CHAPTER 10

CENTRAL PLACE FRACTALS: THEORETICAL GEOGRAPHY IN AN URBAN SETTING

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ABSTRACT

The geometry of close packings has been used to model the structure of a hierarchy of trade areas resulting from interurban competition in complex, abstract systems of cities. When fractal geometry is used instead of the traditional approach, unsolved problems are resolved, new results are exposed, and new research directions are suggested by this deep connection between geometry and geography.

INTRODUCTION

The problem of how to partition space "evenly" in the Euclidean plane is solved abstractly by considering geometric packings of the plane (Coxeter, 1961); thus, one close packing of the plane sees circles, centered on points of an equilateral triangular lattice, expand outward until all interstitial space is filled with colliding circles compressing into regular hexagons. The efficiently arranged hive of the bee exhibits this phenomenon (von Frisch, 1950), as do soap bubbles blown between parallel plates of glass (Boys, 1962; Courant and Robbins, 1958).

The question of how human settlements might share space, even as a purely theoretical question embedded in an abstract geographical space that might resemble the Euclidean plane, has numerous complexities. There are economic issues facing commercial establishments concerning volume of sales and cost of moving goods to market areas, as well as concerns from consumers involving competitive pricing and distance to market. There are topographic issues, which might involve the ease of travel to market. There are political issues, which might involve gerrymandering of, or within, political units. There are social issues, which might involve religious or food preference or any of a number of elements based on cultural tradition. In short, the various components already present in a society, coupled with different perceived needs, might influence decisions as to how settlements share space and how trade areas might take shape around such settlements. These and related issues are topics on which much geographic research has been based. The reader is referred to references by Parr and Jones (1983) and Mulligan (1984) as entries to this literature and to works by Hanjoul et al. (1988) and Griffith (1989), among others, for mathematical interpretations.

CLASSICAL CENTRAL PLACE THEORY: AN ABSTRACT OVERVIEW

The geometry of the classical central place theory, originated by Walter Christaller in the 1930s and corroborated, from a perspective focusing on local rather than on global economic needs, by August Lösch in the 1940s, presents a theoretical model for examining the geographical issue of how cities and towns might share space (Christaller, 1933, 1966; Lösch, 1940, 1954). Michael Dacey's mid-1960s explanation of this geometry provided a clear and compact explanation of its theoretical foundations (Dacey, 1965).

The focus of this paper is on the geometry of classical central place theory and its linkage with self-similar, geometric fractals (Arlinghaus, 1985; Arlinghaus and Arlinghaus, 1989). Because central place theory rests on the notion of hierarchy and because self-similarity does as well, this is a logical place to look for alignment of theoretical concepts from perhaps otherwise disparate disciplines. An effort is made to show how this alignment happens by concentrating on basic issues of central place theory and fractal geometry, thus referring the reader interested in greater mechanical detail to other sources.

GEOMETRY OF THE CLASSICAL CENTRAL PLACE THEORY

The classical central place theory solution to creating a hierarchy of settlements requiring trade areas of different sizes has been to use the close-packing idea (Christaller, 1966; Lösch, 1954). Loosely speaking, large settlements provide a wider variety of goods and services and therefore have a broader support and service area (that is, trade area) than do smaller places. Thus, a net composed of relatively large close-packed hexagonal trade areas, with each hexagon centered on a large settlement as a node, might represent the abstract pattern of how large settlements partition space. Smaller settlements have smaller trade areas, and therefore an associated hexagonal net of smaller cell size, uniformly distributed throughout the net. Any number of nets, each of uniform cell size determined by city or town size (or its rank on some list), may be formed.

To see how the entire configuration of cities and towns of varying sizes might share space, imagine a stack of superimposed layers of close-packed hexagonal nets with cells of fixed diameter within a layer, but with diameter varying from layer to layer (Figure 10.1). Evidently, there are an infinite number of orientations of one layer relative to another within this stack. Because the large settlements also offer the goods and services of the small settlements, a hierarchy, based on nesting of services, becomes evident under orientations that align boundaries. Historically, different styles of orientation have been interpreted in different ways. The examples below display some of the detail of the three most commonly considered cases.

