Closure Operations that Induce Big Cohen-Macaulay Modules and Algebras, and Classification of Singularities

by

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TABLE OF CONTENTS

LIST OF FIGURES

Figure

CHAPTER I

Introduction

There are two ways to study a system of polynomial equations: using geometry, and using algebra. Geometrically, we can work with the set of solutions of the equations, called a variety, as in the curves pictured below in the real plane.

$y = x^2$ $\qquad\qquad$ $y^2 = x^3$

Figure 1.1: Solution sets of polynomial equations

We say that the curve defined by $y = x^2$ is nonsingular, but the curve defined by $y^2 = x^3$ has a singularity at the origin. A singularity can be a cusp as in this case, a crossing, or any shape that causes there to be a non-unique tangent line to the curve.

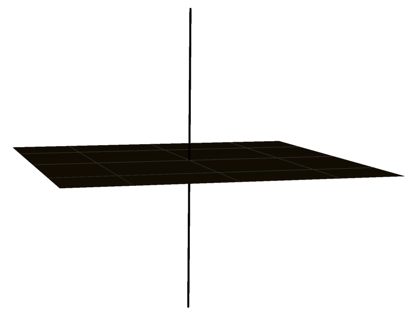
Another way to study systems of polynomial equations is to study the commutative ring of polynomial functions that vanish where the equations hold. For the curve defined by $y = x^2$ pictured above, this gives us the ring $\mathbb{R}[x, y]/(y - x^2)$. In studying general commutative rings, we search for properties similar to those of rings given by systems of polynomial equations. When we find these properties, we can take advantage of the connections between commutative algebra and algebraic geometry. My research focuses on the ring-centric point of view.

Rings that are local are of particular interest. Geometrically, a local ring consists of polynomial functions defined on and around a point on the curve (or higher dimensional variety). For example, we can form the local ring at the origin for the curve defined by $y = x^2$. Algebraically, these functions are given by the local ring $\mathbb{R}[x, y]_{(x,y)}/(y - x^2)$, in which we invert all polynomials that do not vanish at $(0, 0)$. For example, $1 + x$ is now invertible. A local ring has a unique maximal ideal, consisting of all non-invertible elements of the ring.

We call a ring of polynomial functions around a nonsingular point a regular local ring. The ring $\mathbb{R}[x, y]_{(x,y)}/(y - x^2)$ is an example of a regular local ring. However, the local ring $\mathbb{R}[x, y]_{(x,y)}/(y^2 - x^3)$ of $y^2 = x^3$ at the origin is not regular since the origin is a singular point of the corresponding curve.

Both of the examples mentioned above are curves, which are 1-dimensional. An example of a two dimensional variety is the boundary of a sphere. We study varieties in higher dimensions as well, and define the dimension of a coordinate ring to be the dimension of the highest dimensional piece that appears in the corresponding variety. So $\mathbb{R}[x, y]/(y - x^2)$ has dimension 1. For a local ring, we look only at the piece of the geometric object near the point. The curve defined by $y = x^2$ is one-dimensional at every point. However, the set of points where $xz = 0$ and $yz = 0$ in 3-space is a two-dimensional object that is locally one-dimensional at points along the z-axis (see Figure 1.2).

Given the d-dimensional local ring R of a point, a system of parameters on the ring is a set of d elements that vanish exactly at that point. For example, x, y form a system of parameters on $\mathbb{R}[x, y]$ (the xy-plane): the origin is the only solution of



$(xz = 0) \cap (yz = 0)$

Figure 1.2: Two-dimensional variety that is not two-dimensional at every point

$x = 0$ and $y = 0$. Another example of a system of parameters is $x, y - z$ on the ring $\mathbb{R}[x, y]_{(x,y,z)}/(xz, yz)$.

A regular sequence on an R-module M is a set of elements $x_1, \ldots, x_n$ of a ring R such that $M \neq (x_1, \ldots, x_n)M$, x_1 is not a zero-divisor on M, and for each $1 < i \leq d$, the image x_i is not a zero-divisor on $M/(x_1, \ldots, x_{i-1})M$. In particular, we can have regular sequences on R. On $R = \mathbb{R}[x, y]$, x, y form a regular sequence. However, $x, y - z$ is not a regular sequence on $\mathbb{R}[x, y]_{(x,y,z)}/(xz, yz)$, since x is a zero-divisor. Every regular sequence of length $\dim(R)$ is a system of parameters, but not every system of parameters is a regular sequence. We say that a local ring R is Cohen-Macaulay if every system of parameters on R is a regular sequence on R. Note that in the local case, the elements of the regular sequence are all contained in the unique maximal ideal. All regular local rings are Cohen-Macaulay, including $\mathbb{R}[x, y]_{(x,y)}/(y - x^2)$ and $\mathbb{R}[x, y]$. As we saw above though, $\mathbb{R}[x, y, z]_{(x,y,z)}/(xz, yz)$ is not a Cohen-Macaulay ring.

Commutative algebra often involves proving that arbitrary rings have properties similar to those of polynomial rings and other regular local rings. Cohen-Macaulay rings act quite a bit like polynomial rings, and there are many results describing their behavior. One such result states that if R is a Noetherian local ring and S a regular local subring such that R is finitely-generated as a module over S, then R is Cohen-Macaulay if and only if it is a free S-module [BH93].

When a local ring is not Cohen-Macaulay, a number of important results still hold if the ring has a big Cohen-Macaulay module, a module M over the ring, not necessarily finitely-generated, such that every system of parameters on the ring is a regular sequence on the module. For example, the Monomial Conjecture holds on rings with big Cohen-Macaulay modules. This result states that given a system of parameters $x_1, \ldots, x_d$ on a local ring R, we have

$$x_1^t x_2^t \cdots x_k^t \notin (x_1^{t+1}, x_2^{t+1}, \ldots, x_k^{t+1})R \tag{1.1}$$

for any $t \geq 0$ or $1 \leq k \leq d$. Note that it is simple to prove that (1.1) holds on any Cohen-Macaulay ring, since it holds when $x_1, \ldots, x_d$ is a regular sequence.

The Monomial Conjecture is equivalent to a family of conjectures fundamental to commutative algebra, including the Direct Summand Conjecture [Hoc73] and the Canonical Element Conjecture [Hoc83]. These are implied by the existence of big Cohen-Macaulay modules, as are many other conjectures such as the Syzygy Theorem [EG85, Hoc75].

We know that certain local domains over fields of characteristic $p > 0$ (such as finitely-generated algebras over $\mathbb{Z}/p\mathbb{Z}$) have big Cohen-Macaulay modules [HH90]. This was proved using the theory of tight closure developed by Mel Hochster and Craig Huneke [HH88, HH89, HH90, BH93, Hun96, Hoc07] (and many others). This result was then extended to the equal characteristic 0 case (e.g., algebras over $\mathbb{C}[x_1, \ldots, x_d]$) by using reduction to characteristic $p > 0$ to develop a characteristic 0 version of tight closure [HH99].

Tight closure is an example of a closure operation, a map taking each ideal of the ring to an ideal that contains it, with a similar action on submodules of all finitely-generated R-modules (see Definition II.2). In characteristic $p > 0$, tight closure can be defined using the Frobenius endomorphism, which raises elements of R to their pth powers. The idea of the characteristic 0 version of tight closure can be seen from this example: in the case of the characteristic 0 ring $\mathbb{Q}[x, y, z]/(x^3 + y^3 + z^3)$, the image of z^2 is in the tight closure of (x, y) because this is true in $\mathbb{Z}/p\mathbb{Z}[x, y, z]/(x^3+y^3+z^3)$ for almost all primes numbers p.

In characteristic $p > 0$, tight closure can also be defined by taking images of ideals or submodules after tensoring with big Cohen-Macaulay R-algebras (see Theorem II.9). This makes it clear that tight closure is closely tied to the question of the existence of big Cohen-Macaulay modules.

However, there is currently no version of tight closure for rings of mixed characteristic that can be shown to have the properties needed to prove the existence of big Cohen-Macaulay modules. The mixed characteristic case includes localizations of finitely-generated $\mathbb{Z}$-algebras — many of the rings that number theorists study are in this case. A slightly more complicated case involves a polynomial ring over a discrete valuation ring, a ring with a single nonzero prime ideal, which is principal (generated by 1 element). The p-adic integers $\mathbb{Z}_p$ form a discrete valuation ring of mixed characteristic that is heavily studied by number theorists.

In fact, we do not know whether mixed characteristic rings of dimension greater than 3 have big Cohen-Macaulay modules. The existence of big Cohen-Macaulay modules in mixed characteristic rings of dimension 3 was proved by very different methods [Hei02, Hoc02], which do not appear to extend to higher-dimensional rings. As a result, all of the conjectures mentioned above are unknown for rings of mixed characteristic and dimension greater than 3. The obstruction to extending these results to the mixed characteristic case may be viewed as the lack of a good enough version of tight closure on these rings.

There have been many attempts to find a version of tight closure for rings of mixed characteristic, including solid closure [Hoc94], parameter tight closure [Hoc03], diamond closure [HV04], and parasolid closure [Bre03]. In [Die10], Dietz gave a list of axioms for a closure operation such that for a local domain R, the existence of a closure operation satisfying these properties (we call these Dietz closure, see Definition II.17) is equivalent to the existence of a big Cohen-Macaulay module. The idea of the axioms is that they represent the properties that a closure operation needs in order to fulfill the role of tight closure in one of the proofs of the existence of big Cohen-Macaulay modules.

In characteristic $p > 0$, tight closure satisfies these properties for complete rings (e.g. a power series ring over a field like $\mathbb{Z}/p\mathbb{Z}[[x_1, \ldots, x_d]]$), as do plus closure [Smi94] and closures coming from big Cohen-Macaulay modules (as in Definition II.4) [Die10]. The axioms do not depend on the characteristic of the ring, so a closure operation in mixed characteristic could satisfy them. As a result, they provide a way to check potential closure operations in mixed characteristic, as well as a direction in which to look for such a closure operation.

The main idea of [Die10], following the method of [Hoc75], is that it is possible to construct a big Cohen-Macaulay module by building modules that force certain relations to hold on systems of parameters of the ring R. Starting with R, at each stage new elements are added to the module to form the coefficients of the desired relation, and then the quotient by this relation is taken (see Definition II.21 for a more precise explanation). In the direct limit B, every system of parameters is a regular sequence. The Dietz closure is used in lieu of tight closure to show that at each stage, the image of 1 stays out of the image of the maximal ideal of R, so that for any system of parameters $x_1, \ldots, x_d$ on R, $\operatorname{im}(1) \notin \operatorname{im}(x_1, \ldots, x_d)$. This forces $B \neq (x_1, \ldots, x_d)B$.

We study Dietz closures in an effort to understand their properties and their relationship to better-understood closure operations. We prove that they are connected to the singularities of the ring in a similar way to tight closure, and give an additional axiom that allows us to construct big Cohen-Macaulay algebras, big Cohen-Macaulay modules that are also R-algebras. While we do not prove that big Cohen-Macaulay modules (or algebras) exist in any new cases, it is our hope that others will use these results to find a suitable closure operation for rings of mixed characteristic, which will imply the existence of big Cohen-Macaulay modules in that case.

We start with definitions and background information in Chapter II, including some results from [Die10].

In Chapter III, we develop some basic properties of closure operations that are used throughout the paper, including properties of big Cohen-Macaulay module closures. This is followed in Chapter IV by a discussion of properties of closure operations for which there is a smallest closure satisfying the property. In particular, any ring that has a Dietz closure has a smallest Dietz closure, as well as a smallest big Cohen-Macaulay module closure. In certain rings of dimension 2, the smallest big Cohen-Macaulay module closure comes from the S_2-ification of R (see Definition IV.15). Studying the smallest Dietz closure or big Cohen-Macaulay module closure should provide information on the properties of R.

In Chapter V, we prove:

Theorem 1 (Theorem V.1)**.** *Let cl be a Dietz closure on a local domain (R, m). Then cl is contained in cl_B for some big Cohen-Macaulay module B, i.e., for all* *finitely-generated R-modules $N \subseteq M$, $N_M^{cl} \subseteq N_M^{cl_B}$.*

Using this result, we prove:

Theorem 2 (Theorem V.9, Theorem V.11)**.** *Suppose that (R, m) is a local domain that has at least one Dietz closure (in particular, it suffices for R to have equal characteristic and any dimension, or mixed characteristic and dimension at most 3). Then R is regular if and only if all Dietz closures on R are equal to the trivial closure.*

This leads to a family of questions we wish to investigate (see Sections 8.2 and 9.3): how does the prevalence of Dietz closures on a ring R correspond to the singularities of R?

In the proof of Theorem V.11, we see that a particular module of syzygies gives a closure operation not equal to the trivial closure, which we can compute explicitly. In Section 6.3, we use these results to compare Dietz closures to better understood closure operations, proving that all Dietz closures are contained in (liftable) integral closure, and that persistent families of Dietz closures are contained in regular closure.

In Theorem VI.1, we show that integral closure and regular closure are not Dietz closures using a criterion that can be applied more generally. As a corollary of the above theorems, we also conclude that solid closure is not a Dietz closure for rings of equal characteristic 0. Studying the reasons why certain closure operations are or are not Dietz closures provides more information on the pieces that are needed to get a good enough closure operation in mixed characteristic.

Dietz asked whether it was possible to give an additional axiom such that the existence of a Dietz closure satisfying this axiom is equivalent to the existence of a big Cohen-Macaulay algebra. Due to results on the existence of weakly functorial big

Cohen-Macaulay algebras [HH95], one can work with families of big Cohen-Macaulay algebras over a family of rings, making them more useful than big Cohen-Macaulay modules, which do not appear to work well over families of rings. Further, big Cohen-Macaulay algebras are known to exist in every case where big Cohen-Macaulay modules are known to exist.

In this paper, we answer Dietz's question in the positive, by giving an Algebra Axiom, Axiom VII.1. We prove:

Theorem 3 (Theorem VII.3, Corollary VII.12)**.** *A local domain R has a Dietz closure that satisfies the Algebra Axiom if and only if R has a big Cohen-Macaulay algebra.*

In Section 7.2, we prove that many closure operations satisfy this axiom, including tight closure, which is also a Dietz closure, and torsion-free algebra closures, which are not in general Dietz closures. To prove this, we find alternative characterizations of cl-phantom extensions for the closures cl that we discuss. We also show that the big Cohen-Macaulay algebras that we construct using the Algebra Axiom give the same closure operation as those constructed using algebra modifications as in [HH95], and that this closure operation is the smallest big Cohen-Macaulay algebra closure on the ring R.

In Section 8.2, we use these results to compare Dietz closures satisfying the Algebra Axiom to other closure operations. We prove that each such closure is contained in a big Cohen-Macaulay algebra closure. In characteristic $p > 0$, we also show that all such closures are contained in tight closure. As a consequence, weakly F-regular rings (see Definition II.10) have a unique Dietz closure satisfying the Algebra Axiom. These results may lead to further characterizations of the singularities of a ring in terms of these closure operations. We also give an example that shows that not all

Dietz closures satisfy the Algebra Axiom.

We conclude with a list of further questions in Section IX. Interestingly, we do not know whether there is a largest big Cohen-Macaulay module closure, as discussed in Section 9.2.

CHAPTER II

Background

In this section we give the necessary definitions and some notation that will be used throughout the paper. All rings are commutative Noetherian rings with a multiplicative identity element 1. We use the term *local ring* to refer to rings that are Noetherian and have a unique maximal ideal.

Definition II.1. Let (R, m) be a local ring. An R-module B is a (balanced) *big Cohen-Macaulay module* over R if every system of parameters for R is a regular sequence on B, and $mB \neq B$. The word "big" appears because B need not be finitely-generated. A *big Cohen-Macaulay algebra* over R is a big Cohen-Macaulay module over R that is also an R-algebra.

Definition II.2. A *closure operation* cl on a ring R is a map on the submodules of each finitely-generated R-module M, taking each submodule $N \subseteq M$ to another submodule N_M^{cl} of M such that if $N \subseteq N' \subseteq M$ are finitely-generated R-modules,

1. (Extension) $N \subseteq N_M^{\mathrm{cl}}$,
2. (Idempotence) $(N_M^{\mathrm{cl}})_M^{\mathrm{cl}} = N_M^{\mathrm{cl}}$, and
3. (Order-preserving) $N_M^{\mathrm{cl}} \subseteq (N')_M^{\mathrm{cl}}$.

Example II.3. Some straightforward examples of closure operations are:

1. Trivial closure: $N_M^{\mathrm{cl}} = N$ for all finitely-generated R-modules $N \subseteq M$.

2. Improper closure: $N_M^{\mathrm{cl}} = M$ for all finitely-generated R-modules $N \subseteq M$.

3. Local closure: Assume that (R, m) is local. Set $N_M^{\mathrm{cl}} = N + mM$ for all finitely-generated R-modules $N \subseteq M$.

Definition II.4. Suppose that S is an R-module (resp. R-algebra). We can define a closure operation cl_S on R by $u \in N_M^{\mathrm{cl}_S}$ if for all $s \in S$,

$$s \otimes u \in \mathrm{im}(S \otimes N \to S \otimes M)$$

for any $N \subseteq M$ finitely-generated R-modules and $u \in M$. This is called the *closure given by* S. A closure that is given by some R-module (resp. R-algebra) S is called a *module (resp. algebra) closure.*

Remark II.5. If S is an R-algebra, $u \in N_M^{\mathrm{cl}_S}$ if and only if

$$1 \otimes u \in \mathrm{im}(S \otimes N \to S \otimes M).$$

Example II.6. 1. The 0 module gives the improper closure.

2. R gives the trivial closure.

We give a brief definition of tight closure; a more detailed one can be found in [Hoc07].

Definition II.7. Let R be a ring of characteristic $p > 0$. The *Frobenius endomorphism* is the map $F : R \to R$ sending $r \mapsto r^p$. We use F^e for the eth iteration of F, which sends $r \mapsto r^{p^e}$. Let $F_*^e(R)$ denote the abelian group R, viewed as an R-module via the Frobenius endomorphism. For M an R-module, $F_*^e(M)$ will denote $F_*^e(R) \otimes_R M$.

Definition II.8. Let R be a ring of characteristic $p > 0$, and $N \subseteq M$ finitely-generated R-modules. Let F denote the Frobenius functor, and F^e its eth iteration. We say that $u \in N_M^*$, the *tight closure* of N in M, if there is some $c \in R$ not in any minimal prime of R such that $c \otimes u \in \operatorname{im}(F_*^e(N) \to F_*^e(M))$ for all $e \gg 0$. When $M = R$ and $N = I$ is an ideal of R, this definition can be written as $u \in I^*$ if there is some $c \in R$ not in any minimal prime such that $cu^{p^e} \in (i^{p^e} : i \in I)$ for all $e \gg 0$.

We will use the following definition of tight closure later on.

Theorem II.9 [Hoc94, Theorem 11.1]. *Let R be a complete local domain of characteristic $p > 0$ and let $N \subseteq M$ be finitely-generated R-modules. Then $u \in N_M^*$ if and only if there is some big Cohen-Macaulay R-algebra B such that $1 \otimes u \in$ im$(B \otimes_R N \to B \otimes_R M)$.*

Definition II.10 [Hoc07]. We say that a ring R is *weakly F-regular* if all ideals of R are tightly closed. Equivalently, we say that R is *weakly F-regular* if for all finitely-generated R-modules $N \subseteq M$, $N_M^* = N$ (tight closure is equal to the trivial closure on R).

Remark II.11. It is not known whether localizations of a weakly F-regular ring are weakly F-regular. If they are, we say that the ring is F-regular.

Now we define a cl-phantom extension, which is a major component of the results in this thesis.

Definition II.12 [Die10, Definition 2.2]. Let R be a ring with a closure operation cl, M a finitely-generated R-module, and $\alpha : R \to M$ an injective map with cokernel Q. We have a short exact sequence

$$0 \longrightarrow R \xrightarrow{\alpha} M \longrightarrow Q \longrightarrow 0.$$

Let $\epsilon \in \operatorname{Ext}_R^1(Q, R)$ be the element corresponding to this short exact sequence via the Yoneda correspondence. Let $P_\bullet$ be a projective resolution of Q and let $\vee$ denote $\mathrm{Hom}_R(-, R)$. We say that α is a *cl-phantom extension* if a cocycle representing ϵ in $\mathrm{Ext}^1_R(Q, R) \subseteq P_1^\vee$ is contained in $\mathrm{im}(P_0^\vee \to P_1^\vee)^{\mathrm{cl}}_{P_1^\vee}$.

Remark II.13. This definition is independent of the choice of $P_\bullet$ [Die10, Discussion 2.3].

A split map $\alpha : R \to M$ is cl-phantom for any closure operation cl: in this case, the cocycle representing ϵ is in $\mathrm{im}(P_0^\vee \to P_1^\vee)$. We can view cl-phantom extensions as maps that are "almost split" with respect to a particular closure operation.

Notation II.14. We use some notation from [Die10]. Let R be a ring, M a finitely generated R-module, and $\alpha : R \to M$ an injective map with cokernel Q. Let $e_1 = \alpha(1), e_2, \ldots, e_n$ be generators of M such that the images of $e_2, \ldots, e_n$ in Q form a generating set for Q. We have a free presentation for Q,

$$R^m \xrightarrow{\nu} R^{n-1} \xrightarrow{\mu} Q \longrightarrow 0,$$

where μ sends the generators of R^{n-1} to $e_2, \ldots, e_n$ and ν has matrix $(b_{ij})_{2 \le i \le n, 1 \le j \le m}$ with respect to some basis for R^m. We have a corresponding presentation for M,

$$R^m \xrightarrow{\nu_1} R^n \xrightarrow{\mu_1} M \longrightarrow 0,$$

where μ_1 sends the generators of R^n to $e_1, \ldots, e_n$. Using the same basis for R^m as above, ν_1 has matrix $(b_{ij})_{1 \le i \le n, 1 \le j \le m}$ where $b_{1j}e_1 + b_{2j}e_2 + \ldots + b_{nj}e_n = 0$ in M [Die10, Discussion 2.4]. The top row of ν_1 gives a matrix representation of the map $\phi : R^m \to R$ in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\alpha"] & M \arrow[r] & Q \arrow[r] & 0 \\
 & R^m \arrow[u, "\phi"] \arrow[r, "\nu"] & R^{n-1} \arrow[u, "\psi"] \arrow[r, "\mu"] & Q \arrow[u, "\mathrm{id}_Q"] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

In [Die10, Discussion 2.4], Dietz gives an equivalent definition of a phantom extension using the free presentations M and Q given above. While he assumes that R is a complete local domain and that cl satisfies 2 additional properties, these are not needed for all of the results. We restate some of his results in greater generality below.

