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# Ternary expansions of powers of 2

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*Dedicated to Mel Nathanson on his 60th birthday*

Abstract

Erdős asked how frequently $2^n$ has a ternary expansion that omits the digit 2. He conjectured that this holds only for finitely many values of $n$. We generalize this question to consider iterates of two discrete dynamical systems. The first considers truncated ternary expansions of real sequences $x_n(\lambda) = \lfloor \lambda 2^n \rfloor$, where $\lambda > 0$ is a real number, along with its untruncated version, whereas the second considers 3-adic expansions of sequences $y_n(\lambda) = \lambda 2^n$, where $\lambda$ is a 3-adic integer. We show in both cases that the set of initial values having infinitely many iterates that omit the digit 2 is small in a suitable sense. For each nonzero initial value we obtain an asymptotic upper bound as $k \to \infty$ on the number of the first $k$ iterates that omit the digit 2. We also study auxiliary problems concerning the Hausdorff dimension of intersections of multiplicative translates of 3-adic Cantor sets.

## 1. *Introduction*

Erdős [5] asked the question, 'When does the ternary expansion of $2^n$ omit the digit 2?' This happens for $2^0 = (1)_3$, $2^2 = 4 = (11)_3$, and $2^8 = 256 = (100111)_3$. He conjectured that it does not happen for all $n \geqslant 9$ and commented that: 'As far as I can see, there is no method at our disposal to attack this conjecture.' This question was initially studied by Gupta [14], who found by a sieving procedure that there are no other solutions for $n < 4374$. In 1980 Narkiewicz [17] showed that the number

$$N_1(X) := \#\{n \leqslant X : \text{the ternary expansion } (2^n)_3 \text{ omits the digit } 2\}$$

has $N_1(X) \leqslant 1.62 X^{\alpha_0}$ with $\alpha_0 = \log_3 2 \approx 0.63092$. The Erdős question remains open and has appeared in several problem lists, for example, Erdős and Graham [6] and Guy [15, Problem B33]. In this paper we call the 'conjecture of Erdős' the weaker assertion that there are only finitely many exponents $n$ such that the ternary expansion $(2^n)_3$ of $2^n$ omits the digit 2.

This paper considers analogues of the conjecture of Erdős for iterates of two discrete dynamical systems, one acting on the real numbers and the other acting on the 3-adic integers, with an additional degree of freedom given by a parameter $\lambda$ specifying the initial condition. In both dynamical systems the parameter value $\lambda = 1$ recovers the original sequence $\{2^n : n \geqslant 0\}$ of Erdős as a forward orbit of the dynamics.

The first dynamical system is $y \mapsto 2y$ acting on the real numbers, which is a homeomorphism of $\mathbb{R}$ that is an expanding map. It produces a sequence of iterates $y_n = 2^n y_0$ starting from $y_0 = \lambda$. The *real dynamical system* concerns the iterates $y_n$. We also consider an associated dynamical system that gives integers, by applying the floor operator, obtaining the sequence $x_n = \lfloor y_n \rfloor$; that is,

$$x_n = x_n(\lambda) := \lfloor \lambda 2^n \rfloor \quad \text{for } n \geqslant 0. \tag{1.1}$$

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We call this the *truncated real dynamical system*. Strictly speaking, the truncated real dynamical system has forward orbits involving two variables $O^+(\lambda) := \{(y_n(\lambda), x_n(\lambda)) : n \geqslant 0\}$, with $\{y_n(\lambda)\}$ driving the dynamics. However, the expanding nature of the map $y \mapsto 2y$ implies that the integer sequence $\{x_n(\lambda) : n \geqslant 0\}$ contains enough information to uniquely determine the initial condition $\lambda$ of the iteration; here we consider the ternary expansions of the $x_n(\lambda)$.

The second dynamical system is $y \mapsto 2y$ acting on the 3-adic integers $\mathbb{Z}_3$, which is a 3-adic measure-preserving homeomorphism of $\mathbb{Z}_3$. It produces a sequence of iterates $y_n = 2^n y_0$ starting from the initial condition $y_0 = \lambda$. We write

$$y_n = y_n(\lambda) = \lambda 2^n \quad \text{for } n \geqslant 0. \tag{1.2}$$

In this case we study the membership of the values $y_n(\lambda)$ in the subset $\Sigma_{3,\bar{2}}$ of all 3-adic integers whose 3-adic expansion omits the digit 2; this is the multiplicative translate $\frac{1}{2}\Sigma_{3,\bar{1}}$ of the 3-adic analogue $\Sigma_{3,\bar{1}}$ of the classical 'middle-third' Cantor set.

In the real number case, dynamical systems of a related nature have been studied by several authors. Flatto, Lagarias, and Pollington [**9**] introduced a parameter $\lambda$ in similar questions concerning the fractional parts of the sequences $\{\{\lambda\xi^n\}\}$, for fixed $\xi > 1$, with the aim of proving results for the parameter value $\lambda = 1$ by proving universal results that are valid for all parameter values $\lambda > 0$. Recently, Dubickas and Novikas [**4**] considered the prime or compositeness properties of integers occurring in truncated recurrence sequences, including $\lfloor \lambda 2^n \rfloor$ as a particularly simple case. Dubickas [**3**] further extended both of these results to certain $\lambda$ that are real algebraic numbers.

Now we state the main results, and formulate some conjectures.

### 1.1. *Truncated real dynamical system: results*

For the truncated real dynamical system $x_n = \lfloor \lambda 2^n \rfloor$, we show that there is a uniform asymptotic upper bound that is valid for all nonzero $\lambda$ on the number of $n \leqslant X$ for which $(\lfloor \lambda 2^n \rfloor)_3$ omits the digit 2. Let $(k)_3$ denote the ternary digit expansion of the integer $k$.

THEOREM 1.1. *For each $\lambda > 0$, the upper bound*

$$N_\lambda(X) := \#\{n : 1 \leqslant n \leqslant X \text{ and } (\lfloor \lambda 2^n \rfloor)_3 \text{ omits the digit } 2\} \leqslant 25X^{0.9725} \tag{1.3}$$

*holds for all sufficiently large $X \geqslant n_0(\lambda)$.*

In the complementary direction, the function $N_\lambda(X)$ is not always bounded. The next result shows that there exist uncountably many $\lambda > 0$ such that the sequence $x_n(\lambda)$ contains infinitely many integers omitting the digit 2 in their ternary expansion.

