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Scattering From Resonant Slots on a Semi-Infinite Cone[1]

M. A. Plonus[2] and R. F. Goodrich

University of Michigan, Radiation Laboratory, Department of Electrical Engineering, Ann Arbor, Mich. 48108, U.S.A.

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The scattered field, when a plane wave is incident along the axis of a slotted, perfectly conducting, semi-infinite cone, is obtained by a superposition of the scattered field from an unslotted cone and the radiation field from the slot on the cone. The slot is a finite, thin, circumferential slot backed by a cavity which is characterized by an admittance. The induced voltage is represented by a sinusoidal distribution. The amplitude of the induced voltage is then related to the incident field by expressing it in terms of the radiating and load admittances of the slot. Expressions for the radiation admittance and the scattered far fields are derived.

1. Introduction

It is often desired to know the scattering properties of slots which ordinarily serve as radiating apertures. In this paper we will consider a semi-infinite cone with a finite, thin slot in the circumferential direction, as shown in figure 1. The slot is located at a distance a from the apex, subtends an angle $2\phi_0$, has a width of 2Δ, and is backed by a cavity that is characterized by a load admittance Y_l. If the incident wave frequency is such that the slot has a resonant length, a sinusoidal distribution is a valid representation for the induced voltage. However, for slot length other than resonant, this distribution should also be a good approximation. The scattered field from the slotted cone will then be a superposition of the scattered field from an unslotted cone and the radiation field from the slot. The amplitude of the slot voltage is then related to the incident field by expressing it in terms of the radiation and load admittances of the slot.

2. The Scattered Field From a Semi-infinite Cone

Let us first consider an unslotted perfectly conducting semi-infinite cone, whose apex is at the origin of a Cartesian coordinate system (x, y, z), and whose axis is parallel to the z axis. A plane electromagnetic wave is assumed incident in the direction of the negative z axis, and, since there is no loss of generality in taking its electric vector to lie in the x direction, we choose

$$\underline{E}^i = \hat{i}_x e^{ikz} \quad \text{and} \quad \underline{H}^i = -\hat{i}_y Y e^{ikz} \tag{1}$$

where Y is the intrinsic admittance of free space and a time factor $e^{i\omega t}$ has been suppressed.

The total diffracted field can be given in terms of the Debye potentials (u, v) (Goryanov, 1961) as follows:

$$u = \frac{\pi \cos\phi}{ik^2 r} \sum_{n=1}^{\infty} \frac{(2\nu_n + 1)e^{i\frac{\pi\nu_n}{2}}}{\nu_n(\nu_n+1)\sin\pi\nu_n} \frac{P^1_{\nu_n}(-\cos\theta_0)}{\frac{\partial}{\partial\nu_n} P^1_{\nu_n}(\cos\theta_0)} P^1_{\nu_n}(\cos\theta)\psi_{\nu_n}(kr) \tag{2}$$

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[2] On leave from Department of Electrical Engineering, Northwestern University, Evanston, Ill. 60201, U.S.A.

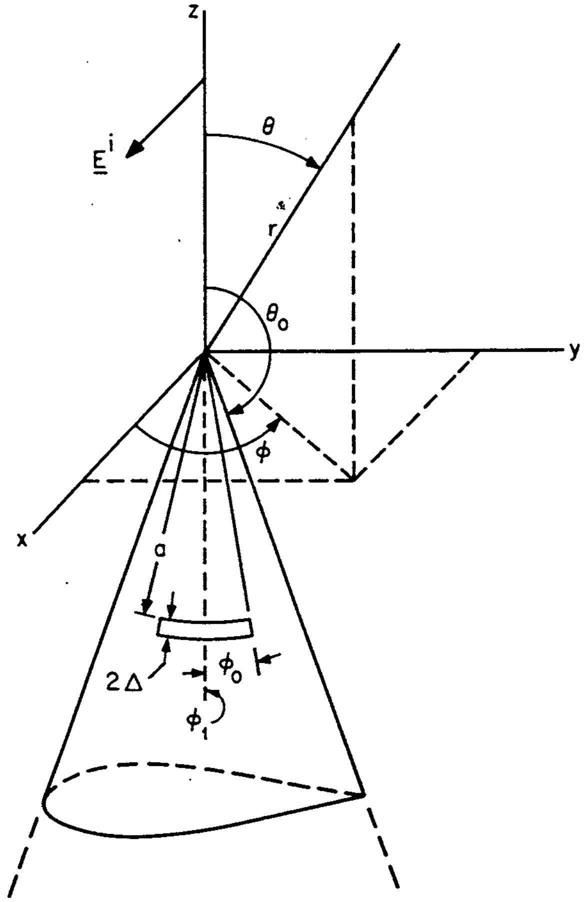

FIGURE 1. *Geometry of semi-infinite cone with a finite slot of angular width $2\phi_0$ which is centered at ϕ_1.*

