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Dispersive stabilization

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Abstract

Ill-posed linear and nonlinear initial value problems may be stabilized, that is, converted to well-posed initial value problems, by the addition of purely nonscalar linear dispersive terms. This is a stability analogue of the Turing instability. This idea applies to systems of quasilinear Schrödinger equations from nonlinear optics.

1. *Introduction*

In nonlinear optics, one commonly encounters coupled systems of scalar Schrödinger equations

$$\partial_t u_j + i\lambda_j \Delta_x u_j = \sum_{k=1}^{N} b_{j,k}(u, \partial_x) u_k, \quad j \in \{1, \ldots, N\}, \quad (t, x) \in \mathbb{R}^{1+d}, \tag{1.1}$$

where the λ_j are real and the $b_{j,k}$ are first-order partial differential operators with coefficients depending smoothly on u (see [**3**] and the references therein). The nonlinear terms usually depend on u and $\overline{u}$ as follows:

$$\partial_t u_j + i\lambda_j \Delta_x u_j = \sum_{k=1}^{N} c_{j,k}(u, \partial_x) u_k + d_{j,k}(u, \partial_x) \overline{u}_k, \tag{1.2}$$

where the $c_{j,k}$ and $d_{j,k}$ are first order in ∂_x. Introducing u and $\overline{u}$ as unknowns reduces to the form (1.1) for a doubled real system.

For the local in time existence of smooth solutions, the easy case is when the first-order part, $B(u, \partial_x)u$ on the right-hand side is symmetric. In this symmetric case there are easy L^2 estimates, followed by H^s estimates obtained by commutations, which imply the local well-posedness of the Cauchy problem for (1.1) in Sobolev spaces $H^s(\mathbb{R}^d)$ for $s > 1 + d/2$.

In many applications, $B(u, \partial_x)$ *is not symmetric* and even more $\partial_t - B(u, \partial_x)$ is not hyperbolic and the Cauchy problem for $\partial_t u - B(u, \partial_x)u = 0$ can be as ill-posed as the Cauchy problem for the Laplacian. However, the Cauchy problem for (1.1) may be well-posed even if it is ill-posed for the first-order part. This is so even though the dispersive terms $i\lambda_j\Delta$ are neither dissipative nor smoothing in the scale of spaces $H^s(\mathbb{R}^d)$. We call this phenomenon *dispersive stabilization*.

Example 1.1. With $x \in \mathbb{R}$ the Cauchy problem for the system,

$$\partial_t u + i\frac{\partial^2 u}{\partial x^2} + \partial_x v = 0, \quad \partial_t v - i\frac{\partial^2 u}{\partial x^2} - \partial_x u = 0,$$

is well-posed in H^s even though the first-order part defines a badly ill-posed initial value problem.

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This is proved by Fourier transformation in x. The amplification matrix is

$$\exp t \begin{pmatrix} i\xi^2 & -i\xi \\ i\xi & -i\xi^2 \end{pmatrix}.$$

For large ξ the matrix in the exponential has purely imaginary eigenvalues close to $\pm i\xi^2$ and is uniformly diagonalizable showing that the amplification matrix is uniformly bounded for $\xi \in \mathbb{R}$ and t belonging to compact sets. The bound grows exponentially in time. The growth comes from $|\xi| \leqslant R$.

The fact that the addition of a term diag $(i\partial_x^2, -i\partial_x^2)$ whose evolution is neutrally stable can stabilize a strongly ill-posed Cauchy problem is not intuitively clear. There are many related results of this sort. The simplest is the following assertion about linear constant coefficient ordinary differential equations in the plane.

Example 1.2. If A and B are 2×2 real matrices, knowing the stability of the origin as an equilibrium of

$$X' = A\,X \quad \text{and} \quad X' = B\,X,$$

one can draw no conclusion about the stability of the equilibrium for $X' = (A + B)X$.

The best known is the *Turing instability* [**15**] for which A and B have eigenvalues with a strictly negative real part, and hence the input dynamics are exponentially stable and the sum dynamics can be unstable. Each of the stable dynamics is dissipative for certain scalar products. When the scalar products are different the Turing instability is possible. One, but not both, of the matrices A and B can be symmetric.

A related example is the two-dimensional wave equation.

Example 1.3. For the system version of the $2 - d$ wave equation,

$$u_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u_y = 0$$

each of the split dynamics

$$u_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u_x = 0, \quad u_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u_y = 0$$

defines a bounded semigroup on $L^\infty(\mathbb{R}^2)$. The first and second conserve respectively

$$\|u_1\|_{L^\infty} \quad \text{and} \quad \|u_2\|_{L^\infty} \quad \textit{and} \quad \|u_1 + u_2\|_{L^\infty} \quad \text{and} \quad \|u_1 - u_2\|_{L^\infty}.$$

The sum defines a dynamics so that the map

$$u(0, x, y) \mapsto u(t, x, y)$$

is unbounded on $L^\infty(\mathbb{R}^2)$ for all $t \neq 0$.

The analysis in this paper resembles Example 1.1. The Fourier transform method is extended using the paradifferential calculus. We do not use the local smoothing properties of Schrödinger equations. The idea is to conjugate $iA - B$ by a change of variable $I + V$ with V of order -1 to a *normal form*

$$(\mathrm{Id} + V)(iA - B)(\mathrm{Id} + V)^{-1} = iA - \widetilde{B} \tag{1.3}$$

up to zeroth-order terms, with $\widetilde{B} = i[V, A] - B$ symmetric. The conjugation (1.3) means that the principal symbols satisfy

$$\sigma_{\widetilde{B}} = \sigma_B + i[\sigma_A, \sigma_V]. \tag{1.4}$$

Equivalently, the energy estimates are obtained using the *pseudodifferential symmetrizers* as follows:

$$S = \mathrm{Id} + V^* + V. \tag{1.5}$$

If the λ_j are pairwise distinct, one can reduce B to its diagonal part to prove the following result.

