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# Strong convergence to the homogenized limit of elliptic equations with random coefficients II

Joseph G. Conlon and Arash Fahim


### Abstract

Consider a discrete uniformly elliptic divergence form equation on the $d \geqslant 3$ dimensional lattice $\mathbf{Z}^d$ with random coefficients. In Conlon and Spencer [*Trans. Amer. Math. Soc.*, http://www.math.lsa.umich.edu/∼conlon/paper/hom10.pdf], rate of convergence results in homogenization and estimates on the difference between the averaged Green's function and the homogenized Green's function for random environments which satisfy a Poincaré inequality were obtained. Here, these results are extended to certain environments in which correlations can have arbitrarily small power law decay. These environments are simply related via a convolution to environments which do satisfy a Poincaré inequality.


## 1. *Introduction*

In this paper, we continue the study of solutions to divergence form elliptic equations with random coefficients begun in [**3**]. In [**3**], we were concerned with solutions $u(x, \eta, \omega)$ to the equation

$$\eta u(x,\eta,\omega) + \nabla^* \mathbf{a}(\tau_x\omega)\nabla u(x,\eta,\omega) = h(x), \quad x \in \mathbf{Z}^d, \ \omega \in \Omega, \tag{1.1}$$

where $\eta > 0$, $\mathbf{Z}^d$ is the $d$-dimensional integer lattice and $(\Omega, \mathcal{F}, P)$ is a probability space equipped with measure-preserving translation operators $\tau_x : \Omega \to \Omega$, $x \in \mathbf{Z}^d$. In (1.1), we take $\nabla$ to be the discrete gradient operator defined by

$$\nabla\phi(x) = (\nabla_1\phi(x), \ldots, \nabla_d\phi(x)), \quad \nabla_i\phi(x) = \phi(x + \mathbf{e}_i) - \phi(x), \tag{1.2}$$

where the vector $\mathbf{e}_i \in \mathbf{Z}^d$ has 1 as the $i$th coordinate and 0 for the other coordinates, $1 \leqslant i \leqslant d$. Then $\nabla$ is a $d$-dimensional *column* operator, with adjoint $\nabla^*$ which is a $d$-dimensional *row* operator.

The function $\mathbf{a} : \Omega \to \mathbf{R}^{d(d+1)/2}$ from $\Omega$ to the space of symmetric $d \times d$ matrices satisfies the quadratic form inequality

$$\lambda I_d \leqslant \mathbf{a}(\omega) \leqslant \Lambda I_d, \quad \omega \in \Omega, \tag{1.3}$$

where $I_d$ is the identity matrix in $d$ dimensions and $\Lambda, \lambda$ are positive constants.

It is well known [**8**, **12**, **15**] that if the translation operators $\tau_x$, $x \in \mathbf{Z}^d$, are ergodic on $\Omega$, then solutions to the random equation (1.1) converge to solutions of a constant coefficient equation under suitable scaling. Thus, suppose that $f : \mathbf{R}^d \to \mathbf{R}$ is a $C^\infty$ function with compact support and, for $\varepsilon$ satisfying $0 < \varepsilon \leqslant 1$, let $u_\varepsilon(x, \eta, \omega)$ be the solution to (1.1) with $h(x) = \varepsilon^2 f(\varepsilon x)$, $x \in \mathbf{Z}^d$. Then $u_\varepsilon(x/\varepsilon, \varepsilon^2\eta, \omega)$ converges with probability 1 as $\varepsilon \to 0$ to a function $u_{\text{hom}}(x, \eta)$, $x \in \mathbf{R}^d$, which is the solution to the constant coefficient elliptic partial differential equation (PDE)

$$\eta u_{\text{hom}}(x,\eta) - \nabla \mathbf{a}_{\text{hom}} \nabla u_{\text{hom}}(x,\eta) = f(x), \quad x \in \mathbf{R}^d, \tag{1.4}$$

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where the $d \times d$ symmetric matrix $\mathbf{a}_{\rm hom}$ satisfies the quadratic form inequality (1.3). This homogenization result can be viewed as a kind of central limit theorem, and our goal in [**3**] was to show that the result can be strengthened for certain probability spaces $(\Omega, \mathcal{F}, P)$. In particular, we extended a result of Yurinskii [**14**] which gives a rate of convergence in homogenization,

$$\sup_{x\in\varepsilon\mathbf{Z}^d} \langle |u_\varepsilon(x/\varepsilon,\varepsilon^2\eta,\cdot) - u_{\rm hom}(x,\eta)|^2\rangle \leqslant C\varepsilon^\alpha \quad \text{for } 0<\varepsilon\leqslant 1. \tag{1.5}$$

Yurinskii's assumption on $(\Omega, \mathcal{F}, P)$ is a quantitative strong mixing condition. To describe it, we first observe that any environment $\Omega$ can be considered to be a set of fields $\omega : \mathbf{Z}^d \to \mathbf{R}^n$ with $n \leqslant d(d+1)/2$, where the translation operators $\tau_x,\ x \in \mathbf{Z}^d$, act as $\tau_x\omega(z) = \omega(x+z),\ z \in \mathbf{Z}^d$ and $\mathbf{a}(\omega) = \tilde{\mathbf{a}}(\omega(0))$ for some function $\tilde{\mathbf{a}} : \mathbf{R}^n \to \mathbf{R}^{d(d+1)/2}$. Now let $\chi(\cdot)$ be a positive decreasing function on $\mathbf{R}^+$ such that $\lim_{q\to\infty}\chi(q) = 0$. The quantitative strong mixing condition is given in terms of the function $\chi(\cdot)$ as follows: For any subsets $A, B$ of $\mathbf{Z}^d$ and events $\Gamma_A,\ \Gamma_B \subset \Omega$, which depend, respectively, only on variables $\omega(x),\ x \in A$, and $\omega(y),\ y \in B$, then

$$|P(\Gamma_A\cap\Gamma_B) - P(\Gamma_A)P(\Gamma_B)| \leqslant \chi\left(\inf_{x\in A, y\in B}|x-y|\right). \tag{1.6}$$

In the proof of (1.5), he requires the function $\chi(\cdot)$ to have power law decay, that is, $\lim_{q\to\infty} q^\beta\chi(q) = 0$ for some $\beta > 0$. Evidently, (1.6) trivially holds if the $\omega(x),\ x \in \mathbf{Z}^d$, are independent variables. Recently, Caffarelli and Souganidis [**2**] have obtained rates of convergence results in the homogenization of fully nonlinear PDE under the quantitative strong mixing condition (1.6). In their case, the function $\chi(q)$ is assumed to decay logarithmically in $q$ to 0, and correspondingly the rate of convergence in homogenization that is obtained is also logarithmic in $\varepsilon$. In their methodology, a stronger assumption on the function $\chi(\cdot)$, for example, power law decay, does not yield a stronger rate of convergence in homogenization.

In [**3**], we followed an approach to the problem of obtaining rates of convergence in homogenization pioneered by Naddaf and Spencer [**11**]. They obtained rate of convergence results under the assumption that a Poincaré inequality holds for the random environment. Specifically, consider the measure space $(\tilde{\Omega}, \tilde{\mathcal{F}})$ of vector fields $\tilde{\omega} : \mathbf{Z}^d \to \mathbf{R}^k$, where $\tilde{\mathcal{F}}$ is the minimal Borel algebra such that each $\tilde{\omega}(x) : \tilde{\Omega} \to \mathbf{R}^k$ is Borel measurable, $x \in \mathbf{Z}^d$. For any $C^1$ function $G : \tilde{\Omega} \to \mathbf{C}$, we denote by $d_{\tilde{\omega}}G(y;\tilde{\omega}) = \partial G(\tilde{\omega})/\partial\tilde{\omega}(y),\ y \in \mathbf{Z}^d$, its gradient. Thus, for fixed $\tilde{\omega} \in \tilde{\Omega}$, the gradient $d_{\tilde{\omega}}G(\cdot;\tilde{\omega})$ is a mapping from $\mathbf{Z}^d$ to $\mathbf{C}^k$, which has Euclidean norm $\|d_{\tilde{\omega}}G(\cdot;\tilde{\omega})\|_2$ in $\ell^2(\mathbf{Z}^d, \mathbf{C}^k)$. A probability measure $\tilde{P}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ satisfies a Poincaré inequality if there is a constant $K_{\tilde{P}} > 0$ such that

$$\mathrm{Var}[G(\cdot)] \leqslant K_{\tilde{P}}\langle \|d_{\tilde{\omega}}G(\cdot;\tilde{\omega})\|_2^2\rangle \quad \text{for all } C^1 \text{ functions } G : \tilde{\Omega} \longrightarrow \mathbf{C}. \tag{1.7}$$

In [**11**], it is assumed that $\tilde{P}$ is translation invariant, that is, the translation operators $\tau_x,\ x \in \mathbf{Z}^d$, acting by $\tau_x\tilde{\omega}(z) = \tilde{\omega}(x+z),\ z \in \mathbf{Z}^d$, are measure preserving, and that the Poincaré inequality (1.7) holds. Rate of convergence results are then obtained provided $\mathbf{a}(\omega) = \tilde{\mathbf{a}}(\tilde{\omega}(0))$ in (1.1), where the function $\tilde{\mathbf{a}} : \mathbf{R}^k \to \mathbf{R}^{d(d+1)/2}$ is $C^1$ and has bounded derivative, in addition to satisfying (1.3).

