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Appendix

In this appendix, we present all the technical proofs. Throughout the proofs we define $R_n^*(x_n, \rho_n)$ and $A_n^*(x_n, \rho_n)$ to be the optimal solutions for the variables R_n and A_n, given that the number of customers at the beginning of period n is x_n and the observed fraction of unhappy customers is ρ_n.

Proof of Lemma 1. The problem we are studying is

$$\max_{x\geq 0, y\geq 0} f(x) + g(y) + E[h(x+y+\epsilon K)], \tag{5}$$

and we have continuity and strict concavity assumptions on the three functions and that both constant K and random variable ϵ are non-negative.

Rewrite (5) as

$$\max_{z\geq 0}\Big(D(z) + E[h(z+\epsilon K)]\Big) \tag{6}$$

with

$$D(z) = \max_{0\leq x\leq z}\Big(f(x) + g(z-y)\Big). \tag{7}$$

The optimization problem in (6) is submodular in (z, K) because ϵ is non-negative, thus the optimal solution, denoted by $z^*(K)$, is decreasing in K. From the problem given in (7), the maximand is supermodular in (x, z) and the constraint $0 \leq x \leq z$ is a lattice, hence the optimal x^* is increasing in z. Considered together, this implies that a smaller value of K results in a larger value of z and a larger value of x. Therefore, $x^*(K)$ is decreasing in K. We rewrite (7) as

$$D(z) = \max_{0\leq y\leq z}\Big(f(z-y) + g(y)\Big). \tag{8}$$

Using this equation (8) and the supermodularity in (z, y) we similarly obtain that the optimal y, denoted by $y^*(K)$, is decreasing in K.

To show that the slopes of the optimal $x^*(K)$ and $y^*(K)$ are between -1 and 0, we argue that the optimal $z^*(K)$ has slope between 0 and -1. This is sufficient to conclude the same about $x^*(K)$ and $y^*(K)$ because, by the fact that each is decreasing in K, and $x^*(K) + y^*(K) = z^*(K)$, it would be impossible for one of $x^*(K)$ and $y^*(K)$ to decrease by more than that of $z^*(K)$.

Suppose that K increases by $c > 0$, but z^* decreases by $d > c$. This condition is formally written as $z^*(K+c) = z^*(K) - d < z^*(K) - c$. We argue such a situation cannot occur because if true, we are able to find a very small $\delta > 0$, such that $z^*(K+c) + \delta$ is a strictly better solution than $z^*(K+c)$. We argue this solution is better by the following inequalities. Note that it is easy to see from (7), that $D(\cdot)$ in equation (6) is convex.

$$\begin{aligned}
& D(Z^*(k)-d+\delta)-D(Z^*(K)-d) \\
< \; & D(z^*(K))-D_n(z^*(K)-\delta) \\
\leq \; & E[h(z^*(K)+\epsilon K-\delta)]-E[h(z^*(K)+\epsilon K)] \\
\leq \; & E[h(z^*(K)-d+\epsilon(K+c))]-E[h(z^*(K)-d+\delta+\epsilon(K+c))],
\end{aligned}$$

where the first inequality comes from the convexity of $D(\cdot)$, the second from the optimality of the solution $z^*(K)$, and the third from the convexity of $h(\cdot)$ along with the fact that we can pick δ small enough so that $c\epsilon - d + \delta \leq 0$. Considering the first and last expressions, we see that

$$E[h(z^*(K)-d+\delta+\epsilon(K+c))]+D(Z^*(k)-d+\delta) > E[h(z^*(K)-d+\epsilon(K+c))]+D(Z^*(K)-d),$$

contradicting the optimality of the original solution.

Thus, the analysis above shows that the optimal $x^*(K)$ and $y^*(K)$ are decreasing in K, but with slopes between -1 and 0.

Proof of Theorem 1. The optimality equation is

$$\begin{aligned}
& V_n(x_n) = E_{\rho_n}\Big[M_n(x_n) \\
& + \max_{0\leq R_n\leq x_n\rho_n, 0\leq A_n, C_n^A(A_n)+C_n^R(R_n)\leq S_n} \Big(-C_n^A(A_n)-C_n^R(R_n)+E[\alpha V_{n+1}(\gamma_n x_n(1-\rho_n)+R_n+A_n)]\Big)\Big].
\end{aligned} \tag{9}$$

First note that, for any given selections of A_n, R_n, and an outcome ρ_n, the objective function of the maximization problem on the right hand side of (10) is increasing in x_n, and the feasible region is strictly larger for larger x_n, thus after maximization it is also increasing in x_n. Then, by the assumption that $M_n(x_n)$ is increasing, we conclude that $V_n(x_n)$ is increasing in x_n.

