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Dynamic Customer Acquisition and Retention Management

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In consulting, finance, and other service industries, customers represent a revenue stream, and must be acquired and retained over time. In this paper, we study the resource allocation problem of a profit maximizing service firm that dynamically allocates its resources towards acquiring new clients and retaining unsatisfied existing ones. The interaction between acquisition and retention in our model is reflected in the cash constraint on total expected spending on acquisition and retention in each period. We formulate this problem as a dynamic program in which the firm makes decisions in both acquisition and retention after observing the current size of its customer base and receiving information about customers in danger of attrition, and we characterize the structure of the optimal acquisition and retention strategy.

We show that when the firm's customer base size is relatively low, the firm should spend heavily on acquisition and try to retain every unhappy customer. However, as its customer base grows, the firm should gradually shift its emphasis from acquisition to retention, and it should also aim to strike a balance between acquisition and retention while spending its available resources. Finally, when the customer base is large enough, it may be optimal for the firm to begin spending less in both acquisition and retention. We also extend our analysis to situations where acquisition or retention success as a function of resources allocated is uncertain and show that the optimal acquisition and retention policy can be surprisingly complex. However, we develop an effective heuristic for that case. This paper aims to provide service managers some analytics principles and effective guidelines on resource allocation between these two significant activities based on their firm's customer base size.

Key words: Dynamic Programming, Service Operations, OM-Marketing Interface

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1. Introduction

Customer retention is a growing concern for firms in many industries. From consulting, to finance, to cable service, customer retention is the key to long term profitability for many companies. Also critical is a firm's ability to acquire new customers in order to build its customer base. These two considerations in parallel naturally lead to the question of how a service firm should manage the trade-off between customer acquisition and retention. Customer acquisition and retention are costly activities for a firm, and the firm may be prone to fail at this process due to a lack of guidance or actionable strategies. For example, during the dot-com boom of the late 1990s and early 2000s many companies spent millions on customer acquisition without proper processes in place for retention.

This paper is derived from industry experience of the first author, who faced this problem while working at a small third-party-financing company. The firm lent money to patients for medical procedures through a network of doctors. Thus, these doctors were considered as the firm's customers because their satisfaction and service usage drove profitability for the firm. A sales force located throughout the United States was tasked with acquiring new customers as well as visiting existing ones to keep them satisfied with the service being provided. The trade-off between acquisition and retention was widely discussed at the company, and its impact on profitability was significant. During this period, the firm heavily emphasized acquisition while experiencing rapid growth. Analysis supported this practice, concluding that time and money were better spent in acquisition. However, as the firm matured, two things happened. First, the efforts in acquisition became futile, because incremental prospects were harder to acquire and less profitable. Second, attrition became a problem because the firm had neglected some of the existing users. Naturally the focus started to shift towards retention, though subsequent analyses indicated that the shift occurred too late. A primary motivating factor for this research is to build a model that helps companies better allocate resources towards acquisition and retention over time.

We consider the acquisition and retention trade-off from the perspective of a service manager. The key research questions relate to the timing and quantity of spend in each of these two areas: How many customers should be targeted and how can the manager appropriately determine the effort that should be spent on acquisition of new accounts versus development of existing accounts? Does the strategy change as the customer base of the firm grows over time? Is there an efficient number of customers for the firm to maintain over time?

Acquisition and retention management is an issue of alignment between marketing and operations, and thus deserves consideration in the operations management and service operations literature. While the specific acquisition and retention tactics themselves may be marketing (or sales) activities, they need to be balanced against operations and service capabilities. The service capabilities affect the ability of the firm to allocate resources towards acquisition and retention. Therefore, acquisition and retention activities should be carefully coordinated between marketing and operations. Furthermore, we make the distinction between specific acquisition and retention strategies and the higher level resource allocation decision about how much capital or effort to spend in these areas. The latter is often an operational decision of a firm.

We study how a service operations manager should dynamically allocate his resources towards acquisition/retention when faced with limited resources (e.g., a limited budget to spend on these activities). We characterize how the acquisition/retention policy dynamically depends on the number of customers of the firm, the number of 'unhappy' customers in danger of canceling service, and the firm's total budget (cash constraint) for acquisition and retention. Our results indicate that early on when the firm has few customers, the firm should spend heavily on acquisition and try to retain every 'unhappy' customer. However, as the customer base of the firm grows, the firm may reach a point where it is not optimal to retain all unhappy customers due to resource constraint on acquisition and retention activities. In this situation the firm needs to carefully strike a balance between acquisition and retention while using up all of the available budget. Finally, when the customer base is large enough, it may be optimal for the firm to begin spending less in both acquisition and retention. This result, i.e., the firm may reach a point in the number of customers it wants to have and curtails its acquisition and retention activities beyond this critical size, may seem unintuitive at first. However, it is driven by the fact that the marginal acquisition and retention costs are increasing in the number of customers acquired and retained, while the marginal increase in revenues is decreasing in the number of customers the firm serves. These are reasonable assumptions since sales forces acquire and retain the easiest prospects in a market first and acquiring and retaining customers gets more costly as the number of customers of the firm increases. For example, a few years ago telecommunication companies such as Sprint decided to 'hang up' their high-maintenance customers, which corresponds to refusing to retain customers beyond a certain point in our model.[1] Another implication of our results is that if the firm can become more efficient in acquisition and retention (by reducing acquisition or retention costs, or finding ways to make its service more valuable to customers so that customers are willing to pay more for service), it will enable the firm to increase its efficient size, which then leads to an increased overall customer base.

As the economy has become more service oriented, the importance of maintaining customer relationships is more critical today than ever before. The goal of this work is to provide structural insights and analysis of the essential trade-offs that occur in managing service industries, through the use of a dynamic decision making model. We begin with a literature review in §2, present the model and results in §3, discuss a model extension with heuristic in §4, before we conclude in §5. Throughout the paper we use the terms increasing and decreasing to mean non-decreasing and non-increasing, respectively. Finally, all mathematical proofs are given in the Appendix.

2. Literature Review

The trade-off between acquisition and retention has been primarily studied in the marketing literature. The novel approach of our work is that we analyze this problem as a dynamic one, which captures the dynamic nature of resource allocation over time. The vast majority of other work is not dynamic. As a result, our approach has system dynamics in the form of state transitions. We also use the machinery of stochastic optimization, in contrast to most papers which use regression, empirical, or deterministic techniques.

In a well known article in Harvard Business Review, Blattberg and Deighton (1996) establish the 'customer equity test' for determining the allocation of resources between acquisition and retention of customers. Using a deterministic model, the main contribution

[1] http://www.foxnews.com/story/2007/07/09/sprint-hangs-up-on-high-maintenance-customers/. Accessed May 2, 2015.

of this work is a simple calculation used to compare acquisition and retention costs with potential benefits.

The marketing literature contains numerous sources analyzing the acquisition and retention trade-off. Reinartz et al. (2005) discuss the problem from a strict profitability perspective using industry data. They find that under-investment in either area can be detrimental to success while over-investment is less costly, and that firms often under-invest in retention. Thomas (2001) discusses a statistical methodology for linking acquisition and retention. Homburg et al. (2009) use a portfolio management approach to maintaining a customer base.

Fruchter and Zhang (2004) is most closely related to our work in that it takes a dynamic approach to analyze the trade-off between acquisition and retention. However, there are fundamental differences between our approach and theirs. In Fruchter and Zhang (2004), there are two firms and a fixed market in which customers use one firm or the other. Acquisition represents converting customers from the other firm while retention is preventing existing customers from switching to a competitor. Furthermore, their model is a differential game in which they make very specific assumptions on how effective acquisition and retention are at generating sales, namely that effectiveness is proportional to the square root of the expenditure. With this special model structure, Fruchter and Zhang (2004) show that equilibrium retention increases in a firm's market share while equilibrium acquisition decreases. Our work does not assume a fixed market, or specific functions that determine the relationship between expenditure and impact, and our work also captures randomness (Fruchter and Zhang (2004) is deterministic). Due to the fact that we do not assume a fixed market where the only way to obtain more customers is to convert them from another company, our insights are also different than Fruchter and Zhang (2004).

A recent paper on customer acquisition and retention from the operations management literature is Dong et al. (2011), and the reader is referred to their introduction for additional references on the problem studied. Dong et al. (2011) consider joint acquisition and retention, and use an incentive mechanism design approach to solve their problem. Additionally, they consider the question of direct versus indirect selling, in which the firm decides whether to use a sales force (for which an incentive is designed) or not. Their problem is static, where decisions are made only once.

Sales force management is a topic well-studied from the incentive-design perspective by others in addition to Dong et al. (2011). It often represents a traditional adverse selection problem where designing a proper incentive structure can be difficult and costly due to the economics concept of *information rent* that must be paid to the sales agent to induce them to truthfully reveal their hidden information. Papers that discuss sales incentives in this context come from both the economics and operations management literature. From the economics literature, important works include Gonik (1978), Grossman and Hart (1983), Holmstrom (1979), and Shavell (1979). These papers set the stage for how moral hazard applies in the sales context and propose potential incentive mechanisms. In the operations literature, sales force incentives have been discussed primarily in the context of inventory-control, and manufacturing. Important references include Chen (2005), Porteus and Whang (1991), and Raju and Srinivasan (1996). These papers do not discuss the trade-off between acquisition and retention.

There also exists a body of literature on customer management from a service and capacity perspective. Hall and Porteus (2000) study a dynamic game model of capacity investment where maintaining sufficient capacity relative to market share drives retention, and excess capacity leads to acquisition. With a special structure for costs and benefits of capacity, they are able to solve explicitly for the subgame perfect equilibrium. Related dynamic game inventory-based competition research includes Ahn and Olsen (2011) and Olsen and Parker (2008). In these papers, retention and acquisition are driven by fill rates, and are not explicit decisions, as in our paper.

There is a broad literature on service failure recovery, which generally finds that it is beneficial to recover dissatisfied customers after a service failure, see for example, Spreng et al. (1995). This literature also discusses the 'service recovery paradox', which says that previously dissatisfied customers respond most strongly to recovery efforts (see Matos et al. (2007)). Our concept of 'retention' is more pro-active, and we do not explicitly model service failures, but instead assume that some customers are 'unhappy' , and we have the option to work on retaining them once we find out their unhappiness. (Our model is especially relevant in many settings such as cable or telecommunications where a set of customers express unhappiness with the service absent a specific explicit service failure which caused the unhappiness). Another related area of literature is organizational learning, where acquisition and retention are framed as exploration and exploitation (as it pertains to new innovations/technologies). A seminal paper in this area is March (1991) and a more recent update is Gupta et al. (2006). The general finding in these papers is that exploitation is beneficial in the short term but can have negative consequences in the long term unless accompanied by exploration. Thus the key to success for a firm is to be ambidextrous across both capabilities of exploration and exploitation.

The main contribution of this paper is to study the acquisition and retention management problem using a dynamic optimization approach. With this approach, we are able to incorporate the dynamic nature of this important resource allocation problem, and derive managerial insights on optimal acquisition and retention related to the system dynamics, e.g., how does a firm's current level of satisfied and dissatisfied customers impact its allocation decisions in acquisition and retention?

