Made in United States of America
Reprinted from GEOGRAPHICAL REVIEW
Vol. 83, No. 2, April 1993

ELECTRONIC GEOMETRY

SANDRA LACH ARLINGHAUS

ABSTRACT. A fractal approach to classical central-place theory is recast in terms of hierarchies of square trade areas. Infinite central-place hierarchies of squares or hexagons can be assigned fractional dimensions according to the extent to which they fill space. The fractional dimensions of hexagonal hierarchies are in all cases less than the square counterparts. When these geometries are interpreted electronically, in terms of square or hexagonal pixels, a higher degree of picture resolution on a cathode-ray tube is possible with a hexagonal lattice than with a square one, especially when pixel boundaries permit some transmission of content.

THE cathode-ray tube functions as a tablet of paper for many persons: they write and arrange text and draw maps on it. Whether this electronic tablet is lined or plain paper is a matter of scale. A blank screen appears unlined, yet monitor resolution depends on the number of electronic rows and columns involved. A single row-and-column entry in a lined electronic matrix serves as a fundamental picture element or pixel, from which text and maps are formed. As cells in a rectangular matrix, pixels are identical, nonoverlapping squares; they form a tiling of the paper, which ensures uniqueness in the designation of location with respect to the grid, and they apparently carry the color content within themselves, as an open set, rather than on their pixel boundaries.

Although the logic by which pixels are linked to one another is purely mathematical, the use of such linkages is often geographical. Geographical information systems (GIS) and text processors with font designers are boundary dwellers (Nystuen 1966) in the realm between geometry and geography: they are anchored in the logic of geometry, yet feed off the spatial content of geography. Researchers have explored the role of GIS in geography (Goodchild and Gopal 1989; Gibson and Lucas 1990; Hall and Gökmen 1990). In contrast, this article explores the role of geography, specifically central-place theory, as a theoretical basis for GIS.

PIXEL SEQUENCES AND CENTRAL-PLACE ANALYSIS

Lines and curves on cathode-ray tubes are formed from sequences of pixels joined together at their edges and corners. The pixel sequence merely suggests the curve: it does not actually produce a correct curve. Reducing the size of the pixel can improve the resolution of the image representing the curve. The extent to which improvement in resolution is possible can be evaluated with a fractal approach to central-place geometry.

● DR. ARLINGHAUS is director of the Institute of Mathematical Geography, 2790 Briarcliff, Ann Arbor, Michigan 48105.

The pixels in a sequence are separated by boundaries. Hence, when smaller square pixels are introduced, more lines separating pixels are likewise introduced. The interior of the pixel, not its boundary, carries the content. Thus, if the process of introducing finer and finer pixel mesh is carried out infinitely, the pixel interiors become smaller, and the entire plane region may become filled with pixel boundary. In this situation, all pixel content is lost. Clearly, improvement in resolution does not continue ad infinitum; at some point the trade-off between fineness in resolution and loss of information content reaches a peak. The question is, does this diminishing-returns dilemma exist independent of the shape of the fundamental pixel unit, that is, independent of the mathematical context in which pixels are embedded? An analysis using fractally generated central-place hierarchies suggests not: some pixel shapes and overlay orientations are better suited than others to reducing clutter from pixel boundaries.

The geometry of central-place theory conjures images of layers of hexagonal nets of varying diameter, superimposed in any of an infinity of orientations (Dacey 1965). The reason for hexagons rather than squares is that a hexagon is the cell shape that gives the tightest packing of cells. It is a well-known imitation of hexagons in nature, from cells in a beehive to coalescing soap bubbles.

Concepts from fractal geometry quantify the extent to which part of the central-place boundary fills its containing space. The fractal approach provides replicable, rigorous support for the earlier intuitive notions as to which hexagonal hierarchies might most reasonably be cast as the ones with marketing, transportation, or administrative orientations (Arlinghaus 1985). This approach also permits the solution of previously unsolved problems lurking beneath the integral surface of classical central-place geometry by assigning fractional dimensions to central-place nets, with that assignment based solely on properties deduced from the number-theoretic characteristics of the hierarchy (Marshall 1975; Arlinghaus and Arlinghaus 1989; Arlinghaus 1990).