The concavity of $V_n(x_n)$ follows by concavity preservation. By Assumptions 1 and 2, on $C_n^A(\cdot)$ and $C_n^R(\cdot)$, and the induction hypothesis on $V_{n+1}(.)$, the objective function of the maximization problem on the right hand side of (10) is jointly concave in (A_n, R_n, x_n). Because the feasible region constitutes a convex set, it follows from Heyman and Sobel (2004) that $V_n(x_n)$ is a concave function.

To characterize the optimal policy, we consider the unconstrained optimization problem by relaxing the constraint in (10) as follows:

$$U_n(x_n,\rho_n) = M_n(x_n) + \max_{0\leq A_n, 0\leq R_n}\Big(-C_n^A(A_n)-C_n^R(R_n)+\alpha V_{n+1}(\gamma_n x_n(1-\rho_n)+R_n+A_n)\Big). \tag{10}$$

We will call this the relaxed problem, and use it for subsequent analysis. Note the difference between this problem and the original problem: problem (10) does not have either constraint $R_n \leq x_n\rho_n$ or $C_n^A(A_n) + C_n^R(R_n) \leq T_n$, and it assumes the fraction of unhappy customer ρ_n is known.

Now we can use Lemma 1 to argue the following property on the relaxed problem: The optimal solution to the problem $U_n(x_n, \rho_n)$, which we denote by $(A_n^{U*}(x_n(1-\rho_n)), R_n^{U*}(x_n(1-\rho_n))$, is decreasing in the expression $x_n(1-\rho_n)$, with slope between 0 and -1. This is an immediate application of Lemma 1, simply by switching from maximization to minimization and corresponding A_n and R_n to x and y and the functions $C_n^A(\cdot)$, $C_n^R(\cdot)$, and $-V_n(\cdot)$ to $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$. The expression $x_n(1-\rho_n)$ plays the role of the constant K.

Based on the problem given in (10) with decreasing solution vector $(A_n^{U*}(x_n(1-\rho_n)), R_n^{U*}(x_n(1-\rho_n))$, we define the following value,

$$K_n = \{w : C_n^A(A_n^{U*}(w)) + C_n^R(R_n^{U*}(w)) = S_n\}, \tag{11}$$

which will be useful in establishing the main result. If such a value K_n does not exist, we we set $K_n = 0$.

Next we consider a second intermediary problem in which we consider only the cash constraint, but not the upper bound on retention. This problem is

$$\begin{aligned} & Y_n(x_n(1-\rho_n)) \\ = \; & M_n(x_n) + \max_{0 \le A_n, 0 \le R_n, C_n^A(A_n) + C_n^R(R_n) \le S_n} \Big(-C_n^A(A_n) - C_n^R(R_n) + \alpha V_{n+1}(\gamma_n x_n(1-\rho_n) + R_n + A_n) \Big). \end{aligned} \tag{12}$$

In what follows we show that the solution to (12) is

$$(A_n^{Y*}(x_n(1-\rho_n)), R_n^{Y*}(x_n(1-\rho_n))) = (A_n^{U*}(x_n(1-\rho_n)), R_n^{U*}(x_n(1-\rho_n))$$

if $x_n(1-\rho_n) \ge K_n$ and otherwise, it is

$$(A_n^{Y*}(x_n(1-\rho_n)), R_n^{Y*}(x_n(1-\rho_n))) = (A_n^{U*}(K_n), R_n^{U*}(K_n)).$$

First consider the case when $x_n(1-\rho_n) \ge K_n$. In this situation, because $A_n^{U*}(\cdot)$ and $R_n^{U*}(\cdot)$ are decreasing, we are able to show the following:

$$C_n^A(A_n^{U*}(x_n(1-\rho_n))) + C_n^R(R_n^{U*}(x_n(1-\rho_n)) \le C_n^A(A_n^{U*}(K_n)) + C_n^R(R_n^{U*}(K_n)) = T_n.$$

Therefore in this case the solution from (10) is feasible for (12), so it is also optimal for (12).