In an abstract geographical space in which movement is equally easy in all directions from every point (an isotropic plain), consider a stack of three layers of hexagonal nets (Figure 10.1). The first layer (layer 1) is composed of small towns, each competing with its nearest neighbors for the space between towns as trade areas. The distance between each pair of rival towns in layer 1 is one unit. Rival towns split evenly the space between them, creating a net of small unit hexagons (dotted lines in Figure 10.1) centered on the rival towns (layer 1 towns will not necessarily be evident in all figures to follow; on occasion, boundaries will obscure them).

The next layer (layer 2) is composed of larger rivals (second largest dots in Figure 10.1) that are next nearest neighbors at a distance of $\sqrt{3}$, when measured relative to the layer 1 rivals (Figure 10.1). In an identical manner, a third layer ("layer 3") of even larger hexagonal trade areas may be created by centering hexagons on the next nearest rivals (largest dots in Figure 10.1). The spacing between rivals in layer 3 is also $\sqrt{3}$ (when that

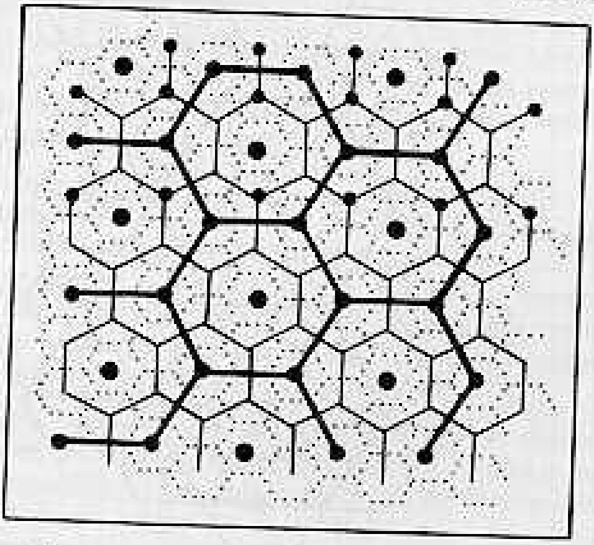

Figure 10.1 A classical central place hierarchy; the orientation is such that each trade area contains the equivalent of three of the next smaller sized trade areas ($K = 3$). Solid lines bound the trade areas of large competing central places (largest dots); dashed lines bound the trade areas of mid-sized competing central places (medium-sized dots, evident near the top of this figure and inferred elsewhere); dotted lines bound the trade areas of smallest competing central places (not shown, but inferred).

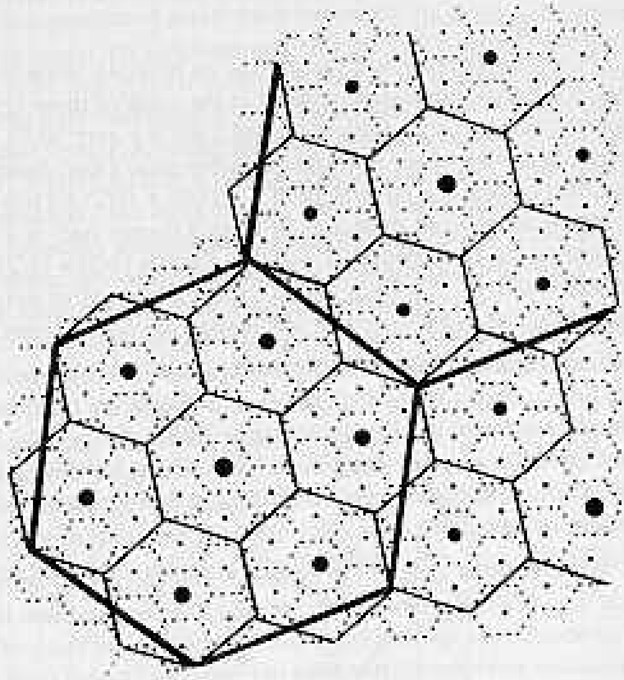

Figure 10.3 A classical central place hierarchy; the orientation is such that each trade area contains the equivalent of seven of the next smaller sized trade areas ($K = 7$). Solid, dashed, and dotted lines bounding trade areas are as described in the caption for Figure 10.1.