THEOREM 1.2. *There exists an infinite sequence $S = \{n_k : k \geqslant 1\}$, satisfying $n_1 = 2$ and*

$$2^{1/14(n_{k-1}+2k-7)} \leqslant n_k \leqslant 2^{27(n_{k-1}+2k+6)}, \tag{1.4}$$

*having the following property: the set of real numbers $\Sigma(S)$ consisting of all $\lambda > 0$ for which all of the integers $\{x_n(\lambda) := \lfloor \lambda 2^n \rfloor : n \in S\}$ have ternary expansions omitting the digit 2 is an uncountable set.*

The set of exponents produced in this theorem forms a very thin infinite set. One can show that (1.4) implies that, for $X \geqslant 2$, its cardinality satisfies

$$\#\{n_k : 1 \leqslant n_k \leqslant X\} \geqslant \log_*(X) - 4, \tag{1.5}$$

in which $\log_*(X)$ denotes the number of iterations of the logarithm function starting at $X$ necessary to get a value smaller than 1. Thus we obtain for all $\lambda \in \Sigma(S)$, that

$$N_\lambda(X) \geqslant \log_*(X) - 4. \tag{1.6}$$

Next we consider properties of the set of $\lambda$ that have infinitely many such integers. We define the *truncated real exceptional set* $\mathcal{E}_T(\mathbb{R}_+)$ by

$$\mathcal{E}_T(\mathbb{R}_+) := \{\lambda > 0 : \text{infinitely many ternary expansions}(\lfloor \lambda 2^n \rfloor)_3 \text{ omit the digit } 2\}. \tag{1.7}$$

We prove the following result.

THEOREM 1.3. *The truncated real exceptional set has Hausdorff dimension*

$$\dim_H(\mathcal{E}_T(\mathbb{R}_+)) = \log_3(2) = \frac{\log 2}{\log 3} \approx 0.63092.$$

*It has nonzero* $\log_3(2)$*-dimensional Hausdorff measure.*

This result gives an indication of why it may be a hard problem to tell whether there are infinitely many exceptional powers of 2 for any particular $\lambda$, such as $\lambda = 1$; namely, it is likely to be a hard problem to decide whether any particular real number belongs to this 'small' exceptional set.

### 1.2. *Real dynamical system: conjecture*

Consider the real dynamical system $y \mapsto 2y$ on $\mathbb{R}_+$, without truncation, having forward orbits $O^+(\lambda) := \{y_n = \lambda 2^n : n \geqslant 0\}$. We define the *real exceptional set* $\mathcal{E}(\mathbb{R}_+)$ by

$$\mathcal{E}(\mathbb{R}_+) := \{\lambda > 0 : \text{infinitely many ternary expansions}(\lambda 2^n)_3 \text{ omit the digit } 2\}. \tag{1.8}$$

This set is much more constrained than the truncated exceptional set $\mathcal{E}_T(\mathbb{R}_+)$ discussed above. As far as we know, it could even be the empty set. The conjecture of Erdős is equivalent to the assertion that $1 \notin \mathcal{E}(\mathbb{R}_+)$.

Concerning this exceptional set, we make the following conjecture.

CONJECTURE 1.4. The real exceptional set

$$\mathcal{E}(\mathbb{R}) := \{\lambda \in \mathbb{R}_+ : \text{infinitely many ternary expansions}(\lambda 2^n)_3 \text{ omit the digit } 2\}$$

has Hausdorff dimension zero.

A stronger form of this conjecture would be that the exceptional set is countable; even stronger would be the assertion that the real exceptional set is empty. Thus, for the moment, there remains the possibility that the conjecture of Erdős might hold for all initial conditions $\lambda > 0$, for the full ternary expansions $(\lambda 2^n)_3$ as real numbers.

Note that, if the real exceptional set is nonempty, then it will necessarily be an infinite set, because it is forward invariant under multiplication by 2, that is, $2\mathcal{E}(\mathbb{R}_+) \subset \mathcal{E}(\mathbb{R}_+)$. It is clearly also forward invariant under multiplication by 3, that is, $3\mathcal{E}(\mathbb{R}_+) \subset \mathcal{E}(\mathbb{R}_+)$. Thus it is forward invariant under two commuting semigroup actions. However, the real exceptional set is not known to be a (topologically) closed set, so that known results implying Hausdorff dimension zero for closed sets invariant under certain commuting semigroup actions cannot be directly applied.

## 1.3. *3-adic dynamical system: results*

For a 3-adic integer $\lambda = \sum_{j=0}^{\infty} d_j 3^j$, with each $d_j \in \{0,1,2\}$, we write $(\lambda)_3 = (\dots d_2 d_1 d_0)_3$ for its 3-adic digital expansion. Our first observation is an upper bound on the number of solutions that are valid for all nonzero $\lambda \in \mathbb{Z}_3$, which extends the result of Narkiewicz [**17**] for $\lambda = 1$, using essentially the same proof.

THEOREM 1.5. *For each nonzero $\lambda \in \mathbb{Z}_3$, the 3-adic integers, and each $X \geqslant 2$, we have*

$$\tilde{N}_\lambda(X) := \#\{n \leqslant X : (\lambda 2^n)_3 \in \mathbb{Z}_3 \text{ omits the digit } 2\} \leqslant 2X^{\alpha_0}, \tag{1.9}$$

*with $\alpha_0 = \log_3 2 \approx 0.63092$.*

Next we study the *3-adic exceptional set*

$$\mathcal{E}(\mathbb{Z}_3) := \{\lambda \in \mathbb{Z}_3 : \text{infinitely many 3-adic expansions } \lambda 2^n \text{ omit the digit } 2\}. \tag{1.10}$$

This set seems hard to study directly, so, as approximations to the 3-adic exceptional set, we define for $k \geqslant 1$ the sequence of sets

$$\mathcal{E}^{(k)}(\mathbb{Z}_3) := \{\lambda \in \mathbb{Z}_3 : \text{at least } k \text{ values of } \lambda 2^n \text{ omit the digit } 2\}. \tag{1.11}$$

These sets clearly form a nested family under inclusion:

$$\mathcal{E}^{(1)}(\mathbb{Z}_3) \supset \mathcal{E}^{(2)}(\mathbb{Z}_3) \supset \mathcal{E}^{(3)}(\mathbb{Z}_3) \supset \dots,$$

and their intersection contains the exceptional set $\mathcal{E}(\mathbb{Z}_3)$. These sets are somewhat easier to study.