THEOREM 1.4. *If the λ_j are real and pairwise distinct and if the diagonal terms $b_{j,j}(u, \partial_x)$ have real coefficients, then locally in time, the Cauchy problem for (1.1) is well-posed in the Sobolev spaces $H^s(\mathbb{R}^d)$ for $s > 1 + d/2$.*

An analogous result for the systems (1.2) is the following.

THEOREM 1.5. *Suppose that the following hold*:
(1) *the λ_j are real and pairwise distinct*;
(2) *the diagonal terms $b_{j,j}(u, \partial_x)$ have real coefficients*;
(3) *$c_{j,k}(u, \partial_x) = c_{k,j}(u, \partial_x)$ for all pairs (j, k) such that $\lambda_j + \lambda_k = 0$.*
Then locally in time, the Cauchy problem for (1.2) is well-posed in the Sobolev spaces $H^s(\mathbb{R}^d)$ with $s > 1 + d/2$.

In the next section we give a more general statement which allows for more general nondiagonal second-order terms. In particular the $\lambda_j\Delta_x$ can be replaced by different second-order elliptic operators $A_j(\partial_x)$. The idea of using pseudodifferential symmetrizers is related to the proof in [**2**], where the symmetry is obtained after differentiation of the equations and clever linear recombination. This amounts to using differential symmetrizers. Our analysis is a systematic exploration of the idea. Because of the quasilinear character of the equations, we use the paradifferential calculus in place of the classical pseudodifferential version. The latter would have sufficed to treat semilinear problems.[†] The paradifferential methods can also be used to treat the strongly nonlinear case $F(u, \partial_x u)$. Such a term is reduced to a quasilinear term by paralinearization (see Section 2).

For the systems case the dispersive terms rotating at different speeds regularize an explosive first-order term. For the scalar case, that is $N = 1$, such a stabilization is not possible. The Cauchy problem for $\partial_t - i\Delta_x + i\partial_{x_1}$ is ill-posed. However, if $\operatorname{Im} b(x)$ satisfies suitable decay assumptions at infinity, then the Cauchy problem for $\partial_t - i\Delta_x + b(x)\cdot\nabla_x$ is well-posed (see [**12**]). Intuitively, the waves propagate to the regions where b is small and are no longer amplified. The proofs use the dispersive and local smoothing properties of Schrödinger equations. This idea has been extensively studied. Some of the foundational papers are [**4**, **6**–**8**, **13**], and the references therein. It would be natural to combine such ideas with those of dispersive stabilization with the goal of extending the local existence to the case where the antisymmetric part of $\widetilde{B}$ has suitable decay at infinity rather than requiring that it vanish. We do not pursue this line of inquiry.

[†] The paradifferential calculus is a convenient and systematic tool for the use of pseudodifferential techniques when the coefficients have limited smoothness.

2. *Statement of the result*

Consider the general equations,

$$\partial_t u + iA(\partial_x)u + B(t,x,u,\partial_x)u = 0, \tag{2.1}$$

with A second order and B first order as follows:

$$A(\partial_x) = \sum_{j,k=1}^{d} A_{j,k}\partial_{x_j}\partial_{x_k}, \tag{2.2}$$

$$B(t,x,u,\partial_x) = \sum_{j=1}^{d} B_j(t,x,u)\partial_{x_j}. \tag{2.3}$$

The matrices $B_j(t,x,u)$ are assumed to be C^∞ functions of $(t, x, \operatorname{Re} u, \operatorname{Im} u)$, so that for each α and bounded $K \subset \mathbb{C}^N$, we have

$$\partial^\alpha_{t,x,\operatorname{Re} u,\operatorname{Im} u} B \in L^\infty([0,T]\times\mathbb{R}^d\times K).$$

With the example (1.1) in mind, we assume that A is smoothly block-diagonalizable.

Assumption 2.1. For all $\xi \in \mathbb{R}^n\backslash\{0\}$, we have that $A(\xi) = \sum A_{j,k}\xi_j\xi_k$ is self-adjoint. Moreover, there are smooth real eigenvalues $\lambda_p(\xi)$ and smooth self-adjoint eigenprojectors $\Pi_p(\xi)$ such that

$$A(\xi) = \sum_p \lambda_p(\xi)\Pi_p(\xi).$$

This assumption is satisfied if A is self-adjoint with eigenvalues of constant multiplicity. The assumption allows for some crossing eigenvalues. The conditions on B involve

$$\operatorname{Im} B := \frac{1}{2i}(B - B^*).$$

Assumption 2.2. For all p and q, there are smooth matrix-valued functions $V_{p,q}(t,x,u,\xi)$ such that

$$\Pi_p(\xi)\operatorname{Im} B(t,x,u,\xi)\,\Pi_q(\xi) = \big(\lambda_p(\xi)-\lambda_q(\xi)\big)V_{p,q}(t,x,u,\xi). \tag{2.4}$$

Remark 2.3. The condition (2.4) holds in $\xi \neq 0$. On open sets where $\lambda_p(\xi) \neq \lambda_q(\xi)$, it is always satisfied as it defines $V_{p,q}$. Assumption 2.2 contains two types of information.