$$v=\frac{\pi \sin \phi Y}{-ik^2 r} \sum_{n=1}^{\infty} \frac{(2\mu_n+1)e^{i\frac{\pi\nu_n}{2}}}{\mu_n(\mu_n+1)\sin \pi\mu_n} \frac{\frac{\partial}{\partial\theta_0} P^1_{\mu_n}(-\cos\theta_0)}{\frac{\partial^2}{\partial\mu_n\partial\theta_0} P^1_{\mu_n}(\cos\theta_0)} P^1_{\mu_n}(\cos\theta)\psi_{\mu_n}(kr) \tag{3}$$

where $P^1_\nu(\cos x)$ is the associated Legendre function, $\psi_\nu(x)$ is the Bessel function connected with the ordinary Bessel function according to

$$\psi_\nu(x)=\sqrt{\frac{\pi x}{2}}J_{\nu+1/2}(x)=xj_\nu(x) \tag{4}$$

and the numbers ν_n and μ_n appear as the eigenvalues of the boundary problem and are determined from the equations

$$P^1_{\nu n}(\cos\theta_0)=0, \qquad \frac{\partial}{\partial\theta_0} P^1_{\mu n}(\cos\theta_0)=0. \tag{5}$$

Senior and Wilcox (1967) have programmed the computation of the roots ν_n and μ_n of

$$P^m_{\nu n}(\cos\theta) \text{ and } \frac{\partial}{\partial\theta} P^m_{\mu n}(\cos\theta)\Big|_{\theta 0}$$

respectively, as well as the values

$$\frac{\partial}{\partial\nu}P_\nu^m(\cos\theta_0)\bigg|_{\nu_n} \text{ and } \frac{\partial}{\partial\mu}\left(\frac{\partial}{\partial\theta}P_\mu^m(\cos\theta)\right)\bigg|_{\mu_n}.$$

The induced current $\underline{J}$ on the cone surface is related to the total magnetic field by

$$\underline{J}=-\hat{i}_0\times\underline{H} \tag{6}$$

so that

$$J_r=-H_\phi, \qquad J_\phi=H_r. \tag{7}$$

The components of an electromagnetic field in a spherical system of coordinates are given by

$$E_r=\left(\frac{\partial^2}{\partial r^2}+k^2\right)(rv) \tag{8a}$$

$$E_\theta=\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(ru)-\frac{i\omega\mu}{\sin\theta}\frac{\partial v}{\partial\phi} \tag{8b}$$

$$E_\phi=\frac{1}{r\sin\theta}\frac{\partial^2}{\partial r\partial\phi}(ru)+i\omega\mu\frac{\partial v}{\partial\theta} \tag{8c}$$

$$H_r=\left(\frac{\partial^2}{\partial r^2}+k^2\right)(rv) \tag{9a}$$

$$H_\theta=\frac{i\omega\epsilon}{\sin\theta}\frac{\partial u}{\partial\phi}+\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(rv) \tag{9b}$$

$$H_\phi=-i\omega\epsilon\frac{\partial u}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial^2}{\partial r\partial\phi}(rv). \tag{9c}$$

The magnetic field on the cone surface can then be given as

$$H_r=Y\sin\phi T_1(r) \tag{10}$$

$$H_\phi=Y\cos\phi T_2(r) \tag{11}$$

where

$$T_1(r)=i\pi\sum_{n=1}^{\infty}\frac{(2\mu+1)e^{i\frac{\pi\mu}{2}}}{\mu(\mu+1)\sin\pi\mu}\frac{\frac{\partial}{\partial\theta_0}P_\mu^1(-\cos\theta_0)}{\frac{\partial^2}{\partial\mu\partial\theta_0}P_\mu^1(\cos\theta_0)}P_\mu^1(\cos\theta_0)[\psi_\mu''(kr)+\psi_\mu(kr)] \tag{12}$$

$$T_2(r)=-\frac{\pi}{kr}\sum_{n=1}^{\infty}\frac{(2\nu+1)e^{i\frac{\pi\nu}{2}}}{\nu(\nu+1)\sin\pi\nu}\frac{P_\nu^1(-\cos\theta_0)}{\frac{\partial}{\partial\nu}P_\nu^1(\cos\theta_0)}\frac{\partial}{\partial\theta_0}P_\nu^1(\cos\theta_0)\psi_\nu(kr)$$

$$+i\frac{\pi}{kr\sin\theta_0}\sum_{n=1}^{\infty}\frac{(2\mu+1)e^{i\frac{\pi\mu}{2}}}{\mu(\mu+1)\sin\pi\mu}\frac{\frac{\partial}{\partial\theta_0}P_\mu^1(-\cos\theta_0)}{\frac{\partial^2}{\partial\mu\partial\theta_0}P_\mu^1(\cos\theta_0)}P_\mu^1(\cos\theta_0)\psi_\mu'(kr) \tag{13}$$

and the primes denote differentiation with respect to the entire argument. From this point on we will delete the subscripts on the eigenvalues; therefore it should be understood that $\mu \equiv \mu_n$ and $\nu \equiv \nu_n$.