We believe it is important to study dynamic acquisition and retention for a firm due to the following reasons. First of all, we argue that a dynamic model cannot be reduced to a single period model as one needs at least two periods for the investment in acquisition and retention to later pay-off. Second, it is the dynamic nature of our model that makes it more representative of reality and more general, and enables us to obtain a richer characterization of optimal acquisition and retention policies' dependence on the size of the customer base, the size of the 'unhappy' customers, as well as the available budget for acqusition/retention.

3. Model and Main Results

We model the acquisition and retention resource allocation problem as an N period finite-horizon dynamic program. The decision period is indexed by n, $n = 1, \ldots, N$. At the beginning of period n, the firm knows its number of customers, x_n, and a random fraction ρ_n of its customers are identified as being at high risk for attrition. For simplicity we call these customers 'unhappy' customers. After observing the number of 'unhappy' customers, the firm decides how many customers to retain, and how many new customers to acquire, decisions we denote by R_n and A_n respectively. Note that $\rho_1, \ldots, \rho_N$ are random variables and ρ_n is realized (and observed) at the beginning of period n. As an example of how this works in practice, it is common in the cable industry for customers to call and ask to disconnect service, or otherwise express discontent. Once these customers are identified, the cable company will make a retention offer with enhanced service or lower pricing. During the same period n, the firm also signs up new acquisition prospects.

In this section we consider the situation in which a firm decides how many of its 'unhappy' customers to retain and how many new customers to acquire, and the firm will spend the necessary resources to implement the decision in the period. Therefore, the outcomes for these decisions are deterministic, A_n (acquisition) and R_n (retention) respectively, while the costs to implement the decisions are random, with average values denoted by $C_n^A(A_n)$ and $C_n^R(R_n)$, respectively. To make the trade-off between acquisition and retention explicit, we have a cash constraint on total expected spending in each period of our model. This constraint is given as $C_n^A(A_n) + C_n^R(R_n) \le S_n$ for a positive number S_n. Because of such a constraint, our problem is a service capacity one in the traditional sense. Note that because our acquisition and retention costs are random, the cash constraint applies to *expected* acquisition and retention costs. This is reasonable because when firms are faced with uncertain costs, they are likely to budget a priori based on expected outcomes.

Because customers represent a revenue stream for the firm, the expected revenue generated during period n, given that the number of customers at the beginning of period n is

x_n, is denoted by $M_n(x_n)$. Note that $M_n(x_n)$ represents the customer revenue minus any variable costs associated with providing service to the customer base of size x_n.

It is also possible for some 'happy' customers to discontinue service even though the firm has no prior indication of their dissatisfaction with the service. We denote the random percentage of 'happy' customers that continue service in period n as $\gamma_n \in [0,1]$ (thus, $1-\gamma_n$ is the fraction of 'happy' customers that discontinue service). At the beginning of the next period, $n+1$, the number of customers evolves according to state transition

$$x_{n+1} = \gamma_n(1-\rho_n)\, x_n + R_n + A_n, \qquad n = 1, 2, \ldots, N-1. \tag{1}$$

Thus, the firm retains γ_n of the'happy' customers and R_n of the 'unhappy' ones, while adding A_n in acquisition. In this section we assume R_n and A_n are deterministic, and we will study the case of uncertain acquisition and uncertain retention in the next section.

Suppose the decision maker uses a discount factor, $\alpha \in (0,1)$, in computing its profit. The objective of the firm is to balance acquisition and retention in each period to maximize its total expected discounted profits. Let $V_n(x_n)$ be the maximum expected total discounted profit from period n until the end of the planning horizon, given that the number of customers at the beginning of period n is x_n. Then the optimality equation is

$$\begin{aligned} V_n(x_n) = M_n(x_n) + E_{\rho_n}\Big[\max_{0 \le A_n, 0 \le R_n \le \rho_n x_n, C_n^A(A_n) + C_n^R(R_n) \le S_n} \Big(& -C_n^A(A_n) - C_n^R(R_n) \\ & + \alpha E_{\gamma_n}\big[V_{n+1}\big(\gamma_n(1-\rho_n)\, x_n + R_n + A_n\big)\big]\Big)\Big]. \end{aligned} \tag{2}$$

The boundary condition is $V_{N+1}(x) \equiv 0$ for all $x \ge 0$, implying that the firm makes profits only through period N.

The optimality equation is described as follows. Suppose x_n is the number of customers at the beginning of period n. The firm earns a revenue related to the size of its customer base in period n, given by $M_n(x_n)$. After observing the number of 'unhappy' customers, $\rho_n x_n$, the firm decides how many 'unhappy' customers to retain and how many new customers to acquire, with respective costs $C_n^R(R_n)$ and $C_n^A(A_n)$. The firm may spend up to S_n total on acquisition and retention. The state at the beginning of the next period is given by equation (1). Since the proportion of 'unhappy' customers is random, we need to take expectation with respect to ρ_n, and then with respect to γ_n. Because the firm's decision is made after realization of the number of 'unhappy' customers, the optimization decision is inside the first expectation in (2).

Assumption 1. *The cost functions for the retention of existing customers and acquisition of new customers given by* $C_n^R(\cdot)$ *and* $C_n^A(\cdot)$ *are increasing and strictly convex with continuous derivatives defined on a domain of* $[0, \infty)$.

More acquisition or retention is always more costly to the firm, thus $C_n^R(\cdot)$ and $C_n^A(\cdot)$ are increasing functions. Assumption 1 also assumes that retention and acquisition costs are both convex in the number of targets captured by the firm in each category. This can be explained as follows. When given targets, sales forces usually acquire or retain the easiest prospects in a market first. As the best prospects are acquired, acquisition and retention grows more difficult and costly. Furthermore, getting more work from a fixed-size sales force could result in overtime and other costs, which also leads to an increasing convex cost function.

Assumption 2. *The expected revenue function* $M_n(x_n)$ *is increasing concave and continuous in* x_n *with domain of* $[0, \infty)$.

The expected revenue is clearly increasing in the number of customers using the firm's service. Here we also assume that it is concave in the number of customers. Larger and higher margin customers are likely to be targeted first in acquisition, so incremental customers will generate less revenue. In the third-party-financing industry, incremental customers tend to be less profitable because they are likely to be smaller and more skeptical of the benefit associated with the service being provided. In addition, as the prospects valuing the service most are acquired, it takes more effort and better terms to successfully acquire more skeptical customers.

Our model complements the landmark work of Blattberg and Deighton (1996) that establishes the 'customer equity test'. In their model, as in ours, acquisition and retention success is a concave function of total spending, customers generate a per-period revenue, and a firm makes acquisition and retention decisions. However, the primary difference is that they assume that acquisition and retention decisions are made only once, and then maintained over the lifetime of a customer. In our model, the firm can dynamically adjust its strategy over time, which enables us to characterize these dynamic decisions as a function of the size of the firm's customer base and the number of its 'unhappy' customers. Also, we allow costs and revenues to have a more general form, while Blattberg and Deighton (1996) assume linear revenues and an exponential spending/outcome relationship. Finally, we have a cash constraint on total expected acquisition and retention activities, as discussed.

Before proceeding to the main result, we present here a preliminary modularity result which will be useful in our characterization of the firm's optimal strategy.

LEMMA 1. *Given continuous and strictly concave functions $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$, a non-negative constant K, and a non-negative random variable ϵ, the optimal solution to the optimization problem*

$$\max_{x\geq 0, y\geq 0} f(x) + g(y) + E[h(x + y + \epsilon K)],$$

denoted by x^* and y^*, are decreasing in K with slopes between 0 and -1.

The result above states that as K varies, the optimal x^* and y^* move in the same direction as each other, and in the opposite direction as K. Note that this is a meaningful result but it does not follow from modularity analysis in Topkis (1998). This is because, the objective function for the optimization problem above is submodular in (x, y, K), but the optimization is maximization. There is no general comparative statics results on maximizing submodular functions when the decisions are not single dimensional (in this case the decision is two-dimensional).

With our technical result in Lemma 1, we are ready to present the main result of this paper. The following theorem states that there exists a ρ_n-dependent threshold $Q_n(\rho_n)$, decreasing in ρ_n, such that when the number of customers at the beginning of period n is less than this threshold, the firm targets every 'unhappy' customer, while the optimal number of acquired new customers is decreasing in the current customer base, i.e., in this range the firm gradually shifts emphasis from acquisition to retention as its customer base grows. The existence of a second ρ_n-dependent threshold $\frac{K_n}{1-\rho_n}$ defines a second region where the firm maximizes allowable resources in acquisition and retention while maintaining flat levels of acquisition and retention spending. Finally when the firm's customer base is larger than both thresholds, the firm begins to target fewer and fewer customers in both acquisition and retention; in this range, both the optimal acquisition and optimal retention are decreasing in the customer base x_n with slope no less than $-(1-\rho_n)$.

THEOREM 1. *Suppose x_n is the number of customers at the beginning of period n, and the proportion of 'unhappy' customers is ρ_n.*

(i) The optimal strategy for period n is determined by a critical number K_n, a decreasing function $Q_n(\rho_n)$, and decreasing curves $R_n^{U*}(\cdot)$, $A_n^{U*}(\cdot)$, and $A_n^{W*}(\cdot)$ of slopes no less than -1, such that a) if $x_n \le Q_n(\rho_n)$, then the firm retains all 'unhappy' customers and sets $(A_n, R_n) = (\min\{A_n^{W*}(x_n), S_n - C_n^R(\rho_n x_n)\}, \rho_n x_n)$; b) if $x_n \in (Q_n(\rho_n), \frac{K_n}{1-\rho_n})$, then the firm uses maximum allowable resource S_n and sets $(A_n, R_n) = (A_n^{U*}(K_n), R_n^{U*}(K_n))$ with $A_n^{U*}(K_n) + R_n^{U*}(K_n) = S_n$; and c) in all other cases $(A_n, R_n) = (A_n^{U*}(x_n(1-\rho_n)), R_n^{U*}(x_n(1-\rho_n)))$.

(ii) There exist increasing functions $Q_n^A(\rho_n)$ and $Q_n^R(\rho_n)$ such that when $x_n \ge Q_n^R(\rho_n)$, the firm does no retention, and when $x_n \ge Q_n^A(\rho_n)$, the firm does no acquisition.

(iii) There exists a critical threshold function $x_n^*(\rho_n)$, decreasing in ρ_n, such that, under the optimal acquisition and retention policies the expected market size of the firm goes up in the next period if the current market size is less than this threshold, and goes down otherwise, i.e.,

$$E[x_{n+1}] - x_n = \begin{cases} \le 0, & \text{if } x_n \ge x_n^*(\rho_n); \\ \ge 0, & \text{if } x_n \le x_n^*(\rho_n). \end{cases}$$

Therefore, under the optimal strategy, the firm will lose customers (in expectation) when above a critical point and add customers (in expectation) when below that same point.