(a) For any $\underline{\xi}$, if $\underline{\lambda}$ is an eigenvalue of $A(\underline{\xi})$ and $\Pi(\underline{\xi})$ the spectral projector, then $\Pi(\underline{\xi})B(t,x,u,\underline{\xi})\Pi(\underline{\xi})$ is self-adjoint. If the eigenvalue remains of constant multiplicity for ξ near $\underline{\xi}$, nothing more needs to be added for this polarization. In particular, if all the distinct eigenvalues $\lambda_p(\xi)$ of $A(\xi)$ have constant multiplicity, then Assumption 2.2 reduces to the condition that the matrices $\Pi_p(\xi)B(t,x,u,\xi)\Pi_p(\xi)$ are self-adjoint.

(b) If the eigenvalue $\underline{\lambda}$ splits into several eigenvalues $\lambda_p(\xi)$ for ξ near $\underline{\xi}$, then the condition (2.4) means that not only $\Pi_p(\xi)\operatorname{Im} B(t,x,u,\xi)\Pi_q(\xi)$ vanishes at $\underline{\xi}$ and on the variety $\{\lambda_p = \lambda_q\}$, but also that $\lambda_p(\xi) - \lambda_q(\xi)$ is a divisor. In particular, if $\widetilde{\Pi}(\xi)$ denotes the spectral projector on the invariant space associated to the eigenvalues close to $\underline{\lambda}$, then this condition is locally satisfied with $V_{p,q} = 0$ whenever $\widetilde{\Pi}(\xi)\,B(t,x,u,\xi)\,\widetilde{\Pi}(\xi)$ is self-adjoint. This is so since

$$0 = \widetilde{\Pi}\operatorname{Im} B\,\widetilde{\Pi} = \sum_{p,q}\Pi_p \operatorname{Im} B\Pi_q, \quad \text{and hence } \Pi_p\operatorname{Im} B\,\Pi_q = \Pi_p\,\widetilde{\Pi}\operatorname{Im} B\,\widetilde{\Pi}\,\Pi_q = 0.$$

REMARK 2.4. There is no assumption on the spectrum of $B(t,x,u,\xi)$. In particular, $\partial_t + B$ may be nonhyperbolic and thus strongly unstable in Hadamard's sense. The dispersive term A has a stabilizing effect, provided that the condition in Assumption 2.2 is satisfied. For this reason models of this type appear often in the descriptions of instabilities, for example, that of Raman. The dispersive stabilization regularizes to a well-posed causal model albeit with the possibility of growth for moderate wave numbers as in Example 1.1.

We show that under the Assumptions 2.1 and 2.2 the Cauchy problem for (2.1) is well-posed in H^s for $s > d/2+1$, locally in time.

THEOREM 2.5. *If Assumptions* 2.1 *and* 2.2 *hold,* $s > d/2 + 1$ *and* $h \in H^s(\mathbb{R}^d)$*, then there exists* $T > 0$ *and a unique solution* $u \in C^0([0,T]; H^s(\mathbb{R}^d))$ *of* (2.1) *with* $u|_{t=0} = h$.

EXAMPLE 2.6 (From [2]). Suppose that A is block diagonal, $A = \operatorname{diag}\{\lambda_p \mathrm{Id}_p\}$, with real $\lambda_p(\xi)$ homogeneous of degree two and $\lambda_p(\xi) \neq \lambda_q(\xi)$ for $p \neq q$ and $\xi \neq 0$. The second assumption is trivially satisfied if the diagonal blocks $B^{p,p}$ vanish.

For the applications, we make explicit the assumptions when the first-order part depends on $\overline{u}$ as follows:

$$\partial_t u + iA(\partial_x)u + B(t,x,u,\partial_x)u + C(t,x,u,\partial_x)\overline{u} = 0. \tag{2.5}$$

Introducing $v = \overline{u}$ as a variable and setting $U = {}^t(u,v)$, the equation reads as follows:

$$\partial_t U + i\mathcal{A}(\partial_x)U + \mathcal{B}(t,x,u,\partial_x)U = 0, \tag{2.6}$$

with

$$\mathcal{A} = \begin{pmatrix} A(\partial_x) & 0 \\ 0 & -A(\partial_x) \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} B & C \\ \overline{C} & \overline{B} \end{pmatrix}. \tag{2.7}$$

In this context, Assumption 2.2 becomes the following.

ASSUMPTION 2.7. For all p and q, we find that $\Pi_p(\xi) \operatorname{Im} B(t,x,u,\xi)\Pi_q(\xi)$ vanishes when $\lambda_p(\xi) = \lambda_q(\xi)$ and $\Pi_p(\xi)\big(C(t,x,u,\xi) - {}^tC(t,x,u,\xi)\big)\Pi_q(\xi)$ vanishes when $\lambda_p(\xi) + \lambda_q(\xi) = 0$. In addition, there are smooth matrices $V_{p,q}(t,x,u,\xi)$ and $W_{p,q}(t,x,u,\xi)$ such that

$$\Pi_p(\operatorname{Im} B)\Pi_q = (\lambda_p - \lambda_q)V_{p,q}, \tag{2.8}$$

$$\Pi_p\big(C - {}^tC\big)\Pi_q = (\lambda_p + \lambda_q)W_{p,q}. \tag{2.9}$$

THEOREM 2.8. *Under Assumptions* 2.1 *and* 2.7*, for* $s > d/2 + 1$ *and* $h \in H^s(\mathbb{R}^d)$*, there exists* $T > 0$ *and a unique solution* $u \in C^0([0,T]; H^s(\mathbb{R}^d))$ *of* (2.5) *with* $u|_{t=0} = h$.

