) = \ell\}.$$

Consider the partition $\mathscr{L}^T$. We may write $\mathscr{L}^T = \{D_\pi : \pi \in L^T\}$ where

$$D_\pi = \bigcap_{t \in T} t \cdot R_{\pi(t)}.$$

Since T is a group, it naturally acts on L^T by shifts: $(t \cdot \pi)(s) = \pi(t^{-1}s)$. For $u \in T$ we have $u \cdot D_\pi = D_{u \cdot \pi}$ since

$$u \cdot D_\pi = \bigcap_{t \in T} ut \cdot R_{\pi(t)} = \bigcap_{t \in T} t \cdot R_{\pi(u^{-1}t)} = D_{u \cdot \pi}.$$

Let $\mathcal{Q} = \{Q_{[\pi]} : \pi \in L^T\}$ be the partition of L^Γ where $[\pi]$ denotes the T-orbit of π and

$$Q_{[\pi]} = \bigcup_{t \in T} D_{t \cdot \pi}.$$

Since $T \cap G = \{1_\Gamma\}$, the G-translates of $\mathcal{Q}$ are mutually independent. As L^T has at least two distinct T-orbits, the factor $G \curvearrowright (Z, \eta)$ associated to $\sigma\text{-alg}_G(\mathcal{Q})$ is isomorphic to a G-Bernoulli shift and is in particular a free action.

By Theorem II.1, there is a factor $\Gamma \curvearrowright (Y, \nu)$ of $(L^\Gamma, \lambda^\Gamma)$ such that $h^{\mathrm{Rok}}_\Gamma(Y, \nu) < \epsilon$ and the action of Γ on Y is free. The T-orbits of Y are finite and partition Y, so there is a Borel set $M' \subseteq Y$ which meets every T-orbit precisely once. Let $\mathcal{F}$ be the Γ-invariant sub-σ-algebra of L^Γ associated to Y, and let $M \in \mathcal{F}$ be the pre-image of M'.

Define $\xi = \{C_\pi : \pi \in L^T\}$ to be the partition of L^Γ defined by

$$C_\pi = \bigcup_{s \in T} s \cdot (D_\pi \cap M).$$

This is indeed a partition of L^Γ since the T-translates of M partition L^Γ and the sets $D_\pi \cap M$ partition M. To add clarification to this definition, we remark that $x_1, x_2 \in L^\Gamma$ lie in the same class of ξ if and only if $s_1^{-1} \cdot x_1$ and $s_2^{-1} \cdot x_2$ lie in the same class of $\mathscr{L}^T$, where $s_1, s_2 \in T$ are defined by the condition $s_1^{-1} \cdot x_1, s_2^{-1} \cdot x_2 \in M$. We observe that $\sigma\text{-alg}_\Gamma(\xi) \vee \mathcal{F} = \mathcal{B}(L^\Gamma)$ since for $\ell \in L$

$$\begin{aligned} R_\ell = \bigcup_{\substack{\pi \in L^T \\ \pi(1_\Gamma)=\ell}} D_\pi &= \bigcup_{s \in T} \bigcup_{\substack{\pi \in L^T \\ \pi(1_\Gamma)=\ell}} \Big(D_\pi \cap s \cdot M \Big) = \bigcup_{s \in T} \bigcup_{\substack{\pi \in L^T \\ \pi(1_\Gamma)=\ell}} s \cdot (D_{s^{-1} \cdot \pi} \cap M) \\ &= \bigcup_{s \in T} \bigcup_{\substack{\pi \in L^T \\ \pi(s^{-1})=\ell}} s \cdot (D_\pi \cap M) = \bigcup_{s \in T} \bigcup_{\substack{\pi \in L^T \\ \pi(s^{-1})=\ell}} \Big(C_\pi \cap s \cdot M \Big). \end{aligned}$$