In (12) and (13) we split off the incident field (Goryanov, 1961) and obtain the scattered far field by replacing $\zeta(kr)$ and its derivative by the leading term of their asymptotic expression for large kr, where $\zeta_\nu(x) = \sqrt{\frac{\pi x}{2}}\, H^{(2)}_{\nu+1/2}(x)$. The scattered far field is then

$$E_\theta^s = i \cos\phi \frac{e^{-ikr}}{kr} S_1^s(\theta) \tag{14}$$

where

$$S_1^s(\theta) = -\sum_{n=1} \left\{ \frac{2\nu+1}{\nu(\nu+1)} \frac{e^{i\nu\pi} \frac{\partial}{\partial\theta} P_\nu^1(\cos\theta)}{\sin\theta_0 \frac{\partial P_\nu(\cos\theta_0)}{\partial\theta_0} \frac{\partial P_\nu(\cos\theta_0)}{\partial\nu}} + \frac{2\mu+1}{\mu(\mu+1)} \frac{e^{i\mu\pi} P_\mu^1(\cos\theta)}{\sin\theta \sin\theta_0 \frac{\partial^2}{\partial\theta_0 \partial\mu} P_\mu(\cos\theta_0)} \right\} \tag{15}$$

and

$$E_\phi^s = -i \sin\theta \frac{e^{-ikr}}{kr} S_2^s(\theta) \tag{16}$$

where

$$S_2^s(\theta) = -\sum_{n=1}^{\infty} \left\{ \frac{2\mu+1}{\mu(\mu+1)} \frac{e^{i\mu\pi} \frac{\partial}{\partial\theta} P_\mu^1(\cos\theta)}{\sin\theta_0 \frac{\partial^2}{\partial\theta_0\partial\mu} P_\mu(\cos\theta_0)} + \frac{2\nu+1}{\nu(\nu+1)} \frac{e^{i\nu\pi} P_\nu^1(\cos\theta)}{\sin\theta \sin\theta_0 \frac{\partial P_\nu(\cos\theta_0)}{\partial\theta_0} \frac{\partial P_\nu(\cos\theta_0)}{\partial\nu}} \right\}. \tag{17}$$

Since

$$\left.\frac{P_n^1(\cos\theta)}{\sin\theta}\right|_{\substack{\theta=0\\ \theta=\pi}} = \pm \left.\frac{\partial}{\partial\theta} P_n^1(\cos\theta)\right|_{\substack{\theta=0\\ \theta=\pi}} \tag{18}$$

we note that

$$S_1^s(0) = S_2^s(0) \tag{19}$$

implying that for backscattering, the field has the same linear polarization as the incident field. In all other directions, however, the field is elliptically polarized and the component cross sections are

$$\sigma_\theta = \frac{\lambda^2}{\pi} \left| S_1^s(\theta) \right|^2 \cos^2\phi \tag{20}$$

$$\sigma_\phi = \frac{\lambda^2}{\pi} \left| S_2^s(\theta) \right|^2 \sin^2\phi \tag{21}$$

with the complete scattering cross section given by

$$\sigma = \sigma_\theta + \sigma_\phi \ . \tag{22}$$

3. The Radiation Field of a Resonant Slot on a Cone

Let us consider a cavity-backed resonant slot centered at ϕ_1. The slot is narrow and in the circumferential direction, i.e., a thin finite circumferential slot defined by $|r'-a| \leq \Delta$, $|\phi'-\phi_1| \leq \phi_0$, where $2\phi_0$ is the angle the slot subtends and the primes denote coordinates on the surface of the cone. If the source frequency is so chosen that the slot is a half-wavelength long, a good representation for the induced slot voltage distribution is a sinusoidal one. Although it is expected that a sinusoidal distribution is still a good approximation for larger slots, and especially for smaller slots it should provide us with useful results. Hence we shall consider

$$E_r(r',\phi') = E_0 \cos\frac{\pi}{2}\,\frac{\phi'-\phi_1}{\phi_0} \text{ for } |\phi'-\phi_1| \leq \phi_0 \tag{23}$$

where the length of a resonant slot is $2\phi_0 a \sin\theta_0 = \lambda/2$. Since the slot width is much smaller than one wavelength, the slot voltage v across the gap can be considered a constant, i.e., $v = -2\Delta E_0$. The distribution at the slot can then be stated as

$$E_r = -\frac{v}{2\Delta}\cos\frac{\pi(\phi'-\phi_1)}{2\phi_0} \tag{24}$$

where v is the maximum slot voltage at the center of the slot. This voltage must be related to the radiation and loading admittance of the slot, which characterize the cavity backing the slot.