We briefly discuss the case of equations with a fully nonlinear right-hand side as follows:

$$\partial_t u + iA(\partial_x)u + F(t,x,u,\partial_x u) = 0, \tag{2.10}$$

where $F(t,x,u,v_1,\dots,v_d)$ is a smooth function of $(t,x,\operatorname{Re} u, \operatorname{Im} u)$ and of $(\operatorname{Re} v_1,\dots,\operatorname{Im} v_d)$. Our analysis relies on a paralinearization of the first-order term, so that the analogues of B

and C are

$$B(t,x,u,v,\xi) = \sum_j \xi_j \frac{\partial F}{\partial v_j}(t,x,u,v), \tag{2.11}$$

$$C(t,x,u,v,\xi) = \sum_j \xi_j \frac{\partial F}{\partial \overline{v}_j}(t,x,u,v) \tag{2.12}$$

with

$$\frac{\partial}{\partial v_j} = \frac{1}{2}\frac{\partial}{\partial \operatorname{Re} v_j} - \frac{i}{2}\frac{\partial}{\partial \operatorname{Im} v_j}, \qquad \frac{\partial}{\partial \overline{v}_j} = \frac{1}{2}\frac{\partial}{\partial \operatorname{Re} v_j} + \frac{i}{2}\frac{\partial}{\partial \operatorname{Im} v_j}$$

as usual. The stability condition is that (2.8) and (2.9) are satisfied with smooth matrices $V_{p,q}(t,x,u,v)$ and $W_{p,q}(t,x,u,v)$. In this case, the Cauchy problem is well-posed in H^s for $s > d/2+2$.

3. *Basic L^2 estimate*

We solve (2.1) by Picard iteration. Consider first the linear problem

$$\partial_t u + iA(\partial_x)u + B(t,x,a,\partial_x)u = f, \quad u_{|t=0} = h, \tag{3.1}$$

where

$$a \in C^0_w([0,T]; H^s(\mathbb{R}^d)), \quad \partial_t a \in C^0_w([0,T]; H^{s-2}(\mathbb{R}^d)) \tag{3.2}$$

with $s > d/2+1$ and $C^0_w([0,T]; H^\sigma)$ denotes the space of functions which are continuous from $[0,T]$ to H^σ equipped with the weak topology.

THEOREM 3.1. *There are functions C_0 and C_1 such that the solution of (3.1) satisfies*

$$\left\|u(t)\right\|_{L^2} \leqslant C_0(K_0)e^{tC_1(K_1)}\Big(\left\|u(0)\right\|_{L^2} + \int_0^t \left\|f(t')\right\|_{L^2} dt'\Big) \tag{3.3}$$

with

$$K_0 := \|a\|_{L^\infty([0,T]\times\mathbb{R}^d)}, \tag{3.4}$$

$$K_1 := \|a\|_{L^\infty([0,T];H^s(\mathbb{R}^d))} + \|\partial_t a\|_{L^\infty([0,T];H^{s-2}(\mathbb{R}^d))}. \tag{3.5}$$

LEMMA 3.2 (Conjugation). *For $|\xi|$ large, there exists a smooth invertible matrix $V_{-1}(t,x,u,\xi)$, homogeneous of degree -1 in ξ, such that*

$$B(t,x,u,\xi) - [A(\xi), V_{-1}(t,x,u,\xi)] \tag{3.6}$$

is self-adjoint and homogeneous of degree 1 in ξ.

Proof. By Assumption 2.2, we see that

$$V_{-1} := i\sum_{p\neq q}\frac{1}{\lambda_p - \lambda_q}\Pi_p(\operatorname{Im} B)\Pi_q$$

is smooth and $[A, V_{-1}] = i\sum \Pi_q(\operatorname{Im} B)\Pi_p = i\operatorname{Im} B$, so that $B - [A, V_{-1}] = \operatorname{Re} B$ is self-adjoint. □

Proof of Theorem 3.1. Use the paradifferential calculus and the notation of Section 5.

(a) For simplicity denote by $B_j(t,x)$ the matrix $B_j(t,x,a(t,x))$ and by $B = B(t,x,a(t,x),\xi)$ the symbol $\sum \xi_j B_j$. Since $s > 1 + d/2$, we have that (3.2) implies that $B_j \in C^0([0,T];H^s)$, $\partial_t B_j \in C^0([0,T];H^{s-2})$ and

$$\|B_j\|_{L^\infty([0,T];H^s(\mathbb{R}^d))} + \|\partial_t B_j\|_{L^\infty([0,T];H^{s-2}(\mathbb{R}^d))} \leqslant C_1(K_1). \tag{3.7}$$

In particular, as a symbol, B belongs to the class $\widetilde{\Gamma}^1_1$ introduced in Definition 5.11. Using the paralinearization Proposition 5.8, we see that $f_1 := B(t,x,\partial_x)u - T_{iB}u$ satisfies

$$\|f_1(t)\|_{L^2} \leqslant C_1(K_1)\|u(t)\|_{L^2}, \tag{3.8}$$

and u satisfies the paralinearized equation

$$\partial_t u + iA(\partial_x)u + T_{iB}u = f - f_1, \quad u_{|t=0} = h. \tag{3.9}$$

