Correspondences between cluster structures

by

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ABSTRACT

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Chair: Sergey Fomin

This thesis introduces quasi-homomorphisms of cluster algebras, a class of maps relating cluster algebras of the same type, but with different coefficients. The definition is given in terms of seed orbits, the smallest equivalence classes of seeds on which the mutation rules for non-normalized seeds are unambiguous. After proving basic structural results, we provide examples of quasi-homomorphisms involving familiar cluster algebras. We construct a quasi-isomorphism between cluster stuctures in Grassmannians and cluster structures in Fock-Goncharov spaces of configurations of affine flags. We explore the related notion of a quasi-automorphism, and compare the resulting group with other groups of symmetries of cluster structures. For cluster algebras from surfaces, we determine the subgroup of quasi-automorphisms inside the tagged mapping class group of the surface.

CHAPTER I

Introduction

Cluster algebras were introduced by Fomin and Zelevinsky [22] with a view towards gaining an explicit understanding of canonical bases in representation theory. As this program has been carried out with ongoing success, cluster algebras have proven to have a rich life of their own. This thesis seeks to add to the conversation: we introduce a notion of a structure-preserving map between cluster algebras which we call a *quasi-homomorphism*, and provide applications of this notion in familiar settings.

Quasi-homomorphisms provide an algebraic framework for discussing maps between cluster algebras of the same cluster type, but with different coefficients. They have connections with essential results in the structural theory of cluster algebras, namely the finite type classification [23] and the separation of additions formula [24]. The related notion of a quasi-automorphism leads to a group of symmetries of cluster structures, which interpolates between other groups of symmetries introduced in [1, 2, 15].

It is natural to investigate the above notions for particular classes of cluster structures. For cluster algebras from surfaces [19, 20, 29], we determine the group of quasi-automorphisms as a subgroup of the tagged mapping class group of the surface [2]. In another application, we construct correspondences between Grassmannians [52] and Fock-Goncharov spaces [15] that induce quasi-isomorphisms between the respective cluster structures. We exploit these quasi-isomorphism to obtain symmetries of the cluster structures which are not otherwise apparent.

A cluster algebra is the algebra generated (inside some ambient field of rational functions in several variables) by certain elements known as *cluster variables.* These cluster variables are grouped into overlapping subsets called *clusters.* The prominent feature of the setup is that the clusters are obtained recursively from one another by a combinatorial process called *mutation* – each cluster variable in any cluster can be mutated, producing a new cluster.

The precise formula for performing a given mutation depends on two additional layers of ingredients, the *exchange matrix* and the *coefficient tuple.* The former is a skew-symmetrizable square matrix with integer entries, while the latter consists of several Laurent monomials in certain *frozen variables.* When a cluster mutates, these ingredients also do. Mutation of exchange matrices is self-contained, i.e., prescribing a single exchange matrix specifies the entire mutation pattern of exchange matrices. On the other hand, the new coefficient tuple obtained in a mutation step depends on *both* the current coefficient tuple and the current exchange matrix. In particular, there is an infinite family of valid coefficient patterns atop any fixed pattern of exchange matrices.

The structural theory of cluster algebras focuses on the combinatorics of clusters. For example, given some description of the set of cluster variables, when do two of them lie in the same cluster? Can one describe the graph whose vertices are the clusters and whose edges correspond to mutations? When does performing all possible sequences of mutations produce only finitely many clusters? This theory has been well developed in the 15 years since cluster algebras were introduced. Despite this, there does not seem to be a consensus on what the “right” notion of a homomorphism between cluster algebras should be – several such notions have arisen in different mathematical settings, see e.g. [1, 2, 9, 10, 50, 51].

Though it is far from obvious from the definitions, the key structural properties of the cluster algebra tend to depend on the underlying pattern of exchange matrices (i.e., not on the choice of coefficient pattern). The definition of quasi-homomorphism is designed to preserve the underlying combinatorial scaffolding while allowing for maps between cluster algebras with different coefficient patterns.

Besides introducing quasi-homomorphisms as a conceptual tool, we also demonstrate that non-trivial quasi-homomorphisms arise in familiar settings. As our main example, we exhibit a quasi-isomorphism between the cluster structure in the homogeneous coordinate ring of a Grassmannian, on the one hand, and the cluster structure in a higher Teichmüller space for a disk with marked points on its boundary (as introduced by Fock and Goncharov in [15]). In [25], we will use this quasi-isomorphism to prove new things about the cluster combinatorics in these spaces.

The thesis is organized as follows:

Chapter II introduces quasi-homomorphisms in the context of the structural theory of cluster algebras. Section 2.1 provides necessary background on non-normalized seed patterns. Example 2.1.11 begins a running example, namely a pair of quasi-isomorphic cluster algebras used to illustrate the basic notions throughout Chapter II. This example is generalized in Appendix A.

Section 2.2 introduces *seed orbits*. These are the smallest equivalence classes on which the non-normalized mutation rule is unambiguous. Proposition 2.2.3 gives a more explicit characterization of such an equivalence class as an orbit with respect to a rescaling action on seeds. Section 2.3 introduces quasi-homomorphisms and their basic properties. We end Section 2.3 by describing the key differences between quasi-homomorphisms and some preexisting notions, specifically *rooted cluster morphisms* [1] and *coefficient specializations* [23, 50, 51].

Section 2.4 treats quasi-homomorphisms between normalized cluster algebras. For cluster algebras of geometric type, we relate quasi-homomorphisms to linear combinations of the rows of an extended exchange matrix, making connections to the separation of additions formula and to gradings on cluster algebras.

Section 2.5 presents the easiest way of specifying a quasi-homomorphism in practice, by checking that a given semifield map sends cluster variables to rescaled cluster variables on a *nerve*. We make use of this result in Section 4.4.

Chapter III introduces the the *quasi-automorphism group* of a cluster algebra and compares it with the cluster modular group of Fock and Goncharov [13] and the group of cluster automorphisms of Assem, Schiffler, and Schramchenko [2]. Section 3.2 illustrates these various groups for cluster algebras from surfaces. The highlight is Theorem 3.2.5 explicitly computing the group of quasi-automorphisms as a subgroup of the tagged mapping class group (barring a small number of exceptional surfaces). In particular, it implies that regardless of the choice of coefficients in such a cluster algebra, the quasi-automorphism group is always a finite index subgroup of the tagged mapping class group. The proofs for Section 3.2 are in Section 3.3

Chapter IV concerns the quasi-isomorphism between the Grassmannian and Fock-Goncharov cluster algebras. Sections 4.1 and 4.2 introduce the respective cluster structures. In Section 4.3 we define the quasi-isomorphism and in Section 4.4 we prove its correctness. Section 4.5 closes with a significant application of the quasi-isomorphism in Section 4.3 – symmetries that are obvious in one space can be moved to another other space using the quasi-isomorphism, giving rise to non-obvious symmetries of cluster structures. We also relate the twist map on the Grassmannian (as given by Marsh and Scott [44]) to more transparent symmetries of Fock-Goncharov spaces.

We close the thesis with several appendices. Appendix A illustrates how Proposition 2.4.4 can be used to create new cluster structures from known ones. Starting from the cluster structure on the Grassmannian, and applying an appropriate map, we obtain a cluster structure on an affine space of *band matrices*.

Appendix B shows another application of the concept of a nerve. There is an argument, relying on the algebraic Hartogs' principle, that is typically used when trying to establish that a given cluster algebra is contained in another algebra. The argument involves checking a condition holds in a cluster and in each of its neighbors [17, Proposition 3.6]. Appenxi B generalizes this argument, showing that it suffices to check these conditions on an arbitrary nerve.

Appendix C discusses another important example – the cluster structure on the *base affine space* [16]. When k is even, this cluster algebra has the same cluster type as a Grassmannian. We give a (not fully fleshed out) description of a quasi-isomorphism between these spaces.

CHAPTER II

Seed orbits and quasi-homomorphisms

Before diving in to the definitions, we give an overview. Each cluster algebra is constructed from a set of data called a *seed pattern*. A seed consists of an exchange matrix B, a coefficient tuple $\mathbf{p}$, and a cluster $\mathbf{x}$. A collection of seeds related to each other by mutations in all possible directions forms a *seed pattern*.

In the most general cluster algebra setup – that of *non-normalized* seed patterns, the mutation recipe does not uniquely specify the new coefficient tuple $\mathbf{p}'$ from $\mathbf{p}$ and B. This ambiguity propagates through iterated mutations, and consequently the set of cluster variables is not uniquely determined by a choice of initial seed.

The usual way to remove this ambiguity is to impose the additional assumption that the coefficient group is endowed with an extra "auxiliary addition" operation (making it into a *semifield*), and then require the corresponding *normalization condition* to hold at every seed (cf. Section 2.4). This assumption is satisfied for the most important examples of cluster algebras, e.g. those studied in Chapter IV.

From our perspective, the difficulty in defining homomorphisms between normalized cluster algebras is rooted in the fact that the construction involves three operations: addition, multiplication, and the auxiliary addition. We suggest instead that even in the usual (i.e., normalized) setting, it is fruitful to consider maps that only preserve the structures intrinsic to non-normalized cluster algebras, ignoring the auxiliary addition. This leads us to consider *seed orbits*, the smallest equivalence classes of seeds on which the non-normalized mutation rules are unambiguous, and to the concept of a mutation pattern of seed orbits. The natural notion of a homomorphism between two such mutation patterns brings us to the definition of a *quasi-homomorphism*, a rational map (more precisely, a semifield homomorphism) that respects the seed orbit structure and commutes with mutations.

2.1 Preliminaries on non-normalized seed patterns

We define non-normalized seed patterns while fixing standard notation. For a number x we let $[x]_+ := \max(x, 0)$. We let $\mathrm{sign}(x)$ equal either -1, 0 or 1 according to whether x is negative, zero, or positive. We denote $\{1, \ldots, n\}$ by $[1, n]$.

The setup begins with a choice of *ambient field of rational functions* $\mathcal{F}$ with coefficients in a *coefficient group* $\mathbf{P}$. The coefficient group is an abelian multiplicative group without torsion. The ambient field is a field of rational functions in n variables with coefficients in $\mathbf{P}$: it is the set of expressions that can be made out of n elements $x_1, \ldots, x_n$ and the elements of of $\mathbf{P}$, using the standard arithmetic operations $+, -, \times$ and $\div$, under the usual notion of equivalence of such rational expressions. The integer n is called the *rank*.

Definition 2.1.1 (Non-normalized seed, [22, 20])**.** Let $\mathbf{P}$ and $\mathcal{F}$ be as above. A *non-normalized seed* in $\mathcal{F}$ is a triple $\Sigma = (B, \mathbf{p}, \mathbf{x})$, consisting of the following three ingredients:

- a skew-symmetrizable $n \times n$ matrix $B = (b_{ij})$,
- a *coefficient tuple* $\mathbf{p} = (p_1^{\pm}, \ldots, p_n^{\pm})$ consisting of $2n$ elements in $\mathbf{P}$,
- a *cluster* $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathcal{F}$, whose elements (called *cluster variables*) are

algebraically independent and freely generate $\mathcal{F}$ over $\mathbb{Q}\mathbf{P}$.

When the exchange matrix B is skew-symmetric (not merely skew-symmetrizable) it defines a quiver $Q(B)$. This quiver has vertex set $[1,n]$. There are directed arrows $i \to j$ in $Q(B)$ if and only if $b_{ij} > 0$, in which case there are b_{ij} such arrows. We sometimes treat the matrix B and the quiver $Q(B)$ as being the same object without comment.

Definition 2.1.2. A *labeled n-regular tree*, $\mathbb{T}_n$, is an n-regular tree with edges labeled by integers so that the set of labels emanating from each vertex is $[1,n]$. We write $t \xrightarrow{k} t'$ to indicate that vertices t, t' are joined by an edge with label k. An isomorphism $\mathbb{T}_n \to \overline{\mathbb{T}}_n$ of labeled trees $\mathbb{T}_n$ and $\overline{\mathbb{T}}_n$ sends vertices to vertices and edges to edges, preserving incidences of edges and the edge labels. Such an isomorphism is uniquely determined by its value at a single vertex $t \in \mathbb{T}_n$.

Definition 2.1.3 (Non-normalized seed pattern, [20, 22])**.** Let $\mathbf{P}$ and $\mathcal{F}$ be as above. A collection of non-normalized seeds in $\mathcal{F}$, with one seed $\Sigma(t) = (B(t), \mathbf{p}(t), \mathbf{x}(t))$ for each $t \in \mathbb{T}_n$, is called a *non-normalized seed pattern* if for each edge $t \xrightarrow{k} t'$, the seeds $\Sigma(t)$ and $\Sigma(t')$ are related by a *mutation in direction k*:

- The matrices $B(t)$ and $B(t')$ are related by a *matrix mutation*

$$b_{ij}(t') = \begin{cases} -b_{ij}(t) & \text{if } i = k \text{ or } j = k \\ b_{ij}(t) + \operatorname{sign}(b_{ik}(t))[b_{ik}(t)b_{kj}(t)]_+ & \text{otherwise,} \end{cases} \tag{2.1}$$

- the coefficient tuples $\mathbf{p}(t)$ and $\mathbf{p}(t')$ are related by

$$p_k^{\pm}(t') = p_k^{\mp}(t) \text{ and} \tag{2.2}$$

$$\frac{p_j^+(t')}{p_j^-(t')} = \begin{cases} \frac{p_j^+(t)}{p_j^-(t}p_k^+(t)^{b_{kj}(t)} & \text{if } b_{kj} \geq 0 \\ \frac{p_j^+(t)}{p_j^-(t)}p_k^-(t)^{b_{kj}(t)} & \text{if } b_{kj} \leq 0 \end{cases} \tag{2.3}$$

when $j \neq k$,

- and the clusters $\mathbf{x}(t)$ and $\mathbf{x}(t')$ are related by

$$x_j(t') = x_j(t) \text{ for } j \neq k \text{, and} \tag{2.4}$$

$$x_k(t)x_k(t') = p_k^+ \prod x_j(t)^{[b_{jk}]_+} + p_k^- \prod x_j(t)^{[-b_{jk}]_+}, \tag{2.5}$$

the latter of which is called an *exchange relation*.

The rules (2.1) through (2.5) are ambiguous, meaning $\Sigma(t')$ is not determined uniquely from $\Sigma(t)$. Indeed, since (2.3) only mentions the ratio $\frac{p_j^+(t')}{p_j^-(t')}$, for each $j \neq k$ one can rescale both of $p_j^+(t')$ and $p_j^-(t')$ by a common element of $\mathbf{P}$ while preserving (2.3). We write $\Sigma \overset{\mu_k}{\leftrightsquigarrow} \Sigma'$ to indicate that two seeds Σ and Σ' are related by a mutation in direction k; this condition is symmetric in Σ and Σ'.

Remark 2.1.4. Non-normalized seed patterns have been around since the inception of cluster algebras [22, 3, 20]. The more restrictive notion of a *normalized seed pattern* is given in Definition 2.4.1. Normalized seeds are much more studied in the literature, where they are usually simply called *seeds*. Thus, we persistently use the adjective *non-normalized* to describe the seed patterns we are considering.

Thinking of (2.5) as a recipe for computing $x_k(t')$ from $\Sigma(t)$, we crucially observe that the computation is *subtraction-free*: the only operations needed are $+, \times$ and $\div$ in $\mathcal{F}$. This motivates the following definition:

Definition 2.1.5 (Ambient semifield)**.** Let $\mathcal{E}$ be a non-normalized seed pattern, and $\mathbf{x}(t)$ one of its clusters. The *ambient semifield*, $\mathcal{F}_{>0} = \mathcal{F}_{>0}(\mathcal{E}) \subset \mathcal{F}$ is the subset of all elements which can be given as a *subtraction-free* rational expression in the elements of $\mathbf{x}(t)$, with coefficients in $\mathbf{P}$. Thus, it is the set of rational functions which can be built out of $x_1(t), \ldots, x_n(t)$ and the elements of $\mathbf{P}$ using the operations $+, \times$ and $\div$ in $\mathcal{F}$.

Since (2.5) is subtraction-free, $\mathcal{F}_{>0}$ is independent of the choice of t (it only depends on $\mathcal{E}$), and every cluster variable for $\mathcal{E}$ lies in $\mathcal{F}_{>0}$. Recall that a *semifield* is an abelian multiplicative group, with an additional binary operation (called the *auxiliary addition*) that is commutative and associative, such that multiplication distrbutes over the auxiliary addition. The ambient semifield is a semifield with respect to the multiplication and addition operations in $\mathcal{F}$, justifying its name. Homomorphisms between semifields are defined in the obvious way. The ambient semifield has the following universality property.

Lemma 2.1.6 ([24, Definition 2.1])**.** *Let $\mathcal{E}$ be a non-normalized seed pattern with coefficient group $\mathbf{P}$ and ambient semifield $\mathcal{F}_{>0}$. Fix a cluster $\mathbf{x}(t)$ in $\mathcal{E}$. Let $\mathcal{S}$ be* any *semifield. Then given a multiplicative group homomorphism $\mathbf{P} \to \mathcal{S}$, and a function $\mathbf{x}(t) \to \mathcal{S}$, there exists a unique semifield homomorphism $\mathcal{F}_{>0} \to \mathcal{S}$ agreeing with the given maps on $\mathbf{P} \cup \mathbf{x}(t)$.*

The following elements of $\mathcal{F}_{>0}$ will play a prominent role in Section 2.2.

Definition 2.1.7 (Hatted variables)**.** Let $\mathcal{E}$ be a non-normalized seed pattern. Let $\hat{\mathbf{y}}(t) = (\hat{y}_1(t), \ldots, \hat{y}_n(t))$ denote the n-tuple of *hatted variables*

$$\hat{y}_j(t) = \frac{p_j^+(t)}{p_j^-(t)} \prod_i x_i(t)^{b_{ij}(t)}, \tag{2.6}$$

obtained by taking the ratio of the two terms on the right hand side of (2.5).

The hatted variables in adjacent seeds determine each other as follows:

Proposition 2.1.8 ([20, Proposition 2.9])**.** *Let $\mathcal{E} = (B(t), \mathbf{p}(t), \mathbf{x}(t))$ be a non-normalized seed pattern with hatted variables $\hat{\mathbf{y}}(t)$. For each edge $t \overset{k}{\to} t'$, the n-tuples*

$\hat{\mathbf{y}}(t)$ and $\hat{\mathbf{y}}(t')$ satisfy

$$\hat{y}_j(t') = \begin{cases} \hat{y}_j(t)^{-1} & \text{if } j = k \\ \hat{y}_j(t)\hat{y}_k(t)^{[b_{kj}(t)]_+}(\hat{y}_k(t)+1)^{-b_{kj}(t)} & \text{if } j \neq k. \end{cases} \tag{2.7}$$

The propagation rule (2.7) takes place in $\mathcal{F}_{>0}$, and only depends on the B matrix.

We recall a few more familiar cluster algebra definitions which are used in later sections.

Definition 2.1.9 (Exchange graph)**.** The *exchange graph* $\mathbf{E}$ associated with a seed pattern $\mathcal{E}$ is the graph whose vertices are the *unlabeled seeds* in $\mathcal{E}$, and whose edges correspond to mutations between these seeds. More precisely, permuting the indices $[1,n]$ in a non-normalized seed commutes with the mutation rules (2.1) through (2.5). The exchange graph is the n-regular graph obtained by identifying vertices $t_1, t_2 \in \mathbb{T}_n$ if the seeds $\Sigma(t_1)$ and $\Sigma(t_2)$ are permutations of each other. The *star neighborhood* $\mathrm{star}(t)$ of a vertex $t \in \mathbf{E}$ is the set of n edges adjacent to it. Rather than being indexed by $[1,n]$, the data in an unlabeled seed $\Sigma(t)$ for $t \in \mathbf{E}$ is indexed by the n seeds adjacent to $\Sigma(t)$, i.e. by the elements of $\mathrm{star}(t)$.

Definition 2.1.10 (Cluster algebra)**.** Let $\mathbf{P}$ be a free abelian multiplicative group of Laurent monomials in certain variables (known as *frozen variables*). Let $\mathcal{E}$ be a seed pattern with coefficients in $\mathbf{P}$. The *cluster algebra* $\mathcal{A}$ associated with $\mathcal{E}$ is the $\mathbb{Z}$-algebra generated by the frozen variables and all of the cluster variables arising in the seeds of $\mathcal{E}$.

Example 2.1.11. We now introduce a pair of affine algebraic varieties $\mathbf{X}$ and $\mathbf{Y}$ and a pair of seed patterns in their respective fields of rational functions. The cluster algebras associated with these seed patterns are the coordinate rings $\mathbb{C}[\mathbf{X}]$ and $\mathbb{C}[\mathbf{Y}]$. Both cluster algebras are of finite Dynkin type A_2.

Let $\mathbf{X} = \hat{\mathrm{Gr}}(3,5)$ be the affine cone over the Grassmann manifold of 3-dimensional planes in $\mathbb{C}^5$. The points in $\mathbf{X}$ are the simple tensors $\{x \wedge y \wedge z : x, y, z \in \mathbb{C}^5\} \subset \bigwedge^3(\mathbb{C}^5)$. The coordinate ring $\mathbb{C}[\mathbf{X}]$ is generated by the *Plücker coordinates* Δ_{ijk}, extracting the coefficient of $e_i \wedge e_j \wedge e_k$ in $x \wedge y \wedge z$, where $e_1, \ldots, e_5$ is the standard basis for $\mathbb{C}^5$.

The cluster structure on $\mathbb{C}[\mathbf{X}]$ is a special case of the one discussed in Section 4.1. The frozen variables are the Plücker coordinates

$$\Delta_{123}, \Delta_{234}, \Delta_{345}, \Delta_{145}, \Delta_{125}. \tag{2.8}$$

There are five cluster variables, listed in (2.9) with cyclically adjacent pairs of cluster variables forming clusters

$$\Delta_{245}, \Delta_{235}, \Delta_{135}, \Delta_{134}, \Delta_{124}. \tag{2.9}$$

The clusters and exchange relations are given in Figure 2.1. All of the other data in the seed pattern can be determined from these. For example, focusing on the seed whose cluster is $(x_1, x_2) = (\Delta_{235}, \Delta_{245})$, from the first and fifth exchange relations in Figure 2.1 follows

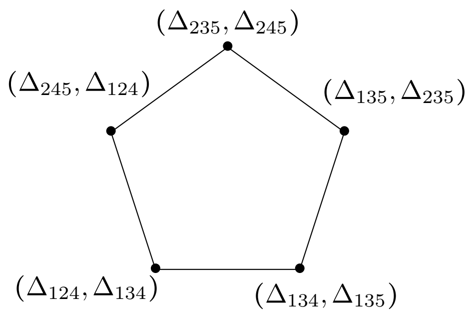
$$(p_1^+, p_1^-, p_2^+, p_2^-) = (\Delta_{125}\Delta_{234}, \Delta_{123}, \Delta_{145}, \Delta_{345}\Delta_{125}) \tag{2.10}$$

$$(\hat{y}_1, \hat{y}_2) = \left(\frac{\Delta_{125}\Delta_{234}}{\Delta_{123}\Delta_{245}}, \frac{\Delta_{145}\Delta_{235}}{\Delta_{345}\Delta_{125}}\right). \tag{2.11}$$

The exchange relations are written so that mutating is moving clockwise in the exchange graph. If a mutation moves counterclockwise, one should swap the order of the two terms in the exchange relation.

Second, let $\mathbf{Y} \cong \mathbb{C}^9$ the affine space of *band matrices* of the form

$$y = \begin{pmatrix} y_{1,1} & y_{1,2} & y_{1,3} & 0 & 0 \\ 0 & y_{2,2} & y_{2,3} & y_{2,4} & 0 \\ 0 & 0 & y_{3,3} & y_{3,4} & y_{3,5} \end{pmatrix}. \tag{2.12}$$



$$\Delta_{245}\Delta_{135} = \Delta_{145}\Delta_{235} + \Delta_{125}\Delta_{345}$$
$$\Delta_{235}\Delta_{134} = \Delta_{234}\Delta_{135} + \Delta_{123}\Delta_{345}$$
$$\Delta_{135}\Delta_{124} = \Delta_{125}\Delta_{134} + \Delta_{123}\Delta_{145}$$
$$\Delta_{134}\Delta_{245} = \Delta_{345}\Delta_{124} + \Delta_{123}\Delta_{345}$$
$$\Delta_{124}\Delta_{235} = \Delta_{123}\Delta_{245} + \Delta_{125}\Delta_{234}$$

Figure 2.1: The exchange graph for $\mathbb{C}[\mathbf{X}]$. The vertices are clusters and edges between vertices are mutations. Each mutation exchanges two cluster variables via an exchange relation listed in the table at top right. The extra data in each seed can be inferred from these exchange relations.