Each $C_\pi \in \xi$ is T-invariant since for $u \in T$ and $\pi \in L^T$ we have

$$u \cdot C_\pi = \bigcup_{s \in T} (us) \cdot (D_\pi \cap M) = C_\pi.$$

Furthermore, ξ is finer than $\mathcal{Q}$ as

$$\begin{aligned} Q_{[\pi]} = \bigcup_{t \in T} D_{t \cdot \pi} &= \bigcup_{s,t \in T} \Big(D_{t \cdot \pi} \cap s \cdot M \Big) = \bigcup_{s,t \in T} \Big(D_{st \cdot \pi} \cap s \cdot M \Big) \\ &= \bigcup_{s,t \in T} s \cdot (D_{t \cdot \pi} \cap M) = \bigcup_{s,t \in T} \Big(C_{t \cdot \pi} \cap s \cdot M \Big) = \bigcup_{t \in T} C_{t \cdot \pi}. \end{aligned}$$

Let $G \curvearrowright (W, \omega)$ be the factor of $(L^\Gamma, \lambda^\Gamma)$ associated to $\sigma\text{-alg}_G(\xi)$. Since ξ is finer than $\mathcal{Q}$, (W, ω) factors onto (Z, η). Thus G acts freely on (W, ω). We have $h_G^{\mathrm{Rok}}(W, \omega) \leq \mathrm{H}(\xi) < \infty$ and thus by assumption $h_G^{\mathrm{Rok}}(W, \omega) \leq h_{sup}^{\mathrm{Rok}}(G) < \log(k)$. Apply Theorem I.3 to get a k-piece generating partition β' for W, and let $\beta \subseteq \sigma\text{-alg}_G(\xi)$ be the pre-image of β'. Then $\xi \subseteq \sigma\text{-alg}_G(\beta)$ and hence

$$\mathcal{B}(L^\Gamma) = \sigma\text{-alg}_\Gamma(\xi) \vee \mathcal{F} \subseteq \sigma\text{-alg}_\Gamma(\beta) \vee \mathcal{F}.$$

We observed that every $C_\pi \in \xi$ is T-invariant. Since G and T commute, it follows that every set in $\sigma\text{-alg}_G(\xi)$ is T-invariant. In particular, each $B \in \beta$ is T-invariant.

Therefore, setting

$$\alpha = \{L^\Gamma \setminus M\} \cup (\beta \restriction M),$$

we have $\beta \subseteq \sigma\text{-alg}_T(\alpha) \vee \mathcal{F}$. Thus $\mathcal{B}(L^\Gamma) = \sigma\text{-alg}_\Gamma(\alpha) \vee \mathcal{F}$. Therefore by Corollary IX.3

$$\begin{aligned}
h_\Gamma^{\mathrm{Rok}}(L^\Gamma, \lambda^\Gamma) &\leq h_\Gamma^{\mathrm{Rok}}(Y, \nu) + h_\Gamma^{\mathrm{Rok}}(L^\Gamma, \lambda^\Gamma \mid \mathcal{F}) \\
&< \epsilon + \mathrm{H}(\alpha \mid \mathcal{F}) \\
&\leq \epsilon + \lambda^\Gamma(M) \cdot \mathrm{H}_M(\alpha) \\
&= \epsilon + \frac{1}{|T|} \cdot \mathrm{H}_M(\beta) \\
&\leq \epsilon + \frac{1}{|T|} \cdot \log(k) \\
&< 2\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we conclude that $h_\Gamma^{\mathrm{Rok}}(L^\Gamma, \lambda^\Gamma) = 0$. □

Corollary XV.9. *Assume that every countably infinite group G admits a free ergodic p.m.p. action with $h_G^{\mathrm{Rok}}(X, \mu) > 0$. Then:*

(i) $h_G^{\mathrm{Rok}}(L^G, \lambda^G) = \mathrm{H}(L, \lambda)$ for all countably infinite groups G and all probability spaces (L, λ).

(ii) All Bernoulli shifts over countably infinite groups have completely positive outer Rokhlin entropy.

(iii) Gottschalk's surjunctivity conjecture and Kaplansky's direct finiteness conjecture are true.

Proof. It follows from Corollary XV.7 and Theorem XV.8 that $h_{sup}^{\mathrm{Rok}}(G) = \infty$ for all countably infinite groups G. By applying Theorems XV.4 and XV.6 we obtain (i). From Corollaries XII.6 and XII.7 we obtain (ii) and (iii). □

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