The optimal strategy stated in Theorem 1 takes an intuitive form. For a relatively small base of customers, the firm should retain each and every 'unhappy' customer. In this region, acquisition is also critical. After this point, there may exist a second region where the firm spends S_n on acquisition and retention, the maximum allowable resource, but not necessarily able to retain every 'unhappy' customer. Finally, as the customer base of the firm grows large enough, it spends less on acquisition and retention. We remark that the second region in the optimal acquisition and retention strategy characterization, i.e., b) of case i), could be an empty set.

From the results of this section, we learn that a firm should shift resources from acquisition to retention as its customer base grows. However, this is only true up to some critical point. After that point, there may first exist a region in which acquisition and retention are constant due to the firm's cash constraint; and when the customer base grows large enough, the optimal strategy for the firm will be to invest less in both acquisition and retention. Note that our result indicates that retention effort is a function of a firm's customer size, an insight that is consistent with much of the marketing literature. However, compared with the marketing literature (e.g. Fruchter and Zhang (2004)), our results are consistent only in the first region, where the firm increases retention as its customer base grows and decreases acquisition. However, when the firm is large enough, our results predict less spending in both acquisition and retention. With our explicit cash constraint on total expected acquisition and retention spending, we also characterize a region in which the constraint is tight so the total acquisition and retention are constant.

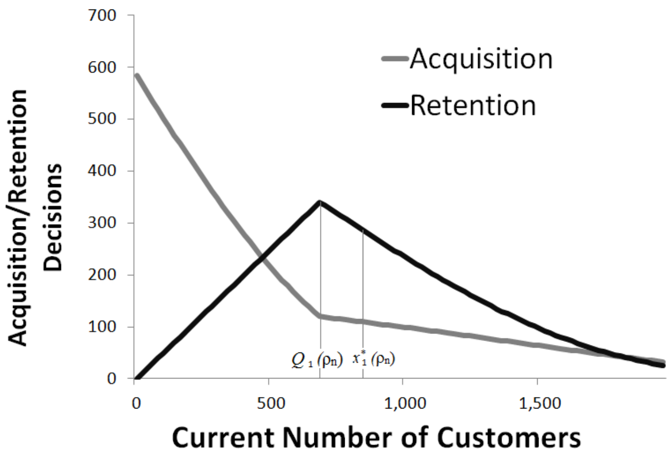
We also want to point out that the threshold function $x_n^*(\cdot)$ in (iii) depends on the period n and is not a constant. The existence of a threshold $x_n^*(\rho_n)$ may appear counter-intuitive at first, but it is to be expected. This is because, the firm's revenue function is increasing concave while its acquisition and retention costs are increasing convex. When the firm's customer base is large enough, the marginal increase in acquisition/retention cost may overgrow the marginal increase in revenue. Note that the existence of a critical threshold of customers for the firm to have for a specific period does not indicate that the firm should not grow its number of customers over time. Nothing about our model and result regarding the existence of the x_n^* (for each period n) prohibits the possibility of firms growing its number of customers over time. First, it may be the case that the x_n^* are high (possibly even infinite) relative to a reasonable customer base, in which case the firm is always targeting growth. Second, the values of x_n^* can increase across periods. We demonstrate this second point with two simple examples below.

In the first example, suppose that the costs of acquisition and retention decrease over time due to experience gained by sales individuals performing these tasks. Even with constant customer revenue rate, we will see that x_n^* are increasing. Let $M_n(x_n) = 10x_n$ with $\rho_n = \gamma_n = 0.5$, for $n = 1, \ldots, 5$. The acquisition and retention costs for the first period are $C_1^A(A_1) = 10000(\frac{A_1}{1000})^2$ and $C_1^R(R_1) = 5000(\frac{A_1}{1000})^2$, and then subsequent costs in periods 2, 3, 4 and 5 are given by 90%, 80%, 70% and 60% of these values respectively. In this case, the x_n^* values are given by $x_1^* = 2200$, $x_2^* = 2480$, $x_3^* = 2770$, $x_4^* = 3110$ and $x_4^* = 3323$ (note that these values typically depend on ρ_n, but in this example ρ_n is constant).

The second example assumes instead that costs are constant but customers become more profitable over time because the firm finds ways to extract additional value through cost efficiencies or cross selling. In this example let $\rho_n = \gamma_n = 0.5$, $C_n^A(A_n) = 10000(\frac{A_n}{1000})^2$ and $C_n^R(R_n) = 5000(\frac{A_n}{1000})^2$ for $n = 1, \ldots, 5$. Revenues increase between periods and are given by $6x_1$, $6x_2$, $6x_3$, $8x_4$, and $10x_5$ respectively for the periods 1 through 5. With these parameters, the x_n^* values are given by $x_1^* = 1231$, $x_2^* = 1234$, $x_3^* = 1722$, $x_4^* = 2100$ and $x_5^* = 2000$, only dropping in the final period of the model.

Thus, our results indicate that if firms want to keep growing its customer base, they need to find ways to reduce their acquisition and retention costs over time or find ways to make customers more valuable over time (by potentially selling them additional services

Figure 1 **Optimal Acquisition and Retention Strategies in terms of number of customers x_1 with fixed $\rho_1 = 0.5$, when $Q_n(\rho_n) > \frac{K_n}{1-\rho_n}$ ($N = 2$, $M_2(x_2) = 10\ln(1+\frac{x_2}{2})$, $S_1 = 10$, $C_1^A(A_1) = (\frac{A_1}{100})^{1.2}$, $\gamma = 1$ with probability 1, and $C_1^R(R_1) = (\frac{R_1}{100})^{1.1}$)**



that the customers may be willing to buy). To the extent that the firm is able to do that, in Theorem 1, the expected number of customers that a firm will have in each period will be higher in expectation than in the previous period.

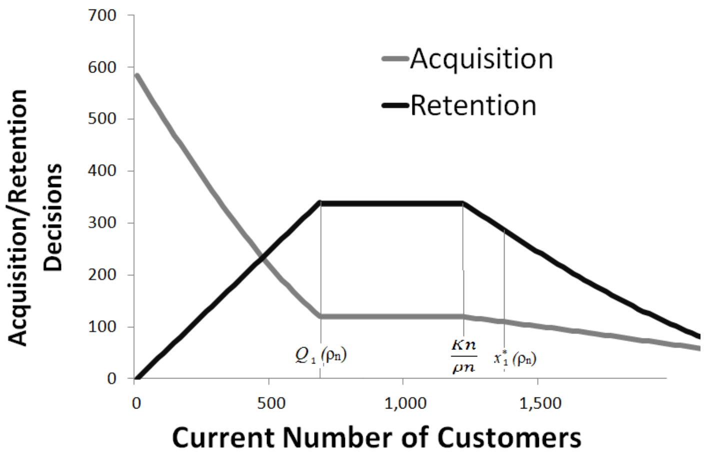
The optimal strategy is demonstrated in Figures 1 and 2, in which we can observe the strategy and how it changes as a function of the customer base, x_n, for a fixed value of ρ_n. The two graphs differ in that the smaller cash-constraint in Figure 2 implies that $Q_n(\rho_n) < \frac{K_n}{1-\rho_n}$, creating a region in which acquisition and retention are flat, because the firm is spending at the maximal level of S_n.

To further characterize the optimal strategy, we need the following result. In this result, we use $(C_n^A)'(0)$ and $(C_n^R)'(0)$ to denote the right derivative of the cost functions at zero.

Lemma 2. *If* $(C_n^A)'(0) \leq (C_n^R)'(0)$, *then* $Q_n^A(\rho_n) \geq Q_n^R(\rho_n)$ *for all* $\rho_n > 0$; *and if* $(C_n^A)'(0) \geq (C_n^R)'(0)$, *then* $Q_n^A(\rho_n) \leq Q_n^R(\rho_n)$ *for all* $\rho_n > 0$. *In particular, if* $(C_n^A)'(0) = (C_n^R)'(0)$, *then* $Q_n^A(\rho_n) = Q_n^R(\rho_n)$ *for all* $\rho_n > 0$.

The implication of Lemma 2 is that the monotone switching curves $Q_n^A(\rho_n)$ and $Q_n^R(\rho_n)$ do not cross, and they are ordered. Thus, for example, when $Q_n^A(\rho_n) \geq Q_n^R(\rho_n)$ is true, the customer base size beyond which the firm does not do any acquisition is always higher than

Figure 2 **Optimal Acquisition and Retention Strategies in terms of number of customers x_1 with fixed $\rho_1 = 0.5$, when $Q_n(\rho_n) < \frac{K_n}{1-\rho_n}$ ($N = 2$, $M_2(x_2) = 10\ln(1+\frac{x_2}{2})$, $S_1 = 4$, $C_1^A(A_1) = (\frac{A_1}{100})^{1.2}$, $\gamma = 1$ with probability 1, and $C_1^R(R_1) = (\frac{R_1}{100})^{1.1}$)**

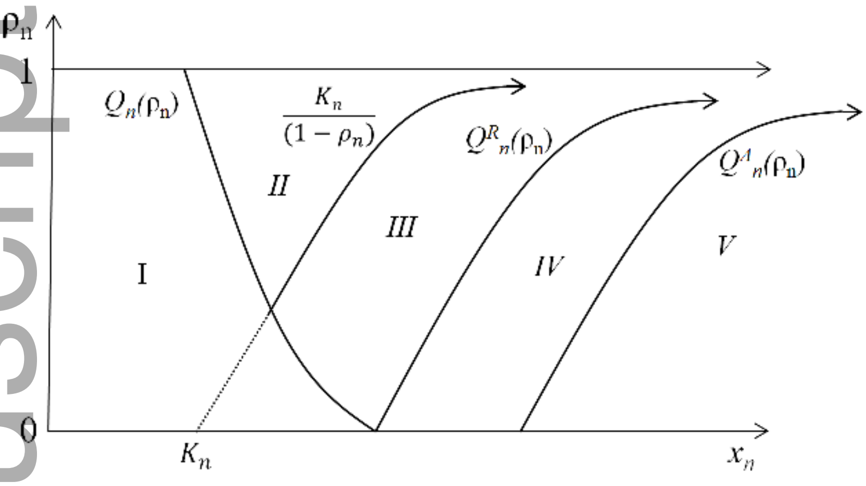


the customer base beyond which the firm does no retention. Lemma 2 allows us to analyze the optimal strategies when both parameters x_n and ρ_n vary. For the purposes of the diagrams, let us assume that $(C_n^A)'(0) < (C_n^R)'((C_n^R)^{-1}(S_n))$ and $(C_n^R)'(0) < (C_n^A)'((C_n^A)^{-1}(S_n))$ such that the firm does both acquisition and retention when the cash constraint is tight, because the marginal cost of acquisition (retention) at zero is smaller than the marginal cost of retention (acquisition) when the cash constraint is tight.

In the first case, i.e., $Q_n^R(\rho_n) \leq Q_n^A(\rho_n)$, the optimal strategy is demonstrated in Figure 3 as a function of x_n and ρ_n. When both x_n and ρ_n are small (region I), the optimal strategy is to retain everyone, and also spend in acquisition. When both become larger, the firm will still spend on both acquisition and retention, but may not retain all 'unhappy' customers (regions II and III). When x_n is still relatively small with a larger ρ_n, the firm should spend up to the cash-constraint maximum (region II); when x_n is large with ρ_n relatively small, the firm will invest in just acquisition (region IV). Finally, when the number of customers is really large, then the firm will spend neither on retention nor acquisition (region V).