Its coordinate ring $\mathbb{C}[\mathbf{Y}]$ contains the *minors* $Y_{I,J}$. Evaluating $Y_{I,J}$ on $y \in \mathbf{Y}$ returns the minor of y occupying rows I and columns J, e.g. $Y_{i,j}(y) = y_{i,j}$ and $Y_{12,23}(y) = y_{1,2}y_{2,3} - y_{1,3}y_{2,2}$. Some of these minors factor, e.g. $Y_{12,13} = Y_{1,1}Y_{2,3}$.

Figure 2.2 shows a seed pattern whose cluster algebra is $\mathbb{C}[\mathbf{Y}]$. The frozen variables are the following minors

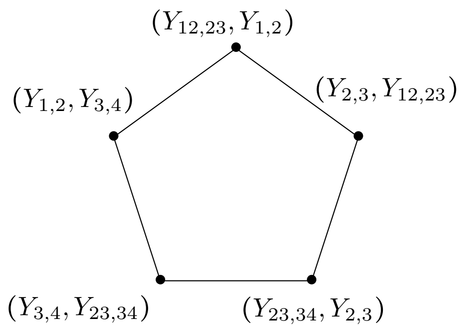
$$Y_{1,1}, Y_{2,2}, Y_{3,3}, Y_{1,3}, Y_{2,4}, Y_{3,5}, Y_{123,234}. \tag{2.13}$$

The cluster variables are listed in (2.14), with cyclically adjacent pairs forming clusters

$$Y_{1,2}, Y_{12,23}, Y_{2,3}, Y_{23,34}, Y_{3,4}. \tag{2.14}$$

2.2 Seed orbits

We introduce seed orbits by first describing them as equivalence classes under a certain equivalence relation on seeds. Proposition 2.2.3 gives another characterization as orbits under an explicit rescaling action.



$$
\begin{aligned}
Y_{1,2}Y_{2,3} &= Y_{12,23} + Y_{2,2}Y_{1,3} \\
Y_{12,23}Y_{23,34} &= Y_{123,234}Y_{2,3} + Y_{2,2}Y_{3,3}Y_{1,3}Y_{2,4} \\
Y_{2,3}Y_{3,4} &= Y_{23,34} + Y_{3,3}Y_{2,4} \\
Y_{23,34}Y_{1,2} &= Y_{2,2}Y_{1,3}Y_{3,4} + Y_{123,234} \\
Y_{3,4}Y_{12,23} &= Y_{3,3}Y_{2,4}Y_{1,2} + Y_{123,234}
\end{aligned}
$$

Figure 2.2: The exchange graph and exchange relations for $\mathbb{C}[\mathbf{Y}]$, mirroring Figure 2.1.

Definition 2.2.1. Let $\vec{k} = (k_1, \dots, k_\ell)$ be a sequence of elements of $[1, n]$. Choosing a base point $t_0 \in \mathbb{T}_n$, such a sequence determines a walk $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_\ell} t_\ell$ in $\mathbb{T}_n$. We say that $\vec{k}$ is *contractible* if this walk starts and ends at the same vertex of $\mathbb{T}_n$, i.e. $t_\ell = t_0$.

Given non-normalized seeds Σ and Σ^*, we write $\Sigma \sim \Sigma^*$ if there is a contractible sequence of mutations from Σ to Σ^*, i.e. a contractible sequence $\vec{k}$ and non-normalized seeds $\Sigma_1, \dots, \Sigma_{\ell-1}$ such that

$$\Sigma = \Sigma_0 \overset{\mu_{k_1}}{\leftrightsquigarrow} \Sigma_1 \overset{\mu_{k_2}}{\leftrightsquigarrow} \Sigma_2 \overset{\mu_{k_3}}{\leftrightsquigarrow} \cdots \overset{\mu_{k_{\ell-1}}}{\leftrightsquigarrow} \Sigma_{\ell-1} \overset{\mu_{k_\ell}}{\leftrightsquigarrow} \Sigma_\ell = \Sigma^*. \tag{2.15}$$

Clearly, $\sim$ is an equivalence relation on non-normalized seeds. Furthermore, it removes the ambiguity present in mutation of non-normalized seeds:

Lemma 2.2.2. *The mutation rule* $\overset{\mu_k}{\leftrightsquigarrow}$ *becomes unambiguous and involutive once it is thought of as a rule on equivalence classes of seeds under* $\sim$. *That is, fixing a* $\sim$*-equivalence class* $\mathfrak{S}$ *and a direction* $k \in [n]$*, the set of seeds*

$$\{\Sigma' \colon \Sigma' \overset{\mu_k}{\leftrightsquigarrow} \Sigma \textit{ for some } \Sigma \in \mathfrak{S}\} \tag{2.16}$$

is again a $\sim$*-equivalence class of seeds.*

We now characterize $\sim$-equivalence classes explicitly. We say two elements $z, x \in \mathcal{F}$ are *proportional*, written $z \asymp x$, if $\frac{z}{x} \in \mathbf{P}$. We emphasize that $\mathbf{P}$ does not include constants, e.g. $-1, 2 \notin \mathbf{P}$, and thus x is not proportional to $-x, 2x$, etc.

Proposition 2.2.3 (Seed orbits)**.** *Let* $\Sigma = (B, \mathbf{p}, \mathbf{x}), \Sigma^* = (B^*, \mathbf{p}^*, \mathbf{x}^*)$ *be non-normalized seeds in* $\mathcal{F}$*, of rank* $n \geq 2$*, with* $\mathbf{x} = (x_i), \mathbf{p} = (p_i^{\pm}), \mathbf{x}^* = (x_i^*), \mathbf{p}^* = ((p^*)_i^{\pm})$*. Then the following are equivalent:*

1. $\Sigma \sim \Sigma^*$.
2. $B = B^*, \hat{\mathbf{y}}(\Sigma) = \hat{\mathbf{y}}(\Sigma^*)$*, and* $x_i \asymp x_i^*$ *for all* i*.*
3. $B = B^*$*, and there exist* $c_1, \dots, c_n, d_1, \dots, d_n \in \mathbf{P}$*, such that*

$$x_j^* = \frac{x_j}{c_j} \tag{2.17}$$

$$(p^*)_j^{\pm} = \frac{p_j^{\pm}}{d_j} \prod c_i^{[\pm b_{ij}]_+}. \tag{2.18}$$

Equations (2.17) and (2.18) define a *rescaling action* of $\mathbf{P}^n \times \mathbf{P}^n$ on non-normalized seeds, denoted by $(\vec{c}, \vec{d}) \cdot \Sigma$ where $(\vec{c}, \vec{d}) \in \mathbf{P}^n \times \mathbf{P}^n$ and Σ is a non-normalized seed. Proposition 2.2.3 says that a $\sim$-equivalence class of non-normalized seeds is precisely a $\mathbf{P}^n \times \mathbf{P}^n$ orbit under this action; we henceforth refer to these equivalence classes as *seed orbits.*

Proof. Conditions (2) and (3) are a re-translation of each other by immediate calculation.

We show (1) implies (2). Defining a seed orbit by (2), this implication follows from the fact that seed orbits are "closed under mutation." More precisely, if Σ and $\Sigma^\dagger = (\vec{c}, \vec{d}) \cdot \Sigma$ are in the same seed orbit and Σ' and $(\Sigma^\dagger)'$ are two seeds satisfying $\Sigma \overset{\mu_k}{\leftrightsquigarrow} \Sigma'$ and $\Sigma^\dagger \overset{\mu_k}{\leftrightsquigarrow} (\Sigma^\dagger)'$, then Σ' and $(\Sigma^\dagger)'$ are in the same seed orbit. By (2.1) and Proposition 2.1.8, we know that $B' = (B^\dagger)'$ and $\hat{\mathbf{y}}' = (\hat{\mathbf{y}}^\dagger)'$, so the claim will follow if we check $(x^\dagger)'_j \asymp x'_j$ for all j. This is obvious when $j \neq k$ from (2.4). When $j = k$, (2.5) for the mutation $\Sigma^\dagger \overset{\mu_k}{\leftrightsquigarrow} (\Sigma^\dagger)'$ says that

$$(x^\dagger)'_k = (x^\dagger_k)^{-1}((p^\dagger)^+_k \prod (x^\dagger_j)^{[b_{jk}]_+} + (p^\dagger)^-_k \prod (x^\dagger_j)^{[-b_{jk}]_+}) \tag{2.19}$$

$$= (x^\dagger_k)^{-1}(p^\dagger)^-_k(1 + \hat{y}_k(\Sigma^\dagger)) \prod (x^\dagger_j)^{[-b_{jk}]_+} \tag{2.20}$$

$$= \frac{c_k}{d_k}(x_k)^{-1}p^-_k(1 + \hat{y}_k(\Sigma)) \prod x_j^{[-b_{jk}]_+} \tag{2.21}$$

$$= \frac{c_k}{d_k}x'_k, \tag{2.22}$$

as desired. Returning to the implication (1) $\Rightarrow$ (2), from the symmetry of $\overset{\mu_k}{\leftrightsquigarrow}$ it follows that Σ is related to itself along any contractible sequence $\vec{k}$. Since seed orbits are closed under mutation, any seed Σ^* related to Σ by a contractible sequence of mutations is therefore in the same seed orbit as Σ.

Now we show (3) implies (1). Let $\hat{c}_j(a) \in \mathbf{P}^n \times \mathbf{P}^n$ denote the vector with $c_j = a$ and all other entries equal to 1, and define similarly $\hat{d}_j(a)$. Clearly, it suffices to show that $\Sigma \sim \hat{c}_j(a) \cdot \Sigma$ and $\Sigma \sim \hat{d}_j(a) \cdot \Sigma$, since rescalings of this type generate $\mathbf{P}^n \times \mathbf{P}^n$.

Seeds of the form $\hat{d}_j(a) \cdot \Sigma$ are equivalent to Σ, as follows by mutating twice in any direction $k \neq j$. For seeds of the form $\hat{c}_j(a) \cdot \Sigma$, let Σ' be any seed satisfying $\Sigma \overset{\mu_j}{\leftrightsquigarrow} \Sigma'$:

$$\Sigma \sim \hat{d}_j(a^{-1}) \cdot \Sigma \tag{2.23}$$

$$\overset{\mu_j}{\leftrightsquigarrow} (\hat{c}_j(a^{-1})\hat{d}_j(a^{-1})) \cdot \Sigma' \tag{2.24}$$

$$\sim (\hat{c}_j(a^{-1})) \cdot \Sigma' \tag{2.25}$$

$$\overset{\mu_j}{\leftrightsquigarrow} \hat{c}_j(a) \cdot \Sigma, \tag{2.26}$$

where (2.24) and (2.26) follow from the calculation in (2.22), and (2.23) and (2.25) are admissible since we already know rescaling by $\hat{d}_j(a)$ preserves equivalence of seeds. Since (2.23) through (2.26) amounts to mutating in direction j twice on seed equivalence classes, it follows that $\Sigma \sim \hat{c}_j(a) \cdot \Sigma$ as desired. $\square$

2.3 Quasi-homomorphisms

We will now give the definition of a quasi-homomorphisms from a seed pattern $\mathcal{E}$ to another seed pattern $\overline{\mathcal{E}}$. We retain the notation of Section 2.1 for all the data in $\mathcal{E}$, and we use bars to denote the analogous quantities in the second pattern $\overline{\mathcal{E}}$. Thus $\overline{\mathcal{E}}$ has coefficient group $\overline{\mathbf{P}}$, ambient field $\overline{\mathcal{F}}$, seeds $\overline{\Sigma}(\overline{t}) = (\overline{B}(\overline{t}), \overline{\mathbf{p}}(\overline{t}), \overline{\mathbf{x}}(\overline{t}))$, hatted variables $\hat{\overline{y}}_j(\overline{t})$, and so on. It is built on a second copy of the n-regular tree, $\overline{\mathbb{T}}_n$.

The motivating observation is the following: since the mutation rules (2.3) through (2.5) are certain algebraic relations in in $\mathcal{F}_{>0}$, they are preserved by a homomorphism of semifields.

Definition 2.3.1 (Quasi-homomorphism)**.** Let $\mathcal{E}$ and $\overline{\mathcal{E}}$ be non-normalized seed patterns. Let $\Psi\colon \mathcal{F}_{>0} \to \overline{\mathcal{F}}_{>0}$ be a semifield homomorphism satisfying $\Psi(\mathbf{P}) \subset \overline{\mathbf{P}}$ (in this case we say Ψ *preserves coefficients*). We say Ψ is a *quasi-homomorphism from $\mathcal{E}$ to $\overline{\mathcal{E}}$* if it maps each seed in $\mathcal{E}$ to a seed that is $\sim$-equivalent to a seed in $\overline{\mathcal{E}}$, in a way that is compatible with mutation. More precisely, Ψ is a quasi-homomorphism if

$$\Psi(\Sigma(t)) \sim \overline{\Sigma}(\overline{t}), \tag{2.27}$$

for all $t \in \mathbb{T}_n$, where $t \mapsto \overline{t}$ is an isomorphism of the labeled trees $\mathbb{T}_n$ and $\overline{\mathbb{T}}_n$, and $\Psi(\Sigma(t)) = (B(t), \Psi(\mathbf{p}), \Psi(\mathbf{x}))$ is the triple obtained by evaluating Ψ on $\Sigma(t)$.

Remark 2.3.2. We say two seed patterns $\mathcal{E}$ and $\overline{\mathcal{E}}$ have the *same cluster type* if their underlying pattern of exchange matrices coincide. We see that quasi-homomorphisms are only defined between seed patterns having the same cluster type.

As motivation for Definition 2.3.1, we imagine a situation where $\mathcal{E}$ is well understood combinatorially, and we would like to understand another seed pattern $\overline{\mathcal{E}}$ by comparing it with $\mathcal{E}$. The requirement (2.27) says that the seeds $\Psi(\Sigma(t))$ mutate "in parallel" with the seeds in $\overline{\mathcal{E}}$, in the sense that their corresponding seeds only differ by the rescalings (2.17) and (2.18).

The following Propositions 2.3.3 and 2.3.4 show two ways in which quasi-homomorphisms are well-behaved. Both of their proofs follow immediately from the observation that applying a semifield homomorphism commutes with mutation.

Proposition 2.3.3. *Let* $\Psi: \mathcal{F}_{>0} \to \overline{\mathcal{F}}_{>0}$ *be a semifield homomorphism satisfying* (2.27) *for* some $t \in \mathbb{T}_n$*. Then* Ψ *is a quasi-homomorphism.*

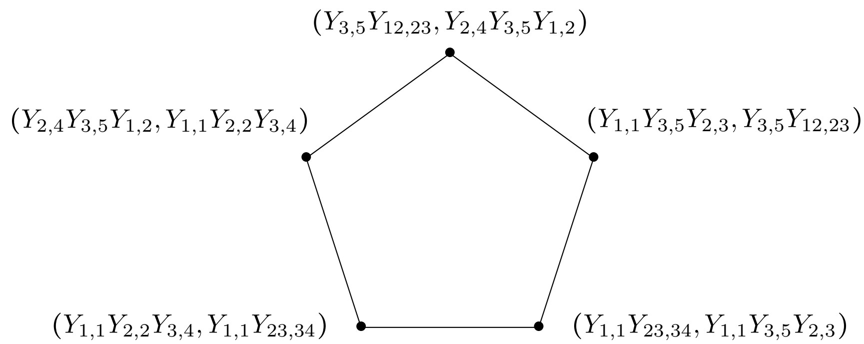
That is, rather than checking that (2.27) holds at *every* $t \in \mathbb{T}_n$, it suffices to check this at a single $t \in \overline{\mathbb{T}}_n$.

Proposition 2.3.4. *Let* Ψ *be a quasi-homomorphism from* $\mathcal{E}$ *to* $\overline{\mathcal{E}}$*. Let* Σ *be a seed in* $\mathcal{E}$*, and let* Σ^* *be a non-normalized seed satisfying* $\Sigma \sim \Sigma^*$*. Then* $\Psi(\Sigma) \sim \Psi(\Sigma^*)$*.*

Proposition 2.3.4 says that quasi-homomorphism preserves ~-equivalence of seeds. Thus, if $\mathfrak{S}(t)$ denotes the seed orbit of $\Sigma(t)$ and ditto for $\overline{\mathfrak{S}}(\bar{t})$ and $\overline{\Sigma}(\bar{t})$, then Ψ maps $\mathfrak{S}(t)$ inside $\overline{\mathfrak{S}}(\bar{t})$ for all t. A quasi-homomorphism is therefore a natural notion of homomorphism between the respective seed orbit patterns $(t, \mathfrak{S}(t))$ and $(\bar{t}, \overline{\mathfrak{S}}(\bar{t}))$.

Now we describe a quasi-homomorphism between the pair of seed patterns in Example 2.1.11.

Example 2.3.5. Given $Y \in \mathbf{Y}$, let $Y[1], Y[2], Y[3] \in \mathbb{C}^5$ denote its rows. There is a surjective map of varieties $F: \mathbf{Y} \to \mathbf{X}$ sending $Y \overset{F}{\mapsto} Y[1] \wedge Y[2] \wedge Y[3]$. It determines a map on cluster algebras $F^*: \mathbb{C}[\mathbf{X}] \to \mathbb{C}[\mathbf{Y}]$ sending $\Delta_{ijk} \mapsto Y_{123,ijk}$. Figure 2.3 shows the non-normalized seed pattern that arises from applying F^* to Figure 2.1 and factoring the cluster variables inside $\mathbb{C}[\mathbf{Y}]$. The seeds in Figure 2.3 are in the same seed orbit as the corresponding seeds in Figure 2.2, and thus F^* is a quasi-homomorphism from $\mathbb{C}[\mathbf{X}]$ to $\mathbb{C}[\mathbf{Y}]$.



$$
\begin{aligned}
Y_{2,4}Y_{3,5}Y_{1,2} \cdot Y_{1,1}Y_{3,5}Y_{2,3} &= Y_{1,1}Y_{3,5}^2Y_{2,4}(Y_{12,23} + Y_{2,2}Y_{1,3}) \\
Y_{3.5}Y_{12,23} \cdot Y_{1,1}Y_{23,34} &= Y_{1,1}Y_{3,5}(Y_{123,234}Y_{2,3} + Y_{2,2}Y_{3,3}Y_{1,3}Y_{2,4}) \\
Y_{1.1}Y_{3.5}Y_{2,3} \cdot Y_{1.1}Y_{2.2}Y_{3,4} &= Y_{1.,1}^2Y_{2,2}Y_{35}(Y_{23,34} + Y_{3,3}Y_{24}) \\
Y_{1.1}Y_{23,34} \cdot Y_{24}Y_{3.5}Y_{1,2} &= Y_{1.1}Y_{24}Y_{3,5}(Y_{2,2}Y_{1,3}Y_{3,4} + Y_{123,234}) \\
Y_{1.1}Y_{2.2}Y_{3,4} \cdot Y_{35}Y_{12,23} &= Y_{1.1}Y_{22}Y_{3,5}(Y_{3,3}Y_{2,4}Y_{1,2} + Y_{123,234})
\end{aligned}
$$

Figure 2.3: The non-normalized seed pattern obtained by applying F^* to the seed pattern in Figure 2.1. The clusters agree with the clusters in Figure 2.2 up to the frozen variables listed in (2.13). Cancelling the common frozen variable factors from both sides of the exchange relations yields the exchange relations in Figure 2.2. It follows that the $\hat{y}$ values are the same in both figures.

Definition 2.3.6. Two quasi-homomorphisms Ψ_1, Ψ_2 from $\mathcal{E}$ to $\overline{\mathcal{E}}$ are called *proportional* if $\Psi_1(\Sigma) \sim \Psi_2(\Sigma)$ for all seeds Σ in $\mathcal{E}$. We say a quasi-homomorphism Ψ from $\mathcal{E}$ to $\overline{\mathcal{E}}$ is a *quasi-isomorphism* if there is a quasi-homomorphism Φ from $\overline{\mathcal{E}}$ to $\mathcal{E}$ such that $\Phi \circ \Psi$ is proportional to the identity map on $\mathcal{F}_{>0}$. We say that Ψ and Φ are *quasi-inverses* of one another.

Once we have a quasi-isomorphism between two seed patterns, we think of them as being essentially "the same." Up to coefficients, the maps in both directions allows us to write the cluster variables in one seed pattern in terms of the cluster variables in the other one.

Remark 2.3.7. The set of seed patterns with quasi-homomorphisms as morphisms is a category. Proportionality is an equivalence relation on the morphisms in this category, and this equivalence relation respects composition of quasi-homomorphisms.

This yields a quotient category whose objects are seed patterns and whose morphisms are proportionality classes of quasi-homomorphisms. A morphism in this quotient category is an isomorphism if and only if one (hence any) of its constituent quasi-homomorphisms is a quasi-isomorphism.

The following lemma provides a simple method for checking a candidate map is a quasi-inverse of a given quasi-homomorphism.

Lemma 2.3.8. *Let Ψ be a quasi-homomorphism from $\mathcal{E}$ to $\overline{\mathcal{E}}$, and $\Phi: \overline{\mathcal{F}}_{>0} \to \mathcal{F}_{>0}$ a semifield map that preserves coefficients and for which $\Phi \circ \Psi(x) \asymp x$ for all cluster variables x in $\mathcal{E}$. Then Ψ and φ are quasi-inverse quasi-isomorphisms.*

Lemma 2.3.8 follows from the more general Proposition 2.5.2 below. In fact, it will suffice to merely check that $\varphi \circ \Psi(x) \asymp x$ for all x lying on a *nerve* (cf. Definition 2.5.1).

Example 2.3.9. Using Lemma 2.3.8 we describe a quasi-inverse G^* for the quasi-homomorphism F^* from Example 2.3.5. Let $G: \mathbf{X} \to \mathbf{Y}$ be the morphism sending $X \in \mathbf{X}$ to the the band matrix

$$G(X) = \begin{pmatrix} \Delta_{145}(X) & \Delta_{245}(X) & \Delta_{345}(X) & 0 & 0 \\ 0 & \Delta_{125}(X) & \Delta_{135}(X) & \Delta_{145}(X) & 0 \\ 0 & 0 & \Delta_{123}(X) & \Delta_{124}(X) & \Delta_{125}(X) \end{pmatrix} \in \mathbf{Y}$$

all of whose entries are Plücker coordinates of X. The coordinate ring map $\mathbb{C}[\mathbf{Y}] \to \mathbb{C}[\mathbf{X}]$ sends $Y_{i,j}$ to the Plücker coordinate in the (i,j) entry of $G(X)$, e.g. $G^*(Y_{1,2}) = \Delta_{245}$.

The matrix $G(X)$ has an interesting property: all of its minors are monomials in the Plücker coordinates of X. In particular, its maximal minors agree with those of

X, up to a multiplicative factor:

$$\Delta_{ijk}(G(X)) = \Delta_{145}(X)\Delta_{125}(X)\Delta_{ijk}(X). \tag{2.28}$$

Thus, $G^* \circ F^*(\Delta_{ijk}) = \Delta_{145}\Delta_{125}\Delta_{ijk} \asymp \Delta_{ijk}$ for each cluster variable Δ_{ijk}. Since G^* preserves coefficients (the only nontrivial check is $G^*(Y_{123,234}) = \Delta_{125}\Delta_{145}\Delta_{234}$), from Lemma 2.3.8 it follows that G^* is a quasi-inverse of F^*.

Remark 2.3.10. A quasi-homomorphism is defined as a map on ambient semifields since these maps transparently preserve the mutation rules (2.1)–(2.5). This should be suitable for most purposes, since one is mostly interested in evaluating a quasi-homomorphism on cluster variables or coefficients. However, the cluster algebra $\mathcal{A}$ is the more familiar algebraic object associated to a seed pattern. If one wants to think of a quasi-homomorphism Ψ as an algebra map of cluster algebras $\mathcal{A} \to \overline{\mathcal{A}}$, one will sometimes need to first localize at (i.e., invert) the frozen variables in $\overline{\mathcal{A}}$.

Definition 2.3.11. A seed is called *indecomposable* if the underlying graph described by its exchange matrix is connected (this graph has vertex set $[1,n]$ and an edge (i,j) if and only if $b_{i,j} \neq 0$). For a seed $\Sigma = (B,\mathbf{p},\mathbf{x})$, the *opposite seed* $\Sigma^{\mathrm{opp}} = (B^{\mathrm{opp}},\mathbf{p}^{\mathrm{opp}},\mathbf{x}^{\mathrm{opp}})$ is the seed defined by $B^{\mathrm{opp}} = -B$, $(p^{\mathrm{opp}})^{\pm}_j = p^{\mp}_j$, and $x^{\mathrm{opp}}_i = x_i$. It satisfies $\hat{y}^{\mathrm{opp}}_j = \frac{1}{\hat{y}_j}$. The operations of restricting to an indecomposable component and replacing a seed by its opposite seed both commute with mutation.