Gloria and Otto [**6**, **7**] have developed much further the methodology of Naddaf and Spencer, under the assumption that the environment satisfies a *weak* Poincaré inequality. This weak Poincaré inequality holds for an environment in which the variables $\mathbf{a}(\tau_x\omega),\ x \in \mathbf{Z}^d$, are independent, whereas the inequality (1.7) in general does not. These papers are concerned with establishing an optimal rate of convergence for finite length scale approximations to the homogenized coefficient $\mathbf{a}_{\rm hom}$ of (1.4). The recent paper [**5**] uses a similar approach to obtain optimal estimates on the variance of $u_\varepsilon(x/\varepsilon, \varepsilon^2\eta, \cdot)$.

If the translation invariant probability measure $\tilde{P}$ is Gaussian, then the measure is determined by the 2-point correlation function $\Gamma : \mathbf{Z}^d \to \mathbf{R}^k \otimes \mathbf{R}^k$ defined by $\Gamma(x) = \langle\tilde{\omega}(x)\tilde{\omega}(0)^*\rangle,\ x \in \mathbf{Z}^d$, where $\tilde{\omega}(\cdot) \in \mathbf{R}^k$ is assumed to be a column vector and the superscript

$*$ denotes adjoint. Defining the Fourier transform of a function $h : \mathbf{Z}^d \to \mathbf{C}$ by

$$\hat{h}(\xi) = \sum_{x \in \mathbf{Z}^d} h(x)\, e^{ix\cdot\xi}, \quad \xi \in [-\pi, \pi]^d, \tag{1.8}$$

one can easily see that the Poincaré inequality (1.7) holds if and only if $\hat{\Gamma} \in L^\infty([-\pi,\pi]^d)$. Hence, if $\Gamma(\cdot)$ is summable on $\mathbf{Z}^d$, then (1.7) holds. Suppose now that, for some $\beta > 0$, the function $\Gamma(x) \simeq 1/|x|^\beta$ for large $|x|$. Then the inequality (1.6) holds for a function $\chi(\cdot)$ with power law decay $\beta$, but the Poincaré inequality does not hold in general unless $\beta > d$.

The main goal of the present paper is to show that the approach to obtaining rate of convergence results in homogenization based on using the Poincaré inequality can be extended to some environments for which $\Gamma(\cdot)$ is not summable. In particular, they include certain Gaussian environments for which $\Gamma(x) \simeq 1/|x|^\beta$ at large $|x|$ and $\beta > 0$ can be arbitrarily small. Hence, our approach bridges a gap between the Yurinskii criterion (1.6), which requires only $\beta > 0$, and the Naddaf–Spencer criterion (1.7), which corresponds to $\beta > d$. The idea is to consider environments defined by $\mathbf{a}(\omega) = \tilde{\mathbf{a}}(\omega(0))$, where $\omega : \mathbf{Z}^d \to \mathbf{R}^n$ is a convolution $\omega(\cdot) = h * \tilde{\omega}(\cdot)$, $\tilde{\omega} \in \tilde{\Omega}$. The function $h : \mathbf{Z}^d \to \mathbf{R}^n \otimes \mathbf{R}^k$ from $\mathbf{Z}^d$ to $n \times k$ matrices is assumed to be $q$ summable for some $q < 2$, and the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ to satisfy the Poincaré inequality (1.7).

In [**3**], we proved rate of convergence results for a massive Euclidean field theory environment $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. The environment consists of fields $\phi : \mathbf{Z}^d \to \mathbf{R}$ with measure $\tilde{P}$ formally given by

$$\exp\left[-\sum_{x \in \mathbf{Z}^d} V(\nabla\phi(x)) + \frac{1}{2} m^2 \phi(x)^2\right] \prod_{x \in \mathbf{Z}^d} d\phi(x) \Big/ \text{normalization}, \tag{1.9}$$

where $V : \mathbf{R}^d \to \mathbf{R}$ is a uniformly convex function and $m > 0$. Then $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ with measure (1.9) satisfies the inequality (1.7). In the Gaussian case when $V(\cdot)$ is quadratic, one has that the correlation function $\langle \phi(x)\phi(0)\rangle = G_{m^2}(x)$, $x \in \mathbf{Z}^d$, where the Green's function $G_\nu(\cdot)$ is the solution to

$$\nu G_\nu(x) + \nabla^* V'' \nabla G_\nu(x) = \delta(x), \quad x \in \mathbf{Z}^d. \tag{1.10}$$

Hence, $\langle \phi(x)\phi(0)\rangle$ decays exponentially in $|x|$ as $|x| \to \infty$. Taking $\omega(\cdot) = h * \phi(\cdot)$ for some $h \in \ell^q(\mathbf{Z}^d)$, we have that

$$\Gamma(x) = \langle \omega(x)\omega(0)\rangle = \sum_{y, y' \in \mathbf{Z}^d} h(x-y)h(-y')G_{m^2}(y-y'), \tag{1.11}$$

and so if $1 \leqslant q \leqslant 2$, then $\langle \omega(0)^2 \rangle < \infty$. If $\beta > 0$ and $h(z) = 1/[1 + |z|^{d/2+\beta/2}]$, $z \in \mathbf{Z}^d$, then $h \in \ell^q(\mathbf{Z}^d)$ for $q > 2d/(d+\beta)$. We easily see from (1.11) that $\Gamma(x) \simeq |x|^{-\beta}$ as $|x| \to \infty$.

The limit as $m \to 0$ of the measure (1.9) is a probability measure $\tilde{P}$ on gradient fields $\tilde{\omega} : \mathbf{Z}^d \to \mathbf{R}^d$, where formally $\tilde{\omega}(x) = \nabla\phi(x)$, $x \in \mathbf{Z}^d$, a result first shown by Funaki and Spohn [**4**]. This massless field theory measure satisfies a Poincaré inequality (1.7) for all $d \geqslant 1$. In the case $d = 1$, the measure has a simple structure since then the variables $\tilde{\omega}(x)$, $x \in \mathbf{Z}$, are independent and identically distributed. For $d \geqslant 3$, the gradient field theory measure induces a measure on fields $\phi : \mathbf{Z}^d \to \mathbf{R}$ which is simply the limit of the measures (1.9) as $m \to 0$. For $d = 1, 2$, the $m \to 0$ limit of the measures (1.9) on fields $\phi : \mathbf{Z}^d \to \mathbf{R}$ does not exist. If $d \geqslant 3$, then $\langle \phi(x)\phi(0)\rangle \simeq |x|^{-(d-2)}$ as $|x| \to \infty$ for the massless field theory. Observe now that

$$\phi(x) = \sum_{y \in \mathbf{Z}^d} [\nabla G_0(x-y)]^* \nabla\phi(y) = h * \tilde{\omega}(x), \quad x \in \mathbf{Z}^d, \tag{1.12}$$

where $G_0(\cdot)$ is the Green's function for (1.10) with $\nu = 0, V'' = I_d$. Since $h : \mathbf{Z}^d \to \mathbf{R}^d$ in (1.12) is $q$ summable for any $q > d/(d-1)$, the environment of massless fields $\phi : \mathbf{Z}^d \to \mathbf{R}$ with $d \geqslant 3$ is of the form $\phi = h * \tilde{\omega}$, where $h : \mathbf{Z}^d \to \mathbf{R}^d$ is $q$ summable for some $q < 2$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ satisfies the Poincaré inequality (1.7).

Rather than attempt to formulate a general theorem for environments $\omega = h * \tilde{\omega}$ where $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ satisfies the Poincaré inequality (1.7), we shall rigorously prove only that the results obtained in [**3**] hold for massless fields $\phi : \mathbf{Z}^d \to \mathbf{R}$ with $d \geqslant 3$. In § 2, we indicate the generality of our argument by showing that the proof of [**3**, Proposition 5.3] formally extends to environments $\omega = h * \tilde{\omega}$. Our first theorem concerns the rate of convergence (1.5) in homogenization.

THEOREM 1.1. *Let $\tilde{\mathbf{a}} : \mathbf{R} \to \mathbf{R}^{d(d+1)/2}$ be a $C^1$ function on $\mathbf{R}$ with values in the space of symmetric $d \times d$ matrices, which satisfies the quadratic form inequality (1.3) and has bounded first derivative $D\tilde{\mathbf{a}}(\cdot)$, so $\|D\tilde{\mathbf{a}}(\cdot)\|_\infty < \infty$. For $d \geqslant 3$, let $(\Omega, \mathcal{F}, P)$ be the probability space of massless fields $\phi(\cdot)$ determined by the limit of the uniformly convex measures (1.9) as $m \to 0$, and set $\mathbf{a}(\cdot)$ in (1.1) to be $\mathbf{a}(\phi) = \tilde{\mathbf{a}}(\phi(0))$, $\phi \in \Omega$. Let $f : \mathbf{R}^d \to \mathbf{R}$ be a $C^\infty$ function of compact support; $u_\varepsilon(x, \eta, \omega)$ be the corresponding solution to (1.1) with $h(x) = \varepsilon^2 f(\varepsilon x), x \in \mathbf{Z}^d$ and $u_{\text{hom}}(x, \eta)$, $x \in \mathbf{R}^d$, be the solution to (1.4). Then there is a constant $\alpha > 0$ depending only on $d, \Lambda/\lambda$ and a constant $C$ depending only on $\eta, d, \Lambda, \lambda, \|D\tilde{\mathbf{a}}(\cdot)\|_\infty, f(\cdot)$ such that (1.5) holds.*

Our second theorem concerns *point-wise* convergence at large length scales of the averaged Green's function for (1.1) to the homogenized Green's function for (1.4), which is uniform as $\eta \to 0$. The averaged Green's function $G_{\mathbf{a},\eta}(x)$, $x \in \mathbf{Z}^d$, for (1.1) is defined by $G_{\mathbf{a},\eta}(x) = \langle u(x, \eta, \cdot)\rangle$, where $h(\cdot)$ in (1.1) is the Kronecker delta function $h(x) = 0$ if $x \neq 0$ and $h(0) = 1$.

