THE ZEROS OF $P'_{n_i}(x_0)$ OF NON-INTEGRAL DEGREE

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1. Introduction. In January, 1951, tables were published [1] on the zeros of $P'_{n_i}(x)$, the Associated Legendre Function of order 1 and non-integral degree. In April, 1951, certain errors were observed [2] in those tables of Reference 1.

This note explains a method devised for use at the Willow Run Research Center which, though not as accurate as the method described in Reference 2, can still yield results suitable for engineering problems. Throughout we will consider the general problem for n_i real and $x_0 = \cos\theta_0$. We will continually use as a numerical example $x_0 = \cos 165°$.

We are interested in finding the values of n_i and $\int_{x_0}^{1} [P'_{n_i}(x)]^2\,dx$ where the n_i's are such that

$$P'_{n_i}(x_0) = 0 \tag{1.1}$$

In order to introduce as many bounds and checks of results as possible, the following asymptotic relations and observations were employed.

When the n_i are large, we obtain from the asymptotic formula [3]:

$$n_{i+1} - n_i = \pi/\theta_0$$

The two limiting cases for flow around a cone or scattering by a cone are $\theta_0 = \pi$ and $\theta_0 = \pi/2$, and for these two limiting cases the n_i differ by 1 and 2, respectively.

Thus for $\theta_0 = 165°$ the maximum value of the differences between successive n_i's is 1.0909. A few asymptotic difference values are shown in Table 1.

In the 90° and 180° cases we will now show that the asymptotic forms are the exact forms for all i.

Consider

$$P^m_{n_i}(0) = \frac{\sqrt{\pi}\,2^m}{\Gamma[\frac{1}{2}(n_i - m) + 1]\Gamma[\frac{1}{2} - \frac{1}{2}(n_i + m)]} \qquad \text{(Ref. 4, p. 63)}$$

Substituting $m = 1$, the case under discussion, one obtains

$$P'_{n_i}(0) = \frac{2\sqrt{\pi}}{\Gamma(\frac{1}{2}n_i + \frac{1}{2})\Gamma(-\frac{1}{2}n_i)}. \tag{1.2}$$

Since $P^m_{n_i}(x) = P^m_{-n_i-1}(x)$ (Ref. 4, p 62) in order not to introduce degenerate forms we will find the critical value of n_i and only consider n_i greater than this critical value.

Thus

$$n_i^c = -n_i^c - 1 \text{ i.e., } n_i^c = -\tfrac{1}{2}$$

Therefore, we will only consider those n_i's such that

$$n_i > -\tfrac{1}{2} \tag{1.3}$$

By (1.1) and (1.2) this would mean, when $\theta_0 = \pi/2$, that

$$\frac{1}{\Gamma(\frac{1}{2}n_i + \frac{1}{2})\Gamma(-\frac{1}{2}n_i)} = 0$$

Thus, either $\frac{1}{2}(n_i + 1)$ or $-\frac{1}{2}n_i$ must be poles of the gamma function $\Gamma(\frac{1}{2}n_i + \frac{1}{2})$ has no poles for $n_i > -\frac{1}{2}$, but $\Gamma(-\frac{1}{2}n_i)$ has poles for $n_i = 0, 2, 4, 6, \cdots$. Therefore

$$n_i = 2i \qquad i \geqq 0 \tag{1.4}$$

In the case of $\theta_0 = 180°$

$$P'_{n_i}(x) = -\tfrac{1}{2}n_i(n_i + 1)(1 - x^2)^{\frac{1}{2}}\,{}_2F_1[1 - n_i\,, n_i + 2; 2; \tfrac{1}{2}(1 - x)] \tag{1.5}$$

(Ref. 4, p. 63)