Now suppose instead that $x_n(1-\rho_n) < K_n$, we want to show, by contradiction, that the optimal solution pair is $(A_n^{U*}(K_n), R_n^{U*}(K_n))$. Suppose for some $x_n(1-\rho_n) < K_n$, we have an optimal strategy of $A_n^{Y*}(x_n(1-\rho_n))$ and $R_n^{Y*}(x_n(1-\rho_n))$ which are not equal to $A_n^{U*}(K_n)$ and $R_n^{U*}(K_n)$ respectively. In the following we show that this would lead to contradiction.

Consider several cases. First, suppose

$$A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)) > A_n^{U*}(K_n) + R_n^{U*}(K_n). \tag{13}$$

This would contradict the optimality of the solution pair $A_n^{U*}(K_n)$ and $R_n^{U*}(K_n)$, by the following arguments:

$$\begin{aligned} & -C_n^A(A_n^{U*}(K_n)) - C_n^R(R_n^{U*}(K_n)) + E[V_{n+1}(\gamma_n K_n + A_n^{U*}(K_n) + R_n^{U*}(K_n))] \\ = & -S_n + E[V_{n+1}(\gamma_n K_n + A_n^{U*}(K_n) + R_n^{U*}(K_n))] \\ < & -S_n + E[V_{n+1}(\gamma_n K_n + A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)))] \end{aligned}$$

$$\le -C_n^A(A_n^{Y*}(x_n(1-\rho_n))) - C_n^R(R_n^{Y*}(x_n(1-\rho_n)))$$
$$+ E[V_{n+1}(\gamma_n K_n + A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)))],$$

where the equality follows form the definition of K_n, the first inequality follows from the strict monotonicity of the value function $V_{n+1}(\cdot)$, and the second follows from the fact that the cash constraint must be satisfied by $A_n^{Y*}(x_n(1-\rho_n))$ and $R_n^{Y*}(x_n(1-\rho_n))$. Looking at the first and last expressions, the firm is strictly better off switching strategies from the pair $A_n^{U*}(K_n)$ and $R_n^{U*}(K_n)$ to the pair $A_n^{Y*}(x_n(1-\rho_n))$ and $R_n^{Y*}(x_n(1-\rho_n))$, which contradictions the optimality of the first solution pair.

Next, suppose that $A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)) < A_n^{U*}(K_n) + R_n^{U*}(K_n)$. We will prove that this contradicts the optimality of the solution pair $A_n^{Y*}(x_n(1-\rho_n))$ and $R_n^{Y*}(x_n(1-\rho_n))$. Note that

$$C_n^A(A_n^{U*}(K_n)) + C_n^R(R_n^{U*}(K_n)) - C_n^A(A_n^{Y*}(x_n(1-\rho_n))) - C_n^R(R_n^{Y*}(x_n(1-\rho_n)))$$
$$\le E[V_{n+1}(\gamma_n K_n + A_n^{U*}(K_n) + R_n^{U*}(K_n))] - E[V_{n+1}(\gamma_n K_n + A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)))]$$
$$< E[V_{n+1}(\gamma_n x_n(1-\rho_n) + A_n^{U*}(K_n) + R_n^{U*}(K_n))]$$
$$-E[V_{n+1}(\gamma_n x_n(1-\rho_n) + A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)))],$$

where the first inequality comes from the optimality of the solution with K_n, and the second inequality follows from the concavity of the value function because $x_n(1-\rho_n) < K_n$. Considering the first and last expressions together, we conclude that $A_n^{U*}(K_n)$ and $R_n^{U*}(K_n)$ is a strictly better solution, contradicting the optimality of the solution pair of $A_n^{Y*}(x_n(1-\rho_n))$ and $R_n^{Y*}(x_n(1-\rho_n))$.