Remark 2.3.12. It can be useful to modify Definition 2.3.1 by also allowing $\Psi(\Sigma(t)) \sim \overline{\Sigma}(\bar{t})^{\mathrm{opp}}$ in (2.27). This technicality doesn't arise when considering quasi-homomorphisms between *different* seed patterns (since we can always replace one of the seed patterns by its opposite without any essential changes). However, this more general definition is useful when discussing maps from a given seed pattern to itself (cf. Section 3.1).

We close this section by explaining the differences between quasi-homomorphisms and preexisting notions of homomorphisms between cluster algebras.

Coefficient specializations were introduced by Fomin and Zelevinsky [24] and studied by Reading [50, 51]. Speaking briefly, a *coefficient specialization* is an algebra map that sends each cluster variable to a cluster variable (and preserves clusters), but whose underlying map on coefficients can be any group homomorphism $\mathbf{P} \to \overline{\mathbf{P}}$.

A category of rooted cluster algebras was introduced by Assem, Dupont, and Shiffler [1]. The objects in this category are rooted seed patterns (i.e., a seed pattern together with a choice of initial seed). A morphism is an algebra map that sends each cluster variable in the initial seed either to a cluster variable or an integer, and sends each frozen variable to either a frozen variable, a cluster variable, or an integer.

The key difference between these notions and quasi-homomorphisms is that a quasi-homomorphism allows for cluster variables to be *rescaled* by an element of $\overline{\mathbf{P}}$. On the other hand, while quasi-homomorphisms are more flexible in allowing for cluster variables to be rescaled and for frozen variables to be sent to *(Laurent) monomials* in the frozen variables, they are also less flexible as they do not allow for unfreezing frozen variables or specializing variables to integers. It probably would not be hard to combine these two notions.

Finally: we formulated Definition 2.3.1 so that quasi-homomorphisms preserve exchange matrices, whereas rooted cluster morphisms allow for the an exchange matrix to be sent to its opposite. To make Definition 2.3.1 more consonant with preexisting notions, we can modify it as in Remark 2.3.12 without significant changes.

2.4 Normalized seed patterns

Now we recall the definition of normalized seed patterns and apply the results of Section 2.2 in the case that $\mathcal{E}$ and $\overline{\mathcal{E}}$ are normalized.

Definition 2.4.1 (Normalized seed pattern)**.** A seed pattern $\mathcal{E}$ as in Definition 2.1.3 is called *normalized* if the coefficient group $\mathbf{P}$ is a semifield, and each coefficient tuple $\mathbf{p}(t)$ satisfies

$$p_j^+(t) \oplus p_j^-(t) = 1 \text{ for all } j, \tag{2.29}$$

where $\oplus$ is the addition in $\mathbf{P}$.

The advantage of this normalization condition is that it makes the mutation rule (2.3), and therefore mutation of normalized seeds, unambiguous. Indeed, (2.3) specifies the ratio $y_j(t') = \frac{p_j^+(t')}{p_j^-(t')}$ in terms of $B(t)$ and $\mathbf{p}(t)$, and there is a unique choice of $p_j^\pm(t') \in \mathbf{P}$ with this ratio and satisfying the normalization condition, namely the pair

$$p_j^+(t') = \frac{y_j(t')}{1 \oplus y_j(t')} \text{ and } p_j^-(t') = \frac{1}{1 \oplus y_j(t')}. \tag{2.30}$$

Furthermore, mutating a normalized seed twice in a given direction is the identity map.

At the same time, the disadvantage is that computing a cluster algebra now involves three operations, the two operations present in $\mathcal{F}_{>0}$ along with $\oplus$ in $\mathbf{P}$. The definition of quasi-homomorphism prioritizes these first two operations. Proposition 2.4.4 says that in the case of a quasi-homomorphism between normalized seed patterns, there is a "separation of additions" phenomenon, separating the addition in $\overline{\mathcal{F}_{>0}}$ from the one in $\overline{\mathbf{P}}$.

Before stating Proposition 2.4.4, we say a little more about normalized seed patterns and Y-*patterns*. In a normalized seed pattern, the tuple of ratios $(y_1(t),\ldots,y_n(t))$ determines the coefficient tuple by (2.30). Accordingly, for normalized seed patterns one keeps track of $y_j(t)$ rather than $p_j^{\pm}(t)$. Rewriting (2.2) and (2.3) in terms of $y_j(t)$ determines a Y-*pattern recurrence in the semifield* $\mathbf{P}$:

$$y_j(t') = \begin{cases} y_j(t)^{-1} & \text{if } j = k \\ y_j(t)y_k(t)^{[b_{kj}(t)]_+}(y_k(t)\oplus 1)^{-b_{kj}(t)} & \text{if } j \neq k. \end{cases} \tag{2.31}$$

A collection of quantities $(B(t),\mathbf{y}(t))_{t\in\mathbb{T}_n}$ satisfying (2.1) and (2.31), with the $\mathbf{y}(t)$ lying in some semifield $\mathcal{S}$, is called a *Y-pattern in the semfield* $\mathcal{S}$. Notice that the concept of semifield is now playing two different roles, either as the ambient semifield $\mathcal{F}_{>0}$ in which the exchange relation calculations take place, or as the coefficient semifield $\mathbf{P}$ used to remove the ambiguity in mutation of seeds. The surprising connection between these two roles is Proposition 2.1.8, which we now recognize as saying that the $(B(t),\hat{\mathbf{y}}(t))$ form a Y-pattern in the ambient semfield $\mathcal{F}_{>0}$.

The most important example of a coefficient semifield arising in applications is the *tropical semifield*.

Definition 2.4.2. A *tropical semifield* is a free abelian multiplicative group in some generators $u_1,\ldots,u_m$, with auxiliary addition $\oplus$ given by

$$\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j,b_j)}.$$

A normalized seed pattern over a tropical semifield is said to be *of geometric type*.

For a seed pattern $\mathcal{E}$ of geometric type, we typically denote the frozen variables by $x_{n+1},\ldots,x_{n+m}$. An *extended cluster* in $\mathcal{E}$ is a union $\mathbf{x}(t)\cup\{x_{n+1},\ldots,x_{n+m}\}$ of a cluster in $\mathcal{E}$ with the set of frozen variables. The data of $(B(t),\mathbf{y}(t))$ is entirely described by an $(n+m)\times n$ matrix $\tilde{B}(t) = (b_{ij}(t))$, called the *extended exchange matrix*. Its top $n\times n$ submatrix is $B(t)$ and is called the *principal part*. Its bottom m rows are called *coefficient rows*. They are specified from the equality $y_j(t) = \prod_{i=1}^{m} x_{n+i}^{b_{n+i,j}(t)}$. The mutation rule (2.31) translates into the rule (2.1) on $\tilde{B}$.

If the principal part of an extended exchange matrix $\tilde{B}$ is skew-symmetric, we typically identify $\tilde{B}$ with its corresponding *ice quiver*. This is the quiver for $\tilde{B}$ drawn so that vertices corresponding to frozen variables are indicated by boxes (cf. Figure 3.2, Figure 4.2).

Example 2.4.3. The seed pattern in Figure 2.1 is of geometric type over the tropical semifield in the frozen variables in (2.8). The same holds for the seed pattern in Figure 2.2 over the frozen variables in (2.13). On the other hand, the seed pattern in Figure 2.3 is not normalized, e.g. the first exchange relation there satisfies $p_2^+ \oplus p_2^- = Y_{1,1}Y_{2,4}Y_{3,5}$.

Now we state Proposition 2.4.4 describing quasi-homomorphisms between normalized seed patterns $\mathcal{E}$ and $\overline{\mathcal{E}}$. It arose during the process of writing a forthcoming book on cluster algebras [21], in proving one direction of the finite type classification (namely, that a cluster algebra with a quiver whose principal part is an orientation of a Dynkin quiver necessarily has only finitely many seeds). We will state it as a recipe for *constructing* a normalized seed pattern from a given one, since we envision this being useful in applications. We illustrate this technique in appendix Section A.

Proposition 2.4.4. *Let $\mathcal{E}$ be a non-normalized seed pattern, with the usual notation. Let $(x_i) = (x_i(t_0))$ be a fixed initial cluster in $\mathcal{E}$. Let $\overline{\mathbf{P}}$ be a semifield, and $\overline{\mathcal{F}}_{>0}$ the semifield of subtraction-free rational expressions in algebraically independent elements $\overline{x}_1,\dots,\overline{x}_n$ with coefficients in $\overline{\mathbf{P}}$.*

Let $\Psi: \mathcal{F}_{>0} \to \overline{\mathcal{F}}_{>0}$ *be a semifield map satisfying* $\Psi(x_i) \asymp \overline{x}_i$*, and let* $c: \mathcal{F}_{>0} \to \overline{\mathbf{P}}$ *be the composition of semifield maps* $\mathcal{F}_{>0} \xrightarrow{\Psi} \overline{\mathcal{F}}_{>0} \xrightarrow{\overline{x}_i \mapsto 1} \overline{\mathbf{P}}$ *where the second map in this composition specializes all* $\overline{x}_i$ *to* 1 *and is the identity on* $\overline{\mathbf{P}}$*.*

Then there is a normalized seed pattern $\overline{\mathcal{E}}$ *in* $\overline{\mathcal{F}}_{>0}$ *with seeds* $(\overline{B}(\bar{t}), \overline{\mathbf{p}}(\bar{t}), \overline{\mathbf{x}}(\bar{t}))$ *satisfying*

$$\overline{B}(\bar{t}) = B(t) \tag{2.32}$$

$$\overline{x}_i(\bar{t}) = \frac{\Psi(x_i(t))}{c(x_i(t))} \tag{2.33}$$

$$\hat{\overline{y}}_i(\bar{t}) = \Psi(\hat{y}_i(t)) \tag{2.34}$$

$$\overline{y}_i(\bar{t}) = c(\hat{y}_i(t)). \tag{2.35}$$

Clearly, Ψ *is a quasi-homomorphism from* $\mathcal{E}$ *to* $\overline{\mathcal{E}}$*.*

Formulas (2.33) through (2.35) can be deduced from [20, Proposition 3.4]) to $\Psi(\mathcal{E})$, but we give a self-contained proof. A similar proof will appear in [21].

Proof. We can define a normalized seed pattern $\overline{\mathcal{E}}$ by requiring that (2.32), (2.33), and (2.35) hold when $t = t_0$, and it follows that (2.34) also holds when $t = t_0$. What remains is to check that (2.33) through (2.35) hold for all t, which means checking the right hand sides satisfy the appropriate recurrences.

We know that the $(B(t), \hat{\mathbf{y}}(t))_{t \in \mathbb{T}_n}$ are a Y-system in $\mathcal{F}_{>0}$. Since maps of semifields preserve the formulas (2.31), it follows that $(B(t), \Psi(\hat{y}(t)))$ is a Y-system in $\overline{\mathcal{F}}_{>0}$. Likewise, $(B(t), c(\hat{y}(t)))$ is a Y-system in $\overline{\mathbf{P}}$. This establishes (2.34) and (2.35).

Finally, we need to check (2.33). Note that (2.5) can be rewritten in the following equivalent form:

$$x_k(t')x_k(t) = \frac{\hat{y}_k(t) + 1}{y_k(t) \oplus 1} \prod_i x_i(t)^{[-b_{ik}(t)]_+}. \tag{2.36}$$

Applying Ψ and c in turn to (2.36), and taking the ratio of the resulting pair of equalities, we obtain

$$\frac{\Psi(x_k(t'))}{c(x_k(t'))}\frac{\Psi(x_k(t))}{c(x_k(t))} = \frac{\Psi(\hat{y}_k(t))+1}{c(\hat{y}_k(t))\oplus 1}\prod_i \left(\frac{\Psi(x_i(t))}{c(x_i(t))}\right)^{[-b_{ik}(t)]_+}, \tag{2.37}$$

which is the version of (2.36) for the seed pattern $\overline{\mathcal{E}}$ using (2.34) and (2.35). To get (2.37) from (2.36), we canceled out a common factor of $\Psi(y_k \oplus 1) = c(y_k \oplus 1)$ from the ratio. □

Example 2.4.5. Beginning with the seed pattern in Figure 2.1, one can construct the normalized seed pattern in Figure 2.2 by first applying the semifield map F^* – obtaining the non-normalized seed pattern in Figure 2.3– and then normalizing by a semifield map $c\colon \mathcal{F}_{>0} \to \overline{\mathbf{P}}$. This map c agrees with F^* on frozen variables and sends a cluster variable x to the frozen variable monomial dividing $F^*(x)$, e.g. $c(\Delta_{235}) = Y_{3,5}$ and $c(\Delta_{245}) = Y_{2,4}Y_{3,5}$.

When both $\mathcal{E}$ and $\overline{\mathcal{E}}$ are of geometric type, constructing a quasi-homomorphism that sends one seed into (the seed orbit of) another seed is a matter of linear algebra:

Corollary 2.4.6. *Let $\Sigma = (\tilde{B}, \{x_1, \ldots, x_n\})$ and $\overline{\Sigma} = (\overline{\tilde{B}}, \{\overline{x}_1, \ldots, \overline{x}_n\})$ be seeds of geometric type, with frozen variables $x_{n+1}, \ldots, x_{n+m}$ and $\overline{x}_{n+1}, \ldots, \overline{x}_{n+\overline{m}}$ respectively. Let $\mathcal{E}$ and $\overline{\mathcal{E}}$ be the respective seed patterns.*

Let Ψ be a quasi-homomorphism from $\mathcal{E}$ to $\overline{\mathcal{E}}$ such that $\Psi(\Sigma) \sim \overline{\Sigma}$. It determines a Laurent monomial map from the x_i to the $\overline{x}_i$. Let M_Ψ denote the matrix of exponents of this monomial map, thus M_Ψ is an $(n+\overline{m}) \times (n+m)$ matrix satisfying $\Psi(x_k) = \prod_{i=1}^{n+\overline{m}} \overline{x}_i^{(M_\Psi)_{ik}}$. Then the extended exchange matrices $\tilde{B}, \overline{\tilde{B}}$ are related by

$$\overline{\tilde{B}} = M_\Psi \tilde{B}. \tag{2.38}$$

In particular, such a quasi-homomorphism Ψ exists if and only if the principal *parts of $\tilde{B}, \overline{\tilde{B}}$ agree, and the (integer) row span of $\tilde{B}$ contains the (integer) row span of $\overline{\tilde{B}}$.*

Proof. Indeed, the (i,j) entry of the left hand side of (2.38) encodes the exponent of $\overline{x}_i$ in $\hat{\overline{y}}_j$, while the (i,j) entry of the right hand side encodes the exponent of $\overline{x}_i$ in $\Psi(\hat{y}_j)$. So (2.38) now follows from (2.34).

The "in particular" statement follows by studying (2.38): the "interesting" rows of M_Ψ are its bottom $\overline{m}$ rows. Each of these rows determines a particular linear combination of the rows of $\tilde{B}$, and these linear combinations can be prescribed arbitrarily by prescribing the exponent of $\overline{x}_i$ in $\Psi(x_j)$ for $1 \le i \le \overline{m}, 1 \le j \le n+m$ using Lemma 2.1.6. □

Remark 2.4.7 (Exchange graphs and separation of additions)**.** The formulas (2.32),(2.33) and (2.35) show that if there is a quasi-homomorphism from $\mathcal{E}$ to $\overline{\mathcal{E}}$, then the exchange graph of $\mathcal{E}$ covers that of $\overline{\mathcal{E}}$. In particular, by Corollary 2.4.6, if the rows of $\tilde{B}$ span $\mathbb{Z}^n$, then the exchange graph for the corresponding cluster algebra $\mathcal{A}(\tilde{B})$ covers the exchange graph of every other cluster algebra $\overline{\mathcal{A}}$ with the same underlying exchange matrix.

This is a natural generalization of the *separation of additions formula* [24, Theorem 3.7] from the case of a quiver with principal coefficients to *any* $\tilde{B}$-matrix whose rows span $\mathbb{Z}^n$. Namely, let $\Sigma_0 = (B_0, \mathbf{y}, \mathbf{x})$ and $\overline{\Sigma_0} = (B_0, \overline{\mathbf{y}}, \overline{\mathbf{x}})$ be a pair of normalized seeds with the same exchange matrix, and suppose Σ_0 has principal coefficients, i.e. $y_i = x_{n+i}$. There is a natural choice of maps Ψ mapping Σ_0 to $\overline{\Sigma_0}$ as in Proposition 2.4.4, defined by $\Psi(x_i) = \overline{x}_i$ and $\Psi(x_{n+i}) = \overline{y}_i$. For this choice of Ψ, formula (2.33) becomes separation of additions: the numerator of [24, Theorem (3.7)] (evaluating the "X polynomial" in $\overline{\mathcal{F}}$) is applying the semifield homomorphism Ψ, while the denominator (specializing the cluster variables to 1 and evaluating the X polynomial in $\overline{\mathbf{P}}$) is applying the semifield map c.

Remark 2.4.8 (Proportionality and gradings)**.** Let $\mathcal{E}$ be a seed pattern of geometric type. We recall briefly the concept of a $\mathbb{Z}^r$*-grading on* $\mathcal{E}$ cf. [33, 34]. Choosing an initial seed $(\tilde{B}, \{x_i\})$ in $\mathcal{E}$, such a choice of grading is determined by a $r \times (n+m)$ grading matrix G satisfying $G\tilde{B} = 0$. The i^{th} column of G determines the grading of x_i as a vector in $\mathbb{Z}^r$, for $1 \leq i \leq n+m$. The condition $G\tilde{B} = 0$ guarantees that every exchange relation (2.5) is homogeneous with respect to this $\mathbb{Z}^r$-grading; this in turn defines the multi-grading of each adjacent cluster variable and thereby each adjacent grading matrix. It can be seen that these adjacent grading matrices again satisfy the left kernel condition, so that the grading propagates to a $\mathbb{Z}^r$-grading on the entire cluster algebra in which the cluster variables and coefficients are homogeneous.

Now we suppose we are given two seeds $\mathcal{E}$ and $\overline{\mathcal{E}}$ of geometric type with notation as in Corollary 2.4.6. Let Ψ_1 and Ψ_2 be a pair of proportional quasi-homomorphisms of $\mathcal{E}$ and $\overline{\mathcal{E}}$. We obtain as in (2.38) matrices M_{Ψ_1} and M_{Ψ_2} such that $M_{\Psi_1}\tilde{B} = M_{\Psi_2}\tilde{B} = \overline{\tilde{B}}$, which implies that $M_{\Psi_1} - M_{\Psi_2}$ defines a $\mathbb{Z}^{\overline{m}}$-grading G on $\mathcal{E}$ (the first n rows of $M_{\Psi_1} - M_{\Psi_2}$ define the trivial grading). Conversely, fixing a quasi-homomorphism Ψ_1 with matrix M_{Ψ_1}, any choice of $\mathbb{Z}^{\overline{m}}$-grading matrix G on $\tilde{B}$ provides a quasi-homomorphism Ψ_2, proportional to Ψ_1, whose matrix is $M_{\Psi_2} = M_{\Psi_1} + G$.

Remark 2.4.9. For simplicity, we stated Corollary 2.4.6 in terms of $\mathbb{Z}$ row spans, but a similar statement holds for $\mathbb{Q}$ row spans. To do this, one enlarges the tropical semifield $\overline{\mathbf{P}}$ to the *Puiseux tropical semifield* consisting of Puiseux monomials with rational exponents in the frozen variables. This is unpleasant from the perspective of cluster algebras as coordinate rings, but is perfectly fine if one is only interested in writing algebraic formulas for cluster variables, etc.

This is foreshadowed in the work of Sherman and Zelevinsky [53, Section 6], which discusses the coefficient-free rank 2 cluster algebra $\mathcal{A}(b,c)$ with exchange matrix $\begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}$. The authors write the cluster variables in any cluster algebra with this B matrix in terms of the cluster variables for $\mathcal{A}(b,c)$. Their formulas involve Puiseux monomials in the frozen variables.

2.5 Nerves

By Proposition 2.3.3, to check that a given semifield map Ψ is a quasi-homomorphism from $\mathcal{E}$ to $\overline{\mathcal{E}}$, it suffices to check that $\Psi(\Sigma(t)) \sim \overline{\Sigma}(\bar{t})$ for *some* pair of seeds $\Sigma(t)$ in $\mathcal{E}$ and $\overline{\Sigma}(\bar{t})$. By Proposition 2.2.3, this means checking that $B(t) = B(\bar{t})$ and $\hat{\mathbf{y}}(t) = \hat{\mathbf{y}}(\bar{t})$, and furthermore $\Psi(x_j(t)) \asymp \overline{x}_j(\bar{t})$ holds for all j. We envision applications where checking the proportionality condition on cluster variables is easy and can be done in *many* seeds t, but checking the equality of exchange matrices or $\hat{\mathbf{y}}$'s is inconvenient. Such an example is given in Section 4.4. Our present goal is to give a criterion that guarantees Ψ is a quasi-homomorphism by only checking proportionality conditions. The relevant concept is that of a *nerve* for a seed pattern.

Definition 2.5.1. Let $\mathcal{E}$ be a seed pattern. A *nerve* $\mathcal{N}$ for $\mathbb{T}_n$, is a connected subgraph of $\mathbb{T}_n$ such that every edge label $k \in [1,n]$ arises at least once in $\mathcal{N}$.

The basic example of a nerve is the star neighborhood of a vertex. We believe that there are many theorems of the form, "if a property holds on a nerve, then it holds on the entire seed pattern." We give an example of such a theorem in the appendix, generalizing the "Starfish Lemma" [17, Proposition 3.6] from a star neighborhood to a nerve.

Proposition 2.5.2. *Let $\mathcal{E}$ and $\overline{\mathcal{E}}$ be non-normalized seed patterns, with respective* *ambient semifields* $\mathcal{F}_{>0}$ *and* $\overline{\mathcal{F}}_{>0}$*. Suppose the seeds in* $\overline{\mathcal{E}}$ *are indecomposable. Let* $\Psi\colon \mathcal{F}_{>0} \to \overline{\mathcal{F}}_{>0}$ *be a semifield homomorphism that preserves coefficients and satisfies* $\Psi(x_j(t)) \asymp \overline{x}_j(\bar{t})$ *for every vertex* t *and label* j *such that* $t \xrightarrow{j} t'$ *is in* $\mathcal{N}$*. Then* Ψ *is a quasi-homomorphism from* $\mathcal{E}$ *to* $\overline{\mathcal{E}}$ *or from* $\mathcal{E}$ *to* $\overline{\mathcal{E}}^{opp}$*.*

In particular applying the proposition when $\mathcal{N}$ is the star neighborhood of a vertex t, to check that Ψ is a quasi-homomorphism, it suffices to check that $\Psi(x_j(t)) \asymp \overline{x}_j(\bar{t})$ for all $j \in [1,n]$, as well as checking $\Psi(x_j(t')) \asymp \overline{x}_j(\overline{t'})$ for each adjacent edge $t \xrightarrow{j} t'$. Lemma 2.3.8 now follows.

Proof. Choose a vertex $t \in \mathcal{N}$. By hypothesis for all j, $\Psi(x_j(t)) = c_j(t)\overline{x}_j(\bar{t})$ for some $c_j(t) \in \overline{\mathbf{P}}$, so we are left checking that $B(t) = B(\bar{t})$ and $\hat{\mathbf{y}}(t) = \hat{\mathbf{y}}(\bar{t})$. Suppose $t \xrightarrow{k} t'$ is an edge in $\mathcal{N}$, then there is a scalar $c_k(t')$ such that $\Psi(x_k(t')) = c_k(t')\overline{x}_k(\overline{t'})$.

The exchange relation defining $\overline{x}_k(\overline{t'})$ in $\overline{\mathcal{E}}$ is

$$\overline{x}_k(\bar{t})\overline{x}_k(\overline{t'}) = \overline{p}_k^+(\bar{t}) \prod \overline{x}_j(\bar{t})^{[\bar{b}_{jk}(\bar{t})]_+} + \overline{p}_k^-(\bar{t}) \prod \overline{x}_k(\bar{t})^{[-\bar{b}_{jk}(\bar{t})]_+}. \tag{2.39}$$

On the other hand, applying Ψ to the relation defining $x_k(t')$ in $\mathcal{E}$ and rearranging yields

(2.40)
$$\overline{x}_k(\bar{t})\overline{x}_k(\overline{t'}) = \frac{1}{c_k(t')c_k(t)}\Big(\Psi(p_k^+(t)) \prod \Psi(x_j(t))^{[b_{jk}(t)]_+} + \Psi(p_k^-(t)) \prod (\Psi(x_j(t)))^{[-b_{jk}(t)]_+}\Big).$$

Abbreviating the two terms on the right hand side of (2.39) as $X+Y$, and the two terms in (2.40) as $Z+W$, we see by algebraic independence in the seed at t that either $X = Z, Y = W$, or $X = W, Y = Z$. Refer to these as Case 1 or Case 2 respectively. By inspection, we see that $\hat{\overline{y}}_k(\bar{t})$ is the ratio $\frac{X}{Y}$, while $\Psi(\hat{y}_k(t))$ is $\frac{Z}{W}$. Thus in Case 1 we deduce that $\Psi(\hat{y}_k(t)) = \hat{\overline{y}}_k(\bar{t})$ and the matrices $B(t)$ and $B(\bar{t})$ have the same k^{th} column. In Case 2 we deduce the same thing once we replace $\overline{\mathcal{E}}$ by $\overline{\mathcal{E}}^{\text{opp}}$.