FRACTAL GENERATION OF THREE CENTRAL PLACE HIERARCHIES

The following transformation shows how to generate entire central place hierarchies from fractal geometry; a single generator, applied to a hexagonal initiator (as shown below) will produce the hierarchies noted in the section above. Different generators produce different hierarchies; however, the transformation remains the same, independent of the shape of the generator or the size of the initiator. The fractal dimension (Mandelbrot, 1983), determined from geometric entities invariant under this transformation, will serve as a single value to characterize each hierarchy (Arlinghaus, 1985).

Fractal Generation of the K = 3 Hierarchy

Consider scaling a two-sided wedge with vertex angle of 120° to span the six sides of a regular hexagon (Figure 10.4). This wedge is the generator, and the hexagon (Figure 10.4a) is the initiator. Apply the generator alternately to the inside and the outside of the initiator. Remove the edges of the initiator, leaving a shape more highly crenulated than the initiator: this is called the first teragon (Figure 10.4b). The highlighting surrounding the first teragon is designed to make the first teragon stand out, and to suggest the inside-outside manner of applying the generator to the initiator. The outer edges of the darkened areas are regions of the previous stage "underfit" by the transformation, thereby suggesting the outline of the shape in the previous frame. When the lines suggested by the symmetry of the first teragon are also included (as dashed lines in the middle frame, Figure 10.4b), this figure comprises three hexagons: three scaled down images of the preceding stage. Next, scale the same wedge-shaped generator to fit the side of the first teragon. Apply it, alternately, inside and outside, and remove the edges of the first teragon. The result is the second teragon, shown in the last frame (Figure 10.4c). The highlighting suggests the boundary of the first teragon, and it illustrates the manner in which the second teragon was derived from the first. When interior lines suggested by the symmetry are added, it becomes clear that the second teragon contains three scaled-down copies of the first teragon (and, therefore, $3 \times 3 = 9$ scaled-down copies of the initiator).

The cell sizes and numerical properties within successive frames of this transformation (of applying a wedge-shaped generator to an initiator) meet the classical central place, $K = 3$, criteria outlined above. Iteration of this transformation permits the rapid generation of the layers of the $K = 3$ hierarchy to any level of detail.

Each fractally generated layer contains hexagonal cells of the correct size so that, when resultant nets are superimposed on the initial hexagon on the point distinguished in the interior of each stage in Figure 10.4, the orientation of the nets is also correct for the $K = 3$ configuration (Figure 10.5). This small stack might be regarded as a $K = 3$ tile (Grünbaum and Shepard, 1987). When this tile and identical copies of it are placed side by side in the plane, they fit perfectly with no overlap and no gap. This is equivalent to using a lattice of hexagonal initiators and applying the inside–outside transformation of Figure 10.4 to the entire set. Figure 10.6 displays the stages of how this transformation might proceed, displaying in increasing detail, from left to right, a mosaic of $K = 3$ tiles. The highlighting is intended to suggest how elements of the tiles fit together (rather than to correspond with previous highlighting used to display different relationships). This fractally generated central place landscape (Figure 10.6, right-hand side) is identical to the classical $K = 3$ net as shown in Figure 10.1.

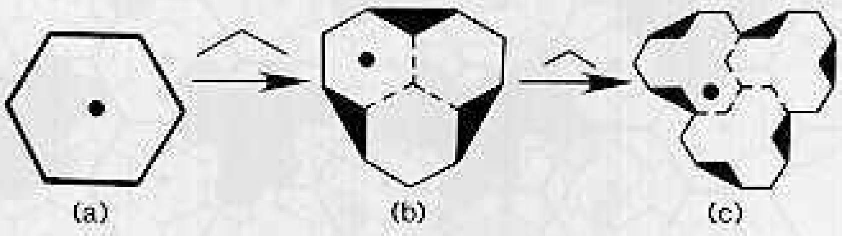

Figure 10.4 Fractal transformation leading to the $K = 3$ central place hierarchy: (a) hexagonal initiator; (b) result of applying the generator (above the left arrow), successively inside and outside, to the initiator; the three adjacent white cells are the first teragon; (c) second teragon, derived from the first teragon, using the scaled-down generator above the right arrow. The dot is distinguished as a point on which the shapes are to be stacked.