The final case is

$$A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)) = A_n^{U*}(K_n) + R_n^{U*}(K_n), \tag{14}$$

but $A_n^{Y*}(x_n(1-\rho_n)) \ne A_n^{U*}(K_n)$ and $R_n^{Y*}(x_n(1-\rho_n)) \ne R_n^{U*}(K_n)$. Let $A_n^{Y*}(x_n(1-\rho_n)) - A_n^{U*}(K_n) = \delta$, so that it is also true that $R_n^{Y*}(x_n(1-\rho_n)) - R_n^{U*}(K_n) = -\delta$. We first prove that

$$-C_n^A(A_n^{U*}(K_n)+\delta) - C_n^R(R_n^{U*}(K_n)-\delta)) < -C_n^A(A_n^{U*}(K_n)) - C_n^R(R_n^{U*}(K_n)))]. \tag{15}$$

Suppose instead that

$$-C_n^A(A_n^{U*}(K_n)+\delta) - C_n^R(R_n^{U*}(K_n)-\delta)) > -C_n^A(A_n^{U*}(K_n)) - C_n^R(R_n^{U*}(K_n)).$$

Then this would contradict the optimality of the solution pair $A_n^{U*}(K_n)$ and $R_n^{U*}(K_n)$, because the alternative solution of $A_n^{U*}(K_n)+\delta$ and $R_n^{U*}(K_n)-\delta$ would have strictly lower cost with identical impact on the expression inside of $V_{n+1}(\cdot)$, due to the given condition that $A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)) = A_n^{U*}(K_n) + R_n^{U*}(K_n)$. Hence, let us next suppose

$$-C_n^A(A_n^{U*}(K_n)+\delta) - C_n^R(R_n^{U*}(K_n)-\delta)) = -C_n^A(A_n^{U*}(K_n)) - C_n^R(R_n^{U*}(K_n)). \tag{16}$$

We propose an alternative solution, with strictly lower cost, again with identical impact on the expression inside of $V_{n+1}(\cdot)$ (thus, again contradicting the optimality of the given solutions). This solution is $A_n^{U*}(K_n)+\frac{\delta}{2}$ and $R_n^{U*}(K_n)-\frac{\delta}{2}$. Then,

$$-C_n^A\left(A_n^{U*}(K_n)+\frac{\delta}{2}\right) - C_n^R\left(R_n^{U*}(K_n)-\frac{\delta}{2}\right)$$

$$
\begin{aligned}
&> -\frac{1}{2}C_n^A(A_n^{U*}(K_n)) - \frac{1}{2}C_n^R(R_n^{U*}(K_n)) - \frac{1}{2}C_n^A(A_n^{U*}(K_n)+\delta) - \frac{1}{2}C_n^R(R_n^{U*}(K_n)-\delta)) \\
&= C_n^A(A_n^{U*}(K_n)) - C_n^R(R_n^{U*}(K_n)),
\end{aligned}
$$

where the inequality follows from the strictly convexity of the cost functions in Assumption 1, and the equality follows from (16). So in this case we again contradict the optimality of $A_n^{U*}(K_n)$ and $R_n^{U*}(K_n)$, because the proposed solution of $A_n^{U*}(K_n)+\frac{\delta}{2}$ and $R_n^{U*}(K_n)-\frac{\delta}{2}$ has strictly lower cost and with identical impact on the expression inside of $V_{n+1}(\cdot)$.

Consequently, we have

$$
\begin{aligned}
&-C_n^A(A_n^{Y*}(x_n(1-\rho_n))) - C_n^R(R_n^{Y*}(x_n(1-\rho_n))) \\
&\qquad\qquad + E[V_{n+1}(x_n(1-\rho_n) + A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)))] \\
&= -C_n^A(A_n^{U*}(K_n)+\delta) - C_n^R(R_n^{U*}(K_n)-\delta)) + E[V_{n+1}(x_n(1-\rho_n) + A_n^{Y*}(K_n) + R_n^{U*}(K_n))] \\
&< -C_n^A(A_n^{U*}(K_n)) - C_n^R(R_n^{U*}(K_n))) + E[V_{n+1}(x_n(1-\rho_n) + A_n^{Y*}(K_n) + R_n^{U*}(K_n))] \\
&= -C_n^A(A_n^{U*}(K_n)) - C_n^R(R_n^{U*}(K_n))) + E[V_{n+1}(x_n(1-\rho_n) + A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n))],
\end{aligned}
$$

where the inequality follows from (15), and the second equality follows from condition (14). This contradicts the optimality of the solution pair $A_n^{Y*}(x_n(1-\rho_n))$ and $R_n^{Y*}(x_n(1-\rho_n))$.