Now apply Lemma 2.5.3. $\square$

Lemma 2.5.3. *Let $\mathcal{Y} = \{\mathbf{y}(t), B(t)\}$ and $\overline{\mathcal{Y}} = \{\overline{\mathbf{y}}(t), \overline{B}(t)\}$ be two Y-patterns whose matrices $B(t)$ are indecomposable. Let $\mathcal{N}$ be a nerve for $\mathbb{T}_n$. Suppose for every vertex $t \in \mathcal{N}$ and label k such that the edge $t \xrightarrow{k} t'$ is in $\mathcal{N}$, one of the following holds*

$$y_k(t) = \overline{y}_k(t) \text{ and } b_{jk}(t) = \overline{b}_{jk}(t) \text{ for all } j \in [1,n], \text{ or} \tag{2.41}$$

$$y_k^{opp}(t) = \overline{y}_k(t) \text{ and } b_{jk}^{opp}(t) = \overline{b}_{jk}(t) \text{ for all } j \in [1,n], \tag{2.42}$$

then $\mathcal{Y} = \overline{\mathcal{Y}}$ or $\mathcal{Y}^{opp} = \overline{\mathcal{Y}}$ accordingly.

Roughly, there are two issues here: first the question of whether Y-patterns can be checked on a nerve (they can), and second whether we are dealing with $\mathcal{Y}$ or $\mathcal{Y}^{\text{opp}}$ (this relies on indecomposability).

Proof. In any Y-pattern, for a given (k,t) pair (not necessarily in $\mathcal{N}$), we will refer to $y_k(t)$ and the k^{th} column of $B(t)$ as the *k-part of the seed at t*. The equations in (2.41) say that $\mathcal{Y}, \overline{\mathcal{Y}}$ have either the same k-parts, or opposite k-parts, for any edge $t \xrightarrow{k} t' \in \mathcal{N}$.

Pick a vertex $t_0 \in \mathcal{N}$, and an edge $k \in \mathcal{N}$ incident to t_0. If necessary, replace $\mathcal{Y}$ by $\mathcal{Y}^{\text{opp}}$ so that the given Y-patterns have the same k-part at t_0. We seek to prove $\mathcal{Y}, \overline{\mathcal{Y}}$ have the same j-part at t_0, for all $j \in [n]$.

Let $t_0 \xrightarrow{k} t_1 \in \mathcal{N}$ be an edge in the nerve incident to t_0. The mutation rules (2.1), (2.31) are involutive and have the property that for any j, the j-part of the seed at t_1 depends only on the j-part and k-part of the seed at t_0. Since the given Y-patterns agree at k, we see that their j-parts agree at t_1 if and only if they agree at t_0. Repeatedly apply this observation, mutating in all possible directions in the nerve, while preserving the fact that the j-parts at $t \in \mathcal{N}$ coincide if and only if they coincide at t_0. Since the nerve is connected and every edge label shows up at least once in $\mathcal{N}$, we conclude that for all j, the j-parts at $\mathcal{Y}$ and $\overline{\mathcal{Y}}$ are either the same or opposite. The connectedness hypothesis assures they are all in fact the same. $\square$

CHAPTER III

The group of quasi-automorphisms

3.1 Quasi-automorphisms and the cluster modular group

A *quasi-automorphism* is a quasi-isomorphism from a given seed pattern $\mathcal{E}$ to itself, cf. Definition 2.3.6. One can think of a quasi-automorphism as a choice of a map describing an automorphism of the pattern of seed orbits associated to $\mathcal{E}$. We will use quasi-automorphisms to define a variant of a group of automorphisms of $\mathcal{E}$, generalizing the group of cluster automorphisms defined for seed patterns with trivial coefficients in [2] while retaining many of the properties of cluster automorphisms (e.g. Proposition 2.3.3, Corollary 2.4.6 and Proposition 2.5.2).

The following example illustrates that the notion of quasi-automorphism is more general than the "naive" notion of a semifield automorphism preserving the seed orbit pattern.

Example 3.1.1. A quasi-automorphism does not have to be an automorphism of semifields. Consider the composition $G^* \circ F^*$ from Example 2.3.9, which is a quasi-automorphism of $\mathbb{C}[\mathbf{X}]$ proportional to the identity map. It rescales each Plücker variable by a product of frozens: $G^* \circ F^*(\Delta_S) = \Delta_{145}\Delta_{125}\Delta_S$. The ambient semifield of $\mathbb{C}[\mathbf{X}]$ has a grading for which every Plücker variable is degree one, and every homogeneous element in the image of $G^* \circ F^*$ has degree a multiple of 3. Thus

$G^* \circ F^*$ cannot be surjective.

Definition 3.1.2. The *quasi-automorphism group* $\mathrm{QAut}_0(\mathcal{E})$ is the set of proportionality classes of quasi-automorphisms of $\mathcal{E}$. This is the automorphism group of $\mathcal{E}$ in the quotient category discussed in Remark 2.3.7.

Remark 3.1.3. Let us call a quasi-automorphism trivial if it is proportional to the identity map. The set of trivial quasi-automorphisms is a monoid (but not usually a group) under composition; the composition $G^* \circ F^*$ from Example 2.3.9 bears witness to this. One way to construct quasi-automorphisms proportional to a given Ψ is to compose $\epsilon_1 \circ \Psi \circ \epsilon_2$ with ϵ_1 and ϵ_2 trivial. It is tempting to try and define $\mathrm{QAut}_0(\mathcal{E})$ purely in terms of thse trivial quasi-automorphisms, without mentioning proportionality. However the relation $\equiv$ defined by $\Psi_1 \equiv \Psi_2$ if $\Psi_2 = \epsilon_1 \circ \Psi \circ \epsilon_2$ is neither symmetric nor transitive, so one cannot form a quotient category using this relation.

We write $\mathrm{QAut}_0(\mathcal{E}) = \mathrm{QAut}_0(\tilde{B})$ when $\mathcal{E}$ is of geometric type and specified by an initial matrix $\tilde{B}$. By Remark 2.4.8, two quasi-automorphisms are proportional to each other if and only if their ratio defines a $\mathbb{Z}^m$-grading on $\mathcal{E}$ (taking exponents of elements of $\mathbf{P}$ to obtain elements of $\mathbb{Z}^m$). Fixing a particular quasi-automorphism Ψ, the number of degrees of freedom in specifying another quasi-automorphism proportional to Ψ is therefore the corank of $\tilde{B}$.

Lemma 3.1.4. *Let $\mathcal{E}$ be a seed pattern of geometric type and Ψ a quasi-homomorphism from $\mathcal{E}$ to itself. Then Ψ is a quasi-automorphism.*

Thus when $\mathcal{E}$ is of geometric type, every quasi-homomorphism Ψ from $\mathcal{E}$ to itself determines an element of $\mathrm{QAut}_0(\mathcal{E})$, i.e. any such Ψ has a quasi-inverse.

Proof. By [3, Lemma 3.2], if two $\tilde{B}$-matrices $\tilde{B}(t_0)$ and $\tilde{B}(\bar{t}_0)$ are in the same mutation class, they are related by a pair of unimodular integer matrices: $\tilde{B}(\bar{t}_0) =$

$M\tilde{B}(t_0)N$, for $M \in \mathrm{GL}_{m+n}(\mathbb{Z})$, and $N \in \mathrm{GL}_n(\mathbb{Z})$.

By Corollary 2.4.6, for a pair of vertices $t_0, \overline{t_0} \in \mathbb{T}_n$, there is a quasi-homomorphism Ψ sending the seed orbit at t_0 to the seed orbit at $\bar{t}_0$ if and only if the principal parts of $\tilde{B}(t_0)$ and $\tilde{B}(\bar{t}_0)$ agree, and the row span of $\tilde{B}(t_0)$ contains the row span of $\tilde{B}(\bar{t}_0)$. By the unimodularity of mutation, this criterion is preserved under swapping the roles of t_0 and $\bar{t}_0$ – if the row span of $\tilde{B}(t_0)$ contains the row span of $\tilde{B}(\bar{t}_0)$ then in fact the two row spans are equal submodules of $\mathbb{Z}^n$. □

We will now recall the definitions of some preexisting groups of automorphisms associated to a seed pattern $\mathcal{E}$. Namely:

- the cluster modular group $\mathrm{CMG}(\mathcal{E})$ of Fock and Goncharov [13], and
- the group $\mathrm{Aut}(\mathcal{E})$ of automorphisms in the category of (rooted) cluster algebras defined by Assem, Dupont and Schiffler [1].

We first present these definitions and then discuss a particular example where all the groups are computed and compared to each other and to the quasi-automorphism group.

Definition 3.1.5 (Cluster modular group [13, Definition 2.14])**.** Let $\mathcal{E}$ be a seed pattern with exchange graph $\mathbf{E}$. The *cluster modular group* $\mathrm{CMG}(\mathcal{E})$ is the group of graph automorphisms $g \in \mathrm{Aut}(\mathbf{E})$ that preserve the exchange matrices. More precisely, recall that the unlabeled seed at vertex $t \in \mathbf{E}$ is indexed not by $[1,n]$ but by the elements of $\mathrm{star}(t)$. Then an element of the cluster modular group is a graph automorphism $g \in \mathrm{Aut}(\mathbf{E})$ satisfying $B(t)_{t',t''} = B(g(t))_{g(t'),g(t'')}$ for all $t \in \mathbf{E}$ and $t', t'' \in \mathrm{star}(t)$. Such a graph automorphism can be determined by choosing a pair of vertices $t_0, \overline{t_0} \in \mathbf{E}$ and an identification of $\mathrm{star}(t_0)$ with $\mathrm{star}(\bar{t}_0)$ under which $B(t_0) = B(\overline{t_0})$.

Remark 3.1.6. Because Definition 3.1.5 is in terms of automorphisms of the exchange graph, the cluster modular group appears to depend on the entire seed pattern $\mathcal{E}$, and not just the underlying exchange matrices in $\mathcal{E}$. However, it is widely believed that the exchange graph – and therefore the cluster modular group – is in fact independent of the choice of coefficients (i.e., it only depends on the exchange matrices, and therefore can be prescribed by giving a single such matrix). This has been proven for skew-symmetric exchange matrices [8].

The quasi-automorphism group is a subgroup of the cluster modular group. Indeed, each quasi-automorphism Ψ determines a cluster modular group element g via $\Psi(\Sigma(t)) \sim \Psi(g(t))$, and proportional quasi-automorphisms determine the same g. Since Ψ preserves exchange matrices and evaluating Ψ commutes with permuting the cluster variables in a seed, the element g produced this way is indeed an element of the cluster modular group.

One can also consider automorphisms in the category of cluster algebras defined in [1]. We reproduce a version of the definition for the sake of convenience.

Definition 3.1.7. Let $\mathcal{E}$ be a seed pattern. We say two seeds Σ_1 and Σ_2 in $\mathcal{E}$ are similar if Σ_2 coincides with Σ_1 after first permuting the frozen variables, and then permuting the indices $[1, n]$ appropriately.

Suppose the exchange matrices in $\mathcal{E}$ are indecomposable. Let $\mathcal{A}$ be its cluster algebra. A $\mathbb{Z}$-algebra map $f\colon \mathcal{A} \to \mathcal{A}$ is an *automorphism* of $\mathcal{E}$ if for every (equivalently, for any) seed Σ in $\mathcal{E}$, $f(\Sigma)$ or $f(\Sigma)^{\mathrm{opp}}$ is similar to a seed in $\mathcal{E}$. We denote the group of automorphisms of $\mathcal{E}$ by $\mathrm{Aut}(\mathcal{E})$.

The elements of $\mathrm{Aut}(\mathcal{E})$ are similar to *strong isomorphisms* [23] but slightly more general since one is allowed to permute the frozen variables.

We say f as in Definition 3.1.7 is a *direct automorphism* or *inverse automorphism* according to whether $f(\Sigma)$ or $f(\Sigma)^{\mathrm{opp}}$ is a seed in $\mathcal{E}$. Let $\mathrm{Aut}^{+}(\mathcal{E}) \subset \mathrm{Aut}(\mathcal{E})$ denote the subgroup of direct automorphisms. By similar reasoning to [2, Theorem 2.11], this subgroup has index two in $\mathrm{Aut}(\mathcal{E})$ if each seed Σ in $\mathcal{E}$ is mutation-equivalent to Σ^{opp}; otherwise $\mathrm{Aut}^{+}(\mathcal{E}) = \mathrm{Aut}(\mathcal{E})$.

An important special case of Definition 3.1.7 is when $\mathcal{E}$ has trivial coefficients in which case the group $\mathrm{Aut}(\mathcal{E})$ is the group of *cluster automorphisms* [2]. When $\mathcal{E}$ has trivial coefficients, we have $\mathrm{Aut}^{+}(\mathcal{E}) = \mathrm{CMG}(\mathcal{E})$. Furthermore, a direct cluster automorphism is the same as a quasi-automorphism in this case.

We can summarize the containments between the preceding groups as

$$\mathrm{Aut}^{+}(\mathcal{E}) \subset \mathrm{QAut}_0(\mathcal{E}) \subset \mathrm{CMG}(\mathcal{E}) \stackrel{?}{=} \mathrm{Aut}^{+}(\mathcal{E}_{\mathrm{triv}}). \tag{3.1}$$

where $\mathcal{E}$ is a seed pattern and $\mathcal{E}_{\mathrm{triv}}$ is the seed pattern obtained from $\mathcal{E}$ by trivializing its coefficients. The equality $\mathrm{CMG}(\mathcal{E}) \stackrel{?}{=} \mathrm{Aut}^{+}(\mathcal{E}_{\mathrm{triv}})$ depends on the belief that $\mathrm{CMG}(\mathcal{E}) = \mathrm{CMG}(\mathcal{E}_{\mathrm{triv}})$, cf. Remark 3.1.6. The group $\mathrm{Aut}(\mathcal{E}_{\mathrm{triv}})$ contains all of the groups in (3.1), and the group $\mathrm{Aut}(\mathcal{E})$ doesn't sit nicely with the rest of the containments when $\mathrm{Aut}^{+}(\mathcal{E}) \subsetneq \mathrm{Aut}(\mathcal{E})$.

Remark 3.1.8. Suppose $\tilde{B}$ is an extended exchange matrix whose rows span $\mathbb{Z}^n$, and $\mathcal{E} = \mathcal{E}(\tilde{B})$ is the corresponding seed pattern of geometric type. By Corollary 2.4.6, the containment $\mathrm{QAut}_0(\mathcal{E}) \subset \mathrm{CMG}(\mathcal{E})$ is an equality in this case.

We next illustrate the differences between the groups in (3.1) using a particular cluster algebra associated with a bordered marked surface. Basic notions and references concerning this class of cluster algebras are given in Section 3.2.

Example 3.1.9. Let $(\mathbf{S}, \mathbf{M})$ be an annulus with two marked points on each boundary component cf. Figure 3.1. We have colored the marked points either black or white to aid in describing the automorphism groups below.

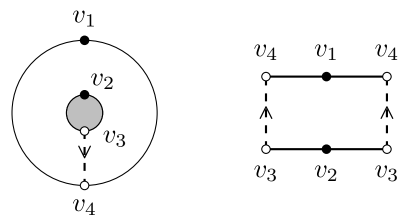


Figure 3.1: An annulus with two marked points on each boundary component. At right, we show a "flat form" of this annulus obtained by cutting along the dashed line.

The cluster modular group $\mathrm{CMG}(\mathbf{S},\mathbf{M})$ for a cluster algebra associated with this annulus coincides with the *mapping class group* of the annulus (see Proposition 3.2.2 below). This group has the following explicit description: let ρ be the (isotopy class of) the homeomorphism of $\mathbf{S}$ that rotates the inner boundary of the annulus clockwise by a half-turn. Let τ be the clockwise half-turn of the outer boundary. Let σ be the homeomorphism represented by a 180 degree turn of the flat form of the annulus; it swaps the inner and outer boundary components. Then the elements ρ, τ, and σ generate the cluster modular group. The group has a presentation $\mathrm{CMG}(\mathbf{S},\mathbf{M}) = \langle \rho, \tau, \sigma \colon (\rho\tau)^2 = \sigma^2 = 1, \rho\tau = \tau\rho, \sigma\rho = \tau\sigma \rangle$ with respect to these generators. It is a central extension $1 \mapsto \mathbb{Z}/2\mathbb{Z} \mapsto \mathrm{CMG} \mapsto \mathrm{Dih}_\infty \mapsto 1$ of the infinite dihedral group $\mathrm{Dih}_\infty = \langle r, s \colon s^2 = (sr)^2 = 1 \rangle$ by $\mathbb{Z}/2\mathbb{Z} = \langle \rho\tau \rangle$, using the map $\sigma \mapsto s, \rho \mapsto r, \tau \mapsto r^{-1}$.

Figure 3.2 gives a choice of lamination L and triangulation T, as well as the quivers $\tilde{B}(T), \tilde{B}(\rho(T))$ and $\tilde{B}(\rho^2(T))$. Let $\mathcal{A}$ be the corresponding cluster algebra with frozen variable x_L, $\mathcal{E}$ its seed pattern, and $\mathcal{F}_{>0}$ its ambient semifield.

The cluster modular group element $\sigma\rho\tau$ permutes the arcs in T. It induces an automorphism of the quiver $\tilde{B}(T)$, and therefore an element of $\mathrm{Aut}(\mathcal{E})$.

The quivers $\tilde{B}(T)$ and $\tilde{B}(\rho^2(T))$ are neither isomorphic nor opposite, so there

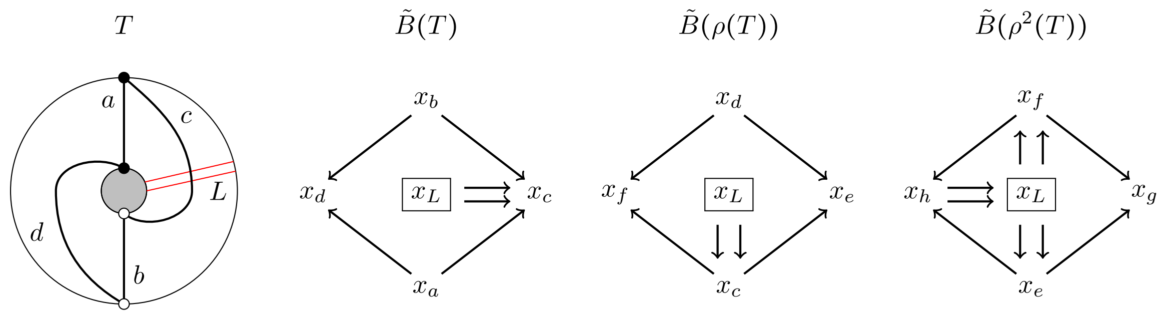


Figure 3.2: A lamination L consisting of two copies of the same curve on the annulus, determining a single frozen variable x_L. We have also drawn a triangulation T of this annulus by the arcs a, b, c, d. The quivers $\tilde{B}(T), \tilde{B}(\rho(T))$, and $\tilde{B}(\rho^2(T))$ are shown at right, where the extra arcs are $e = \rho^2(a), f = \rho^2(b), g = \rho^2(c), h = \rho^2(d)$. The values of $\hat{y}$ in each quiver are read off as the Laurent monomial "incoming variables divided by outgoing variables."

is no strong automorphism sending the seed at T to the seed at $\rho^2(T)$. Likewise, there is no strong automorphism between T and $\rho^{\pm 4}(T), \rho^{\pm 6}(T), \rho^{\pm 8}(T)$, and so on. However, there *is* a quasi-automorphism relating these seeds, which we describe now. It is the semifield map $\Psi\colon \mathcal{F}_{>0} \to \mathcal{F}_{>0}$ defined by $\Psi(x_L) = x_L$ as well as $\Psi(x_\gamma) = x_L^{-1} \cdot x_{\rho^2(\gamma)}$ for $\gamma = a, b, c, d$. It sends each $\hat{y}$ for $\Sigma(T)$ to the corresponding $\hat{y}$ for $\Sigma(\rho^2 \cdot T)$, defining a quasi-automorphism of $\mathcal{E}$ whose map on seed orbits is ρ^2. It has a simple global description on cluster variables which can be checked inductively by performing appropriate mutations away from T. Namely, for each arc γ let $\iota(\gamma, L)$ denote the number of times γ crosses the two curves in L. For example, $\iota(a, L) = 0$ and $\iota(c, L) = 1$. Then

$$\Psi(x_\gamma) = x_L^{\iota(\gamma, L) - \iota(\rho^2(\gamma), L)} \cdot x_{\rho^2(\gamma)} \tag{3.2}$$

for all arcs γ in the annulus. The power of x_L on the right hand side of (3.2) is always equal to $0, 1$, or -1. It is also simple to describe quasi-automorphisms realizing σ and $\rho\tau$.

Perhaps surprisingly, the seeds at T and $\rho(T)$ are *not* related by a quasi-automorphism. Indeed, the values of $\hat{y}$ are equal at the top and bottom vertices of $\tilde{B}(T)$ in Figure

3.2, but they are not equal in $\tilde{B}(\rho(T))$. The same holds for T and $\tau(T)$.

Putting all of this together, there is only one nontrivial strong automorphism of $\mathcal{E}$, namely the element $\sigma\rho\tau$. On the other hand, the quasi-automorphism group is infinite, generated by ρ^2, σ and $\rho\tau$. It is a direct product $\mathrm{Dih}_\infty \times \mathbb{Z}/2\mathbb{Z}$. It is an index two subgroup of $\mathrm{CMG}(\mathbf{S}, \mathbf{M})$, namely the kernel of the map $\mathrm{CMG} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ that computes the parity of the number of black marked points sent to a white marked point.

Example 3.1.9 suggests that although the cluster modular group $\mathrm{CMG}(\mathcal{E})$ may be strictly larger than the quasi-automorphism group $\mathrm{QAut}_0(\mathcal{E})$, the gap between these groups is not so large. Indeed, Section 3.2 establishes that for seed patterns associated with surfaces, $\mathrm{QAut}_0(\mathcal{E})$ is always a finite index subgroup of the cluster modular group.

3.2 Quasi-automorphisms of cluster algebras from surfaces

In this section we place Example 3.1.9 in context via results valid for any cluster algebra associated to a marked bordered surface as in [15, 19, 20, 28]. We describe quasi-automorphisms of these cluster algebras in terms of the tagged mapping class group of the marked surface.

We follow the setup and notation in [20]. Let $\mathcal{A}(\mathbf{S}, \mathbf{M}, \mathbf{L})$ denote the cluster algebra of geometric type determined by the triple $(\mathbf{S}, \mathbf{M}, \mathbf{L})$. Here $\mathbf{S}$ is an oriented bordered surface with a nonempty set $\mathbf{M}$ of *marked points*. The marked points reside either in the interior of $\mathbf{S}$ (we call these *punctures*) or in $\partial\mathbf{S}$ (we call these *cilia*). The set of punctures is $\overline{\mathbf{M}}$. We disallow a few possibilities for $(\mathbf{S}, \mathbf{M})$, namely a sphere with three or fewer punctures, an n-gon when $n < 4$, and a once-punctured monogon. The choice of coefficients is specified by a *multi-lamination* $\mathbf{L} = (L_1, \ldots, L_m)$, an

m-tuple of (integral unbounded measured) laminations on $(\mathbf{S},\mathbf{M})$. Each lamination L_i consists of a finite number of *curves* in $(\mathbf{S},\mathbf{M})$.

The cluster variables in $\mathcal{A}(\mathbf{S},\mathbf{M})$ are indexed by *tagged arcs* γ, the set of which we denote by $\mathbf{A}^{\bowtie}(\mathbf{S},\mathbf{M})$. The seeds in $\mathcal{A}(\mathbf{S},\mathbf{M},\mathbf{L})$ are indexed by *tagged triangulations* T of $(\mathbf{S},\mathbf{M})$. The extended exchange matrix $\tilde{B}(T)$ for a seed has the *signed adjacency matrix* $B(T)$ as its principal part, and has the *shear coordinate vector* $\vec{b}(T,L_i)$ of the lamination L_i with respect to T as its i^{th} row of coefficients.

The exchange graph of the resulting cluster algebra is independent of the choice of coefficients [20, Corollary 6.2]. We let $\mathrm{CMG}(\mathbf{S},\mathbf{M})$ denote the corresponding cluster modular group. It is closely related to the following geometrically defined group.