THEOREM 1.2. *With the same environment as in the statement of Theorem 1.1, let $G_{\mathbf{a}_{\text{hom}},\eta}(x)$, $x \in \mathbf{R}^d$, be the Green's function for the homogenized equation (1.4). Then there are constants $\alpha, \gamma > 0$ depending only on $d$ and the ratio $\Lambda/\lambda$ of the constants $\lambda, \Lambda$ of (1.3), and a constant $C$ depending only on $\|D\tilde{\mathbf{a}}(\cdot)\|_\infty, \Lambda/\lambda, d$ such that*

$$|G_{\mathbf{a},\eta}(x) - G_{\mathbf{a}_{\text{hom}},\eta}(x)| \leqslant \frac{C}{\Lambda(|x|+1)^{d-2+\alpha}}\, e^{-\gamma\sqrt{\eta/\Lambda}|x|}, \quad x \in \mathbf{Z}^d - \{0\}, \tag{1.13}$$

$$|\nabla G_{\mathbf{a},\eta}(x) - \nabla G_{\mathbf{a}_{\text{hom}},\eta}(x)| \leqslant \frac{C}{\Lambda(|x|+1)^{d-1+\alpha}}\, e^{-\gamma\sqrt{\eta/\Lambda}|x|}, \quad x \in \mathbf{Z}^d - \{0\}, \tag{1.14}$$

$$|\nabla\nabla G_{\mathbf{a},\eta}(x) - \nabla\nabla G_{\mathbf{a}_{\text{hom}},\eta}(x)| \leqslant \frac{C}{\Lambda(|x|+1)^{d+\alpha}}\, e^{-\gamma\sqrt{\eta/\Lambda}|x|}, \quad x \in \mathbf{Z}^d - \{0\}, \tag{1.15}$$

*provided $0 < \eta \leqslant \Lambda$.*

It was shown in [**3**] that Theorem 1.2 follows once one has established some regularity properties of the Fourier transform of the averaged Green's function $G_{\mathbf{a},\eta}(\cdot)$. We establish these properties (Hypothesis 3.1) in § 3 for the massless field theory environment. As observed in [**3**], the proof of Theorem 1.1 follows along the same lines as the proof of Theorem 1.2, and is somewhat simpler. We therefore have omitted its proof here. The problem of determining the optimal value of $\alpha$ in (1.5) is a subtle one. In our proof for an environment $\omega = h * \tilde{\omega}$ with $h(\cdot)$ being $q$ summable with $q < 2$, the exponent $\alpha$ depends on $q$ as well as the ellipticity ratio $\Lambda/\lambda$ for the PDE (1.1). If $q \to 2$, then $\alpha \to 0$ in our approach, which corresponds to $\alpha \to 0$ when $\beta \to 0$ in the Yurinskii approach.

## 2. *Variance estimate on the solution to a PDE on $\Omega$*

We recall some definitions from [3]. For $\xi \in \mathbf{R}^d$ and $1 \leqslant j \leqslant d$, we define the $\xi$ derivative of a measurable function $\psi : \Omega \to \mathbf{C}$ in the $j$ direction by $\partial_{j,\xi}$, and its adjoint by $\partial_{j,\xi}^*$, where

$$\begin{aligned} \partial_{j,\xi}\psi(\omega) &= e^{-i\mathbf{e}_j\cdot\xi}\psi(\tau_{\mathbf{e}_j}\omega) - \psi(\omega), \\ \partial_{j,\xi}^*\psi(\omega) &= e^{i\mathbf{e}_j\cdot\xi}\psi(\tau_{-\mathbf{e}_j}\omega) - \psi(\omega). \end{aligned} \tag{2.1}$$

We also define a $d$-dimensional column $\xi$ gradient operator $\partial_\xi$ by $\partial_\xi = (\partial_{1,\xi}, \dots, \partial_{d,\xi})$, which has adjoint $\partial_\xi^*$ given by the row operator $\partial_\xi^* = (\partial_{1,\xi}^*, \dots, \partial_{d,\xi}^*)$. Let $\mathcal{H}(\Omega)$ be the Hilbert space of measurable functions $\Psi : \Omega \to \mathbf{C}^d$ with norm $\|\Psi\|_{\mathcal{H}(\Omega)}$ given by $\|\Psi\|^2_{\mathcal{H}(\Omega)} = \langle |\Psi(\cdot)|_2^2\rangle$, where $|\cdot|_2$ is the Euclidean norm on $\mathbf{C}^d$. Then there is a unique row vector solution $\Phi(\xi,\eta,\omega) = (\Phi_1(\xi,\eta,\omega),\dots,\Phi_d(\xi,\eta,\omega))$ to the equation

$$\eta\Phi(\xi,\eta,\omega) + \partial_\xi^*\mathbf{a}(\omega)\partial_\xi\Phi(\xi,\eta,\omega) = -\partial_\xi^*\mathbf{a}(\omega), \quad \eta > 0, \;\; \xi \in \mathbf{R}^d, \;\; \omega \in \Omega, \tag{2.2}$$

such that $\Phi(\xi,\eta,\cdot)v \in L^2(\Omega)$ for any $v \in \mathbf{C}^d$. Furthermore, $\Phi(\xi,\eta,\cdot)v \in L^2(\Omega)$ satisfies the inequality

$$\eta\|\Phi(\xi,\eta,\cdot)v\|^2_{L^2(\Omega)} + \lambda\|\partial_\xi\Phi(\xi,\eta,\cdot)v\|^2_{\mathcal{H}(\Omega)} \leqslant \Lambda^2|v|^2/\lambda. \tag{2.3}$$

Letting $\mathcal{P}$ denote the projection orthogonal to the constant function, our generalization of [3, Proposition 5.3] is as follows.

PROPOSITION 2.1. *Suppose that $\mathbf{a}(\cdot)$ in (2.2) is given by $\mathbf{a}(\omega) = \tilde{\mathbf{a}}(\omega(0))$, where $\tilde{\mathbf{a}} : \mathbf{R}^n \to \mathbf{R}^{d(d+1)/2}$ is a $C^1$ $d\times d$ symmetric matrix-valued function satisfying the quadratic form inequality (1.3) and $\|D\tilde{\mathbf{a}}(\cdot)\|_\infty < \infty$. The random field $\omega : \mathbf{Z}^d \to \mathbf{R}^n$ is a convolution $\omega(\cdot) = h * \tilde{\omega}(\cdot)$ of an $n\times k$ matrix-valued function $h : \mathbf{Z}^d \to \mathbf{R}^n \otimes \mathbf{R}^k$ and a random field $\tilde{\omega} : \mathbf{Z}^d \to \mathbf{R}^k$. The function $h$ is assumed to be $p_0$ summable for some $p_0$ with $1 \leqslant p_0 < 2$ and the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ of the fields $\tilde{\omega} : \mathbf{Z}^d \to \mathbf{R}^k$ to satisfy the Poincaré inequality (1.7). Then there exists $p_1$ depending only on $d, \Lambda/\lambda, p_0$ and satisfying $1 < p_1 < 2$, such that, for $g \in L^p(\mathbf{Z}^d, \mathbf{C}^d \otimes \mathbf{C}^d)$ with $1 \leqslant p \leqslant p_1$ and $v \in \mathbf{C}^d$,*

$$\left\| \mathcal{P}\sum_{x\in\mathbf{Z}^d} g(x)\partial_\xi\Phi(\xi,\eta,\tau_x\cdot)v \right\|_{\mathcal{H}(\Omega)} \leqslant \frac{CK_{\tilde{P}}\|D\tilde{\mathbf{a}}(\cdot)\|_\infty|v|}{\Lambda}\|h\|_{p_0}\|g\|_p, \tag{2.4}$$

*where $K_{\tilde{P}}$ is the Poincaré constant in (1.7) and $C$ is a constant depending only on $d, n, k, \Lambda/\lambda, p_0$.*

*Proof.* From (1.7), we have that

$$\left\| \mathcal{P}\sum_{x\in\mathbf{Z}^d} g(x)\partial_\xi\Phi(\xi,\eta,\tau_x\cdot)v \right\|^2_{\mathcal{H}(\Omega)} \leqslant K_{\tilde{P}} \sum_{z\in\mathbf{Z}^d} \left\langle \left| \frac{\partial}{\partial\tilde{\omega}(z)} \sum_{x\in\mathbf{Z}^d} g(x)\partial_\xi\Phi(\xi,\eta,\tau_x\cdot)v \right|_2^2 \right\rangle. \tag{2.5}$$

From the chain rule, we see that

$$\frac{\partial}{\partial\tilde{\omega}(z)}\partial_\xi\Phi(\xi,\eta,\tau_x\cdot)v = \sum_{y\in\mathbf{Z}^d} \left[\frac{\partial}{\partial\omega(y)}\partial_\xi\Phi(\xi,\eta,\tau_x\cdot)v\right] h(y-z). \tag{2.6}$$