The potentials Π and Π^* which describe radiation from a cone when the aperture fields have r components only are given by (Bailin and Silver, 1956):

$$\Pi = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{-ik(2\nu+1)P_\nu^m(\cos\theta)}{\nu(\nu+1)(1+\delta_{0m})\pi\sin\theta_0\,\partial P_\nu^m(\cos\theta_0)/\partial\nu}$$

$$\cdot\int_{a-\Delta}^{a+\Delta}\int_{\phi_1-\phi_0}^{\phi_1+\phi_0} E_r(r',\phi')\cos m(\phi-\phi')j_\nu(kr_<)h_\nu^{(2)}(kr_>)r'\sin\theta_0\,dr'\,d\phi' \tag{25}$$

where $r_>$, $r_<$ symbolize the larger and smaller of the coordinates r, r', respectively, and

$$\Pi^* = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{Ym(2\mu+1)P_\mu^m(\cos\theta)}{\mu(\mu+1)\sin^2\theta_0\,\partial^2 P_\mu^m(\cos\theta_0)/\partial\theta_0\partial\mu\,\pi(1+\delta_{0m})}.$$

$$\cdot\int_{a-\Delta}^{a+\Delta}\int_{\phi_1-\phi_0}^{\phi_1+\phi_0} E_r(r',\phi')\Gamma_1(r,r')\sin m(\phi'-\phi)\,dr'\,d\phi' \tag{26}$$

where
$$\Gamma_1(r,r') = \begin{cases} j_\mu(kr)\zeta_\mu'(kr') & r<r' \\ j_\mu(kr')\zeta_\mu'(kr')h_\mu^{(2)}(kr)/h_\mu^{(2)}(kr') & r>r'. \end{cases}$$

The primes denote derivatives with respect to the entire argument and $\zeta(x) = xh_\mu^{(2)}(x)$. Since both TM and TE modes are excited by the slot distribution (24), we have for the components of the electromagnetic field

$$E_\theta = \frac{1}{r}\frac{\partial^2}{\partial r \partial\theta}(r\Pi) - \frac{i\omega\mu}{r\sin\theta}\frac{\partial}{\partial\phi}(r\Pi^*)$$

$$E_\phi = \frac{1}{r\sin\theta}\frac{\partial^2}{\partial r\partial\phi}(r\Pi) + \frac{i\omega\mu}{r}\frac{\partial}{\partial\theta}(r\Pi^*)$$

$$E_r = \left(\frac{\partial^2}{\partial r^2} + k^2\right) r\Pi$$

$$H_\theta = \frac{i\omega\epsilon}{r\sin\theta}\frac{\partial}{\partial\phi}(r\Pi) + \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(r\Pi^*)$$

$$H_\phi = -\frac{i\omega\epsilon}{r}\frac{\partial}{\partial\theta}(r\Pi) + \frac{1}{r\sin\theta}\frac{\partial^2}{\partial r\partial\phi}(r\Pi^*)$$

$$H_r = \left(\frac{\partial^2}{\partial r^2} + k^2\right) r\Pi^*. \tag{27}$$

For the slot distribution given by (24) the integrations give

$$\frac{\pi}{\phi_0}A = \int_{\phi_1-\phi_0}^{\phi_1+\phi_0} \cos\frac{\pi(\phi'-\phi_1)}{2\phi_0}\cos m(\phi-\phi')\,d\phi' = \frac{\pi}{\phi_0}\frac{\cos m(\phi-\phi_1)\cos\dfrac{m\pi}{2ka\sin\theta_0}}{(ka\sin\theta_0)^2 - m^2} \tag{28}$$

where $ka\sin\theta_0 = \pi/(2\phi_0)$, and

$$\Gamma_1(r,a) = \frac{1}{2\Delta}\int_{a-\Delta}^{a+\Delta}\Gamma_1(r,r')\,dr' = \begin{cases} h_\mu^{(2)}(kr)j_\mu(ka)\zeta_\mu'(ka)/h_\mu^{(2)}(ka) & r>a \\ j_\mu(kr)\zeta_\mu'(ka) & r<a \end{cases} \tag{29}$$

$$\frac{\pi}{\phi_0}B = \int_{\phi_1-\phi_0}^{\phi_1+\phi_0}\cos\frac{\pi(\phi'-\phi_1)}{2\phi_0}\sin m(\phi'-\phi)\,d\phi' = \frac{\pi}{\phi_0}\frac{\sin m(\phi_1-\phi)\cos m\phi_0}{(\pi/2\phi_0)^2 - m^2}. \tag{30}$$