TABLE 1

θ_0	π/θ_0
90°	2.0000
165°	1.0909
170°	1.0588
175°	1 0286
180°	1 0000

We observe as $x \to -1$ that the right hand side of (1.5) becomes indeterminate; thus, to evaluate it at this point we must determine

$$\lim_{z=(\pi-\theta)\to+0} P'_{n_i}(\cos\theta) \tag{1.6}$$

$$= -\tfrac{1}{2}n_i(n_i + 1)z\,{}_2F_1(1 - n_i\,, n_i + 2, 2;\{1 - \tfrac{1}{4}z^2 + 0(z^4)\})$$

Since the ${}_2F_1$ diverges for $x = -1$ in (1.5), and since we can expand (1.6) in a Maclaurin series in z, and since z times it also diverges, then a priori, (1.6) diverges. In order that (1.6) be consistent with (1.1), ${}_2F_1[1 - n_i\,, n_i + 2; 2; \frac{1}{2}(1 - x)]$ must cut off. Hence, either $(1 - n_i)$ or $(n_i + 2)$ should be zero or a negative integer. By (1.3), $(n_i + 2)$ is always $> 3/2$; thus, $(1 - n_i)$ must equal zero or a negative integer, i.e.,

$$n_i = i \qquad i = 1, 2, 3, \cdots. \tag{1.7}$$

Thus, we observe that the asymptotic form for the differences (Table 1) is the exact form for all $(n_{i+1} - n_i)$ in the case of $\theta_0 = 90°$ and $180°$.

A consequence of MacDonald's theorem, "as θ increases from 0 to π, any zero diminishes" (Ref. 5), is that the differences between successive zeros, for the same value of θ, decrease with increasing θ from 0 to π.

For $\theta_0 = 165°$ the maximum difference between the successive zeros is 1.0909 and the minimum difference is 1; for $\theta_0 = 170°$ the maximum difference is 1.0588

2. The Method of Solution. Schelkunoff [6, p. 514] expanded the Legendre unction in a Taylor expansion

$$P_{n'+z_{n'}}(x) = P_{n'}(x) + 2z_{n'}[\tfrac{1}{2}P_{n'}(x) \log \tfrac{1}{2}(1 + x) + S'_{n'}] \tag{2.1}$$

where (with $[n']$ = largest integer in n')

$$S'_{n'} = \sum_{s=1}^{[n']} \frac{(-1)^s(n' + s)!}{(s!)^2(n' - s)!}\left\{\frac{1}{n' + s} + \cdots + \cdots + \frac{1}{n' + 1}\right\}\left(\frac{1 - x}{2}\right)^s \tag{2.1a}$$

Another form for the S'_n is

$$S'_{n'} = (-1)^{[n']} \sum_{s=0}^{[n']} \frac{(-1)^s(n + s)!}{(s!)^2(n - s)!}\left\{\frac{1}{s + 1} + \cdots + \frac{1}{s + n'}\right\}\left(\frac{1 + x}{2}\right)^s \tag{2.1b}$$

Differentiating (2.1) with respect to x we obtain

$$\frac{dP_{n'+z_{n'}}(x)}{dx} = \frac{dP_{n'}(x)}{dx} + z_{n'}\left\{\frac{dP_{n'}(x)}{dx} \log \frac{1 + x}{2} + \frac{P_{n'}(x)}{1 + x} + 2\frac{dS'_{n'}}{dx}\right\} \tag{2.2}$$

Setting $x = x_0$ and remembering that $P'_{n_i}(x_0) = 0$ we obtain

$$z_{n'} = \left[\frac{-\dfrac{dP_{n'}(x)}{dx}}{\dfrac{P_{n'}(x)}{1 + x} + \dfrac{dP_{n'}(x)}{dx} \log \dfrac{1 + x}{2} + 2\dfrac{dS'_{n'}}{dx}}\right]_{x=x_0} \tag{2.3}$$

where $n_i = n' + z_{n'}$

Equation (2.3) is a relationship between Legendre Functions and Associated Legendre Functions and a finite sum $dS'_{n'}/dx$. To make it numerically useful one must be able to evaluate the functions on the right without having to sum slowly convergent series.