We consider the problem of estimating the Hausdorff dimension of the sets $\mathcal{E}^{(k)}(\mathbb{Z}_3)$ (with respect to the 3-adic metric) and show the following result.

THEOREM 1.6. (i) *The exceptional set $\mathcal{E}^{(1)}(\mathbb{Z}_3)$ has Hausdorff dimension*

$$\dim_H(\mathcal{E}^{(1)}(\mathbb{Z}_3)) = \alpha_0 \approx 0.63092. \tag{1.12}$$

(ii) *The exceptional set $\mathcal{E}^{(2)}(\mathbb{Z}_3)$ has Hausdorff dimension bounded as follows:*

$$\frac{1}{2}\log_3(2) \leqslant \dim_H(\mathcal{E}^{(2)}(\mathbb{Z}_3)) \leqslant \frac{1}{2}. \tag{1.13}$$

(iii) *The exceptional set $\mathcal{E}^{(3)}(\mathbb{Z}_3)$ has positive Hausdorff dimension bounded as follows:*

$$\frac{1}{6}\log_3 2 \leqslant \dim_H(\mathcal{E}^{(3)}(\mathbb{Z}_3)) \leqslant \dim_H(\mathcal{E}^{(2)}(\mathbb{Z}_3)). \tag{1.14}$$

This result is only a beginning of the study of $\dim_H(\mathcal{E}^{(k)})$ for general $k$. The (not necessarily closed) set $\mathcal{E}^{(k)}(\mathbb{Z}_3)$ is a countable union of closed sets $\mathcal{C}(2^{m_1}, 2^{m_2}, \dots, 2^{m_k})$ consisting of those $\lambda$ for which $\{\lambda 2^{m_j} : 1 \leqslant j \leqslant k\}$ all have 3-adic expansions that omit the digit 2. One can use this to obtain upper and lower bounds on the Hausdorff dimension of these sets by analysing the Hausdorff dimension of the individual sets $\mathcal{C}(2^{m_1}, 2^{m_2}, \dots, 2^{m_k})$. These sets are intersections of multiplicative translates of the 3-adic Cantor set, which we discuss in Section 1.4. In Theorem 1.6 the upper bound in (ii) is deduced using Theorem 1.8 below.

It is not clear whether $\dim_H(\mathcal{E}^{(k)}(\mathbb{Z}_3)) > 0$ for all $k \geqslant 1$. Proving or disproving this assertion already seems a subtle question.

Since $\mathcal{E}(\mathbb{Z}_3) \subseteq \mathcal{E}^{(k)}(\mathbb{Z}_3)$ for each $k \geqslant 1$, any upper bound on the Hausdorff dimension of $\mathcal{E}^{(k)}(\mathbb{Z}_3)$ gives an upper bound for the Hausdorff dimension of the 3-adic exceptional set $\mathcal{E}(\mathbb{Z}_3)$.

Each condition $\lambda 2^{m_j} \in \Sigma_{3,\bar{2}}$ imposes more constraints, apparently lowering the Hausdorff dimension. This motivates the following conjecture concerning the 3-adic exceptional set $\mathcal{E}(\mathbb{Z}_3)$, which parallels Conjecture 1.4.

CONJECTURE 1.7. The 3-adic exceptional set

$$\mathcal{E}(\mathbb{Z}_3) := \{\lambda \in \mathbb{Z}_3 : \text{infinitely many 3-adic expansions } \lambda 2^n \text{ omit the digit } 2\}$$

has Hausdorff dimension zero.

As in the real dynamical system case, we do not know much about this exceptional set, except that it contains 0. Again, the conjecture of Erdős is equivalent to the assertion that $1 \notin \mathcal{E}(\mathbb{Z}_3)$. The 3-adic exceptional set $\mathcal{E}(\mathbb{Z}_3)$ is forward invariant under multiplication by 2 and 3, but is not known to be a closed set.

### 1.4. *Intersection of multiplicative translates of Cantor sets: results*

The study of the exceptional sets $\mathcal{E}^{(k)}(\mathbb{Z}_3)$ leads to auxiliary questions concerning the Hausdorff dimensions of intersections of multiplicative translates of the 3-adic Cantor set $\Sigma_{3,\bar{2}}$, defined by

$$\Sigma_{3,\bar{2}} := \{\lambda \in \mathbb{Z}_3 : \text{the 3-adic expansion } (\lambda)_3 \text{ omits the digit } 2\}. \tag{1.15}$$

For integers $1 \leqslant M_1 < M_2 < \ldots < M_k$ we study the *multiplicative intersection sets*

$$\begin{aligned} \mathcal{C}(M_1, M_2, \ldots, M_k) &:= \{\lambda \in \mathbb{Z}_3 : (M_j\lambda)_3 \text{ omits the digit 2 for } 1 \leqslant j \leqslant k\} \\ &= \bigcap_{j=1}^{k} \left(\frac{1}{M_j}\Sigma_{3,\bar{2}}\right). \end{aligned} \tag{1.16}$$

These sets are closed sets. The standard 'middle third' Cantor set

$$\Sigma_{3,\bar{1}} := \{\lambda \in \mathbb{Z}_3 : \text{the 3-adic digit expansion } (\lambda)_3 \text{ omits the digit } 1\} \tag{1.17}$$

has $\Sigma_{3,\bar{1}} = 2\Sigma_{3,\bar{2}}$, so that all of the results given below for $\Sigma_{3,\bar{2}}$ convert to equivalent results for multiplicative translates of $\Sigma_{3,\bar{1}}$.

Multiplicative intersection sets arise in studying sets $\mathcal{E}^{(k)}(\mathbb{Z}_3)$ because they are given by countable unions of such sets, namely

$$\mathcal{E}^{(k)}(\mathbb{Z}_3) = \bigcup_{0 \leqslant m_1 < m_2 < \ldots < m_k} \mathcal{C}(2^{m_1}, 2^{m_2}, \ldots, 2^{m_k}).$$

What can be said about the Hausdorff dimension of sets $\mathcal{C}(M_1, M_2, \ldots, M_k)$? This dimension depends in a complicated manner on the 3-adic expansions of the $M_i$, and leads to various problems that seem interesting in their own right.