Classical Hexagonal and Square Hierarchies

One way to derive the classical central-place hierarchy is as layers of hexagonal nets representing trade areas between competing villages and cities: villages have relatively small trade areas, and cities have larger ones. Thus a net of unit hexagons surrounds villages spread evenly, as points on a triangular lattice, across a minimal abstract environment. Similar layers emerge with trade areas centered on large central places (Dacey 1965). Under the hexagonal scheme, the village located at P (Fig. 1**a**) has six nearest neighbors, each sharing evenly with P the interstitial space. The boundaries from this sharing of space form the small hexagonal trade area immediately surrounding P. Each of the other villages, in turn, competes with its nearest neighbors, which creates a network of unit hexagonal cells.

If P enlarges its capacity as a central place and its nearest neighbors do not do so but the next nearest do so, a larger hexagon would be produced around P as its trade area (middle-sized hexagons in Fig. **1a**). This area contains the equivalent of three unit trade areas: the entire small hexagonal trade area of P and one-third of each of six others, one of which is shaded. This relationship, known as the Löschian number, is constant throughout the hierarchy between successive layers; the distance between competing centers within a layer is the square root of three (Lösch 1954; Christaller 1966). The invariance of three in this specific orientation of hexagonal nets is often emphasized by referring to it as a $K = 3$ hierarchy.

The same style of analysis can be used on an underlying lattice that is square, not triangular. Any village selected as a distinguishing point (lattice point) on Figure **1b** has four nearest neighbors with which it competes for interstitial space. This level of competition produces a net of square unit areas surrounding each village. When a village expands its capacities, it competes with its next nearest neighbors, four of them, for space. An enlarged trade area results that contains the equivalent of two small ones: the full square trade area around the distinguishing point and quarters of four others (one of which is shaded). As a parallel to the classical notion, this hierarchy might be denoted a $J = 2$ hierarchy. Hexagonal hierarchies beyond $K = 3$ that are generally familiar to geographers include the $K = 4$ and $K = 7$ hierarchies (Figs. 1**c** and 1**e**); their square-lattice counterparts are $J = 4$ and $J = 5$ (Figs. **1d** and **1f**). When the sharing of trade areas is used as the basic concept, the generalization from hexagonal to square hierarchy is straightforward.

It is easy to insert a natural coordinate system into either a triangular or a square lattice. In the former the axes are oblique, at angles of 60° and 120° (Fig. 2**a**); in the latter they are orthogonal (Fig. 2**b**). When each lattice point is assigned a pair of coordinates (x, y), relative to these axes, a K or J value may be associated with each ordered pair, based on the distance from (x, y) to the origin: in the case of the square lattice, $J = x^2 + y^2$; in the case of the hexagonal lattice, $K = x^2 + xy + y^2$. Thus, with any given ordered pair of integers, it is possible to find both J and K values associated with that point. If $(x, y) = (2, 3)$, then $J = 2^2 + 3^2 = 4 + 9 = 13$, and $K = 2^2 + (2)(3) + 3^2 = 4 + 6 + 9 = 19$. As hierarchies associated with the lattice point (2, 3), each next-larger square trade area would contain the equivalent of thirteen small trade areas, and each next-larger hexagonal trade area would contain the equivalent of nineteen small ones.

Fractal Hierarchies

Fractal geometry generates central-place nets (Arlinghaus 1991, 1993). The fractal approach produces all the classical results and far more as well. An advantage of using fractals to characterize central-place geometry is that the fractal approach assigns a single number to an entire hierarchy that measures