Hence, using the translation invariance of the probability measure $\tilde{P}$ on $\tilde{\Omega}$, we conclude from (2.5) and (2.6) that

$$\left\| \mathcal{P} \sum_{x\in\mathbf{Z}^d} g(x)\partial_\xi \Phi(\xi,\eta,\tau_x\cdot)v \right\|^2_{\mathcal{H}(\Omega)} \leqslant K_{\tilde{P}} \sum_{z\in\mathbf{Z}^d} \left\langle \left| \sum_{x\in\mathbf{Z}^d} g(x) \sum_{y\in\mathbf{Z}^d} \left[\tau_{-z}\frac{\partial}{\partial\omega(y)}\partial_\xi\Phi(\xi,\eta,\tau_x\cdot)v\right] h(y-z)\right|^2_2 \right\rangle. \tag{2.7}$$

For a differentiable function $\psi:\Omega\to\mathbf{C}$, we denote its gradient by $d_\omega\psi:\mathbf{Z}^d\times\Omega\to\mathbf{C}$ so that $d_\omega\psi(y;\omega)=\partial\psi(\omega)/\partial\omega(y),\ y\in\mathbf{Z}^d,\omega\in\Omega$. The gradient operator $d_\omega$ does not commute with the translation operators $\tau_x,\ x\in\mathbf{Z}^d$, and in fact we have that

$$\frac{\partial}{\partial\omega(y)}\psi(\tau_x\omega) = d_\omega\psi(y-x;\tau_x\omega), \quad x,y\in\mathbf{Z}^d. \tag{2.8}$$

Defining now the function $u:\mathbf{Z}^d\times\Omega\to\mathbf{C}^k$ by

$$u(z,\omega) = e^{-iz\cdot\xi}\sum_{y\in\mathbf{Z}^d}\left[d_\omega\Phi(y;\xi,\eta,\tau_z\omega)v\right]h(y+z), \tag{2.9}$$

we conclude from (2.8) that

$$\nabla u(x-z,\omega) = e^{i(z-x)\cdot\xi}\sum_{y\in\mathbf{Z}^d}\left[\tau_{-z}\frac{\partial}{\partial\omega(y)}\partial_\xi\Phi(\xi,\eta,\tau_x\omega)v\right]h(y-z). \tag{2.10}$$

Hence, (2.7) becomes

$$\left\| \mathcal{P} \sum_{x\in\mathbf{Z}^d} g(x)\partial_\xi \Phi(\xi,\eta,\tau_x\cdot)v \right\|^2_{\mathcal{H}(\Omega)} \leqslant K_{\tilde{P}} \sum_{z\in\mathbf{Z}^d} \left\langle \left| \sum_{x\in\mathbf{Z}^d} g(x)\, e^{i(x-z)\cdot\xi}\nabla u(x-z,\cdot)\right|^2_2 \right\rangle. \tag{2.11}$$

In [**3**], we also defined the $\xi$ derivative of a measurable function $\psi:\mathbf{Z}^d\times\Omega\to\mathbf{C}$ in the $j$ direction by $D_{j,\xi}$, and its adjoint by $D^*_{j,\xi}$, where

$$\begin{aligned} D_{j,\xi}\psi(x,\omega) &= e^{-i\mathbf{e}_j\cdot\xi}\psi(x-\mathbf{e}_j,\tau_{\mathbf{e}_j}\omega)-\psi(x,\omega),\\ D^*_{j,\xi}\psi(x,\omega) &= e^{i\mathbf{e}_j\cdot\xi}\psi(x+\mathbf{e}_j,\tau_{-\mathbf{e}_j}\omega)-\psi(x,\omega). \end{aligned} \tag{2.12}$$

The corresponding $d$-dimensional column $\xi$ gradient operator $D_\xi$ is then given by $D_\xi=(D_{1,\xi},\dots,D_{d,\xi})$, and it has adjoint $D^*_\xi$ given by the row operator $D^*_\xi=(D^*_{1,\xi},\dots,D^*_{d,\xi})$. We see from (2.8) that these operators satisfy the identity

$$\frac{\partial}{\partial\omega(y)}\partial_\xi\psi(\omega) = D_\xi d_\omega\psi(y;\omega), \quad y\in\mathbf{Z}^d,\ \omega\in\Omega, \tag{2.13}$$

for differentiable functions $\psi:\Omega\to\mathbf{C}$. A similar relationship holds for the adjoints $\partial^*_\xi, D^*_\xi$. Hence, on taking the gradient of equation (2.2) with respect to $\omega(\cdot)$, we conclude from (2.13) that

$$\begin{aligned} &\eta d_\omega\Phi(y;\xi,\eta,\omega)v + D^*_\xi\tilde{\mathbf{a}}(\omega(0))D_\xi d_\omega\Phi(y;\xi,\eta,\omega)v\\ &\qquad = -D^*_\xi[\delta(y)D\tilde{\mathbf{a}}(\omega(0))\{v+\partial_\xi\Phi(\xi,\eta,\omega)v\}] \quad \text{for } y\in\mathbf{Z}^d,\ \omega\in\Omega. \end{aligned} \tag{2.14}$$

Evidently (2.14) holds with $\omega\in\Omega$ replaced by $\tau_z\omega$ for any $z\in\mathbf{Z}^d$. We now multiply (2.14) with $\tau_z\omega$ in place of $\omega$ on the right by $e^{-iz\cdot\xi}h(y+z)$ and sum with respect to $y\in\mathbf{Z}^d$. It then follows from (2.9) that

$$\eta u(z,\omega)+\nabla^*\tilde{\mathbf{a}}(\omega(z))\nabla u(z,\omega) = -\nabla^* f(z,\omega), \tag{2.15}$$

where the function $f : \mathbf{Z}^d \times \Omega \to \mathbf{C}^d \otimes \mathbf{C}^k$ is given by the formula

$$f(z,\omega) = D\tilde{\mathbf{a}}(\omega(z))\{v + \partial_\xi \Phi(\xi,\eta,\tau_z\omega)v\}\, e^{-iz\cdot\xi} h(z). \tag{2.16}$$

Now from (2.3), it follows that $\partial_\xi \Phi(\xi,\eta,\cdot)v \in \mathcal{H}(\Omega)$ and $\|\partial_\xi \Phi(\xi,\eta,\cdot)v\|_{\mathcal{H}(\Omega)} \leqslant \Lambda|v|/\lambda$. Hence, if $h \in L^2(\mathbf{Z}^d, \mathbf{R}^n \otimes \mathbf{R}^k)$, then the function $f$ is in $L^2(\mathbf{Z}^d \times \Omega, \mathbf{C}^d \otimes \mathbf{C}^k)$ and $\|f\|_2 \leqslant \|D\tilde{\mathbf{a}}(\cdot)\|_\infty(1 + \Lambda/\lambda)|v|\|h\|_2$. We see from (2.15) that if $f \in L^2(\mathbf{Z}^d \times \Omega, \mathbf{C}^d \otimes \mathbf{C}^k)$, then $\nabla u$ is in $L^2(\mathbf{Z}^d \times \Omega, \mathbf{C}^d \otimes \mathbf{C}^k)$ and $\|\nabla u\|_2 \leqslant \|f\|_2/\lambda$. It follows then from (2.11) and Young's inequality for convolutions [**13**] that (2.4) holds with $p_0 = 2$ and $p = 1$.

To prove the inequality for some $p > 1$, we use a version of Meyer's theorem [**10**] for solutions of elliptic equations on $\mathbf{Z}^d$. Lattice versions of Meyer's theorem were already used in [**11**] and more recently in [**7**]. For any $1 < q < \infty$, we consider the function $f$ as a mapping $f : \mathbf{Z}^d \to L^2(\Omega, \mathbf{C}^d \otimes \mathbf{C}^k)$ with norm defined by

$$\|f\|_q^q = \sum_{z\in\mathbf{Z}^d} \|f(z,\cdot)\|_2^q, \tag{2.17}$$

where $\|f(z,\cdot)\|_2$ is the norm of $f(z,\cdot) \in L^2(\Omega, \mathbf{C}^d \otimes \mathbf{C}^k)$. It was observed in [**13**] that the Calderon–Zygmund theorem applies to Fourier multiplier operators of functions on $\mathbf{R}^d$ with range in a Hilbert space. One can similarly see that it applies to Fourier multiplier operators of functions on $\mathbf{Z}^d$ with range in a Hilbert space. We conclude therefore that there exists $q_0$ depending only on $d, \Lambda/\lambda$ with $1 < q_0 < 2$ such that if $\|f\|_{q_0} < \infty$, then $\|\nabla u\|_q \leqslant 2\|f\|_q/\lambda$ for $q_0 \leqslant q \leqslant 2$. If $h$ is $p_0$ integrable with $p_0 < 2$, then we can take $\max[p_0, q_0] = q_1 \leqslant q \leqslant 2$. It follows again from (2.11) and Young's inequality for convolutions [**13**] that (2.4) holds with $p_1 = 2q_1/(3q_1 - 2)$. □