If we consider n' either an integer or an odd multiple of $\frac{1}{2}$, we can evaluate the functions on the right of (2.3) quite easily. Some tables exist for the Legendre Polynomials and their derivatives (Ref. 7 and 8) When n' is considered an odd multiple of $\frac{1}{2}$, several methods of obtaining tables are available when one utilizes Hall's observation (Ref. 9) that the $\frac{1}{2}$ integer P's can be expressed in terms of the complete elliptic integrals, K and E, of modulus $\frac{1}{2}(1 - x_0)$.

$$\begin{aligned} P_{-\frac{1}{2}}(x) &= (2/\pi)K \\ P_{\frac{1}{2}}(x) &= (2/\pi)(2E - K) \end{aligned} \tag{2.4}$$

(Ref. 9, p. 927)

These functions have been tabulated (Ref. 10). There are many ways to obtain the other $\frac{1}{2}$ integer Legendre and Associated Legendre Functions. One could use the recursion relationship between the Legendre and Associated Legendre Functions. In Section 4 these functions for $x_0 = \cos 165°$ are evaluated, and the existing tables were used as checks.

To improve our results we added a third term to the Taylor expansion (2.1) and thus instead of (2.3) we solved a resulting quadratic equation in $z_{n'}$.

Thus, to obtain the zeros of

$$P'_{n_i}(x_0) = [-\sqrt{1 - x^2}(dP_{n_i}(x)/dx)]_{x=x_0} = 0 \tag{2.5}$$

we set $n_i = n' + z_{n'}$ and compute two columns (one for $n' = j + \frac{1}{2}$ and one for $n' = j; j = 1, 2, 3, \cdots$), as shown in Table 2.

Table 2 is obtained from a Taylor expansion in z which was terminated after three terms, thus committing an error proportional to z cubed. The obvious criteria then is to choose a_k if $| z_k'' | < | z_k' |$ or a_k' if $| z_k'' | > | z_k' |$.

A method of predicting which part of Table 2 should be used, even before the n_i's are computed, is available for most of the n_i's for any given angle. We shall proceed to illustrate this for $x_0 = \cos 165°$.

TABLE 2

Zeros (n_i)	A $j + \frac{1}{2} \pm Z_j''$	A' $j \pm Z_j'$
n_1	a_1	a_1'
n_2	a_2	a_2'
n_k	a_k	a_k'

where $z_j'' = z_{j+\frac{1}{2}}$ & $z_j' = z_j$

In Reference 2 it was seen that the integer parts of the zeros could be determined from the Associated Legendre Functions. The example given in that reference was

$$\tag{2.6} \begin{aligned} \frac{1}{\sin 165°} P_{13}' (\cos 165°) &= 6.50140 \\ \frac{1}{\sin 165°} P_{14}' (\cos 165°) &= -.86327 \\ \frac{1}{\sin 165°} P_{15}' (\cos 165°) &= -5.23851 \end{aligned}$$

and

$$\tag{2.7} 1.0321 > n_1 > 1.0316$$

From (2.6) we note that a change in sign exists between P_{13}' (cos 165°) and P_{14}' (cos 165°) but not between P_{14}' (cos 165°) and P_{15}' (cos 165°). Thus, $P_{n_i}' (x_0) \neq 0$ for $14 \leqq n_i \leqq 15$. Since by (2.7) $n_1 \approx 1.032$ and since the largest difference is 1.0909, then each n_i for $1 \leqq i \leqq 13$ has the number i to the left of the decimal point. For $14 \leqq i \leqq N$, where N is at least 23, the n_i has the number $(i + 1)$ to the left of the decimal point.