The second case, i.e., when $Q_n^R(\rho_n) \geq Q_n^A(\rho_n)$, is depicted in Figure 4. As in the previous case, when both x_n and ρ_n are small (region I), the optimal strategy is to retain everyone, and spend some in acquisition. However, now there is another region (region II), when x_n

Figure 3 **Case I - Optimal Acquisition and Retention Strategies in terms of number of customers x_n and percentage 'unhappy' ρ_n when $Q_n^R(\rho_n) \leq Q_n^A(\rho_n)$ (Region I - Retain all 'unhappy' and do some acquisition, Region II - Retention and Acquisition up to the cash constraint, Region III - Some retention, some acquisition, Region IV - Only acquisition, Region V - No spending)**



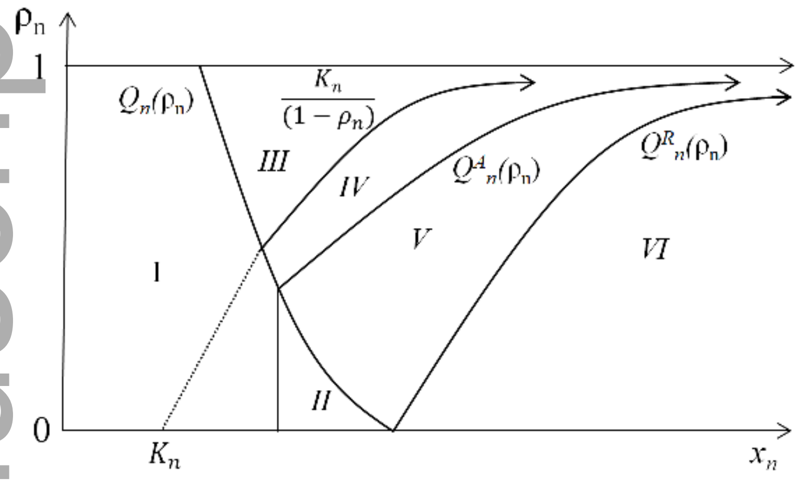
is larger, in which the firm may retain everyone, but not spend anything in acquisition. The firm spends on both acquisition and retention for relatively large ρ_n and small x_n (regions III and IV); spending up to the cash constraint when x_n is smaller and ρ_n larger (region III). When x_n is large with ρ_n relatively small, the firm will invest only in retention (region V). Finally, the firm invests in neither acquisition nor retention when the number of customers is really large (region VI).

Lemma 2 allows us to develop detailed guidelines to firms on how to allocate their valuable resources towards acquisition and retention in different ranges of the size of their customer base and the fraction of dissatisfied customers (see Figures 3 and 4), and these results are much more detailed and intricate than provided in the previous literature. Indeed, as shown in Figures 3 and 4, the results indicate that there may be as many as 6 different regions in which the firm adapts different strategies based on these parameters.

In practice, one may think that the firm would always dedicate some resource towards retention. However, this is not true in general. In the following, we present a sufficient condition under which this is true.

Corollary 1. *If there exists a positive number $\kappa > 0$ such that* $\lim_{x_{n+1}\to\infty} M'_{n+1}(x_{n+1}) \geq$

Figure 4 **Case II - Optimal Acquisition and Retention Strategies in terms of number of customers x_n and percentage 'unhappy' ρ_n when $Q_n^R(\rho_n) \geq Q_n^A(\rho_n)$ (Region I - Retain all 'unhappy' and do some acquisition, Region II - Retain all 'unhappy' with no acquisition, Region III - Retention and Acquisition up to the cash constraint, Region IV - Some retention, some acquisition, Region V - Only retention, Region VI - No spending)**



$\kappa \geq \frac{(C_n^R)'(0)}{\alpha}$, *then* $Q_n^R(\rho_n) = \infty$ *and the firm will always do some retention, as long as* $\rho_n x_n > 0$ *(there are 'unhappy' customers) and* $(C_n^R)'(0) < (C_n^A)'((C_n^A)^{-1}(S_n))$ *(maximal cash constraint spending involves some retention).*

Essentially, the firm will always retain some customers as long as the discounted marginal benefit from an additional customer is always higher than the cost to retain one customer. This is a reasonable and intuitive finding. Similarly, the following corollary establishes a sufficient condition under which the firm always does some acquisition.

Corollary 2. *If there exists a positive number* $\kappa > 0$ *such that* $\lim_{x_{n+1} \to \infty} M'_{n+1}(x_{n+1}) \geq \kappa \geq \frac{(C_n^A)'(0)}{\alpha}$, *then* $Q_n^A(\rho_n) = \infty$ *and the firm will always have some acquisition, as long as* $(C_n^A)'(0) < (C_n^R)'((C_n^R)^{-1}(S_n))$ *(maximal cash constraint spending involves some acquisition).*

We highlight that our results are critically dependent upon the fact that we have a *dynamic* model. Suppose that the firm's revenue function is linear in the number of customers it has at the end of a single period (static) problem and assume for simplicity that the firm faces no cash constraint. Then with linear payouts and convex acquisition and retention costs, the optimal strategy would be determined by a simple order-up-to policy, in which the optimal acquisition is A^* with the optimal retention given by $\min(R^*, \rho_n x_n)$. For this model the optimal retention is increasing in the size of the firm's customer base and is never decreasing. Also, regardless of how many customers the firm already has, the firm aims to add a constant number. However, with this same set of assumptions (linear per-customer revenue and convex costs) and a dynamic model, the optimal strategy is instead what we have characterized in Theorem 1, in which the optimal strategy is state-dependent and we have a region in which retention decreases, and also regions in which acquisition decreases. In this way it is the *expected future retention costs* that cause the firm to experience diminishing returns with additional customers, thus making acquisition and retention spending less appealing when x_n is large (i.e. the region we characterize in which both acquisition and retention spending decrease). This nuance of our result highlights the value of studying a dynamic model while also distinguishing our work from others.

Remark 1. We note that our results can be extended to a scenario in which the cost of acquisition depends on both the number acquired A_n and the number of current customers x_n if there is no cash constraint on total acquisition and retention spending. In this case, we need the acquisition function $C_n^A(A_n, x_n)$ to be jointly convex and supermodular in (A_n, x_n),. This is a relatively strong assumption but it is satisfied by some functions, e.g., when the cost function is separable with convex functions $f_n(\cdot)$ and $g_n(\cdot)$, such that $C_n^A(A_n, x_n) = g_n(A_n) + f_n(x_n)$. With these assumptions and without the cash constraint, we are able to replicate all the results in the paper.

4. Model Extension

The formulation in Section 3 fits in environments where retaining or acquiring customers requires a lot of personal interactions. For example, in the health care finance industry one of the authors worked in, sales people paid visits to customers who intended to discontinue service and sales staff knew whether retention or acquisition had been successful. Thus sales staff would be given targets on how many customers to retain and could keep working until their targets were met. However, in many applications it is common that both costs *and* outcomes are random for acquisition and/or retention. Such situations occur when the results of acquisition or retention efforts are realized after certain time. For example, in the magazine subscription industry, acquisition and retention are done through mails, and success would not be realized immediately.

In this section, we consider this generalization of our model in which outcomes in acquisition or retention may be stochastic, meaning that a confirmed success in acquisition or retention is not always possible at the time the effort is made.

4.1. Stochastic Retention and Acquisition

Let ϵ_n^1, and ϵ_n^2 be the random success rates for the firm in retention and acquisition respectively. Then the state transition for this system is

$$x_{n+1} = \gamma_n x_n(1-\rho_n) + \epsilon_n^1 R_n + \epsilon_n^2 A_n, \qquad n = 1, 2, \ldots, N-1,$$

and the optimality equation is

$$V_n(x_n) = E_{\rho_n}\Big[M_n(x_n) + \max_{0\le A_n, 0\le R_n\le \rho_n x_n, C_n^A(A_n)+C_n^R(R_n)\le S_n}\Big(-C_n^A(A_n) - C_n^R(R_n) + \alpha E[V_{n+1}(\gamma_n x_n(1-\rho_n) + \epsilon_n^1 R_n + \epsilon_n^2 A_n)]\Big)\Big]. \quad (3)$$

With the same boundary condition as before ($V_{N+1}(x) \equiv 0$), we have the following results for this model.

Theorem 2. *(i) The optimal strategy is defined by three state-dependent switching curves,* $R_n^{Y*}(x_n, \rho_n)$, $A_n^{Y*}(x_n, \rho_n)$, *and* $A_n^{W*}(x_n, \rho_n)$, *such that*

a) if $R_n^{Y*}(x_n, \rho_n) \le \rho_n x_n$, *the optimal strategy is to set* $(A_n, R_n) = (A_n^{Y*}(x_n, \rho_n), R_n^{Y*}(x_n, \rho_n))$;

b) otherwise, the firm sets $(A_n, R_n) = (A_n^{W*}(x_n, \rho_n), \rho_n x_n)$.

(ii) The switching curves $A_n^{Y*}(\cdot,\cdot), R_n^{Y*}(\cdot,\cdot)$ *and* $A_n^{W*}(\cdot,\cdot)$ *are not necessarily monotone in* x_n *or* ρ_n, *and parts (ii) and (iii) of Theorem 1 do not hold for this model.*

The lack of monotonicity in the switching curves indicates that the optimal policy for acquisition and retention no longer has a nice, or intuitive structure. The following example illustrates some of these phenomena; the optimal acquisition and retention strategies are given in Figure 5, which is in contrast to the optimal policy structures from Theorem 1 displayed in Figure 1 and 2.

Example 2. This example is generated by modifying data from (Chen et al. 2012). Again we consider a problem with two periods, $N = 2$, hence $V_2(\cdot) = M_2(\cdot)$. The revenue function is piece-wise linear and concave, where the firm makes 14 dollars for each customer up to 500, and half a dollar for customers thereafter, i.e.,

$$M_2(x_2) = \begin{cases} 14x_2, & \text{if } x_2 \leq 500; \\ 7000 + 0.5(x_2 - 500), & \text{if } x_2 > 500, \end{cases}$$

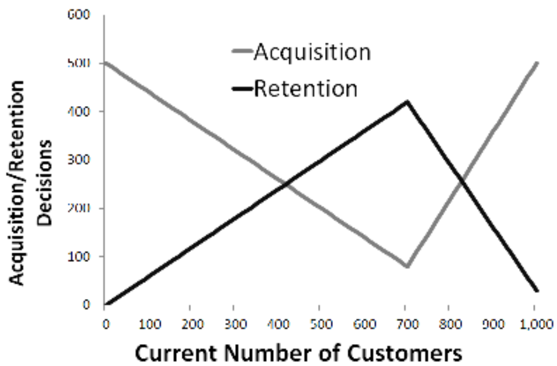
and the acquisition and retention costs are linear:

$$C_1(A_1) = 2.5A_1, \qquad D_1(R_1) = 3.6R_1.$$

The three random variables are assumed to be discrete: $\gamma_n = 0$ and 1 with probabilities 1/2 and 1/2; $\epsilon_n^1 = 0.5$ and 1 with probabilities 1/2 and 1/2; and $\epsilon_n^2 = 0.2$ and 1 with probabilities 1/2 and 1/2. We fix parameter $\rho_1 = 0.6$ and assume that S_1 is sufficiently large such that the cash constraint does not factor into the decision-making. We study how the strategy varies in the initial number of customers at the beginning of period 1, x_1.