THEOREM 1.8. *Let $M$ be a positive integer that is not a power of 3. Let $\Sigma_{3,\bar{2}}$ be the ternary Cantor set. Then the Hausdorff dimension of $\mathcal{C}(1, M) = \Sigma_{3,\bar{2}} \cap (1/M)\Sigma_{3,\bar{2}}$ satisfies*

$$\dim_H(\mathcal{C}(1, M)) \leqslant \frac{1}{2}. \tag{1.18}$$

We do not know if this bound is sharp. However, it is possible to show that

$$\dim_H(\mathcal{C}(1, 7)) = \log_3\left(\frac{1 + \sqrt{5}}{2}\right) \approx 0.438.$$

For lower bounds on the Hausdorff dimension of such sets, we give the following simple sufficient condition for positivity of the Hausdorff dimension.

THEOREM 1.9. *Let $1 \leqslant M_1 < M_2 < \ldots < M_k$ be positive integers. Suppose that there is a positive integer $N$ belonging to the 3-adic Cantor set $\Sigma_{3,\bar{2}} \cup \mathbb{Z}$ such that all of the integers $NM_i$ satisfy*

$$NM_i \in \Sigma_{3,\bar{2}} \cap \mathbb{Z}, \quad 1 \leqslant j \leqslant k. \tag{1.19}$$

*Then*

$$\dim_H(\mathcal{C}(M_1, M_2, \ldots, M_k)) \geqslant \frac{\log_3(2)}{\lceil \log_3(NM_k) \rceil}. \tag{1.20}$$

This result is proved by the direct construction of a Cantor set of positive Hausdorff dimension inside $\mathcal{C}(M_1, M_2, \ldots, M_k)$. It gives a possible approach to obtaining a nonzero lower bound for $\dim_H(\mathcal{E}^{(k)}(\mathbb{Z}_3))$ for $k = 4$ or larger, if suitable $M_i = 2^{n_i}$ can be found that fulfil its hypotheses. However, it can be shown that the sufficient condition of Theorem 1.9 is not necessary; for example, $N = 1$ and $M_1 = 1$ and $M_2 = 52$ does not satisfy the hypothesis of this theorem, nevertheless $\mathcal{C}(1, 52)$ has positive Hausdorff dimension. Thus further strengthenings of this approach may be possible.

Determining the structure and Hausdorff dimension of the sets $\mathcal{C}(M_1, \ldots, M_k)$ leads to many open problems.

PROBLEM 1.10. Let

$$\mathcal{M}_C := \{M \geqslant 1 : \text{there exist integers } N_1, N_2 \in \Sigma_{3,\bar{2}} \text{ with } N_1 M = N_2\}.$$

Obtain upper and lower bounds for the number of integers $1 \leqslant M \leqslant X$ in $\mathcal{M}_C$.

PROBLEM 1.11. Let

$$\mathcal{M}_H := \{M \geqslant 1 : \dim_H(\mathcal{C}(1, M) > 0\}.$$

Obtain upper and lower bounds for the number of integers $1 \leqslant M \leqslant X$ in $\mathcal{M}_H$.

These are different problems, because it can be shown that the inclusion $\mathcal{M}_C \subset \mathcal{M}_H$ is strict.

### 1.5. *Generalized Erdős conjecture*

We formulate the following strengthening of Erdős's original question, by analogy with a conjecture of Furstenberg [**11**, Conjecture 2′], which is reviewed in Section 5.

CONJECTURE 1.12. Let $p$ and $q$ be multiplicatively independent positive integers, that is, all $\{p^i q^j : i \geqslant 0, j \geqslant 0\}$ are distinct. Then the base $q$ expansions of the powers $\{(p^n)_q : n \geqslant 1\}$ have the property that any given finite pattern $P = a_1 a_2 \ldots a_k$ of consecutive $q$-ary digits occurs in $(p^n)_q$, for all sufficiently large $n \geqslant n_0(P)$.

Conjecture 1.12 generalizes Erdős's original problem, which is the special case $p = 2$ and $q = 3$ with the single pattern $P = 2$. Furstenberg's conjecture concerns $d$-ary expansions of $\{(p^n)_d : n \geqslant 1\}$ with $d = pq$ in which $p$ and $q$ are multiplicatively independent, that is, his conjecture would apply to the 6-adic expansion $\{(2^n)_6 : n \geqslant 0\}$, rather than the 3-adic expansion above.

This conjecture might more properly be formulated as a question, since we present no significant new evidence in its favour. However, we think that any mechanism that forces a single pattern to appear from some point on should apply without exception to all patterns.

### 1.6. *Summary*

First, this paper places the original Erdős problem in a more general dynamical context. The two dynamical generalizations seem to give restrictions on the original Erdős question of roughly equal strength, as formulated in Theorems 1.1 and 1.5; that is, they each reduce the number of candidate $1 \leqslant n \leqslant X$ to at most $X^c$ for some $0 < c < 1$. What is interesting is that these arguments appear to use 'independent' information about the ternary expansions of $2^n$. The method used for the real dynamical system estimates the number of $n$ with $(2^n)_3$ omitting 2 in its $\log_3 X$ most significant ternary digits, whereas for the 3-adic dynamical system the method estimates the omission of 2 in the $\log_3 X$ least significant ternary digits of $2^n$. Heuristically, the most significant digits and least significant digits seem uncorrelated; this is the 'independence' referred to above. Furthermore, since the ternary expansion $(2^n)_3$ has about $\alpha_0 n$ ternary digits, the vast number of digits in the middle of the expansion $(2^n)_3$ are not exploited in either method; only a logarithmically small proportion of the available digits in the ternary expansion $(2^n)_3$ is considered.

It seems a challenging problem to find a method that effectively combines the two approaches to find better upper bounds on $N_1(X)$ than that given by Narkiewicz. Can one obtain an upper bound of $O(X^{\beta})$ for some $\beta < \log_3 2$ in this way? Can one show that the high-order digits and the low-order digits in the ternary expansion $(2^n)_3$ are 'uncorrelated' in some quantifiable way?

Second, we put forward Conjectures 1.4 and 1.7, which assert Hausdorff dimension zero of exceptional sets. These seem more approachable questions than the original question of Erdős. A much harder question seems to be to resolve whether the exceptional sets $\mathcal{E}(\mathbb{R}_+)$ and $\mathcal{E}(\mathbb{Z}_3)$ might be countable or finite.