## 3. *Proof of Theorem* 1.2

The basic approach of [**3**] is to use the fact that the solution to (1.1) can be expressed by a Fourier inversion formula in terms of the solution to the equation

$$\eta\Phi(\xi,\eta,\omega) + \mathcal{P}\partial_\xi^* \mathbf{a}(\omega)\partial_\xi \Phi(\xi,\eta,\omega) = -\mathcal{P}\partial_\xi^* \mathbf{a}(\omega), \quad \eta > 0,\ \ \xi \in \mathbf{R}^d,\ \ \omega \in \Omega, \tag{3.1}$$

where $\mathcal{P}$ is the projection orthogonal to the constant. It is easy to see that, just like the solution to (2.2), the solution to (3.1) also satisfies the inequality (2.3). If $\xi = 0$, then the solution $\Phi(\xi,\eta,\omega)$ to (2.2) has zero mean, so $\langle \Phi(0,\eta,\cdot)\rangle = 0$. Hence, the solutions to (2.2) and (3.1) coincide if $\xi = 0$ but are in general different. For $\xi \in \mathbf{R}^d$ and $\eta > 0$, let $e(\xi) \in \mathbf{C}^d$ be the vector $e(\xi) = \partial_\xi 1$ and $q(\xi,\eta)$ be the $d \times d$ matrix

$$q(\xi,\eta) = \langle \mathbf{a}(\cdot)\rangle + \langle \mathbf{a}(\cdot)\partial_\xi \Phi(\xi,\eta,\cdot)\rangle, \tag{3.2}$$

where $\Phi(\xi,\eta,\omega)$ is the solution to (3.1). The solution to (1.1) is then given in [**3**] by the formula

$$u(x,\eta,\omega) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \frac{\hat{h}(\xi)\, e^{-i\xi\cdot x}}{\eta + e(\xi)^* q(\xi,\eta) e(\xi)} [1 + \Phi(\xi,\eta,\tau_x\omega)e(\xi)]\, d\xi, \quad x \in \mathbf{Z}^d,\ \ \omega \in \Omega. \tag{3.3}$$

If the environment $(\Omega, \mathcal{F}, P)$ is ergodic, then the limit $\lim_{\eta\to 0} q(0,\eta) = \mathbf{a}_{\mathrm{hom}}$ exists, and $\mathbf{a}_{\mathrm{hom}}$ is the diffusion matrix for the homogenized equation (1.4). It follows from (3.3) that the Fourier transform $\hat{G}_{\mathbf{a},\eta}(\xi),\ \xi \in [-\pi,\pi]^d$, of the averaged Green's function $G_{\mathbf{a},\eta}(x),\ x \in \mathbf{Z}^d$, is given by the formula

$$\hat{G}_{\mathbf{a},\eta}(\xi) = 1/[\eta + e(\xi)^* q(\xi,\eta) e(\xi)] \quad \text{for } \xi \in [-\pi,\pi]^d. \tag{3.4}$$

In [**3**], it was shown (see especially § 7) that Theorem 1.2 is a consequence of the following.

Hypothesis 3.1. For $\xi \in \mathbf{C}^d$ denote its real part by $\Re\xi \in \mathbf{R}^d$ and its imaginary part by $\Im\xi \in \mathbf{R}^d$ so that $\xi = \Re\xi + i\Im\xi$. Then there exist positive constants $C_1$ and $\alpha \leqslant 1$ depending only on $d$ and $\Lambda/\lambda$, such the function $q(\xi,\eta)$, $\xi \in \mathbf{R}^d, \eta > 0$, has an analytic continuation to the region $|\Im\xi| \leqslant C_1\sqrt{\eta/\Lambda}$ and

$$\|q(\xi',\eta') - q(\xi,\eta)\| \leqslant C\Lambda[|\xi'-\xi|^\alpha + |(\eta'-\eta)/\Lambda|^{\alpha/2}],$$
$$0 < \eta \leqslant \eta' \leqslant \Lambda, \quad \xi',\xi \in \mathbf{C}^d \text{ with } |\Im\xi|, |\Im\xi'| \leqslant C_1\sqrt{\eta/\Lambda}, \tag{3.5}$$

where $C$ is a constant depending on the environment and the function $\mathbf{a}(\cdot)$.

Here, we will prove that Hypothesis 3.1 holds for the massless field theory environment $(\Omega, \mathcal{F}, P)$ of Theorem 1.1. To do this, we recall some operators defined in [**3**]. For any $g \in \mathcal{H}(\Omega)$, let $\psi(\xi,\eta,\omega)$ be the solution to the equation

$$\frac{\eta}{\Lambda}\psi(\xi,\eta,\omega) + \partial_\xi^*\partial_\xi\psi(\xi,\eta,\omega) = \partial_\xi^* g(\omega), \quad \eta > 0, \ \xi \in \mathbf{R}^d, \ \omega \in \Omega. \tag{3.6}$$

The operator $T_{\xi,\eta}$ on $\mathcal{H}(\Omega)$ is defined by $T_{\xi,\eta}g(\cdot) = \partial_\xi\psi(\xi,\eta,\cdot)$. It also has the representation

$$T_{\xi,\eta}g(\omega) = \sum_{x\in\mathbf{Z}^d} \{\nabla\nabla^* G_{\eta/\Lambda}(x)\}^* \exp[-ix\cdot\xi] g(\tau_x\omega), \quad \omega \in \Omega, \tag{3.7}$$

where $G_\nu(\cdot)$ is the Green's function defined by (1.10) with $V''(\cdot) \equiv I_d$. It easily follows from (3.6) that $T_{\xi,\eta}$ is a bounded operator on $\mathcal{H}(\Omega)$ with $\|T_{\xi,\eta}\|_{\mathcal{H}(\Omega)} \leqslant 1$, provided $\xi \in \mathbf{R}^d, \eta > 0$. Furthermore, by Conlon and Spencer [**3**, Lemma 2.1] the function $\xi \to T_{\xi,\eta}$ from $\mathbf{R}^d$ to the Banach space of bounded linear operators on $\mathcal{H}(\Omega)$ has an analytic continuation to a strip $|\Im\xi| < C\sqrt{\eta/\Lambda}$, where $C$ is a constant depending only on $d$. Let $\mathbf{b}$ be the $d\times d$ matrix-valued function $\mathbf{b}(\omega) = I_d - \mathbf{a}(\omega)/\Lambda$, $\omega \in \Omega$, whence (1.3) implies the quadratic form inequality $0 \leqslant \mathbf{b}(\cdot) \leqslant (1-\lambda/\Lambda)I_d$. We define, for $\eta > 0, r = 1,2,\ldots$, and $\Im\xi \in \mathbf{R}^d$ with $|\Im\xi| < C\sqrt{\eta/\Lambda}$, an operator $T_{r,\eta,\Im\xi}$ from functions $g : \mathbf{Z}^d \to \mathbf{C}^d \otimes \mathbf{C}^d$ to periodic functions $T_{r,\eta,\Im\xi}g : [-\pi,\pi]^d \times \Omega \to \mathbf{C}^d \otimes \mathbf{C}^d$ by

$$T_{r,\eta,\Im\xi}g(\Re\xi,\cdot) = \sum_{x\in\mathbf{Z}^d} g(x)\tau_x\mathcal{P}\mathbf{b}(\cdot)[\mathcal{P}T_{\xi,\eta}\mathbf{b}(\cdot)]^{r-1}, \quad \text{where } \xi = \Re\xi + i\Im\xi. \tag{3.8}$$

For $1 \leqslant p < \infty$, let $\ell^p(\mathbf{Z}^d, \mathbf{C}^d \otimes \mathbf{C}^d)$ be the Banach space of $d\times d$ matrix-valued functions $g : \mathbf{Z}^d \to \mathbf{C}^d \otimes \mathbf{C}^d$ with norm $\|g\|_p$ defined by

$$\|g\|_p^p = \sup_{v\in\mathbf{C}^d:|v|=1} \sum_{x\in\mathbf{Z}^d} |g(x)v|_2^p, \tag{3.9}$$

where $|g(x)v|_2$ is the Euclidean norm of the vector $g(x)v \in \mathbf{C}^d$. We similarly define the space $L^\infty([-\pi,\pi]^d \times \Omega, \mathbf{C}^d \otimes \mathbf{C}^d)$ of $d\times d$ matrix-valued functions $g : [-\pi,\pi]^d \times \Omega \to \mathbf{C}^d \otimes \mathbf{C}^d$ with norm $\|g\|_\infty$ defined by

$$\|g\|_\infty = \sup_{v\in\mathbf{C}^d:|v|=1} \left[\sup_{\zeta\in[-\pi,\pi]^d} \|g(\zeta,\cdot)v\|_{\mathcal{H}(\Omega)}\right]. \tag{3.10}$$

Since $\|T_{\xi,\eta}\|_{\mathcal{H}(\Omega)} \leqslant 1$ if $\xi \in \mathbf{R}^d, \eta > 0$, then it follows from (3.7) and (3.8) that if $\Im\xi = 0$, then $T_{r,\eta,\Im\xi}$ is a bounded operator from $\ell^1(\mathbf{Z}^d, \mathbf{C}^d \otimes \mathbf{C}^d)$ to $L^\infty([-\pi,\pi]^d \times \Omega, \mathbf{C}^d \otimes \mathbf{C}^d)$ with norm $\|T_{r,\eta,\Im\xi}\|_{1,\infty} \leqslant (1-\lambda/\Lambda)^r$. In the following, we show that $T_{r,\eta,\Im\xi}$ is a bounded operator from $\ell^p(\mathbf{Z}^d, \mathbf{C}^d \otimes \mathbf{C}^d)$ to $L^\infty([-\pi,\pi]^d \times \Omega, \mathbf{C}^d \otimes \mathbf{C}^d)$ for some $p > 1$ in the case of the environment of Theorem 1.1 and estimate its norm $\|T_{r,\eta,\Im\xi}\|_{p,\infty}$. This extends [**3**, Lemma 5.1] to the massless field case.