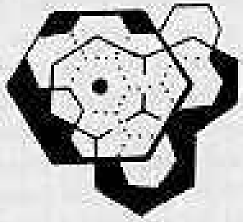

Figure 10.5 Initiator, first teragon, and second teragon from Figure 10.4 are stacked on the distinguished point, forming a tile of overlain nets with the correct $K = 3$ relationship.

Fractal Generation of the K = 4 Hierarchy

In the same vein, when a three-sided trapezoidal generator, with included angles of 120°, is scaled to span the sides of a hexagonal initiator (Figure 10.7a) and applied successively inside and outside the initiator, the resulting first teragon contains four self-similar copies of the initiator (Figure 10.7b). Rescaling the generator to fit the sides of the first teragon and applying it in the inside–outside scheme produces a second teragon (Figure 10.7c) containing four scaled-down copies of the first teragon (and, therefore, $4 \times 4 = 16$ scaled-down copies of the initiator). (This pattern is not unique to central place fractals; see Holroyd, 1983, for example). When the teragons are stacked on the initiator, on the point distinguished in Figure 10.7, a tile showing three layers of the $K = 4$ central place hierarchy emerges with cells of the correct size and in the proper relative orientation (Figure 10.8). When these tiles are placed side by side (that is, when the transformation of Figure 10.7 is applied to a net of hexagonal initiators), a fractally generated $K = 4$ central place landscape emerges (Figure 10.9) that is identical to the $K = 4$ net (Figure 10.2).

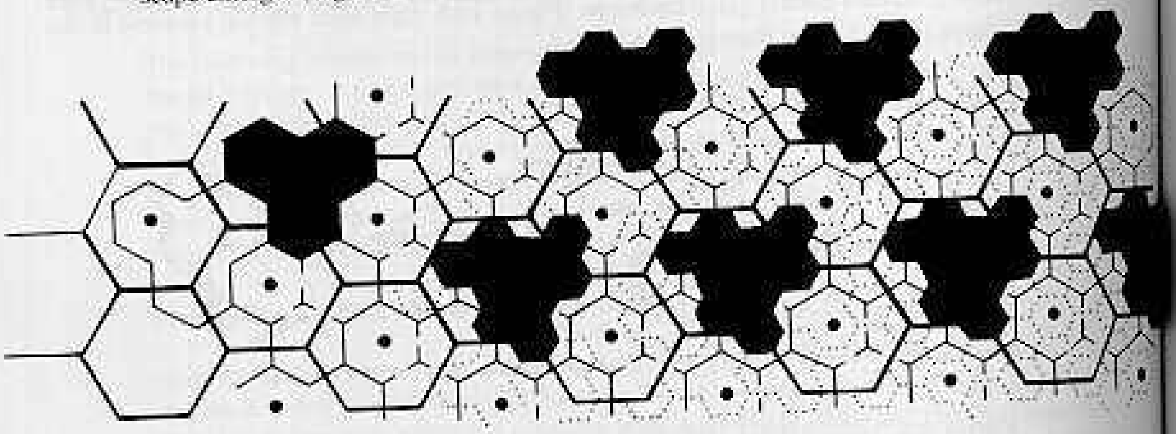

Figure 10.6 A fractally generated central place landscape. The blackened regions suggest how tiles fit together to cover the plane. The level of detail, shown as a continuum, increases from left to right. The white portion of the right side displays detail identical to that of the $K = 3$ central place hierarchy in Figure 10.1.

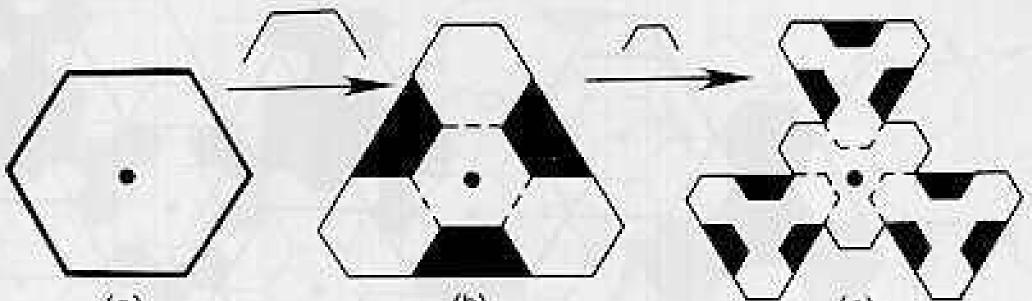

Figure 10.7 Fractal transformation leading to the $K = 4$ central place hierarchy. Details of explanation in text and in caption for Figure 10.4.