Summarizing the analysis above, we have shown that the solution to (12) is given by $(A_n^{Y*}(x_n(1-\rho_n)), R_n^{Y*}(x_n(1-\rho_n)) = (A_n^{U*}(x_n(1-\rho_n)), R_n^{U*}(x_n(1-\rho_n))$ if $x_n(1-\rho_n) \geq K_n$, and it is $(A_n^{Y*}(x_n(1-\rho_n)), R_n^{Y*}(x_n(1-\rho_n))) = (A_n^{U*}(K_n), R_n^{U*}(K_n))$ otherwise.

We are now ready to prove Theorem 1. We first prove (i). Note that the relaxed problem (12) represents the optimization in problem (10) without constraint $R_n \leq \rho_n x_n$. Writing the optimization problem as sequential optimization of A_n and R_n respectively, it follows from the joint concavity in (A_n, R_n) that the objective function after optimizing A_n is a concave function of R_n. Since the resulting objective function after optimizing A_n in (12) is concave in R_n with maximizer $R_n^{Y*}(x_n(1-\rho_n))$, it is clear that the optimal solution of the original value function in (2) is $R_n^{Y*}(x_n(1-\rho_n))$ when $R_n^{Y*}(x_n(1-\rho_n)) \leq \rho_n x_n$ and , and otherwise it is $\rho_n x_n$.

Because $R_n^{Y*}(x_n(1-\rho_n)) \geq 0$ is decreasing in x_n, as x_n increases, there must exist a unique point where $R_n^{Y*}(x_n(1-\rho_n)) = x_n\rho_n$, which establishes the existence of $Q_n(\rho_n)$ from the theorem, defined by

$$
Q_n(\rho_n) = \sup\left\{x_n \geq 0;\ \rho_n x_n \leq R_n^{Y*}(x_n(1-\rho_n))\right\}, \tag{17}
$$

such that as $x_n \leq Q_n(\rho_n)$ it holds that $R_n^{Y*}(x_n(1-\rho_n)) > \rho_n x_n$; while if $x_n > Q_n(\rho_n)$ then $R_n^{Y*}(x_n(1-\rho_n)) \leq \rho_n x_n$.

Combining this insight with the characterization of the optimal policy $R_n^{Y*}(x_n(1-\rho_n))$ given above yields the optimal retention policy as stated in the theorem, which is to set R_n to $\rho_n x_n$ if $x_n \leq Q_n(\rho_n)$, set R_n to $R_n^{U*}(K_n)$ if $x_n \in (Q_n(\rho_n), \frac{K_n}{\rho_n})$ and set R_n to $R_n^{U*}(x_n(1-\rho_n))$ otherwise. Note that the second region might be empty.

To find the optimal acquisition strategy, we let $A_n^{W^*}(.)$ be defined as the maximizer of

$$W_n(x_n,\rho_n) = \max_{0\le A_n}\Big(-C_n^A(A_n) + E[V_{n+1}(\gamma_n x_n(1-\rho_n) + \rho_n x_n + A_n)]\Big). \tag{18}$$

By the same analysis as above, it can be seen that $A_n^{W*}(x_n,\rho_n)$ is also decreasing in x_n but with slope no less than -1. Note that on the range $x_n \le Q_n(\rho_n)$, the optimization problem for A_n in (2) can be written as

$$\max_{0\le A_n}\Big(-C_n^A(A_n) - C_n^R(\rho_n x_n) + \alpha V_{n+1}(\gamma_n x_n(1-\rho_n) + \rho_n x_n + A_n)\Big),$$

and its optimal solution is $A_n^{W*}(x_n,\rho_n)$ just defined in (18).

With these quantities defined, we can discuss the optimal acquisition strategy on the regions discussed above. When $x_n \le Q_n(\rho_n)$ and $R_n = \rho_n x_n$, then the optimal A_n is $A_n^{W*}(x_n,\rho_n)$, conditional on that it is within cash constraint. If not, by the convexity of the problem given in (18), the best solution is at the truncated solution. Thus, the optimal A_n on this region is $\min\{A_n^{W*}(x_n,\rho_n), T_n - C_n^R(\rho_n x_n)\}$. When $x_n \in (Q_n(\rho_n), K_n/\rho_n)$, the optimal solution is the one discussed in problem (12), which is $A_n^{U*}(K_n)$. In all other cases, the optimal solution is given by $A_n^{U*}((1-\rho_n)x_n)$.