The optimal strategies are presented in Figure 5. One can see that acquisition is no longer decreasing in x_1, which was our insight for the previous model. The intuition for this phenomenon is the following. When x_1 is small, the firm prefers the more certain strategy of retention, and invests up to the upper bound of the constraint on retention. The firm prefers the certain strategy because that increases the chances to get to $x_2 = 500$, which is where the marginal customer value changes. For high x_1, the firm already has good chance of getting up to $x_1 = 500$, so it starts to prefer acquisition, which is more uncertain, but slightly more cost effective. For this reason, we see that acquisition increases while retention decreases. This lack of monotonicity is not surprising given the results in the literature for inventory models with random yield and two suppliers (see Chen et al. (2012)).

Figure 5 **Optimal Acquisition and Retention Strategy for variable x_1 and $\rho_1 = 0.6$ for Stochastic Retention and Acquisition Model**



4.2. A Heuristic

For the model presented in Section 3, the optimal acquisition and retention strategies had monotone properties (in the number of customers x_n) that led to a nice policy structure. Optimal acquisition was decreasing in a firm's market share while optimal retention was first increasing, possibly flat on a middle region, and then decreasing. However, in the extension with random retention and acquisition outcomes (in addition to the retention and acquisition costs), the optimal strategy no longer exhibits these properties. For this reason, we develop a heuristic policy for the situation where retention and acquisition outcomes are random. Toward that end, rather than random variables ϵ_1 and ϵ_2, we instead propose to use $E[\epsilon_1]$ and $E[\epsilon_2]$ in the optimality equation. Therefore, the heuristic model is

$$V_n(x_n) = E_{\rho_n}\Big[M_n(x_n) + \max_{0\le A_n, 0\le R_n\le \rho_n x_n, C_n^A(A_n)+C_n^R(R_n)\le S_n}\Big(-C_n^A(A_n) - C_n^R(R_n) + \alpha E_{\gamma_n}[V_{n+1}(\gamma_n x_n(1-\rho_n) + E[\epsilon_n^1]R_n + E[\epsilon_n^2]A_n)]\Big)\Big] \quad (4)$$

with the same boundary condition as before ($V_{N+1}(\cdot) = 0$). Using the same argument given for Theorem 1, it can be seen that the heuristic model from equation (4) has optimal solution structure exactly the same as that given in Theorem 1. Thus, we propose using the solution to (4) as a heuristic for the problem with stochastic acquisition and retention outcomes.

Table 1 **Testing Scenario Overview ($N=5$ and for all n, $\rho_n=0.1$ and 0.2 with probabilities 0.5 and 0.5, $C_n^A(A_n)=A_n$ if $A_n\le 100$, and $C_n^A(A_n)=100+5(A_n-100)$ if $A_n>100$, and S_n sufficiently large. γ_n, ϵ_n^1 and ϵ_n^2 are distributed with equal probability across given values)**

Parameter	Option 1	Option 2	Option 3
Revenue Function	$6x_n^{0.65}$	$6x_n^{0.8}$	-
Retention Cost	$0.8R_n$	R_n	$1.2R_n$
γ_n Distribution (discrete uniform)	$\{0.8,0.9\}$	$\{0.8,0.9,1.0\}$	$\{0.7,0.8,0.9,1.0\}$
ϵ_n^1 Distribution (discrete uniform)	$\{0.6,0.9\}$	$\{0.5,0.7,0.9\}$	$\{0.6,0.7,0.8,0.9\}$
ϵ_n^2 Distribution (discrete uniform)	$\{0.5,0.8\}$	$\{0.4,0.6,0.8\}$	$\{0.4,0.5,0.6,0.7\}$

To understand the performance of this heuristic approach, we conducted a numerical study on a number of different scenarios, and computed the performance of the heuristic as compared to an optimal strategy. The parameters that we used in our numerical study are summarized in Table 1. For all scenarios, we consider a five-period problem ($N=5$), and assume, for all n, that $\rho_n=0.1$ and 0.2 with probabilities 1/2 and 1/2, $C_n^A(A_n)=A_n$ if $A_n\le 100$, and $C_n^A(A_n)=100+5(A_n-100)$ if $A_n>100$. We also assume that the random variables γ_n, ϵ_n^1 and ϵ_n^2 are distributed with equal probability across a number of different values (discrete uniform distribution). Finally, we assume that the expected cash constraint is $S_n=400$ in each period of the model.

By varying the different parameters, we tested 162 different scenarios. We summarize the results of the numerical study in Table 2. For each scenario, we determined the average error (across a number of different possible starting states), and the worst error.

Table 2 **Testing Summary**

Metric	Performance
Average Average Error	0.04 %
Average Worst Error	0.31 %
Worst Average Error	0.09 %
Worst Worst Error	0.61 %

In each of the scenarios we tested, the average error was well under one tenth of a percent with a maximum error under one percent. This indicates that our heuristic performs very well with reasonable functions for acquisition/retention costs and benefits.

Although our numerical study indicates that the heuristic generally performs well, it is possible to also find examples where it does not. Consider once again our numerical example in Subsection 4.1. We compared the optimal strategy for this problem to the heuristic across scenarios in which the number of starting customers x_1 varied between 0 and 2000. In this case, the average error of the heuristic was 13.9 percent across all scenarios with the worst error equal to 20.7 percent.

We note however that this is a highly contrived example where the firm's revenue function has a severe discontinuity: the firm makes 14 dollars per customer up to 500 customers but only 0.5 thereafter. Furthermore, while acquisition is less costly in expectation, retention outcomes are less random. Therefore when we replace the random outcomes by their expectations, the heuristic only uses acquisition. This is to its detriment because the sharp discontinuity in the revenue function rewards a strategy in which retention is used so long as the number of customers is below 500. However, we were only able to generate examples where the heuristic performed so poorly when our parameters had such drastic changes and we believe that such situations are less likely in practice. In fact, with a similar example and a more smooth revenue function, we see that the heuristic performs well, a finding consistent with the results from our more comprehensive computational testing. Suppose instead of making \$14 up to 500 customers and then \$0.50 thereafter, the firm's revenue function is piece-wise linear with decreasing marginal customer values of 14,12,11,10,9,8,7.5,7,6.5,6,5.5,5,4.5,4,3.5,3,2.5,2,1.5,1 (all in \$) for the twenty 25-customer increments from 0 to 500 (i.e. the firm makes \$14 for each of the first 25 customers, then \$12 for each of the next 25, then \$11 for the next 25, etc.). Above 500, we assume the marginal customer value is \$0.50 as before. In this case the heuristic performs well, with the average profit across all cases equal to 0.75 percent and the worst error observed across all cases equal to 3.42 percent. Again because firms' revenue functions are likely smooth in practice, we think this example is more realistic than the one where the firm's marginal revenue from customers went from \$14 to \$0.5 at a single sharp point. More generally, our numerical tests found that it is possible to find cases where the suggested heuristic does not work well, but to generate these cases, one needed very sharp jumps in the revenue, acquisition cost and retention cost functions. As seen in the above example, more gradual jumps in these functions resulted in a pretty good performance for the heuristic.

5. Conclusion

Maintaining and growing a base of profitable customers is critical to the success of many companies across different industries. To succeed, companies need to appropriately allocate resources to the retention of existing customers and to acquisition of new ones. In this paper we develop a model to analyze this problem which captures the practical interactive dynamic decision-making process. Existing literature has focused on the acquisition and retention trade-off using regression, empirical analysis, or static optimization. This work is unique in that it provides a *dynamic* optimization perspective on the resource allocation trade-off between customers acquisition and retention. Because customer relationships evolve over time, we believe the paper makes a meaningful contribution to the literature.

With some plausible assumptions on the costs of acquisition and retention and the revenue generated from customers, we obtain some interesting structural properties for the optimal strategy, which then provide important insights to the firm's optimal solution. For a small firm undergoing initial growth in its customer base, our results emphasize the critical importance of customer retention; the firm should spend heavily on both channels, while shifting resources from acquisition to retention during this initial growth. In practice, we believe that many firms undervalue retention during initial growth of its customer base and overemphasize acquisition. If this were to occur, acquisition can be undermined by the loss of existing customers, stalling growth. When a firm gets larger, there may exist a region in which the spending in acquisition and retention is flat because the firm is spending at the maximal amount dictated by the cash constraint. Finally, when its customer base is large enough the firm begins to invest less in both acquisition and retention. The reason for this is that when retention efforts become prohibitively expensive, the firm accepts that it might lose some customers, rather than spending a lot of resource to try and keep every customer. This important result is consistent with some observations in the telecommunication industries. In practice, some customers may be so expensive to keep satisfied that it no longer makes sense for the firm to continue retaining every one of them, if the customer base is large enough. However, we also did find conditions that enable the firm to continue growing: namely, if the firm can reduce acquisition and retention costs over time, or if the firm can increase the value of its customers by convincing the customers to buy more services, then it is optimal for the firm to continue growing its customer base over time. We also discussed an extension to our model where acquisition and retention outcomes (as well as their costs) are random. This case results in a much more complex optimal policy structure and we developed an effective heuristic policy for that.

There is significant opportunity for additional research from the operations management community on the topic of customer acquisition and retention management. For example, it is often the case in practice that multiple firms target the same pool of prospective customers, and one would need to apply game theory to study the dynamic decision making and competition of the firms. There is also the possibility of incorporating other sales management decisions into the framework of the acquisition and retention trade-off. For example, one may consider joint decisions on acquisition, retention, and sales compensation design, or joint decisions on acquisition, retention, and hiring or laying-off employees. Such models would extend our work to consider other strategic aspects of the dynamic acquisition and retention management problem.

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Appendix

In this appendix, we present all the technical proofs. Throughout the proofs we define $R_n^*(x_n, \rho_n)$ and $A_n^*(x_n, \rho_n)$ to be the optimal solutions for the variables R_n and A_n, given that the number of customers at the beginning of period n is x_n and the observed fraction of unhappy customers is ρ_n.

Proof of Lemma 1. The problem we are studying is

$$\max_{x\geq 0, y\geq 0} f(x) + g(y) + E[h(x + y + \epsilon K)], \tag{5}$$

and we have continuity and strict concavity assumptions on the three functions and that both constant K and random variable ϵ are non-negative.