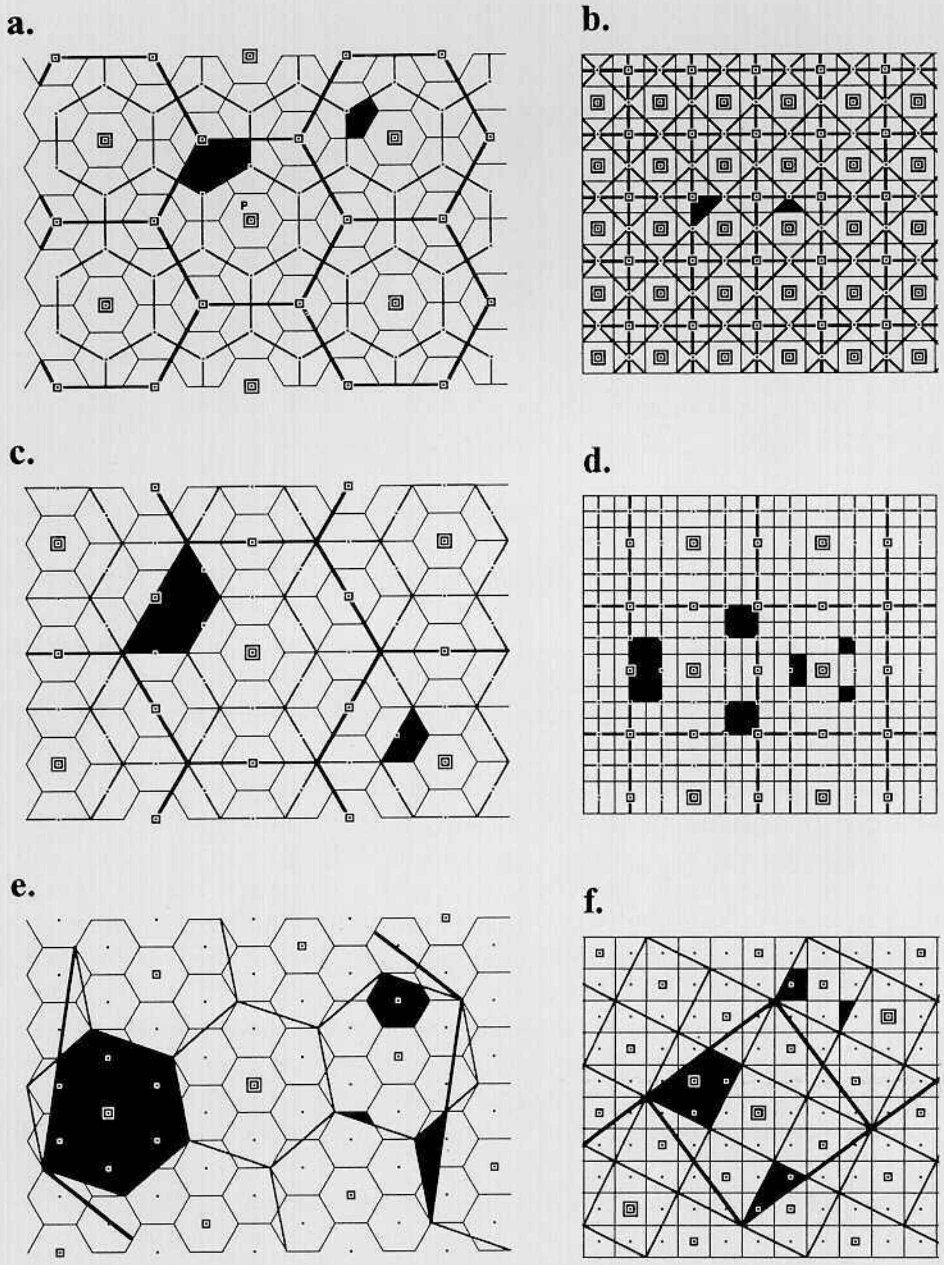

FIG. 1—**a**: hexagonal $K = 3$ hierarchy. **b**: square $J = 2$ hierarchy. **c**: hexagonal $K = 4$ hierarchy. **d**: square $J = 4$ hierarchy. **e**: hexagonal $K = 7$ hierarchy. **f**: square $J = 5$ hierarchy.

the extent to which boundary fills space. The classical approach assigns a single number to the hierarchy that measures the relationship of trade-area sizes between adjacent layers of the hierarchy; the fractal approach uses this relationship, in combination with the idea of a fractional dimension, to