Fractal Generation of the K = 7 Hierarchy

When a three-sided, zigzag generator with included angles of 120° is scaled to fit the sides of a hexagonal initiator (Figure 10.10a) and applied in the inside–outside pattern, a first teragon containing seven smaller hexagons is generated (Figure 10.10b). Repeating the procedure produces a second teragon containing seven scaled-down copies of the first teragon (and hence $7 \times 7 = 49$ self-similar copies of the initiator) (Figure 10.10c). When these configurations of Figure 10.10, which exhibit the $K = 7$ characteristics, are superimposed on the point distinguished in Figure 10.10, a portion of a $K = 7$ three-layer stack is formed (Figure 10.11). When this and copies of it are used to tile a plane, the $K = 7$ central place landscape emerges (Figure 10.12) in the same manner it would have had the generator been applied to an entire net of hexagonal initiators (Figures 10.12 and 10.3 are identical).

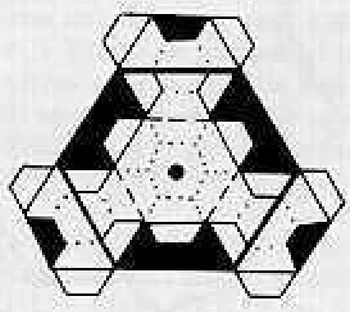

Figure 10.8 Initiator, first teragon, and second teragon from Figure 10.7 are stacked on the distinguished point, forming a tile of overlain nets with the correct $K = 4$ relationship.

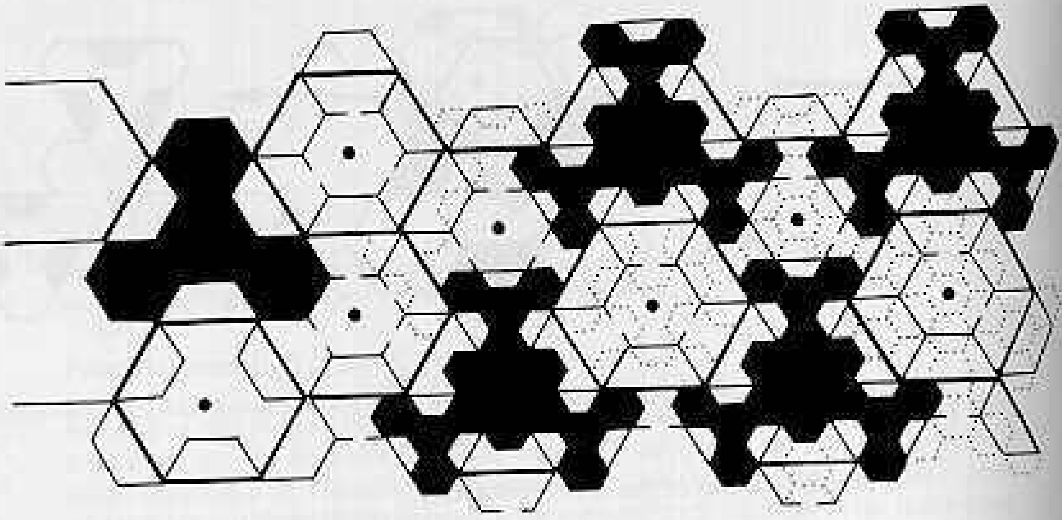

Figure 10.9 A fractally generated central place landscape. The blackened regions suggest how tiles fit together to cover the plane. The level of detail, shown as a continuum, increases from left to right. The white portion of the right side displays detail identical to that of the $K = 4$ central place hierarchy in Figure 10.2.