Since we have observed maximum and minimum differences for cos 165°

(1.0909 and 1) we can obtain more information from the fact that no change in sign occurs between P'_{14} (cos 165°) and P'_{15} (cos 165°); namely,

$$
\begin{aligned}
&16.1818 > n_{15} > 16, \ | z'_{15} | < .1818 \\
&15.0909 > n_{14} > 15, \ | z'_{14} | < .0909 \\
&14 > n_{13} > 13.9091, \ | z'_{13} | < .0909 \\
&13 > n_{12} > 12.8182, \ | z'_{12} | < .1818
\end{aligned} \tag{2.8}
$$

and also, from (2.7)

$$
\begin{aligned}
&1.0321 > n_1 > 1.0316 \\
&2.1230 > n_2 > 2.0316 \\
&3.2139 > n_3 > 3.0316
\end{aligned} \tag{2.9}
$$

By (2 9), for at least the first three zeros, column A′ of Table 2 is better than column A. By (2.8), for n_{12} , n_{13} , n_{14} , and n_{15} , again column A′ is better. The same sort of reasoning can be applied to the $P'_{i + \frac{1}{2}}(x)$ to tell when column A of Table 2 should be used.

Since there is no change in sign between $P'_{7.5}(x_0)$ and $P'_{8\ 5}(x_0)$ and no change in sign between $P'_{19\ 5}(x_0)$ and $P'_{20\ 5}(x_0)$, it can be concluded that

$$
\begin{aligned}
&9.6818 > n_9 > 9.5, \ 8.5909 > n_8 > 8.5, \\
&7.5 > n_7 > 7.4091, \ 6.5 > n_6 > 6.3182
\end{aligned} \tag{2 10}
$$

and

$$
\begin{aligned}
&21.6818 > n_{20} > 21.5, \ 20.5909 > n_{19} > 20.5, \\
&19.5 > n_{18} > 19.4091, \ 18.5 > n_{17} > 18.3182.
\end{aligned} \tag{2 11}
$$

and thus for the zeros indicated in (2 10) and (2.11), the results of column A in Table 2 should be used.

Then there will be very few cases where one should average between the tabulated data of columns A and A′ in Table 2 In Section 4 in our numerical example we shall follow this type of reasoning and select either the value in column A or that in column A′. It could be pointed out that instead of using a table with two columns and thus creating a scientific selection process, one could use either column A or A′ and add more terms to the Taylor expansion Not only would this add more work than the suggested method (and also create places of increased error) but also would remove redundancy from the system. Since these computations were carried out on hand calculating machines, whenever we have a choice between two methods of almost equal over-all accuracy we choose the method with the most redundancy, a valuable means of checking computations. In this method we always have two values of n_i although one of these values is discarded in the final result The two values serve as a check inasmuch as we would immediately be alerted if the two values were very different.

The integrals $\int_1^{x_0} [P_{n_i}(x)]^2 \, dx$, $\int_1^{x_0} [P'_{n_i}(x)]^2 \, dx$, and $\int_1^{x_0} [P'_{m_i}(x)]^2 \, dx$ have been of much interest in our electromagnetic scattering work (Ref. 11). In Reference 11, it was shown that

$$\int_1^{x_0} [P'_{n_i}(x)]^2\,dx = n_i(n_i + 1) \int_1^{x_0} [P_{n_i}(x)]^2\,dx \tag{2.12}$$

where the n_i are the zeros of $P'_{n_i}(x_0) = 0$. It was also shown that

$$\int_1^{x_0} [P_{n_i}(x)]^2\,dx = \frac{-(1 - x_0^2)}{2n_i + 1} P_{n_i}(x_0) \left[\frac{\partial^2 P_y(x)}{\partial x \partial y}\right]_{y=n_i,\ x=x_0} \tag{2.13}$$

In addition it was shown that when the m_i are zeros of $[dP'_{m_i}(x)/dx]_{x=x_0} = 0$ we obtain

$$\begin{aligned}\int_{x_0}^{1} [P'_{m_i}(x)]^2\,dx &= m_i(m_i + 1)(x_0^2 - 1)P_{m_i}(x_0) \\ &\cdot \left[\frac{P_{m_i}(x)}{x} + \frac{m_i(m_i + 1)}{x(2m_i + 1)} \frac{\partial P_y(x)}{\partial y} + \frac{1}{2m_i + 1} \frac{\partial^2 P_y(x)}{\partial x \partial y}\right]_{y=m_i,\ x=x_0}\end{aligned} \tag{2.14}$$

With respect to these integrals we can employ the methods described in this section to the right hand members of (2.13) and (2.14). In Section 3 the requisite algebra for (2.13) is shown and in Section 4 the results of its application to the case $x_0 = \cos 165°$ are given.