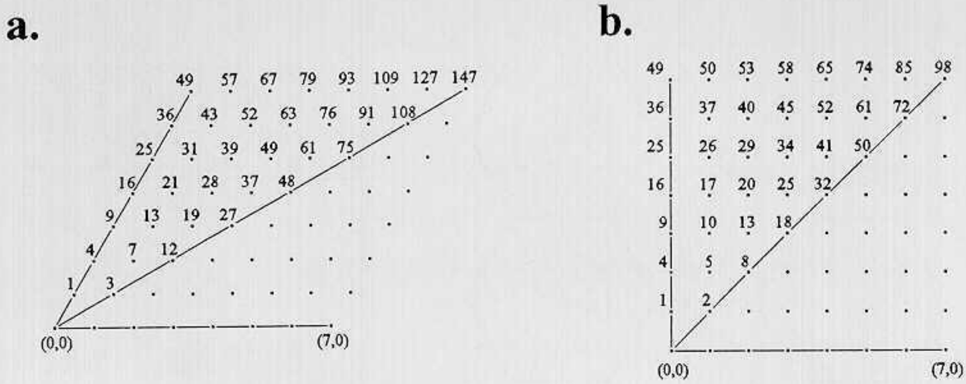

FIG. 2—**a**: sector of a triangular lattice and selected coordinates. **b**: sector of a square lattice and selected coordinates.

generate a different universal constant of the hierarchy. To understand the uses to which this new constant might be put, one must have some appreciation for the mechanics of the fractal approach for producing classical hierarchies.

The general strategy is to select a shape, or generator, that, when applied to a hexagonal or square initiator, will yield hexagonal or square cells of the next-smaller size that are in correct central-place orientation relative to the original initiator (Fig. 3). In Figure 3**a**, the two-sided wedge-shaped generator is applied to a hexagonal initiator, by replacing the hexagonal initiator sides with generator sides, alternately pointing the generator toward the interior of the hexagon and then toward the outside of the hexagon. Portions of the underlying figure thus underfit and overfit the original shaded hexagon. After the first iteration, a cluster that outlines three hexagons emerges; the second iteration produces nine hexagonal cells. Each of these frames corresponds to a portion of a layer in the $K = 3$ central-place hierarchy. The fit is exact, when the layers are superimposed on the distinguished point. Iteration can be carried out infinitely using successively smaller and smaller generators. When this is done, a dimension can be assigned to the hierarchy. The dimension measures the extent to which the solid boundary fills the region.

I refer to clusters of this sort, which outline size and position for sets of polygons of the same shape as the previous stage, as boundary animals. Changing the scale of the generator will produce successive boundary animals representing layers of the hierarchy and illustrating the invariance, in the case of Figure 3**a**, of the value 3 throughout the successive application of the generator. The middle boundary animal in Figure 3**a** outlines three hexagons; the right-hand boundary animal there outlines three of the threefold boundary animals from the central frame. Each stage in the fractal iteration corresponds exactly to a layer in the $K = 3$ central-place hierarchy. Figures 3**c** and 3**e** suggest how fractal generators may be applied to hexagonal