LEMMA 3.1. *Let $(\Omega,\mathcal{F},P)$ be an environment of massless fields $\phi:\mathbf{Z}^d\to\mathbf{R}$ with $d\geqslant 3$, and $\tilde{\mathbf{a}}:\mathbf{R}\to\mathbf{R}^{d(d+1)/2}$ be as in the statement of Theorem 1.1. Set $\mathbf{a}(\phi)=\tilde{\mathbf{a}}(\phi(0))$, $\phi\in\Omega$. Then there exists $p_0(\Lambda/\lambda)$ with $1<p_0(\Lambda/\lambda)<2$ depending only on $d$ and $\Lambda/\lambda$, and positive constants $C_1(\Lambda/\lambda),C_2(\Lambda/\lambda)$ depending only on $d$ and $\Lambda/\lambda$ such that*

$$\|T_{r,\eta,\Im\xi}\|_{p,\infty}\leqslant\frac{C_2(\Lambda/\lambda)r\|D\tilde{\mathbf{a}}(\cdot)\|_\infty}{\Lambda\sqrt{\lambda}}(1-\lambda/\Lambda)^{(r-1)/2}\quad\text{for } 0<\eta\leqslant\Lambda,$$
$$|\Im\xi|<C_1(\Lambda/\lambda)\sqrt{\eta/\Lambda},\tag{3.11}$$

*provided $1\leqslant p\leqslant p_0(\Lambda/\lambda)$.*

*Proof.* It will be sufficient for us to bound $\|T_{r,\eta,\Im\xi}g\|_\infty$ in terms of $\|g\|_p$ for $g:\mathbf{Z}^d\to\mathbf{C}^d\otimes\mathbf{C}^d$ of finite support. Let $Q$ be a cube in $\mathbf{Z}^d$ containing the support of the function $g(\cdot)$ and $(\Omega_Q,\mathcal{F}_Q,P_{Q,m})$ be the probability space of periodic functions $\phi:Q\to\mathbf{R}$ with measure

$$\exp\left[-\sum_{x\in Q}V(\nabla\phi(x))+\frac{1}{2}m^2\phi(x)^2\right]\prod_{x\in Q}d\phi(x)\bigg/\text{normalization},\tag{3.12}$$

where we assume $m>0$ and $V:\mathbf{R}^d\to\mathbf{R}$ is $C^2$ with $\mathbf{a}(\cdot)=V''(\cdot)$ satisfying the quadratic form inequality (1.3). We denote by $\tilde{\Omega}_Q$ the space of periodic fields $\tilde{\omega}:Q\to\mathbf{R}^d$ and let $F:\tilde{\Omega}_Q\times\Omega_Q\to\mathbf{C}$ be a $C^1$ function which, for some constants $A,B$, satisfies the inequality

$$|F(\tilde{\omega},\phi)|+|d_{\tilde{\omega}}F(y;\tilde{\omega},\phi)|+|d_\phi F(y;\tilde{\omega},\phi)|\leqslant A\exp[B\{\|\tilde{\omega}\|_2+\|\phi\|_2\}],$$
$$y\in Q,\ \ \tilde{\omega}\in\tilde{\Omega}_Q,\ \ \phi\in\Omega_Q.\tag{3.13}$$

Let $\langle\cdot\rangle_{\Omega_Q,m}$ denote expectation with respect to the measure (3.12) and note that the Hessian of $\sum_{x\in Q}V(\nabla\phi(x))+\frac{1}{2}m^2\phi(x)^2$ is bounded below in the quadratic form sense by the constant operator $-\lambda\Delta+m^2$. It follows then from the Brascamp–Lieb inequality [**1**] that

$$\begin{aligned}
&\mathrm{Var}_{\Omega_Q,m}[F(\nabla\phi,\phi)]\\
&\quad\leqslant\langle[\nabla^*d_{\tilde{\omega}}F(\nabla\phi,\phi)+d_\phi F(\nabla\phi,\phi)]^*(-\lambda\Delta+m^2)^{-1}[\nabla^*d_{\tilde{\omega}}F(\nabla\phi,\phi)+d_\phi F(\nabla\phi,\phi)]\rangle_{\Omega_Q,m}\\
&\quad\leqslant 2\langle[\nabla^*d_{\tilde{\omega}}F(\nabla\phi,\phi)]^*(-\lambda\Delta+m^2)^{-1}[\nabla^*d_{\tilde{\omega}}F(\nabla\phi,\phi)]\rangle_{\Omega_Q,m}\\
&\qquad+2\langle[d_\phi F(\nabla\phi,\phi)]^*(-\lambda\Delta+m^2)^{-1}[d_\phi F(\nabla\phi,\phi)]\rangle_{\Omega_Q,m}.
\end{aligned}\tag{3.14}$$

We conclude from (3.14) that the Poincaré inequality

$$\mathrm{Var}_{\Omega_Q,m}[F(\nabla\phi,\phi)]\leqslant\frac{2}{\lambda}\langle\|d_{\tilde{\omega}}F(\nabla\phi,\phi)\|_2^2\rangle_{\Omega_Q,m}+\frac{2}{m^2}\langle\|d_\phi F(\nabla\phi,\phi)\|_2^2\rangle_{\Omega_Q,m}\tag{3.15}$$

holds. We will show using (3.15) that $\|T_{r,\eta,\Im\xi}g\|_\infty$ is bounded in terms of $\|g\|_p$ if the environment is the probability space $(\Omega_Q,\mathcal{F}_Q,P_{Q,m})$. The result will then follow by taking first $Q\to\mathbf{Z}^d$ and then $m\to 0$.

Let us suppose that the cube $Q$ is centered at the origin in $\mathbf{Z}^d$ with side of length $L$, where $L$ is an even integer. Let $G_\nu:\mathbf{Z}^d\to\mathbf{R}$ be the solution to (1.10) with $V''(\cdot)=I_d$. Then there exist positive constants $C,\gamma$ depending only on $d$ such that $G_\nu$ satisfies the inequality

$$G_\nu(x)+(|x|+1)|\nabla G_\nu(x)|\leqslant\frac{C}{(|x|+1)^{d-2}}\,e^{-\gamma\sqrt{\nu}|x|}\quad\text{for } d\geqslant 3,\ \ 0<\nu\leqslant 1,\ \ x\in\mathbf{Z}^d.\tag{3.16}$$

The inequality (3.16) can be proved by using the Fourier inversion formula (see [**9**] and references therein). We denote by $G_{\nu,Q}:Q\to\mathbf{R}$ the corresponding Green's function for the periodic lattice $Q$, so

$$G_{\nu,Q}(x)=\sum_{n\in\mathbf{Z}^d}G_\nu(x+Ln),\quad x\in Q.\tag{3.17}$$

Then any periodic function $\phi : Q \to \mathbf{R}$ can be written as

$$\phi(x) = \sum_{y\in Q} [\nabla G_{\nu,Q}(x-y)]^* \nabla\phi(y) + \sum_{y\in Q} \nu G_{\nu,Q}(x-y)\phi(y), \quad x \in Q. \tag{3.18}$$

We take $\nu = 1/L^2$ in (3.18) to obtain a representation

$$\phi(\cdot) = h_Q * \tilde{\omega}(\cdot) + k_Q * \phi(\cdot), \tag{3.19}$$

where $h_Q = [h_{Q,1}, \dots, h_{Q,d}]$ is a row vector and the operation $*$ denotes convolution on the periodic lattice $Q$. It follows from (3.16) and (3.17) that if $q > d/(d-1)$, then there is a constant $C_q$ depending only on $q, d$ such that $\|h_Q\|_q \leqslant C_q$. Similarly, if $q \geqslant 1$ and $q \neq d/(d-2)$, then $\|k_Q\|_q \leqslant C_q / \min[L^{d(1-1/q)}, L^2]$.

We first prove (3.11) when $r = 1$. For the environment $(\Omega_Q, \mathcal{F}_Q, P_{Q,m})$, we have from (3.19) that

$$T_{1,\eta,\Im\xi}\, g(\Re\xi, \phi) = \sum_{x\in Q} g(x) \mathcal{P}\tilde{\mathbf{b}}(h_Q * \tilde{\omega}(x) + k_Q * \phi(x)). \tag{3.20}$$

Let $\mathcal{H}_m(\Omega_Q)$ be the Hilbert space of functions $f : \Omega_Q \to \mathbf{C}^d$ which are square integrable with respect to the measure $P_{Q,m}$. It follows from (3.15) that if $v \in \mathbf{C}^d$, then the norm of $T_{1,\eta,\Im\xi}\, g(\Re\xi, \cdot)v \in \mathcal{H}_m(\Omega_Q)$ is bounded as

$$\begin{aligned} \|T_{1,\eta,\Im\xi}\, g(\Re\xi, \cdot)v\|^2_{\mathcal{H}_m(\Omega_Q)} \leqslant{}& \frac{2}{\lambda} \sum_{z\in Q} \sum_{j=1}^{d} \left\| \sum_{x\in Q} g(x) h_{Q,j}(x-z) D\tilde{\mathbf{b}}(\phi(x))v \right\|^2_{\mathcal{H}_m(\Omega_Q)} \\ &+ \frac{2}{m^2} \sum_{z\in Q} \left\| \sum_{x\in Q} g(x) k_Q(x-z) D\tilde{\mathbf{b}}(\phi(x))v \right\|^2_{\mathcal{H}_m(\Omega_Q)}. \end{aligned} \tag{3.21}$$