Third, our analysis leads to a variety of interesting auxiliary problems in combinatorial number theory. These concern the Hausdorff dimension of intersections of multiplicative translates of 3-adic Cantor sets. These Hausdorff dimensions depend in a complicated arithmetic way on the values of the integer multipliers. These sets seem worthy of further study.

Finally, we observe analogies with work of Furstenberg [**10**, **11**] on actions of multiplicative semigroups and intersections of Cantor sets. These analogies suggest Conjecture 1.12.

### 1.7. *Contents and notation*

The contents of the remainder of the paper are as follows. In Section 2 we prove results for the truncated real dynamical system. In Section 3 we prove results for the 3-adic dynamical system. In Section 4 we establish auxiliary results on the Hausdorff dimensions of intersections of a finite number of multiplicative translates (by positive integers) of the 3-adic Cantor set, and include several examples. These results are used to complete the proofs of one result in Section 3. In Section 5 we discuss the work of Furstenberg. This includes a conjecture that motivates Conjecture 1.12, and his formulation of a notion of transversality of semigroup actions on a compact space and implications for intersections of Cantor sets. In the concluding Section 6 we describe the history associated with Erdős's original question.

NOTATION 1.13. Let

$$\{\{x\}\} := x - \lfloor x \rfloor = x \pmod 1$$

denote the fractional part of a real number $x$. Let

$$\langle\langle x\rangle\rangle := \left\{\left\{x+\frac{1}{2}\right\}\right\}-\frac{1}{2}$$

denote the (signed) distance of $x$ to the nearest integer.

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## 2. *Real dynamical systems: proofs*

We consider the sequence of real numbers $x_n^* := \lambda 2^n$ and consider the associated integers

$$x_n(\lambda) = \lfloor x_n^* \rfloor.$$

On taking logarithms to base 3, we have

$$\log_3 x_n^* = \log_3 \lambda + n \log_3 2 = m_n + w_n,$$

in which $m_n = \lfloor \log_3 x_n^* \rfloor$ is the integer part and $w_n := \log_3 x_n^* \pmod 1$ is the fractional part, with $0 \leqslant w_n < 1$. Now the digits in the ternary expansion of $x_n(\lambda)$ are completely determined by knowledge of the real number $w_n$, since $x_n(\lambda) = 3^{m_n} 3^{w_n}$; so they are the first $m_n$ ternary digits in the ternary expansion of $3^{w_n}$, since multiplication by $3^{m_n}$ simply shifts ternary digits to the left without changing them.

On the other hand, the sequence of $w_n$ forms an orbit under iteration of the map $T : [0,1] \mapsto [0,1]$ given by

$$T(w) = w + \log_3 2 \pmod 1, \tag{2.1}$$

on taking the initial condition $w_0 = \log_3 \lambda$, with $w_{n+1} = T(w_n)$. Since $\alpha_0 = \log_3 2$ is irrational, the map $T$ is an irrational rotation on the torus $\mathbb{R}/\mathbb{Z}$, which is known to be uniquely ergodic. In particular, every forward orbit of iteration of $T$ is uniformly distributed (mod 1), with the convergence rate to uniform distribution determined by properties of the continued fraction expansion of $\alpha_0$. We now examine the consequences of this property for the ternary expansions of $x_n^*$.

First, the leading ternary digits of $3^{w_n}$ confine the position of $w_n$ in the interval $[0,1]$ to a small subinterval. The property of omitting the digit 2 in a leading digit of a ternary expansion of $x_n$ will prohibit $w_n$ from certain subintervals in $[0,1]$; the allowed subintervals will have small measure. Using the fact that the distribution of $w_n$ (mod 1) approaches the uniform distribution fairly rapidly, one can show that most $w_n$ have some leading digit that is a 2; Theorem 1.1 is deduced using this idea, where the number $k$ of leading digits used will depend on the interval $[1, X]$ considered.

Second, one uses a construction that selects a rapidly growing set of values of $n = n_k$, chosen using the continued fraction expansion of $\alpha_0$, in such a way as to permit each $w_{n_k}$ to fall in a 'good' interval where the initial ternary digits for a large set of short intervals have $x_{n_k}(\lambda)$s with ternary expansions avoiding any 2s. A recursive interval construction, which modifies $\lambda$ slightly at each stage while not disturbing the initial ternary digits already selected, produces the sets in Theorem 1.2. Finally, we use a quantitative version of such an interval construction producing the set of Hausdorff dimension $\alpha_0$ in Theorem 1.3.

### 2.1. *Diophantine approximation lemmas*

We begin with two preliminary lemmas, the first on the spacings of multiples of an irrational number (modulo one) and the second on Diophantine approximation properties of $\alpha_0 = \log_3 2$.

LEMMA 2.1. *Let $\theta$ be irrational and consider the $N+1$ numbers*

$$\{x + j\theta \ (\mathrm{mod}\ 1) : 0 \leqslant j \leqslant N\},$$

*viewed as subdividing the torus $\mathbb{R}/\mathbb{Z}$ (the interval $[0,1]$ with endpoints identified) into $N+1$ subintervals ('steps').*

(i) *These subintervals take at most three distinct lengths. If three different lengths occur, say $L_1, L_2$, and $L_3$, then one of them is the sum of the other two, say $L_1 + L_2 = L_3$.*

(ii) *Let $\theta$ have the continued fraction expansion $\theta = [a_0, a_1, a_2, \ldots]$, with partial quotients $a_i$ and convergents $p_n/q_n$, whose denominators satisfy $q_{n+1} = a_{n+1}q_n + q_{n-1}$. Write uniquely*

$$N = (j+1)q_n + q_{n-1} + k, \quad 0 \leqslant k \leqslant q_n - 1, \tag{2.2}$$

*with $0 \leqslant j \leqslant a_{n+1} - 1$. Then the subintervals have lengths*

$$\begin{aligned} L_1 &= |\langle\langle q_n\theta\rangle\rangle|, \\ L_2 &= |\langle\langle q_{n-1}\theta\rangle\rangle + (j+1)\langle\langle q_n\theta\rangle\rangle|, \\ L_3 &= |\langle\langle q_{n-1}\theta\rangle\rangle + j\langle\langle q_n\theta\rangle\rangle|, \end{aligned}$$

*and occur with multiplicities $jq_n + q_{n-1} + k + 1, k+1$, and $q_n - (k+1)$, respectively. Here $L_3 = L_1 + L_2$, and $L_1 < L_2$ if $0 \leqslant j \leqslant a_{n+1} - 2$, whereas $L_2 < L_1$ if $j = a_{n+1} - 1$. Intervals of size $L_3$ do not occur if and only if $k = q_n - 1$.*

(iii) *For $N = q_{n+1} - 1$, there occur intervals of exactly two lengths $L_1$ and $L_2$ as above, and these lengths satisfy*

$$L_2 < L_1 < 2L_2. \tag{2.3}$$

*Proof.* Statements (i) and (ii) have a long history, which is detailed in Slater [**22**]. In particular, (ii) implies (i) and the formulas in (ii) appear in Slater [**22**, Equation (33), p. 1120]. The ordering of $L_1$ and $L_2$ follows from the fact that the $\langle\langle q_n\theta\rangle\rangle$ alternate in sign with successive $n$.

