NOTE ON THE ZEROS OF $(dP^1_{m_i}(x)/dx)|_{x=x_0}$

By K. M. Siegel, J. W. Crispin, R. E. Kleinman and H. E. Hunter

In an earlier article the authors [1] presented a method for the determination of the positive elements of the set

$$\{n_i \mid P^1_{n_i}(x_0) = 0\}, \tag{1}$$

the zeros of the Associated Legendre Function of the first kind of order 1 and non-integral degree, and the evaluation of the associated integrals

$$\int_{x_0}^1 [P_{n_i}(x)]^2\, dx, \qquad (x = \cos\theta). \tag{2}$$

This note presents an extension of the process to the determination of the positive elements of the set

$$\{m_i \mid dP^1_{m_i}(x_0)/dx = 0\}, \tag{3}$$

and the evaluation of the integrals

$$\int_{x_0}^1 [P^1_{m_i}(x)]^2\, dx, \qquad (x = \cos\theta). \tag{4}$$

Prior to the discussion of the method employed for finding the m_i, let us first examine a variation on MacDonald's formula which will permit a quick approximation to the first m_i for a given θ_0. By differentiating, with respect to μ, equation (14) of Reference 2 one obtains

$$\begin{aligned} dP^{-1}_n(-\mu)/d\mu &= [\cos (n-1)\pi](dP^{-1}_n(\mu)/d\mu) \\ &\qquad - (2/\pi)[\sin (n-1)\pi](dQ^{-1}_n(\mu)/d\mu), \end{aligned} \tag{5}$$

where $-\mu = \cos\theta$ and the Q is the Associated Legendre Function of the second kind.

If we assume that $n = m_1$, using (3) and recursion formulas [p. 61, ref. 3], equation (5) reduces to

$$\begin{aligned} \tan (n-1+k)\pi \\ &= \pi[n(n+1)\tan^2 \tfrac{1}{2}(\pi-\theta_0)][n+2-(n+1)\cos(\pi-\theta_0)]/ \\ &\qquad [n-(n+1)\cos(\pi-\theta_0)], \quad (k = 0, -1, \cdots). \end{aligned} \tag{6}$$

For values of θ_0 less than and nearly equal to π, if, for a given n, k can be chosen so that $(n-1+k)\pi$ is small, then (6) reduces to the approximation formula.

$$n - 1 + k = -n(n+1)(\pi-\theta_0)^2/4. \tag{7}$$

The application of this formula to the case of $x_0 = \cos 165°$ results in the approximation $m_1 = .967$, which is determined by taking $k = 0$.

The method employed in finding the elements of the set (3) is very similar to the method discussed in reference 1 relative to the elements of the set (1). We assume that m_i is of the form $j + z_j$ or $j + \frac{1}{2} + z_j$ ($j = 0, 1, 2, \cdots$) and use the same Taylor expansions employed in reference 1. With the boundary condition stated in (3) rather than the one in (1) we obtain the following quadratic in z_j

$$
\begin{aligned}
&\{\tfrac{1}{2}y(y+1)[\partial^2 P_y(x)/\partial y^2] + (2y+1)[\partial P_y(x)/\partial y] - \tfrac{1}{2}x[\partial^3 P_y(x)/\partial x \partial y^2]\}z_j^2 \\
(8) \qquad &+ \{y(y+1)[\partial P_y(x)/\partial y] + (2y+1)[P_y(x)] - x(\partial^2 P_y(x)/\partial x \partial y]\}z_j \\
&\qquad + \{y(y+1)[P_y(x)] - x[\partial P_y(x)/\partial x]\} = 0
\end{aligned}
$$

where the coefficients are evaluated at $x = x_0$ and $y = j$ if the m_i is assumed to be of the form $j + z_j$ or $y = j + \frac{1}{2}$ if the m_i is assumed to be of the form $j + \frac{1}{2} + z_j$.

Thus equation (8) provides a method for the determination of the z_j and thus a method of obtaining two approximations of each m_i. This is completely analogous to the process employed in determining the elements of the set (1), [ref. 1], and it is worth noting that the work involved in finding the n_i's provides the computations required to evaluate the coefficients in equation (8). The final choice between the two approximations, is based, as in reference 1, upon the smaller value of $|z_j|$.

In reference 1 it was shown that certain bounds and checks could be readily established for the n_i's. Similar checks are available in this case and are obtained from the relation

$$
(9) \qquad (1 - x_0^2)^{3/2}[dP_n^1(x)/dx]_{x=x_0} = n[(n + 1 - nx_0^2)P_n(x_0) - x_0 P_{n-1}(x_0)],
$$

which can be obtained from the recursion formulas appearing on p. 62 of reference 3. Using the values of $P_n(x_0)$ and $P_{n-1}(x_0)$ determined for $n = \frac{1}{2}K$ (K = an integer) by the methods presented in reference 1 one is able to establish intervals of width $\frac{1}{2}$ which contain the m_i's.