THE DIMENSION OF CENTRAL PLACE HIERARCHIES

Mandelbrot's (1983) formula for calculating the dimension of fractals

$$D = \frac{\ln(N)}{\ln(\frac{1}{r})} \tag{10.1}$$

where N is generally a count of the number of units of size $1/r$, may be used to calculate a single value to characterize each central place hierarchy (Arlinghaus, 1985; Arlinghaus and Arlinghaus, 1989). Two constants that are independent of size of trade area within the central place fractal construction for a single K value (but which change for different K values) are the number of generator sides and the spacing between rival centers. Thus, when N is taken to be the number of generator sides, and $1/r$ is taken to be the spacing between rival centers (the square root of K), Mandelbrot's D will yield a value to associate with each K value that represents the extent to which space would "fill-in" if the transformation were carried out infinitely.

Thus, for the K = 3 central place fractal, $N = 2$ and $1/r$ is $\sqrt{3}$, so $D = 1.2618595$. For the $K = 4$ central place fractal, $N = 3$ and $1/r$ is $\sqrt{4}$, so $D = 1.5849625$. For the $K = 7$ central place fractal, $N = 3$ and $1/r$ is $\sqrt{7}$, so $D = 1.1291501$. The higher the value of D is the greater the penetration of surrounding space by network lines. Thus, fractional dimensions systematically support our subjective impressions that the boundaries of the $K = 4$ system are more complicated that those of the $K = 7$ system.

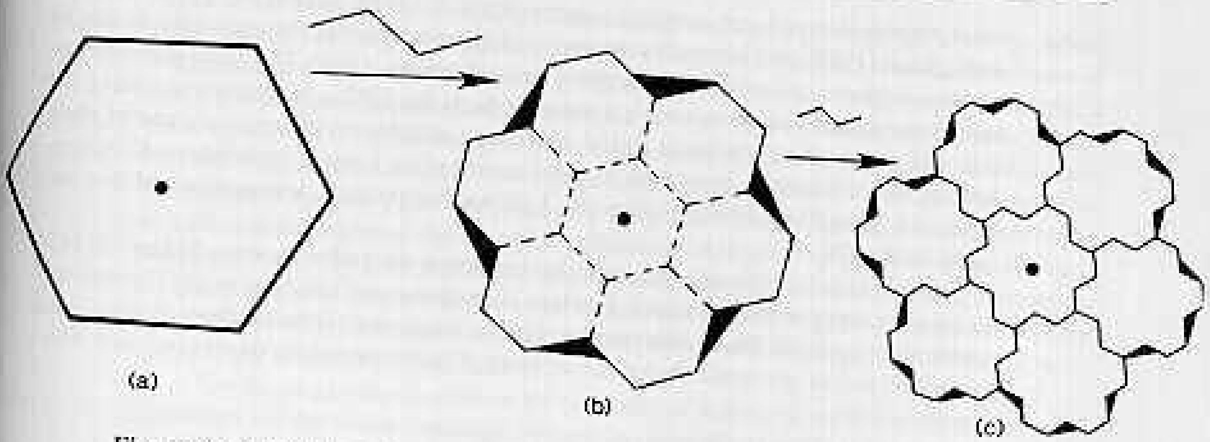

Figure 10.10 Fractal transformation leading to the $K = 7$ central place hierarchy. Details of explanation in text and in caption for Figure 10.4.

GENERALIZATIONS TO HIGHER *K* VALUES

It is possible to construct a central place fractal and to calculate its fractal dimension for all central place *K* values. What is difficult is to choose a generator from the set of infinite possibilities that will yield the desired result. How to do this was conjectured in Arlinghaus (1985); number theoretic properties of the *K* values provide all the information required to produce a generator that will guarantee the correct cell size, shape, and orientation for corresponding nets once a particular *K* has been selected. John U. Marshall proved a theorem in 1975 showing that integers with prime divisors with specified number-theo-