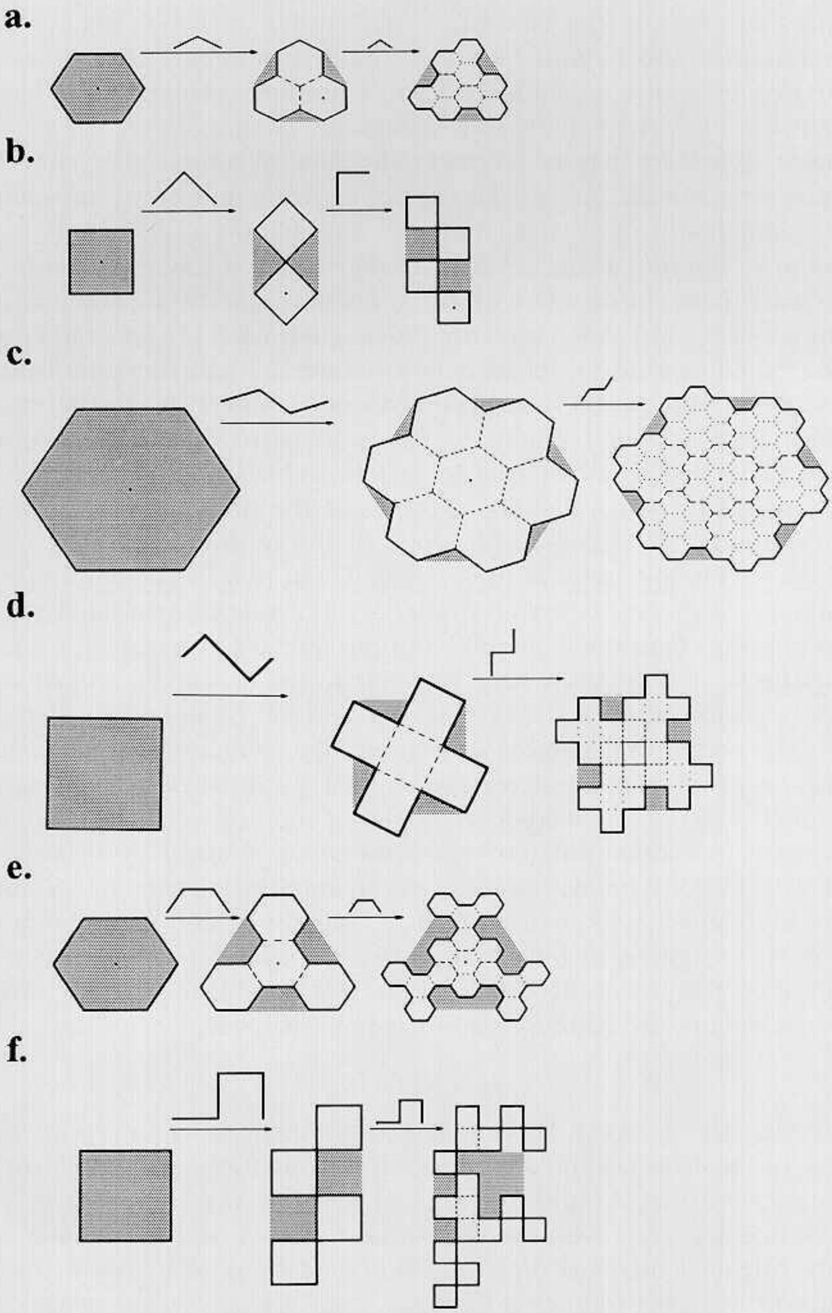

FIG. 3—Fractally generated hierarchies. **a**: $K = 3$ hexagonal. **b**: $J = 2$ square. **c**: $K = 7$ hexagonal. **d**: $J = 5$ square. **e**: $K = 4$ hexagonal. **f**: $J = 4$ square.

initiators to generate the classical $K = 7$ and $K = 4$ hierarchies. Fractal generators may also be found for any Löschian number; the requirements for finding them are dependent only on number-theoretic considerations (Arlinghaus 1985; Arlinghaus and Arlinghaus 1989).

Fractal geometry may be tailored to fit classical central-place concepts; it may also be extended into a central-place environment based on squares as the fundamental unit. In the latter case, the manner of application of the generator to the initiator is, once again, to remove the sides of the initiator and replace them alternately with the generator underfitting and overfitting the initiator (Fig. 3**b**). A wedge with a 90° angle is used as a generator applied to a square to illustrate how to produce a hierarchy. Again, the square initiator is shaded throughout the sequence of three frames to make the underfit-overfit strategy apparent. The center frame of Figure 3**b** contains two smaller squares generated from the one to its left; the boundary animal, in solid lines, consists of two scaled-down copies of the previous stage. The right frame contains two scaled-down copies of the central frame. The classical constant of this hierarchy is thus $J = 2$. These two hexagonal and square hierarchies, with generators of similar style, are both associated with the lattice point (1, 1) in the triangular and hexagonal sectors of Figure 2.