Lemma II.15 [Die10, Lemma 2.10]. *Let R be a ring possessing a closure operation cl. Let M be a finitely generated module, and let $\alpha : R \to M$ be an injective map. Let notation be as above. Then α is a cl-phantom extension of R if and only if the vector $(b_{11}, \ldots, b_{1m})^{tr}$ is in $B^{cl}_{R^m}$, where B is the R-span in R^m of the vectors $(b_{i1}, \ldots, b_{im})^{tr}$ for $2 \le i \le n$.*

Definition II.16. Let N, M, and W be finitely-generated R-modules with $N \subseteq M$, and let cl be a closure operation on R.

1. Functorial property: We say that cl is *functorial* if given a homomorphism $f : M \to W$,
$$f(N^{\mathrm{cl}}_M) \subseteq f(N)^{\mathrm{cl}}_W.$$
2. Semi-residual property: We say that cl is *semi-residual* if whenever $N^{\mathrm{cl}}_M = N$, $0^{\mathrm{cl}}_{M/N} = 0$.
3. Faithful property: Assume that (R, m) is local. We say that cl is *faithful* if $m^{\mathrm{cl}}_R = m$.

Definition II.17 [Die10]. Let (R, m) be a fixed local domain and let N, M, and W be arbitrary finitely generated R-modules with $N \subseteq M$. A closure operation cl is called a *Dietz closure* if it is functorial, semi-residual, faithful, and satisfies the generalized colon-capturing axiom:

Let $x_1, \ldots, x_{k+1}$ be a partial system of parameters for R, and let $J = (x_1, \ldots, x_k)$. Suppose that there exists a surjective homomorphism $f : M \to R/J$ and $v \in M$ such that $f(v) = x_{k+1} + J$. Then $(Rv)^{\mathrm{cl}}_M \cap \ker f \subseteq (Jv)^{\mathrm{cl}}_M$.

Remark II.18. The axioms originally included the assumption that $0^{\mathrm{cl}}_R = 0$, but this is implied by the other axioms [Die15].

We need to assume that R is local for this generalization of [Die10, Lemma 2.11]:

Lemma II.19. *Let (R, m) be a local ring possessing a closure operation cl that is functorial, semi-residual, and faithful. If M is a finitely generated R-module such that $\alpha : R \to M$ is cl-phantom, then $\alpha(1) \notin mM$.*

We restate the main result of [Die10], removing the assumption that R is complete. The proof works without change.

Theorem II.20 [Die10, Theorems 3.16 and 4.2]. *Let R be a local domain. Then R has a Dietz closure if and only if it has a big Cohen-Macaulay module.*

Definition II.21 [HH94c, Discussion 5.15]. Let R be local and M an R-module. A *parameter module modification* of M is a map

$$M \to M' = \frac{M \oplus Rf_1 \oplus \ldots \oplus Rf_k}{R(u \oplus x_1 f_1 \oplus \ldots \oplus x_k f_k)},$$

where $x_1, \ldots, x_{k+1}$ is part of a system of parameters for R and $u_1, \ldots, u_k, u$ are elements of M such that

$$x_{k+1} u = x_1 u_1 + \ldots + x_k u_k.$$

Remark II.22. In his proof that a Dietz closure cl can be used to construct a big Cohen-Macaulay module [Die10], Dietz shows that a parameter module modification of a cl-phantom extension of R is also a cl-phantom extension of R, following the method used by Hochster and Huneke to prove that rings of characteristic $p > 0$ have big Cohen-Macaulay modules [Hoc75, HH94c]. One could replace generalized colon-capturing with any axiom that implies that given a cl-phantom extension $\alpha : R \to M$ and a parameter module modification $M \to M'$, the map $R \to M'$ is still a cl-phantom extension. However, we do not know of a good candidate to replace the generalized colon-capturing property.

We use a result on phantom extensions from [HH94c, Section 5], given in the notation of Notation II.14.

Lemma II.23. *[HH94c, Lemma 5.6a and c] Let*

$$0 \longrightarrow R \xrightarrow{\alpha} M \longrightarrow Q \longrightarrow 0$$

be an exact sequence. Letting $P_\bullet$ be a projective resolution for Q, we get a commutative diagram with vertical maps induced by the identity map on Q:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{\alpha} & M & \longrightarrow & Q & \longrightarrow & 0 \\ \uparrow & & \uparrow{\scriptstyle \phi} & & \uparrow & & \uparrow{\scriptstyle id} & & \\ P_2 & \longrightarrow & P_1 & \xrightarrow{d} & P_0 & \longrightarrow & Q & \longrightarrow & 0. \end{array}$$

By definition, α is cl-phantom if and only if

$$\phi \in im(Hom_R(P_0, R) \to Hom_R(P_1, R))^{cl}_{Hom_R(P_1,R)}.$$

1. *For each $c \in R$, the image of $c\phi$ is a coboundary in $H^1(Hom_R(P_\bullet, R))$ if and only if there is a map $\gamma : M \to R$ such that $\gamma\alpha = c(id_R)$.*

2. *Let S be an R-algebra, and $G_\bullet$ a projective resolution for $S \otimes_R Q$ that ends*

$$\ldots \to S \otimes P_1 \to S \otimes P_0 \to S \otimes Q \to 0.$$

The sequence

$$0 \longrightarrow R \xrightarrow{\alpha} M \longrightarrow Q \longrightarrow 0$$

remains exact upon tensoring with S *if and only if* $id_S \otimes_R \phi \in Hom_S(S \otimes_R P_1, S)$ *is a 1-cocycle in* $Hom_S(G_\bullet, S)$*, in which case* $id_S \otimes_R \phi$ *represents the extension over* S *given by the sequence*

$$0 \longrightarrow S \xrightarrow{id_S \otimes \alpha} S \otimes_R M \longrightarrow S \otimes_R Q \longrightarrow 0.$$

CHAPTER III

Module Closures

We prove a number of properties of module closures, most of which will be used in later chapters. The major result is Proposition III.8, which allows us to compare module closures. We end with a discussion of the properties of big Cohen-Macaulay module closures (as in Definition II.4), which are particularly relevant to the rest of this thesis.

Lemma III.1. *Let R be a ring possessing a closure operation cl. In the following, N, N', and $N_i \subseteq M_i$ are all R-submodules of the finitely generated R-module M.*

(a) Suppose that cl is functorial and semi-residual. Let $N' \subseteq N \subseteq M$. Then $u \in N_M^{cl}$ if and only if $u + N' \in (N/N')_{M/N'}^{cl}$.

(b) Suppose that cl is functorial, $\mathcal{I}$ is a finite set, $N = \bigoplus_{i \in \mathcal{I}} N_i$, and $M = \bigoplus_{i \in \mathcal{I}} M_i$. Then $N_M^{cl} = \bigoplus_{i \in \mathcal{I}} (N_i)_{M_i}^{cl}$.

(c) Let $\mathcal{I}$ be any set. If $N_i \subseteq M$ for all $i \in \mathcal{I}$, then $\left(\bigcap_{i \in \mathcal{I}} N_i\right)_M^{cl} \subseteq \bigcap_{i \in \mathcal{I}} (N_i)_{M_i}^{cl}$.

(d) Let $\mathcal{I}$ be any set. If N_i is cl-closed in M for all $i \in \mathcal{I}$, then $\bigcap_{i \in \mathcal{I}} N_i$ is cl-closed in M.

(e) $(N_1 + N_2)_M^{cl} = \left((N_1)_M^{cl} + (N_2)_M^{cl}\right)_M^{cl}$.

(f) Suppose that cl is functorial. Let $N \subseteq N' \subseteq M$. Then $N_{N'}^{cl} \subseteq N_M^{cl}$.

(g) Suppose that R is a domain, cl is functorial, $0_R^{cl} = 0$, and M is a torsion-free R-module. Then $0_M^{cl} = 0$.

(h) Suppose that (R, m) is local and cl is functorial, semi-residual, and faithful. Then $N_M^{cl} \subseteq N + mM$.

In particular, if cl is a Dietz closure on a local domain R, then it satisfies all of the properties given above.

Proof. Parts (a) to (e) are proved in [Die10, Lemma 1.2].

For part (f), let $f : N' \to M$ be the inclusion map. Then since cl is functorial,

$$N_{N'}^{\mathrm{cl}} = f(N_{N'}^{\mathrm{cl}}) \subseteq f(N)_M^{\mathrm{cl}} = N_M^{\mathrm{cl}}.$$

For part (g), notice that $M \hookrightarrow R^s$ for some $s > 0$. By part (f), $0_M^{\mathrm{cl}} \subseteq 0_{R^s}^{\mathrm{cl}}$. By part (b), $0_{R^s}^{\mathrm{cl}} = \bigoplus 0_R^{\mathrm{cl}} = 0$.

For part (h), we first prove that for F a finitely-generated free module, $(mF)_F^{\mathrm{cl}} = mF$. By part (a), this is equivalent to $0_{F/mF}^{\mathrm{cl}} = 0$. Let $u \in 0_{F/mF}^{\mathrm{cl}}$ be nonzero. Then there exists a map $\phi : F/mF \to R/m$ with $\phi(u) \neq 0$. Since cl is functorial, $\phi(u) \in 0_{R/m}^{\mathrm{cl}} = 0$ (since $m_R^{\mathrm{cl}} = m$), which is a contradiction. Hence $0_{F/mF}^{\mathrm{cl}} = 0$.

By part (a), it suffices to show that $0_M^{\mathrm{cl}} \subseteq mM$. Let

$$F_1 \longrightarrow F_0 \xrightarrow{\pi} M \longrightarrow 0$$

be part of a minimal free resolution of M. Then $\mathrm{im}(F_1) \subseteq mF_0$. This implies that $\mathrm{im}(F_1)_{F_0}^{\mathrm{cl}} \subseteq (mF_0)_{F_0}^{\mathrm{cl}} = mF_0$. By part (a), $0_M^{\mathrm{cl}} = \pi(\mathrm{im}(F_1)_{F_0}^{\mathrm{cl}})$. We have

$$0_M^{\mathrm{cl}} = \pi(\mathrm{im}(F_1)_{F_0}^{\mathrm{cl}}) \subseteq \pi(mF_0) = m\pi(F_0) = mM,$$

as desired. $\square$

Lemma III.2. *Let R be a ring and S an R-module or R-algebra. Then cl_S is functorial and semi-residual. Hence cl_S has properties (a)-(f) of Lemma III.1. Further, for $N \subseteq M$ finitely generated R-modules, cl_S satisfies*

$$I^{cl_S} N_M^{cl_S} \subseteq (IN)_M^{cl_S}$$

for all $I \subseteq R$. In particular, $yN_M^{cl_S} \subseteq (yN)_M^{cl_S}$ for all $y \in R$.

Remark III.3. If R is a domain, then this Lemma implies that cl_S is semi-prime as in [Eps12].

Proof. First we show that cl_S is functorial and semi-residual. Suppose that $N \subseteq M$ and W are finitely generated R-modules, and $f : M \to W$ is an R-module homomorphism. Let $u \in N_M^{\mathrm{cl}_S}$. We will show that $f(u) \in f(N)_W^{\mathrm{cl}_S}$. For every $s \in S$, $s \otimes u \in \mathrm{im}(S \otimes N \to S \otimes M)$. Applying $\mathrm{id}_S \otimes_R f$, we get $s \otimes f(u) \in \mathrm{im}(S \otimes f(N) \to S \otimes W)$ for every $s \in S$. So cl_S is functorial.

Suppose $N_M^{\mathrm{cl}_S} = N$. We will show that $0_{M/N}^{\mathrm{cl}_S} = 0$. Let $\bar{u} \in 0_{M/N}^{\mathrm{cl}_S}$. Then for every $s \in S$, $s \otimes \bar{u} = 0$ in $S \otimes M/N$. Since $S \otimes \_$ is right exact, $S \otimes M/N \cong (S \otimes M)/(S \otimes N)$. Thus $s \otimes u \in \mathrm{im}(S \otimes N \to S \otimes M)$. Since this holds for every $s \in S$, $u \in N_M^{\mathrm{cl}_S} = N$. Thus $\bar{u} = 0$ in M/N. So cl_S is semi-residual.

Now we prove that

$$I^{\mathrm{cl}_S} N_M^{\mathrm{cl}_S} \subseteq (IN)_M^{\mathrm{cl}_S}$$

for all $I \subseteq R$. Suppose that $u \in N_M^{\mathrm{cl}_S}$ and $y \in I^{\mathrm{cl}_S}$. Then for every $s \in S$,

$$s \otimes u \in \mathrm{im}(S \otimes N \to S \otimes M),$$

and $ys \in IS$. In particular, for every $s \in S$,

$$s \otimes yu = ys \otimes u = i_1(s_1 \otimes u) + i_2(s_2 \otimes u) + \ldots + i_n(s_n \otimes u)$$

for some $i_1, \ldots, i_n \in I$, $s_1, \ldots, s_n \in S$. But each $i_j(s_j \otimes u) = s_j \otimes i_j u \in \operatorname{im}(S \otimes IN \to S \otimes M)$. Hence $yu \in (IN)_M^{\mathrm{cl}_S}$.

The last statement follows, because

$$yN_M^{\mathrm{cl}_S} \subseteq (y)^{\mathrm{cl}_S} N_M^{\mathrm{cl}_S} \subseteq (yN)_M^{\mathrm{cl}_S}$$

by the previous statement. □

The following lemma allows us to generalize the idea of an algebra closure.

Lemma III.4. *Let $\mathcal{S}$ be a directed family of R-algebras. We can define a closure operation $cl_{\mathcal{S}}$ by $u \in N_M^{cl_{\mathcal{S}}}$ if for some $S \in \mathcal{S}$, $u \in N_M^{cl_S}$.*

Proof. To see that $N_M^{\mathrm{cl}_{\mathcal{S}}}$ is a submodule of M, let $u, v \in N_M^{\mathrm{cl}_{\mathcal{S}}}$. It is clear that for any $r \in R$, $ru \in N_M^{\mathrm{cl}_{\mathcal{S}}}$. To see that $u + v \in N_M^{\mathrm{cl}_{\mathcal{S}}}$, note that there is some $S, S' \in \mathcal{S}$ such that $u \in N_M^{\mathrm{cl}_S}$ and $v \in N_M^{\mathrm{cl}_{S'}}$. Since $\mathcal{S}$ is a directed family, there is some $T \in \mathcal{S}$ such that S, S' both map to T. We will have $1 \otimes u, 1 \otimes v \in \operatorname{im}(T \otimes N \to T \otimes M)$, so $1 \otimes (u + v) \in \operatorname{im}(T \otimes N \to T \otimes M)$. Hence $u + v \in N_M^{\mathrm{cl}_T} \subseteq N_M^{\mathrm{cl}_{\mathcal{S}}}$.

The extension and order-preserving properties of a closure operation are not difficult to prove. We prove the idempotence property. Let $u \in (N_M^{\mathrm{cl}_{\mathcal{S}}})_M^{\mathrm{cl}_{\mathcal{S}}}$. Then for some $S \in \mathcal{S}$, $1 \otimes u \in \operatorname{im}(S \otimes N_M^{\mathrm{cl}_{\mathcal{S}}} \to S \otimes M)$, say $1 \otimes u = \sum_{i=1}^n s_i \otimes u_i$ with the $u_i \in N_M^{\mathrm{cl}_{\mathcal{S}}}$. For each i, there is some $S_i \in \mathcal{S}$ such that $u_i \in N_M^{\mathrm{cl}_{S_i}}$. There is some $T \in \mathcal{S}$ such that each S_i maps to T. Hence $1 \otimes u \in \operatorname{im}(T \otimes N \to T \otimes M)$. □

Definition III.5. Let R be a complete local domain and let $\mathcal{B}$ be the family of big Cohen-Macaulay algebras of a ring R. By a result of Dietz [Die07], when R has characteristic $p > 0$, this is a directed family of algebras, and so we can define a closure operation $\mathrm{cl}_{\mathcal{B}}$ as in Definition III.4. More generally, the family of big Cohen-Macaulay algebras generates a closure operation as described in Definition III.6 below. In either case, we call this the *big Cohen-Macaulay algebras closure.*

When $\mathcal{S}$ is any family (not necessarily directed) of R-modules, it still generates a closure operation:

Definition III.6. Let $\mathcal{S}$ be a family of R-modules. For $N \subseteq M$ finitely-generated R-modules, we define $\mathrm{cl}_{\mathcal{S}}$ as follows:

1. Let $N_M^{\mathrm{cl}_1}$ be the submodule of M generated by the elements $u \in M$ such that $u \in N_M^{\mathrm{cl}_S}$ for some $S \in \mathcal{S}$.

2. Let $N_M^{\mathrm{cl}_{k+1}} = (N_M^{\mathrm{cl}_k})_M^{\mathrm{cl}_1}$.

3. Since R is Noetherian, the chain of $N_M^{\mathrm{cl}_k}$ will eventually stabilize. Set $N_M^{\mathrm{cl}_{\mathcal{S}}}$ equal to the stable value of this chain.

By [Eps12, Construction 3.1.5], $\mathrm{cl}_{\mathcal{S}}$ is a closure operation.

Proposition III.7. *Let cl be a closure operation that commutes with finite direct sums (in particular, it is enough to assume that cl is functorial). Suppose the map $R \to M$ that sends $1 \mapsto u$ is cl-phantom, as is the map $R \to N$ that sends $1 \mapsto v$. Then the map $f : R \to (M \oplus N)/(u \oplus -v)$ that sends $1 \mapsto (u, 0) = (0, v)$ is cl-phantom, too. Further, any phantom extension $R \to Q$ that factors through both M and N factors through $(M \oplus N)/(u \oplus -v)$ as well.*

Note: If f split, we would have $M = R \oplus M_0$, $N = R \oplus N_0$, and $(M \oplus N)/(u \oplus -v) = R \oplus (M_0 \oplus N_0)$.

Proof. The last statement is automatic from the definition of a push-out. The cokernel of f that sends $1 \mapsto (u, 0) = (0, v)$ is the direct sum of the cokernels of the maps $R \to M$ and $R \to N$, and the direct sum of free resolutions $P_\bullet$ and $P'_\bullet$, respectively, of these cokernels gives us a free resolution of the cokernel of f. If $\phi : P_1 \to R$ and $\phi' : P'_1 \to R$ are maps induced by the identity map on the cokernels, then the hypothesis tells us that

$$\phi \in (\operatorname{im}(\operatorname{Hom}(P_0, R) \to \operatorname{Hom}(P_1, R)))^{\mathrm{cl}}_{\operatorname{Hom}(P_1,R)}$$

and

$$\phi' \in (\operatorname{im}(\operatorname{Hom}(P_0', R) \to \operatorname{Hom}(P_1', R)))^{\mathrm{cl}}_{\operatorname{Hom}(P_1',R)}\,.$$

Since cl commutes with direct sums, we get

$$\phi \oplus \phi' \in (\operatorname{im}(\operatorname{Hom}(P_0 \oplus P_0', R) \to \operatorname{Hom}(P_1 \oplus P_1', R)))^{\mathrm{cl}}_{\operatorname{Hom}(P_1 \oplus P_1',R)}\,,$$

as desired. □

The following Proposition is a key component of the proofs of Proposition IV.14, Proposition IV.16, Theorem IV.20, and Lemma V.4, all of which compare closure operations.

Proposition III.8. *Let S and T be R-modules such that for each $t \in T$, there is a map $S \to T$ whose image contains t. Then $cl_S \subseteq cl_T$.*

Proof. Suppose that $N \subseteq M$ are finitely-generated R-modules, and that $u \in N_M^{\mathrm{cl}_S}$. We will show that $u \in N_M^{\mathrm{cl}_T}$. Since $u \in N_M^{\mathrm{cl}_S}$, for each $s \in S$, $s \otimes u \in \operatorname{im}(S \otimes N \to S \otimes M)$. Let $t \in T$. Then there is some map $f : S \to T$ whose image contains t, say $s' \mapsto t$. There is some element y of $S \otimes N$ that maps to $s' \otimes u$ in $S \otimes M$. The image $(f \otimes \mathrm{id})(y)$ of y in $T \otimes N$ maps to $t \otimes u$ in $T \otimes M$, by the commutativity of the following diagram:

$$\begin{array}{ccc} S \otimes N & \longrightarrow & S \otimes M \\ {\scriptstyle f\otimes\mathrm{id}}\big\downarrow & & {\scriptstyle f\otimes\mathrm{id}}\big\downarrow \\ T \otimes N & \longrightarrow & T \otimes M \end{array}$$

Hence $t \otimes u \in \operatorname{im}(T \otimes N \to T \otimes M)$ for every $t \in T$, which implies that $u \in N_M^{\mathrm{cl}_T}$. □

Remark III.9. The assumption in Proposition III.8 can be replaced by the assumption that some (possibly infinite) direct sum of copies of S maps onto T.

Proposition III.10. *Let S and T be R-modules. Then $cl_{S\oplus T} = cl_S \cap cl_T$.*

Proof. Suppose that $N \subseteq M$ are finitely-generated R-modules, and $u \in N_M^{\mathrm{cl}_{S\oplus T}}$. Then for each $(s,t) \in S \oplus T$,

$$(s,t)\otimes u \in \mathrm{im}((S\oplus T)\otimes N \to (S\oplus T)\otimes M) = \mathrm{im}(S\otimes N \to S\otimes M)\oplus \mathrm{im}(T\otimes N \to T\otimes M).$$

So $s \otimes u$ is in the first image, and $t \otimes u$ is in the second. Thus $u \in N_M^{\mathrm{cl}_S} \cap N_M^{\mathrm{cl}_T}$. If $u \in N_M^{\mathrm{cl}_S} \cap N_M^{\mathrm{cl}_T}$, then for each $s \in S$, $s\otimes u \in \mathrm{im}(S\otimes N \to S\otimes M)$ and for each $t \in T$, $u \in \mathrm{im}(T \otimes N \to T \otimes M)$. Hence $(s,t) \otimes u \in \mathrm{im}((S \oplus T) \otimes N \to (S \oplus T) \otimes M)$. $\square$

Proposition III.12 gives an additional property of module closures.