One might observe that the algebra required to put (2.14) in a similar computational form would be no more difficult, although it would be more lengthy.

3. The Computational Forms. In Section 2 the method of approach used in finding the a_k and a'_k of Table 2 was described, in this section we display the details of the computational method

For column A′ of Table 2 we start with

$$P_{j + z'_j}(x) = \left[P_y(x) + z'_j \frac{\partial P_y(x)}{\partial y} + \tfrac{1}{2} z'^2_j \frac{\partial^2 P_y(x)}{\partial y^2}\right]_{y=j} \tag{3.1a}$$

and for column A of the table we start with

$$P_{j + \frac{1}{2} + z''_j}(x) = \left[P_y(x) + z''_j \frac{\partial P_y(x)}{\partial y} + \tfrac{1}{2} z''^2_j \frac{\partial^2 P_y(x)}{\partial y^2}\right]_{y=j+\frac{1}{2}} \tag{3.1b}$$

Differentiating these equations with respect to x, evaluating at $x = x_0$, and under the stipulation that $n_i = j + z'_j$ in (3.1a) and $n_i = j + \frac{1}{2} + z''_j$ in (3.1b), we obtain the quadratic equations referred to in Section 2. Solving them for z_j we obtain, in the first case

$$\begin{aligned} z'_j = \Bigg[&\left\{\frac{\partial^3 P_y(x)}{\partial x \partial y^2}\right\}^{-1} \\ &\cdot \left\{-\frac{\partial^2 P_y(x)}{\partial x \partial y} \pm \sqrt{\left(\frac{\partial^2 P_y(x)}{\partial x \partial y}\right)^2 - 2\left(\frac{\partial P_y(x)}{\partial x}\right)\left(\frac{\partial^3 P_y(x)}{\partial x \partial y^2}\right)}\right\}\Bigg]_{y=j,\ x=x_0}\end{aligned} \tag{3.2a}$$

and in the second case

$$
\begin{aligned}
z_j'' &= \left[\left\{\frac{\partial^3 P_y(x)}{\partial x \partial y^2}\right\}^{-1}\right. \\
&\cdot \left.\left\{-\frac{\partial^2 P_y(x)}{\partial x \partial y} \pm \sqrt{\left(\frac{\partial^2 P_y(x)}{\partial x \partial y}\right)^2 - 2\left(\frac{\partial P_y(x)}{\partial x}\right)\left(\frac{\partial^3 P_y(x)}{\partial x \partial y^2}\right)}\right\}\right]_{y=j+\frac{1}{2},\ x=x_0}
\end{aligned} \tag{3.2b}
$$

The expressions that were computed in order to find the z_j associated with a given j or $j + \frac{1}{2}$ are those shown in (3.2a) and (3 2b). However, before we comment on them, let us turn our attention to the integral, referred to in Section 2,

$$
\int_{x_0}^{1} [P_{n_i}(x)]^2\, dx = \frac{1 - x_0^2}{2n_i + 1} P_{n_i}(x_0) \left[\frac{\partial^2 P_y(x)}{\partial x \partial y}\right]_{y=n_i,\ x=x_0} \tag{3.3}
$$