Rewrite (5) as

$$\max_{z\geq 0}\Big(D(z) + E[h(z + \epsilon K)]\Big) \tag{6}$$

with

$$D(z) = \max_{0\leq x\leq z}\Big(f(x) + g(z - y)\Big). \tag{7}$$

The optimization problem in (6) is submodular in (z, K) because ϵ is non-negative, thus the optimal solution, denoted by $z^*(K)$, is decreasing in K. From the problem given in (7), the maximand is supermodular in (x, z) and the constraint $0 \leq x \leq z$ is a lattice, hence the optimal x^* is increasing in z. Considered together, this implies that a smaller value of K results in a larger value of z and a larger value of x. Therefore, $x^*(K)$ is decreasing in K. We rewrite (7) as

$$D(z) = \max_{0\leq y\leq z}\Big(f(z - y) + g(y)\Big). \tag{8}$$

Using this equation (8) and the supermodularity in (z, y) we similarly obtain that the optimal y, denoted by $y^*(K)$, is decreasing in K.

To show that the slopes of the optimal $x^*(K)$ and $y^*(K)$ are between -1 and 0, we argue that the optimal $z^*(K)$ has slope between 0 and -1. This is sufficient to conclude the same about $x^*(K)$ and $y^*(K)$ because, by the fact that each is decreasing in K, and $x^*(K) + y^*(K) = z^*(K)$, it would be impossible for one of $x^*(K)$ and $y^*(K)$ to decrease by more than that of $z^*(K)$.

Suppose that K increases by $c > 0$, but z^* decreases by $d > c$. This condition is formally written as $z^*(K + c) = z^*(K) - d < z^*(K) - c$. We argue such a situation cannot occur because if true, we are able to find a very small $\delta > 0$, such that $z^*(K + c) + \delta$ is a strictly better solution than $z^*(K + c)$. We argue this solution is better by the following inequalities. Note that it is easy to see from (7), that $D(\cdot)$ in equation (6) is convex.

$$\begin{aligned}&D(Z^*(k)-d+\delta)-D(Z^*(K)-d)\\&< D(z^*(K))-D_n(z^*(K)-\delta)\\&\leq E[h(z^*(K)+\epsilon K-\delta)]-E[h(z^*(K)+\epsilon K)]\\&\leq E[h(z^*(K)-d+\epsilon(K+c))]-E[h(z^*(K)-d+\delta+\epsilon(K+c))],\end{aligned}$$

where the first inequality comes from the convexity of $D(\cdot)$, the second from the optimality of the solution $z^*(K)$, and the third from the convexity of $h(\cdot)$ along with the fact that we can pick δ small enough so that $c\epsilon-d+\delta\leq 0$. Considering the first and last expressions, we see that

$$E[h(z^*(K)-d+\delta+\epsilon(K+c))]+D(Z^*(k)-d+\delta)>E[h(z^*(K)-d+\epsilon(K+c))]+D(Z^*(K)-d),$$

contradicting the optimality of the original solution.

Thus, the analysis above shows that the optimal $x^*(K)$ and $y^*(K)$ are decreasing in K, but with slopes between -1 and 0.

Proof of Theorem 1. The optimality equation is

$$\begin{aligned}V_n(x_n)&=E_{\rho_n}\Big[M_n(x_n)\\&+\max_{0\leq R_n\leq x_n\rho_n,0\leq A_n,C_n^A(A_n)+C_n^R(R_n)\leq S_n}\Big(-C_n^A(A_n)-C_n^R(R_n)+E[\alpha V_{n+1}(\gamma_n x_n(1-\rho_n)+R_n+A_n)]\Big)\Big].\end{aligned}\tag{9}$$

First note that, for any given selections of A_n, R_n, and an outcome ρ_n, the objective function of the maximization problem on the right hand side of (10) is increasing in x_n, and the feasible region is strictly larger for larger x_n, thus after maximization it is also increasing in x_n. Then, by the assumption that $M_n(x_n)$ is increasing, we conclude that $V_n(x_n)$ is increasing in x_n.

The concavity of $V_n(x_n)$ follows by concavity preservation. By Assumptions 1 and 2, on $C_n^A(\cdot)$ and $C_n^R(\cdot)$, and the induction hypothesis on $V_{n+1}(.)$, the objective function of the maximization problem on the right hand side of (10) is jointly concave in (A_n,R_n,x_n). Because the feasible region constitutes a convex set, it follows from Heyman and Sobel (2004) that $V_n(x_n)$ is a concave function.

To characterize the optimal policy, we consider the unconstrained optimization problem by relaxing the constraint in (10) as follows:

$$U_n(x_n,\rho_n)=M_n(x_n)+\max_{0\leq A_n,0\leq R_n}\Big(-C_n^A(A_n)-C_n^R(R_n)+\alpha V_{n+1}(\gamma_n x_n(1-\rho_n)+R_n+A_n)\Big).\tag{10}$$

We will call this the relaxed problem, and use it for subsequent analysis. Note the difference between this problem and the original problem: problem (10) does not have either constraint $R_n\leq x_n\rho_n$ or $C_n^A(A_n)+C_n^R(R_n)\leq T_n$, and it assumes the fraction of unhappy customer ρ_n is known.

Now we can use Lemma 1 to argue the following property on the relaxed problem: The optimal solution to the problem $U_n(x_n,\rho_n)$, which we denote by $(A_n^{U*}(x_n(1-\rho_n)), R_n^{U*}(x_n(1-\rho_n))$, is decreasing in the expression $x_n(1-\rho_n)$, with slope between 0 and -1. This is an immediate application of Lemma 1, simply by switching from maximization to minimization and corresponding A_n and R_n to x and y and the functions $C_n^A(\cdot)$, $C_n^R(\cdot)$, and $-V_n(\cdot)$ to $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$. The expression $x_n(1-\rho_n)$ plays the role of the constant K.

Based on the problem given in (10) with decreasing solution vector $(A_n^{U*}(x_n(1-\rho_n)), R_n^{U*}(x_n(1-\rho_n))$, we define the following value,

$$K_n = \{w : C_n^A(A_n^{U*}(w)) + C_n^R(R_n^{U*}(w)) = S_n\}, \tag{11}$$

which will be useful in establishing the main result. If such a value K_n does not exist, we we set $K_n = 0$.

Next we consider a second intermediary problem in which we consider only the cash constraint, but not the upper bound on retention. This problem is

$$\begin{aligned} &Y_n(x_n(1-\rho_n)) \\ &= M_n(x_n) + \max_{0\le A_n, 0\le R_n, C_n^A(A_n)+C_n^R(R_n)\le S_n} \Big(-C_n^A(A_n) - C_n^R(R_n) + \alpha V_{n+1}(\gamma_n x_n(1-\rho_n) + R_n + A_n)\Big). \end{aligned} \tag{12}$$

In what follows we show that the solution to (12) is

$$(A_n^{Y*}(x_n(1-\rho_n)), R_n^{Y*}(x_n(1-\rho_n))) = (A_n^{U*}(x_n(1-\rho_n)), R_n^{U*}(x_n(1-\rho_n))$$

if $x_n(1-\rho_n) \ge K_n$ and otherwise, it is

$$(A_n^{Y*}(x_n(1-\rho_n)), R_n^{Y*}(x_n(1-\rho_n))) = (A_n^{U*}(K_n), R_n^{U*}(K_n)).$$

First consider the case when $x_n(1-\rho_n) \ge K_n$. In this situation, because $A_n^{U*}(\cdot)$ and $R_n^{U*}(\cdot)$ are decreasing, we are able to show the following:

$$C_n^A(A_n^{U*}(x_n(1-\rho_n))) + C_n^R(R_n^{U*}(x_n(1-\rho_n)) \le C_n^A(A_n^{U*}(K_n)) + C_n^R(R_n^{U*}(K_n)) = T_n.$$

Therefore in this case the solution from (10) is feasible for (12), so it is also optimal for (12).

Now suppose instead that $x_n(1-\rho_n) < K_n$, we want to show, by contradiction, that the optimal solution pair is $(A_n^{U*}(K_n), R_n^{U*}(K_n))$. Suppose for some $x_n(1-\rho_n) < K_n$, we have an optimal strategy of $A_n^{Y*}(x_n(1-\rho_n))$ and $R_n^{Y*}(x_n(1-\rho_n))$ which are not equal to $A_n^{U*}(K_n)$ and $R_n^{U*}(K_n)$ respectively. In the following we show that this would lead to contradiction.

Consider several cases. First, suppose

$$A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)) > A_n^{U*}(K_n) + R_n^{U*}(K_n). \tag{13}$$

This would contradict the optimality of the solution pair $A_n^{U*}(K_n)$ and $R_n^{U*}(K_n)$, by the following arguments:

$$\begin{aligned} &-C_n^A(A_n^{U*}(K_n)) - C_n^R(R_n^{U*}(K_n)) + E[V_{n+1}(\gamma_n K_n + A_n^{U*}(K_n) + R_n^{U*}(K_n))] \\ &= -S_n + E[V_{n+1}(\gamma_n K_n + A_n^{U*}(K_n) + R_n^{U*}(K_n))] \\ &< -S_n + E[V_{n+1}(\gamma_n K_n + A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)))] \end{aligned}$$

$$\leq -C_n^A(A_n^{Y*}(x_n(1-\rho_n))) - C_n^R(R_n^{Y*}(x_n(1-\rho_n)))$$
$$+ E[V_{n+1}(\gamma_n K_n + A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)))],$$

where the equality follows form the definition of K_n, the first inequality follows from the strict monotonicity of the value function $V_{n+1}(\cdot)$, and the second follows from the fact that the cash constraint must be satisfied by $A_n^{Y*}(x_n(1-\rho_n))$ and $R_n^{Y*}(x_n(1-\rho_n))$. Looking at the first and last expressions, the firm is strictly better off switching strategies from the pair $A_n^{U*}(K_n)$ and $R_n^{U*}(K_n)$ to the pair $A_n^{Y*}(x_n(1-\rho_n))$ and $R_n^{Y*}(x_n(1-\rho_n))$, which contradictions the optimality of the first solution pair.

Next, suppose that $A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)) < A_n^{U*}(K_n) + R_n^{U*}(K_n)$. We will prove that this contradicts the optimality of the solution pair $A_n^{Y*}(x_n(1-\rho_n))$ and $R_n^{Y*}(x_n(1-\rho_n))$. Note that

$$\begin{aligned}
&C_n^A(A_n^{U*}(K_n)) + C_n^R(R_n^{U*}(K_n)) - C_n^A(A_n^{Y*}(x_n(1-\rho_n))) - C_n^R(R_n^{Y*}(x_n(1-\rho_n)))\\
\leq\ & E[V_{n+1}(\gamma_n K_n + A_n^{U*}(K_n) + R_n^{U*}(K_n))] - E[V_{n+1}(\gamma_n K_n + A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)))]\\
<\ & E[V_{n+1}(\gamma_n x_n(1-\rho_n) + A_n^{U*}(K_n) + R_n^{U*}(K_n))]\\
&\quad - E[V_{n+1}(\gamma_n x_n(1-\rho_n) + A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)))],
\end{aligned}$$

where the first inequality comes from the optimality of the solution with K_n, and the second inequality follows from the concavity of the value function because $x_n(1-\rho_n) < K_n$. Considering the first and last expressions together, we conclude that $A_n^{U*}(K_n)$ and $R_n^{U*}(K_n)$ is a strictly better solution, contradicting the optimality of the solution pair of $A_n^{Y*}(x_n(1-\rho_n))$ and $R_n^{Y*}(x_n(1-\rho_n))$.