Definition III.11 [Eps12]. A closure operation cl is hereditary if given R-modules $N \subseteq N' \subseteq M$, $N_M^{\mathrm{cl}} \cap N' = N_{N'}^{\mathrm{cl}}$.

Proposition III.12. *Let S be a flat R-module and $cl = cl_S$. Then given finitely-generated R-modules $N, W \subseteq M$,*

$$N_M^{cl} \cap W = N_{N+W}^{cl} \cap W.$$

In particular, cl is hereditary.

Proof. The right side is contained in the left side by part (f) of Lemma III.1. To get the other direction, suppose that $m \in N_M^{\mathrm{cl}} \cap W$. Then for each $s \in S$,

$$s \otimes m \in \mathrm{im}(S \otimes_R N \to S \otimes_R M),$$

say $s \otimes m = \sum_i s_i \otimes n_i$, with each $s_i \in S$ and $n_i \in N$.

$$\begin{array}{ccc} S \otimes N & \longrightarrow & S \otimes (N+W) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\ S \otimes N & \longrightarrow & S \otimes M \end{array}$$

The element $s_i \otimes n_i$ maps to y_i in $S \otimes (N + W)$. Set $y = \sum y_i$. Since $m \in N$, we can view $s \otimes m$ as an element of $S \otimes (N + W)$. If $y = s \otimes m$ in $S \otimes (N + W)$, then we are done. If not, then by the commutativity of the diagram, $y - (s \otimes m)$ maps to 0 in $S \otimes M$. But since S is flat, the kernel of $S \otimes (N + W) \to S \otimes M$ is 0. Hence $y = s \otimes m$ in $S \otimes (N + W)$, as desired.

To get the final statement, note that if $N \subseteq N' \subseteq M$, the result gives $N_M^{\mathrm{cl}} \cap N' = N_{N'}^{\mathrm{cl}}$. □

3.1 Properties of Big Cohen-Macaulay Module Closures

We give several useful properties of big Cohen-Macaulay module closures that will be used later on.

Definition III.13. Let *cl* be a closure operation on a ring R.

1. We say that cl satisfies *colon-capturing* if for every partial system of parameters $x_1, \ldots, x_{k+1}$ on R,

$$(x_1, \ldots, x_k) : x_{k+1} \subseteq (x_1, \ldots, x_k)^{\mathrm{cl}}.$$

2. We say that cl satisfies *strong colon-capturing, version A*, if for every partial system of parameters $x_1, \ldots, x_k$ on R,

$$(x_1^t, x_2, \ldots, x_k)^{\mathrm{cl}} : x_1^a \subseteq (x_1^{t-a}, x_2, \ldots, x_k)^{\mathrm{cl}}$$

   for all $a < t$.

3. We say that cl satisfies *strong colon-capturing, version B*, if for every partial system of parameters $x_1, \ldots, x_{k+1}$ on R,

$$(x_1, \ldots, x_k)^{\mathrm{cl}} : x_{k+1} \subseteq (x_1, \ldots, x_k)^{\mathrm{cl}}.$$

   This is a stronger condition than colon-capturing.

Proposition III.14. *Let B be a big Cohen-Macaulay module over a local domain R. Then the module closure cl_B satisfies strong colon-capturing, version A.*

Proof. Let $x_1, \ldots, x_k$ be a partial system of parameters on R. Suppose that $a < t$, and that $u \in (x_1^t, x_2, \ldots, x_k)^{\mathrm{cl}} : x_1^a$. In other words, for each $b \in B$,

$$ux_1^a b \in (x_1^t, \ldots, x_k)B,$$

say $ux_1^a b = x_1^t b_1 + x_2 b_2 + \ldots + x_k b_k$. Then $x_1^a(ub - x_1^{t-a} b_1) \in (x_2, \ldots, x_k)B$. Since B is a big Cohen-Macaulay module, this implies that $ub - x_1^{t-a} b_1 \in (x_2, \ldots, x_k)B$. Hence $ub \in (x_1^{t-a}, x_2, \ldots, x_k)B$. Since this holds for each $b \in B$, $u \in (x_1^{t-a}, x_2, \ldots, x_k)^{\mathrm{cl}_B}$. $\square$

Proposition III.15. *Let B be a big Cohen-Macaulay module over R and $x_1, \ldots, x_{k+1}$ a partial system of parameters on R. Then $(x_1, \ldots, x_k)^{cl_B} : x_{k+1} \subseteq (x_1, \ldots, x_k)^{cl_B}$, i.e., cl_B satisfies strong colon-capturing, version B. In particular, cl_B satisfies colon-capturing.*

Proof. Suppose that $v \in R$ such that $vx_{k+1} \in (x_1, \ldots, x_k)^{\mathrm{cl}_B}$. Then for each $b \in B$, $vx_{k+1}b \in (x_1, \ldots, x_k)B$. Equivalently, $x_{k+1}(vb) \in (x_1, \ldots, x_k)B$. Since $x_1, \ldots, x_{k+1}$ form part of a system of parameters on R, they form a regular sequence on B. Hence $vb \in (x_1, \ldots, x_k)B$. As we proved this for an arbitrary $b \in B$, $v \in (x_1, \ldots, x_k)^{\mathrm{cl}_B}$, as desired. $\square$

CHAPTER IV

Smallest Closures

In this chapter, we show that many properties of closure operations extend to intersections of closure operations, so that there is a smallest closure satisfying these properties under certain conditions. We describe the smallest big Cohen-Macaulay module closure on a general local domain, and give a specific module (the S_2-ification of R) that gives this closure in dimension 2. Finally, we construct the smallest module closure containing a given closure.

4.1 Intersection Stable Properties

Given a set $\{\mathrm{cl}_\lambda\}_{\lambda \in \Lambda}$ of closure operations, their intersection $\bigcap_{\lambda \in \lambda} \mathrm{cl}_\lambda$ is also a closure operation [Eps12, Construction 3.1.3].

Definition IV.1. Given a property P of a closure operation, we call P *intersection stable* if whenever cl_λ satisfies P for every $\lambda \in \Lambda$, $\bigcap_{\lambda \in \Lambda} \mathrm{cl}_\lambda$ also satisfies P.

The following lemma is immediate:

Lemma IV.2. *Suppose that P is an intersection stable property of a closure operation and that R has a closure operation satisfying P. Then R has a smallest closure operation satisfying P.*

Proposition IV.3. *1. Functoriality is intersection stable.*

2. *Semi-residuality is intersection stable on sets of closures that are functorial.*

3. *When R is local, faithfulness and generalized colon-capturing are intersection stable.*

Proof. 1. Let $\{\mathrm{cl}_\lambda\}_{\lambda \in \Lambda}$ be a family of closure operations, and

$$\mathrm{cl} = \bigcap_{\lambda \in \Lambda} \mathrm{cl}_\lambda.$$

If each cl_λ is functorial, $f : M \to W$ is an R-module map, and $N \subseteq M$ is a submodule, then $f(N_M^{\mathrm{cl}}) \subseteq f(N_M^{\mathrm{cl}_\lambda}) \subseteq f(N)_W^{\mathrm{cl}_\lambda}$ for each λ. Thus $f(N_M^{\mathrm{cl}}) \subseteq \bigcap_\lambda f(N)_W^{\mathrm{cl}_\lambda} = f(N)_W^{\mathrm{cl}}$, as desired. $\square$

2. Suppose that $N_M^{\mathrm{cl}} = N$, and that for each λ, cl_λ is functorial and semi-residual. We will show that $0_{M/N}^{\mathrm{cl}} = 0$. Suppose that $\bar{u} \in 0_{M/N}^{\mathrm{cl}}$. Then for each λ, $\bar{u} \in 0_{M/N}^{\mathrm{cl}_\lambda}$. By Lemma III.1, $u \in N_M^{\mathrm{cl}_\lambda}$ if and only if $\bar{u} \in 0_{M/N}^{\mathrm{cl}_\lambda}$. Hence $u \in N_M^{\mathrm{cl}_\lambda}$ for each λ, which implies that $u \in N_M^{\mathrm{cl}} = N$. Thus $\bar{u} = 0$, and so cl is semi-residual. $\square$

3. It is clear that faithfulness is intersection stable.

Suppose that cl_λ satisfies generalized colon-capturing for each λ and that $x_1, \ldots, x_{k+1}$ is part of a system of parameters for R, $J = (x_1, \ldots, x_k)$, and $f : M \twoheadrightarrow R/J$ such that there is some $v \in M$ with $f(v) = x_{k+1} + J$. We need to show that $(Rv)_M^{\mathrm{cl}} \cap \ker(f) \subseteq (Jv)_M^{\mathrm{cl}}$. Since $(Rv)_M^{\mathrm{cl}} \cap \ker(f) \subseteq (Rv)_M^{\mathrm{cl}_\lambda} \cap \ker(f) \subseteq (Jv)_M^{\mathrm{cl}_\lambda}$ for each λ, cl satisfies generalized colon-capturing. $\square$

$\square$

Corollary IV.4. *If a local domain R has a Dietz closure, then it has a smallest Dietz closure.*

In the case of a Cohen-Macaulay ring, the smallest Dietz closure is the trivial closure. However, we do not know what it looks like in more generality.

Remark IV.5. Colon-capturing is a useful property for a closure operation to have, but it is not enough on its own to guarantee that a closure operation can be used to construct big Cohen-Macaulay modules. For example, the closure $N_M^{\mathrm{cl}} = M$ for all $N \subseteq M$ finitely-generated R-modules captures colons, but is too large to be useful.

Lemma IV.6. *Colon-capturing is an intersection stable property.*

Proof. This is immediate from Definition III.13. □

Lemma IV.7. *Strong colon-capturing, version A, as in Definition III.13 is intersection stable.*

Proof. To see this, notice that if $x_1, \ldots, x_k$, t, and a are as in the definition of strong colon-capturing, version A, then

$$(x_1^t, x_2, \ldots, x_k)^{\mathrm{cl}} :_R x_1^a \subseteq (x_1^t, x_2, \ldots, x_k)^{\mathrm{cl}_\lambda} :_R x_1^a \subseteq (x_1^{t-a}, x_2, \ldots, x_k)^{\mathrm{cl}_\lambda}$$

for each λ. Hence $(x_1^t, x_2, \ldots, x_k)^{\mathrm{cl}} :_R x_1^a \subseteq (x_1^{t-a}, x_2, \ldots, x_k)^{\mathrm{cl}}$. □

Remark IV.8. A similar proof works for strong colon-capturing, version B.

Definition IV.9. Given a class of rings, a *closure operation on the class* consists of an assignment of a closure operation to each ring.

The property defined below is one of the important properties of tight closure, particularly when combined with colon-capturing. If cl is defined on a class of rings, then we would like to find the smallest closure operation as above (if any such exist) that satisfies colon-capturing and also satisfies the following property:

Definition IV.10. A closure operation cl on a class of rings with a class of ring maps between them is *persistent for change of rings* if whenever $R \to S$ is a map in the class, and $N \subseteq M$ are finitely generated R-modules, then $\mathrm{im}(S \otimes_R N_M^{\mathrm{cl}} \to S \otimes_R M) \subseteq (\mathrm{im}(S \otimes_R N \to S \otimes_R M))_{S \otimes_R M}^{\mathrm{cl}}$.

Tight closure is persistent for change of rings and satisfies colon-capturing on classes of complete local domains with ring homomorphisms (or local ring homomorphisms) between them [HH94a].

Remark IV.11. The trivial closure on any class of rings and ring maps between them is persistent for change of rings, but is colon-capturing if and only if R is Cohen-Macaulay. The improper closure satisfies both properties on any class of rings and ring maps between them.

Proposition IV.12. *Persistence for change of rings is an intersection stable property.*

Proof. Suppose that cl_λ are closure operations on a class of rings with a class of ring maps between them that are persistent for change of rings. Let $\mathrm{cl} = \bigcap_{\lambda \in \Lambda} \mathrm{cl}_\lambda$. We will show that cl is persistent for change of rings. Let $R \to S$ be a map in the class, and suppose that $u \in N_M^{\mathrm{cl}}$. Our goal is to show that $1 \otimes u \in (\mathrm{im}(S \otimes_R N \to S \otimes_R M))^{\mathrm{cl}}_{S \otimes_R M}$. By definition of cl, $u \in N_M^{\mathrm{cl}_\lambda}$ for every $\lambda \in \Lambda$. Since each cl_λ is persistent for change of rings, this implies that

$$1 \otimes u \in (\mathrm{im}(S \otimes_R N \to S \otimes_R M))^{\mathrm{cl}_\lambda}_{S \otimes_R M}$$

for every $\lambda \in \Lambda$. Hence $1 \otimes u \in (\mathrm{im}(S \otimes_R N \to S \otimes_R M))^{\mathrm{cl}}_{S \otimes_R M}$. □

Corollary IV.13. *The category of all complete local domains (with local maps or with all ring maps between them) has a smallest closure operation that is colon-capturing and persistent for change of rings.*

Proof. This follows immediately from Lemma IV.6, Remark IV.11, and Proposition IV.12. □

Not much is known about the smallest persistent, colon-capturing closure operation on a class of rings, except when all of the rings in the class are Cohen-Macaulay, in which case it is the trivial closure.

4.2 Smallest Big Cohen-Macaulay Module Closure

Given a big Cohen-Macaulay module B over a local domain R, we get a module closure cl_B. In [Die10], Dietz proves that cl_B is a Dietz closure. We can define a new closure operation by intersecting all closures cl_B for which B is a big Cohen-Macaulay module. Since the property of being a Dietz closure is intersection stable, this is also a Dietz closure. As we prove below, it is also a big Cohen-Macaulay module closure.

Proposition IV.14. *Let R be a local domain, and let B be a big Cohen-Macaulay module constructed using the method of [Die10]. If B' is any big Cohen-Macaulay module over R, then $cl_B \subseteq cl_{B'}$. As a result, the module closure cl_B is the smallest big Cohen-Macaulay module closure on R.*

Proof. Let B be a big Cohen-Macaulay module constructed as described in [Die10], and B' an arbitrary big Cohen-Macaulay module. Then for each map $R \to B'$, we can construct a map $B \to B'$ that takes the image of 1 in B to the image of 1 in B' via the given map $R \to B'$. To get this map, we start with the map $R \to B'$. Let $M_0 = R$, $M_1, \ldots, M_t, \ldots$ be as in [Die10]. If we already have maps from $M_0 = R, M_1, \ldots,$ and M_t to B', we extend the map to M_{t+1} as follows:

$$M_{t+1} = (M \oplus Rf_1 \oplus \ldots \oplus Rf_k)/(u \oplus x_1 f_1 \oplus \ldots x_k f_k)$$

for some $u \in M_t$ and partial system of parameters $x_1, \ldots, x_k$ for R such that

$$x_{k+1} u = x_1 m_1 + \ldots + x_k m_k$$

is a bad relation in M_t. Since B' is a big Cohen-Macaulay module, the image of u in B' under the map already constructed is in $(x_1, \ldots, x_k)B'$, say $u = x_1b_1 + \ldots + x_kb_k$ with $b_1, \ldots, b_k \in B'$. We extend our map $M_t \to B'$ to a map from M_{t+1} to B' by sending $f_i \mapsto b_i$. Take the direct limit of this system of maps $M_t \to B'$ as $t \to \infty$ to get the desired map $B \to B'$. Since we can start with any map $R \to B'$, every element of B' is in the image of a map constructed this way. Hence Proposition III.8 implies that $\mathrm{cl}_B \subseteq \mathrm{cl}_{B'}$. $\square$

In certain rings of dimension 2, we know more about the smallest big Cohen-Macaulay module closure.

Definition IV.15 [HH94b]. For R a local domain, the S_2-*ification* of R is the unique smallest extension of R in its fraction field that satisfies Serre's condition S_2, if such a ring exists. When it exists, it can be constructed by adding to R all elements $f \in \mathrm{Frac}(R)$ such that some height 2 ideal of R multiplies f into R.

Proposition IV.16. *Let R be a local domain of dimension 2 that has an S_2-ification S. Then the module closure cl_S is the smallest big Cohen-Macaulay module closure on R.*

Proof. Let B be a big Cohen-Macaulay module constructed by the method of [Die10], so that cl_B is the smallest big Cohen-Macaulay module closure on R. Since S is Cohen-Macaulay when R has dimension 2, we know that $\mathrm{cl}_B \subseteq \mathrm{cl}_S$. By Proposition III.8, it is enough to show that for any map $R \to B$, $1 \mapsto u$, we have a map $S \to B$ whose image contains u. To do this, we need to extend the map from R to S by defining it on elements $f \in \mathrm{Frac}(R)$ such that some height 2 ideal of R multiplies f into R. Let f be such an element. Since $\dim(R) = 2$, there is some system of parameters x, y for R such that $xf, yf \in R$. Then the map is already defined on

xf, yf, say $xf \mapsto v$, $yf \mapsto w$. The element xyf must map to yv, but also must map to xw, so $yv = xw$. Since x, y is a regular sequence on B, $v = xv_0$ and $w = yw_0$ for some $v_0, w_0 \in B$. Then $xyv_0 = yv = xw$, so $w = yv_0$. Hence $yv_0 = yw_0$, which implies that $v_0 = w_0$. Thus $f \mapsto v_0$ is a well-defined extension of the map $R \to B$. Further, 1_S maps to u, so this is the map we need to see that $\mathrm{cl}_S \subseteq \mathrm{cl}_B$. $\square$

Example IV.17. Let $R = k[[x^4, x^3y, xy^3, y^4]]$. The S_2-ification S of R must contain x^2y^2, since $x^4(x^2y^2) = (x^3y)^2 \in R$ and $y^4(x^2y^2) = (xy^3)^2 \in R$. In fact, S is the subring $k[[x^4, x^3y, x^2y^2, xy^3, y^4]]$ of $k[[x, y]]$. Since $(x^3y)^2 = x^4(x^2y^2)$ in S, $(x^3y)^2 \in (x^4)_R^{\mathrm{cl}_S}$. Similarly, $(xy^3)^2 \in (y^4)_R^{\mathrm{cl}_S}$. Hence $(x^3y)^2 \in (x^4)_R^{\mathrm{cl}}$ and $(xy^3)^2 \in (y^4)_R^{\mathrm{cl}}$ for every Dietz closure cl on R.

4.3 Smallest module closure containing another closure

Given a closure operation cl on R, we construct the smallest module closure containing cl. This will be used in Chapter V to prove that every Dietz closure is contained in a big Cohen-Macaulay module closure. To construct the smallest module closure containing a given closure, we use a second type of module modification.

Definition IV.18. Let cl be a closure operation on R, $G \subseteq R^s$ a submodule of a finitely-generated free R-module generated by

$$e_1 = (e_{11}, \ldots, e_{1s}), \ldots, e_k = (e_{k1}, \ldots, e_{ks}),$$

and let $v = (v_1, \ldots, v_s) \in G_{R^s}^{\mathrm{cl}} - G$. A *containment module modification* of an R-module M relative to an element $x \in M$ is a map

$$M \to M' = \frac{M \oplus Rf_1 \oplus \ldots \oplus Rf_k}{R(v_1 x \oplus e_{11} f_1 \oplus \ldots \oplus e_{k1} f_k, \ldots, v_s x \oplus e_{1s} f_1 \oplus \ldots \oplus e_{ks} f_k)}.$$

Proposition IV.19. *Let R be a ring, W an R-module, and cl a closure operation on R that is functorial and semi-residual. Then there is an R-module S with a map*

$\phi : W \to S$ such that $cl \subseteq cl_S$, and for any R-module T such that $cl \subseteq cl_T$ and any map $\psi : W \to T$, we have a map $\gamma : S \to T$ such that $\psi = \gamma \circ \phi$.

Proof. To create such an S, we apply containment module modifications to finitely-generated submodules of W. First we show that we have a direct limit system of containment module modifications. Given a finite set of modules $G_1, \ldots, G_t$ with $G_i \subseteq R^{s_i}$, and for each i, a finite set of elements $v_{i1}, v_{i2}, \ldots, v_{i\ell_i} \in (G_i)^{\mathrm{cl}}_{R^{s_i}} - G_i$, we can apply finitely many containment module modifications to a finitely-generated submodule $W_0 \subseteq W$ to get a module W_1 such that for each $1 \leq i \leq t$ and $1 \leq j \leq \ell_i$,

$$\mathrm{im}(v_{ij} \otimes W_0 \to R^{s_i} \otimes W_1) \subseteq \mathrm{im}(G_i \otimes W_1 \to R^{s_i} \otimes W_1).$$

Then we apply finitely many containment module modifications to W_1, forcing

$$\mathrm{im}(v_{ij} \otimes W_1 \to R^{s_i} \otimes W_2) \subseteq \mathrm{im}(G_i \otimes W_2 \to R^{s_i} \otimes W_2)$$

for all i, j. Repeating this process infinitely many times, we get a module W_∞ that is the direct limit of the W_r and such that

$$\mathrm{im}(v_{ij} \otimes W_\infty \to R^s \otimes W_\infty) \subseteq \mathrm{im}(G_i \otimes W_\infty \to R^s \otimes W_\infty)$$

for all i, j. We have a map $W_0 \to W_\infty$ since each containment module modification comes with a map from W_0.