(b) Similarly, use the simplified notation $V(t,x,\xi) = V_{-1}(t,x,a(t,x),\xi)\zeta(\xi)$, where $\zeta \in C^\infty(\mathbb{R}^d)$ vanishes near the origin and is equal to 1 for $|\xi| \geqslant 1$. Note that $V \in \widetilde{\Gamma}^{-1}_1$ and that, for all α, there are functions $C_{0,\alpha}$ and $C_{1,\alpha}$ such that, for all $t \in [0,T]$ and $\xi \in \mathbb{R}^d$, we have

$$\big\|\partial_\xi^\alpha V(t,\cdot,\xi)\big\|_{L^\infty} \leqslant C_{0,\alpha}(K_0)(1+|\xi|)^{\mu-|\alpha|}, \tag{3.10}$$

$$\big\|\partial_\xi^\alpha \partial_t V(t,\cdot,\xi)\big\|_{H^{s-2}} \leqslant C_{1,\alpha}(K_1)(1+|\xi|)^{\mu-|\alpha|}. \tag{3.11}$$

Use a symmetrizer as follows:

$$\Sigma := \mathrm{Id} + T_V + (T_V)^* + \gamma(1-\Delta_x)^{-1}. \tag{3.12}$$

By Proposition 5.2 and Remark 5.7, there exists a constant $C_0(K_0)$ that depends only on K_0 such that

$$\big\|T_V u(t)\big\|_{H^1} \leqslant C_0(K_0)\|u(t)\big\|_{L^2}.$$

Therefore we have

$$\big(\Sigma u, u\big)_{L^2} \geqslant \big\|u\big\|^2_{L^2} - 2C_0(K_0)\big\|u\big\|_{L^2}\big\|u\big\|_{H^{-1}} + \gamma\big\|u\big\|^2_{H^{-1}}.$$

Choose $\gamma = \gamma(K_0)$ so that

$$\big(\Sigma u, u\big)_{L^2} \geqslant \tfrac{1}{2}\big\|u\big\|^2_{L^2}. \tag{3.13}$$

Then, with another constant $C_0(K_0)$, we have

$$\big\|\Sigma u(t)\big\|_{L^2} \leqslant C_0(K_0)\big\|u(t)\big\|_{L^2}. \tag{3.14}$$

(c) Compute

$$\frac{d}{dt}\big(\Sigma(t)u(t), u(t)\big)_{L^2} = 2\,\mathrm{Re}\,\big(\Sigma\partial_t u, u\big)_{L^2} + \big([\partial_t, \Sigma]u, u\big)_{L^2}. \tag{3.15}$$

By Lemma 5.9 and Proposition 5.12, $[\partial_t, \Sigma] = [\partial_t, T_V] + [\partial_t, T_V]^*$ is bounded from L^2 to L^2 and

$$\big([\partial_t, \Sigma]u(t), u(t)\big)_{L^2} \leqslant C_1(K_1)\|u(t)\big\|^2_{L^2}. \tag{3.16}$$

Next, observe that $T_V A(\partial_x) = A(\partial_x)T_V + [T_V, A(\partial_x)]$, that $A(\partial_x) = -T_{A(\xi)}$ and that $[T_V, A(\partial_x)] - T_{[A,V]}$ is of order zero. Therefore, the equation and the symbolic calculus of Proposition 5.5 imply that

$$\Sigma\partial_t u = -iA(\partial_x)u + i\big(A(\partial_x)T_V + (T_V)^*A(\partial_x) - T_{\widetilde{B}}\big)u + \Sigma f + f_2,$$

where $\widetilde{B}(t,x,\xi) = B(t,x,\xi) - [A(t,x,\xi), V(\xi)] \in \widetilde{\Gamma}^1_1$ and f_2 satisfies an estimate similar to (3.8). By Lemma 3.2, $\widetilde{B}$ is self-adjoint for $|\xi| \geqslant 2$, and hence Proposition 5.6 implies that

$$\mathrm{Re}\,\big(iT_B u(t), u(t)\big)_{L^2} \leqslant C_1(K_1)\|u(t)\big\|^2_{L^2}.$$

Since $A(\partial_x)$ is self-adjoint, we conclude that

$$\frac{d}{dt}\big(\Sigma(t)u(t),u(t)\big)_{L^2} \leqslant 2\big\|\Sigma f(t)\big\|_{L^2}\big\|u(t)\big\|_{L^2} + C_1(K_1)\big\|u(t)\big\|^2_{L^2}. \tag{3.17}$$

Equations (3.13) and (3.14) imply estimate (3.3). □

4. *Sobolev estimates and nonlinear existence*

It is seen that H^s estimates for the linearized equation (3.1) are obtained by differentiating the equation. The commutators $[\partial_x^\alpha, B(t,x,a,\partial_x)]u$ are estimated by standard nonlinear estimates as in the analysis of first-order hyperbolic equations. Because $s > d/2+1$, for $|\alpha| \leqslant s$, one has

$$\Big\|\big[\partial_x^\alpha, B(t,x,a,\partial_x)\big]u(t)\Big\|_{L^2} \leqslant C_1(K_1)\big\|u(t)\big\|_{H^s}. \tag{4.1}$$

This implies the following estimates.

PROPOSITION 4.1. *There are functions C_0 and C_1 such that smooth solutions of (3.1) satisfy*

$$\big\|u(t)\big\|_{H^s} \leqslant C_0(K_0)e^{tC_1(K_1)}\Big(\big\|u(0)\big\|_{H^s} + \int_0^t \big\|f(t')\big\|_{H^s}\,dt'\Big) \tag{4.2}$$

with K_0 and K_1 defined in (3.4) and (3.5).

As in the hyperbolic theory, these estimates imply the following strong continuity result.