The final case is

$$A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)) = A_n^{U*}(K_n) + R_n^{U*}(K_n), \tag{14}$$

but $A_n^{Y*}(x_n(1-\rho_n)) \neq A_n^{U*}(K_n)$ and $R_n^{Y*}(x_n(1-\rho_n)) \neq R_n^{U*}(K_n)$. Let $A_n^{Y*}(x_n(1-\rho_n)) - A_n^{U*}(K_n) = \delta$, so that it is also true that $R_n^{Y*}(x_n(1-\rho_n)) - R_n^{U*}(K_n) = -\delta$. We first prove that

$$-C_n^A(A_n^{U*}(K_n)+\delta) - C_n^R(R_n^{U*}(K_n)-\delta)) < -C_n^A(A_n^{U*}(K_n)) - C_n^R(R_n^{U*}(K_n)))]. \tag{15}$$

Suppose instead that

$$-C_n^A(A_n^{U*}(K_n)+\delta) - C_n^R(R_n^{U*}(K_n)-\delta)) > -C_n^A(A_n^{U*}(K_n)) - C_n^R(R_n^{U*}(K_n)).$$

Then this would contradict the optimality of the solution pair $A_n^{U*}(K_n)$ and $R_n^{U*}(K_n)$, because the alternative solution of $A_n^{U*}(K_n)+\delta$ and $R_n^{U*}(K_n)-\delta$ would have strictly lower cost with identical impact on the expression inside of $V_{n+1}(\cdot)$, due to the given condition that $A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)) = A_n^{U*}(K_n) + R_n^{U*}(K_n)$. Hence, let us next suppose

$$-C_n^A(A_n^{U*}(K_n)+\delta) - C_n^R(R_n^{U*}(K_n)-\delta)) = -C_n^A(A_n^{U*}(K_n)) - C_n^R(R_n^{U*}(K_n)). \tag{16}$$

We propose an alternative solution, with strictly lower cost, again with identical impact on the expression inside of $V_{n+1}(\cdot)$ (thus, again contradicting the optimality of the given solutions). This solution is $A_n^{U*}(K_n)+\frac{\delta}{2}$ and $R_n^{U*}(K_n)-\frac{\delta}{2}$. Then,

$$-C_n^A\left(A_n^{U*}(K_n)+\frac{\delta}{2}\right) - C_n^R\left(R_n^{U*}(K_n)-\frac{\delta}{2}\right)$$

$$
\begin{aligned}
&> -\frac{1}{2}C_n^A(A_n^{U*}(K_n)) - \frac{1}{2}C_n^R(R_n^{U*}(K_n)) - \frac{1}{2}C_n^A(A_n^{U*}(K_n)+\delta) - \frac{1}{2}C_n^R(R_n^{U*}(K_n)-\delta)) \\
&= C_n^A(A_n^{U*}(K_n)) - C_n^R(R_n^{U*}(K_n)),
\end{aligned}
$$

where the inequality follows from the strictly convexity of the cost functions in Assumption 1, and the equality follows from (16). So in this case we again contradict the optimality of $A_n^{U*}(K_n)$ and $R_n^{U*}(K_n)$, because the proposed solution of $A_n^{U*}(K_n)+\frac{\delta}{2}$ and $R_n^{U*}(K_n)-\frac{\delta}{2}$ has strictly lower cost and with identical impact on the expression inside of $V_{n+1}(\cdot)$.

Consequently, we have

$$
\begin{aligned}
&-C_n^A(A_n^{Y*}(x_n(1-\rho_n))) - C_n^R(R_n^{Y*}(x_n(1-\rho_n))) \\
&\qquad + E[V_{n+1}(x_n(1-\rho_n) + A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)))] \\
&= -C_n^A(A_n^{U*}(K_n)+\delta) - C_n^R(R_n^{U*}(K_n)-\delta)) + E[V_{n+1}(x_n(1-\rho_n) + A_n^{Y*}(K_n) + R_n^{U*}(K_n))] \\
&< -C_n^A(A_n^{U*}(K_n)) - C_n^R(R_n^{U*}(K_n))) + E[V_{n+1}(x_n(1-\rho_n) + A_n^{Y*}(K_n) + R_n^{U*}(K_n))] \\
&= -C_n^A(A_n^{U*}(K_n)) - C_n^R(R_n^{U*}(K_n))) + E[V_{n+1}(x_n(1-\rho_n) + A_n^{Y*}(x_n(1-\rho_n)) + R_n^{Y*}(x_n(1-\rho_n)))],
\end{aligned}
$$

where the inequality follows from (15), and the second equality follows from condition (14). This contradicts the optimality of the solution pair $A_n^{Y*}(x_n(1-\rho_n))$ and $R_n^{Y*}(x_n(1-\rho_n))$.

Summarizing the analysis above, we have shown that the solution to (12) is given by $(A_n^{Y*}(x_n(1-\rho_n)), R_n^{Y*}(x_n(1-\rho_n)) = (A_n^{U*}(x_n(1-\rho_n)), R_n^{U*}(x_n(1-\rho_n))$ if $x_n(1-\rho_n) \geq K_n$, and it is $(A_n^{Y*}(x_n(1-\rho_n)), R_n^{Y*}(x_n(1-\rho_n))) = (A_n^{U*}(K_n), R_n^{U*}(K_n))$ otherwise.

We are now ready to prove Theorem 1. We first prove (i). Note that the relaxed problem (12) represents the optimization in problem (10) without constraint $R_n \leq \rho_n x_n$. Writing the optimization problem as sequential optimization of A_n and R_n respectively, it follows from the joint concavity in (A_n, R_n) that the objective function after optimizing A_n is a concave function of R_n. Since the resulting objective function after optimizing A_n in (12) is concave in R_n with maximizer $R_n^{Y*}(x_n(1-\rho_n))$, it is clear that the optimal solution of the original value function in (2) is $R_n^{Y*}(x_n(1-\rho_n))$ when $R_n^{Y*}(x_n(1-\rho_n)) \leq \rho_n x_n$ and , and otherwise it is $\rho_n x_n$.

Because $R_n^{Y*}(x_n(1-\rho_n)) \geq 0$ is decreasing in x_n, as x_n increases, there must exist a unique point where $R_n^{Y*}(x_n(1-\rho_n)) = x_n\rho_n$, which establishes the existence of $Q_n(\rho_n)$ from the theorem, defined by

$$
Q_n(\rho_n) = \sup\left\{x_n \geq 0;\ \rho_n x_n \leq R_n^{Y*}(x_n(1-\rho_n))\right\}, \tag{17}
$$

such that as $x_n \leq Q_n(\rho_n)$ it holds that $R_n^{Y*}(x_n(1-\rho_n)) > \rho_n x_n$; while if $x_n > Q_n(\rho_n)$ then $R_n^{Y*}(x_n(1-\rho_n)) \leq \rho_n x_n$.

Combining this insight with the characterization of the optimal policy $R_n^{Y*}(x_n(1-\rho_n))$ given above yields the optimal retention policy as stated in the theorem, which is to set R_n to $\rho_n x_n$ if $x_n \leq Q_n(\rho_n)$, set R_n to $R_n^{U*}(K_n)$ if $x_n \in (Q_n(\rho_n), \frac{K_n}{\rho_n})$ and set R_n to $R_n^{U*}(x_n(1-\rho_n))$ otherwise. Note that the second region might be empty.

To find the optimal acquisition strategy, we let $A_n^{W^*}(.)$ be defined as the maximizer of

$$W_n(x_n,\rho_n) = \max_{0\leq A_n}\Big(-C_n^A(A_n) + E[V_{n+1}(\gamma_n x_n(1-\rho_n)+\rho_n x_n + A_n)]\Big). \tag{18}$$

By the same analysis as above, it can be seen that $A_n^{W*}(x_n,\rho_n)$ is also decreasing in x_n but with slope no less than -1. Note that on the range $x_n \leq Q_n(\rho_n)$, the optimization problem for A_n in (2) can be written as

$$\max_{0\leq A_n}\Big(-C_n^A(A_n) - C_n^R(\rho_n x_n) + \alpha V_{n+1}(\gamma_n x_n(1-\rho_n)+\rho_n x_n + A_n)\Big),$$

and its optimal solution is $A_n^{W*}(x_n,\rho_n)$ just defined in (18).

With these quantities defined, we can discuss the optimal acquisition strategy on the regions discussed above. When $x_n \leq Q_n(\rho_n)$ and $R_n = \rho_n x_n$, then the optimal A_n is $A_n^{W*}(x_n,\rho_n)$, conditional on that it is within cash constraint. If not, by the convexity of the problem given in (18), the best solution is at the truncated solution. Thus, the optimal A_n on this region is $\min\{A_n^{W*}(x_n,\rho_n), T_n - C_n^R(\rho_n x_n)\}$. When $x_n \in (Q_n(\rho_n), K_n/\rho_n)$, the optimal solution is the one discussed in problem (12), which is $A_n^{U*}(K_n)$. In all other cases, the optimal solution is given by $A_n^{U*}((1-\rho_n)x_n)$.

The argument that $Q_n(\rho_n)$ is decreasing follows from the fact that $R_n^{Y*}(x_n(1-\rho_n))$ is decreasing in $x_n(1-\rho_n)$ with slope between -1 and 0, and the definition of $Q_n(\rho_n)$ in (17). To see that, suppose ρ_n were to increase by a positive number $s>0$, then $\rho_n x_n$ would increase by $s x_n$, while $R_n^{Y*}(x_n(1-\rho_n))$ would increase by a value between 0 and $s x_n$. Therefore to reach equality once again, one would need to *decrease* x_n. This establishes that $Q_n(\rho_n)$ is decreasing in ρ_n.

We next prove (ii). From part (i), we know that the optimal decision in acquisition is decreasing in x_n. Therefore, either eventually $A_n^*(x_n,\rho_n)=0$, or this value is infinite, establishing the existence of $Q_n^A(\rho_n)$ (possibly infinity). Likewise, retention spending is first increasing, and then decreasing, so eventually $R_n^*(x_n,\rho_n)$ may reach 0, showing that $Q_n^R(\rho_n)$ exists (also possibly infinity). Both are increasing in ρ_n, because the curves $R_n^{U*}(x_n(1-\rho_n))$ and $A_n^{U*}(x_n(1-\rho_n))$ are increasing in ρ_n.

To establish part (iii), we need to argue that the following expression

$$x_{n+1} - x_n = x_n(1-\rho_n)\gamma_n + R_n^*(x_n,\rho_n) + A_n^*(x_n,\rho_n) - x_n \tag{19}$$

is decreasing in x_n for any given ρ_n and γ_n, where $R_n^*(x_n,\rho_n)$ and $A_n^*(x_n,\rho_n)$ are the optimal retention and optimal acquisition decision of the original problem, which are given according to cases above. Since $A_n^*(x_n,\rho_n)$ is decreasing while $R_n^*(x_n,\rho_n)$ first increases with slope ρ_n and then decreases, we conclude that the terms combined must be decreasing in x_n.