Consider all finite sets $\mathcal{G} = \{G_1, \ldots, G_t, v_{11}, v_{12}, \ldots, v_{1\ell_1}, v_{21}, v_{22}, \ldots, v_{t\ell_t}\}$ with $G_i \subseteq R^{s_i}$ and finitely many elements $v_{i1}, \ldots, v_{i\ell_i} \in (G_i)^{\mathrm{cl}}_{R^s} - G_i$ for each $1 \leq i \leq t$, and also all finitely-generated submodules W_0 of W. Suppose that $\mathcal{G} \subseteq \mathcal{G}'$ are two such sets, that $W_0 \subseteq W_0'$ are finitely-generated submodules of W, and that W_∞ and W_∞' are corresponding direct limit modules constructed from W_0 using $\mathcal{G}$ and from W_0' from $\mathcal{G}'$, respectively, using the process described above. We build a map $W_\infty \to W_\infty'$, starting with the map $W_0 \subseteq W_0' \to W_\infty'$.

It suffices to demonstrate that the map can be extended to a single containment module modification. Let P be an intermediate module in the direct limit system of W_∞ with a map $P \to W'_\infty$, $v = v_{ij} \in \mathcal{G}$ for some i, j, $e_1, \dots, e_k$ be the generators of $G = G_i$, and $x \in Q$ as in Definition IV.18. We need to specify the images of $f_1, \dots, f_k$ in W'_∞. Since $v \otimes W'_\infty \subseteq G \otimes W'_\infty$, $vx = e_1 w_1 + e_2 w_2 + \dots + e_k w_k$ for some $w_1, \dots, w_k \in W'_\infty$. Then the map that sends $f_i \mapsto w_i$ is a well-defined extension of the map $P \to W'_\infty$. Hence we have a map $W_\infty \to W'_\infty$ for any $\mathcal{G} \subseteq \mathcal{G}'$.

The W_∞ form a partially ordered set via $W_\infty \leq W'_\infty$ if the corresponding finite sets satisfy $\mathcal{G} \subseteq \mathcal{G}'$. This is a directed set, using the maps $W_\infty \to W'_\infty$ we constructed above. Let S be the direct limit. By the set-up above, we have a well-defined map $\phi : W \to S$. We are now done proving that for submodules G of finitely-generated free R-modules R^s, $G^{\mathrm{cl}}_{R^s} \subseteq G^{\mathrm{cl}_S}_{R^s}$.

Suppose that $N \subseteq M$ are arbitrary finitely-generated R-modules. We will show that $N^{\mathrm{cl}}_M \subseteq N^{\mathrm{cl}_S}_M$. There is some s for which $M/N \cong R^s/G$, where G is a submodule of R^s. Let $u \in N^{\mathrm{cl}}_M$. By Lemma III.1, part (a), $\bar{u} \in 0^{\mathrm{cl}}_{M/N} \cong 0^{\mathrm{cl}}_{R^s/G}$. Applying the Lemma again, any lift v of $\mathrm{im}(\bar{u})$ to R^s is in $G^{\mathrm{cl}}_{R^s}$, which is contained in $G^{\mathrm{cl}_S}_{R^s}$ by the previous paragraph. Applying the Lemma twice more, we get $\bar{u} \in 0^{\mathrm{cl}_S}_{M/N}$, which implies that $u \in N^{\mathrm{cl}_S}_M$.

Now suppose that T is an R-module such that $\mathrm{cl} \subseteq \mathrm{cl}_T$, and we have a map $\psi : W \to T$. Let $\phi : W \to S$ be as above. For any intermediate module P in the direct limit system of S, let ϕ_P be the corresponding map $W \to P$. Suppose that we have a map $\gamma_P : P \to T$ such that $\psi = \gamma_P \circ \phi_P$. We demonstrate how to extend the map to a map $\gamma_{P'} : P' \to T$ such that $\psi = \gamma_{P'} \circ \phi_{P'}$ when P' is a containment module modification of P. We have:

$$P \to P' = \frac{P \oplus Rf_1 \oplus \dots \oplus Rf_k}{R(v_1 x \oplus e_{11} f_1 \oplus \dots \oplus e_{k1} f_k, \dots, v_s x \oplus e_{1s} f_1 \oplus \dots \oplus e_{ks} f_k)},$$

where $x \in P$, and $v, e_1, \ldots, e_k$ are as in Definition IV.18. We need to specify the images of the f_i. Since $\mathrm{cl} \subseteq \mathrm{cl}_T$, $vx \in (e_1, \ldots, e_k)T$, say $vx = e_1 t_1 + \ldots + e_k t_k$. Then sending $f_i \mapsto t_i$ gives us a well-defined extension of γ_P such that $\psi = \gamma_{P'} \circ \phi_{P'}$. Since S is a direct limit of such containment module modifications, we get a map $\gamma : S \to T$ such that $\psi = \gamma \circ \phi$. □

Theorem IV.20. *Let R be a ring and cl a closure operation on R that is functorial and semi-residual. Then if we set $W = R$ and construct a module S as in Proposition IV.19, cl_S is the smallest module closure containing cl, i.e., if T is any R-module such that $\mathrm{cl} \subseteq \mathrm{cl}_T$, we have $\mathrm{cl}_S \subseteq \mathrm{cl}_T$. In particular, if cl is a module closure, then $\mathrm{cl} = \mathrm{cl}_S$ (conversely, if cl is not a module closure, then $\mathrm{cl} \subsetneq \mathrm{cl}_S$).*

Proof. By Proposition IV.19, for every R-module map $R \to T$, we have a map $S \to T$ that agrees with the original map on the image of R. So for every element $t \in T$, we have a map $S \to T$ whose image contains t. By Proposition III.8, this implies that $\mathrm{cl}_S \subseteq \mathrm{cl}_T$. □

CHAPTER V

A Connection between Dietz Closures and Singularities

In this section, we show that for any local domain R that has a Dietz closure, R is regular if and only if all Dietz closures on R are equal to the trivial closure. First, we prove a result on the relationship between general Dietz closures and big Cohen-Macaulay module closures.

Theorem V.1. *Let cl be a Dietz closure on a local domain (R, m). Then cl is contained in cl_B for some big Cohen-Macaulay module B.*

Proof. Let cl be a Dietz closure on R. To construct B, we use both parameter module modifications and containment module modifications. First, we construct a big Cohen-Macaulay module S_1 using parameter module modifications as in [Die10]. We apply containment module modifications to S_1 as in Proposition IV.19 to get a module S_2 such that $\text{cl} \subseteq \text{cl}_{S_2}$ and a map $S_1 \to S_2$, and then we use parameter module modifications to construct an R-module S_3 such that every system of parameters on R is a regular sequence on S_3 and a map $S_2 \to S_3$. We repeat these two constructions countably many times, getting maps

$$R = S_0 \to S_1 \to S_2 \to S_3 \to \dots$$

The direct limit B is an R-module such that $\text{cl} \subseteq \text{cl}_B$ and every system of parameters on R is a regular sequence on B. We need to show that $\mathrm{im}(1) \notin mB$ when we apply the map $R \to B$ that is the direct limit of the maps $R \to S_i$.

We follow the proof of [HH95, Proposition 3.7]. If $\mathrm{im}(1) \in mB$, then there is a finitely-generated R-module P with $1 \in mP$ such that P maps to B.

Claim: There is an R-module W constructed from R by taking finitely many module modifications (of either or both types) such that the map $P \to B$ passes through W.

Proof of Claim. Given any finitely-generated R-module P with a map $P \to B$, there is some $i > 0$ for which $\mathrm{im}(P) \subseteq S_i$. Then there is also a finite sequence of containment module modifications and parameter module modifications of S_{i-1} giving a module W_{i-1} such that the map $P \to B$ passes through W_{i-1}. We use induction on the value of i. If $i = 1$, then the result is immediate. Suppose the result holds for $i = 1, 2, \ldots, k-1$, and let S be a module gotten from S_{k-1} by applying a finite sequence of module modifications, such that $\mathrm{im}(P) \subseteq S$. By induction, there is an R-module W_{k-1} that is constructed from R by taking finitely many module modifications, and such that $\mathrm{im}(P \cap S_{k-1}) \subseteq W_{k-1}$. Any element of P not in S_{k-1} must come from one of the module modifications applied to S_{i-1} to get S. So when we apply the same sequence of module modifications to W_{k-1}, we get an R-module W_k that is constructed by applying finitely many module modifications to R and such that $\mathrm{im}(P) \subseteq W_k$. □

Further, if we apply any finite sequence of module modifications to R to get a module W, we have a map $W \to B$, constructed in the same way as the maps $W_\infty \to W'_\infty$ in the proof of Proposition IV.19 and the maps $M_t \to B'$ in the proof of Proposition IV.14. Therefore, $\mathrm{im}(1) \in mB$ if and only if $\mathrm{im}(1) \in mW$, where W

is an R-module obtained by applying finitely many module modifications to R. We will show that we cannot have $\mathrm{im}(1) \in mW$. To do this, we show that if we have a cl-phantom map $R \to M$, and we apply a single module modification to M to get M', the resulting map $R \to M'$ is cl-phantom. Hence $\mathrm{im}(1) \notin mM'$.

Assume $R \xrightarrow{\alpha} M$ is a phantom extension of R. If we apply a parameter module modification to M, we know that the resulting map $\alpha' : R \to M'$ is phantom by [Die10]. In the following Lemma, we show that $\alpha' : R \to M'$ is phantom when we apply a containment module modification to M. Hence by Lemma II.19, $\alpha'(1) \notin mM'$. This guarantees that in the limit, $mB \neq B$. □

Lemma V.2. *Suppose that (R, m) is a local domain and cl is a Dietz closure on R that is functorial and semi-residual, and such that $0_R^{cl} = 0$. Suppose that $\alpha : R \to M$ is a phantom extension, and let M' be a containment module modification of M. Then $\alpha' : R \to M'$ is a phantom extension.*

Proof. Let $v = (v_1, \ldots, v_s) \in G_{R^s}^{\mathrm{cl}} - G$ for some nonzero submodule $G \subseteq R^s$ (as $0_{R^s}^{\mathrm{cl}} = 0$ by assumption), and let $x \in M$. Let u be the image of 1 in M. Taking a single module modification, we get

$$M' = \frac{M \oplus Rf_1 \oplus \ldots \oplus Rf_k}{R\left(v_1 x \oplus e_{11} f_1 \oplus \ldots \oplus e_{k1} f_k, \ldots, v_s x \oplus e_{1s} f_1 \oplus \ldots \oplus e_{ks} f_k\right)}.$$

First, we need to show that the composite map $\alpha' : R \to M \to M'$ is injective. Let $F = \mathrm{Frac}(R)$. Then $F \to F \otimes_R M$ is injective, and it suffices to show that $F \to F \otimes M'$ is injective, i.e. that it is nonzero (if $R \to M'$ were not injective, applying $F\otimes$ would preserve this). We claim that $v \in \mathrm{im}(F \otimes G \to F^s)$. To see that this is true, notice that by Lemma III.1, $0_{R^s/G}^{\mathrm{cl}}$ is contained in the torsion part of R^s/G. Hence $v \in G_{R^s}^{\mathrm{cl}}$ implies that $\bar{v}$ is a torsion element of R^s/G. Hence $\bar{v} = 0$ in $F^s/(F \otimes G)$, which implies that $v \in \mathrm{im}(F \otimes G \to F^s)$. Then the relations we kill to get $F \otimes M'$ already hold in $F \otimes M$, so there is a retraction $F \otimes M' \to F \otimes M$. This implies that $F \otimes M \to F \otimes M'$ is injective, and so $F \to F \otimes M'$ is injective, as desired.

Remark V.3. In the special case $s = 1$, we can show that the map $M \to M'$ sending each element $y \mapsto y \oplus 0 \oplus \ldots \oplus 0$ is injective. If $y \mapsto 0$, then $y \oplus 0 \oplus \ldots \oplus 0 = r(vx \oplus r_1 f_1 \oplus \ldots \oplus r_k f_k)$ in $M \oplus Rf_1 \oplus \ldots \oplus Rf_k$, for some $r \in R$. We may assume without loss of generality that some r_i is nonzero, say r_1. Then $rr_1 f_1 = 0$, so $rr_1 = 0$. Since R is a domain, $r = 0$. So $y = rvx = 0$.

Following Notation II.14 and [Die10, Discussion 2.4], pick a generating set $w_1, \ldots, w_n$ for M such that $w_1 = u$ and $w_n = x$. Then the images of $w_2, \ldots, w_n$ form a generating set for Q. Let

$$R^m \xrightarrow{\nu} R^{n-1} \xrightarrow{\mu} Q \longrightarrow 0$$

be a free presentation of Q, where μ sends the generators of R^{n-1} to $w_2, \ldots, w_n$, respectively. We can choose a basis for R^m such that ν is given by the $(n-1) \times m$ matrix $(b_{ij})_{2 \le i \le n, 1 \le j \le m}$. As in [Die10], we construct the diagram

$$\begin{tikzcd}
R^m \arrow[r, "\nu_1"] \arrow[d, "\mathrm{id}"] & R^n \arrow[r, "\mu_1"] \arrow[d, "\pi"] & M \arrow[r] \arrow[d] & 0 \\
R^m \arrow[r, "\nu"] & R^{n-1} \arrow[r, "\mu"] & Q \arrow[r] & 0,
\end{tikzcd}$$

where π kills the first generator of R^n and the rows are exact. The map μ_1 sends the generators of R^n to $w_1, \ldots, w_n$, respectively, and ν_1 has matrix $(b_{ij})_{1 \le i \le n, 1 \le j \le m}$ with respect to the same basis for R^m used to give ν.

Now we construct corresponding resolutions for M' and Q'. M' has k new generators and s new relations, as does Q', so we get the following diagram:

$$\begin{array}{ccccccc}
R^{m+s} & \xrightarrow{\nu_1'} & R^{n+k} & \xrightarrow{\mu_1'} & M' & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
R^{m+s} & \xrightarrow{\nu'} & R^{n-1+k} & \xrightarrow{\mu'} & Q' & \longrightarrow & 0
\end{array}$$

The maps μ' and μ_1' take the generators of Q' and M' to $\overline{w_2}, \ldots, \overline{w_n}, f_1, \ldots, f_k$ and $w_1, \ldots, w_n, f_1, \ldots, f_k$, respectively. The map π kills the first generator of R^{n+k}. The map ν_1' can be given by the matrix

$$\left(\begin{array}{c|c}
 & 0 \\
\nu_1 & \vdots \\
 & 0 \\
 & v \\
\hline
 & e_1 \\
0 & \vdots \\
 & e_k
\end{array}\right),$$

and ν' is this matrix with the top row removed.

The rows of this diagram are exact. We demonstrate the exactness at R^{n+k}. To see that $\mu_1' \circ \nu_1' = 0$, we observe that $\mu_1 \circ \nu_1 = 0$, and for all i, $v_i x + e_{1i} f_1 + \ldots + e_{ki} f_k = 0$ in M'. To see that $\ker(\mu_1') \subseteq \operatorname{im}(\nu_1')$, suppose that $\mu_1'(a_1, \ldots, a_{n+k})^{\mathrm{tr}} = 0$. Then

$$\begin{aligned}
a_1 w_1 + \ldots + a_n w_n + a_{n+1} f_1 + \ldots + a_{n+k} f_k = r_1(v_1 x + e_{11} f_1 + \ldots + e_{k1} f_k) \\
+ r_2(v_2 x + e_{12} f_1 + \ldots + e_{k2} f_k) \\
+ \ldots + r_s(v_s x + e_{1s} f_1 + \ldots + e_{ks} f_k)
\end{aligned}$$

in $M \oplus Rf_1 \oplus \ldots \oplus Rf_k$, for some $r_1, \ldots, r_s \in R$. So

$$a_1 w_1 + \ldots + a_{n-1} w_{n-1} + (a_n - \sum_{i=1}^{s} r_s v) x = 0,$$

and

$$a_{n+i} = \sum_{j=1}^{s} r_j e_{ij}$$

for $1 \leq i \leq k$. This implies that $(a_1, \ldots, a_{n-1}, a_n - \sum_{i=1}^{s} r_s v, 0, \ldots, 0)^{\mathrm{tr}}$ is in the image of the first m columns of ν_1', and $(0, \ldots, 0, \sum_{i=1}^{s} r_s v, a_{n+1}, \ldots, a_{n_k})^{\mathrm{tr}}$ is in the image of the last s columns of ν_1'. Hence $(a_1, \ldots, a_{n+k})^{\mathrm{tr}}$ is in the image of ν_1', as desired.

By Lemma II.15, α' is phantom if and only if the top row of ν_1' is in the cl-closure of the span of the other rows. Denote the top row of ν_1 by $\boldsymbol{x}$, the bottom row by $\boldsymbol{y}$, and the span of the middle rows by H. Then α' is phantom if and only if

$$\boldsymbol{x} \oplus 0 \in (R(\boldsymbol{y} \oplus v) + (H \oplus \mathbf{0}) + (\mathbf{0} \oplus G))^{\mathrm{cl}}_{R^{m+s}}.$$

But since α is phantom, $\boldsymbol{x} \in (R\boldsymbol{y} + H)^{\mathrm{cl}}_{R^m}$. Hence $\boldsymbol{x} \oplus \mathbf{0} \in (R\boldsymbol{y} + H)^{\mathrm{cl}}_{R^m} \oplus \mathbf{0}$, and we have

$$\begin{aligned}(R\boldsymbol{y} + H)^{\mathrm{cl}}_{R^m} \oplus \mathbf{0} &= (R\boldsymbol{y} + H)^{\mathrm{cl}}_{R^m} \oplus 0^{\mathrm{cl}}_{R^s} \\ &= ((R\boldsymbol{y} + H) \oplus \mathbf{0})^{\mathrm{cl}}_{R^{m+s}} \\ &= ((R\boldsymbol{y} \oplus \mathbf{0}) + (H \oplus \mathbf{0}))^{\mathrm{cl}}_{R^{m+s}}\end{aligned}$$

We want to show that this is contained in $(R(\boldsymbol{y} \oplus v) + (H \oplus \mathbf{0}) + (\mathbf{0} \oplus G))^{\mathrm{cl}}_{R^{m+s}}$. We have

$$\begin{aligned}(R\boldsymbol{y} \oplus 0) + (H \oplus \mathbf{0}) &\subseteq R(\boldsymbol{y} \oplus v) + (H \oplus \mathbf{0}) + (\mathbf{0} \oplus G^{\mathrm{cl}}_{R^s}) \\ &= (R\boldsymbol{y} \oplus v) + (H \oplus \mathbf{0}) + ((\mathbf{0} \oplus G))^{\mathrm{cl}}_{R^{m+s}} \\ &\subseteq (R(\boldsymbol{y} \oplus v) + (H \oplus \mathbf{0}))^{\mathrm{cl}}_{R^{m+s}} + (\mathbf{0} \oplus G)^{\mathrm{cl}}_{R^{m+s}}.\end{aligned}$$

Thus

$$\begin{aligned}((R\boldsymbol{y} \oplus 0) + (H \oplus \mathbf{0}))^{\mathrm{cl}}_{R^{m+s}} &\subseteq \left((R(\boldsymbol{y} \oplus v) + (H \oplus \mathbf{0}))^{\mathrm{cl}}_{R^{m+s}} + (\mathbf{0} \oplus G)^{\mathrm{cl}}_{R^{m+s}}\right)^{\mathrm{cl}}_{R^{m+s}} \\ &= (R(\boldsymbol{y} \oplus v) + (H \oplus \mathbf{0}) + (\mathbf{0} \oplus G))^{\mathrm{cl}}_{R^{m+s}}\end{aligned}$$

by Lemma III.1. Therefore, α' is phantom. $\square$

It turns out that the closure operation cl_B from Theorem V.1 is the smallest big Cohen-Macaulay module closure containing cl, the initial Dietz closure.

Lemma V.4. *Let notation be as in Theorem V.1. Given a big Cohen-Macaulay-module B' such that $cl \subseteq cl_{B'}$, $cl_B \subseteq cl_{B'}$.*

Proof. For any map $R \to B'$, we construct a map $B \to B'$. We already know from the proof of Proposition IV.14 how to extend the map $M \to B'$ to a map $M' \to B'$, where M' is a parameter module modification of M. We need to know how to extend the map when

$$M' = \frac{M \oplus Rf_1 \oplus \ldots \oplus Rf_k}{R\,(v_1 x \oplus e_{11} f_1 \oplus \ldots \oplus e_{k1} f_k, \ldots, v_s x \oplus e_{1s} f_1 \oplus \ldots \oplus e_{ks} f_k)}.$$

Since $(v_1, \ldots, v_s) \in G_{R^s}^{\mathrm{cl}}$, for each $b' \in B'$, $(v_1, \ldots, v_s) \otimes b' \in \mathrm{im}(G \otimes B' \to R^s \otimes B')$. In particular, for each $1 \leq i \leq s$, $v_i x = e_{1i} b_1 + e_{2i} b_2 + \ldots + e_{ki} b_k$ where $b_1, \ldots, b_k \in B'$. Define the map $M' \to B'$ by sending $f_i \mapsto b_i$.

Now for every map $R \to B'$ sending $1 \mapsto u$, we have a map $B \to B'$ whose image contains u. So by Proposition III.8, $\mathrm{cl}_B \subseteq \mathrm{cl}_{B'}$. $\square$

Question V.5. 1. *Are all Dietz closures big Cohen-Macaulay module closures, or any kind of module closure? If not, is there a nice way of characterizing the difference between Dietz closures that are big Cohen-Macaulay module closures and those that are not?*

2. *If we use only containment module modifications as in Proposition IV.19, are there useful hypotheses that guarantee that the constructed module S is a big Cohen-Macaulay module?*

We use the following definition in our proof that Dietz closures are the trivial closure on regular rings.

Definition V.6. Given a closure operation cl, a ring R is *weakly cl-regular* if for $N \subseteq M$ finitely generated R-modules, $N_M^{\mathrm{cl}} = N$. This is a generalization of Definition II.10.

Remark V.7. It is equivalent to say that $I_R^{\mathrm{cl}} = I$ for all ideals I of R. This follows from an argument in [HH90].

Proposition V.8. *Let cl be a closure operation on a regular local ring (R, m) that satisfies*

1. *if $x_1, \ldots, x_k$ is part of a system of parameters for R, then $(x_1^t, x_2, \ldots, x_k)^{cl} :_R x_1^a \subseteq (x_1^{t-a}, x_2, \ldots, x_k)^{cl}$ (strong colon-capturing, version A),*
2. *$m^{cl} = m$, and*
3. *if $N' \subseteq N \subseteq M$ are finitely-generated R-modules, then $(N')^{cl}_N \subseteq (N')^{cl}_M$.*

Then R is weakly cl-regular.