The potentials can then be expressed as

$$\Pi = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{ika\nu(2\nu+1)P_\nu^m(\cos\theta)}{\phi_0\nu(\nu+1)(1+\delta_{0m})\partial P_\nu^m(\cos\theta_0)/\partial\nu}\, j_\nu(kr_<)h_\nu^{(2)}(kr_>)A \tag{31}$$

where $r_>$, $r_<$ symbolize the larger and smaller of the coordinates r, a respectively; $\delta_{0m}=0$ for $m\neq 0$, and $\delta_{0m}=1$ for $m=0$; and

$$\Pi^* = \sum_m^\infty\sum_n^\infty \frac{Yvm(2\mu+1)\,P_\mu^m(\cos\theta)\Gamma_1(r,a)B}{\mu(\mu+1)\sin^2\theta_0\partial^2 P_\mu^m(\cos\theta_0)/\partial\theta_0\partial\mu(1+\delta_{0m})}. \tag{32}$$

From (27) the magnetic field H_θ on the surface is given by

$$H_\phi = YvT_2^\zeta(r,\phi) \tag{33}$$

where $$T_2^r(r, \phi) = \sum_m^\infty \sum_n^\infty \frac{A}{\phi_0(1+\delta_{0m})} \left[\frac{k^2 a(2\nu+1)\partial P_\nu^m(\cos\theta_0)/\partial\theta_0}{\nu(\nu+1)\partial P_\nu^m(\cos\theta_0)/\partial\nu}\right.$$

$$\left. \cdot j_\nu(kr_<)h_\nu^{(2)}(kr_>) + \frac{m^2(2\mu+1)P_\mu^m(\cos\theta_0)\Gamma'(r, a)}{r\sin^3\theta_0\mu(\mu+1)\partial^2 P_\mu^m(\cos\theta_0)/\partial\theta_0\partial\mu}\right] \tag{34}$$

where $$\Gamma'(r, a) = \frac{\partial}{\partial r} r\Gamma_1(r, a) = \begin{cases} \zeta_\mu'(kr)j_\mu(ka)\zeta_\mu'(ka)/h_\mu^{(2)}(ka) & r > a \\ \psi_\mu'(kr)\zeta_\mu'(ka) & r < a \end{cases}.$$

In the far field where $r \gg a$ we have

$$h_\nu^{(2)}(kr) = \frac{e^{-ikr}}{kr} e^{i(\nu+1)\pi/2}.$$

With this substitution the potentials can be expressed as

$$\Pi = -v\frac{e^{-ikr}}{kr} \sum_m^\infty \sum_n^\infty \frac{(2\nu+1)e^{i\nu\pi/2}P_\nu^m(\cos\theta)}{\nu(\nu+1)\phi_0(1+\delta_{0m})\partial P_\nu^m(\cos\theta_0)/\partial\nu} \psi_\nu(ka)A \tag{35}$$

and $$\Pi^* = ivY\frac{e^{-ikr}}{kr} \sum_m^\infty \sum_n^\infty \frac{Bm(2\mu+1)P_\mu^m(\cos\theta)e^{i\mu\pi/2}j_\mu(ka)\zeta_\mu'(ka)}{\phi_0(1+\delta_{0m})\sin^2\theta_0\mu(\mu+1)\partial^2 P_\mu^m(\cos\theta_0)/\partial\theta_0\partial\mu h_\mu^{(2)}(ka)}. \tag{36}$$

The components of the electric field can now be obtained. The θ-component is

$$E_\theta^r = \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(r\Pi) - \frac{ik}{Y\sin\theta}\frac{\partial}{\partial\phi}\Pi^* = ikv\frac{e^{-ikr}}{kr} S_1^r(\theta, \phi) \tag{37}$$

where $$S_1^r(\theta, \phi) = \frac{1}{\phi_0}\sum_m^\infty \sum_n^\infty \frac{A}{(1+\delta_{0m})}\left[\frac{(2\nu+1)e^{i\nu\pi/2}\partial P_\nu^m(\cos\theta)/\partial\theta}{\nu(\nu+1)\partial P_\nu^m(\cos\theta_0)/\partial\nu}\psi_\nu(ka)\right.$$

$$\left. - \frac{im^2(2\mu+1)P_\mu^m(\cos\theta)e^{i\mu\pi/2}j_\mu(ka)\zeta_\mu'(ka)}{\sin\theta\sin^2\theta_0\mu(\mu+1)\partial^2 P_\mu^m(\cos\theta_0)/\partial\theta_0\partial\mu h_\mu^{(2)}(ka)}\right]. \tag{38}$$