Definition 3.2.1 (Tagged mapping class group [2])**.** Let $(\mathbf{S},\mathbf{M})$ be a bordered marked surface that is not a closed surface with exactly one puncture. A *tagged mapping class* for $(\mathbf{S},\mathbf{M})$ is a pair $g=(f,\psi)$, where

- f is an element of the *mapping class group* of $(\mathbf{S},\mathbf{M})$ – i.e. f is an orientiation-preserving homeomorphism of $\mathbf{S}$ mapping $\mathbf{M}$ to itself setwise, considered up to isotopies of $\mathbf{S}$ that fix $\mathbf{M}$ pointwise, and
- $\psi\colon\overline{\mathbf{M}}\to\{\pm1\}$ is a function from the set of punctures to $\{\pm1\}$.

When $(\mathbf{S},\mathbf{M})$ is a closed surface with one puncture p, we make the same definition but impose $\psi(p)=1$ since tagged versions of arcs are not in the cluster algebra. The tagged mapping classes comprise the *tagged mapping class group*, denoted $\mathcal{MG}_{\bowtie}(\mathbf{S},\mathbf{M})$.

We understand $\mathcal{MG}_{\bowtie}(\mathbf{S},\mathbf{M})$ by its action on tagged arcs $\gamma\in A^{\bowtie}(\mathbf{S},\mathbf{M})$. A tagged mapping class $g=(f,\psi)$ acts on γ by first performing the homeomorphism f to γ, and then changing the tag of any end of γ incident to a puncture p for which

$\psi(p) = -1$. The resulting action on tagged triangulations preserves the signed adjacency matrices, and embeds $\mathcal{MG}_{\bowtie}(\mathbf{S},\mathbf{M})$ as a subgroup of $\mathrm{CMG}(\mathbf{S},\mathbf{M})$, cf. [2]. The following result is due to Bridgeland and Smith [4], building on the work of Gu [32].

Proposition 3.2.2. *The tagged mapping class group* $\mathcal{MG}_{\bowtie}(\mathbf{S},\mathbf{M})$ *coincides with the cluster modular group* $\mathrm{CMG}(\mathbf{S},\mathbf{M})$*, unless* $(\mathbf{S},\mathbf{M})$ *is a sphere with four punctures, a once-punctured square, or a digon with one or two punctures.*

Thus barring these exceptional cases, two tagged triangulations of $(\mathbf{S},\mathbf{M})$ have isomorphic quivers precisely when they are related by an element of the tagged mapping class group (see [4, Proposition 8.5] and the subsequent discussion; see also [2, Conjecture 1]). In the exceptional cases listed in Proposition 3.2.2, the tagged mapping class group is a proper finite index subgroup of the cluster modular group.

Motivated by Proposition 3.2.2, we set out to describe, for various choices of coefficients $\mathbf{L}$, the quasi-automorphism group $\mathrm{QAut}_0(\mathbf{S},\mathbf{M},\mathbf{L})$ from Definition 3.1.2 as a subgroup of the tagged mapping class group. The main ingredient in our answer is a black-white coloring similar to one in the examples from Section 3.1.

Definition 3.2.3. The *even components* of $(\mathbf{S},\mathbf{M})$ are the punctures $C \in \overline{M}$ as well the as boundary components $C \subset \partial\mathbf{S}$ having an even number of cilia. We let r denote the number of even components, and label the even components $C_1,\dots,C_r$. For each even boundary component $C \subset \partial\mathbf{S}$, we color the cilia on C black or white so that the colors alternate, i.e. adjacent cilia have opposite colors.

Using the black-white coloring in Definition 3.2.3, each tagged mapping class $g = (f,\psi)$ determines an $r\times r$ signed permutation matrix π_g whose entries are indexed by the even components. The (i,j) entry of π_g is 0 unless $f(C_i) = C_j$. If $C_j \subset \partial\mathbf{S}$ is a boundary component and $f(C_i) = C_j$, then the (i,j) entry is $+1$ if f sends black cilia on C_i to black cilia on C_j, and is -1 if f sends black cilia on C_i to white cilia on C_j. When C_i and C_j are punctures, the sign of the (i,j) entry is the sign $\psi(C_j)$. Not all signed permutation matrices will arise in this way since f can only permute components that have the same number of cilia.

Definition 3.2.4. Let L be a lamination. For each curve α in L, Figure 3.3 shows how to assign a sign to an end of α that either lands on even boundary component or spirals around a puncture. At a puncture, the sign is chosen according to whether α spirals counterclockwise or clockwise into the puncture. At a boundary component, the sign is chosen according to whether the nearest neighboring cilium in the clockwise direction along C is black or white. An end on an odd component has zero sign. The *pairing* $p(L;C)$ of a lamination L with the even component C is the sum of all the signs associated to L, i.e. the sum over all curves α in L of the signs of the two ends of α. We let $\vec{p}(L) = (p(L;C_i))_{i=1,\ldots,r} \in \mathbb{Z}^r$ denote the vector of pairings of L with the even components.

Example 3.2.8 works out these signs for the annulus from Example 3.1.9.

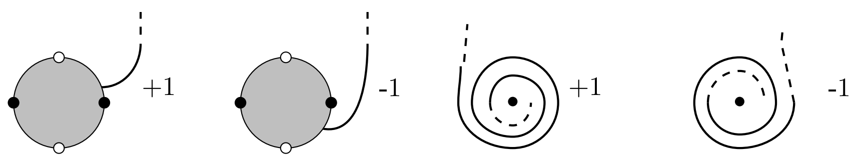


Figure 3.3: The conventions for assigning signs to each end of a curve that lands on an even boundary component (in this case, a boundary component with 4 cilia) or spirals around a puncture. The pairing $p(L,C)$ is obtained by adding up all of these signs.

In addition to acting on tagged arcs, $\mathcal{MG}_{\bowtie}(\mathbf{S},\mathbf{M})$ also acts on laminations L. A tagged mapping class $g = (f,\psi)$ acts by first performing the homeomorphism f to L, and then changing the direction of spiral at each puncture p for which $\psi(p) = -1$. This action preserves shear coordinates in the sense that $\vec{b}(T,L) = \vec{b}(g(T),g(L))$ for a triangulation T and lamination L. It is easy to see that g acts on the vector of pairings by the matrix π_g, i.e. $\vec{p}(g(L)) = \pi_g \cdot \vec{p}(L)$ for a lamination L.

The next theorem is the main result of this section, describing $\mathrm{QAut}(\mathbf{S},\mathbf{M},\mathbf{L})$ inside the marked mapping class group in very concrete terms.

Theorem 3.2.5. *Suppose* $(\mathbf{S},\mathbf{M})$ *is not one of the four exceptional surfaces in Proposition 3.2.2. Let* $\mathbf{L}$ *be a multi-lamination. Let* $V_{\mathbf{L}} = \mathrm{span}(\{\vec{p}(L) \colon L \in \mathbf{L}\}) \subset \mathbb{Z}^r$ *be the submodule spanned by the vectors of pairings associated to the laminations* L *in* $\mathbf{L}$*. Then*

$$\mathrm{QAut}_0(\mathbf{S},\mathbf{M},\mathbf{L}) = \{g \in \mathcal{MG}_{\bowtie}(\mathbf{S},\mathbf{M}) \colon \pi_g(V_{\mathbf{L}}) = V_{\mathbf{L}}\}. \tag{3.3}$$

We prove Theorem 3.2.5 in Section 3.3. The subgroup of $\mathcal{MG}_{\bowtie}(\mathbf{S},\mathbf{M})$ described in (3.3) only depends on the endpoint behavior of laminations – it doesn't mention the topology of the surface, or how much curves wrap around the holes and handles of the surface. The map $g \mapsto \pi_g$ is a group homomorphism from $\mathcal{MG}_{\bowtie}(\mathbf{S},\mathbf{M})$ to the group of signed permutation matrices. The subgroup in (3.3) is an inverse image of the subgroup of signed permutation matrices that fix $V_{\mathbf{L}}$ and therefore is always finite index in $\mathcal{MG}_{\bowtie}(\mathbf{S},\mathbf{M})$.

Corollary 3.2.6. *Let* g *be a tagged mapping class. If* π_g *is plus or minus the identity matrix, then* $g \in \mathrm{QAut}_0(\mathbf{S},\mathbf{M},\mathbf{L})$ *for* any choice *of multi-lamination* $\mathbf{L}$*. Otherwise,* $g \notin \mathrm{QAut}_0(\mathbf{S},\mathbf{M},\mathbf{L})$ *for* some choice *of* $\mathbf{L}$.

Remark 3.2.7. The tagged mapping classes in Corollary 3.2.6 are those that fix all even components setwise, and furthermore either preserve the black-white coloring of ends of curves, or simultaneously swap all colors. This group is generated by the following four types of elements (see [11] for generators of the mapping class group): Dehn twists about simple closed curves, homeomorphisms that permute odd components, fractional Dehn twists rotating the cilia on a given boundary component by two units, and the *tagged rotation*. This last element is the one that simultaneously changes tags at all punctures and rotates all boundary components by one unit. It was studied in [6], where it was shown to coincide with the shift functor of a 2-Calabi-Yau cluster category associated with the surface.

Proof. If $\pi_g = \pm 1$, then π_g clearly preserves $V_{\mathbf{L}}$ regardless of the choice of $\mathbf{L}$ and by Theorem 3.2.5 g is in $\mathrm{QAut}_0(\mathbf{S},\mathbf{M},\mathbf{L})$ for any $\mathbf{L}$.

If $\pi_g \neq \pm 1$, think of π_g as a signed permutation σ of $\{\pm 1, \dots, \pm r\}$ in the usual way. If there is any index $i \in [1,r]$ such that $\sigma(i) \neq \pm i$, then let L be a lamination consisting of a curve with two black ends on C_i, satisfying $p(L;C) = 2$. If there is no such index i, we can choose a pair of indices $i,j \in [1,r]$ such that $\sigma(i) = -i$ but $\sigma(j) = j$. In this case we let L be a lamination consisting of a curve connecting the even components C_i and C_j by a curve that is black at both ends. In both of these two cases, we see that $\pi_g(\vec{p}(L))$ is not in the span of $\vec{p}(L)$ and by Theorem 3.2.5, $g \notin \mathrm{QAut}_0(\mathbf{S},\mathbf{M},\mathbf{L})$. $\square$

Example 3.2.8. We order the boundary components in Figure 3.2 so that the inner boundary is first. The vector of pairings for the lamination L in Figure 3.2 is $\vec{p}(L) = (-2,-2)$. Then ρ and τ act on the vector of parings by swapping the sign of the first or second component respectively, and σ acts by permuting the first and second component. The description of $\mathrm{QAut}_0(\mathbf{S},\mathbf{M},\mathbf{L})$ in Example 3.1.9 matches the one in Theorem 3.2.5. The subgroup of elements described in Corollary 3.2.6 is a a direct product $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generated by ρ^2 and $\rho\tau$. It has index 4 in the cluster modular group.

Remark 3.2.9. Theorem 3.2.5 can be modified in the case that $(\mathbf{S},\mathbf{M})$ is one of the exceptional surfaces in Proposition 3.2.2. Namely, the left hand side of (3.3) merely describes the subgroup of $\mathrm{QAut}_0(\mathbf{S},\mathbf{M},\mathbf{L})$ consisting of tagged mapping classes (that is, ignoring the exotic symmetries). For particular choices of coefficients, the "extra" elements of the cluster modular group might also be inside QAut_0.

3.3 Proofs for Section 3.2

The key result of this section is Proposition 3.3.4 describing quasi-homomorphisms of cluster algebras from surfaces. Theorem 3.2.5 follows from it as a special case.

Let $\mathbf{B}(\mathbf{S},\mathbf{M})$ denote the set of *boundary segments* connecting adjacent cilia in ∂S. There is an especially natural choice of multi-lamination $\mathbf{L}_{\text{boundary}} = (L_\beta)_{\beta\in\mathbf{B}(\mathbf{S},\mathbf{M})}$ with one frozen variable for each boundary segment (see e.g. in [20, Remark 15.8]). Lemma 3.3.1 expresses the shear coordinates of certain laminations in terms of the extended exchange matrix determined by $\mathbf{L}_{\text{boundary}}$. It is patterned after [20, Lemma 14.3].

Lemma 3.3.1. *Let L be a lamination none of whose curves has an end that spirals at a puncture. Given an arc γ the* transverse measure *of γ in L is the minimal number of intersections of γ with the curves in L. We denote it by $l(\gamma, L)$. For a boundary segment $\beta \in \mathbf{B}(\mathbf{S},\mathbf{M})$, we similarly let $l(\beta, L)$ denote the number of ends of the curves in L on β. We let $\vec{l}(T, L) = (l(\star, L))_{\star\in T\cup\mathbf{B}(\mathbf{S},\mathbf{M})}$ be the row vector containing all of these transverse measures. Then*

$$-2\vec{b}(T, L) = \vec{l}(T, L)\tilde{B}(T, \mathbf{L}_{boundary}). \tag{3.4}$$

Proof. We check that the γ_0 components of the left and right hand sides of (3.3.1) are equal, where $\gamma_0 \in T$. Let $\gamma_1, \ldots, \gamma_4$ be the quadrilateral containing γ_0 (number in clockwise order). Some of the γ_i may be boundary segments. Each time α shears across the quadrilateral in an 'S' crossing, it contributes +1 to the left hand side, while contributing $-\frac{1}{2}(-1 + -1)$ to the right hand side. And so on. □

This argument is like [20, Lemma 14.13] but simpler because our we are not dealing with spirals at the puncture, for which $l(T,L)=\infty$. Comparing (3.4) with (2.38), we see that if $\mathbf{L}$ is any multi-lamination none of whose curves spiral at punctures, then there is a quasi-homomorphism from $\mathcal{A}(\mathbf{S},\mathbf{M},\mathbf{L}_{\text{boundary}})$ to $\mathcal{A}(\mathbf{S},\mathbf{M},\mathbf{L})$. A version of Lemma 3.3.1 allowing for spirals at punctures would involve the extended exchange matrix $\tilde{B}(\overline{T},\overline{L})$ on the *fully opened surface* $(\overline{\mathbf{S},\mathbf{M}})$, where $\overline{L}$ and $\overline{T}$ are lifts of L and T to the opened surface (see [20, Sections 9,14] for details). The corresponding version of (3.4) determines a quasi-homomorphism from $\mathcal{A}(\overline{\mathbf{S},\mathbf{M}},\overline{\mathbf{L}})$ to $\mathcal{A}(\mathbf{S},\mathbf{M},\mathbf{L})$.

Our next step is to to describe the row span of the signed adjacency matrices $B(T)$. It strengthens [19, Theorem 14.3] which states that the corank of $B(T)$ is the number of even components. The description requires associating a sign to the ends of arcs $\gamma\in A^{\bowtie}(\mathbf{S},\mathbf{M})$, in a similar fashion as was done for the ends of curves in Definition 3.2.4. Namely if γ has an endpoint on a boundary component $C\subset\partial\mathbf{S}$, the endpoint gets sign ±1 if its endpoint is a black or white cilium respectively. If the endpoint is on a puncture $C\in\overline{M}$, the sign is ±1 if the end is plain or tagged respectively. An endpoint on an odd component gets sign 0. The pairing $p(\gamma;C)$ of γ with C is the sum of the signs of the ends of γ that reside on C, and the vector of pairings is $\vec{p}(\gamma)=(p(\gamma;C_i))_{i=1,\dots,r}$. This pairing satisfies $p(\gamma;C)=p(L_\gamma;C)$ where L_γ is the *elementary lamination* determined by γ (cf. [20, Definition 17.2], see also [51, Section 5]).

Lemma 3.3.2. *For a tagged triangulation T, let $\mathbb{Q}(T)=\mathbb{Q}(\gamma\colon\gamma\in T)$ be $\mathbb{Q}$-vector space of row vectors with entries indexed by $\gamma\in T$. Let $\mathbb{Q}(T)^*=\mathbb{Q}(\gamma^*\colon\gamma\in T)$ be the dual space of column vectors with entries indexed by the dual basis $\{\gamma^*\colon\gamma\in T\}$.*

For each even component C, consider the column vector

$$R_C = \sum_{\gamma \in T} p(\gamma; C)\gamma^* \in \mathbb{Q}(T)^*. \tag{3.5}$$

Then a vector $\vec{a} \in \mathbb{Q}(T)$ is in the the row span of $B(T)$ if and only if the dot product $\vec{a} \cdot R_C$ vanishes for all C.

Said differently, the map $\gamma \mapsto p(\gamma; C_i)$ for C_i an even component determines a $\mathbb{Z}$-grading on the coefficient-free cluster algebra $\mathcal{A}(\mathbf{S}, \mathbf{M})$. These gradings form a *standard $\mathbb{Z}^r$-grading* as i varies from $1, \ldots, r$ (a standard grading is one that spans the kernel of the B-matrices, see [33]). We will not rely on gradings in what follows. Lemma 3.3.2 is proved at the end of this section.

For a vector $\vec{a} \in \mathbb{Q}(T)$, the *residue of $\vec{a}$ around C in T* is the dot product $\vec{a} \cdot R_C = \sum_{\gamma \in T}(\gamma, C)a_\gamma$. Writing $\vec{a}$ as the shear coordinate of a lamination L, the residue has the following simple description.

Lemma 3.3.3. *Let L be a lamination and C an even component. Then the residue of $\vec{b}(T, L)$ around C is the pairing $p(L; C)$ from Definition 3.2.4.*

Proof. The residue is computed in terms of the shear coordinates of arcs adjacent to C. To compute these shear coordinates, rather than considering the entire surface, we can focus on the set of triangles having at least one vertex on C. Lifting to a finite cover of $\mathbf{S}$ perhaps (in order to remove interesting topology nearby C that is irrelevant to computing the residue) this union of triangles will either be a triangulated annulus (when $C \subset \partial\mathbf{S}$ is a boundary component) or a once-punctured n-gon for some n (if C is a puncture). We call this set of triangles the *annular neighborhood* of C. Even when L consists of a single curve, the intersection of L with this annular neighborhood might consist of several curves. By the linearity of shear coordinates and residues, it suffices to consider the case that L consists of a single curve in the annular neighborhood.

When C is a puncture, its annular neighborhood is a punctured disc with a triangulation all of whose arcs are radii joining the puncture to the boundary of the disc. By inspection, a curve L contributes nonzero residue at C if and only it spirals at C, and the value of this residue is ± 1 according to whether it spirals counterclockwise or clockwise respectively as claimed.

When C is a even boundary component, we compute the residue of $\vec{b}(T, L)$ using the right hand side of (3.3.1). We split up this right hand side into two terms by splitting up $\vec{l}(T, L)$ as a concatenation of $(l(\gamma, L)_{\gamma \in T}$ and $(l(\beta, L)_{\beta \in \partial \mathbf{S}}$, and performing the matrix multiplication with $\tilde{B}$ in block form. The first term in this expression has zero residue around C since it is a linear combination of the rows of $B(T)$. What's left over is a sum

$$-\frac{1}{2} \sum_{\gamma \in T, \beta \in \mathbf{B}(\mathbf{S}, \mathbf{M})} p(\gamma; C) l(\beta, L) B_{\beta, \gamma}. \tag{3.6}$$

We claim the sum above evaluates to $p(L; C)$. For the sum to be nonzero, L must have a nonzero end at some segment $\beta = [v_{i-1}, v_i]$ with v_{i-1}, v_i in clockwise order. This segment β is contained in a unique triangle in T. Call the other two sides in this triangle γ_{i-1} and γ_i, whose endpoint on C is v_{i-1} and v_i respectively. There are cases according to whether either of these sides is a boundary segment. If neither is, then $p(\gamma_{i-1}; C) B_{\beta, \gamma_{i-1}} = p(\gamma_i; C) B_{\beta, \gamma_i}$ is ± 1 according to whether v_i is white or black. The total contribution to (3.6) is $p(C; L)$. In the degenerate case that γ_i is a boundary segment, it does not contribute to (3.6), but $p(\gamma_{i-1}; C) = 2$ and this effect is cancelled out, and so on. $\square$

Proposition 3.3.4. *Suppose* $(\mathbf{S}, \mathbf{M})$ *is not among the four listed exceptions in Proposition 3.2.2. Let* $\mathbf{L}, \mathbf{L}'$ *be multi-laminations on* $(\mathbf{S}, \mathbf{M})$ *and recall the submodules* $V_{\mathbf{L}}$ *and* $V_{\mathbf{L}'}$ *from Theorem 3.2.5. Let* $g \in \mathcal{MG}_{\bowtie}(\mathbf{S}, \mathbf{M})$ *be a tagged mapping class and* π_g

the corresponding signed permutation matrix. The following are equivalent:

- *there is a quasi-homomorphism* Ψ *from* $\mathcal{A}(\mathbf{S},\mathbf{M},\mathbf{L})$ *to* $\mathcal{A}(\mathbf{S},\mathbf{M},\mathbf{L}')$ *whose map on tagged triangulations is* $T \mapsto g(T)$ *(that is,* $\Psi(\Sigma(T)) \sim \overline{\Sigma}(g(T))$*),*
- $V_{\mathbf{L}'} \subset \pi_g(V_{\mathbf{L}})$*.*

Proof of Proposition 3.3.4. A quasi-homomorphism Ψ from $\mathcal{A}(\mathbf{S},\mathbf{M},\mathbf{L})$ to $\mathcal{A}(\mathbf{S},\mathbf{M},\mathbf{L}')$ is determined by a pair of tagged triangulations T and T', such that $\tilde{B}(T,\mathbf{L})$ and $\tilde{B}(T',\mathbf{L}')$ are related as in (2.38). Since the principal parts of these matrices agree, by Proposition 3.2.2 there is a tagged mapping class g such that $g(T) = T'$. Furthermore, for each lamination $L' \in \mathbf{L}'$, the vector $\vec{b}(T',L')$ must be in $\operatorname{span}(\{\vec{b}(T,L) : L \in \mathbf{L}\})$. Since $\vec{b}(T',L') = \vec{b}(T,g^{-1}(L'))$, by Lemma 3.3.2, it is equivalent to find a linear combination of $\vec{b}(T,g^{-1}(L'))$ and $\{\vec{b}(T,L) : L \in \mathbf{L}\}$ that has zero residue around every even component. Proposition 3.3.4 follows now from Lemma 3.3.3 and the fact that g acts on a vector of pairings by the matrix π_g. □

Proof of Lemma 3.3.2. Restating the Lemma, we seek to show that the R_C form a basis for the dual space to the row span.We begin by verifying each of these vectors pair to zero with the row span.

First, we check this when C is a puncture. We begin with the case that all of the arcs in T are untagged at C. We need to check that $\sum_{\gamma\in T} p(\gamma;C)B(T)_{\gamma',\gamma}$ vanishes for each $\gamma' \in T$. Indeed, letting L be the lamination consisting of a tiny simple closed curve contractible to C, the shear coordinate vector $\vec{b}(T,L)$ is clearly 0. Now we apply (3.3.1) for this choice of L: the γ'^{th} component of (3.3.1) says $0 = \sum_{\gamma\in T} p(\gamma;C)B(T)_{\gamma,\gamma'}$ as desired, using the fact that $l(\gamma,L) = 0$ if γ is a boundary segment. The argument when all arcs are tagged at C is identical. If C is incident to exactly two arcs, namely the plain and tagged version of the same arc, then R_C

follows from [19, Definition 9.6] (or a calculation in a once-punctured digon).

Second we check this when $C \subset \partial\mathbf{S}$ is a boundary component. Number the cilia on C by $v_1, \ldots, v_{2m}$. For each $i \in [1, 2m]$, let L_i be a tiny lamination contractible to v_i – its two endpoints are on the two boundary segments adjacent to v_i. Again, $\vec{b}(T, L_i)$ is clearly 0 and in particular $\sum_{v_i \text{ black}} \vec{b}(T, L_i) = \sum_{v_i \text{ white}} \vec{b}(T, L_i)$. Summing over the corresponding right hand sides of (3.3.1), again performing the matrix multiplication in (3.3.1) in block form as in the argument for Lemma 3.3.3, the terms corresponding to boundary segments are present in both the sum over black v_i and the sum over white v_i. Canceling these common terms, we get the equality $\sum_{v_i \text{ black}, \gamma \in T} l(\gamma, L_i) B(T)_{\gamma,\gamma'} = \sum_{v_i \text{ white}} l(\gamma, L_i) B(T)_{\gamma,\gamma'}$ for all γ', which says $\sum_{\gamma \in T} p(\gamma, C) B(T)_{\gamma,\gamma'} = 0$ for all γ' as desired.

Thus all of the R_C pair to zero with the row span of $B(T)$. We will now show that they are linearly independent, which completes the proof since they have the expected size by [19, Theorem 14.3].

Consider a linear relation of the form

$$\sum a_C R_C = 0. \tag{3.7}$$

We define scalars a_v for all marked points $v \in \mathbf{M}$ as follows: if v is a puncture C, then $b_v = a_C$. If v is a cilium residing on an even component C, then $b_v = \pm a_C$, with $\pm$ sign consistent with the black-white coloring on C. If v is a cilium on an odd component, we set $a_v = 0$.