Proof. Let $N \subseteq M$ be finitely-generated R-modules, and let $x_1, \ldots, x_d$ be regular parameters for R (i.e., $(x_1, \ldots, x_d) = m$). Since $N = \bigcap_s (N + m^s M)$, by Lemma III.1 it suffices to show that $N + m^s M$ is cl-closed in M for each s. Fix a value of s. By the same Lemma, we may replace M by $M/(N + m^s M)$ and show that 0 is cl-closed in this module instead. Since M now has finite length, for some t, $I_t = (x_1^{t+1}, x_2^{t+1}, \ldots, x_d^{t+1})$ kills M, and so M is an R/I_t-module. Now I_t is m-primary, so R/I_t is 0-dimensional. Additionally, R is regular and $x_1, \ldots, x_d$ form a system of parameters, so R/I_t is Gorenstein. Hence R/I_t is injective as a module over itself and is also the only indecomposable injective R/I_t-module. This implies that $M \hookrightarrow (R/I_t)^h$ for some $h \geq 0$. Now it suffices to show that I_t is cl-closed in R, as then 0 is cl-closed in $(R/I_t)^h$. Since $0 \subseteq M \subseteq (R/I_t)^h$, this implies that $0_M^{\mathrm{cl}} \subseteq 0_{(R/I_t)^h}^{\mathrm{cl}} = 0$.

We show that I_t is cl-closed in R for all t. Let $x = x_1 x_2 \cdots x_d$. Since $(x_1, \ldots, x_d) = m$, 1 generates the socle in $R/I_0 = R/m$. Then x^t generates the socle in R/I_t for $t \geq 1$. So if I_t is not cl-closed, we must have $x^t \in (I_t)^{\mathrm{cl}}_R$. Thus it suffices to show that $x^t \notin (I_t)^{\mathrm{cl}}_R$.

Suppose that $x^t \in (I_t)^{\mathrm{cl}}_R$. Then

$$x_1^t (x_2^t \cdots x_d^t) \in (x_1^{t+1}, \ldots, x_d^{t+1})^{\mathrm{cl}}.$$

By hypothesis (1) on cl,

$$x_2^t \cdots x_d^t \in (x_1, x_2^{t+1}, \ldots, x_d^{t+1})^{\mathrm{cl}}.$$

Using this hypothesis again,

$$x_3^t \cdots x_d^t \in (x_1, x_2, x_3^{t+1}, \ldots, x_d^{t+1})^{\mathrm{cl}}.$$

Continuing in this manner, we see that

$$x_d^t \in (x_1, x_2, \ldots, x_{d-1}, x_d^{t+1})^{\mathrm{cl}},$$

and taking one more step, that $1 \in (x_1, \ldots, x_d)^{\mathrm{cl}}$. However, $m^{\mathrm{cl}} = m$, so this is a contradiction. Therefore, $(I_t)^{\mathrm{cl}}_R = I_t$ for all t, which finishes the proof that $N^{\mathrm{cl}}_M = N$ for all submodules N of finitely-generated R-modules M. $\square$

Theorem V.9. *Dietz closures are all equal to the trivial closure on regular local rings.*

Proof. Earlier, we showed that any Dietz closure is contained in a big Cohen-Macaulay module closure and that big Cohen-Macaulay module closures satisfy strong colon-capturing, version A. Since they are Dietz closures, they satisfy the other two properties required to use Proposition V.8. Therefore, Dietz closures are the trivial closure on regular rings. $\square$

Remark V.10. It is also possible to show that big Cohen-Macaulay module closures are the trivial closure on regular rings by noting that a big Cohen-Macaulay module B over a regular ring is faithfully flat [HH92], so that ideals and submodules of finitely-generated modules are "contracted" from B.

Theorem V.11. *Suppose that (R, m, K) is a local domain that has at least one Dietz closure (in particular, it suffices for R to have equal characteristic and any dimension, or mixed characteristic and dimension at most 3), and that all Dietz closures on R are equal to the trivial closure. Then R is regular.*

Proof. Since R has a big Cohen-Macaulay module B that gives a Dietz closure cl_B equal to the trivial closure, R is Cohen-Macaulay. We show that R is also approximately Gorenstein. If $\dim(R) \geq 2$, then $\operatorname{depth}(R) \geq 2$, so this follows from [Hoc77]. If $\dim(R) = 0$, then R is a field, which is approximately Gorenstein. If $\dim(R) = 1$, then the integral closure S of R is a big Cohen-Macaulay algebra for R. Let $b/a \in S$. We have $b \in (a)^{\mathrm{cl}_S}$, but cl_S must be the trivial closure on R, so $b \in (a)$. Hence $S = R$, and so R is normal. By [Hoc77], R is approximately Gorenstein.

Let $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_t \supseteq \ldots$ be a sequence of m-primary ideals such that each R/I_t is Gorenstein and the I_t are cofinal with the powers of m. Let $E = E_R(K)$, the injective hull of K over R. Then E is equal to the increasing union $\bigcup_t \operatorname{Ann}_E(I_t)$. Further, each $\operatorname{Ann}_E(I_t)$ is isomorphic to $E_{R/I_t}(K) \cong R/I_t$, so we have injective maps $R/I_t \to R/I_{t+1}$ for each $t \geq 1$. Let u_1 be a generator of the socle in R/I_1. For $t \geq 1$, let u_{t+1} be the image of u_t in R/I_{t+1}, which will generate the socle in R/I_{t+1}.

Suppose that M is a finitely-generated Cohen-Macaulay module with no free summand. We will show that M is equal to the increasing union of $I_tM : u_t$, so that $u_tM \subseteq I_tM$ for $t \gg 1$. This will imply that M gives us a Dietz closure not equal to the trivial closure. To see that the union is increasing, suppose that

$v \in I_tM : u_t$. Then $u_tv \in I_tM$. Applying the map $I_tM \to I_{t+1}M$ induced by the map $R/I_t \to R/I_{t+1}$, we see that $u_{t+1}v \in I_{t+1}M$.

Suppose that $M \neq \bigcup_t I_tM : u_t$. Then we can pick $v \in M - \bigcup_t I_tM : u_t$. For every $t \geq 1$, $u_tv \notin I_tM$. Consider the map $R \to M$ given by multiplication by v. Since R is local and M is finitely-generated, this splits if and only if $E \to E \otimes M$ is injective. But this is true if and only if $R/I_t \to M/I_tM$ is injective for all $t \gg 1$. For any t, $u_t \mapsto u_tv \notin I_{t+1}M$, so the socle of R/I_t is not contained in the kernel of the map $R/I_t \to M/I_tM$. Hence $R/I_t \to M/I_tM$ is injective, which implies that $R \to M$ splits. This contradicts our assumption that M had no free summand.

If R is not regular, then since R is Cohen-Macaulay, $\mathrm{syz}^d(k)$ is a finitely-generated Cohen-Macaulay module that is not free. Then it has some minimal direct summand (which can't be written as a nontrivial direct sum) that is not free. This gives us a Dietz closure not equal to the trivial closure on R. Therefore, R must be regular. $\square$

Remark V.12. By a result of [Dut89], $\mathrm{syz}^d(k)$ has no free summand when R is not regular, so we can use $\mathrm{syz}^d(k)$ instead of a minimal direct summand of it.

The following is a corollary to the proof of Theorem V.11.

Corollary V.13. *Let R be a local domain with at least one Dietz closure. Suppose that R has a finitely-generated Cohen-Macaulay module B with no free summands and that R is approximately Gorenstein but not regular. Then R has a Dietz closure cl_B that is not equal to the trivial closure.*

R satisfies these hypotheses when it is Cohen-Macaulay, $\dim(R) \neq 1$, *and R is not regular. Alternatively, it suffices for R to be complete but not regular. If R is Cohen-Macaulay of dimension not equal to 1 but is not regular, $B = syz^d(k)$ gives a nontrivial closure on R. In particular, if R has equal characteristic,* $\dim(R) \neq 1$,

and R is weakly F-regular but not regular, cl_B is nontrivial on R.

CHAPTER VI

Further Studies of Dietz Closures

6.1 Proofs that Certain Closures are not Dietz Closures

Dietz gives some examples of Dietz closures, as well as some closures that fail to be Dietz closures. Understanding why certain closure operations fail to be Dietz closures adds to our understanding of Dietz closures, and may help us find a good closure operation for rings of mixed characteristic. The following result gives one way for a closure operation to be "too big" to be a Dietz closure.

Theorem VI.1. *Let R be a local domain with $x_1, \ldots, x_k$ part of a system of parameters for R and $(x_1 \cdots x_k)^t \in (x_1^{t+1}, x_2^{t+1}, \ldots, x_k^{t+1})^{cl}$ for some $t \geq 0$ and closure operation cl. Then cl is not a Dietz closure.*

Proof. Suppose that cl is a Dietz closure. Then by Theorem V.1, there is a big Cohen-Macaulay module B such that $\mathrm{cl} \subseteq \mathrm{cl}_B$. Then we have

$$(x_1 \cdots x_k)^t \in (x_1^{t+1}, \ldots, x_k^{t+1})^{\mathrm{cl}} \subseteq (x_1^{t+1}, \ldots, x_k^{t+1})^{\mathrm{cl}_B}.$$

By Proposition III.14, this implies that

$$(x_2 \cdots x_k)^t \in (x_1, x_2^{t+1}, \ldots, x_k^{t+1})^{\mathrm{cl}_B},$$

which implies that

$$(x_3 \cdots x_k)^t \in (x_1, x_2, x_3^{t+1}, \ldots, x_k^{t+1})^{\mathrm{cl}_B},$$

and so on until

$$1 \in (x_1, \ldots, x_k)^{\mathrm{cl}_B}.$$

But $(x_1, \ldots, x_k)^{\mathrm{cl}_B} \subseteq m^{\mathrm{cl}_B} = m$. As $B \not\subseteq (x_1, \ldots, x_k)B$, this is a contradiction. Therefore, cl is not a Dietz closure. □

The following is not a new result, but it is interesting that it follows directly from Theorem VI.1.

Corollary VI.2. *Suppose that a local domain R has a Dietz closure cl. Then the Monomial Conjecture holds on R.*

Next we use Theorem VI.1 to prove that certain closure operations are not Dietz closures.

Corollary VI.3. *Integral closure is not a Dietz closure on R if* $\dim(R) \geq 2$.

Proof. Let x, y be part of a system of parameters for R. We always have $xy \in \overline{(x^2, y^2)}$, so by Theorem VI.1, integral closure is not a Dietz closure. □

Definition VI.4 [McD99]. We define *regular closure* on a ring R by $u \in N_M^{\mathrm{reg}}$ if for every regular R-algebra S, $u \in N_M^{\mathrm{cl}_S}$, where $N \subseteq M$ are finitely-generated R-modules and $u \in M$.

Lemma VI.5. *Let $R = k[[x, y, z]]/(x^3 + y^3 + z^3)$, where char$(k) \neq 3$. Then $(x, y)^t \subseteq (x^t, y^t)^{reg}$.*

Proof. In [HH93], Hochster and Huneke show that $z \in (x, y)^{\mathrm{reg}}$ but $z \notin (x, y)^*$. To do this, they reduce to the case of maps to complete regular local rings with algebraically closed residue field and show that any solution (a, b, c) of $u^3 + v^3 + w^3 = 0$ in S has the form $(\alpha d, \beta d, \gamma d)$, where $d \in S$ and (α, β, γ) is a solution of the same equation such that either at least two of α, β, and γ are units, or all three are 0.

If all three are 0, then clearly $(x, y)^t S \subseteq (x^t, y^t)S$. If at least one of α or β is a unit, then $(x^t, y^t)S = (d^t)$, which must contain $(x, y)^t S$. Since these are the only possible cases, we have $(x, y)^t S \subseteq (x^t, y^t)S$ for any regular R-algebra S. Hence $(x, y)^t \subseteq (x^t, y^t)^{\mathrm{reg}}$. □

Corollary VI.6. *Regular closure may fail to be a Dietz closure.*

Proof. Consider the ring $R = k[[x, y, z]]/(x^3 + y^3 + z^3)$, where $\mathrm{char}(k) \neq 3$. In this ring, $xy \in (x^2, y^2)^{\mathrm{reg}}$ by Lemma VI.5. □

Remark VI.7. By the exact argument used in Lemma VI.5, UFD closure (consider all R-algebras that are UFD's, rather than the regular R-algebras) may fail to be a Dietz closure.

The following result is a consequence of Theorem V.9.

Theorem VI.8. *For rings of equal characteristic 0, solid closure is not always a Dietz closure. In particular, solid closure is not a Dietz closure on regular local rings containing the rationals with dimension at least 3.*

Proof. By Theorem V.9, Dietz closures are equal to the trivial closure on regular rings. By [Hoc94, Corollary 7.24] and [Rob94], if R is a regular local ring containing the rationals with dimension at least 3, then solid closure is not the trivial closure on R. Hence solid closure is not a Dietz closure on these rings. □

6.2 Full Extended Plus Closure

We do not know whether Heitmann's mixed characteristic plus closure, full extended plus closure, and full rank one closure [Hei02] are Dietz closures, even in dimension 3. To discuss this question, we first extend the definition of full extended plus closure (epf) to finitely generated modules. The other definitions can be extended similarly.

Definition VI.9. Let R be a mixed characteristic local domain, whose residue field has characteristic p. Let $N \subseteq M$ be finitely generated modules over R. We define the *full extended plus closure* of N in M by $u \in M$ is in N_M^{epf} if there is some $c \neq 0 \in R$ such that for all $n \in \mathbb{Z}_+$,

$$c^{1/n} \otimes u \in \operatorname{im}(R^+ \otimes N + R^+ \otimes p^n M \to R^+ \otimes M).$$

Proposition VI.10. *For R a local domain of mixed characteristic p, full extended plus closure is a closure operation that is functorial, semi-residual, and faithful, and $0_R^{epf} = 0$.*

Proof. It is easy to prove the extension and order-preserving properties. To see that epf satisfies idempotence, making it a closure operation, let $u \in (N_M^{\mathrm{epf}})_M^{\mathrm{epf}}$. Then there is some $c \neq 0$ in R such that

$$c^{1/n} \otimes u \in \operatorname{im}(R^+ \otimes N_M^{\mathrm{epf}} + R^+ \otimes p^n M \to R^+ \otimes M)$$

for all n, say

$$c^{1/n} \otimes u = \sum_i r_i \otimes y_i + \sum_j s_j \otimes p^n m_j,$$

with $r_i, s_j \in R^+$, $y_i \in N_M^{\mathrm{epf}}$, and $m_j \in M$. For each i, there is some nonzero $d_i \in R$ such that

$$d_i^{1/n} \otimes y_i \in \operatorname{im}(R^+ \otimes N + R^+ \otimes p^n M \to R^+ \otimes M).$$

Then

$$c^{1/n} \cdot \Pi_i d_i^{1/n} \otimes u = \Pi_i d_i^{1/n} \left(\sum_i r_i \otimes y_i + \sum_j s_j \otimes p^n m_j \right) \in \operatorname{im}(R^+ \otimes N + R^+ \otimes p^n M \to R^+ \otimes M).$$

Since $c \cdot \Pi_i d_i$ is a nonzero element of R, this proves that $u \in N_M^{\mathrm{epf}}$.

To check that epf is functorial, let $f : M \to W$ be an R-module homomorphism and $N \subseteq M$. Let $u \in N_M^{\mathrm{epf}}$. Then there is some nonzero $c \in R$ such that

$$c^{1/n} \otimes u \in \mathrm{im}(R^+ \otimes N + R^+ \otimes p^n M \to R^+ \otimes M)$$

for every $n > 0$. Apply f. This tells us that

$$c^{1/n} \otimes f(u) \in \mathrm{im}(R^+ \otimes f(N) + R^+ \otimes p^n W \to R^+ \otimes W)$$

for every $n > 0$, which implies that $f(u) \in f(N)_W^{\mathrm{epf}}$.

Next, suppose that $N_M^{\mathrm{epf}} = N$. We will show that $0_{M/N}^{\mathrm{epf}} = 0$. Let $\bar{u} \in 0_{M/N}^{\mathrm{epf}}$, where $u \in M$. Then there is some nonzero $c \in R$ with

$$c^{1/n} \otimes \bar{u} \in \mathrm{im}(R^+ \otimes p^n(M/N) \to R^+ \otimes M).$$

But $R^+ \otimes p^n(M/N)$ is isomorphic to $p^n(R^+ \otimes M)/(R^+ \otimes N)$, which tells us that

$$c^{1/n} \otimes u \in \mathrm{im}(R^+ \otimes p^n M + R^+ \otimes N \to R^+ \otimes M).$$

This implies that $u \in N_M^{\mathrm{epf}} = N$, so $\bar{u} = 0$ in M/N.

To see that $m_R^{\mathrm{epf}} = m$, let $u \in m_R^{\mathrm{epf}}$. Then

$$c^{1/n} u \in (m, p^n)R^+$$

for some nonzero $c \in R$ (using the ideal version of the definition of epf) and for all n. Since $p^n \in m$, $c^{1/n}u \in mR^+$ for all n. If $u \notin m$, then $c^{1/n} \in mR^+$ for all n. But we can extend the m-adic valuation on R to a $\mathbb{Q}$-valued valuation on R^+. The order of $c^{1/n}$ will be $\frac{1}{n}\mathrm{ord}(c)$. So this is impossible.

Now let $u \in 0_R^{\mathrm{epf}}$. Then $c^{1/n}u \in p^n R^+$ for some $c \neq 0$ in R and for all n. Let ord denote a $\mathbb{Q}$-valued valuation on R^+ that extends the m-adic valuation on R. Let $s = \mathrm{ord}(c)$ and $t = \mathrm{ord}(p)$. Then we must have $s/n + \mathrm{ord}(u) \geq nt$ for all n. This implies that $u = 0$. $\square$

A similar argument works for mixed characteristic plus closure and for full rank one closure.

If at least one of these closures is a Dietz closure in dimension 3, this would tie the results of [Hei02, Hoc02] in to the results of this paper. If they are not Dietz closures in dimension 3, this would imply that the Dietz axioms are stronger than they need to be–there could be a weaker set of axioms that would be sufficient for the proof of the Direct Summand Conjecture in mixed characteristic rings.

6.3 Connections between Dietz closures and other closure operations

We show that Dietz closures are contained in (liftable) integral closure. This is proved for ideals in [Die05] with the added assumption that the closures are persistent for change of rings, but we do not need this assumption here. This result gives a weak upper bound for all Dietz closures; we give better bounds for certain classes of Dietz closures in Proposition VI.17 and Theorem VIII.4.

Theorem VI.11. *Let R be a domain and $cl = cl_M$ where M is a solid module over R. Then $I^{cl} \subseteq \bar{I}$ for every ideal I of R.*

Proof. Since M is solid, there is some nonzero map $f : M \to R$, with image $\mathfrak{a}$, a nonzero ideal of R. Suppose that $I \subseteq J \subseteq I^{\mathrm{cl}}$. Then $JM = IM$. Applying f, we get $J\mathfrak{a} = I\mathfrak{a}$. Since R is a domain, $\mathfrak{a}$ is a finitely-generated, torsion-free R-module. By the lemma below, $J \subseteq \bar{I}$. □

Lemma VI.12 [HS06]. *Suppose that $I \subseteq J$ are ideals of a domain R such that $IM = JM$ for some finitely-generated, torsion-free R-module M. Then $J \subseteq \bar{I}$.*

Corollary VI.13. *Let R be a complete local domain and B a big Cohen-Macaulay module over R. Then $I^{cl_B} \subseteq \bar{I}$ for all ideals I of R.*

Proof. By [Hoc94, Proposition 10.5], B is a solid module over R. Hence by Theorem VI.11, $I^{\mathrm{cl}_B} \subseteq \bar{I}$ for every ideal I of R. □

There are several ways to extend integral closure to modules. Here we use liftable integral closure, denoted $\vdash$, as defined by Epstein and Ulrich.

Definition VI.14 [EU14]. Let G be a submodule of a finitely-generated free R-module F, let S be the symmetric algebra over R defined by F, and let T be the subring of S induced by the inclusion $G \subseteq F$. Observe that S is $\mathbb{N}$-graded and generated in degree 1 over R, and that T is an $\mathbb{N}$-graded subring of S, also generated in degree 1 over R. We define the *integral closure* G_F^- of G in F to be the degree 1 part of the integral closure of the subring T of S.

Now let $N \subseteq M$ be finitely-generated R-modules. Take a finitely-generated free module F and a surjection $\pi : F \to M$, and let $G = \pi^{-1}(N)$. We define the *liftable integral closure* of N in M by

$$N_M^{\vdash} = \pi(G_F^-).$$

Proposition VI.15. *Let R be a domain and $cl = cl_M$ where M is a solid R-module. Then for all finitely-generated free modules F over R and all submodules G of F, $G_F^{cl} \subseteq G_F^{\vdash}$.*

Proof. Let F be a free module of rank h over R and $G \subseteq F$. Let $S = \mathrm{Sym}(F) \cong R[x_1, \ldots, x_h]$, I the ideal generated by the image of G in S, and $\widetilde{M} = S \otimes_R M$. We will show that G_F^{cl} is contained in the degree one piece of $I_S^{\mathrm{cl}_{\widetilde{M}}}$.

Suppose that $u \in G_F^{\mathrm{cl}}$. Then for every $m \in M$, $m \otimes u \in \mathrm{im}(M \otimes G \to M^h)$. This implies that $m \otimes u \otimes 1 \in \mathrm{im}(M \otimes_R G \otimes_R S \to M^h \otimes_R S)$. By associativity and commutativity of tensor, $M \otimes_R G \otimes_R S \cong \widetilde{M} \otimes_S I \cong I\widetilde{M}$. This isomorphism takes $m \otimes u \otimes 1 \mapsto u(1 \otimes m)$. Then $u(s \otimes m) \in I\widetilde{M}$ for all $s \in S$, $m \in M$, which implies that $u \in I_S^{\mathrm{cl}_{\widetilde{M}}}$. Since $u \in G$, its image in S is of degree 1.