The θ-component is $$E_\phi^r = \frac{1}{r\sin\theta}\frac{\partial^2}{\partial r\partial\phi}(r\Pi) + \frac{ik}{Y}\frac{\partial}{\partial\phi}\Pi^* = -ikv\frac{e^{-ikr}}{kr}S_2^r(\theta, \phi) \tag{39}$$

where $$S_2^r(\theta, \phi) = \frac{1}{\phi_0}\sum_m^\infty \sum_n^\infty \frac{mB}{(1+\delta_{0m})}\left[\frac{(2\nu+1)e^{i\nu\pi/2}P_\nu^m(\cos\theta)}{\sin\theta\nu(\nu+1)\partial P_\nu^m(\cos\theta_0)/\partial\nu}\psi_\nu(ka)\right.$$

$$\left. - \frac{i(2\mu+1)\partial P_\mu^m(\cos\theta)/\partial\theta e^{i\mu\pi/2}j_\mu(ka)\zeta_\mu'(ka)}{\sin^2\theta_0\mu(\mu+1)\partial^2 P_\mu^m(\cos\theta_0)/\partial\theta_0\partial\mu h_\mu^{(2)}(ka)}\right]. \tag{40}$$

4. The Radiation and Loading Admittances

To calculate the radiation and loading admittance, we will first define admittance as twice the ratio of the complex power flow across the aperture to the square of the maximum voltage. The complex power radiated is

$$W_r = -\frac{1}{2}\int_{a-\Delta}^{a+\Delta}\int_{\phi_1-\phi_0}^{\phi_1+\phi_0} \underline{E}^s \times \underline{\tilde{H}}^r \cdot \hat{i}_\theta a \sin\theta_0 d\phi' dr' \tag{41}$$

where $\sim$ denotes the complex conjugate. Substituting for the aperture field from (24)

$$W_r = \frac{a \sin\theta_0 v}{4\Delta}\int_{a-\Delta}^{a+\Delta}\int_{\phi_1-\phi_0}^{\phi_1+\phi_0} \cos\left[\frac{\pi}{2\phi_0}(\phi'-\phi_1)\right]\tilde{H}_\phi^r d\phi' dr'. \tag{42}$$

The radiation admittance is then

$$Y_r = 2\tilde{W}/v^2 = \frac{Ya \sin\theta_0}{2\Delta}\int\int \cos\left[\frac{\pi}{2\phi_0}(\phi'-\phi_1)\right] T_2^r(r', \phi') d\phi' dr' \tag{43}$$

where (33) was substituted for H_ϕ^r. To evaluate the admittance we will need the following integrations (see appendix):

$$\int_{\phi_1-\phi_0}^{\phi_1+\phi_0} \cos m(\phi'-\phi_1)\cos\left[\frac{\pi}{2\phi_0}(\phi'-\phi_1)\right] d\phi' = \frac{\pi}{\phi_0}\frac{\cos m\phi_0}{(\pi/2\phi_0)^2 - m^2} \tag{44}$$

$$\int_{a-\Delta}^{a+\Delta} \Gamma'(r', a) dr' = \Delta\zeta_\mu'(ka)[\psi_\mu'(ka) + \zeta_\mu'|ka) j_\mu(ka)/h_\mu^{(2)}(ka)] + 0(\Delta^2) \approx \Delta\gamma_\mu. \tag{45}$$

The radiation admittance becomes then

$$Y_r = Y\sum_m^\infty\sum_n^\infty \frac{\pi \sin\theta_0}{(1+\delta_{0m})\phi_0^2}\left(\frac{\cos m\phi_0}{(\pi/2\phi_0)^2 - m^2}\right)^2 \left[\frac{k^2a^2(2\nu+1)\partial P_\nu^m(\cos\theta_0)/\partial\theta_0}{\nu(\nu+1)\partial P_\nu^m(\cos\theta_0)/\partial\nu} j_\nu(ka) h_\nu^{(2)}(ka) + \frac{m^2(2\mu+1)P_\mu^m(\cos\theta_0)\gamma_\mu}{2\sin^3\theta_0 \mu(\mu+1)\partial^2 P_\mu^m(\cos\theta_0)/\partial\theta_0\partial\mu}\right]. \tag{46}$$