Now consider any vertices v_1, v_2 forming an edge in the triangulation T. We claim

$$a_{v_1} + a_{v_2} = 0. \tag{3.8}$$

Indeed, if v_1, v_2 are the endpoints of an arc $\gamma \in T$, the γ^{th} component of the relation (3.7) is $a_{v_1} + a_{v_2}$ by construction, and (3.8) holds. If they are the endpoints of a boundary segment, then (3.8) clearly holds.

However, in any given triangle in T with vertices v_1, v_2, v_2, the only way for (3.8) to hold for all 3 of the pairs $(v_1, v_2), (v_1, v_3), (v_2, v_3)$ is if $a_{v_1} = a_{v_2} = a_{v_3} = 0$. Varying the vertex and triangle containing it, this establishes that all $a_v = 0$ for all $v \in \mathbf{M}$, and hence all $a_C = 0$, as desired. $\square$

CHAPTER IV

Grassmannian and Fock-Goncharov cluster algebras

This chapter presents a quasi-isomorphism between cluster structures on Grassmannians and those on Fock-Goncharov configuration spaces of affine flags. Both of these types of spaces arise as quotients with respect to an action by an algebraic group. To bypass subtleties involved in taking such quotients, we work birationally. This allows us to only consider generic configurations of vectors and affine flags below, which is sufficient for our purposes (i.e., for constructing a quasi-isomorphism).

Throughout Chapter IV, we fix a k-dimensional complex vector space V together with a non-degenerate skew-symmetric volume form. We denote by $\bigwedge(V) = \bigoplus_a \bigwedge^a(V)$ the exterior algebra on V. The volume form is a nonzero element $\omega^* \in \bigwedge^k(V)$ in the dual to the top exterior power. It determines a nonzero element $\omega \in \Lambda^k(V)$ satisfying $\omega^*(\omega) = 1$.

The exterior product $\bigwedge^a(V) \otimes \bigwedge^b(V) \to \bigwedge^{a+b}(V)$ will be denoted by multiplication, i.e. $v \wedge w = vw$. We will refer to the exterior product of two (or several) anti-symmetric tensors as their *join* (we will also make use of a *meet* operation $\bigcap$ on such tensors, cf. Definition 4.3.2).

4.1 Grassmannian cluster algebras

Let $\mathrm{Gr}(k,n)$ denote the Grassmann manifold of k-dimensional subspaces of a fixed n-dimensional complex vector space. We denote by $\hat{\mathrm{Gr}}(k,n)$ the affine cone over $\mathrm{Gr}(k,n)$ with respect to its *Plücker embedding* $\mathrm{Gr}(k,n) \hookrightarrow \mathbb{CP}^{\binom{n}{k}-1}$. Its field of fractions, $\mathbb{C}(\hat{\mathrm{Gr}}(k,n))$, is generated by the Plücker coordinates Δ_S ranging over subsets $S \subset [1,n]$ of size k.

A *configuration of n vectors in V* is a point in the space $\mathrm{SL}(V)\backslash V^n$, i.e. an ordered sequence of n vectors in V considered up to simultaneous $\mathrm{SL}(V)$ action. The spaces $\mathrm{SL}(V)\backslash V^n$ and $\hat{\mathrm{Gr}}(k,n)$ are the same up to genericity, so their fields of rational functions coincide. For $S = \{s_1, \dots, s_k\} \subset [1,n]$, the value of the Plücker coordinate Δ_S at a configuration $p \in \mathrm{SL}(V)\backslash V^n$ is obtained by evaluating ω^* on the join of the vectors labeled by S. That is

$$\Delta_S(p) = \omega^*(v_{s_1} \cdots v_{s_k}) \tag{4.1}$$

where $(v_1, \dots, v_n)$ is a choice of vectors representing p. Thus, to specify a rational map into $\mathbb{C}(\hat{\mathrm{Gr}}(k,n))$, it suffices to provide an $\mathrm{SL}(V)$-equivariant function defined on vectors $v_1, \dots, v_n$ in V. This is how we deal with $\mathbb{C}(\hat{\mathrm{Gr}}(k,n))$ and its Plücker coordinates throughout Chapter IV.

J. Scott [52] provided a cluster structure in $\mathbb{C}(\hat{\mathrm{Gr}}(k,n))$ Its frozen variables are those Plücker coordinates Δ_S whose column set S is cyclically consecutive modulo n. A frozen variable whose columns "wrap around" modulo n are written with their indices increasing. For example, Δ_{1678} (rather than Δ_{6781}) is a frozen variable in $\mathbb{C}(\mathrm{Gr}(4,8))$.

Remark 4.1.1. Each non-frozen Plücker coordinate is a cluster variable. The combinatorial notion criterion specifying when two Plücker coordinates reside in a cluster together is provided by the notion of *weak separation* [48, 52]. A simple choice of seed in this cluster structure can be found in [52, Theorem 1]. After deleting the frozen variables, the quiver for this seed is a $(k-1)\times(n-k-1)$ rectangular grid, all of whose "small squares" are oriented.

Remark 4.1.2. We list the Grassmannians with finitely many seeds (i.e., those *of finite type*) and with finitely many exchange matrices (i.e., those of *finite mutation type*):

- $\mathbb{C}(\hat{\mathrm{Gr}}(2,n+3))$ has finite Dynkin type A_n.
- $\mathbb{C}(\hat{\mathrm{Gr}}(3,6)), \mathbb{C}(\hat{\mathrm{Gr}}(3,7))$ and $\mathbb{C}(\hat{\mathrm{Gr}}(3,8))$ have finite Dynkin types D_4, E_6, E_8 respectively.
- $\mathbb{C}(\hat{\mathrm{Gr}}(3,9))$ and $\mathbb{C}(\hat{\mathrm{Gr}}(4,8))$ are of infinite type but of finite mutation type
- The cluster algebras $\mathbb{C}(\hat{\mathrm{Gr}}(k,n))$ and $\mathbb{C}(\hat{\mathrm{Gr}}(n-k,n))$ can be identified with each other via a cluster automorphism (induced by the complementation map on Plücker coordinates). Thus, $\mathbb{C}(\hat{\mathrm{Gr}}(n+1,n+3))$ has cluster type A_n and so on.

4.2 Fock-Goncharov cluster algebras

Here we introduce the second family of cluster algebras of our interest. An anti-symmetric tensor $v \in \bigwedge^a(V)$ is called *simple* if it can be written as a join of several vectors, i.e. $v = v_1 \cdots v_a \in \bigwedge^a(V)$. For simple tensors $v \in \bigwedge^a(V)$ and $w \in \bigwedge^{a+1}(V)$, we say that v *divides* w if the latter tensor can be written as a join $w = vv'$ for some $v' \in V$.

Definition 4.2.1. An *affine flag in* V (also called a *decorated flag in* V) is a sequence

$$F_{(1)}, F_{(2)}, \ldots, F_{(k)} = \omega, \tag{4.2}$$

of simple anti-symmetric tensors $F_{(a)} \in \bigwedge^a(V)$, each one dividing the next. A *configuration of r affine flags in V* is an r-tuple of affine flags in V considered up to simultaneous $\mathrm{SL}(V)$ action. We denote the space of such configurations by $\mathrm{FG}(k,r)$, and its field of rational functions by $\mathbb{C}(\mathrm{FG}(k,r))$.

Remark 4.2.2. A simple anti-symmetric tensor $v \in \bigwedge^a(V)$ determines a subspace

$$\overline{v} = \{w \in V : wv = 0\} \subset V. \tag{4.3}$$

This subspace has dimension a. If v and v' are two simple tensors satisfying $\overline{v} = \overline{v'}$, then v and v' are scalar multiples of one another inside $\bigwedge^a(V)$. For an affine flag as in (4.2), the chain of subspaces $\overline{F_{(1)}}, \overline{F_{(2)}}, \dots, \overline{F_{(k)}}$ is a complete flag in V. An affine flag can therefore be thought of as an ordinary flag equipped with a choice of volume form on each subspace in the flag.

Fock and Goncharov provided birational coordinates on $\mathrm{FG}(k,r)$ which we describe now. We begin by focusing on a triple of flags (i.e. the case $r=3$). Consider a point $p \in \mathrm{FG}(k,3)$ represented by a triple of affine flags (F_1,F_2,F_3). Let (a,b,c) be a triple of nonnegative integers satisfying $a+b+c=k$, at least two of which are positive. We define the *Fock-Goncharov coordinate* by

$$\Delta_{a,b,c,}(p) = \omega^*(F_{1,(a)}F_{2,(b)}F_{3,(c)}). \tag{4.4}$$

We note that the right-hand side of (4.4) does not depend on the choice affine flags (F_1,F_2,F_3) representing p.

Following [15], we arrange these Fock-Goncharov coordinates in a triangular array, drawing directed arrows between adjacent entries so that every small triangle in the diagram is oriented counterclockwise, cf. Figure 4.1.

$$\begin{tikzcd}
 & & \Delta_{301} \arrow[dr] & & \Delta_{310} \arrow[ll] & & \\
 & \Delta_{202} \arrow[dr] & & \Delta_{211} \arrow[ll] \arrow[ur] \arrow[dr] & & \Delta_{220} \arrow[ll] & \\
\Delta_{103} \arrow[dr] & & \Delta_{112} \arrow[ll] \arrow[ur] \arrow[dr] & & \Delta_{121} \arrow[ll] \arrow[ur] \arrow[dr] & & \Delta_{130} \arrow[ll] \\
 & \Delta_{013} \arrow[ur] & & \Delta_{022} \arrow[ll] \arrow[ur] & & \Delta_{031} \arrow[ll] \arrow[ur] &
\end{tikzcd}$$

Figure 4.1: A triangular array of Fock-Goncharov coordinates for $\mathrm{FG}(4,3)$ (i.e. 3 affine flags in 4-space). The directed edges will serve as a fragment of a quiver. The three "corners" of the triangle are not included in the array, as they correspond to (a,b,c) with two entries equal to 0.

Now we return to the general case of r affine flags.

Definition 4.2.3. Consider a convex r-gon with its vertices numbered $1,\ldots,r$ in clockwise fashion. For each triangulation T of this r-gon we define a seed $\Sigma(T) = (\tilde{\mathbf{x}}(T), Q_k(t))$ in $\mathbb{C}(\mathrm{FG}(k,r))$ as follows. Let $x < y < z$ be the vertices of a triangle in T. We denote by $\Delta_{a,b,c}(x,y,z) \in \mathbb{C}(\mathrm{FG}(k,r))$ the Fock-Goncharov coordinates coming from this triangle. Notice that if two triangles in T share an edge, then the Fock-Goncharov coordinates on this shared edge agree as functions on $\mathrm{FG}(k,r)$. The extended cluster $\tilde{\mathbf{x}}(T)$ is the union of the Fock-Goncharov coordinates taken over the various triangles in T. The Fock-Goncharov coordinates sitting on the boundary of the r-gon serve as frozen variables. The ice quiver $Q_k(T)$ for this seed is obtained by gluing together the quiver fragments from each triangle in T, using the directed edges indicated Figure 4.1. See Figure 4.2 for an example.

Theorem 4.2.4 (Fock-Goncharov)**.** *Each extended cluster $\tilde{\mathbf{x}}(T)$ provides a system of birational coordinates on $\mathrm{FG}(k,r)$. The seeds $\Sigma(T)$, as T varies over the set of triangulations of the r-gon, are related to each other by sequences of mutations. Consequently they give rise to a well-defined cluster structure inside $\mathbb{C}(\mathrm{FG}(k,r))$.*

Remark 4.2.5. The space $\mathrm{FG}(k,r)$ comes out of the approach to higher Teichmüller

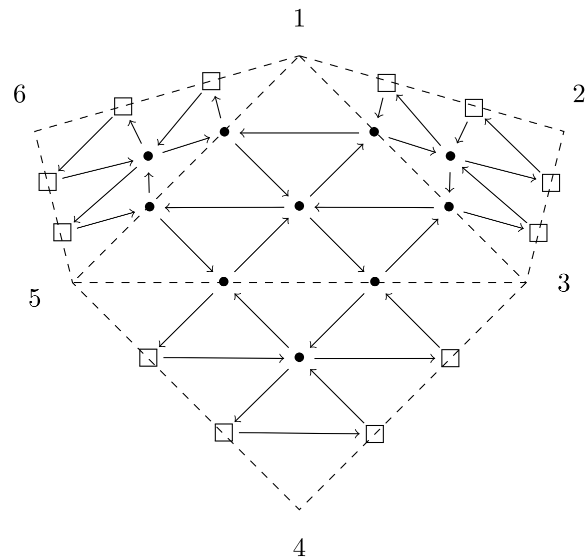


Figure 4.2: A Fock-Goncharov seed $\Sigma(T)$ inside $\mathbb{C}(\mathrm{FG}(3,6)$, i.e. for 6 affine flags in 3-space. T is the triangulation of the hexagon indicated in dashed lines. The ice quiver $Q_3(T)$ is indicated by the directed edges drawn inside the hexagon. The 12 frozen variables lie on the boundary of the hexagon. There are 10 cluster variables.

theory developed in [15]. Let G be a semisimple Lie group. Let S be a surface with boundary and with at least one marked point on each boundary component. Fock and Goncharov defined a moduli space $\mathcal{A}_{G,S}$ of *decorated twisted G-local systems on S*. They produced several systems of birational coordinates on $\mathcal{A}_{G,S}$, one such system for each triangulation of S, giving rise to a cluster structure in $\mathbb{C}(\mathcal{A}_{G,S})$. Theorem 4.2.4 is a special case of this construction when $G = \mathrm{SL}_k$ and S is a disk with r marked points on its boundary.

The following is the main result of Chapter IV.

Theorem 4.2.6. *The cluster structures in $\mathbb{C}(\hat{\mathrm{Gr}}(k,rk))$ and $\mathbb{C}(\mathrm{FG}_{2r}(\mathrm{SL}_k))$ described above are quasi-isomorphic to each other.*

In particular, whenever the number of affine flags in a Fock-Goncharov configuration space is even, the corresponding cluster algebra has Grassmannian cluster type. Theorem 4.2.6 is made explicit in Theorems 4.3.1 and 4.3.5, wherein the required quasi-isomorphism is described in concrete terms.

Remark 4.2.7. It is natural to look for more results in the spirit of Theorem 4.2.6. As a first step, one might ask if the cluster structures in $\mathbb{C}(\hat{\mathrm{G}}\mathrm{r}(k,n))$ and $\mathbb{C}(\mathrm{FG}(k,r))$ are quasi-isomorphic for choices of r and n not covered by Theorem 4.2.6. These cluster algebras have rank $(k-1)(n-k-1)$ and $(k-1)(\frac{rk}{2}-k-1)$ respectively. Hence it is necessary that $n = \frac{rk}{2}$, and one of r or k is even. As long as r is even we are in the situation of Theorem 4.2.6. When $r = 3$ and k is even we are in the situation considered in the next remark. We are curious whether odd $r \geq 5$ lead to any new quasi-isomorphisms. The smallest open instance asks whether the pair of cluster algebras $\mathbb{C}(\hat{\mathrm{G}}\mathrm{r}(4,10))$ and $\mathbb{C}(\mathrm{FG}_5(\mathrm{SL}_4))$ have the same cluster type. Both of these are of infinite mutation type.

Remark 4.2.8. Let $U \subset \mathrm{SL}_k \cong \mathrm{SL}((V)$ be the maximal unipotent subgroup consisting of upper triangular matrices with diagonal entries equal to 1. Let GL_k/U denote the *base affine space*. Its coordinate ring $\mathbb{C}[\mathrm{GL}_k/U]$ is generated by the *flag minors* (i.e., those minors whose column set is left-justified), and there is a standard cluster structure [16] on $\mathbb{C}[\mathrm{GL}_k/U]$. The cluster structure on $\mathrm{FG}(k,3)$ can be obtained from the cluster structure on GL_k/U by setting certain frozen variables equal to 1. In the special case that $k = 2j$ is even, the cluster algebras $\mathbb{C}(\mathrm{FG}(2j,3))$ and $\mathbb{C}(\mathrm{GL}_{2j}/U)$ have the same cluster type as a Grassmannian $\mathbb{C}(\hat{\mathrm{G}}\mathrm{r}(j,3j))$. This coincidence is implied by work of Ladkani [41] which establishes a derived equivalence between a certain algebra associated to the grid quiver for $\mathbb{C}(\hat{\mathrm{G}}\mathrm{r}(k,3k))$, and a certain algebra associated to the the triangular quiver for base affine space. By Theorem 4.2.6, this cluster type also occurs in $\mathbb{C}(\mathrm{FG}(k,6))$.

It seems a natural problem to write down a quasi-isomorphism bearing witness to this coincidence, i.e. a quasi-isomorphism between base affine space GL_{2j}/U and the

Grassmannian $\hat{\mathrm{Gr}}(j,3j)$ (or the configuration space of six affine flags). We partially carry this out in Appendix C.

4.3 The quasi-isomorphism

We begin defining the maps giving rise to the quasi-isomorphism in Theorem 4.2.6.

Let $(v_1,\ldots,v_{rk})$ be a generic tuple of vectors in V. Focusing on the first k vectors, we produce a pair of "opposite" affine flags whose sequences of simple tensors are

$$v_1, v_1v_2, \ldots, v_1v_2\cdots v_{k-1} \tag{4.5}$$

$$v_k, v_{k-1}v_k, \ldots, v_2v_3\cdots v_{k-1}. \tag{4.6}$$

Grouping the vectors $(v_1,\ldots,v_{rk})$ into r groups of size k and repeating the procedure (4.5) and (4.6) in each group, we obtain a $2r$-tuple of affine flags. This procedure is $\mathrm{SL}(V)$-equivariant and descends to a rational map

$$\Psi\colon \mathrm{SL}(V)\backslash(V^{rk}) \dashrightarrow \mathrm{FG}(k,2r). \tag{4.7}$$

Theorem 4.3.1. *The induced map $\Psi^*\colon \mathbb{C}(\mathrm{FG}(k,2r)) \to \mathbb{C}(\hat{\mathrm{Gr}}(k,rk))$ is a quasi-isomorphism. We describe its quasi-inverse in Theoerem 4.3.5.*

Notice that (4.5) only produces an affine flag when the join $v_1v_2\cdots v_k$ is nonzero, i.e the map Ψ is only defined away from a Zariski closed subset on which certain frozen variables vanish (this is why we use a dashed arrow in (4.7)).

To define a quasi-inverse to Ψ^*, we make use of the *Grassmann-Cayley algebra*, i.e. the exterior algebra endowed the following additional *meet* operation:

Definition 4.3.2. Let $v = v_1\cdots v_a$ and $w = w_1\cdots w_b$ be simple tensors with $a+b \geq k$. Their *meet* $v \cap w \in \bigwedge^{a+b-k}(V)$ is defined by

$$v \cap w = \sum_{\sigma} \mathrm{sign}(\sigma)\omega^*(v_{\sigma(1)}\cdots v_{\sigma(k-b)}w_1\cdots w_b)v_{\sigma(k-b+1)}\cdots v_{\sigma(a)}. \tag{4.8}$$

The sum in (4.8) is over permutations σ, in the symmetric group on a letters, satisfying $\sigma(1) < \sigma(2) < \cdots < \sigma(k-b)$. We extend $\cap$ to a bilinear map $\cap\colon \bigwedge^a(V) \otimes \bigwedge^b(V) \to \bigwedge^{a+b-k}(V)$.

Remark 4.3.3. Our notation differs from that in [54], in which the join and meet are denoted by $\vee$ and $\wedge$ respectively. We prefer ours because it does not clash with the usual meaning of the symbol $\wedge$.

Lemma 4.3.4 ([54])**.** *The meet operation has the following properties:*

- *associativity,*
- *commutativity up to a sign, i.e.* $v \cap w = (-1)^{(k-a)(k-b)} w \cap v$ *for* $v \in \bigwedge^a, w \in \bigwedge^b$*,*
- *the meet of simple tensors is again simple. The meet* $v \cap w$ *is nonzero if and only if the subspaces* $\overline{v}$ *and* $\overline{w}$ *span* V*, in which case* $\overline{v \cap w} = \overline{v} \cap \overline{w}$ *is their intersection, cf. Remark 4.2.2.*

We begin describing the quasi-inverse to Ψ^*. For a pair of affine flags (F, G) we produce a list of k vectors in the following way

$$(F, G) \mapsto F_{(1)}, F_{(2)} \cap G_{(k-1)}, F_{(3)} \cap G_{(k-2)}, \cdots, F_{(k-1)} \cap G_{(2)}, G_{(1)}. \tag{4.9}$$

The rule in (4.9) is $\mathrm{SL}(V)$-equivariant.

Now consider a $2r$-tuple of affine flags $(F_1, \ldots, F_{2r})$. Grouping the flags in pairs, and applying (4.9) in each pair, we obtain an rk-tuple of vectors. This descends to a map

$$\Phi\colon \mathrm{FG}(k, 2r) \to \mathrm{SL}(V)\backslash(V^{rk}). \tag{4.10}$$

Theorem 4.3.5. *The induced map* $\Phi^*\colon \mathbb{C}(\hat{\mathrm{G}}\mathrm{r}(k, rk)) \to \mathbb{C}(\mathrm{FG}(k, 2r))$ *is a quasi-inverse to the map in Theorem 4.3.1.*

Remark 4.3.6. Let $n = rk$. The spaces $\mathrm{FG}(k, 2r)$ and $\hat{\mathrm{Gr}}(k, rk)$ are manifolds of dimension $k(n-k)+(n-2r+1)$ and $k(n-k)+1$ respectively. The map (4.10) should have fibers of dimension $n-2r$. For a point $p \in \mathrm{FG}(k, 2r)$ and $\lambda \in \mathbb{C}^*$, rescaling $F_{1,(a+1)}$ by λ and $F_{2,(k-a)}$ by λ^{-1} (for any $1 \leq a \leq k-2$) does not change the value of $\Phi(p)$. This is $k-2$ degrees of freedom in each pair of flags, leading to $r(k-2) = n-2r$ degrees of freedom in each fiber.

Although the maps Ψ and Φ behave nicely with respect to cluster structures, they are less well behaved geometrically. The map Ψ is not injective. Likewise, not every point in the Grassmannian has a nonempty fiber with respect to Φ. Theorem 4.2.6 implies that these statements are better behaved on the open subsets defined by non-vanishing of frozen variables.

Remark 4.3.7. An affine flag in 2-space is the same as a nonzero vector. In the case $k = 2$, the maps in Theorems 4.3.1 and 4.3.5 are the identity map.

Remark 4.3.8. When $k = 3$, we can think of vectors (respectively flags) as points (respectively a point and a line containing it) in the projective space $\mathbb{P}^2$. A configuration of n vectors determines an n-gon in $\mathbb{P}^2$, and a configuration of n flags determines a pair of polygons, one inscribed inside the other. The maps Ψ and Φ induce maps on these geometric objects. For example, when $r = 2$, Ψ sends the six points $P_1, \ldots, P_6$ to the four flags $(P_1, \overline{P_1 P_2}), (P_3, \overline{P_2, P_3}), (P_4, \overline{P_4 P_5}), (P_6, \overline{P_5, P_6})$. The map Φ recovers P_2 by $P_2 = \overline{P_1 P_2} \cap \overline{P_2, P_3}$. Passing from configurations in $\mathbb{C}^3$ to configurations in $\mathbb{P}^2$ is related to moving from the cluster $\mathcal{A}$-space to the cluster $\mathcal{X}$-space in the sense of Fock and Goncharov.

This map on geometric objects was already given by Morier-Genoud, Ovsienko, and Tabachnikov in [46, Section 4.6], although they focused on the case of convex polygons (this amounts to considering points whose coordinates in the cluster $\mathcal{X}$

space are positive).

4.4 Proof of Theorems 4.3.1 and 4.3.5

Before proving Theorems 4.3.1 and 4.3.5, we would like to comment on some of the difficulties involved.

Let $f = \Delta_{a,b,c}(x,y,z) \in \mathbb{C}(\mathrm{FG}(k,2r))$ be a Fock-Goncharov coordinate in a seed $\Sigma(T)$ given by a triangulation. Then $\Psi^*(f)$ is a Plücker coordinate. However, if all of $a,b,c > 0$, and if g denotes the cluster variable obtained by mutating at f, then $\Psi^*(g)$ is not usually even proportional to a Plücker coordinate. That is, $\Psi^*(g)$ is not a recognizable cluster variable in $\mathbb{C}(\hat{\mathrm{Gr}}(k,rk))$. By considering an appropriate *nerve* for $\mathbb{C}(\mathrm{FG}(k,2r))$ (cf. Definition 2.5.1), we will avoid discussing such g.