In order to find both the n_i's and the values of these integrals we observe that in addition to the forms required in the right hand members of (3.2a) and (3.2b) we will need $P_{n_i}(x_0)$ and $[\partial^2 P_y(x)/\partial x \partial y]_{y=n_i,\ x=x_0}$. After the z_j's have been obtained, the $P_{n_i}(x_0)$ are determined through (3.1a) and (3.1b). As for $[\partial^2 P_y(x)/\partial x \partial y]_{y=n_i,\ x=x_0}$, we consider

$$
\left.\frac{\partial P_y(x)}{\partial y}\right|_{y=n_j+z_j} = \left[\frac{\partial P_y(x)}{\partial y} + z_j \frac{\partial^2 P_y(x)}{\partial y^2} + \tfrac{1}{2} z_j^2 \frac{\partial^3 P_y(x)}{\partial y^3} + \cdots\right]_{y=n_j} \tag{3.4}
$$

and

$$
P_{n_j+z_j}(x) = \left[Py(x) + z_j \frac{\partial P_y(x)}{\partial y} + \tfrac{1}{2} z_j^2 \frac{\partial^2 P_y(x)}{\partial y^2} + \cdots\right]_{y=n_j} \tag{3 4a}
$$

where $n_j = j$ in the one case and $n_j = j + \frac{1}{2}$ in the other.

Combining these two equations, differentiating with respect to x, and then solving for $[\partial^2 P_y(x)/\partial x \partial y]_{y=n_j+z_j}$ we obtain, with the boundary condition $P'_{n_i}(x_0) = 0$ which is equivalent to $\partial P_{n_i}(x)/\partial x \mid_{x=x_0} = 0$,

$$
\left.\frac{\partial^2 P_y(x)}{\partial x \partial y}\right|_{y=n_j+z_j,\ x=x_0} \cong \left\{-\frac{1}{z}\frac{\partial P_{n_j}(x)}{\partial x} + \frac{z_j}{2}\frac{\partial^3 P_y(x)}{\partial x \partial y^2}\right\}_{y=n_j,\ x=x_0} \tag{3.5}
$$

Now examination of (3.1a), (3.1b), (3.2a), (3.2b) and (3 5) shows what quantities require computation. For both the $j + z_j'$ and $j + \frac{1}{2} + z_j''$ methods, we start with the form

$$
\begin{aligned}
P_y(x) &= \frac{\sin y\pi}{\pi} \sum_{s=0}^{\infty} \frac{(-1)^s (y+s)!}{(s!)^2 (y-s)!} \\
&\cdot \left\{\log\frac{1+x}{2} + \psi(y+s) + \psi(y-s) - 2\psi(s)\right\}\left(\frac{1+x}{2}\right)^s \\
&+ \cos y\pi \sum_{s=0}^{\infty} \frac{(-1)^s (y+s)!}{(s!)^2 (y-s)!}\left(\frac{1+x}{2}\right)^s \qquad \text{(Ref. 12, p. 54)}
\end{aligned} \tag{3.6}
$$

which is transformed by the use of the relation

$$
\psi(-z) = \psi(z-1) + \pi \cot \pi z \qquad \text{(Ref. 13, p. 19)} \tag{3.7}
$$

into

$$
\begin{aligned}
P_y(x) &= \frac{\sin y\pi}{\pi} \sum_{s=0}^{j} \frac{(-1)^s (y+s)!}{(s!)^2 (y-s)!} \\
&\quad \cdot \left\{ \log \frac{1+x}{2} + \psi(y+s) + \psi(y-s) - 2\psi(s) \right\} \left(\frac{1+x}{2} \right)^s \\
&\quad + \cos y\pi \sum_{s=0}^{j} \frac{(-1)^s (y+s)!}{(s!)^2 (y-s)!} \left(\frac{1+x}{2} \right)^s \\
&\quad + \frac{\sin y\pi}{\pi} \sum_{s'=0}^{\infty} \frac{(-1)^{s'+j+1}}{[(s'+j+1)!]^2} \frac{(y+s'+j+1)!}{(y-j-1-s')!} \\
&\quad \cdot \left\{ \log \frac{1+x}{2} + \psi(y+s'+j+1) + \psi(j+s'-y) - 2\psi(s'+j+1) \right\} \\
&\quad \cdot \left(\frac{1+x}{2} \right)^{s'+j+1}
\end{aligned} \tag{3.8}
$$