The argument that $Q_n(\rho_n)$ is decreasing follows from the fact that $R_n^{Y*}(x_n(1-\rho_n))$ is decreasing in $x_n(1-\rho_n)$ with slope between -1 and 0, and the definition of $Q_n(\rho_n)$ in (17). To see that, suppose ρ_n were to increase by a positive number $s > 0$, then $\rho_n x_n$ would increase by sx_n, while $R_n^{Y*}(x_n(1-\rho_n))$ would increase by a value between 0 and sx_n. Therefore to reach equality once again, one would need to *decrease* x_n. This establishes that $Q_n(\rho_n)$ is decreasing in ρ_n.

We next prove (ii). From part (i), we know that the optimal decision in acquisition is decreasing in x_n. Therefore, either eventually $A_n^*(x_n,\rho_n) = 0$, or this value is infinite, establishing the existence of $Q_n^A(\rho_n)$ (possibly infinity). Likewise, retention spending is first increasing, and then decreasing, so eventually $R_n^*(x_n,\rho_n)$ may reach 0, showing that $Q_n^R(\rho_n)$ exists (also possibly infinity). Both are increasing in ρ_n, because the curves $R_n^{U*}(x_n(1-\rho_n))$ and $A_n^{U*}(x_n(1-\rho_n))$ are increasing in ρ_n.

To establish part (iii), we need to argue that the following expression

$$x_{n+1} - x_n = x_n(1-\rho_n)\gamma_n + R_n^*(x_n,\rho_n) + A_n^*(x_n,\rho_n) - x_n \tag{19}$$

is decreasing in x_n for any given ρ_n and γ_n, where $R_n^*(x_n,\rho_n)$ and $A_n^*(x_n,\rho_n)$ are the optimal retention and optimal acquisition decision of the original problem, which are given according to cases above. Since $A_n^*(x_n,\rho_n)$ is decreasing while $R_n^*(x_n,\rho_n)$ first increases with slope ρ_n and then decreases, we conclude that the terms combined must be decreasing in x_n.

When $x_n = 0$, the firm can only gain customers, and then the change in number of customers is decreasing for $x_n > 0$. Therefore there must exist a non-zero point $x_n^*(\rho_n)$ such that

$$E[x_{n+1}] - x_n = \begin{cases} \le 0 & \text{if } x_n \ge x_n^*(\rho_n); \\ \ge 0 & \text{if } x_n \le x_n^*(\rho_n). \end{cases}$$

This completes the proof of the optimal strategy. Note that it is possible that $x_n^*(\rho_n) = \infty$.

Proof of Lemma 2.

We prove by contradiction. Suppose that $(C_n^A)'(0) < (C_n^R)'(0)$, but $Q_n^A(\rho_n) < Q_n^R(\rho_n)$ for some ρ_n. This implies that for such a ρ_n, and values of $x_n \in (Q_n^A, Q_n^R)$, the firm has a strategy where $A_n^* = 0$ with $R_n^* > 0$. In this case, we show that there exists a small value $\delta > 0$, such that a better solution is $A_n = \delta$, with $R_n = R_n^* - \delta$. Because this strategy has the same impact in $V_{n+1}(\cdot)$, we need only argue that it has lower cost. The new solution would surely satisfy the cash constraint if it is indeed lower cost.

First we observe that because $C_n^R(\cdot)$ is strictly convex, and $(C_n^A)'(0) < (C_n^R)'(0)$, it holds that as $\delta > 0$ is small enough we have

$$C_n^R(R_n^*) - C_n^R(R_n^* - \delta) > C_n^R(\delta) - C_n^R(0) > C_n^A(\delta) - C_n^A(0). \tag{20}$$

These inequalities show the existence of solution $A_n = \delta$ and $R_n = R_n^* - \delta$, which as strictly lower cost, and same impact on future periods. This contradicts the original optimality of our solution. A symmetric argument establishes that $(C_n^A)'(0) > (C_n^R)'(0)$ implies that $Q_n^A(\rho_n) \leq Q_n^R(\rho_n)$.