To evaluate the integrals (4) we begin with

$$
\begin{aligned}
(10) \qquad \int_{x_0}^{1} [P_{m_i}^1(x)]^2\, dx = &\left\{ m_i(m_i + 1)(x_0^2 - 1)P_{m_i}(x_0) \right\} \cdot \\
&\left\{ \frac{P_y(x)}{x} + \frac{y(y+1)}{x(2y+1)} \cdot \frac{\partial P_y(x)}{\partial y} - \frac{1}{2y+1} \frac{\partial^2 P_y(x)}{\partial x \partial y} \right\}_{\substack{x=x_0 \\ y=m_i}}
\end{aligned}
$$

[p. 20, ref. 4]

where the m_i's are determined by the process discussed above and thus are of the form $j + z_j$ or $j + \frac{1}{2} + z_j$. For simplicity the integrals are evaluated by both methods in each case; however the final choice is based upon the same criterion used in connection with the m_i's themselves.

The determination of the computational forms required for the evaluation of these integrals is accomplished by continuing the procedure of approximation outlined above and in reference 1. The forms evolved for this purpose are

$$
\begin{aligned}
P_{m_i}(x_0) &= P_{N+z_j}(x_0) = \left[\sum_{s=0}^{M+1} \frac{1}{s!}\frac{\partial^s P_y(x)}{\partial y^s} z_j^s\right]_{\substack{x=x_0\\ y=N}} \\
(11)\qquad \left.\frac{\partial P_y(x)}{\partial y}\right|_{\substack{x=x_0\\ y=m_i}} &= \left[\sum_{s=0}^{M} \frac{1}{s!}\frac{\partial^{s+1} P_y(x)}{\partial y^{s+1}} z_j^s\right]_{\substack{x=x_0\\ y=N}} \\
\left.\frac{\partial^2 P_y(x)}{\partial x \partial y}\right|_{\substack{x=x_0\\ y=m_i}} &= \left[\sum_{s=0}^{M} \frac{1}{s!}\frac{\partial^{s+2} P_y(x)}{\partial x \partial y^{s+1}} z_j^s\right]_{\substack{x=x_0\\ y=N}}
\end{aligned}
$$

where $N = j$ in one case and $j + \frac{1}{2}$ in the other. The expressions appearing in the right hand members of (11) are all obtainable by means of the methods of reference 1. When the computed value of z_j is close to zero, $M = 1$ suffices in the sums appearing in (11). However, for values of z_j close to $\frac{1}{4}$ (which, in the final analysis, is the largest value one must consider for z_j) a greater number of terms in these sums should be used, i.e. $M \geqq 2$.

Upon applying this method to the case of $\theta_0 = 165°$ the results indicate that the two values of the zeros obtained bound the true value. For example, the values of m_4 computed by this technique were found to be 3.8900 and 3.8597. If bounds on m_4 are obtained using the hypergeometric series method discussed in reference 2, one finds that $3.86 < m_4 < 3\ 89$. A similar situation occurred in connection with the n_i's of reference 1

In the following table the results obtained for the $\theta_0 = 165°$ case are given

$x_0 = \cos 165°$

i	m_i		$\int_{x_0}^{1} [P^1_{m_i}(x)]^2\, dx$		
			$M = 1$		$M = 2$
	$j + z_j$ Method	$j + \frac{1}{2} + z_j$ Method	$j + z_j$ Method	$j + \frac{1}{2} + z_j$ Method	$j + \frac{1}{2} + z_j$ Method
1	9673*	—	1 3640*	—	—
2	1 9198*	1 8850	2 4827*	3.3509	—
3	2 8894*	2 8572	3 5179*	4 7478	—
4	3 8900*	3 8597	4.4494*	6 3861	—
5	4 9180*	4 8872	5.2909*	8 6650	—
6	5 9657*	5 9336	6 1148*	12 076	—
7	7.0264*	7 0900	7 0519*	12.754	—
8	8 0940*	8 1436	8 2651*	12 847	—
9	9 1638*	9.2013	9 9351*	12 670	—
10	10 233*	10 264	12 270	12 394	10 089*
11	11.302	11 333*	15 608	12 206	11 539*
12	12 373	12 410*	20 505	12 294	12.032*
13	13 450	13 492*	27 915	12 887	12 830*
14	14 626	14 576*	23 618	14 204	13 710*
15	15 694	15 658*	21 489	16.422	14 108*

* Indicates preferred value.

and some of the integrals computed for both $M = 1$ and $M = 2$ are presented for comparison purposes. In computing the integrals with $M = 1$, this process yields values which are larger than the true values; in fact the envelope of the resulting curves would more accurately define the true values. It has been found that, due to the periodic changes in the total error, in many cases it is only necessary to compute a few terms with increased accuracy ($M = 2$), while the $M = 1$ cases suffice for most engineering computations.

There has been no attempt to smooth out the entries which appear in the table since such an attempt would imply information relative to the values not presented by the method itself. The authors would again like to point out that this method is not as accurate as certain other known processes. However, it does yield results much more economically than any other known standard technique.

It also should be pointed out that large scale digital machines might be applied to obtaining tables with a large number of entries. In the usual electromagnetic theory scattering problem, one usually uses a summing technique which results in the determination of which terms are dominant and then one only resorts to computing a few of these Legendre Functions and their normalizing integrals. When complete tables come into existence, they undoubtedly will have been obtained by large scale digital machines. In the intervening period, an approximation technique such as this is necessary. It is clear that the method presented in this note can be applied to other functions as well as the Legendre Functions.

REFERENCES

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