When $x_n = 0$, the firm can only gain customers, and then the change in number of customers is decreasing for $x_n > 0$. Therefore there must exist a non-zero point $x_n^*(\rho_n)$ such that

$$E[x_{n+1}] - x_n = \begin{cases} \leq 0 & \text{if } x_n \geq x_n^*(\rho_n); \\ \geq 0 & \text{if } x_n \leq x_n^*(\rho_n). \end{cases}$$

This completes the proof of the optimal strategy. Note that it is possible that $x_n^*(\rho_n) = \infty$.

Proof of Lemma 2.

We prove by contradiction. Suppose that $(C_n^A)'(0) < (C_n^R)'(0)$, but $Q_n^A(\rho_n) < Q_n^R(\rho_n)$ for some ρ_n. This implies that for such a ρ_n, and values of $x_n \in (Q_n^A, Q_n^R)$, the firm has a strategy where $A_n^* = 0$ with $R_n^* > 0$. In this case, we show that there exists a small value $\delta > 0$, such that a better solution is $A_n = \delta$, with $R_n = R_n^* - \delta$. Because this strategy has the same impact in $V_{n+1}(\cdot)$, we need only argue that it has lower cost. The new solution would surely satisfy the cash constraint if it is indeed lower cost.

First we observe that because $C_n^R(\cdot)$ is strictly convex, and $(C_n^A)'(0) < (C_n^R)'(0)$, it holds that as $\delta > 0$ is small enough we have

$$C_n^R(R_n^*) - C_n^R(R_n^* - \delta) > C_n^R(\delta) - C_n^R(0) > C_n^A(\delta) - C_n^A(0). \tag{20}$$

These inequalities show the existence of solution $A_n = \delta$ and $R_n = R_n^* - \delta$, which as strictly lower cost, and same impact on future periods. This contradicts the original optimality of our solution. A symmetric argument establishes that $(C_n^A)'(0) > (C_n^R)'(0)$ implies that $Q_n^A(\rho_n) \leq Q_n^R(\rho_n)$.

We finally consider the case $(C_n^A)'(0) = (C_n^R)'(0)$, and prove that in this case it must hold that $Q_n^A(\rho_n) = Q_n^R(\rho_n)$ for all $\rho_n > 0$. Suppose $Q_n^A(\rho_n) \neq Q_n^R(\rho_n)$ for some ρ_n. Without loss of generality, suppose $0 \leq Q_n^A(\rho_n) < Q_n^R(\rho_n)$. This implies that there exists an $x_n \in (Q_n^A(\rho_n), Q_n^R(\rho_n))$, such that $R_n^*(x_n, \rho_n) > 0$ and $A_n^*(x_n, \rho_n) = 0$. We claim that there exists a small number $\delta > 0$, such that a solution with $R_n = R_n^*(x_n, \rho_n) - \delta$, and $A_n = \delta$ is *strictly* superior. This would contradict the optimality of the original solution.

Observe that by the strict convexity of $C_n^R(\cdot)$, we have that:

$$(C_n^R)'(R_n^*(x_n, \rho_n)) > (C_n^R)'(0) = (C_n^A)'(0).$$

Therefore, by continuity we can find a small $\delta > 0$ such that

$$(C_n^R)'(R_n^*(x_n, \rho_n) - \delta) > (C_n^A)'(\delta).$$

This implies, by convexity of $C_n^R(\cdot)$ and $C_n^A(\cdot)$, that

$$C_n^R(R_n^*(x_n, \rho_n)) - C_n^R(R_n^*(x_n, \rho_n) - \delta) > C_n^A(\delta) - C_n^A(0).$$

Since solutions $R_n = R_n^*(x_n, \rho_n) - \delta$ and $A_n = \delta$ have the same impact to future periods, this proves that the proposed solution has strictly lower cost, contradicting the optimality of the original solution. A symmetric argument holds to contradiction if it were true that $0 \leq Q_n^R(\rho_n) < Q_n^A(\rho_n)$.

Proof of Corollary 1.

The fact that $\lim_{x_{n+1} \to \infty} M'_{n+1}(x_{n+1}) \geq \kappa > 0$, allows us to prove that the value function is κ increasing in period $n+1$, meaning that $V_{n+1}(x_{n+1} + s) - V_{n+1}(x_{n+1}) \geq s\kappa$ for any $s > 0$. We can see this from the value function as follows.

$$E[V_{n+1}(x_{n+1} + s)] - E[V_{n+1}(x_{n+1})] = E[M_{n+1}(x_{n+1} + s)] - E[M_{n+1}(x_{n+1})]$$

$$
\begin{aligned}
&+E\Big[\max_{0\le R_{n+1}\le (x_{n+1}+s)\rho_{n+1},0\le A_{n+1}}(-C^A_{n+1}(A_{n+1})-C^R_{n+1}(R_{n+1})\\
&+\alpha E[V_{n+2}((x_{n+1}+s)(1-\rho_{n+1})\gamma_{n+1}+R_{n+1}+A_{n+1})])\Big]\\
&-E\Big[\max_{0\le R_{n+1}\le x_{n+1}\rho_{n+1},0\le A_{n+1}}(-C^A_{n+1}(A_{n+1})-C^R_{n+1}(R_{n+1})\\
&+\alpha E[V_{n+2}(x_{n+1}(1-\rho_{n+1})\gamma_{n+1}+R_{n+1}+A_{n+1})])\Big]\\
&\ge s\kappa,
\end{aligned}
$$

where the last inequality comes from the fact that $M_{n+1}(x_{n+1}+s)-M_{n+1}(x_{n+1})\ge s\kappa$, while the other terms are non-negative, because $V_{n+2}(\cdot)$ is increasing, and the case with $s+x_{n+1}$ has a larger feasible region.

By contradiction we now show that a point at which $R^*_n(x_n,\rho_n)=0$ can never exist unless $\rho_n x_n=0$ or $(C^R_n)'(0)\ge (C^A_n)'((C^A_n)^{-1}(S_n))$, because the firm is better off by spending a small incremental amount more in retention. Suppose, on the contrary, it holds that the optimal strategies $(R^*_n(x_n,\rho_n),A^*_n(x_n,\rho_n))$ has $R^*_n(x_n,\rho_n)=0$. We will show that in this case there exists a small $\delta>0$ such that the solution would be improved if $R^*_n(x_n,\rho_n)=\delta$, contradicting the optimality of the original solution. This additional small increase is feasible because of our two conditions, which say that the retention constraint is not tight with given decision (because $\rho_n x_n>0$, and neither is the cash constraint (because if $(C^R_n)'(0)<(C^A_n)'((C^A_n)^{-1}(S_n))$ with total acquisition and retention spending at S_n, the firm could save by shifting some money from acquisition to retention). Using the fact that $V_{n+1}(\cdot)$ is κ increasing, we have

$$
\begin{aligned}
C^R_n(\delta)-C^R_n(0) &< \delta\alpha\kappa &(21)\\
&\le \alpha E[V_{n+1}(x_n(1-\rho_n)\gamma_n+\delta+A^*_n(x_n,\rho_n))]-\alpha E[V_{n+1}(x_n(1-\rho_n)\gamma_n+A^*_n(x_n,\rho_n))]. &(22)
\end{aligned}
$$

These inequalities follow from the fact that $C^R_n(\cdot)$ is strictly convex, $(C^R_n)'(0)\le\alpha\kappa$, and $V_{n+1}(\cdot)$ is κ increasing, as we have discussed.

The inequalities (21) implies that a strategy of no retention and $A^*_n(x_n,\rho_n))$ is acquisition is strictly dominated by one with the same acquisition and a small amount $\delta>0$ in retention, contradicting the optimality of former solution. This implies $Q^R_n(\rho_n)=\infty$.

Proof of Corollary 2.

Due the symmetric relationship between A_n and R_n, similar argument as those of Corollary 1 can be used to prove this result. We omit the details.

Proof of Theorem 2.

The optimality equation for this more general case is

$$
\begin{aligned}
&V_n(x_n)\\
=\ &E_{\rho_n}\Big[M_n(x_n)\\
&+\max_{0\le A_n,0\le R_n\le\rho_n x_n,C^A_n(A_n)+C^R_n(R_n)\le S_n}\Big(-C^A_n(A_n)-C^R_n(R_n)+\alpha E\big[V_{n+1}(\gamma_n x_n(1-\rho_n)+\epsilon^1_n R_n+\epsilon^2_n A_n)\big]\Big)\Big].
\end{aligned}
$$

The objective function of the maximization problem above is easily seen to be jointly concave in (A_n,R_n,x_n), and the constraint is a convex set of (A_n,R_n,x_n), hence it follows from the preservation property that $V_n(x_n)$

is concave in x_n. By induction, it is also easy to show that $V_n(x_n)$ is increasing in x_n, since both the objective function and the feasible region in the optimization are increasing in x_n. Consider the relaxed problem that, for any realization of ρ_n,

$$\begin{aligned} & Y_n(x_n,\rho_n) \\ = \ & M_n(x_n) + \max_{0\le A_n, 0\le R_n, C_n^A(A_n)+C_n^R(R_n)\le S_n} \Big(-C_n^A(A_n) - C_n^R(R_n) + \alpha E\big[V_{n+1}(\gamma_n x_n(1-\rho_n) + \epsilon_n^1 R_n + \epsilon_n^2 A_n)\big]\Big) \\ = \ & M_n(x_n) + \max_{0\le R_n, C_n^R(R_n)\le S_n} \Big\{-C_n^R(R_n) + g\big((1-\rho_n)x_n, R_n\big)\Big\}, \end{aligned}$$

where

$$g((1-\rho_n)x_n, R_n) = \max_{0\le A_n, C_n^A(A_n)+C_n^R(R_n)\le S_n} \Big(-C_n^A(A_n) + \alpha E\big[V_{n+1}(\gamma_n x_n(1-\rho_n) + \epsilon_n^1 R_n + \epsilon_n^2 A_n)\big]\Big)$$

is jointly concave in $((1-\rho_n)x_n, R_n)$. Therefore, if the optimal $R_n^{Y*}(x_n,\rho_n) < \rho_n x_n$, then the solution the the relaxed problem is feasible, thus it optimal. Otherwise by joint concavity, the optimal solution is $(A_n, R_n) = (A_n^{W*}(x_n,\rho_n), \rho_n x_n)$, where A_n^{W*} is the optimal solution of

$$W_n(x_n,\rho_n) = \max_{0\le A_n, C_n^A(A_n)\le S_n - C_n^R(\rho_n x_n)} \Big(-C_n^A(A_n) + \alpha E\big[V_{n+1}(\gamma_n x_n(1-\rho_n) + \epsilon_n^1 \rho_n x_n + \epsilon_n^2 A_n)\big]\Big). \qquad (23)$$

This finishes the proof for Theorem 2.