Since $d \geqslant 3$, we can choose $q$ such that $d/(d-1) < q < 2$ and $q \neq d/(d-2)$. It then follows from (3.21) and Young's inequality for convolutions that, for $p = 2q/(3q-2) > 1$,

$$\|T_{1,\eta,\Im\xi}\, g(\Re\xi, \cdot)v\|^2_{\mathcal{H}_m(\Omega_Q)} \leqslant C_q \|g\|_p^2 \|D\tilde{\mathbf{b}}(\cdot)\|^2_\infty |v|^2 \left[\frac{1}{\lambda} + \frac{1}{m^2 L^{a(q)}}\right], \tag{3.22}$$

where $a(q) = 2\min[d(1-1/q), 2]$. Let $(\Omega, \mathcal{F}, P_m)$ be the probability space of fields $\phi : \mathbf{Z}^d \to \mathbf{R}$ with measure $P_m$ given by (1.9). Proposition 5.1 of [**3**] enables us to take the limit of (3.22) as $Q \to \mathbf{Z}^d$ to obtain the inequality

$$\|T_{1,\eta,\Im\xi}\, g(\Re\xi, \cdot)v\|^2_{\mathcal{H}_m(\Omega_{\mathbf{Z}^d})} \leqslant C_q \|g\|_p^2 \|D\tilde{\mathbf{b}}(\cdot)\|^2_\infty |v|^2/\lambda, \tag{3.23}$$

for the environment $(\Omega, \mathcal{F}, P_m)$. Finally, [**3**, Proposition 6.1] enables us to take the limit of (3.23) as $m \to 0$ provided $d \geqslant 3$. We have proved (3.11) when $r = 1$.

To prove the result for $r > 1$, we consider the environment $(\Omega_Q, \mathcal{F}_Q, P_{Q,m})$ and write as in [**3**]

$$T_{r,\eta,\Im\xi}\, g(\Re\xi, \phi)v = \mathcal{P} \sum_{x\in Q} g(x) \tilde{\mathbf{b}}(\phi(x)) \partial_\xi F_r(\xi, \eta, \tau_x\phi), \quad \phi(\cdot) \in \Omega_Q. \tag{3.24}$$

For $\xi \in \mathbf{R}^d, \eta > 0$, the functions $F_r(\xi, \eta, \phi)$ are defined inductively by

$$\begin{aligned} &\frac{\eta}{\Lambda} F_r(\xi,\eta,\phi) + \partial_\xi^* \partial_\xi F_r(\xi,\eta,\phi) = \mathcal{P}\partial_\xi^*[\tilde{\mathbf{b}}(\phi(0))\partial_\xi F_{r-1}(\xi,\eta,\phi)], \quad r > 2, \\ &\frac{\eta}{\Lambda} F_2(\xi,\eta,\phi) + \partial_\xi^* \partial_\xi F_2(\xi,\eta,\phi) = \mathcal{P}\partial_\xi^*[\tilde{\mathbf{b}}(\phi(0))v]. \end{aligned} \tag{3.25}$$

From [**3**, Lemma 2.1], it follows that for fixed $\eta > 0$ the function $F_r(\xi, \eta, \phi)$, $\xi \in \mathbf{R}^d$, has an analytic continuation into the strip $|\Im\xi| < C_1\sqrt{\eta/\Lambda}$ for some constant $C_1$ depending only on

$d$. Furthermore, $\partial_\xi F_r \in \mathcal{H}_m(\Omega_Q)$ and

$$\|\partial_\xi F_r(\xi,\eta,\cdot)\|_{\mathcal{H}_m(\Omega_Q)} \leqslant (1-\lambda/\Lambda)^{r-1}[1+C_2|\Im\xi|^2/(\eta/\Lambda)]^{r-1}|v| \quad \text{for } |\Im\xi| < C_1\sqrt{\eta/\Lambda},\ \ r \geqslant 2, \tag{3.26}$$

where the constant $C_2$ depends only on $d$. Note that (3.26) implies that $\|T_{r,\eta,\Im\xi}\|_{1,\infty}$ is finite, provided $|\Im\xi| < C_1\sqrt{\eta/\Lambda}$.

Using the representation (3.19) for $\phi(\cdot)$, we can consider the $F_r,\ r \geqslant 2$, defined by (3.25) as functions of $\tilde\omega(\cdot)$ and $\phi(\cdot)$, which we denote by $\tilde F_r(\xi,\eta,\tilde\omega,\phi)$. Observe now that, for $1 \leqslant j \leqslant d$,

$$\begin{aligned}\frac{\partial}{\partial\tilde\omega_j(z)} &\sum_{x\in Q} g(x)\tilde{\mathbf{b}}(h_Q * \tilde\omega(x) + k_Q * \phi(x))\partial_\xi \tilde F_r(\xi,\eta,\tau_x\tilde\omega,\tau_x\phi)\\ &= \sum_{x\in Q} g(x)h_{Q,j}(x-z)D\tilde{\mathbf{b}}(\phi(x))\partial_\xi F_r(\xi,\eta,\tau_x\phi)\\ &\quad + \sum_{x\in Q} g(x)\tilde{\mathbf{b}}(\phi(x))\frac{\partial}{\partial\tilde\omega_j(z)}\partial_\xi\tilde F_r(\xi,\eta,\tau_x\tilde\omega,\tau_x\phi).\end{aligned} \tag{3.27}$$

For $\xi \in \mathbf{R}^d$ and $u : Q \to \mathbf{C}$, we denote by $\nabla_\xi u : Q \to \mathbf{C}^d$ the function $\nabla_\xi u(z) = [\nabla_{1,\xi}u(z),\dots,\nabla_{j,\xi}u(z)],\ z \in Q$, where $\nabla_{j,\xi}u(z) = e^{-i\mathbf{e}_j\cdot\xi}u(z+\mathbf{e}_j) - u(z),\ z \in Q,\ j = 1,\dots,d$. Now let $u_{r,j} : \mathbf{R}^d \times \mathbf{R}^+ \times Q \times \Omega_Q \to \mathbf{C}$ be given by the formula

$$u_{r,j}(\xi,\eta,z,\phi) = \sum_{y\in Q} d_\phi F_r(y;\xi,\eta,\tau_z\phi)h_{Q,j}(y+z). \tag{3.28}$$

Then, as in (2.6) and (2.10), we have that

$$\begin{aligned}\nabla_\xi u_{r,j}(\xi,\eta,x-z,\phi) &= \sum_{y\in Q} \tau_{-z}\frac{\partial}{\partial\phi(y)}\partial_\xi F_r(\xi,\eta,\tau_x\phi)h_{Q,j}(y-z)\\ &= \tau_{-z}\frac{\partial}{\partial\tilde\omega_j(z)}\partial_\xi\tilde F_r(\xi,\eta,\tau_x\tilde\omega,\tau_x\phi).\end{aligned} \tag{3.29}$$

Similarly to (2.15) and (2.16), we see that $u_{r,j}(\xi,\eta,z,\phi)$ satisfies the equation

$$\frac{\eta}{\Lambda}u_{r,j}(\xi,\eta,z,\phi) + \nabla_\xi^*\nabla_\xi u_{r,j}(\xi,\eta,z,\phi) = \mathcal{P}\nabla_\xi^* f_{r,j}(\xi,\eta,z,\phi), \tag{3.30}$$

where the function $f_{r,j} : \mathbf{R}^d \times \mathbf{R}^+ \times Q \times \Omega_Q \to \mathbf{C}^d$ is given by the formula

$$\begin{aligned}f_{2,j}(\xi,\eta,z,\phi) &= D\tilde{\mathbf{b}}(\phi(z))vh_{Q,j}(z),\\ f_{r,j}(\xi,\eta,z,\phi) &= D\tilde{\mathbf{b}}(\phi(z))\partial_\xi F_{r-1}(\xi,\eta,\tau_z\phi)h_{Q,j}(z) + \tilde{\mathbf{b}}(\phi(z))\nabla_\xi u_{r-1,j}(\xi,\eta,z,\phi), \quad r > 2.\end{aligned} \tag{3.31}$$

Suppose that $\xi \in \mathbf{R}^d,\ \eta > 0$ and $g : Q \to \mathbf{C}^d$ is a periodic function on $Q$. We define the function $\tilde T_{\xi,\eta}g : Q \to \mathbf{C}^d$ by $\tilde T_{\xi,\eta}g(z) = \nabla_\xi u(z),\ z \in Q$, where $u : Q \to \mathbf{C}$ is the solution to the equation

$$\frac{\eta}{\Lambda}u(z) + \nabla_\xi^*\nabla_\xi u(z) = \nabla_\xi^* g(z), \quad z \in Q. \tag{3.32}$$

It follows easily from (3.32) that the norm of $\tilde T_{\xi,\eta}$ acting on $\ell^2(Q,\mathbf{C}^d)$ satisfies $\|\tilde T_{\xi,\eta}\|_2 \leqslant 1$. Observing that (3.32) is a special case of (3.6), we apply [**3**, Lemma 2.1]. Hence, there are positive constants $C_1, C_2$ depending only on $d$ such that the function $\xi \to \tilde T_{\xi,\eta}$ from $\mathbf{R}^d$ to linear maps on $\ell^2(Q,\mathbf{C}^d)$ has an analytic continuation to the region $|\Im\xi| \leqslant C_1\sqrt{\eta/\Lambda}$ and $\|\tilde T_{\xi,\eta}\|_2 \leqslant (1 + C_2|\Im\xi|^2/[\eta/\Lambda])$ in this region. We can also adapt the proof of the Calderon–Zygmund theorem [**13**] to further conclude that if $|\Im\xi| \leqslant C_1\sqrt{\eta/\Lambda}$, then the norm of $\tilde T_{\xi,\eta}$ on $\ell^q(Q,\mathbf{C}^d)$ for $1 < q < \infty$ satisfies the inequality $\|\tilde T_{\xi,\eta}\|_q \leqslant (1 + C_2|\Im\xi|^2/[\eta/\Lambda])(1+\delta(q))$, where $\delta(q)$ depends only on $d, q$ and $\lim_{q\to 2}\delta(q) = 0$.