Starting with (3.8) each of the derivatives required is determined and approximated by taking a finite number of terms in the corresponding series. The number of terms one would use would depend upon the number of significant digits required. It was found that the first $(j + 1)$ terms of each series would would give sufficient accuracy for our purposes. For example in finding $[\partial P_y(x)/\partial y]_{y=j+\frac{1}{2},\, x=x_0}$, when $j = 1$ the error committed by restricting the sum to the first $(j + 1)$ terms is less than .009 while for $j = 3$ the error is less than 3×10^{-6}.

With forms determined for the derivatives, the equations (3.2a) and (3.2b) could then be employed to determine the values of z_j'' and z_j'. Then the values of the zeros are determined by

$$
n_i = z_j' + j \qquad \text{where } i = j \text{ for } 1 \leqq i \leqq 13, \quad i = j - 1 \text{ for } 14 \leqq i \leqq 23 \tag{3.9a}
$$

and

$$
n_i = j + \tfrac{1}{2} + z_j'' \tag{3.9b}
$$

To determine the values of the integrals both the derivatives and the Legendre and the Associated Legendre Functions of integer degree are required. These Legendre and Associated Legendre Functions of integer degree were expressed in polynomial form so that their values could be determined by substitution. With these determined, the integrals are placed in computational form (see (3.5) and (3.3))

The results that one obtains when this scheme is used are illustrated by the example which is discussed in the following section

4. Results for $x_0 = \cos 165°$. The values for Legendre and Associated Legendre Functions of integer and half-integer degree are tabulated in Table 3.

Table 4 contains the results of our computation.

TABLE 3

Values of Legendre Functions for $x_0 = \cos 165$

y	$P_y(x_0)$	$-P_y'(x_0)/\sqrt{1-x_0^2}$	y	$P_y(x_0)$	$-P_y'(x_0)/\sqrt{1-x_0^2}$
1	− 96592 58263	+1 00	− 5	2.1854 43649 8	
2	+.89951 90528	−2 8977 77479	5	− 88034 27550	
3	− 80416 39229	+5.4975 95264	1 5	.40531 3187	−10 946 2606
4	+ 68469 54388	−8 5269 24940	2 5	− 098198 308	11 586 5587
5	− 54712 58747	+11 659 85421	3 5	− 12690 551	−11.535 451
6	+ 39830 59910	−14 545 30956	4 5	29429 8791	10 571 315
7	−.24554 10452	+16 837 83210	5 5	−.41302 4224	−8 5924 6265
8	+ 09618 43272 4	−18 228 42524	6 5	48750 2435	5.6150 241
9	+.04276 78470 9	+18.472 96566	7 5	− 52104 2564	−1.7674 284
10	− 16505 59738	−17 415 83615	8.5	51721 7324	−2.7216 569
11	+ 26548 99921	+15.006 79021	9 5	− 48040 2367	7 5424 833
12	− 34021 56675	−11.309 56633	10 5	.41591 3951	−12 329 704
13	+ 38689 98145	+6 5013 98521	11 5	−.32992 1717	16.692 590
14	− 40482 44602	− 86327 13353	12 5	22922 4586	−20 247 8253
15	+ 39488 56155	−5.2385 10824	13 5	−.12094 3791	22 652 4294
16	− 35949 81090	+11 378 18274	14 5	012172 729	−23 634 2515
17	+ 30241 35834	−17 101 94842	15.5	090383 6368	23.017 611
18	− 22846 39061	+21 962 65816	16.5	−.18075 1621	−20 741 975
19	+.14324 66015	−25 555 11295	17 5	25398 9737	16 872 056
20	−.05277 21934 5	+27 549 27562	18 5	−.30642 7865	−11.598 345
			19.5	33582 976	5.2278 00
			20.5	− 34146 94	1 8349