Since S is a domain and $\widetilde{M}$ is solid over S, $I^{\mathrm{cl}_{\widetilde{M}}} \subseteq \bar{I}$ for all ideals I of S. This implies that u is contained in the degree 1 piece of $\bar{I}$, and hence $u \in G_F^{\vdash}$. $\square$

Theorem VI.16. *Let R be a domain and $cl = cl_M$ where M is a solid module over R. Then cl is contained in liftable integral closure. In particular, if R is a complete local domain, all big Cohen-Macaulay modules closures on R are contained in liftable integral closure. This implies that all Dietz closures on R are contained in liftable integral closure.*

Proof. Let $L \subseteq N$ be finitely-generated modules over R, and let $\pi : F \to N$ be a surjection of a finitely-generated free module F onto M. Let $K = \pi^{-1}(L)$. Let $u \in L_N^{\mathrm{cl}}$. Then by Lemma III.1, any lift $\tilde{u}$ of u to F is contained in K_F^{cl}. By Proposition VI.15, $\tilde{u} \in K_F^{\vdash}$. Hence $u \in L_N^{\vdash}$. $\square$

Recall that a family of closure operations cl on a class of rings and maps between them is persistent for change of rings if given any $R \to S$ in the class, and $N \subseteq M$ finitely-generated R-modules,

$$S \otimes_R N_M^{\mathrm{cl}} \subseteq (S \otimes_R N)_{S \otimes_R M}^{\mathrm{cl}}.$$

Proposition VI.17. *Let cl denote a Dietz closure defined on a class of local domains and ring maps between them such that for each ring R in the class, all regular R-algebras and their structure maps are also contained in the class. If cl is persistent for change of rings, then it is contained in regular closure.*

Proof. Suppose that $u \in I^{\mathrm{cl}}$. Then in any map to a regular ring S, $u \in (IS)^{\mathrm{cl}} = IS$ by persistence. So $u \in I^{\mathrm{reg}}$. $\square$

CHAPTER VII

Big Cohen-Macaulay Algebras

In [Die10], Dietz asked whether it was possible to give a characterization of Dietz closures that induced big Cohen-Macaulay algebras. Below, I answer this question positively.

There are many reasons to prefer big Cohen-Macaulay algebras to big Cohen-Macaulay modules; one is the ability to compare big Cohen-Macaulay algebra closures on a family of rings. Suppose that we have the following commutative diagram:

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

with $R \to S$ a local map of local domains, B an R-algebra, and C an S-algebra. Then if $u \in N_M^{\mathrm{cl}_B}$, $1 \otimes u \in (S \otimes_R N)^{\mathrm{cl}_C}_{S \otimes_R M}$. This property is a special case of being persistent for change of rings.

Proof of Persistence. By assumption, $1 \otimes u \in \operatorname{im}(B \otimes_R N \to B \otimes_R M)$. We show that

$$1 \otimes (1 \otimes u) \in \operatorname{im}(C \otimes_S (S \otimes_R N) \to C \otimes_S (S \otimes_R M)).$$

We can identify $C \otimes_S (S \otimes_R N)$ with $C \otimes_R N$, and $C \otimes_S (S \otimes_R M)$ with $C \otimes_R M$. Under these identifications, $1 \otimes (1 \otimes u) \mapsto 1 \otimes u$. So our goal is to show that

$$1 \otimes u \in \operatorname{im}(C \otimes_R N \to C \otimes_R M).$$

There is some element $d \in B \otimes_R N$ that maps to $1 \otimes u$ in $B \otimes_R M$. Then $d \mapsto d'$, an element of $C \otimes_R N$. Then by the commutativity of the diagram

$$\begin{array}{ccc} B \otimes_R N & \longrightarrow & B \otimes_R M \\ \downarrow & & \downarrow \\ C \otimes_R N & \longrightarrow & C \otimes_R M, \end{array}$$

d' is an element of $C \otimes_R N$ that maps to $1 \otimes u \in C \otimes_R M$. Thus $1 \otimes u \in (S \otimes_R N)^{\mathrm{cl}_C}_{S \otimes_R M}$. □

This implies that big Cohen-Macaulay algebra closures are persistent for change of rings in any case where we can build a commutative diagram as above, with B and C big Cohen-Macaulay algebras. We have a diagram of this form when R and S are of equal characteristic and $R \to S$ is permissible [HH95, Discussion and Definition 3.8].

Now we describe the extra condition needed to get big Cohen-Macaulay algebras. Suppose that we add the following to the list of axioms for a Dietz closure cl:

Axiom VII.1 (Algebra Axiom)**.** If $R \xrightarrow{\alpha} M$, $1 \mapsto e_1$ is cl-phantom, then the map $R \xrightarrow{\alpha'} \mathrm{Sym}^2(M)$, $1 \mapsto e_1 \otimes e_1$ is cl-phantom.

Remark VII.2. While the Algebra Axiom as stated uses a map to $\mathrm{Sym}^2(M)$, the axiom is more easily understood using the isomorphism to $\mathrm{Sym}^{\leq 2}(M)/(1-e_1)\mathrm{Sym}^{\leq 1}(M)$. This is the module consisting of all elements of $\mathrm{Sym}(M)$ in R, M, and $\mathrm{Sym}^2(M)$, with the following relations: for $r \in R$, $r \sim re_1 \sim re_1 \otimes e_1$, and for $m \in M$, $m \sim m \otimes e_1$. The map α' in the Algebra Axiom is cl-phantom if and only if the map $R \to \mathrm{Sym}^{\leq 2}(M)/(1-e_1)\mathrm{Sym}^{\leq 1}(M)$ sending $1 \mapsto e_1 \otimes e_1$ is cl-phantom, since these modules are isomorphic. To see that the modules are isomorphic, first notice that we can identify any element of $\mathrm{Sym}^{\leq 2}(M)/(1-e_1)\mathrm{Sym}^{\leq 1}(M)$ with an element of $\mathrm{Sym}^2(M)$ by tensoring with copies of e_1. Next we show that

$$(1-e_1)\mathrm{Sym}^{\leq 1}(M)\bigcap \mathrm{Sym}^2(M)=0.$$

Given $m\in M$, $(1-e_1)m=m-e_1\otimes m\in \mathrm{Sym}^2(M)$. Since $m\in \mathrm{Sym}^1(M)$ and $m\otimes e_1\in \mathrm{Sym}^2(M)$, the only way for them to be equal is to have $m=0$. This works similarly for $r\in R$. So we have the desired isomorphism. This holds when we replace 2 by 2^k for any $k\geq 0$ as well.

The axiom will be used to show that when we take the direct limit of the $\mathrm{Sym}^{2k}(M)$, the image of 1 stays out of the image of m. When we view this direct limit as a direct limit of the $\mathrm{Sym}^{\leq 2^k}(M)/(1-e_1)\mathrm{Sym}^{\leq 2^k-1}(M)$, we get

$$\begin{aligned}\varinjlim \mathrm{Sym}^{\leq 2^k}(M)/(1-e_1)\mathrm{Sym}^{\leq 2^k-1}(M)&=\varinjlim \mathrm{Sym}^{\leq n}(M)/(1-e_1)\mathrm{Sym}^{\leq n-1}(M)\\&=\mathrm{Sym}(M)/(1-e_1)\mathrm{Sym}(M).\end{aligned}$$

Theorem VII.3. *If a local domain R has a Dietz closure cl that satisfies the Algebra Axiom, then R has a big Cohen-Macaulay algebra.*

Remark VII.4. Note that if S is an R-algebra, and we have an R-module M and an R-module map $f:M\to S$, we can extend the map to a map from $\mathrm{Sym}^2(M)\to S$ via $m\otimes n\mapsto f(m)f(n)$. If we also have a map $R\to M$, $1\mapsto e$, then we can extend f to a map from

$$\mathrm{Sym}^{\leq 2^k}(M)/(1-e)\mathrm{Sym}^{\leq 2^k-1}(M)$$

to S, since $f(e)$ is equal to the image of 1 in S under the composition of maps $R\to M\to S$.

Proof. We construct a big Cohen-Macaulay module B_1 as in [Die10] with a map $R\to B_1$, $1\mapsto e$, and then take $\mathrm{Sym}(B)/(1-e)\mathrm{Sym}(B)$. We repeat these two steps infinitely many times, and take the direct limit B. This will be an R-algebra such that every system of parameters on R is a regular sequence on B. We need to show that $mB \neq B$.

At any intermediate stage M, after we have applied module modifications and taken symmetric powers, there is always a map $R \to M$ that factors through all previous intermediate modules. It suffices to show that $\mathrm{im}(1) \notin mM$. By the arguments of Theorem V.1, [HH95, Proposition 3.7], and Remark VII.4, if $\mathrm{im}(1) \in mB$ then there is some M obtained from R by a finite sequence of module modifications and finite symmetric powers as in Remark VII.2 for which $\mathrm{im}(1) \in mM$. However, M is a cl-phantom extension of R by [Die10] and the Algebra Axiom. Thus Lemma II.19 implies that $\mathrm{im}(1) \notin mM$. Hence $\mathrm{im}(1) \notin mB$, which implies that B is a big Cohen-Macaulay algebra for R. $\square$

7.1 A description of the Algebra Axiom in terms of a presentation of $\mathbf{Sym}^2(M)$

In this section, we provide a presentation for $\mathrm{Sym}^2(M)$ built from a presentation of M. This technique, used by Dietz in [Die10], allows us to work with specific matrices when studying maps $R \to \mathrm{Sym}^2(M)$ that may be phantom.

Let $\alpha : R \to M$ be an injective map sending $1 \mapsto e_1$. We use the notation of Notation II.14. In particular, $Q = \mathrm{coker}(\alpha)$ and B is the matrix $(b_{ij})_{1 \le i \le n, 1 \le j \le m}$ of the map ν_1 with respect to the basis $e_1, \ldots, e_n$ of R^n and the chosen basis of R^m. We have a map

$$\alpha' : R \to \mathrm{Sym}^2(M),$$

taking $1 \mapsto e_1 \otimes e_1$. Denote the cokernel by Q'. This is isomorphic to $\mathrm{Sym}^2(M)/(R(e_1 \otimes e_1))$.

To get a presentation for $\mathrm{Sym}^2(M)$, we start with the map $R^{m^2} \to R^{n^2}$ given by the matrix $B \otimes B$, and then add in the columns needed for the symmetry relations.

There are $\frac{n^2-n}{2}$ of these columns, one for each pair $i < j$, with an entry equal to 1 in the row corresponding to $e_i \otimes e_j$, an entry equal to -1 in the row corresponding to $e_j \otimes e_i$, and 0's elsewhere. Call the corresponding map ν_1'.

To get a presentation for Q', we use this matrix with the top row removed. Call this matrix ν'. We use this presentation to get the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{\alpha'} & \mathrm{Sym}^2(M) & \longrightarrow & Q' & \longrightarrow & 0 \\
\uparrow & & \phi' \uparrow & & \psi' \uparrow & & \mathrm{id} \uparrow & & \\
F_2 & \longrightarrow & R^{m^2+\frac{n^2-n}{2}} & \xrightarrow{\nu'} & R^{n^2-1} & \xrightarrow{\mu'} & Q' & \longrightarrow & 0
\end{array}$$

Let $\oplus$ denote horizontal concatenation of matrices, and B_i the ith row of the matrix B. The map ϕ' is given by the row matrix $(B_1 \otimes B_1) \oplus 0^{\frac{n^2-n}{2}}$, i.e.,

$$(b_{11}B_1 \; b_{12}B_1 \; \ldots \; b_{1n}B_1 \; 0 \; \ldots \; 0),$$

which is the first row of ν_1'.

The map α' is phantom if and only if $\mathrm{im}(\phi'^{\mathrm{tr}}) \subseteq \mathrm{im}(\nu'^{\mathrm{tr}})^{\mathrm{cl}}$. We can rewrite this statement as:

$$\begin{aligned}
(B_1 \otimes B_1) \oplus 0^{\frac{n^2-n}{2}} \in & \left(\sum_{i=2}^{n} \left((B_i \otimes B_i) \oplus 0^{\frac{n^2-n}{2}} \right) \right. \\
& \left. + \sum_{1 \le i < j \le n} ((B_i \otimes B_j) \oplus f_{i,j} - (B_j \otimes B_i) \oplus f_{i,j}) \right)^{\mathrm{cl}}_{R^{m^2+\frac{n^2-n}{2}}},
\end{aligned}$$

where $f_{i,j}$ is the vector of length $\frac{n^2-n}{2}$ with an entry equal to 1 in the $\left(\sum_{\ell=1}^{i-1}(n-\ell)\right)+(j-i)$th spot and 0's elsewhere.

7.2 Proofs that the Algebra Axiom holds for many closure operations

In this section, we demonstrate that the Algebra Axiom is a weak condition on a closure operation by proving that it holds for large classes of closures, most of which are not Dietz closures.

7.2.1 Tight Closure

Let R be a reduced ring of characteristic $p > 0$, and let $*$ denote tight closure.

For tight closure, we prove the axiom using the following equivalent definition of a phantom extension:

Proposition VII.5 [HH94c, Proposition 5.8]. *Given a short exact sequence $0 \to R \xrightarrow{\alpha} M \to Q \to 0$, α is *-phantom if and only if there is some $c \in R^\circ$ such that for all sufficiently large e, there exist maps $\gamma_e : F^e(M) \to F^e(R) = R$ such that $\gamma_e \circ F^e(\alpha) = c \cdot id_R$.*

Proposition VII.6. *If an injective map $\alpha : R \to M$ sending $1 \mapsto u$ is *-phantom, then so is the map $\alpha' : R \to Sym^2(M)$ sending $1 \mapsto u \otimes u$. As a result, * satisfies the Algebra Axiom.*

Proof. Since $R \xrightarrow{\alpha} M$ is *-phantom, we have maps γ_e as described above. Notice that $F^e(\mathrm{Sym}^2(M))$ is the symmetric tensor product $\mathrm{Sym}^2(F^e(M))$. For any e for which γ_e exists, define a map $\delta_e : F^e(\mathrm{Sym}^2(M)) \to R$ by $\delta_e(m^q \otimes n^q) = \gamma_e(m^q)\gamma_e(n^q)$. (To see that this is well-defined, define it from the tensor product first, then notice that $\delta_e(m^q \otimes n^q) = \delta_e(n^q \otimes m^q)$.) Since $\delta_e(F^e(\alpha))(1) = \delta_e(e_1^q \otimes e_1^q) = c^2$, and c does not depend on the choice of e, $R \xrightarrow{\alpha'} \mathrm{Sym}^2(M)$ is *-phantom. $\square$

7.2.2 Algebra Closures

Let R be a ring and $\mathscr{A}$ a directed family of R-algebras, so that given $A, A' \in \mathscr{A}$, there is a $B \in \mathscr{A}$ that both A and A' map to. We can define a closure operation using $\mathscr{A}$ as in Definition III.4.

Example VII.7. All algebra closures cl_A are closures of this type, with $\mathscr{A} = \{A\}$. In particular, if R is a domain, plus closure is a closure of this type, with $\mathscr{A} = \{R^+\}$.

Example VII.8. If R is a complete local domain and we let $\mathscr{A}$ be the set of solid algebras of R (algebras A such that $\mathrm{Hom}_R(A, R) \neq 0$), we get solid closure [Hoc94].

To show that the axiom holds for algebra closures, we give an equivalent definition of cl-phantom for these closures that is easier to work with.

Lemma VII.9. *Let $\alpha : R \hookrightarrow M$ be an R-module homomorphism. Let A be an R-algebra, and W the multiplicative set of non-zerodivisors of R. If*

1. *all elements of W are non-zerodivisors on A*
2. *and $W^{-1}A$ embeds in a free $W^{-1}R$-module,*

then $id_A \otimes \alpha : A \to A \otimes_R M$ is injective. In particular, if R is a domain and A is a torsion-free algebra over R, then $A \hookrightarrow A \otimes_R M$.

Proof. For all finitely-generated R-submodules A' of A, $W^{-1}A'$ embeds in a free $W^{-1}R$-module, F. The map $W^{-1}R \to W^{-1}M$ is injective, and this holds when we replace $W^{-1}R$ by F. Since $F \to F \otimes_R M$ is injective, so is $W^{-1}A' \to W^{-1}A' \otimes_R M$. Since elements of W are non-zerodivisors on A, and hence on A', this implies that $A' \to A' \otimes_R M$ is injective. Hence $A \to A \otimes_R M$ is injective. □

Proposition VII.10. *Suppose that every $A \in \mathscr{A}$ satisfies the hypotheses of Lemma VII.9. Then an injective map $\alpha : R \to M$ is cl-phantom if and only if for some $A \in \mathscr{A}$ there is a map $\gamma : A \otimes M \to A$ such that $\gamma \circ (id_A \otimes \alpha) = id_A$, i.e., if and only if $id_A \otimes \alpha$ splits.*

Proof. By Lemma VII.9, $\mathrm{id}_A \otimes \alpha : A \to A \otimes_R M$ is injective.

We use the notation of Lemma II.23. By Lemma II.23, since tensoring with A preserves the exactness of

$$0 \longrightarrow R \xrightarrow{\alpha} M \longrightarrow Q \longrightarrow 0,$$

$\mathrm{id}_A \otimes \phi$ is a cocycle in $\mathrm{Hom}_A(G_\bullet, A)$ representing the short exact sequence

$$0 \longrightarrow A \longrightarrow A \otimes M \longrightarrow A \otimes Q \longrightarrow 0.$$

The map α is cl-phantom if and only if $\phi \in \mathrm{im}(\mathrm{Hom}_R(P_0, R) \to \mathrm{Hom}_R(P_1, R))^{\mathrm{cl}}_{\mathrm{Hom}_R(P_1,R)}$. This holds if and only if $\mathrm{id}_A \otimes \phi \in \mathrm{im}(\mathrm{Hom}_A(G_0, A) \to \mathrm{Hom}_A(G_1, A))$, i.e. if and only if $\mathrm{id}_A \otimes \phi$ is a coboundary in $H^1(\mathrm{Hom}_A(G_\bullet, A))$. By Lemma II.23, this holds if and only if there is a map $\gamma : A \otimes M \to A$ such that $\gamma \circ (\mathrm{id}_A \otimes \alpha) = \mathrm{id}_A$. □

Proposition VII.11. *Let R be a domain, and cl be a closure operation coming from a directed family of torsion-free algebras $\mathscr{A}$. Suppose that a map $\alpha : R \hookrightarrow M$ sending $1 \mapsto u$ is cl-phantom. Then the map $\alpha' : R \to Sym^2(M)$ sending $1 \mapsto u \otimes u$ is also cl-phantom.*

Proof. Since α is cl-phantom, for some $A \in \mathscr{A}$, there is a map $\gamma : A \otimes M \to A$ such that $\gamma \circ (\mathrm{id}_A \otimes \alpha) = \mathrm{id}_A$. Define $\lambda : A \otimes \mathrm{Sym}^2(M) \to A$ by $\lambda(a \otimes (m \otimes n)) = a\gamma(m)\gamma(n)$. Let u be the image of 1 in M. Then

$$(\lambda \circ (\mathrm{id}_A \otimes \alpha'))(1) = \lambda(1 \otimes (u \otimes u)) = 1.$$

Hence α' is cl-phantom. □

We emphasize the following corollary:

Corollary VII.12. *Let B be a big Cohen-Macaulay algebra over a local domain. Then cl_B is a Dietz closure that satisfies the Algebra Axiom.*

Proof. Since B is a big Cohen-Macaulay module, cl_B is a Dietz closure by [Die10]. Since B is torsion-free, cl_B also satisfies the Algebra Axiom. □

The above relies on the elements of $\mathscr{A}$ being algebras, rather than modules. We do not know of a simpler condition for a map to be a cl-phantom extension when cl is a module closure, though we have the following:

Lemma VII.13. *Let R be a domain, W a torsion-free R-module, and $\alpha : R \hookrightarrow M$ an R-module map with M finitely-generated. If*

$$\alpha' = (id_W \otimes \alpha) : W \to W \otimes M$$

splits or is pure, then α is cl_W-phantom.

Proof. Let notation be as in Notation II.14. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\alpha"] & M \arrow[r] & Q \arrow[r] & 0 \\
P_2 \arrow[r] \arrow[u] & P_1 \arrow[r, "d"] \arrow[u, "\phi"] & P_0 \arrow[r] \arrow[u, "\psi"] & Q \arrow[r] \arrow[u, "\mathrm{id}"] & 0.
\end{tikzcd}$$

By definition, α is cl_W-phantom if and only if

$$\phi \in (\mathrm{im}(\mathrm{Hom}_R(P_0, R) \to \mathrm{Hom}_R(P_1, R)))^{\mathrm{cl}_W}_{\mathrm{Hom}_R(P_1,R)}.$$

This holds if and only if, for every $w \in W$,

$$w \otimes \phi \in \mathrm{im}(W \otimes \mathrm{Hom}_R(P_0, R) \to W \otimes \mathrm{Hom}_R(P_1, R)).$$

We can identify $W \otimes \mathrm{Hom}_R(P_0, R)$ with $\mathrm{Hom}_R(P_0, W)$ and $W \otimes \mathrm{Hom}_R(P_1, R)$ with $\mathrm{Hom}_R(P_1, W)$. Under this identification, $w \otimes \phi \mapsto \phi_w$, where $\phi_w(y) = \phi(y)w \in W$. So α is cl_W-phantom if and only if for every $w \in W$, $\phi_w = \lambda_w \circ d$ for some $\lambda_w : P_0 \to W$. We have the following commutative diagram for each $w \in W$:

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r, "\alpha'"] & W \otimes_R M \arrow[r] & W \otimes_R Q \arrow[r] & 0 \\
P_2 \arrow[r] \arrow[u] & P_1 \arrow[r, "d"] \arrow[u, "\phi_w"] & P_0 \arrow[r] \arrow[u, "\psi_w"] & Q \arrow[r] \arrow[u, "\mathrm{id}_w"] & 0.
\end{tikzcd}$$

Here $\psi_w(y) = w \otimes \psi(y)$ and $\mathrm{id}_w(z) = w \otimes z$. We know that α' is injective because W is torsion-free.