Remark 4.4.1. Theorem 4.2.6 has as its corollary the fact that $\mathbb{C}(\hat{\mathrm{Gr}}(k,rk))$ and $\mathbb{C}(\mathrm{FG}_{2r}(\mathrm{SL}_k))$ are of the same cluster type. One advantage of our style of proof (and of Proposition 2.5.2) is that we obtain this corollary *without performing any explicit quiver mutations*. This corollary is otherwise not immediately obvious. For example, there are only 3 combinatorially different triangulations of a hexagon, hence only three quivers $Q_k(T)$ for $\mathbb{C}(\mathrm{FG}_6(\mathrm{SL}_k))$ arising from a triangulation. None of these is a familiar quiver for the Grassmannian (i.e., a variant on a grid quiver).

The definition of Fock-Goncharov coordinate can be extended in an obvious way to obtain invariants of four flags, giving rise to functions $\Delta_{a,b,c,d}(w,x,y,z) \in \mathrm{FG}(k,r)$ for any choices of $a+b+c+d = k$ and $\{w < x < y < z\} \subset [1,r]$. We call these functions quadruple invariants.

Proposition 4.4.2. *Suppose $r \geq 4$. There is a nerve $\mathcal{N}$ for $\mathbb{C}(\mathrm{FG}(k,r))$, such that every cluster variable on $\mathcal{N}$ is a quadruple (or triple) invariant.*

Proof. Let T be a triangulation of the r-gon. Fock and Goncharov described some Plücker-like relations amongst quadruple invariants [15, Equation 10.3]. Letting $\overline{a} = (a,b,c,d)$ denote a nonnegative integer solution to $a+b+c+d = k-2$, these relations are

$$\Delta_{\overline{a}+(1,0,1,0)}\Delta_{\overline{a}+(0,1,0,1)} = \Delta_{\overline{a}+(1,1,0,0)}\Delta_{\overline{a}+(0,0,1,1)} + \Delta_{\overline{a}+(1,0,0,1)}\Delta_{\overline{a}+(0,1,1,0)}, \tag{4.11}$$

If a Fock-Goncharov coordinate $x \in \mathbf{x}(T)$ lies on a shared edge between two triangles in T, then its corresponding vertex in $Q_k(T)$ has valence four. The exchange relation mutating x out of $\mathbf{x}(T)$ is of the form (4.11).

If a Fock-Goncharov coordinate $x \in \mathbf{x}(T)$ lies in the interior of a triangle of T, pick a quadrilateral containing this face function. There are two different triangulations of this quadrilateral, one of which is used in T. Fock and Goncharov gave a sequence of mutations between these two triangulations [15, Section 10]. Each exchange relation in this mutation sequence is of the form (4.11) – thus every cluster variable that arises during this mutation sequence is a quadruple invariant – and every face function in the quadrilateral is exchanged at least once. □

The next lemma is used in the proof that Φ^* and Ψ^* are quasi-inverses. It's a calculation from the definitions.

Lemma 4.4.3. *For any pair of affine flags, F, G, the following hold*

$$\bigwedge_{\ell=1}^{j} F_{(\ell)} \cap G_{(k-\ell+1)} = F_{(j)} \prod_{\ell=2}^{j} [F_{(\ell-1)} \cap G_{(k-\ell+1)}] \tag{4.12}$$

$$\bigwedge_{\ell=1}^{j} G_{(k-j+\ell)} \cap F_{(j-\ell+1)} = F_{(j)} \prod_{j=1}^{\ell-1} [G_{i,(k-j+\ell)} \cap F_{(j-\ell)}]. \tag{4.13}$$

We use $\bigwedge_{\ell=1}^{j}$ to denote a join of j vectors, taken from left to right. We use the product symbol on the right-hand side of (4.12)*, since the elements in the product are scalars.*

Now we prove Theorems 4.3.1 and 4.3.5.

Proof. We use Lemma 2.3.8 and Proposition 2.5.2. First we check that both maps preserve coefficients. From the definitions, Ψ^* applied to a frozen variable $\Delta_{a,k-a}(x,x+1) \in \mathbb{C}(\mathrm{FG}_{2r}(\mathrm{SL}_k))$ is a frozen variable in $\mathbb{C}(\hat{\mathrm{Gr}}(k,rk))$. On the other hand, Φ^* applied to a frozen variable in $\mathbb{C}(\hat{\mathrm{Gr}}(k,rk))$ is a product of $k-1$ frozen variables in $\mathbb{C}(\mathrm{FG}(k,r))$. This follows by repeatedly applying (4.12) and (4.13). For example, when $r=2, k=4$ we have that

$$\Phi^*(\Delta_{1234}) = [\Delta_{1,3}(1,2)][\Delta_{2,2}(1,2)][\Delta_{3,1}(1,2)] \tag{4.14}$$

$$\Phi^*(\Delta_{2345}) = [\Delta_{2,2}(1,2)][\Delta_{3,1}(1,2)][\Delta_{3,1}(2,3)] \tag{4.15}$$

$$\Phi^*(\Delta_{3456}) = [\Delta_{3,1}(1,2)][\Delta_{2,2}(2,3)][\Delta_{1,3}(3,4)] \tag{4.16}$$

$$\Phi^*(\Delta_{4567}) = [\Delta_{1,3}(2,3)][\Delta_{1,3}(3,4)][\Delta_{2,2}(3,4)], \tag{4.17}$$

and so on.

Let T be a triangulation of the $2r$-gon. Then $\Psi^*(\mathbf{x}(T))$ is a collection of Plücker coordinates in $\mathbb{C}(\hat{\mathrm{Gr}}(k,rk))$. We claim this collection is weakly separated, hence forms a cluster. Let $S_1, S_2 \subset [1,rk]$ be a pair of k-subsets. A *witness* preventing their weak separation is a choice of $a<b<c<d$ such that $a,c \in S_1$ and $b,d \in S_2$. For a collection of Fock-Goncharov coordinate x inside a single triangle, all of the Plücker coordinates $\Psi^*(x)$ consist of at most three disjoint cyclic intervals. If y is another coordinate in this triangle, then the cyclic intervals in $\Psi^*(x)$ and $\Psi^*(y)$ are nested (meaning for each cyclic interval in $\Psi^*(x)$, either it contains or is contained in a cyclic interval for $\Psi^*(y)$). There can be no witness preventing weak separation in this case (this would be false if we were considering 4 disjoint cyclic intervals). For Fock-Goncharov coordinates x and y lying in different triangles, weak separation

is even more clear: the three disjoint cyclic intervals will be "far away" from each other and can't lead to a witness.

Now we consider the nerve $\mathcal{N}$ for $\mathbb{C}(\mathrm{FG}(k,r))$ constructed in Proposition 4.4.2. Evaluating Ψ^* at any cluster variable in $\mathcal{N}$ yields a Plücker coordinate, so the conditions of Proposition 2.5.2 are satisfied (in fact: $\Psi^*(x)$ is *equal* to a cluster variable on the nerve, not merely proportional to a cluster variable). Applying Ψ^* to the Fock-Goncharov exchange relation (4.11) produces the corresponding 2-term Plücker relation; thus clusters in $\mathcal{N}$ are sent to clusters in $\hat{\mathrm{Gr}}(k,rk)$ in a way that is compatible with mutation.

To complete the proof we need to check that $\Phi^* \circ \Psi^* = (\Psi \circ \Phi)^*$ is proportional to the identity map on $\mathbb{C}(\mathrm{FG}(k,2r))$. Indeed, the composition $\Psi \circ \Phi$ maps the pair of affine flags (F_1, F_2) to a new pair of affine flags (F_1', F_2'). Using (4.12) and (4.13) repeatedly, one sees that the extensors $F'_{1,(j)}$ and $F_{1,(j)}$ differ by a scalar multiple that is a monomial in the frozen variables. Ditto for $F'_{2,(j)}$ and $F_{2,(j)}$ and so on. Continuing with the example $r = 2, k = 4$, the affine flags F_1' and F_2' are given by the following sequence of simple tensors

$$F_1' = F_{1,(1)}, [F_{1,(1)}F_{2,(3)}]F_{1,(2)}, [F_{1,(1)}F_{2,(3)}][F_{1,(2)}F_{2,(2)}]F_{1,(3)} \tag{4.18}$$

$$F_2' = F_{2,(1)}, [F_{1,(3)}F_{2,(1)}]F_{2,(2)}, [F_{1,(3)}F_{2,(2)}][F_{1,(2)}F_{2,(2)}]F_{2,(3)}. \tag{4.19}$$

Since the various extensors only differ by scalar multiples in the frozen variables, the corresponding quadruple invariants only differ by a monomial in the frozen variables, i.e. $\Phi^* \circ \Psi^*(x) \asymp x$ on the nerve. $\square$

4.5 Symmetries and the cluster modular group

In this section, we recall certain symmetries of the cluster structures in Grassmannians and Fock-Goncharov spaces. Using the quasi-isomorphism we have established between these spaces, a transparent symmetry in one space becomes a surprising symmetry in the other.

Following Fock and Goncharov, let $s_G = (-\mathrm{Id})^{k-1} \in G = \mathrm{SL}_k$. It either plus or minus the identity matrix according to whether k is odd or even.

Definition 4.5.1. Let X be one of the configuration spaces $\mathrm{SL}(V)\backslash V^n$ or $\mathrm{FG}(k,n)$. The *twisted cyclic shift* on X is the map induced by the map on n-tuples

$$(X_1,\dots,X_n) \mapsto (X_2,\dots,X_n,s_G(X_1)). \tag{4.20}$$

We denote by ρ and P the twisted cyclic shift of vectors or affine flags respectively.

Let ρ^* and P^* denote the pullbacks to $\mathbb{C}(\hat{\mathrm{Gr}}(k,n))$ and $\mathbb{C}(\mathrm{FG}_n(\mathrm{SL}_k))$ respectively. Note the factor s_G in (4.20) is necessary to fix signs, for example in $\mathbb{C}(\hat{\mathrm{Gr}}(4,8))$ we have $\rho^*(\Delta_{1238}) = -\Delta_{2341} = \Delta_{1234}$.

The following proposition is a basic feature of the cluster structures on $\mathbb{C}(\hat{\mathrm{Gr}}(k,n)$ and $\mathbb{C}(\mathrm{FG}(k,r))$.

Proposition 4.5.2. *Each of the pullbacks $\rho^* \in \mathrm{End}(\mathbb{C}(\hat{\mathrm{Gr}}(k,n)]), P^* \in \mathrm{End}(\mathbb{C}(\mathrm{FG}(k,r)])$ is a cluster automorphism of the corresponding cluster structure.*

That is, these maps permute the frozen variables, cluster variables, and clusters in the respective cluster algebra.

In addition to the cyclic shift, there is another cluster automorphism of the cluster structure on $\mathrm{FG}(k,r)$ induced by the *Hodge star*. We give a proof of this in Proposition 4.5.6. After we had written down the details, we learned this result was given independently in recent work of Goncharov and Shen [31] and Le [42] (see also Henriques [36]).

To define the Hodge star, we endow V with a Hermitian inner product $\langle,\rangle$. One then obtains an inner product on the space of simple tensors of size a: for $v = v_1 \cdots v_a$

and $w = w_1 \cdots w_a$,

$$\langle v, w \rangle = \det(\langle v_i, w_j \rangle)_{i,j=1,\dots,a}. \tag{4.21}$$

Definition 4.5.3. The *Hodge star* (also known as the *generalized cross product*) is the unique linear map $H\colon \bigwedge(V) \to \bigwedge(V)$ satisfying

$$v \cap (H(w)) = \langle v, w \rangle \omega \tag{4.22}$$

for all simple tensors $v, w \in \bigwedge^a(V)$.

The Hodge star restricts to a linear isomorphism $H\colon \bigwedge^a(V) \to \bigwedge^{k-a}(V)$ for all a. It is an involution up to sign: $H \circ H$ rescales $\bigwedge^a(V)$ by a factor of $(-1)^{a(k-a)} = (-1)^{a(k-1)}$. Thus $H \circ H$ acts on $\bigwedge^a(V)$ by the matrix s_G. These and other basic properties of H can be found in [43].

Again let $U \subset \mathrm{SL}_k \cong \mathrm{SL}((V)$ be the maximal unipotent subgroup of upper triangular matrices with 1's on the diagonal. We can identify the space of affine flags with SL_k/U in the following way: from the definition, an affine flag F can be represented by a sequence of vectors $v_1, \dots, v_k$ satisfying $F_{(a)} = v_1 \cdots v_a$. Thinking of these vectors as the columns of a matrix $M \in \mathrm{SL}_k$, right multiplication by U does not change the corresponding affine flag. Conversely, every matrix M' representing F is of the form Mu for some $u \in U$.

Lemma 4.5.4. *Let F be an affine flag given by simple tensors $F_{(1)}, \dots, F_{(k-1)}$. The sequence of tensors*

$$H(F_{(k-1)}), H(F_{(k-2)}), \dots, H(F_{(1)}) \tag{4.23}$$

defines an affine flag.

Representing F by a matrix in $M \in \mathrm{SL}_k$, the dual affine flag is represented by the matrix $(M^t)^{-1} w_0 \in \mathrm{SL}_k$ where $w_0 \in \mathrm{SL}_k$ is the antidiagonal matrix $w_0 = \begin{pmatrix} 0 & 0 & 0 & \iddots \\ 0 & 0 & 1 & \cdots \\ 0 & -1 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \end{pmatrix}$.

It follows that the Hodge star has a well-defined action on $\mathrm{FG}(k,r)$, and $H \circ H$ acts by the identity map on $\mathrm{FG}(k,r)$.

Proof. First we explain why $H(M) = (M^t)^{-1}w_0$. Let $M[1],\ldots,M[k]$ be the columns of M. Choose a standard orthonormal basis $e_1,\ldots,e_k$ for V and let $\{e_S\}_S$ denote the standard basis for $\bigwedge^a(V)$, indexed by subsets $S = \{s_1,\ldots,s_a\} \subset [1,k]$. It suffices to verify (4.22) when $v = e_S$ and $w = H(M[1]\cdots M[a])$. The right hand side of (4.22) is

$$\langle e_S, M[1]\cdots M[a]\rangle = \det(M_{S,[1,a]}), \tag{4.24}$$

the minor of M using the first a columns and the rows in S. By the generalization of Cramer's rule, this minor is expressible as a minor of the inverse matrix as follows. For $1 \le i \le a$, let N be the matrix in which column s_i of M^{-1} is replaced by column i of the identity matrix (that is, replaced by the column vector e_i). Then

$$\det(M_{S,[1,a]}) = \det(N). \tag{4.25}$$

After performing column operations (which do not change the value of $\det(N)$), we can modify N so that its first a rows are $e_{s_1},\ldots,e_{s_a}$, so that the join of the first a rows is e_S. These column operations do not change the bottom $(k-a)$ rows of N. Thus,

$$\det(N) = e_S \wedge ((M^t)^{-1}[a+1]\cdots \wedge (M^t)^{-1}[k]) \tag{4.26}$$

where the transpose is necessary to switch from rows to columns. Comparing with (4.22), it follows

$$H(M[1]\cdots M[a]) = (M^t)^{-1}[a+1]\cdots(M^t)^{-1}[k]. \tag{4.27}$$

Thus, the simple tensors for $H(M)$ are obtained by reading the columns of $(M^t)^{-1}$ from right to left; multiplying by w_0 accounts for this.

□

For calculations involving H, we rely on the following well-known property [26]:

Lemma 4.5.5. *Let $x \in \bigwedge^a(V), y \in \bigwedge^b(V)$. Then*

$$H(xy) = H(x) \cap H(y). \tag{4.28}$$

Informally, we evaluate H^* by replacing $F_{i,(a)}$ with $F_{i,(k-a)}$ and swapping the meet and join operations.

Proposition 4.5.6. *The induced map H^* on $\mathbb{C}(\mathrm{FG}(k,r))$ is a cluster automorphism.*

Proof. The pullback $H^*(\Delta_{a,b,c}(x,y,z))$ evaluates on a point $p \in \mathrm{FG}(k,r)$ via

$$H^*(\Delta_{a,b,c}(x,y,z))(p) = F_{x,(k-a)}(p) \cap F_{y,(k-b)}(p) \cap F_{z,(k-c)}(p). \tag{4.29}$$

In particular, $H^*(F_{x,(a)}F_{x+1,(k-a)}) = F_{x,(k-a)} \cap F_{x+1,(a)} = F_{x,(k-a)}F_{x+1,(a)}$, which says that H^* permutes the frozen variables.

Let $\mathbf{x}(T)$ be a cluster given by a triangulation T. To finish the proof, we need to show the "dual cluster" $H^*(\mathbf{x}(T))$ is actually a cluster in $\mathbb{C}(\mathrm{FG}(k,r))$. That is, we need to provide an appropriate sequence of mutations from $\mathbf{x}(T)$ to $H^*(\mathbf{x}(T))$. In fact, this can be done "triangle by triangle." More precisely, if x,y,z are vertices of a triangle in T, we will describe a sequence of mutations, μ, that replaces the cluster variable $\Delta_{a,b,c}(x,y,z)$ by the cluster variable $H^*(\Delta_{a,c,b}(x,y,z))$. Each mutation in this sequence occurs at a location (a,b,c) with $a,b,c > 0$ inside the triangle formed by x,y,z. Notice that the cluster variables on the edge of the triangle formed by x,y,z are permuted by H^*, so we are done once we describe the sequence μ.

For a triple (a,b,c), we will refer to its first coordinate, a, as its *height*. We denote by μ_a the simultaneous mutation at all interior vertices (a,b,c) whose height is a.

Then μ is the sequence of mutations

$$(4.30) \qquad \mu_{k-1} \circ (\mu_{k-2}\mu_{k-1}) \circ \cdots \circ (\mu_2 \cdots \mu_{k-2}\mu_{k-1}) \circ (\mu_1 \cdots \mu_{k-2}\mu_{k-1}).$$

Note that μ_a is only well-defined when there are no edges in the quiver between vertices at height a. As we will see, this is the case throughout (4.30).

For a point $p \in \mathrm{FG}(k,r)$, we let (F,G,H) denote the three flags describing p at the vertices x,y,z. The additional cluster variables that show up when performing the mutations in (4.30) are certain mixtures of joins and meets which we will denote by $\Delta_{s,t;u,v}$ where $s \geq t$ and $s+t+u+v = 2k$. They are defined by

$$(4.31) \qquad \Delta_{s,t;u,v}(p) = F_{(s)} \cap (F_{(t)}G_{(u)}) \cap H_{(v)},$$

and they satisfy a relation

$$(4.32) \qquad \Delta_{s,t;u,v}\Delta_{s-1,t-1;u+1,v+1} = \Delta_{s-1,t;u+1,v}\Delta_{s,t-1;u,v+1} + \Delta_{s-1,t;u,v+1}\Delta_{s,t-1;u+1,v}.$$

Notice that $\Delta_{k,t;u,v}$ is a triple invariant and $\Delta_{s,0;u,v}$ is the image of a triple invariant under H^*. Notice also that $\Delta_{s,t;u,v}$ factors as $\Delta_{s,0,v}\Delta_{t,u,0}$ when $s+v = t+u = k$, and that $\Delta_{s,t;u,v} = \Delta_{s,u,0}\Delta_{t,0,v}$ when $s+u = t+v = k$.

Every mutation in (4.30) is a consequence of (4.32): the "typical" relation is exactly of the form (4.32), and the more complicated-looking exchange relations follow from (4.32) by factoring one or two of the four terms on the right-hand side. We can think of (4.30) as a composition of $k-2$ rounds, each of which is indicated by parentheses in (4.30). At the beginning of round r, the invariant at location (a,b,c) is $\Delta_{k-r+1,a-r+1;b+r-1,c+r-1}$, and mutating at this vertex produces the new cluster variable $\Delta_{k-r,a-r;b+r,c+r}$. The cluster variable at height r becomes $H^*(\Delta_{a,c,b})$ as claimed. Throughout the mutation sequence (4.30), the quiver transforms in a predictable way. By inspection using [38], it is straightforward to see that the quiver at each instant gives the exchange relation (4.32) (or its factored version) as claimed. $\square$

Intertwining with the maps Φ^* and Ψ^*, every endomorphism of $\mathbb{C}(\mathrm{FG}(k,2r))$ determines an endomorphism self-map of $\mathbb{C}(\hat{\mathrm{Gr}}(k,rk))$. Marsh and Scott [44] have defined a version of the twist map on the Grassmannian. We do not reproduce the definition [44, Definition 2.1] here. This twist map arises from rotation and duality of affine flags:

Proposition 4.5.7. *The pullback* $(\Phi \circ P^{-1} \circ H \circ \Psi)^* \in \mathrm{End}(\mathbb{C}[\hat{\mathrm{Gr}}(k,rk)])$ *is the Marsh-Scott twist map on* $\mathbb{C}(\hat{\mathrm{Gr}}(k,rk))$.

Proof. Let $n = rk$ and $V_1,\ldots,V_n$ be a list of n vectors. Evaluating the composition $\Phi \circ P^{-1} \circ H \circ \Psi$ on this list produces another list of n vectors, namely

(4.33)

$$H(V_{n-k+1}\cdots V_n), H(V_{n-k+3}\cdots V_n) \cap H(V_1), \ldots, H(V_1\cdots V_{k-1}), \ldots, H(V_{n-k+1}\cdots V_{n-1}).$$

We indicated the first, second, k^{th}, and n^{th} term in this list in (4.33). Using $H(V_{n-k+3}\cdots V_n) \cap H(V_1) = H(V_{n-k+3}\cdots V_n V_1)$, and so on, one sees that the ℓ^{th} term in this list is $H(V_{\ell-k+1}\cdots V_{\ell-1})$. This description agrees with [44, Definition 2.1]. □

We now have at our disposal three quasi-automorphisms of the Grassmannian cluster algebra: the cyclic shift ρ^* as well as the intertwined versions of P^* and H^*. They determine elements $\dot{\rho}, \dot{P},$ and $\dot{H} \in \mathrm{CMG}(\mathbb{C}[\hat{\mathrm{Gr}}(k,rk)])$ in the cluster modular group.

Proposition 4.5.8. *The elements* $\dot{\rho}, \dot{P}, \dot{H} \in \mathrm{CMG}(\mathbb{C}(\hat{\mathrm{Gr}}(k,rk)))$ *satisfy the follow-*

ing relations:

$$\dot{H}^2 = \dot{\rho}^{rk} = 1 \tag{4.34}$$

$$\dot{\rho}^k = \dot{P}^2 \tag{4.35}$$

$$\dot{P}\dot{H} = \dot{H}\dot{P} \tag{4.36}$$

$$\dot{\rho}\dot{P^{-1}}\dot{H} = \dot{P^{-1}}\dot{H}\dot{\rho}. \tag{4.37}$$

Proof. The first three relations are obvious, e.g. that $\rho^k = \Psi \circ P^2 \circ \Phi$ follows from the way Φ and Ψ were defined. Similarly, $\dot{P}$ and $\dot{H}$ commute since P and H commute inside $\mathrm{End}(\mathrm{FG}(k,r))$. The last relation follows from Proposition 4.5.7 and the fact that the Marsh-Scott twist commutes with the twisted cyclic shift (or by an easy direct check). □

The group of symmetries in Proposition 4.5.8 can be used to understand the cluster combinatorics in $\mathbb{C}(\hat{\mathrm{Gr}}(k,n))$ and $\mathbb{C}(\mathrm{FG}(k,n))$. In a forthcoming preprint [25] we will prove:

Theorem 4.5.9. *Suppose $k = 3$ and $r \geq 3$. Then the relations in Proposition 4.5.8 give a presentation for the subgroup generated by $\dot{\rho}, \dot{P}, \dot{H}$ inside the cluster modular group.*

The group in Theorem 4.5.9 surjects onto a free product of cyclic groups $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} \cong \mathrm{PSL}_2(\mathbb{Z})$. It has exponential growth. Acting on cluster variables in $\mathbb{C}(\hat{\mathrm{Gr}}(3,3r))$ using this group, Theorem 4.5.9 can be used to give infinite amount for conjectures of Fomin and Pylyavskyy [17, 18] explicitly describing the cluster combinatorics for $\mathbb{C}(\hat{\mathrm{Gr}}(3,3r))$ in terms of Kuperberg's basis of *non-elliptic web invariants* [40]. We are hopeful that our methods can be used to settle the Fomin-Pylyavskyy conjectures completely for $\hat{\mathrm{Gr}}(3,9)$.

In a different direction, for any k (not only $k = 3$) there is a generalized notion of an SL_k-*web* inside $\mathbb{C}(\hat{\mathrm{Gr}}(k,n))$ and $\mathbb{C}(\mathrm{FG}(k,r))$. Although these higher rank webs are less well understood, one expects them to play a role in the cluster combinatorics of these higher-rank Grassmannians. We will prove [25]:

Theorem 4.5.10. *The quasi-isomorphisms Ψ^* and Φ^* send SL_k web invariants in one space to web invariants in the other.*

Acting by the group of symmetries in Proposition 4.5.8, we obtain a large group of cluster variables that are also web invariants (up to frozen variable factors). This is a first step towards understanding non-Plücker cluster variables inside $\mathbb{C}(\hat{\mathrm{Gr}}(k,n))$ when $k \geq 4$.