To determine the loading admittance, the power that enters the slot must be known. It is here that v will be related to the incoming wave, for it is the incident field that induces the slot voltage. Since the component of the current that excites the slot is given by

$$J_r(r, \phi) = \cos\phi J_r(r) \tag{47}$$

a slot placed at $\phi \approx \pi/2$ will have little voltage induced and hence will scatter little. The power entering the slot is

$$W = \int_{a-\Delta}^{a+\Delta}\int_{\phi_1-\phi_0}^{\phi_1+\phi_0} \frac{1}{2}(\underline{E} \times \underline{\tilde{H}}) \cdot \hat{i}_\theta a \sin\theta_0 d\phi' dr'. \tag{48}$$

Substituting the slot field (24), we have

$$W = -\frac{va \sin\theta_0}{4\Delta}\int\int \cos\left[\frac{\pi}{2\phi_0}(\phi'-\phi_1)\right]\tilde{H}_\phi d\phi' dr'. \tag{49}$$

The load admittance is then

$$Y_\ell = -\frac{a \sin \theta_0}{2\Delta v} \int\int \cos\left[\frac{\pi}{2\phi_0}(\phi' - \phi_1)\right] (H_\phi^r + H_\phi^i + H_\phi^s)\, d\phi' dr'$$

$$= -Y_r - \frac{a \sin \theta_0}{2\Delta v} \int\int \cos\left[\frac{\pi}{2\phi_0}(\phi' - \phi_1)\right] (H_\phi^i + H_\phi^s)\, d\phi' dr'. \tag{50}$$

The total magnetic field of the unslotted cone is given by (11). The load admittance is then

$$Y_\ell = -Y_r - \frac{Ya \sin \theta_0}{2\Delta v} \int\int \cos\left[\frac{\pi}{2\phi_0}(\phi' - \phi_1)\right] \cos \phi' T_2(r') d\phi' dr'$$

$$= -Y_r - \frac{Y\pi a \sin \theta_0 \cos \phi_1 \cos \phi_0 T_2(a)}{v\phi_0[(\pi/2\phi_0)^2 - 1]}. \tag{51}$$

This permits us now to solve for v.

Thus the induced slot voltage is

$$v = -\frac{Y\pi a \sin \theta_0 \cos \phi_1 \cos \phi_0 T_2(a)}{(Y_r + Y_\ell)\phi_0[(\pi/2\phi_0)^2 - 1]}. \tag{52}$$

This expression when used in (24) now gives the entire electric field distribution in the slot. It is seen that in this approximation, a slot placed at $\phi_1 = \pi/2$ does not scatter since the incident field does not excite the assumed symmetric mode (24) in the slot. In reality, some weak scattering will be produced by a slot at $\phi_1 = \pi/2$, since an odd mode of small amplitude will be excited. Although the component of the surface current (11) that excites the slot is equal to zero at $\phi = \pi/2$, it increases to a small but finite value either side of $\phi = \pi/2$. For a slot of finite length, an odd mode

$$E_\theta = \frac{v_1}{\delta a} \sin \pi(\phi' - \phi_0)$$

would then be induced. The radiation from such a slot distribution can be readily obtained by following the procedure of the previous sections. If this were done, one would find that $v_1 \ll v$, that v_1 is largest for $\phi_1 = \pi/2$, and that it vanishes as ϕ_1 approaches zero. Therefore, it is the even mode considered here which gives rise to the dominant scatter. The expression (52) for v also shows that the induced amplitude v is a maximum when the slot is placed at $\phi_1 = 0$. From symmetry considerations one can also deduce that for slot positions near $\phi_1 = 1$ the even mode becomes an accurate representation for induced voltage, with the error increasing as $\phi_1 = \pi/2$ is approached.

One should also note that v is indeterminate for $\phi_0 = \pi/2$, which corresponds to slots with a length of a half circumference. However, the limit as $\phi_0 \to \pi/2$ does exist.

5. The Total Scattered Field

The total scattered far field can be written as the superposition of the scattered fields from the unslotted cone and the radiated field from the slotted one. Using (14) and (37) we obtain for the θ component

$$E_\theta = i\frac{e^{-ikr}}{kr} [\cos \phi S_1^s(\theta) + kv S_1^r(\theta, \phi)] \tag{53}$$

and similarly with (16) and (39) we obtain the ϕ-component

$$E_\phi = -i\frac{e^{-ikr}}{kr}[\sin\phi S_2^s(\theta) + kvS_2^r(\theta,\phi)]. \tag{54}$$