Suppose that α' splits. Then there is some map $\beta : W \otimes M \to W$ such that $\beta \circ \alpha' = \mathrm{id}_W$. For $w \in W$, define $\lambda_w : P_0 \to W$ to be $\beta \circ \psi_w$. Then we have:

$$\phi_w = \beta \circ \alpha' \circ \phi_w = \beta \circ \psi_w \circ d = \lambda_w \circ d.$$

In the case that α' is pure, given any $u_1, \ldots, u_k \in W \otimes M$, we have a splitting of $W \to \mathrm{im}(W) + Ru_1 + \ldots + Ru_k$. In particular, for each $w \in W$, we have a map $\beta_w : \mathrm{im}(\psi_w) \to W$ such that $\beta_w \circ \alpha' = \mathrm{id}_W$. Then we can define $\lambda_w : P_0 \to W$ to be $\beta_w \circ \psi_w$. $\square$

Remark VII.14. More generally, there are two ways to think about cl-phantom maps, where $\mathrm{cl} = \mathrm{cl}_W$ is a module closure. First, using notation as above, notice that α is cl-phantom if and only if

$$W \otimes \phi \subseteq \mathrm{im}(W \otimes \mathrm{Hom}_R(P_0, R) \to W \otimes \mathrm{Hom}_R(P_1, R)).$$

Then this holds if and only if for each $w \in W$ there are maps $\lambda_i : W \otimes d(P_1) \to W$ and elements $w_i \in W$ such that $w \otimes \phi = \sum \lambda_i \circ (w_i \otimes d)$.

Second, we can identify $W \otimes \mathrm{Hom}_R(P_i, R)$ with $\mathrm{Hom}_R(P_i, W)$, since P_0, P_1 are free. Under this identification, $w \otimes \phi$ becomes ϕ_w, the map that sends $z \mapsto \phi(z)w$. Then the statement that α is phantom is equivalent to the existence of maps $\lambda_w : P_0 \to W$ such that $\phi_w = \lambda_w \circ d$ for each $w \in W$.

These maps may not glue together away from the image of d, since in general $W \otimes \mathrm{Hom}(P_i, R)$ is not isomorphic to $\mathrm{Hom}(W \otimes P_i, W)$. When they glue together, the map $\mathrm{id}_W \otimes \alpha$ is split.

7.2.3 Heitmann's Closure Operations for Mixed Characteristic Rings

We prove that the Algebra Axiom holds for the extended plus closure as defined in Definition VI.9, as well as for the other closure operations defined by Heitmann in

[Hei02]. These closure operations do not quite fit the pattern of the closure operations above–they are not module or algebra closures. We assume that R is a domain.

We focus on a definition of phantom for the full extended plus closure, as full rank one closure will be similar. In this case, we need a new version of Lemma II.23. Let $\alpha : R \to M$ be an injective map, and use the notation of Notation II.14. Notice that if we have a map $\gamma : M \to R$ such that $\gamma \circ \alpha = c^{1/n}\mathrm{id}_R$ for every $n \in \mathbb{Z}_+$, then by Lemma II.23,

$$c^{1/n}\phi \in \mathrm{im}(R^+ \otimes B \to R^+ \otimes \mathrm{Hom}_R(P_1, R)),$$

where B is the module of coboundaries in $\mathrm{Hom}_R(P_1, R)$. This image is contained in

$$\mathrm{im}(R^+ \otimes B + R^+ \otimes p^n\mathrm{Hom}_R(P_1, R) \to R^+ \otimes \mathrm{Hom}_R(P_1, R)).$$

Since this holds for every n, $\phi \in B^{\mathrm{epf}}_{\mathrm{Hom}_R(P_1,R)}$. However, the reverse implication is no longer true. Instead we get the following result:

Lemma VII.15. *Let R, α, ϕ, $P_\bullet$, etc. be as above, and B the submodule of coboundaries in $Hom_R(P_1, R)$. For every $c \in R - \{0\}$ and for every $n \in \mathbb{Z}_+$,*

$$c^{1/n}\phi \in im(B + p^n Hom_R(P_1, R) \to Hom_R(P_1, R))$$

if and only if there is a map $\gamma : M \to R/p^nR$ such that $\gamma \circ \alpha = \overline{c^{1/n} id_R}$ where $\bar{}$ denotes image modulo p^n.

Proof. Observe that $c^{1/n}\phi$ is in this image if and only if there exist $\lambda : P_0 \to R$, $\delta : P_1 \to R$ such that

$$c^{1/n}\phi = (\lambda \circ d) + p^n\delta.$$

This holds if and only if

$$c^{1/n}\phi - (\lambda \circ d) \in p^n\mathrm{Hom}_R(P_1, R).$$

This is true if and only if the map

$$\overline{c^{1/n}\mathrm{id}_R} \oplus \overline{\lambda} : R \oplus P_0 \to R/p^n R$$

kills $\{\phi(u) - d(u) : u \in P_1\}$. Giving this map is equivalent to giving a map

$$\gamma : (R \oplus P_0)/\{c^{1/n}\phi(u) - d(u) : u \in P_1\} \to R/p^n R$$

such that $\gamma \circ \alpha = \overline{c^{1/n}\mathrm{id}_R}$, and

$$M \cong (R \oplus P_0)/\{c^{1/n}\phi(u) - d(u) : u \in P_1\}.$$

□

This lemma allows us to give an alternate definition of epf-phantom.

Proposition VII.16. *A map $R \xrightarrow{\alpha} M$ is epf-phantom if there is some $c \in R - \{0\}$ such that for every $n \in \mathbb{Z}_+$, there is a map $\gamma_n : R^+ \otimes M \to R^+/p^n R^+$ such that $\gamma_n \circ \alpha^+ = \overline{c^{1/n} id_{R^+}}$, where $\alpha^+ = id_{R^+} \otimes \alpha$ and $^-$ denotes image modulo p^n.*

Proof. Notice that α^+ is injective, so we can apply Lemma VII.15 with R^+, id_{R^+}, etc. By the lemma, γ_n exists if and only if

$$c^{1/n}\phi \in \mathrm{im}(R^+ \otimes B + R^+ \otimes p^n \mathrm{Hom}_R(P_1, R) \to R^+ \otimes \mathrm{Hom}_R(P_1, R)).$$

So we have a map γ_n for each n if and only if $c^{1/n}\phi \in B^{\mathrm{epf}}_{\mathrm{Hom}_R(P_1,R)}$, i.e., if and only if ϕ is epf-phantom. □

Remark VII.17. The result for full rank one closure is very similar–in this case, we have maps $\gamma_{\epsilon,n}$, where $n \in \mathbb{Z}_+$ and $\epsilon > 0$.

Proposition VII.18. *If a map $\alpha : R \to M$ sending $1 \mapsto u$ is epf-phantom, then so is the map $\alpha' : R \to Sym^2(M)$ sending $1 \mapsto u \otimes u$.*

Proof. Suppose that $\alpha : R \to M$ is phantom. Then there is a $c \in R - \{0\}$ such that for every $n \in \mathbb{Z}_+$, there is a map $\gamma_n : R^+ \otimes M \to R^+/p^nR^+$ with $\gamma_n \circ \alpha^+ = \overline{c^{1/n}\mathrm{id}_R}$. We need to find an appropriate $d \in R - \{0\}$ to show that $\alpha' : R \to \mathrm{Sym}^2(M)$ is phantom. Let $d = c^2$. Define $\gamma_n' : R^+ \otimes \mathrm{Sym}^2(M) \to R^+/p^nR^+$ by

$$\gamma_n'(s \otimes (m \otimes m')) = \bar{s}\gamma_n(m)\gamma_n(m').$$

Then

$$\gamma_n'(\alpha^+(1)) = \gamma_n'(1 \otimes (e_1 \otimes e_1)) = \gamma_n(e_1)^2 = \overline{c^{2/n}} = \overline{(c^2)^{1/n}} = \overline{d^{1/n}},$$

as desired. □

7.2.4 Mixed characteristic pullback tight closure

Definition VII.19. Let R be a ring of mixed characteristic $p > 0$, and define a closure cl on R by $u \in N_M^{\mathrm{cl}}$ if $\bar{u} \in (N/pN)^*_{M/pM}$, where the asterisk denotes tight closure in the characteristic p setting. We call this closure *pullback tight closure.*

For the rest of this section, assume that R is reduced.

Lemma VII.20. *Suppose that R/pR is reduced, and $\bar{\alpha} : R/pR \to M/pM$ is injective. Then $F^e(\overline{\alpha}) : F^e(R/pR) \to F^e(M/pM)$ is injective for all $e \geq 0$.*

Proof. Replace R by R/pR, and M by M/pM. By assumption, R is reduced. Let W be the multiplicative system of non-zerodivisors of R, so that $Q = W^{-1}R$ is the total quotient ring of R. The map $Q \to W^{-1}M$ is injective. Since Q is a product of fields, $W^{-1}M$ is a product of vector spaces over these fields, and so $Q \to W^{-1}M$ splits. Hence $F^e(Q) \to F^e(Q^{-1}M)$ is injective for all $e \geq 0$. The restriction of this map to $F^e(R)$ has image in $F^e(M)$, and will still be injective, as desired. □

We then get the following lemma:

Lemma VII.21. *Let R be a ring of mixed characteristic $p > 0$, and let cl denote pullback tight closure as defined above. Suppose that $\alpha : R \to M$ and $\overline{\alpha} : R/pR \to M/pM$ are injective, and R/pR is reduced. Then α is cl-phantom if and only if there is some $c \in R/pR - \{0\}$ such that for every $e \geq 0$, there is a map $\gamma_e : F^e(M/pM) \to R/pR$ such that $\gamma_e \circ F^e(\overline{\alpha}) = c \cdot id_{R/pR}$.*

Proof. Let $P_\bullet$ be a resolution of $Q = M/\mathrm{im}(R)$. Then we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "\alpha"] & M \arrow[r] & Q \arrow[r] & 0 \\ P_2 \arrow[r] \arrow[u] & P_1 \arrow[r, "d"] \arrow[u, "\phi"'] & P_0 \arrow[r] \arrow[u] & Q \arrow[r] \arrow[u, "\mathrm{id}"'] & 0 \end{tikzcd}$$

Taking the tensor product of this diagram with $F^e(R/pR)$, the top row remains exact by assumption. By Lemma II.23, γ_e exists if and only if $cF^e(\overline{\phi})$ is a coboundary. So γ_e exists for all sufficiently large e if and only if for each $e \gg 0$,

$$c\mathrm{im}(F^e(\overline{\phi})) \subseteq \mathrm{im}(\mathrm{Hom}_{R/pR}(P_0/pP_0, R/pR) \to \mathrm{Hom}_{R/pR}(P_1/pP_1, R/pR)).$$

This holds if and only if

$$\phi \in (\mathrm{im}(\mathrm{Hom}(P_0, R) \to \mathrm{Hom}(P_1, R)))^{\mathrm{cl}}_{\mathrm{Hom}(P_1,R)},$$

i.e., if and only if α is phantom by the homological definition. $\square$

Proposition VII.22. *Let R be a ring of mixed characteristic $p > 0$ and let cl denote pullback tight closure. If $\alpha : R \to M$ is cl-phantom, $\overline{\alpha} : R/pR \to M/pM$ is injective, and R/pR is reduced, then $\alpha' : R \to Sym^2(M)$ is cl-phantom.*

Proof. Using Lemma VII.21, we can define $\gamma'_e : F^e(\mathrm{Sym}^2(M)/p\mathrm{Sym}^2(M)) \to R/pR$ by

$$\gamma'_e(\overline{m}^q \otimes \overline{n}^q) = \gamma_e(\overline{m}^q)\gamma_e(\overline{n}^q),$$

where $\gamma_e : F^e(M/pM) \to R/pR$ is as in the lemma. Notice that $\mathrm{Sym}^2(M)/p\mathrm{Sym}^2(M) \cong \mathrm{Sym}^2(M/pM)$, which allows us to use the maps γ_e to define γ'_e. $\square$

Note that this closure operation is not generally a Dietz closure.

Example VII.23. Let $R = V[x_2, \dots, x_d]$, where (V, pV) is a discrete valuation ring. Then 0 is not closed in R: for any $u \in pR \neq 0$, $\bar{u} \in 0^*_{R/pR} = 0$. Since $0^{\mathrm{cl}}_R = 0$ for all Dietz closures cl, pullback tight closure is not a Dietz closure.

7.2.5 Closures constructed from other closures

The results below describe cases in which if every closure operation in a family satisfies the Algebra Axiom, so does a closure constructed from the family. The constructions are among those that appear in [Eps12]. We use the notation of Lemma II.23.

Proposition VII.24. *The Algebra Axiom is intersection stable.*

Proof. Let $\{\mathrm{cl}_\lambda\}_{\lambda \in \Lambda}$ be a set of closure operations, and define the closure operation cl by $N_M^{\mathrm{cl}} = \cap_{\lambda \in \Lambda} N_M^{\mathrm{cl}_\lambda}$. Suppose that every cl_λ satisfies the Algebra Axiom. We will show that cl also satisfies it.

Suppose that $\phi \in \mathrm{im}(\mathrm{Hom}(P_0, R) \to \mathrm{Hom}(P_1, R))^{\mathrm{cl}}$. It suffices to show that this forces $\psi \in \mathrm{im}(\mathrm{Hom}(G_0, R) \to \mathrm{Hom}(G_1, R))^{\mathrm{cl}}$. By our supposition, we know that

$$\phi \in \mathrm{im}(\mathrm{Hom}(P_0, R) \to \mathrm{Hom}(P_1, R))^{\mathrm{cl}_\lambda}$$

for each $\lambda \in \Lambda$. Since each cl_λ satisfies the axiom, $\psi \in \mathrm{im}(\mathrm{Hom}(G_0, R) \to \mathrm{Hom}(G_1, R))^{\mathrm{cl}_\lambda}$ for all λ, which immediately gives us the result we want. $\square$

Proposition VII.25. *Let $\{cl_\lambda\}_{\lambda \in \Lambda}$ be a directed set of closure operations satisfying the axiom. Then the closure operation cl defined by $N_M^{cl} = \sum_{\lambda \in \Lambda} N_M^{cl_\lambda}$ also satisfies this axiom.*

Proof. Note that since R is Noetherian and M is finitely generated over R, for each $N \subseteq M$ there is a $\lambda \in \Lambda$ such that $N_M^{\mathrm{cl}} = N_M^{\mathrm{cl}_\lambda}$ [Eps12]. Suppose that

$$\phi \in \mathrm{im}(\mathrm{Hom}(P_0, R) \to \mathrm{Hom}(P_1, R))^{\mathrm{cl}}.$$

Then for some $\lambda \in \Lambda$,

$$\phi \in \mathrm{im}(\mathrm{Hom}(P_0, R) \to \mathrm{Hom}(P_1, R))^{\mathrm{cl}_\lambda}.$$

Hence

$$\psi \in \mathrm{im}(\mathrm{Hom}(G_0, R) \to \mathrm{Hom}(G_1, R))^{\mathrm{cl}_\lambda} \subseteq \mathrm{im}(\mathrm{Hom}(G_0, R) \to \mathrm{Hom}(G_1, R))^{\mathrm{cl}}.$$

Hence the axiom holds for cl. ∎

Proposition VII.26. *Let $\phi : R \to S$ be a ring map, and cl' a closure operation on S satisfying the Algebra Axiom. Define a closure operation cl on R by*

$$N_M^{cl} = \left\{ x \in M : 1 \otimes x \in (im(S \otimes_R N \to S \otimes_R M))_{S \otimes_R M}^{cl'} \right\}.$$

If S satisfies the hypotheses of Lemma VII.9 (in particular, R a domain and S torsion-free works), then cl satisfies the Algebra Axiom as well.

Remark VII.27. We call closures defined in this way pullback closures. Mixed characteristic pullback tight closure as in Definition VII.19 is one example of this type of closure, with $S = R/pR$ and $\mathrm{cl}' = *$.

Proof. Assume that $\alpha : R \to M$ is cl-phantom. By Lemma VII.9, $\mathrm{id}_S \otimes \alpha : S \to S \otimes_R M$ is injective. We claim that it is cl'-phantom. Since α is cl-phantom, using Notation II.14,

$$\phi \in (\mathrm{im}(\mathrm{Hom}_R(P_0, R) \to \mathrm{Hom}_R(P_1, R)))_{\mathrm{Hom}_R(P_1, R)}^{\mathrm{cl}}.$$

this implies that

$$1 \otimes \phi \in (\mathrm{im}(S \otimes_R \mathrm{Hom}_R(P_0, R) \to S \otimes_R \mathrm{Hom}_R(P_1, R)))^{\mathrm{cl}'}_{S \otimes_R \mathrm{Hom}_R(P_1, R)}.$$

We have $S \otimes_R \mathrm{Hom}_R(P_i, R) \cong \mathrm{Hom}_S(S \otimes P_i, S)$ for $i = 0, 1$. This isomorphism takes $1 \otimes \phi \to \mathrm{id}_S \otimes \phi$. Thus

$$\mathrm{id}_S \otimes \phi \in (\mathrm{im}(\mathrm{Hom}_S(S \otimes P_0, S) \to \mathrm{Hom}_S(S \otimes P_1, S)))^{\mathrm{cl}'}_{\mathrm{Hom}_S(S \otimes P_1, S)}.$$

By Lemma II.23, since $\mathrm{id}_S \otimes \alpha : S \to S \otimes_R M$ is injective, this implies that $\mathrm{id}_S \otimes \alpha$ is cl'-phantom. Since cl' satisfies the Algebra Axiom, this implies that the map $(\mathrm{id}_S \otimes \alpha)' : S \to \mathrm{Sym}^2(S \otimes_R M)$ is also phantom. Using the notation from Lemma II.23 applied to $(\mathrm{id}_S \otimes \alpha)'$, this implies that $\mathrm{id}_S \otimes \psi \in \mathrm{Hom}_S(S \otimes G_0, S)^{\mathrm{cl}'}_{\mathrm{Hom}_S(S \otimes G_1, S)}$. We have $\mathrm{Hom}_S(S \otimes G_i, S) \cong S \otimes \mathrm{Hom}_R(G_i, R)$, and the isomorphism takes $\mathrm{id}_S \otimes \psi \mapsto 1 \otimes \psi$. By the definition of cl, this tells us that cl satisfies the Algebra Axiom. $\square$

One special case is the case where cl' is the trivial closure on S, which is Construction 3.1.1 from [Eps12]. The resulting closure on R is the algebra closure cl_S:

$$N_M^{\mathrm{cl}} = \{x \in M : 1 \otimes x \in \mathrm{im}(S \otimes N \to S \otimes M)\}.$$

We proved in Proposition VII.11 that this closure operation satisfies the Algebra Axiom when S is torsion-free over a domain R.

7.3 Partial algebra modifications and phantom extensions

In this section we show that if cl is a Dietz closure on R satisfying the Algebra Axiom, a partial algebra modificationof a cl-phantom extension of R is also a cl-phantom extension of R. Partial algebra modifications are found in [HH95, Hoc07] as part of a proof of the existence of big Cohen-Macaulay algebras in characteristic $p > 0$. In order to define a partial algebra modification, we first define an algebra modification.

Definition VII.28. Let $x_1, \ldots, x_{k+1}$ be part of a system of parameters for R and $s_1x_1 + \ldots + s_kx_k = sx_{k+1}$ a relation on an R-algebra S. An *algebra modification* of S is

$$S' = S[X_1, \ldots, X_k]/(s + x_1X_1 + \ldots + x_kX_k)S$$

Compare this to a parameter module modification of R as given in Definition II.21–the main difference is that the algebra modification preserves the algebra structure. However, an algebra modification is not generally a module-finite extension of R, and so we work with partial algebra modifications, where we limit ourselves to elements of low degree.

Definition VII.29. Let $x_1, \ldots, x_{k+1}$, S, and $s_1, \ldots, s_k, s \in S$ be as above. A *partial algebra modification* of S is

$$S' = S[X_1, \ldots, X_k]_{\leq n}/(s + x_1X_1 + \ldots + x_kX_k)S[X_1, \ldots, X_k]_{\leq n-1},$$

where $S[X_1, \ldots, X_k]_{\leq n}$ is the set of elements of $S[X_1, \ldots, X_k]$ of total degree $\leq n$. This is a finitely-generated R-module, but not an R-algebra.

Remark VII.30. A partial algebra modification of S is not an R-algebra, but the direct limit of the partial algebra modifications as n increases is an R-algebra, actually the algebra modification S' of S given in Definition VII.28. As a result, proving that partial algebra modifications of a cl-phantom extension of R are cl-phantom extensions of R will imply that algebra modifications of cl-phantom extensions of R are lim cl-phantom in the sense of Definition VII.32.

Lemma VII.31. *Let R be a local domain, cl a Dietz closure on R, M a finitely-generated R-module with $\alpha : R \to M$ a cl-phantom extension, and $x_1, \ldots, x_{k+1}$ part of a system of parameters for R. Suppose that $x_{k+1}m_{k+1} = \sum_{i=1}^k x_im_i$ for some*

$m_1, \ldots, m_{k+1} \in M$. Let

$$M' = M[X_1, \ldots, X_k]_{\leq 1}/RF,$$

where

$$F = m_{k+1} - \sum_{i=1}^{k} x_i X_i.$$

By $M[X_1, \ldots, X_k]_{\leq 1}$ we denote the module generated by all $m \in M$, $X_1, \ldots, X_k$, and mX_i for $m \in M$ and $1 \leq i \leq k$. We have a map $\alpha' : R \to M \to M'$, where the map $M \to M'$ takes $m \mapsto m$. Then $\alpha' : R \to M'$ is a cl-phantom extension.