(iii) Let $N = q_{n+1} - 1$. If $a_n \geqslant 2$ then the decomposition (2.2) is

$$N = (a_{n+1} - 1)q_n + q_{n-1} + (q_n - 1),$$

with $k = q_n - 1$ and $j = a_{n+1} - 1$. Now (ii) says that there are steps of exactly two lengths $L_1$ and $L_2$ given by

$$\begin{aligned} L_1 &= |\langle\langle q_n\theta\rangle\rangle|, \\ L_2 &= |\langle\langle q_{n-1}\theta\rangle\rangle + (a_{n+1} - 1)\langle\langle q_n\theta\rangle\rangle|, \end{aligned}$$

and $L_2 < L_1$. Next we have

$$\langle\langle q_{n+1}\theta\rangle\rangle = \langle\langle q_{n-1}\theta\rangle\rangle + a_{n+1}\langle\langle q_n\theta\rangle\rangle = (\langle\langle q_{n-1}\theta\rangle\rangle + (a_{n+1} - 1)\langle\langle q_n\theta\rangle\rangle) + (\langle\langle q_n\theta\rangle\rangle).$$

Since $\langle\langle q_{n+1}\theta\rangle\rangle$ and $\langle\langle q_n\theta\rangle\rangle$ have opposite signs, and

$$|\langle\langle q_{n+1}\theta\rangle\rangle| \leqslant L_2,$$

we must have

$$L_2 < L_1 = L_2 + |\langle\langle q_{n+1}\theta\rangle\rangle| < 2L_2.$$

(The fact that $\theta$ is irrational gives the strict inequality at the last step.)

There remains the case $a_{n+1} = 1$. Now we find that the decomposition (2.2) is

$$N = q_n + q_{n-1} - 1 = a_n q_{n-1} + q_{n-2} + (q_{n-1} - 1),$$

with $k = q_{n-1} - 1$ and $j = a_{n-1} - 1$. As before, there are intervals of exactly two lengths

$$L_1 = |\langle\langle q_{n-1}\theta\rangle\rangle|,$$
$$L_2 = |\langle\langle q_{n-2}\theta\rangle\rangle + (a_n - 1)\langle\langle q_{n-1}\theta\rangle\rangle|,$$

with $L_2 < L_1$. We deduce, as in the case $a_{n+1} \geqslant 2$, that

$$L_2 < L_1 = L_2 + |\langle\langle q_n\theta\rangle\rangle| < 2L_2,$$

as required. □

The point of Lemma 2.1 for our application is that for the choice $N = q_n - 1$ the points $\{x + j\theta \pmod 1) : 0 \leqslant j \leqslant N\}$ are very close to uniformly spaced on the interval $[0, 1]$. The next result obtains information on the denominators $q_n$ of the continued fraction convergents of the irrational number $\alpha_0$.

Lemma 2.2. *For the irrational number $\alpha_0 = \log_3 2$ the following hold.*

(i) *For all $q \geqslant 1$ and all integer $p$, there holds the Diophantine inequality*

$$\left|\alpha_0 - \frac{p}{q}\right| \geqslant \frac{1}{1200}\frac{1}{q^{c_0+1}}, \tag{2.4}$$

*with $c_0 = 13.3$.*

(ii) *The denominators $q_n$ of the continued fraction convergents $p_n/q_n$ of $\alpha_0$ satisfy*

$$q_n \leqslant 1200(q_{n-1})^{c_0}. \tag{2.5}$$

*Proof.* (i) The existence of a bound of this general form, apart from the precise constants, follows from Baker's results on linear forms in logarithms [**2**, Theorem 3.1], applied to the linear form $\Lambda = k + q\log 2 - p\log 3$, taking $k = 0$, and noting that its height $B := \max\{|p|, q\} \leqslant 2q$.

The particular bound (2.4) is obtained from a result of Simons and de Weger [**21**, Lemma 12], who show, for $k \geqslant 1$ and all integers $l$, that

$$|(k + l)\log 2 - k\log 3| > \exp(-13.3(0.46057))k^{-13.3} > \frac{1}{484}k^{-13.3}.$$

Their result is proved using a transcendence result of Rhin [**18**, Proposition, p. 160] for linear forms in two logarithms. We may suppose that $k < k + l < 1.6k$, and obtain

$$\left|\log_3 2 - \frac{k}{k+l}\right| > \frac{1}{\log 3}\exp(-13.3(0.46057))(k + l)^{-1}k^{-13.3} \geqslant \frac{1}{1200}(k + l)^{-14.3},$$

which on taking $p = k$ and $q = k + l$ gives the needed bound.

(ii) Since $\alpha_0$ lies in the interval between two successive continued fraction convergents $p_{n-1}/q_{n-1}$ and $p_n/q_n$, we obtain using (2.4) that

$$\frac{1}{q_n q_{n-1}} = \left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| = \left|\alpha_0 - \frac{p_{n-1}}{q_{n-1}}\right| + \left|\alpha_0 - \frac{p_n}{q_n}\right| \geqslant \frac{1}{1200}\frac{1}{(q_{n-1})^{c_0+1}}.$$

Multiplying by $1200q_n(q_{n-1})^{c_0+1}$ gives (2.5). □

## 2.2. *Interval constructions*

We now construct sets where the truncated real dynamical system has infinitely many solutions with $(\lfloor\lambda 2^n\rfloor)_3$ omitting the digit 2.