We finally consider the case $(C_n^A)'(0) = (C_n^R)'(0)$, and prove that in this case it must hold that $Q_n^A(\rho_n) = Q_n^R(\rho_n)$ for all $\rho_n > 0$. Suppose $Q_n^A(\rho_n) \neq Q_n^R(\rho_n)$ for some ρ_n. Without loss of generality, suppose $0 \leq Q_n^A(\rho_n) < Q_n^R(\rho_n)$. This implies that there exists an $x_n \in (Q_n^A(\rho_n), Q_n^R(\rho_n))$, such that $R_n^*(x_n, \rho_n) > 0$ and $A_n^*(x_n, \rho_n) = 0$. We claim that there exists a small number $\delta > 0$, such that a solution with $R_n = R_n^*(x_n, \rho_n) - \delta$, and $A_n = \delta$ is *strictly* superior. This would contradict the optimality of the original solution.

Observe that by the strict convexity of $C_n^R(\cdot)$, we have that:

$$(C_n^R)'(R_n^*(x_n, \rho_n)) > (C_n^R)'(0) = (C_n^A)'(0).$$

Therefore, by continuity we can find a small $\delta > 0$ such that

$$(C_n^R)'(R_n^*(x_n, \rho_n) - \delta) > (C_n^A)'(\delta).$$

This implies, by convexity of $C_n^R(\cdot)$ and $C_n^A(\cdot)$, that

$$C_n^R(R_n^*(x_n, \rho_n)) - C_n^R(R_n^*(x_n, \rho_n) - \delta) > C_n^A(\delta) - C_n^A(0).$$

Since solutions $R_n = R_n^*(x_n, \rho_n) - \delta$ and $A_n = \delta$ have the same impact to future periods, this proves that the proposed solution has strictly lower cost, contradicting the optimality of the original solution. A symmetric argument holds to contradiction if it were true that $0 \leq Q_n^R(\rho_n) < Q_n^A(\rho_n)$.

Proof of Corollary 1.

The fact that $\lim_{x_{n+1} \to \infty} M'_{n+1}(x_{n+1}) \geq \kappa > 0$, allows us to prove that the value function is κ increasing in period $n+1$, meaning that $V_{n+1}(x_{n+1} + s) - V_{n+1}(x_{n+1}) \geq s\kappa$ for any $s > 0$. We can see this from the value function as follows.

$$E[V_{n+1}(x_{n+1} + s)] - E[V_{n+1}(x_{n+1})] = E[M_{n+1}(x_{n+1} + s)] - E[M_{n+1}(x_{n+1})]$$

$$
\begin{aligned}
&+E\Big[\max_{0\le R_{n+1}\le (x_{n+1}+s)\rho_{n+1},0\le A_{n+1}}(-C_{n+1}^A(A_{n+1})-C_{n+1}^R(R_{n+1})\\
&+\alpha E[V_{n+2}((x_{n+1}+s)(1-\rho_{n+1})\gamma_{n+1}+R_{n+1}+A_{n+1})])\Big]\\
&-E\Big[\max_{0\le R_{n+1}\le x_{n+1}\rho_{n+1},0\le A_{n+1}}(-C_{n+1}^A(A_{n+1})-C_{n+1}^R(R_{n+1})\\
&+\alpha E[V_{n+2}(x_{n+1}(1-\rho_{n+1})\gamma_{n+1}+R_{n+1}+A_{n+1})])\Big]\\
&\ge s\kappa,
\end{aligned}
$$

where the last inequality comes from the fact that $M_{n+1}(x_{n+1}+s)-M_{n+1}(x_{n+1})\ge s\kappa$, while the other terms are non-negative, because $V_{n+2}(\cdot)$ is increasing, and the case with $s+x_{n+1}$ has a larger feasible region.

By contradiction we now show that a point at which $R_n^*(x_n,\rho_n)=0$ can never exist unless $\rho_n x_n=0$ or $(C_n^R)'(0)\ge (C_n^A)'((C_n^A)^{-1}(S_n))$, because the firm is better off by spending a small incremental amount more in retention. Suppose, on the contrary, it holds that the optimal strategies $(R_n^*(x_n,\rho_n),A_n^*(x_n,\rho_n))$ has $R_n^*(x_n,\rho_n)=0$. We will show that in this case there exists a small $\delta>0$ such that the solution would be improved if $R_n^*(x_n,\rho_n)=\delta$, contradicting the optimality of the original solution. This additional small increase is feasible because of our two conditions, which say that the retention constraint is not tight with given decision (because $\rho_n x_n>0$, and neither is the cash constraint (because if $(C_n^R)'(0)<(C_n^A)'((C_n^A)^{-1}(S_n))$ with total acquisition and retention spending at S_n, the firm could save by shifting some money from acquisition to retention). Using the fact that $V_{n+1}(\cdot)$ is κ increasing, we have

$$
\begin{aligned}
C_n^R(\delta)-C_n^R(0) &< \delta\alpha\kappa &(21)\\
&\le \alpha E[V_{n+1}(x_n(1-\rho_n)\gamma_n+\delta+A_n^*(x_n,\rho_n))]-\alpha E[V_{n+1}(x_n(1-\rho_n)\gamma_n+A_n^*(x_n,\rho_n))]. &(22)
\end{aligned}
$$