Proof. We return to the notation of Notation II.14. As Dietz does in [Die10], we build a resolution of Q', where $Q' = M'/\mathrm{im}(R)$. Let $w_1, w_2, \ldots, w_n$ be a set of generators for M, not necessarily minimal, with $w_1 = \alpha(1)$ and $w_n = m_{k+1}$. Then a presentation of Q' is

$$R^{m(k+1)+1} \xrightarrow{\nu'} R^{nk+n-1} \xrightarrow{\mu'} Q' \longrightarrow 0,$$

where ν' is given by the matrix

$$\left(\begin{array}{cccc|c} & & & & 0 \\ \nu & 0 & \dots & \dots & \vdots \\ & & & & 0 \\ & & & & 1 \\ \hline & & & & x_1 \\ & & & & 0 \\ 0 & \nu_1 & 0 & \dots & \vdots \\ & & & & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ & & & & x_k \\ & & & & 0 \\ 0 & \dots & 0 & \nu_1 & \vdots \\ & & & & 0 \end{array}\right),$$

The corresponding matrix ν_1' in a presentation of M' is this matrix with the top row of ν_1 followed by 0's added to the top. Note that there are m columns for each of $1, X_1, \dots, X_k$, and one additional column for the relation given by F.

To see that M' is a cl-phantom extension of R, we first need to show that α' is injective. It is enough to show that $\beta : M \to M'$ is injective. Suppose that $u \in M$ maps to 0. Then $u = r(m_{k+1} - \sum_{i=1}^{k} x_i X_i)$. This forces $rx_i = 0$ for all i. Since R is a domain, $r = 0$. But then $u = rm_{k+1} = 0$.

To finish, it suffices to show that the top row of ν_1' is in the closure of the image of the other rows. Let $\boldsymbol{x}, \boldsymbol{y}$, and H be as in [Die10, Notation 3.5], let $I = (x_1, \dots, x_k)$, and let E_{X^α} denote the $n \times m(k+1)$ matrix that has an $n \times n$ identity matrix in the columns corresponding to X^α and 0's elsewhere. Then we need to show that $\boldsymbol{x}E_1 \oplus 0$ is contained in

$$\left((HE_1 \oplus 0) + R(\boldsymbol{y}E_1 \oplus 1) + \sum_{i=1}^{k} \left(R(\boldsymbol{x}E_{X_i} \oplus x_i) + (HE_{X_i} \oplus 0) + R(\boldsymbol{y}E_{X_i} \oplus 0)\right)\right)^{\mathrm{cl}}_{R^{m(k+1)+1}}.$$

By the proof of [Die10, Proposition 3.15] and Lemma III.1.b, we have

$$\boldsymbol{x}E_1 \oplus 0 \in ((HE_1 \oplus 0) + I(\boldsymbol{y}E_1 \oplus 0))^{\mathrm{cl}}_{R^{m(k+1)+1}}.$$

So it suffices to show that $(HE_1 \oplus 0) + I(\boldsymbol{y}E_1 \oplus 0)$ is contained in the closure of

$$(HE_1 \oplus 0) + R(\boldsymbol{y}E_1 \oplus 1) + \sum_{i=1}^{k} \left(R(\boldsymbol{x}E_{X_i} \oplus x_i) + (HE_{X_i} \oplus 0) + R(\boldsymbol{y}E_{X_i} \oplus 0)\right). \tag{7.1}$$

It is clear that $(HE_1 \oplus 0)$ is in (7.1). To see that $I(\boldsymbol{y}E_1 \oplus 0)$ is in the closure of (7.1), let $r \in I$, say $r = -r_1x_1 - \ldots - r_kx_k$. Then

$$r(\boldsymbol{y}E_1 \oplus 0) = r(\boldsymbol{y}E_1 \oplus 1) + r_1(\boldsymbol{x}E_{X_1} \oplus x_1) + \ldots + r_k(\boldsymbol{x}E_{X_k} \oplus x_k) - r_1(\boldsymbol{x}E_{X_1} \oplus 0) - \ldots - r_k(\boldsymbol{x}E_{X_k} \oplus 0).$$

The only parts not obviously contained in the closure of (7.1) are the $r_i(\boldsymbol{x}E_{X_i} \oplus 0)$. However, since α is phantom, by [Die10, Lemma 3.14] and Lemma III.1.b, we have

$$\boldsymbol{x}E_{X_i} \oplus 0 \in ((HE_{X_i} \oplus 0) + R(\boldsymbol{y}E_{X_i} \oplus 0))^{\mathrm{cl}}_{R^{m(k+1)+1}},$$

which is sufficient. $\square$

Definition VII.32. [Die05, Definition 5.2.8] Let $\alpha : R \to P$ be an injective map, where P may not be finitely-generated over R, and cl a closure operation on R. We say that $R \to P$ is a *lim cl-phantom extension* if for all $M \subseteq P$ such that $\alpha(R) \subseteq M$, $R \to M$ is a cl-phantom extension.

Proposition VII.33. *Let cl be a closure operation on R that is functorial. Let P be an R-module, not necessarily finitely generated, that is a direct limit of R-modules*

$$R \to M_1 \to M_2 \to \ldots \to P$$

such that $R \hookrightarrow M_i$ is cl-phantom for all $i \geq 1$. Then P is a lim cl-phantom extension of R.

Proof. Let M be a finitely-generated R-module such that $R \hookrightarrow M \hookrightarrow P$. Since M is finitely-generated, there is some i such that $\operatorname{im}(M) \subseteq \operatorname{im}(M_i)$. By [Die05, Lemma 7.3.3.b], since $R \to M_i$ is cl-phantom, so is $R \to M$. □

Proposition VII.34. *Let R be a local domain and let cl be a Dietz closure that satisfies the Algebra Axiom. Suppose that T is an R-algebra, $\alpha : R \to T$ is an R-algebra homomorphism that is a cl-phantom extension, and T' is a partial algebra modification of T. Then the map $\alpha' : R \to T'$ is cl-phantom.*

Proof. By Lemma VII.31, if $T' = T[X_1, \ldots, X_k]_{\leq N}/FT[X_1, \ldots, X_k]_{\leq N-1}$ with $N \leq 1$, then the result is immediate. If not, let $T_1 = T[X_1, \ldots, X_k]_{\leq 1}/FR$. By Lemma VII.31, $R \to T_1$ is cl-phantom. Since cl satisfies the Algebra Axiom, $R \to \operatorname{Sym}^2(T_1)$ is also cl-phantom, and so $R \to \operatorname{Sym}(T_1)$ is lim cl-phantom (since T_1 is an R-algebra, the image of 1 in T is 1, so the direct limit of the $\operatorname{Sym}^{2^k}(T_1)$ is $\operatorname{Sym}(T_1)$). However, $\operatorname{Sym}(T_1) \cong T[X_1, \ldots, X_k]/(F)$. So we have

$$R \to T' \to \operatorname{Sym}(T_1),$$

where the map $R \to \operatorname{Sym}(T_1)$ is cl-phantom. By Proposition VII.33, $R \to T'$ is cl-phantom. □

CHAPTER VIII

Connections with Smallest Closures and Other Closure Operations

8.1 Smallest big Cohen-Macaulay algebra closure

In this section we show that the closure we get from a big Cohen-Macaulay algebra constructed as in the proof of Theorem VII.3 is the same as the closure we get from a big Cohen-Macaulay algebra constructed using algebra modifications [HH95, Hoc07], and that both are the smallest big Cohen-Macaulay algebra closure on the ring.

By Proposition VII.24, the Algebra Axiom is intersection stable as defined in Definition IV.1.

Corollary VIII.1. *If R has a Dietz closure that satisfies the Algebra Axiom, then it has a smallest such closure.*

This is immediate from Proposition VII.24. We do not know whether this closure is a big Cohen-Macaulay algebra closure, but we do have a smallest big Cohen-Macaulay algebra closure for R.

Proposition VIII.2. *If R has a big Cohen-Macaulay algebra (equivalently, a Dietz closure that satisfies the Algebra Axiom), then it has a smallest big Cohen-Macaulay algebra closure. This closure is equal to the closure cl_B where B is constructed as in Theorem VII.3. It is also equal to the closure cl_B where B is constructed using* *algebra modifications as in [HH95, Hoc07].*

Proof. For the second statement, let B be a big Cohen-Macaulay algebra constructed by the method of Theorem VII.3, and B' another big Cohen-Macaulay algebra for R. We show that for any element of B', there is a map $B \to B'$ whose image contains that element. This is enough by Proposition III.8. Let $R \to B'$ be any map of R-modules. We construct a map $B \to B'$ that extends this map. If at any stage, we have a map $M \to B'$, and we take a module modification of M, the map extends as in Proposition IV.14. If we have a map $M \to B'$, and the map $R \to M$ takes $1 \mapsto u$, it extends to a map $\mathrm{Sym}(M)/(1-u)\mathrm{Sym}(M) \to B'$ as B' is an R-algebra. Hence starting with the map $R \to B'$, we can construct a map $B \to B'$ with the necessary properties. This implies both the first and second statements of the Proposition.

For the last statement, it suffices to show that if B is a big Cohen-Macaulay algebra constructed with algebra modifications, cl_B is also the smallest big Cohen-Macaulay algebra closure. Let B' be any big Cohen-Macaulay algebra. We show that for any element of B', there is a map $B \to B'$ whose image contains that element. We start with any R-module map $R \to B'$. Suppose that we have a map $S \to B'$, and that we take an algebra modification

$$S' = S[X_1, \ldots, X_k]/FS[X_1, \ldots, X_k],$$

where $s - x_1X_1 - \ldots - x_kX_k$, $x_1, \ldots, x_{k+1}$ are part of a system of parameters for R, and $sx_{k+1} = s_1x_1 + \ldots + s_kx_k$ is a bad relation in S. Since B' is a big Cohen-Macaulay algebra for R, $s \in (x_1, \ldots, x_k)B'$, say

$$s = x_1b_1 + \ldots + x_kb_k.$$

Then we can extend the map $S \to B'$ to S' by sending $X_i \mapsto b_i$. This gives us a well-defined map $S' \to B'$. Hence we have a map $B \to B'$ whose image includes the image of the original map $R \to B'$. $\square$

8.2 Dietz Closures Satisfying the Algebra Axiom in characteristic $p > 0$

In this section, we prove that in characteristic $p > 0$, any Dietz closure that satisfies the Algebra Axiom is contained in tight closure. As a consequence of this result, we show that there exist Dietz closures that do not satisfy the Algebra Axiom, and that there is a nonzero test ideal for the Dietz closures that do satisfy the Algebra Axiom.

Proposition VIII.3. *Let R be a local domain and cl be a Dietz closure on R that satisfies the Algebra Axiom. Then cl is contained in a big Cohen-Macaulay algebra closure cl_B.*

Proof. This proof follows the method of the proof of Theorem V.1. We can construct B by first constructing a big Cohen-Macaulay algebra as in Chapter VII. Then we can use the containment module modifications to create a module whose closure contains cl. At every stage, we have preserved $1 \notin \mathrm{im}(m)$. Repeating these two steps infinitely many times, we get a big Cohen-Macaulay algebra B such that $\mathrm{cl} \subseteq \mathrm{cl}_B$. $\square$

Theorem VIII.4. *Let R be a complete local domain (or analytically irreducible excellent local domain) of characteristic $p > 0$, and cl a Dietz closure on R that satisfies the Algebra Axiom. Then cl is contained in tight closure (*).*

Proof. In characteristic $p > 0$, Theorem II.9 implies that tight closure is equal to the closure $\mathrm{cl}_{\mathcal{B}}$ given in Definition III.5. Since by Proposition VIII.3, cl is contained in cl_B for some big Cohen-Macaulay algebra B, and cl_B is contained in $\mathrm{cl}_{\mathcal{B}}$, we have $\mathrm{cl} \subseteq *$. $\square$

Note that in equal characteristic 0 it is known that $*\mathrm{EQ}$, big equational tight closure, is contained in $\mathrm{cl}_{\mathcal{B}}$ but it is not known whether they are equal, so we cannot currently prove Theorem VIII.4 in this case.

The following Corollaries are immediate from this result.

Corollary VIII.5. *Let R be a complete local domain of characteristic $p > 0$. Then tight closure is the largest Dietz closure satisfying the Algebra Axiom on R.*

Corollary VIII.6. *Let R be a complete local domain of characteristic $p > 0$, and suppose that R is weakly F-regular. Then all Dietz closures on R that satisfy the Algebra Axiom are equal to the trivial closure.*

Theorem VIII.4 also allows us to prove that the Algebra Axiom is independent of the Dietz axioms.

Theorem VIII.7. *The Dietz Axioms do not imply the Algebra Axiom, i.e., there exist Dietz closures that do not satisfy the Algebra Axiom.*

Proof. Let (R, m, k) be a complete local domain of characteristic $p > 0$ that is weakly F-regular but not regular and has dimension d. By Corollary V.13, R has a Dietz closure that is not the trivial closure, $\mathrm{cl} = \mathrm{cl}_{\mathrm{syz}^d(k)}$. If cl satisfied the Algebra Axiom, then by Corollary VIII.6 it would be equal to the trivial closure, which is a contradiction. Hence cl is a Dietz closure on R that does not satisfy the Algebra Axiom. □

Definition VIII.8. Let cl be a closure operation on a ring R. Define the *cl-test ideal* of R by

$$\tau_{\mathrm{cl}}(R) = \bigcap_{N \subseteq M \text{ f.g.}} N : N_M^{\mathrm{cl}}.$$

This is a definition that extends the notion of a test ideal for tight closure, inspired by [EU14]. In the case below, they could prove to be interesting objects to study.

Lemma VIII.9. *Let R be a complete local domain of characteristic $p > 0$, and cl a Dietz closure on R that satisfies the Algebra Axiom. Then the cl-test ideal of R is nonzero.*

Proof. By [HH90], R has at least one nonzero test element for tight closure. Since cl $\subseteq *$, this will also be a test element for cl. Hence the cl-test ideal of R is nonzero. $\square$

This notion of a test ideal should lead to further connections between Dietz closures on a ring R and the singularities of R. We discuss this further in Chapter IX.

CHAPTER IX

Further Questions

9.1 Examples of Cohen-Macaulay Module Closures

In the proof of Theorem V.11, we showed that if a local domain (R, m, k) is Cohen-Macaulay but not regular, $\mathrm{cl}_{\mathrm{syz}^d(k)}$ is a Dietz closure for R that is not equal to the trivial closure. We give another class of Dietz closures not equal to the trivial closure, which can only occur when R is not regular.

Example IX.1. Let $R = k[[x^2, xy, y^2]]$. Then $M = (x^2, xy)$ is a non-maximal Cohen-Macaulay module over R (it has height=depth=1). Let $I = (x^4, x^3y, xy^3, y^4)$ and $J = (x^4, x^3y, x^2y^2, xy^3, y^4)$. Then $I \subsetneq J$, but

$$I(x^2, xy) = (x^6, x^5y, x^3y^3, x^2y^4, x^5y, x^4y^2, x^2y^4, xy^5) = J(x^2, xy).$$

So $I^{\mathrm{cl}_M} = J^{\mathrm{cl}_M}$.

Example IX.2. Let $R = k[[x, y, u, v]]/(xy - uv)$. Then $M = (x, u)$ is a non-maximal Cohen-Macaulay module over R. Let $I = (y^2, v^2)$ and $J = (yv)$. Then $I \neq J$, but $IM = JM = (xyv, yuv)$, so $I^{\mathrm{cl}_M} = J^{\mathrm{cl}_M}$.

In addition, if we let $I = (x^2, u^2)$ and $J = (x^2, xu, u^2)$, then $IM = JM = (x^3, x^2u, xu^2, u^3)$, even though $I \neq J$.

This gives rise to a more general class of examples: suppose that (x, u) is a non-principal ideal that is a Cohen-Macaulay module (height 1, depth 1), and $xu \notin$

(x^2, u^2). Then $(x^2, u^2)(x, u) = (x^2, xu, u^2)(x, u)$.

Example IX.3. Let $R = k[[x, y, z]]/(x^3 + y^3 + z^3)$, with the characteristic of k not equal to 3. Then $(x, y+z)$ is a height 1 prime of depth 1. Since $x(y+z) \notin (x, y+z)^2$, we are in the case above.

All of these examples are Gorenstein rings, so in each case the canonical module (a maximal Cohen-Macaulay module) is equal to the ring.

Question IX.4. *If R is not Gorenstein and has a canonical module ω, then ω is a Cohen-Macaulay module for R with no free summand. Hence by the proof of Theorem V.11, cl_ω is not the trivial closure on R. How else might we characterize this closure?*

Question IX.5. *How can we describe the smallest big Cohen-Macaulay module closure on a ring R explicitly, when R is not Cohen-Macaulay?*

Question IX.6. *If R has only finitely many indecomposable Cohen-Macaulay modules, what can we say about the resulting family of Dietz closures (the closures from each of the indecomposable Cohen-Macaulay modules and intersections of these closures)?*

9.2 Largest Big Cohen-Macaulay Module Closure

We do not know whether there is a largest Dietz closure. If there is one, then by Theorem V.1 it will also be the largest big Cohen-Macaulay module closure. Hence there is a largest big Cohen-Macaulay module closure if and only if there is a largest Dietz closure.

Proposition IX.7. *If Dietz closures on a local domain R form a directed set, then the sum of all Dietz closures is equal to the largest Dietz closure.*

Proof. Let D denote the sum of all Dietz closures. To see that it is a Dietz closure (it will be a closure operation, by [Eps12]), we use the fact [Eps12] that since R is Noetherian, for any particular $N \subseteq M$ finitely-generated R-modules, there is some Dietz closure cl such that $N_M^{\mathrm{cl}} = N_M^{\mathrm{D}}$.

Functorial: Let $f : M \to W$ be a map of R-modules, and $N \subseteq M$. Let cl be a Dietz closure such that $N_M^{\mathrm{cl}} = N_M^{\mathrm{D}}$. Then $f(N_M^{\mathrm{D}}) = f(N_M^{\mathrm{cl}}) \subseteq f(N)_W^{\mathrm{cl}} \subseteq f(N)_W^{\mathrm{D}}$.

Semi-residual: Suppose that $N_M^{\mathrm{D}} = N$. Then $N_M^{\mathrm{cl}} = N$ for every Dietz closure cl. Hence $0_{M/N}^{\mathrm{cl}} = 0$ for every Dietz closure cl, which implies that $0_{M/N}^{\mathrm{D}} = 0$.

Faithful: We must have $m^{\mathrm{D}} = m$, since m is cl-closed for any Dietz closure cl.

Generalized colon-capturing: With R, v, and J as in the statement of Axiom 4, let cl be a Dietz closure such that $(Rv)_M^{\mathrm{D}} = (Rv)_M^{\mathrm{cl}}$. Then $(Rv)_M^{\mathrm{D}} \cap \ker(f) = (Rv)_M^{\mathrm{cl}} \cap \ker(f) \subseteq (Jv)_M^{\mathrm{cl}} \subseteq (Jv)_M^{\mathrm{D}}$. □

So to prove that there is a largest Dietz closure, it suffices to show that Dietz closures form a directed set. To do this, it would be enough to show that given 2 Dietz closures cl and cl$'$, we can construct a big Cohen-Macaulay module B such that $\mathrm{cl}, \mathrm{cl}' \subseteq \mathrm{cl}_B$. It is not clear that if we perform a modification that is cl-phantom, then one that is cl'-phantom, that im(1) stays out of the image of m, so we do not know of a way to construct such a big Cohen-Macaulay module.

Question IX.8. *Is there a largest big Cohen-Macaulay module closure? What module gives this closure?*

9.3 Further Connections Between Dietz Closures and Singularities

Question IX.9. *If R has a unique Dietz closure (not necessarily the trivial closure), is R regular? What does the assumption that R has few Dietz closures tell us about the singularities of R? More generally, are rings with more distinct Dietz closures* *more singular than rings with fewer distinct Dietz closures?*

Question IX.10. *Can we characterize the singularities of a ring R using properties of its family of Dietz closures, or of a particular Dietz closure on R? In particular, can we do this using properties of the test ideal from Definition VIII.8? In characteristic $p > 0$, we know that the test ideal is nonzero when R is complete, but is it nonzero in other characteristics?*

Question IX.11. *By Definitions V.6 and VIII.8, a ring R is weakly cl-regular when $\tau_{cl}(R) = R$. What rings are weakly cl-regular?*

We can define a big test ideal $\tau_{b,cl}(R)$ by removing the assumption that the modules are finitely-generated from Definition VIII.8, and say that a ring R is strongly cl-regular when $\tau_{b,cl}(R) = R$. What rings are strongly cl-regular? Are some or all weakly cl-regular rings strongly cl-regular?

9.4 More Questions

Question IX.12. *When do two indecomposable big Cohen-Macaulay modules give the same closure operation? When do two modules give the same closure operation? We have a sufficient condition from Proposition III.8, but can we find a necessary one? What do we learn about modules from studying them up to equality of their closure operations?*

Question IX.13. *When R is a complete local domain that is not Cohen-Macaulay, what is the smallest closure operation that satisfies colon-capturing and is persistent for change of rings?*

Question IX.14. *Are all Dietz closures big Cohen-Macaulay module closures? Are they module closures at all? There exist examples of Dietz closures that are not* *obviously module closures [AB12]. If there exist Dietz closures that are not big Cohen-Macaulay module closures, is there a useful way of characterizing these? Additionally, if a module closure is a Dietz closure, is the module a big Cohen-Macaulay module?*

Question IX.15. *Is the smallest Dietz closure a big Cohen-Macaulay module closure?*

Question IX.16. *Do the Dietz axioms imply strong colon-capturing, version A or strong colon-capturing, version B, or vice versa?*

Question IX.17. *Are all versal big Cohen-Macaulay modules (big Cohen-Macaulay modules that map to every other big Cohen-Macaulay module) the direct limit of some set of module modifications? In particular, is every big Cohen-Macaulay module that gives the smallest big Cohen-Macaulay module closure the direct limit of a set of module modifications?*

Question IX.18. *Is full extended plus closure a Dietz closure, either in dimension 3 or in general?*

Question IX.19. *In equal characteristic 0, is (big) equational tight closure a big Cohen-Macaulay algebra closure?*

Question IX.20. *How should one extend the notion of a Dietz closure to rings that are not domains? Rings that are not local?*

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