Figures 3**c** and 3**d** show how to generate the hexagonal and square hierarchies associated with (1, 2), the $K = 7$ and $J = 5$ hierarchies. Both these hierarchies employ a generator with three edges; alteration of inside-outside application of the generator creates a boundary animal that envelops, in the hexagonal case, seven scaled-down copies of the previous stage and, in the square case, five scaled-down copies of the previous stage. In the same vein, Figures 3**e** and 3**f** demonstrate how to generate fractally the hexagonal and square hierarchies associated with (0, 2), the $K = 4$ and $J = 4$ hierarchies. Each of these figures, however, displays only the first stages of an infinite sequence. As the process becomes infinite, it is possible to consider the extent to which the infinite boundary animals will fill space.

Space Filling

Fractals offer a way of looking at a continuous spectrum of dimensions. Most persons are accustomed to thinking of dimension in integral leaps, yet they easily see that a highly crenulated line fills more space than does a straight line segment, even though both are of Euclidean dimension one. A way to calculate this sort of partial filling of space is to use a fractional dimension, F, in the following formula:

$$F = (\log n)/(0.5 \log K),$$

where K and n are related in some natural fashion (Mandelbrot 1983). To see the extent to which fractally generated central-place layers fill space as the number of layers becomes infinite, use n as the number of edges in the generator and K or J as the number of self-similar shapes created by it, the constant of the hierarchy (Arlinghaus 1985). For example, the $K = 7$ hierarchy has a generator with three sides, so $n = 3$, $K = 7$, and $F = 1.129$. For the $J = 5$ hierarchy, $n = 3$, $J = 5$, and $F = 1.365$. The boundary animals of the $K = 7$ hierarchy, after infinite iteration, fill less space than do those of the $J = 5$ hierarchy. Table I shows comparisons for various J and K values. In all cases, given a particular (x, y), the fractional dimension is less for the hexagonal hierarchy than it is for the corresponding square hierarchy.

TABLE I—SPACE FILLING FOR SQUARE AND HEXAGONAL HIERARCHIES

LATTICE COORDINATES	CONSTANTS		DIMENSION	
	K	J	SQUARES	HEXAGONS
(1, 1)	3	2	2.0	1.262
(1, 2)	7	5	1.365	1.129
(0, 2)	4	4	2.0	1.585
(0, 3)	9	9	1.465	1.262
(0, 4)	16	16	1.5	1.161
(0, 5)	25	25	1.365	1.209
(0, 6)	36	36	1.387	1.161
(0, 7)	49	49	1.318	1.129
(0, 8)	64	64	1.333	1.153
(0, 9)	81	81	1.290	1.131
(0, 10)	100	100	1.301	1.114
(0, 11)	121	121	1.270	1.129
(0, 12)	144	144	1.279	1.116

The prevalent electronic geometry using square units relies, for refinement in resolution, on a $J = 4$ approach. Infinite improvement in resolution using this scheme would result in a total loss of pixel content, because the dimension of the boundary animal is two. The screen would be entirely filled by boundary, and a total failure to transmit information would result. Despite this difficulty, one advantage of the $J = 4$ approach over others is that refinement in resolution keeps layers in the same orientation. There is no twisting, an issue likely of interest given current scanning and printer technology.