PROPOSITION 4.2. *Suppose that a satisfies (3.2), $f \in L^1([0,T],H^s)$ and $h \in H^s$. If $u \in C^0_w([0,T];H^s)$ is a solution of (3.1), then $u \in C^0([0,T],H^s)$.*

Proof. With $J_\varepsilon = (1-\varepsilon\Delta_x)^{-1}$, one checks that $J_\varepsilon u$ satisfies

$$\partial_t J_\varepsilon u + iA(\partial_x)J_\varepsilon u + B(t,x,a,\partial_x)J_\varepsilon u = f_\varepsilon, \quad J_\varepsilon u_{|t=0} = J_\varepsilon h, \tag{4.3}$$

with $f_\varepsilon \to f$ in $L^1([0,T],H^s)$. Applying the estimates to $J_\varepsilon u$ shows that $\{J_\varepsilon u\}$ is Cauchy and thus convergent in $C^0([0,T],H^s)$. Therefore $u \in C^0([0,T],H^s)$. □

Turn to the proof of the main result. More details can be found in [**10**].

Proof of Theorem 2.5. (i) To solve (3.1) for a satisfying (3.2), use the mollified equations

$$\partial_t u^\varepsilon + iA(\partial_x)J_\varepsilon u^\varepsilon + B(t,x,a,\partial_x)J_\varepsilon u^\varepsilon = f, \quad u_{|t=0} = h, \tag{4.4}$$

where $J_\varepsilon = (1-\varepsilon\Delta_x)^{-1}$. For fixed ε, this is a linear ordinary differential equation in H^s since $A(D_x)J_\varepsilon$ and BJ_ε are bounded. One checks that the proof of the estimates (4.2) for the solutions of (3.1) immediately extends to the solutions of (4.4), because $\{J_\varepsilon\}$ is a bounded family of pseudodifferential operators of degree 0, and the new commutators they generate are remainders in the symbolic calculus developed in Section 3. Therefore, the u^ε are uniformly bounded in $C^0([0,T];H^s)$. The equation shows that they are bounded in $C^1([0,T],H^{s-2})$.

Extracting a subsequence and passing to the weak limit yields a solution $u \in C^0_w([0,T],H^s)$. By Proposition 4.2, we have $u \in C^0([0,T],H^s)$.

(ii) Solve the nonlinear equation using the iteration scheme

$$\partial_t u_{n+1} + iA(\partial_x)u_{n+1} + B(t,x,u_n\partial_x)u_{n+1} = 0, \quad u_{n+1|t=0} = h. \tag{4.5}$$

Using the estimate (4.2), one proves that there exists $T > 0$ such that the sequence $\{u_n\}$ is bounded in $C^0([0,T], H^s)$ and in $C^1([0,T], H^{s-2})$. Knowing this bound in high norm, one checks that the sequence u_n converges in a low norm $C^0([0,T]; L^2)$. Passing to the limit gives a solution of (2.1) $u \in C^0_w([0,T], H^s)$ that also belongs to $C^1([0,T], H^{s-2})$. Using Proposition 4.2, one obtains that $u \in C^0([0,T], H^s)$. □

5. *Handbook of paradifferential calculus*

The symmetrizers are paradifferential operators in the variables x, depending on the parameter t. This section reviews the paradifferential calculus extended to the case of time-dependent symbols.

5.1. *The spatial calculus*

Consider operators on $\mathbb{R}^d$. The variables are denoted by x and the frequency variables by ξ.

Definition 5.1 (Symbols). Let $\mu \in \mathbb{R}$. Then we have the following:

(i) Γ^μ_0 denotes the space of locally L^∞ functions $a(x,\xi)$ on $\mathbb{R}^d \times \mathbb{R}^d$ which are C^∞ with respect to ξ and such that, for all $\alpha \in \mathbb{N}^d$, there exists a constant C_α such that

$$\forall (x,\xi), \quad |\partial^\alpha_\xi a(x,\xi)| \leqslant C_\alpha (1+|\xi|)^{\mu-|\alpha|}; \tag{5.1}$$

(ii) Γ^μ_1 denotes the space of symbols $a \in \Gamma^\mu_0$ such that, for all j, we have that $\partial_{x_j} a \in \Gamma^\mu_0$.

The paradifferential calculus in $\mathbb{R}^d$ was introduced by Bony [**1**] (see also [**5**, **9**, **11**, **14**]). The reference [**10**] gives a detailed account of the time-dependent results needed here. The calculus associates operators T_a to symbols $a \in \Gamma^\mu_0$. They act in the scale of Sobolev spaces $H^s(\mathbb{R}^d)$. Moreover, there is a symbolic calculus of order one for symbols in Γ^μ_1. Recall the definition which is needed later on.

Consider a C^∞ function $\psi(\eta,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ such that we have the following.

(1) There are ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2 < 1$ and

$$\begin{aligned} &\psi(\eta,\xi) = 1 \quad \text{for } |\eta| \leqslant \varepsilon_1(1+|\xi|),\\ &\psi(\eta,\xi) = 0 \quad \text{for } |\eta| \geqslant \varepsilon_2(1+|\xi|). \end{aligned} \tag{5.2}$$

(2) For all $(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n$, there exists $C_{\alpha,\beta}$ such that

$$\forall (\eta,\xi,\gamma), \quad |\partial^\alpha_\eta \partial^\beta_\xi \psi(\eta,\xi)| \leqslant C_{\alpha,\beta}(1+|\xi|)^{-|\alpha|-|\beta|}. \tag{5.3}$$