*Proof of Theorem* 1.1. Let $\lambda > 0$. We study for $1 \leqslant n \leqslant X$ the ternary expansion of

$$x_n = x_n(\lambda) = \lfloor \lambda 2^n \rfloor.$$

We will study the first $k$ leading ternary digits of the $\{x_n : 1 \leqslant n \leqslant X\}$, where we choose $k$ as follows. If $p_j/q_j$ are the convergents of the continued fraction expansion of $\alpha_0 = \log_3 2$, then choose $l$ such that $q_{l-1} < X \leqslant q_l$, and then choose $k$ to be the number of ternary digits in $q_{l-1}$, so that $3^{k-1} < q_{l-1} \leqslant 3^k$. Note that $k = \lceil \log_3 q_{l-1} \rceil \leqslant \lceil \log_3 X \rceil$.

We now set $w_n := \log_3(\lambda 2^n) \pmod 1$, with $0 \leqslant w_n < 1$, so that

$$w_n = n\alpha_0 + \log_3 \lambda \pmod 1. \tag{2.6}$$

We now observe that where $w_n$ falls in the interval $[0, 1)$ specifies the first $k$ ternary digits in the ternary expansion of $3^{w_n}$, with $1 \leqslant 3^{w_n} < 3$, and we can partition the interval $[0, 1)$ into half-open intervals corresponding to each such ternary expansion. Consider a ternary expansion

$$\mathbf{b} = [b_0 b_1 \dots b_{k-1}]_3, \quad b_i \in \{0, 1, 2\}, \ \ b_0 \neq 0,$$

of length $k$, noting that there are $2 \cdot 3^{k-1}$ such expansions. Set

$$\beta(\mathbf{b}) = \sum_{j=0}^{k-1} \frac{b_j}{3^j}, \tag{2.7}$$

for which $1 \leqslant \beta(\mathbf{b}) < 3$, and associate the following subinterval of $[0, 1)$:

$$J(\mathbf{b}) := \left[\log_3 \beta(\mathbf{b}), \log_3 \left(\beta(\mathbf{b}) + \frac{1}{3^{k-1}}\right)\right). \tag{2.8}$$

These $2 \cdot 3^{k-1}$ subintervals partition $[0, 1)$, from $J([10 \dots 0]_3) = [\log_3(1), \log_3(1 + 1/3^{k-1}))$ to $J([22 \dots 2]_3) = [\log_3(3 - 1/3^{k-1}), \log_3 3)$.

We claim that the following conditions (C1) and (C2) are equivalent for $x_n$ with $3^m \leqslant x_n \leqslant 3^{m+1}$, with $m \geqslant k$:

(C1) $x_n$ has a ternary expansion having the $k$ leading digits $\mathbf{b} = [b_0 b_1 \dots b_{k-1}]_3$, that is, $x_n = \sum_{j=0}^{m} b_j 3^{m-j}$, for some $(b_{k+1}, \dots, b_m)$;

(C2) $w_n = \log_3 x_n \pmod 1$ has $w_n \in J(\mathbf{b})$.

The claim follows because the definition of $J(\mathbf{b})$ specifies the $k$-leading ternary digits of $3^{w_n}$, while $x_n = 3^m 3^{w_n}$ and the effect of multiplying by $3^m$ simply shifts all ternary digits $m$ places to the left without changing the leading digits.

Next we note that the intervals $J(\mathbf{b})$ all have the same length to within a factor of 3, namely

$$\frac{1}{3^k} \leqslant |J(\mathbf{b})| \leqslant \frac{1}{3^{k-1}}. \tag{2.9}$$

This holds using

$$|J(\mathbf{b})| = \log\left(\beta(\mathbf{b}) + \frac{1}{3^{k-1}}\right) - \log(\beta(\mathbf{b})) = \int_{\beta(\mathbf{b})}^{\beta(\mathbf{b})+(1/3^{k-1})} \frac{dx}{x},$$

and the bounds (2.9) follow since $\frac{1}{3} \leqslant 1/x \leqslant 1$.

Next we examine the $w_n$ in consecutive blocks of length $N = q_{l-1} - 1$, that is, the set $\{w_n : j(q_{l-1} - 1) \leqslant n < (j+1)(q_{l-1} - 1)\}$. By (2.6) we may apply Lemma 2.1(iii) to this sequence of numbers, to infer that the spacings between them are of two lengths $L_1$ and $L_2$ that satisfy $L_2 < L_1 < 2L_2$. In particular, since $3^{k-1} \leqslant q_{l-1} \leqslant 3^k$, these block sizes satisfy

$$\frac{1}{2 \cdot 3^k} \leqslant \frac{1}{2(q_{l-1} - 1)} \leqslant L_1 < L_2 \leqslant \frac{2}{q_{l-1} - 1} \leqslant \frac{2}{3^{k-1}}.$$

We conclude using (2.9) that each subinterval $J(\mathbf{b})$ contains at most six points $w_n$ from this block. Thus at most six values of $n$ in $j(q_{l-1} - 1) \leqslant n < (j+1)(q_{l-1} - 1)$ give an $x_n$ having the given initial $k$-digit ternary expansion $\mathbf{b} = [b_0 b_1 \dots b_{k_1}]_3$.

We know that there are exactly $2^{k-1}$ values of $\mathbf{b} = [b_0 b_1 \dots b_{k_1}]_3$ that omit the ternary digit 2, so the above shows that there are at most $6 \cdot 2^{k-1}$ values of $n$ in each such block giving an $x_n$ whose initial $k$ ternary digits avoid 2. There are $\lfloor X/(q_{l-1} - 1)\rfloor + 1$ such blocks covering all $1 \leqslant n \leqslant X$, and hence we conclude that there are at most

$$\begin{aligned} M := 6 \cdot 2^{k-1} \left( \frac{X}{q_{l-1} - 1} + 1 \right) &\leqslant 6 \cdot 2^{k-1} \left( \frac{X}{3^{k-1}} + 1 \right) \\ &\leqslant 6 \left( \left(\frac{2}{3}\right)^{k-1} X + 2^{k-1} \right) \leqslant 12 \left(\frac{2}{3}\right)^{k-1} X \end{aligned}$$

values of $x_n$ whose initial $k$ ternary digits omit the digit 2. (In the last inequality we used $X \geqslant q_{l-1} > 3^{k-1}$.)