As noted in [**13**], one can extend the results of the Calderon–Zygmund theorem to operators on functions with values in a Hilbert space. Let $L^q(Q,\mathcal{H}_m(\Omega_Q))$ be the Banach space of functions $g: Q\to\mathcal{H}_m(\Omega_Q)$ with norm

$$\|g\|_q^q=\sum_{x\in Q}\|g(x)\|^q_{\mathcal{H}_m(\Omega_Q)}. \tag{3.33}$$

We define $g_{r,j,\xi,\eta}: Q\to\mathcal{H}_m(\Omega_Q)$ and $h_{r,j,\xi,\eta}: Q\to\mathcal{H}_m(\Omega_Q)$ by

$$g_{r,j,\xi,\eta}(z)=f_{r,j}(\xi,\eta,z,\cdot),\quad h_{r,j,\xi,\eta}(z)=\nabla_\xi u_{r,j}(\xi,\eta,z,\cdot),\quad z\in Q. \tag{3.34}$$

From (3.26) and (3.31), it follows that if $|\Im\xi|\leqslant C_1/\sqrt{\eta/\Lambda}$, then

$$\begin{aligned}&\|g_{2,j,\xi,\eta}\|_q\leqslant C\|D\tilde{\mathbf{b}}(\cdot)\|_\infty\|h_{Q,j}\|_q|v|,\\ &\|g_{r,j,\xi,\eta}\|_q\leqslant C\|D\tilde{\mathbf{b}}(\cdot)\|_\infty\|h_{Q,j}\|_q(1-\lambda/\Lambda)^{r-2}[1+C_2|\Im\xi|^2/(\eta/\Lambda)]^{r-2}|v|\\ &\qquad\qquad+(1-\lambda/\Lambda)\|h_{r-1,j,\xi,\eta}\|_q\quad\text{if } r>2,\end{aligned} \tag{3.35}$$

where $C$ depends only on $d$. We see from the Hilbert space version of the Calderon–Zygmund theorem (see [**13**, p. 45]) applied to (3.30) that, for $q>1$, there is a constant $\delta(q)\geqslant 0$ such that

$$\|h_{r,j,\xi,\eta}\|_q\leqslant[1+\delta(q)][1+C_2|\Im\xi|^2/(\eta/\Lambda)]\|g_{r,j,\xi,\eta}\|_q\quad\text{and}\quad\lim_{q\to2}\delta(q)=0. \tag{3.36}$$

It follows then from (3.35) and (3.36) that

$$\|g_{r,j,\xi,\eta}\|_q\leqslant \mathrm{Cr}\|D\tilde{\mathbf{b}}(\cdot)\|_\infty\|h_{Q,j}\|_q[1+\delta(q)]^{r-2}(1-\lambda/\Lambda)^{r-2}[1+C_2|\Im\xi|^2/(\eta/\Lambda)]^{r-2}|v|, \tag{3.37}$$

where $C$ depends only on $d$.

From (3.27) and (3.29), we see that

$$\begin{aligned}&\frac12\left\|\frac{\partial}{\partial\tilde\omega_j(z)}\sum_{x\in Q}g(x)\tilde{\mathbf{b}}(h_Q*\tilde\omega(x)+k_Q*\phi(x))\partial_\xi\tilde F_r(\xi,\eta,\tau_x\tilde\omega,\tau_x\phi)\right\|^2_{\mathcal{H}_m(\Omega_Q)}\\ &\quad\leqslant\left\|\sum_{x\in Q}g(x)h_{Q,j}(x-z)D\tilde{\mathbf{b}}(\phi(x-z))\partial_\xi F_r(\xi,\eta,\tau_{x-z}\phi)\right\|^2_{\mathcal{H}_m(\Omega_Q)}\\ &\quad+\left\|\sum_{x\in Q}g(x)\tilde{\mathbf{b}}(\phi(x-z))h_{r,j,\xi,\eta}(x-z,\phi)\right\|^2_{\mathcal{H}_m(\Omega_Q)}.\end{aligned} \tag{3.38}$$

Observe now from (3.26) and Young's convolution inequality for functions with values in a Hilbert space that

$$\begin{aligned}&\sum_{z\in Q}\left\|\sum_{x\in Q}g(x)h_{Q,j}(x-z)D\tilde{\mathbf{b}}(\phi(x-z))\partial_\xi F_r(\xi,\eta,\tau_{x-z}\phi)\right\|^2_{\mathcal{H}_m(\Omega_Q)}\\ &\quad\leqslant C[\|D\tilde{\mathbf{b}}(\cdot)\|_\infty\|g\|_p\|h_{Q,j}\|_q(1-\lambda/\Lambda)^{r-1}[1+C_2|\Im\xi|^2/(\eta/\Lambda)]^{r-1}|v|]^2,\end{aligned} \tag{3.39}$$

where $p=2q/(3q-2)$ with $1\leqslant q\leqslant 2$ and $C$ depends only on $d$. We can bound the second term on the right-hand side of (3.38) similarly. Thus, from (3.36) and (3.37) we conclude that

$$\begin{aligned}&\sum_{z\in Q}\left\|\sum_{x\in Q}g(x)\tilde{\mathbf{b}}(\phi(x-z))h_{r,j,\xi,\eta}(x-z,\phi)\right\|^2_{\mathcal{H}_m(\Omega_Q)}\\ &\quad\leqslant C[r\|D\tilde{\mathbf{b}}(\cdot)\|_\infty\|g\|_p\|h_{Q,j}\|_q[1+\delta(q)]^{r-1}(1-\lambda/\Lambda)^{r-1}[1+C_2|\Im\xi|^2/(\eta/\Lambda)]^{r-1}|v|]^2,\end{aligned} \tag{3.40}$$

where $p = 2q/(3q-2)$ with $1 \leqslant q \leqslant 2$ and $C$ depends only on $d$.

We can argue now as in the $r=1$ case to establish (3.11) for $r \geqslant 2$ by choosing $q < 2$ and $|\Im\xi| \leqslant C_2(\Lambda/\lambda)\sqrt{\eta/\Lambda}$ to satisfy $[1+\delta(q)](1-\lambda/\Lambda)[1+C_2|\Im\xi|^2/(\eta/\Lambda)] \leqslant (1-\lambda/\Lambda)^{1/2}$. We obtain then an estimate on the first term on the right-hand side of (3.15) which is uniform as $Q \to \mathbf{Z}^d$. By essentially repeating our argument, we also see that the second term on the right-hand side of (3.15) vanishes as $Q \to \mathbf{Z}^d$. Finally, (3.11) follows by letting $m \to 0$. □

*Proof of Hypothesis* 3.1. We assume that $(\xi,\eta)$ and $(\xi',\eta')$ are as in the statement of Hypothesis 3.1. Let $g : \mathbf{Z}^d \to \mathbf{C}^d \otimes \mathbf{C}^d$ be the function defined by

$$g(x) = \{\nabla\nabla^* G_{\eta'/\Lambda}(x)\}^* \, e^{-ix\cdot\xi'} - \{\nabla\nabla^* G_{\eta/\Lambda}(x)\}^* \, e^{-ix\cdot\xi}, \tag{3.41}$$

where the Green's function $G_\nu(\cdot)$ is the solution to (1.10) with $V''(\cdot) \equiv I_d$. It follows from (3.7) and [**3**, Lemma 2.1] that the constant $C_1 > 0$ in (3.5) can be chosen depending only on $d$ and $\Lambda/\lambda$ so that

$$\|[q(\xi',\eta') - q(\xi,\eta)]v\| \leqslant C_2\Lambda \sum_{r=1}^{\infty} \|T_{r,\eta,\Im\xi}\, g(\Re\xi,\cdot)v\|_{\mathcal{H}(\Omega)} \quad \text{for } |\Im\xi|, |\Im\xi'| \leqslant C_1\sqrt{\eta/\Lambda}, \tag{3.42}$$

where $C_2$ is a constant depending only on $d, \Lambda/\lambda$. We can see from (3.41) that there is a constant $C_1$ depending only on $d$ such that if $|\Im\xi|, |\Im\xi'| \leqslant C_1\sqrt{\eta/\Lambda}$, then the function $g(\cdot)$ is in $\ell^p(\mathbf{Z}^d, \mathbf{C}^d \otimes \mathbf{C}^d)$ for any $p > 1$. Furthermore, if $0 \leqslant \alpha \leqslant 1$ and $p > d/(d-\alpha)$, then $\|g(\cdot)\|_p$ satisfies the inequality

$$\|g(\cdot)\|_p \leqslant C_p[|\xi'-\xi|^\alpha + |(\eta'-\eta)/\Lambda|^{\alpha/2}], \tag{3.43}$$

where the constant $C_p$ depends only on $d, p$. The Hölder continuity (3.5) for sufficiently small $\alpha > 0$ follows from (3.42), (3.43) and Lemma 3.1. □

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*Joseph G. Conlon and Arash Fahim*
*Department of Mathematics*
*University of Michigan*
*Ann Arbor, MI 48109-1109*
*USA*

conlon@umich.edu
fahimara@umich.edu