TABLE 4

Results for the case $x_0 = \cos 165°$

$n_i = j + z_j'$	$\int_{x_0}^{1} [P_{n_i}(x)]^2 dx$	$n_i = j + \frac{1}{2} + z_j''$	$\int_{x_0}^{1} [P_{n_i}(x)]^2 dx$
1.03158 √	.62837 √	1.08929	.98926
2.08361 √	37588 √	2.13167	56654
3 14588 √	.27853 √	3.18431	.36562
4.21190 √	.23211 √	4 24367	.25318
5.27900	20834	5.31014 √	.18467 √
6.34761	.19719	6.38439 √	.14195 √
7.42048	19522	7.46564 √	.11637 √
8.60072	.17093	8 55028 √	.10286 √
9 67183	.13271	9.63336 √	.09742 √
10 74310	.10509	10.71202 √	.09686 √
11.81852 √	.08513 √	11.78681	09933
12 90018 √	.07148 √	12.86075	.10428
13.98719 √	.06345 √	13 93859	.11224
15.07534 √	.06034 √	15.12111	.09485
16 15973 √	.06085 √	16 19451	.07691
17.23867 ⊗	06368 ⊗	17 26897 ⊗	06338 ⊗
18 31399	06816	18 34815 √	.05344 √
19 38967	.07433	19 43351 √	04689 √
20 47144	.08297	20 52279 √	.04374 √

√ indicates preferred value (according to criterion of Section #2).
⊗ indicates need for averaging (according to criterion of Section #2).

In conclusion it should be pointed out that no effort has been made to smooth out results as the major virtue of this method is the convenience and speed of computation.

The series for the Legendre Functions involved are very slowly convergent and any effort made to "reinforce" unreliable entries in Table 4 would imply the knowledge of a correct entry (the calculation of which we wish to by-pass in obtaining answers to physical problems).

REFERENCES

1 P A Carrus and C G Treuensfels, "Tables of Roots and Incomplete Integrals of Associated Legendre Functions of Fractional Orders " Journal of Mathematics and Physics, Vol XXIX No 4, January 1951

2. K. M Siegel, D. M Brown, H E Hunter, H A Alperin, C. W. Quillen, "The Zeros of the Associated Legendre Functions $P_n^m(\mu')$ of Non-integral Degree," Willow Run Research Center, University of Michigan, April, 1951

3. E W Hobson, "The Theory of Spherical and Ellipsoidal Harmonics, Cambridge University Press, 1931

4 W. Magnus & F Oberhettinger, "Formulas and Theorems for the Special Functions of Mathematical Physics," Chelsea Publishing Company, New York, 1949.

5. H. M MacDonald, "Zeros of Spherical Harmonic $P_n^m(\mu)$ Considered as a Function of n," Proceedings of London Mathematical Society, No I, Vol. XXXI (1900).

6 S A Schelkunoff, "Theory of Antennas of Arbitrary Size and Shape," Proceedings of IRE, Vol 29, 1941

7 S D Hamren, "Scattering from Spheres," Institute of Engineering Research, University of California, June, 1950

8 "Tables of Functions of First and Second Partial Derivatives of Legendre Polynomials," Engineering Research Institute, University of Michigan, November, 1951

9 R N Hall, "The Application of Non-integral Legendre Functions to Potential Problems," Journal of Applied Physics, Vol 20, October 1949

10. Oberhettinger and Magnus, "Anwendung Der Elliptischen Functionen in Physik und Technik," Julius Springer, Berlin, 1949.

11. K. M. Siegel and H. Alperin, "Scattering by a Semi-infinite Cone," Willow Run Research Center, University of Michigan, UMM-87, to be published.

12. S. A. Schelkunoff, "Electromagnetic Waves," D Van Nostrand, Inc New York, 1943

13. Jahnke & Emde, "Tables of Functions with Formulae and Curves," Dover Publications, New York, 1945

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