The hypothesis $r \geq 3$ in Theorem 4.5.9 is essential. Namely:

Proposition 4.5.11. *The elements $\dot{P}, \dot{\rho}$, and $\dot{D}$ generate a finite subgroup inside the cluster modular group for $\mathbb{C}(\hat{\mathrm{Gr}}(k,2k))$.*

Proof. We call a non-frozen Plücker coordinate Δ_S *special* if S is a disjoint union of two cyclic intervals. The cluster in alluded to in Remark 4.1.1 consists entirely of special Plücker coordinates. Clearly, ρ^* preserves the set of special Plücker coordinates, and by [44, Proposition 3.3], so does the twist map (ignoring frozen variable factors). By a direct computation, using similar reasoning to (4.12) and (4.13), one computes that $\dot{H}$ also preserves the set of special Plücker coordinates, and therefore so must $\dot{P}$. □

Example 4.5.12. As a particular case of Proposition 4.5.11, we consider $\hat{\mathrm{Gr}}(3,6)$, which is quasi-isomorphic to $\mathrm{FG}(3,4)$. These cluster algebras have finite cluster type D_4, which also arises in the cluster algebra associated to a once-punctured square (cf. Chapter III). There is a simple dictionary between cluster variables for $\mathbb{C}(\mathrm{FG}(3,4))$ and arcs in the once-punctured square. Rotating the square one unit corresponds to rotation of affine flags. Changing tags at the puncture corresponds to duality for affine flags. Recall (cf. Proposition 3.2.2) that the cluster modular group for this cluster type has "extra" elements not arising as symetries of the once-punctured square. These extra symmetries are generated by the intertwined version of ρ^* inside $\mathbb{C}(\mathrm{FG}(3,4))$.

Proposition 4.5.11 should be expected when $k = 2$ or $k = 3$ since the cluster structures in $\hat{\mathrm{Gr}}(2,4)$ and $\hat{\mathrm{Gr}}(3,6)$ are of finite type. However, it is discouraging when considering $\hat{\mathrm{Gr}}(4,8)$: since $\hat{\mathrm{Gr}}(4,8)$ has infinitely many clusters but only finitely many exchange matrices, the cluster modular group must be infinite. When $k \geq 4$, it turns out that there are *several* fundamentally different choices of quasi-isomorphisms between the cluster structures on $\hat{\mathrm{Gr}}(k,rk)$ and $\mathrm{FG}(k,2r)$ (these different choices do not seem to be related to one another by composing with known quasi-automorphisms). Understanding these extra quasi-isomorphisms should be a first step towards setting up an appropriate group of symmetries for Grassmanians $\hat{\mathrm{Gr}}(k,n)$ with $k \geq 4$.

APPENDICES

APPENDIX A

Grassmannians and band matrices

We extend the constructions in Example 2.3.5 and Example 2.3.9 from the case $(k,n) = (2,5)$ to general (k,n). Let $\mathbf{X} = \hat{\mathrm{Gr}}(n-k,n)$. Let $\mathbf{Y} \cong \mathbb{C}^{(n-k)(k+1)}$ be the affine space of $(n-k) \times n$ *band matrices of width* $k+1$, i.e. the set of matrices Y whose entries $y_{i,j}$ are zero unless $i \le j \le i+k$.

We introduce a useful sign convention: if S is any set of $(n-k)$ natural numbers, we let Δ_S denote the Plücker coordinate obtained by first reducing all the elements of S to their least positive residue modulo n, sorting these residues, and then taking the corresponding Plücker coordinate. If there are fewer than $(n-k)$ distinct elements in S modulo n, then Δ_S is zero.

The coordinate ring $\mathbb{C}[\mathbf{Y}]$ contains minors $Y_{I,J}$ for $I \subset [1, n-k]$, $J \subset [1,n]$ subsets of the same size denoting row and column indices respectively. It is a polynomial ring in the coordinate functions $Y_{i,j}$, $1 \le i \le j \le i+k \le n$. The following elements will serve as frozen variables in $\mathbb{C}[\mathbf{Y}]$:

We denote by $\overline{\mathbf{P}}$ the tropical semifield in these frozen variables.

Just as in Example 2.3.5, there is a morphism of varieties $F: \mathbf{Y} \to \mathbf{X}$ sending $Y \in \mathbf{Y}$ to the simple tensor $Y[1] \wedge \cdots \wedge Y[n-k] \in \mathbf{X}$, where the $Y[i]$ are the rows of Y. Its pullback $F^*: \mathbb{C}[\mathbf{X}] \to \mathbb{C}[\mathbf{Y}]$ is defined by

$$F^*(\Delta_S) = Y_{[1,n-k],S}. \tag{A.1}$$

Letting Δ_S be any non-frozen Plücker coordinate, one sees that

$$F^*(\Delta_S) = c(S) \cdot Y_{I(S),J(S)} \tag{A.2}$$

where $c(S) \in \overline{\mathbf{P}}$ and $Y_{I(S),J(S)}$ is a non-frozen irreducible row-solid minor in $\mathbb{C}[\mathbf{Y}]$. The map $\Delta_S \mapsto Y_{I(S),J(S)}$ is a bijection between the non-frozen Plücker coordinates in $\mathbb{C}[\mathbf{X}]$ and the non-frozen irreducible row-solid minors in $\mathbb{C}[\mathbf{Y}]$.

Just as in Example 2.3.9, there is a morphism of varieties $G\colon \mathbf{X} \to \mathbf{Y}$ sending $X \in \mathbf{X}$ to the band matrix whose whose (i,j) entry is a certain Plücker coordinate evaluated on X:

$$G(X)_{i,j} = \Delta_{[i+k+1,n+i-1]\cup j}(X). \tag{A.3}$$

Since the Plücker coordinate on the right hand side of (A.3) is only nonzero when $i \leq j \leq i+k$, $G(X)$ is indeed a point in $\mathbf{Y}$. The map on coordinate rings is

$$G^*(Y_{i,j}) = \Delta_{[i+k+1,n+i-1]\cup j}. \tag{A.4}$$

Theorem A.1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be the varieties above. Let* $\overline{\mathcal{F}}_{>0}$ *be the semfield of subtraction-free expressions in the* $Y_{i,j}$*. Let* $F^*\colon \mathcal{F}_{>0} \to \overline{\mathcal{F}}_{>0}$ *be the map on semfields given by* (A.1)*, and* $G^*\colon \overline{\mathcal{F}}_{>0} \to \mathcal{F}_{>0}$ *be the map given by* (A.4)*. Then* $\mathbb{C}[\mathbf{Y}]$ *is a cluster algebra of geometric type, and the maps* F^* *and* G^* *provide a quasi-isomorphism. All of the irreducible row-solid minors in* $\mathbb{C}[\mathbf{Y}]$ *are cluster or frozen variables.*

Proof. The proof relies on the following well-known properties of the cluster structure on $\mathbb{C}[\mathbf{X}]$:

1. There exist clusters in $\mathbb{C}[\mathbf{X}]$ consisting entirely of Plücker coordinates (called *Plücker clusters*). Every non-frozen Plücker coordinate shows up in at least one of these clusters.

2. The set of Plücker clusters are connected to one another by mutations whose exchange relations are *short Plücker relations*, of the form $\Delta_{S\cup ik}\Delta_{Sj\ell} = \Delta_{Sij}\Delta_{Sk\ell} + \Delta_{Sjk}\Delta_{Si\ell}$, for $i < j < k < l$ and Sij denotes the union $S \cup \{i, j\}$).

3. For certain Plücker clusters, every neighboring cluster is again a Plücker cluster.

The first two of these facts are consequences of the technology of plabic graphs and square moves, the third fact follows by considering the Plücker seed whose quiver is the grid quiver [52, Theorem 1].

We can now state Theorem 2.3.5 more carefully. Let Σ be any Plücker cluster in $\mathbb{C}[\mathbf{X}]$. For each Plücker coordinate $x_i \in \Sigma$, let $\overline{x}_i$ be the irreducible row-solid minor in $\mathbb{C}[\mathbf{Y}]$ related to x_i as in (A.2). We will see below that $\{\overline{x}_i : x_i \in \Sigma\}$ are algebraically independent generators for $\overline{\mathcal{F}}_{>0}$ over $\overline{\mathbf{P}}$. Assuming this has been proved, applying the construction in Proposition 2.4.4, we obtain a semifield map $c_\Sigma : \mathcal{F}_{>0} \to \overline{\mathbf{P}}$ and a normalized seed $F^*(\Sigma)$ whose cluster variables are $\overline{x}_i = \frac{F^*(x_i)}{c_\Sigma(x_i)}$ as in (2.33). The semifield map c_Σ satisfies $c_\Sigma(\Delta_S) = c(S)$ for all $S \in \Sigma$, where $c(S)$ is defined by (A.2). The *key claim* is that in fact $c_\Sigma(\Delta_S) = c(S)$ holds for all Δ_S, and therefore the semifield map c_Σ does not depend on Σ.

From this key claim, it follows that the seeds $F^*(\Sigma)$ are all related to each other by mutation using Proposition 2.4.4. Furthermore, every non-frozen irreducible row-solid minor is a cluster variable in $\overline{\mathcal{E}}$ by (2.33). Since this includes all of the $Y_{i,j}$, this shows that the $F^*(\Sigma)$ are indeed seeds – each seed has the expected size necessary to form a transcendence basis for $\mathbb{C}(\mathbf{Y})$, the seeds are all related to each other by mutation, and their union clearly contains a generating set for the field of fractions. The cluster algebra for the resulting seed pattern clearly contains $\mathbb{C}[\mathbf{Y}]$. The opposite containment follows from the Algebraic Hartogs' argument on a starfish cf. Section B, using Fact (3) above.

Thus it remains to check the key claim that $c_\Sigma(\Delta_S) = c(S)$ for all non-frozen S. By Fact (2), it suffices to check that c preserves the short Plücker relations, i.e.

$$c(Sik)c(Sj\ell) = c(Sij)c(Sk\ell) \oplus c(Sjk)c(Si\ell). \tag{A.5}$$

Verifying (A.5) is a direct piecewise check: the exponent of $Y_{a,a}$ in the left hand side of (A.5) is $0, 1, 2$ according to whether neither, one of, or both of Sik and $Sj\ell$ contain the interval $[1, a]$. Performing the similar computation for Sij and $Sk\ell$, as well as Sjk and $Si\ell$, and taking the minimum of their respective answers, gives the exponent of $Y_{a,a}$ in the right hand side, and we claim these left and right hand exponents are always equal. This can be done by a case analysis: let E be the largest number such that $[1, E] \subset Sijkl$. If $E < i$, then both sides return a 2 if $a \le E$, and 0 otherwise. If $i \le E < j$, then both sides return a 2 if $a < i$, return a 1 if $i \le a \le E$, and return a 0 otherwise. If $j \le E$ then both sides return a 2 if $a < i$, return a 1 if $i \le a < j$, and return a 0 otherwise. A similar calculation checks that the exponents of $Y_{a,a+k}$ match up in both sides of (A.5).

Finally we check that G^* is a quasi-inverse to F^*. By Lemma 2.3.8, we only need to see that G^* preserves coefficients and that $G^* \circ F^*$ is proportional to the identity. It suffices to check that $G^* \circ F^*(\Delta_S) \asymp \Delta_S$ for every Δ_S. This follows from the determinantal identity Lemma A.2 below, applied to $G^* \circ F^*(\Delta_S) = G^*(Y_{[1,n-k],S})$.

□

Lemma A.2. *Let $I = [a, a+s-1]$ be some consecutive subset of $[n-k]$, and J a subset of $[a, a+s-1+k]$ of size s. Then for $X \in \mathbb{C}[\mathbf{X}]$,*

$$Y_{I,J}(G(X)) = \left(\prod_{i=a}^{a+s-2} \Delta_{[i+k+1,n+i]}(X)\right) \cdot \Delta_{[a+k+s,n+a-1]\cup J}(X) \tag{A.6}$$

Notice that the first product on the right hand side of (A.6) is a monomial in the frozen Plücker coordinates.

Proof. Proceed by induction on s. It's clear when $s = 1$. For $s > 1$, we will need a Plücker relation

(A.7)

$$\Delta_{[a+k+1,n+a]}(X)\Delta_{[a+k+s,n+a-1]\cup J}(X) = \sum_{\ell=1}^{s}(-1)^{\ell+1}\Delta_{[a+1+k,n+a-1]\cup j_\ell}(X)\Delta_{[a+k+s,n+a]\cup(J-j_\ell)}(X),$$

see e.g. [27, Section 9.1, Exercise 1]. Let $J = \{j_1,\dots,j_s\}$ with $j_1 < j_2 < \cdots < j_s$. Assuming (A.6) holds for smaller values of s, we expand along the first row to see

$$\begin{aligned}
Y_{I,J}(G(X)) &= \sum_{\ell=1}^{s}(-1)^{\ell+1}Y_{a,j_\ell}(G(X))Y_{(I-a),(J-j_\ell)}(G(X))\\
&= \sum_{\ell=1}^{s}(-1)^{\ell+1}\Delta_{[a+k+1,n+a-1]}(X)\Big(\prod_{i=a+1}^{a+s-2}\Delta_{i+k+1,n+i}(X)\Big)\cdot\Delta_{[a+k+s,n+a]\cup(J-j_\ell)}\\
&= \Big(\prod_{i=a+1}^{a+s-2}\Delta_{i+k+1,n+i}(X)\Big)\sum_{\ell=1}^{s}(-1)^{\ell+1}\Delta_{[a+1+k,n+a-1]\cup j_\ell}(X)\Delta_{[a+k+s,n+a]\cup(J-j_\ell)}(X),
\end{aligned}$$

and the result follows using (A.7). □

Remark A.3. In the case $k = 2$, our construction is the "motivating example" considered by Yang and Zelevinsky [55]. They establish that the homogeneous coordinate ring of a certain SL_{n+1}-double Bruhat cell is a Dynkin type A_n cluster algebra with principal coefficients. The elements of this double Bruhat cell are $(n+1)\times(n+1)$ band matrices of width 3. Their example follows from ours by setting certain frozen variables equal to 1, and setting $Y_{1,1}$ and $Y_{n-k,n}$ equal to 0.

We also remark that it is already known that the Grassmannian cluster algebras are quasi-isomorphic to a polynomial ring, by a fairly uninteresting quasi-isomorphism. Indeed, we can realize the affine space of $(n-k)\times k$ matrices as the closed subvariety of $\hat{\mathrm{G}}\mathrm{r}(n-k,n)$ defined by specializing the frozen variable $\Delta_{[1,n-k]}$ to 1, and this specialization is a quasi-isomorphism. The resulting cluster structure on the polynomial ring is unrelated to the one we have given in this section.

APPENDIX B

The starfish lemma on a nerve

We give the appropriate generalization of the Starfish Lemma [17, Proposition 3.6] from a star neighborhood to a nerve. Our proof follows the proof of the Starfish Lemma in [21], with appropriate modifications.

Let R be a domain. We say two elements $r, r' \in R$ are *coprime* if they are not contained in the same prime ideal of height 1. When R is a unique factorization domain, every pair of non-associate irreducible elements are coprime.

Proposition B.1. *Let $\mathcal{N}$ be a nerve in $\mathbb{T}_n$. Let $\mathcal{R}$ be a $\mathbb{C}$-algebra and a Noetherian normal domain. Let $\mathcal{E}$ be a seed pattern of geometric type, satisfying the following:*

- *all frozen variables are in $\mathcal{R}$*
- *for each vertex $t \in \mathcal{N}$, the cluster $\mathbf{x}(t) \subset \mathcal{R}$, and the cluster variables $x \in \mathbf{x}(t)$ are pairwise coprime elements of $\mathcal{R}$;*
- *for each edge $t \overset{k}{\to} t'$ in $\mathcal{N}$, the cluster variables $x_k(t)$ and $x_k(t')$ are pairwise coprime.*

Then the cluster algebra $\mathcal{A}$ defined by $\mathcal{E}$ satisfies $\mathcal{A} \subset \mathcal{R}$.

The proof relies on the following two lemmas, the first of which is a standard fact from commutative algebra. For a prime ideal P, let $R_P = R[(R \backslash P)^{-1}]$ denote the localization of R away from P.

Lemma B.2 ([45, Theorem 11.5])**.** *For a normal Noetherian domain R, the natural inclusion $R \subset \bigcap_{\mathrm{ht}\ P=1} R_P$ (intersection over height one primes) is an equality.*

Lemma B.3. *With hypotheses as in Proposition B.1, let P be a height one prime ideal in R. Then at least one of the products*

$$\prod_{x \in \mathbf{x}(t), t \in \mathcal{N}} x \tag{B.1}$$

is not in P.

Proof. By the coprimeness in each cluster $t \in \mathcal{N}$, at most one of the cluster variables x in a product (B.1) satisfies $x \in P$. We will show that for at least one t, *none* of the cluster variables is in P, establishing our claim since P is prime. Pick any vertex $t_0 \in \mathcal{N}$, and suppose the cluster variable $x_i \in P$. Given an edge $t_0 \xrightarrow{j} t_0' \subset \mathcal{N}$ where $j \neq i$, the cluster variable $x_j(t_0') \notin P$ by the coprimality assumption in the cluster at t_0'. Repeatedly applying this assumption while mutating along the nerve, using the connectedness hypothesis and the fact that every edge label shows up in the nerve, we finally arrive at a vertex $t \in \mathcal{N}$ such that the edge $t \xrightarrow{i} t' \subset \mathcal{N}$, and all of the extended cluster variables $x_j \in \tilde{\mathbf{x}}(t)$ with $j \neq i$ are not in P. By the coprimeness assumption along edge i, we see $x_i(t') \notin P$, and the cluster at t' is one where the product (B.1) is not in P. □

Proof of Proposition B.1. We need to prove each cluster variable z is in R. By Lemma B.2, it suffices to show $z \in R_P$ for any height one prime P. Indeed, by Lemma B.3 there is a cluster $t \in \mathcal{N}$ such that $\prod_{x \in \mathbf{x}(t)} x \notin P$. By the Laurent Phenomenon, z is a Laurent polynomial in the elements of $\mathbf{x}(t)$, with coefficients in $\mathbb{C}[x_{n+1}, \ldots, x_{n+m}]$. In particular, $z \in R_P$, as desired. □

APPENDIX C

Base affine space

As mentioned in Remark 4.2.8, the cluster structure in GL_{2k}/U has the same cluster type as that in $\hat{\mathrm{Gr}}(k,3k)$ and $\mathrm{FG}(k,6)$. The goal of this appendix is to suggest a quasi-isomorphism between $\mathbb{C}[\mathrm{GL}_{2k}/U]$ and $\mathbb{C}(\mathrm{FG}(k,6))$. The cluster structure in GL_{2k}/U has certain seeds indexed by *pseudoline arrangements*; each pseudoline arrangement determines an extended cluster consisting entirely of flag minors. The cluster for a given pseudoline arrangement is determined by the various *chambers* in the pseudoline arrangement. See [16] for basic examples.

Example C.1. We will begin with $\mathbb{C}[\mathrm{GL}_4/U]$ and $\mathbb{C}[\hat{\mathrm{Gr}}(2,6)]$. For $S \subset [1,4]$, we denote by $f_S \in \mathbb{C}[\mathrm{SL}_4/U]$ the corresponding flag minor. Thus if $p \in \mathrm{SL}_4$ is a matrix, then $f_S(p)$ is the determinant of the submatrix of p whose rows are given by S and whose columns are left-justified.

From any $p \in \mathrm{SL}_4$, we can construct the following 2×6 matrix:

$$\begin{pmatrix} 1 & 0 & -f_{234}(p) & -f_{23}(p) & -f_{24}(p) & -f_2(p) \\ 0 & 1 & f_{134}(p) & f_{13}(p) & f_{14}(p) & f_1(p) \end{pmatrix}. \tag{C.1}$$

The six column vectors of this matrix give us 6 vectors in $\mathbb{C}^2$. Pulling back the recipe in (C.1) we obtain a map $\mathbb{C}[\hat{\mathrm{Gr}}(2,6)] \to \mathbb{C}[\mathrm{GL}_4(U)]$. By a finite calculation, one checks this map is a quasi-homomorphism.

Now we describe the quasi-inverse to the map in (C.1). Let v be a 2×6 matrix. It determines a point in GL_4/U represented by the lower triangular matrix

$$\begin{pmatrix} \Delta_{16}(v) & 0 & 0 & 0 \\ \Delta_{26}(v) & \Delta_{12}(v) & 0 & 0 \\ \Delta_{46}(v) & \Delta_{14}(v) & \Delta_{34}(v) & 0 \\ \Delta_{56}(v) & \Delta_{15}(v) & \Delta_{35}(v) & \Delta_{45} \end{pmatrix}. \tag{C.2}$$

Pulling back the map in (C.2) determines a map on coordinate rings $\mathbb{C}[\mathrm{GL}_4(U)] \to \mathbb{C}[\hat{\mathrm{Gr}}(2,6)]$. It is quasi-inverse to the map in (C.1).

In fact, we are able to guess the correct generalization of (C.2), i.e. one giving rise to a quasi-homomorphism $\mathbb{C}[\mathrm{GL}_{2k}(U)] \to \mathbb{C}[\mathrm{FG}(k,6)]$. We will state it as a conjecture because we have not verified all of the necessary determinantal identities.

Conjecture C.2. *Let $p \in \mathrm{FG}_6(\mathrm{SL}_k)$. Recall the map in* (4.9) *which produces k vectors from 2 flags. Applying this map to the pair of affine flags $(F_1(p)), F_2(p))$, and then to the pair $(F_4(p), F_5(p))$, we obtain a list*

$$R_1(p), \dots, R_{2k}(p) \tag{C.3}$$

of $2k$ vectors in V (the letter R will stand for "row").

There is a dual procedure to (4.9)*, producing elements in $\bigwedge^{k-1}(V)$ rather than V. Namely*

$$(F, G) \mapsto F_{(k-1)}, F_{(k-2)}G_{(1)}, \dots, F_{(1)}G_{(k-2)}, G_{(k-1)}. \tag{C.4}$$

Applying (C.4) *in turn to the pairs $(F_6(p), F_1(p))$ and $(F_3(p), F_4(p))$ we obtain a list*

$$C_1, \dots, C_{2k} \in \bigwedge^{k-1}(V) \tag{C.5}$$

of $(k-1)$-forms (the letter C stands for "column").

Finally, for any $p \in \mathrm{FG}_6(\mathrm{SL}_k)$ let $M(p)$ be the lower triangular matrix whose entries consist of joins $R_i \wedge C_j$ for $i \le j$. Then the pullback of the map $p \mapsto M(p)$ is a quasi-homomorphism $\mathbb{C}[\mathrm{GL}_{2k}(U)] \to \mathbb{C}[\mathrm{FG}_6(\mathrm{SL}_k)]$.

Example C.3. To verify Conjecture C.2 when $k = 3$, we obtain six vectors

$$F_{1,(1)}, F_{1,(2)} \cap F_{2,(2)}, F_{2,(1)}, F_{4,(1)}, F_{4,(2)} \cap F_{5,(2)}, F_{5,(1)} \tag{C.6}$$

as well as six 2-forms

$$F_{6,(2)}, F_{6,(1)}F_{1,(1)}, F_{1,(2)}, F_{3,(2)}, F_{3,(1)}F_{4,(1)}, F_{4,(1)}. \tag{C.7}$$

The lower triangular matrix obtained by taking their corresponding joins has as its first five columns:

(C.8)
$$\begin{pmatrix} F_{11}F_{62} & 0 & 0 & 0 & 0 \\ F_{12} \cap F_{22)} \cap F_{62)} & (F_{11}F_{22})(F_{12}F_{61}) & 0 & 0 & 0 \\ F_{21}F_{62} & F_{11}F_{21}F_{61} & F_{12}F_{21} & 0 & 0 \\ F_{41}F_{62} & F_{11}F_{41}F_{61} & F_{12}F_{41} & F_{32}F_{41} & 0 \\ F_{42} \cap F_{52} \cap F_{62} & F_{41} \cap F_{52} \cap (F_{11}F_{61}) & F_{12}F_{42} \cap F_{52} & F_{12}F_{42} \cap F_{52} & (F_{41}F_{52})(F_{31}F_{42}) \\ F_{51}F_{62} & F_{11}F_{51}F_{61} & F_{12}F_{51} & F_{32}F_{51} & F_{31}F_{41}F_{51} \end{pmatrix}.$$

We have checked by hand (using webs and skein relations) that the corresponding algebra sends cluster variables to rescaled cluster variables on an appropriate nerve. The minors in this nerve are those whose row and column sets are consecutive, and whose column set includes either column 1 or column 2.

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