These inequalities follow from the fact that $C_n^R(\cdot)$ is strictly convex, $(C_n^R)'(0)\le\alpha\kappa$, and $V_{n+1}(\cdot)$ is κ increasing, as we have discussed.

The inequalities (21) implies that a strategy of no retention and $A_n^*(x_n,\rho_n))$ is acquisition is strictly dominated by one with the same acquisition and a small amount $\delta>0$ in retention, contradicting the optimality of former solution. This implies $Q_n^R(\rho_n)=\infty$.

Proof of Corollary 2.

Due the symmetric relationship between A_n and R_n, similar argument as those of Corollary 1 can be used to prove this result. We omit the details.

Proof of Theorem 2.

The optimality equation for this more general case is

$$
\begin{aligned}
&V_n(x_n)\\
=\ &E_{\rho_n}\Big[M_n(x_n)\\
&+\max_{0\le A_n,0\le R_n\le\rho_n x_n,C_n^A(A_n)+C_n^R(R_n)\le S_n}\Big(-C_n^A(A_n)-C_n^R(R_n)+\alpha E\big[V_{n+1}(\gamma_n x_n(1-\rho_n)+\epsilon_n^1 R_n+\epsilon_n^2 A_n)\big]\Big)\Big].
\end{aligned}
$$

The objective function of the maximization problem above is easily seen to be jointly concave in (A_n,R_n,x_n), and the constraint is a convex set of (A_n,R_n,x_n), hence it follows from the preservation property that $V_n(x_n)$

is concave in x_n. By induction, it is also easy to show that $V_n(x_n)$ is increasing in x_n, since both the objective function and the feasible region in the optimization are increasing in x_n. Consider the relaxed problem that, for any realization of ρ_n,

$$\begin{aligned} & Y_n(x_n, \rho_n) \\ = & M_n(x_n) + \max_{0 \le A_n, 0 \le R_n, C_n^A(A_n) + C_n^R(R_n) \le S_n} \Big(-C_n^A(A_n) - C_n^R(R_n) + \alpha E\big[V_{n+1}(\gamma_n x_n(1-\rho_n) + \epsilon_n^1 R_n + \epsilon_n^2 A_n)\big]\Big) \\ = & M_n(x_n) + \max_{0 \le R_n, C_n^R(R_n) \le S_n} \Big\{ -C_n^R(R_n) + g\big((1-\rho_n)x_n, R_n\big)\Big\}, \end{aligned}$$

where

$$g((1-\rho_n)x_n, R_n) = \max_{0 \le A_n, C_n^A(A_n) + C_n^R(R_n) \le S_n} \Big(-C_n^A(A_n) + \alpha E\big[V_{n+1}(\gamma_n x_n(1-\rho_n) + \epsilon_n^1 R_n + \epsilon_n^2 A_n)\big]\Big)$$

is jointly concave in $((1-\rho_n)x_n, R_n)$. Therefore, if the optimal $R_n^{Y*}(x_n, \rho_n) < \rho_n x_n$, then the solution the the relaxed problem is feasible, thus it optimal. Otherwise by joint concavity, the optimal solution is $(A_n, R_n) = (A_n^{W*}(x_n, \rho_n), \rho_n x_n)$, where A_n^{W*} is the optimal solution of

$$W_n(x_n, \rho_n) = \max_{0 \le A_n, C_n^A(A_n) \le S_n - C_n^R(\rho_n x_n)} \Big(-C_n^A(A_n) + \alpha E\big[V_{n+1}(\gamma_n x_n(1-\rho_n) + \epsilon_n^1 \rho_n x_n + \epsilon_n^2 A_n)\big]\Big). \tag{23}$$

This finishes the proof for Theorem 2.