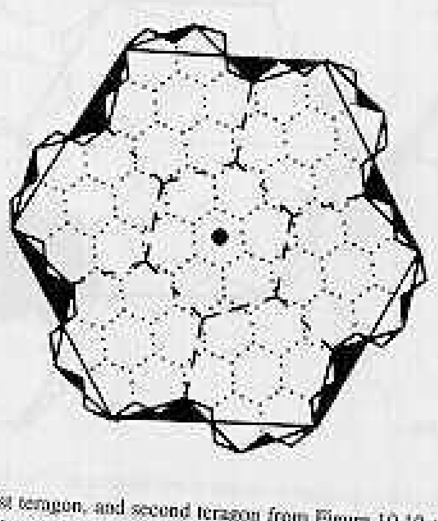

Figure 10.11 Initiator, first teragon, and second teragon from Figure 10.10 are stacked on the distinguished point, forming a tile of overlain nets with the correct $K = 7$ relationship.

retic properties could not serve as central place K values (Marshall, 1975). William Arlinghaus (1989) cast Marshall's theorem in a different number-theoretic light so that he might also prove its converse (Arlinghaus and Arlinghaus, 1989). This latter theorem permitted the author to demonstrate a systematic basis for 1985 conjectures concerning the size and shape of central place fractal generators (and fractal dimensions), and to show how to select "correct" generators to yield central place fractals for arbitrary K (Arlinghaus and Arlinghaus, 1989). Figure 10.13 shows, for example, a generator and first teragon for $K = 76$.

As a side benefit, this theorem also permitted the author to solve Dacey's (1965) open questions concerning which K values have nonunique solutions to the Diophantine generating equation. The reader interested in the mechanics of the solutions to these open questions and of generator choice for arbitrary K is referred to the aforementioned references.

DIRECTIONS FOR FURTHER RESEARCH

A natural, but mechanical, next step would be to automate this procedure to yield clear graphic, computerized displays of central place fractals. The reasons for doing so are twofold. First, it is clear that hexagonal trade areas do not exist as real-world entities anymore than do contour lines. The contour line is an approximation to a level line of a surface.

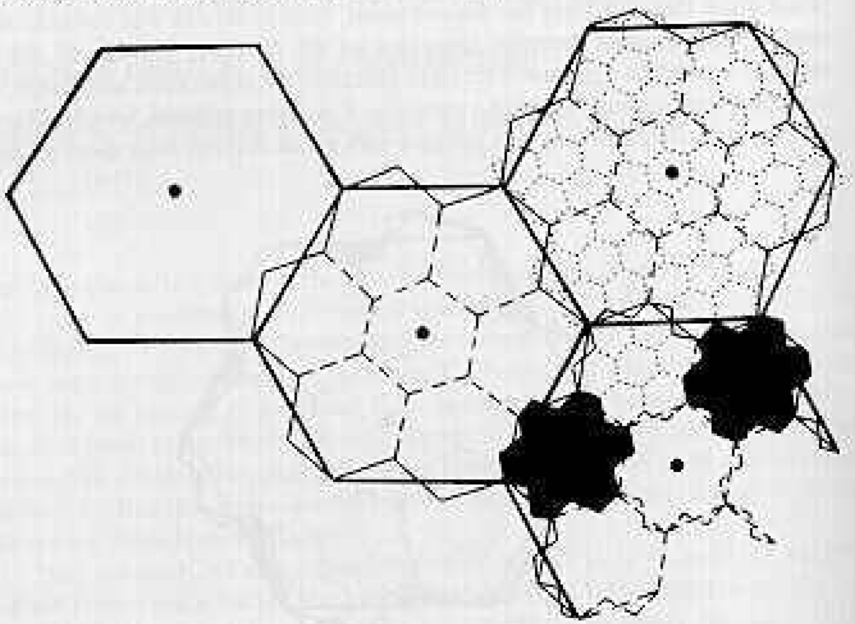

Figure 10.12 A fractally generated central place landscape. The blackened regions suggest how tiles fit together to cover the plane. The level of detail shown increases from left to right. The white portion of the right side displays detail identical to that of the $K = 7$ central place hierarchy in Figure 10.3.

Because the earth is not a regular mathematical surface, equations that describe actual mountains do not exist. This fact does not, however, deny the need to study techniques of integration in order to have a sound grasp of what contouring is all about. Similarly, a sound grasp of central place fractals, from the abstract to the visual, might lead in a number of directions: (1) If we had a computerized system with hexagonal pixels, central place fractals would address directly issues associated with the filling in of shapes formed from these pixels. The theory presented here might be immediately applicable to software based on a pixel arrangement of this sort. (2) Furthermore, it might be productive to modify central place fractals by deriving them from initiators of other shapes: squares to correspond to square pixels, or any number of other shapes of regular polygons or polyhedra. These extensions provide innovative ways of recursively subdividing space for GIS data management and multiscale spatial analysis.