The two preceeding equations represent the solution to the scattered field of a slotted, semi-infinite cone for a plane-wave incident along the axis of the cone. They can be used, for example, to solve for the loading admittance when scattering in certain directions is to be minimized. If zero scattering is desired in the (θ, ϕ) direction, we can set E_θ and E_ϕ equal to zero and obtain two solutions for Y_ℓ, which are

$$Y_\ell = -Y_r + \frac{Y\pi ka \sin\theta_0 \cos\phi_1 \cos\phi_0 T_2(a)}{\phi_0[(\pi/2\phi_0)^2 - 1]\cos\phi}\,\frac{S_1^r(\theta,\phi)}{S_1^s(\theta)} \tag{55}$$

$$Y_\ell = -Y_r + \frac{Y\pi ka \sin\theta_0 \cos\phi_1 \cos\phi_0 T_2(a)}{\phi_0[(\pi/2\phi_0)^2 - 1]\sin\phi}\,\frac{S_2^r(\theta,\phi)}{S_2^s(\theta)}. \tag{56}$$

These two statements will give the admittance for Y_ℓ if

$$\frac{S_1^r(\theta,\phi)}{\cos\phi S_1^s(\theta)} = \frac{S_2^r(\theta,\phi)}{\sin\phi S_2^s(\theta)}. \tag{57}$$

If (57) can be satisfied, zero scattering is obtained in the (θ, ϕ) direction.

6. Concluding Remarks

For a discussion of loading by reactive, semiactive, or active slots, the reader should consult Liepa and Senior (1966). The impedance Y_ℓ which characterizes the cavity backing the slot can usually be obtained from waveguide theory once its physical dimensions are known. For example, if the backing is a piece of waveguide terminated in its characteristic impedance, the cone surface currents would excite the TE_{10} mode in the guide. The loading admittance, using the definition of admittance adapted in this paper would then be (Ramo, Whinnery, and Van Duzer, 1965)

$$Y_\ell^{TE} = \frac{\phi_0 a \sin\theta_0}{\Delta}\left[1 - \left(\frac{\pi}{2\phi_0 ka \sin\theta_0}\right)^2\right]^{1/2}. \tag{58}$$

Note that this admittance is purely reactive for slots slightly shorter than $\lambda/2$, and resistive for $2\phi_0 a \sin\theta_0 > \lambda/2$. For $2\phi_0 a \sin\theta_0 = \lambda/2$ it is zero; however, including losses and other effects it is small but not zero. If the backing cavity is a piece of shorted waveguide of length d, the load admittance would be

$$Y_\ell = j\cot\beta d Y_\ell^{TE}. \tag{59}$$

Most likely, the slot on the cone surface will be backed by a waveguide with a cross section larger than that of the slot. The slot could also coincide with a narrow radiating slot in the broad face of a backing ractangular waveguide, mounted along the inside surface of the cone. In these cases the determination of Y_ℓ is more complicated, but an abundance of published material exists which can be readily applied to this case, as for example, Oliner (1957).

We have tacitly assumed that the slot backing, say a waveguide, can be described in terms of a load admittance which is a "lumped" parameter which carries the essential information about the interior fields vis-a-vis the scattering problem. To describe the interior in any greater detail would add great complication and in fact would be a restriction to a single problem depending upon the geometry and electrical properties of the interior. Such an approach has been carried out by various authors (Liepa and Senior, 1966) leading to the justification of our lumped approach.

On the basis of experiments with slots on a finite cone-sphere (Goodrich et al., 1967), the semi-infinite cone results can be applied to the finite cone provided the slot is directly illuminated by the incident field, the slot lies more than two wavelengths forward of the cone termination, and provided the scattered field is observed directly or if not directly then in the forward hemisphere of the cone.

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7. Appendix

For computational purposes it might be convenient to retain the slot width dependence Δ explicitly in the expression for the radiation admittance Y_r given by (46). Performing the integration of (45), we obtain

$$\int_{a-\Delta}^{a+\Delta} \Gamma'(r', a)dr' = \frac{\zeta_\mu'(ka)}{k}\left\{\psi_\mu(ka) - \psi_\mu[k(a-\Delta)] + \frac{\psi_\mu(ka)}{\zeta_\mu(ka)}\left[\zeta_\mu[k(a+\Delta)] - \zeta_\mu(ka)\right]\right\}. \quad \text{(A.1)}$$

If we were to expand $\psi[k(a-\Delta)]$ and $\zeta[k(a+\Delta)]$ in a Taylor series and substitute the first two terms in the above right-hand side, the result would be the right-hand side of (45), namely $\Delta\gamma_\mu$. The Δ dependence can now be shown explicitly in Y_r' by substituting for the first factor γ_μ in (46) the expression

$$\gamma_\mu = \frac{\zeta_\mu'(ka)}{k\Delta}\{.\ .\ .\} \quad \text{(A.2)}$$

where the braces are those of (A.1).

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