If both the hexagonal and the square geometric environments are viewed as composed of crenulated boundary-animal outlines suggesting the central-place layers, the hexagonal environment is the one that permits infinite iteration with least loss of pixel content. When zooming in by reducing pixel size and improving resolution by reducing pixel size are the only considerations, the hexagonal pixel environment is a better choice than the square pixel environment. This is all the more so when one considers the possibility that parts of pixel boundaries might carry color content or information. The boundary-animal outlines that fill only small amounts of space (for example, 1.129 for $K = 7$) could then become the entire boundary outlines for groupings of hexagonal pixels. The $K = 7$ boundary animal is the outline of a snowflake; when pixel boundaries carry no content, this outline suggests positions for interior pixel boundaries. However, when these interior pixel boundaries carry content, the boundary animal completely defines a portion of the pixel space rather than merely suggesting it. The hexagonal hierarchy associated with (1, 2) has boundaries that fill only 1.129 dimensions of a two-dimensional space, but the corresponding square hierarchy has boundaries that fill 1.365 dimensions in a two-dimensional space. With increasing resolution and boundaries that carry content, more information is once again lost in the square electronic geometry than in the hexagonal electronic geometry. Formulators of mapping technology that involves pixels with boundaries carrying information might be well advised to note this sort of theoretical problem of trade-off in resolution and content, as well as its solution, before creating state-of-the-art programs.

Closing Comments

A continuing increase in resolution can bring about a black-hole-like collapse of an image, with the pixel boundaries choking out pixel content. When pixel boundaries carry information, it is possible to ensure that only low-dimensional boundary animals remain as resolution increases indefinitely. This improvement can be executed in either a square or a hexagonal electronic geometry. The dimensions are lower in the latter.

At an even broader scale, one might also look for this sort of application in hooking computers together as parallel processing units. When central places are thought of as central processing units of electronic information, an underlying geometry for finding shortest paths through networks linking multiple points might emerge. In an electronic environment with the hexagonal pixel as the fundamental unit, the 120° intersection points would correspond exactly to the requirements for finding shortest-distance or Steiner networks linking multiple locations. The branching or Steiner points in an electronic configuration might then correspond to locations at which to jump from one hexagonal lattice of fixed cell size to another of different cell size, or from one machine to another.

An even richer electronic geometry might see layers of pixels of all shapes and sizes, not simply a single shape as a fundamental unit, interacting, even overlapping, to produce dazzling arrays of superfine resolution. At this point, the spatial theory underlying GIS must become tightly intertwined with research that builds theorems in topological spaces, some of which might contain half-open sets as pixels with partial boundaries carrying information.

References

Arlinghaus, S. 1985. Fractals take a central place. *Geografiska Annaler* 67B(2):83–88.

———. 1990. Fractal geometry of infinite pixel sequences: "super-definition" resolution? *Solstice* 2(2):48–53.

———. 1991. Tiling according to the administrative principle. Essays in mathematical geography—III, 16–25. Ann Arbor, Mich.: Institute of Mathematical Geography.

———. 1993. Central place fractals: theoretical geography in an urban setting. Fractals in geography, eds. N. Siu-Ngan Lam and L. De Cola, in press. Englewood Cliffs, N.J.: Prentice-Hall.

Arlinghaus, S., and W. Arlinghaus. 1989. Fractal theory of central place geometry: a Diophantine analysis of fractal generators for arbitrary Löschian numbers. *Geographical Analysis* 21:103–121.

Christaller, W. 1966. Central places in southern Germany. Englewood Cliffs, N.J.: Prentice-Hall.
Dacey, M. F. 1965. Geometry of central place theory. *Geografiska Annaler* 47:111–124.
Gibson, L., and D. Lucas. 1990. Vectorization of raster images using hierarchical methods. Paper available from Interactive Systems Corp., 5500 S. Sycamore St., Littleton, CO 80120.
Goodchild, M. F., and S. Gopal. 1989. Accuracy of spatial databases. London: Taylor and Francis.
Hall, R. W., and M. Gökmen. 1990. Rectangular/hexagonal tesselation transforms and parallel shrinking. Paper TR-SP-90-004, available from Department of Electrical Engineering, University of Pittsburgh, Pittsburgh, PA 15261.
Lösch, A. 1954. Economics of location. New Haven: Yale University Press.
Mandelbrot, B. 1983. Fractal geometry of nature. San Francisco: W. H. Freeman.
Marshall, J. U. 1975. Löschian numbers as a problem in number theory. *Geographical Analysis* 7: 421–426.
Nystuen, J. D. 1966. Effects of boundary shape and the concept of local convexity. Papers, Michigan Inter-University Community of Mathematical Geographers, 10:3–24, Ann Arbor.