Second, in an effort to remove the condition of polygonal regularity for the initiator, a geometry similar to this one might be based on centrally symmetric hexagons (ones with opposite sides parallel and of equal length, in pairs, or Dirichlet regions) (Kasner and Newman, 1956). Because centrally symmetric hexagons can be derived from triangulations of the plane as well of more general mathematical spaces (Spanier, 1966), it might also follow that the corresponding geometry exhibits properties parallel to those in the central place fractal geometry. Such extensions may suggest new ways of generating useful nested irregular networks from point data for interpolation and visualization. Research on all these topics is currently underway and some has been a continuing effort over the past 16 years.

The above outlines some theoretical directions for future research. We might also wonder what other real-world issues are likely to prompt responses of this sort: (*1*) The notion of estuary action describes a "crossing" of the land–water boundary, much as the discriminant of a quadratic form in mathematics describes a "crossing" of the x axis; this

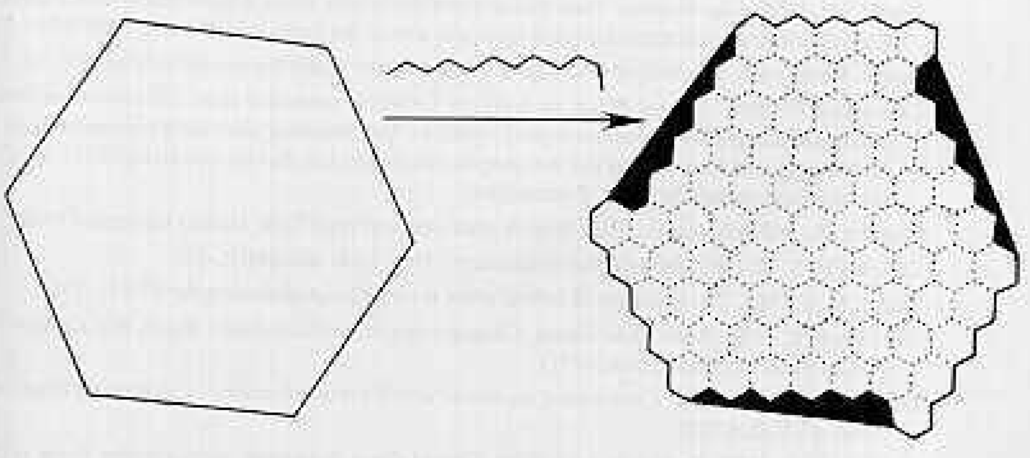

Figure 10.13 A generator and first teragon for $K = 76$; each larger trade area contains 76 of the next smaller trade areas.

might suggest the appropriateness of looking for a linkage between algebra and geometry, such as (but not necessarily limited to) the fractal model, in order to gain insight into problems of acceptable levels of salinity in fresh water. For example, the shape of the Suisun Bay area is similar to the shape of the larger San Francisco Bay area containing it (Arlinghaus and Nystuen, 1986, unpublished paper). Thus, we might use this noted self-similarity to look for corresponding self-similarity of estuary action in these bodies of water that might reflect levels of salinity. (2) The notion of self-similarity and corresponding ideas from fractal geometry might be applied to the idea of measuring rates of energy transference across a soil profile through the spaces between self-similar particles within that profile (Outcalt, 1972; Gleason et al., 1986; Arlinghaus and Nystuen, 1986, unpublished paper; Wheatcraft and Tyler, 1988).

With theory, methodology, application, and possible extension of application in hand, the discovery process will have come full circle. As is characteristic of scientific theory in general, and of spatial theory in particular (dealt with by Christaller, 1966; Lösch, 1954; Bunge, 1966; and others), this discovery process relies on a set of mathematical theorems to capture elements critical to that (spatial) theory. The promise of the wide range of pure mathematical theory (of which fractals are a small part) untapped by geographers offers exciting vistas in spatial theory.

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