For instance, with $N \geqslant 3$, one can consider

$$\psi_N(\eta,\xi) = \sum_{k=0}^{+\infty} \chi_{k-N}(\eta)\varphi_k(\xi), \tag{5.4}$$

where $\chi \in C^\infty_0(\mathbb{R}^d)$ satisfies $0 \leqslant \chi \leqslant 1$ and

$$\chi(\xi) = 1 \ \text{ for } |\xi| \leqslant 1.1, \quad \chi(\xi) = 0 \ \text{ for } |\xi| \geqslant 1.9, \tag{5.5}$$

and for $k \in \mathbb{Z}$, we have

$$\chi_k(\xi) = \chi\big(2^{-k}\xi\big) \tag{5.6}$$

and

$$\varphi_0 = \chi_0 \quad \text{and} \quad \text{for } k \geqslant 1 \ \ \varphi_k = \chi_k - \chi_{k-1}. \tag{5.7}$$

A function ψ satisfying (5.2) and (5.3) is called an *admissible cut-off*. Consider next $G^\psi(\cdot,\xi)$ the inverse Fourier transform of $\psi(\cdot,\xi)$. For $a \in \Gamma_0^\mu$ define

$$\sigma_a^\psi(x,\xi) := \int G^\psi(x-y,\xi)\, a(y,\xi)\, dy \tag{5.8}$$

or equivalently on the Fourier side in x

$$\widehat{\sigma}_a^\psi(\eta,\xi) = \psi(\eta,\xi)\,\widehat{a}(\eta,\xi). \tag{5.9}$$

The symbol σ belongs to Γ_0^μ and thus to Hörmander's class $S_{1,1}^\mu$. The paradifferential operator T_a^ψ is defined by

$$T_a^\psi u(x) := \frac{1}{(2\pi)^n}\int e^{i\xi\cdot x}\,\sigma_a^\psi(x,\xi)\,\widehat{u}(\xi)\,d\xi. \tag{5.10}$$

We collect here the main results.

Proposition 5.2 (Action). *Suppose that ψ is an admissible cut-off.*

(i) *When $a(\xi)$ is a symbol independent of x, the operator T_a^ψ is equal to the Fourier multiplier $a(D)$.*

(ii) *For all $a \in \Gamma_0^\mu$ and $s \in \mathbb{R}$, we see that T_a^ψ is a bounded operator from $H^s(\mathbb{R}^d)$ to $H^{s-\mu}(\mathbb{R}^d)$.*

Proposition 5.3. *If ψ_1 and ψ_2 are two admissible cut-offs, then for all $a \in \Gamma_0^\mu$ and $s \in \mathbb{R}$, we have that $T_a^{\psi_1} - T_a^{\psi_2}$ is a bounded operator from $H^s(\mathbb{R}^d)$ to $H^{s-\mu+1}(\mathbb{R}^d)$.*

Remark 5.4. This proposition implies that the choice of ψ is essentially irrelevant in our analysis, as in [**1**]. To simplify notation, make a definite choice of ψ, for instance, $\psi = \psi_N$ with $N = 3$ as in (5.4) and use the notation T_a for T_a^ψ.

Proposition 5.5 (Symbolic calculus). *Consider $a \in \Gamma_1^\mu$ and $b \in \Gamma_1^{\mu'}$. Then $ab \in \Gamma_1^{\mu+\mu'}$ and for all $s \in \mathbb{R}$, we find that $T_a \circ T_b - T_{ab}$ is bounded from $H^s(\mathbb{R}^d)$ to $H^{s-\mu-\mu'+1}(\mathbb{R}^d)$.*

If b is independent of x, then $T_a \circ T_b = T_{ab}$.

These results extend to matrix-valued symbols and operators.

Proposition 5.6 (Adjoints). *Consider a matrix-valued symbol $a \in \Gamma_1^\mu$. We denote by $(T_a)^*$ the adjoint operator of T_a in $L^2(\mathbb{R}^d)$ and by $a^*(x,\xi)$ the adjoint of the matrix $a(x,\xi)$. Then $(T_a)^* - T_{a^*}$ is bounded from $H^s(\mathbb{R}^d)$ to $H^{s-\mu+1}(\mathbb{R}^d)$.*

Remark 5.7. The norm of the operators acting in the indicated Sobolev spaces are uniformly bounded when the symbols a and b belong to bounded subsets of the symbol classes.

Bounded functions of x are particular examples of symbols in the class Γ_0^0, independent of the frequency variables ζ. In this case, T_a is called a *paraproduct* in [**1**].

Proposition 5.8 (Paralinearization). *There is a constant C such that for all $a \in W^{1,\infty}$ and all $u \in L^2(\mathbb{R}^d)$, we have*

$$\left\|a\partial_{x_j}u - T_a\partial_{x_j}u\right\|_{L^2} \leqslant C\|a\|_{W^{1,\infty}}\|u\|_{L^2}.$$

5.2. *The time-dependent case*

Consider functions of $(t,x) \in [0,T] \times \mathbb{R}^n$ as functions of t with values in various spaces of functions of x. In particular, denote by T_a the operator acting on u so that, for each fixed t, we have $(T_a u)(t) = T_{a(t)} u(t)$.

$$T_a u(t,x) := \frac{1}{(2\pi)^n} \int e^{i\xi\cdot x}\, \sigma_a(t,x,\xi)\, \widehat{u}(\xi)\, d\xi. \tag{5.11}$$

with

$$\sigma_a(t,x,\xi) := \int G(x-y,\xi)\, a(t,y,\xi)\, dy. \tag{5.12}$$

This definition shows that formally

$$[\partial_t, T_a] = T_{\partial_t a}. \tag{5.13}$$

This yields easy estimates when $\partial_t a \in L^\infty$. With the lower bound $s > 1 + d/2$ in Theorem 2.5, a_t may be less regular since in the equation (2.1), we see that ∂_t has the weight of two spatial derivatives. This is why we introduce a slight extension.

Using the Littlewood–Paley decomposition

$$u = \sum_{k=0}^{+\infty} \Delta_k u \quad \text{with } \widehat{\Delta_k u} := \varphi_k \hat{u}, \tag{5.14}$$

as in (5.7), *the Besov space $B_\infty^{-1,\infty}$ is defined as the space of tempered distributions u such that*

$$\|u\|_{B_\infty^{-1,\infty}} = \sup_k\ 2^{-k} \|\Delta_k u\|_{L^\infty} < +\infty. \tag{5.15}$$

This space appears in the analysis because of the following embedding.

Lemma 5.9. *Functions $u \in H^s$ belong to $B_\infty^{-1,\infty}$ when $s > d/2 - 1$.*

In the spirit of Definition 5.1, introduce the following notation.

Definition 5.10 (Γ^μ_{-1}). For $\mu \in \mathbb{R}$, we see that Γ^μ_{-1} denotes the space of distributions $a(x,\xi)$ on $\mathbb{R}^d \times \mathbb{R}^d$ which are C^∞ with respect to ξ with values in $B_\infty^{-1,\infty}$ and such that, for all $\alpha \in \mathbb{N}^d$, there is a constant C_α such that

$$\forall \xi, \quad \|\partial_\xi^\alpha a(\cdot,\xi)\|_{B_\infty^{-1,\infty}} \leqslant C_\alpha\, (1+|\xi|)^{\mu-|\alpha|}. \tag{5.16}$$

Definition 5.11 (Time-dependent symbols). For $\mu \in \mathbb{R}$ and $T > 0$, we have the following.

(i) We denote by $\widetilde{\Gamma}^\mu_0$ the space of locally continuous functions $a(t,x,\xi)$ on $[0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ which are C^∞ with respect to ξ and such that the family $\{a(t,\cdot,\cdot); t \in [0,T]\}$ is bounded in Γ^μ_0.

(ii) We denote by $\widetilde{\Gamma}^\mu_1$ the space of symbols $a \in \widetilde{\Gamma}^\mu_0$ such that we have the following:

  (a) the family $\{a(t,\cdot,\cdot); t \in [0,T]\}$ is bounded in Γ^μ_1;

  (b) the family $\{\partial_t a(t,\cdot,\cdot); t \in [0,T]\}$ is bounded in Γ^μ_{-1}.

For $a \in \widetilde{\Gamma}^\mu_0$, the operator T_a is defined by (5.11) and the Propositions 5.2, 5.5 and 5.8 apply for fixed t, yielding estimates that are uniform in t (see Remark 5.7). The commutation with ∂_t is treated as follows.

PROPOSITION 5.12. *For $a \in \widetilde{\Gamma}_1^\mu$, the commutator $[\partial_t, T_a]$ maps $C^0([0,T]; H^s)$ to $C^0([0,T]; H^{s-\mu-1})$ and for all $t \in [0,T]$ there exists a constant C such that*

$$\big\| [\partial_t, T_a] u(t) \big\|_{H^{s-\mu-1}} \leqslant C \|u\|_{H^s}. \tag{5.17}$$

Moreover, the constant C depends only on the supremum for $t \in [0,T]$ of a finite number of seminorms

$$\sup_\xi (1+|\xi|)^{|\alpha|-\mu} \big\| \partial_\xi^\alpha a(\cdot,\xi) \big\|_{B^{-1,\infty}_\infty}. \tag{5.18}$$

Proof. We denote by $[\partial_t, T_a]$ the operator with symbol $\partial_t \sigma_a$. One has

$$\partial_t \sigma_a(t,\cdot,\xi) = \sum_{k=0}^{+\infty} S_{k-N}(D_x)\big(\partial_t a(t,\cdot,\xi)\big)\varphi_k(\xi), \tag{5.19}$$

where $S_j(D_x)$ is the Fourier multiplier with symbol $\chi_j(\xi)$. By assumption, $\big\|\partial_t a(t,\cdot,\xi)\big\|_{B^{-1,\infty}_\infty} \lesssim (1+|\xi|)^\mu$, and hence

$$\big\| S_{k-N}(D_x)\big(\partial_t a(t,\cdot,\xi)\big\|_{L^\infty} \lesssim 2^k (1+|\xi|)^\mu.$$

On the support of φ_k, the frequency ξ is of order $|\xi| \approx 2^k$. Therefore

$$\big|\partial_t \sigma_a(t,x,\xi)\big| \lesssim (1+|\xi|)^{\mu+1},$$

and

$$\big|\partial_\xi^\beta \partial_t \sigma_a(t,x,\xi)\big| \lesssim (1+|\xi|)^{\mu+1-|\beta|}.$$

By construction of σ_a, it follows that the x-Fourier transform $\partial_t \hat{\sigma}_a(t,\eta,\xi)$ of $\partial_t \sigma_a(t,x,\xi)$ is supported in $|\eta| \leqslant \varepsilon(1+|\xi|)$ for some $\varepsilon > 0$. Therefore uniformly for $t \in [0,T]$, we have that $\partial_t \sigma_a \in \widetilde{\Gamma}_0^{\mu+1}$ and therefore $(\partial_t \sigma_a)(t,x,D_x)$ is bounded from H^s to $H^{s-\mu-1}$ for all s. □

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