It remains to upper bound $M$ as a function of $X$. Using Lemma 2.2(ii), we have

$$X \leqslant q_l \leqslant 1200(q_{l-1})^{c_0} \leqslant 1200(3^k)^{c_0},$$

with $c_0 = 13.3$. We apply this bound to obtain

$$\left(\frac{3}{2}\right)^k = \left(3^{c_0 k}\right)^{\log_3(3/2) c_0^{-1}} \geqslant \left(\frac{1}{1200} X\right)^{((1-\alpha_0)/c_0)}.$$

Here $1/37 < (\log_3(3/2)) c_0^{-1} = (1 - \alpha_0)/c_0 \leqslant 1/36$, so we obtain

$$\left(\frac{2}{3}\right)^k \leqslant (1200)^{1-\alpha_0/c_0} X^{-((1-\alpha_0)/c_0)}.$$

Substituting this into the definition of $M$, we obtain

$$M \leqslant 18 \left(\frac{2}{3}\right)^k X \leqslant 18 \cdot (1200)^{1/36} X^{1-(1-\alpha_0)/c_0} \leqslant 25 X^{36/37} \leqslant 25 X^{0.9725}$$

and the result follows. □

*Proof of Theorem* 1.2. We will construct a rapidly increasing sequence of integers $S_0 = \{m_k : k \geqslant 1\}$ having the form

$$m_k = l_0 + l_1 + \dots + l_k \tag{2.10}$$

such that there is an uncountable set of real numbers $\tilde{\Sigma}$ such that *all* of the following numbers $\lambda \in \Sigma$ have the following property: for each $k \geqslant 1$, the integer $M_k := \lfloor \lambda 2^{m_k} \rfloor$ has a ternary expansion that omits the digit 1.

We now claim that all of the integers $N_k := \lfloor \lambda 2^{m_k - 1} \rfloor$ have ternary expansions $(N_k)_3$ that omit the digit 2. This holds because for each $N_k$ either $M_k = 2N_k$ or $M_k = 2N_k + 1$, but $M_k$ is necessarily an even integer since all of its ternary digits are 0 or 2, so we must have $M_k = 2N_k$. Thus $N_k$ has only digits 0 and 1 in its ternary expansion, so we have, for $S = \{m_k - 1 : k \geqslant 1\}$, that

$$\tilde{\Sigma} \subset \Sigma(S) := \{\lambda : (\lfloor \lambda 2^{n_k} \rfloor)_3 \text{ omits the digit } 2\}.$$

Hence $\Sigma(S)$ is an uncountable set.

We choose the $l_k$ recursively, taking $l_0 = m_0 = 0$ and successively choosing $l_k$ to be the smallest integer satisfying $l_k \geqslant 2k$ with

$$0 < \{\{\log_3 2^{l_k}\}\} = \{\{l_k \alpha_0\}\} < 2^{-m_{k-1} - 2k - 4}, \tag{2.11}$$

with $m_{k-1} = l_0 + l_1 + \dots + l_{k-1}$. We set

$$r_k := \lfloor l_k \alpha_0 \rfloor, \quad \alpha_0 = \log_3 2.$$

The condition $l_k \geqslant 2k$ ensures that $r_k \geqslant k$. Then we have

$$2^{l_k} = 3^{l_k\alpha_0} = 3^{r_k + \{\{l_k\alpha_0\}\}} = 3^{r_k}3^{\{\{l_k\alpha_0\}\}}.$$

Using $e^x \leqslant 1+2x$ for $0 \leqslant x \leqslant 1$, we have

$$3^{\{\{l_k\alpha_0\}\}} = e^{\{\{l_k\alpha_0\}\}\log 3} \leqslant 1 + 2\log 3\{\{l_k\alpha_0\}\} \leqslant 1 + \frac{2\log 3}{2^{m_{k-1}+2k+4}}.$$

Thus we obtain

$$3^{r_k} < 2^{l_k} < 3^{r_k}\left(1 + \frac{2\ln 3}{2^{m_{k-1}+2k+4}}\right) \leqslant 3^{r_k}\left(1 + \frac{1}{3^{(m_{k-1}+2k+2)\alpha_0}}\right). \tag{2.12}$$

This says that the ternary expansion $(2^{l_k})_3$ has leading digit 1 followed by a string of at least $(m_{k-1}+2k+2)\alpha_0$ trailing zeros.

Given this choice of $\{l_k : k \geqslant 1\}$, we define the set $\Sigma$ to consist of all real numbers

$$\tilde{\Sigma} := \left\{\lambda := \sum_{k=0}^{\infty}\frac{d_k}{2^{m_k}} : \lambda \text{ is admissible}\right\}, \tag{2.13}$$

where $\lambda$ is called *admissible* if, for all $k \geqslant 1$, it has the following two properties.

(P1) The digit $d_k$ satisfies

$$0 \leqslant d_k \leqslant 3^{r_k} - 3^{r_k-k}. \tag{2.14}$$

(P2) Let $\lambda_k := \sum_{j=0}^{k} d_j/2^{m_j}$. Then the integer

$$M_k := \lambda_k 2^{m_k} \tag{2.15}$$

has a ternary expansion $(M_k)_3$ that omits the digit 1.

Claim 2.3. *Any* $\lambda = \sum_{j=0}^{\infty} d_j/2^{m_j}$ *with all* $d_k$ *satisfying* (P1) *satisfies*

$$1 \leqslant \lambda < 2 \tag{2.16}$$

*and*

$$M_k = \lambda_k 2^{m_k} = \lfloor \lambda 2^{m_k}\rfloor \quad \textit{for all } k \geqslant 1. \tag{2.17}$$

*Proof.* To prove the claim, we observe that (P1) gives

$$\begin{aligned} 1 \leqslant \lambda &\leqslant 1 + \sum_{k=1}^{\infty}\frac{1}{2^{m_{k-1}}}\left(\frac{3^{r_k} - 3^{r_k-k}}{2^{l_k}}\right) \\ &\leqslant 1 + \sum_{k=1}^{\infty}\frac{1}{2^{m_{k-1}}}(1 - 3^{-k}) < 2. \end{aligned} \tag{2.18}$$

Next, (P1) gives

$$\begin{aligned} 0 \leqslant \lambda - \lambda_k &= \sum_{j=k+1}^{\infty}\frac{d_j}{2^{m_j}} = \frac{1}{2^{m_k}}\left(\sum_{j=k+1}^{\infty}\frac{d_j}{2^{m_j-m_k}}\right) \\ &\leqslant \frac{1}{2^{m_k}}\left(\sum_{j=k+1}^{\infty}\left(1-\frac{1}{3^j}\right)\frac{1}{2^{m_{j-1}-m_k}}\right) \\ &\leqslant \frac{1}{2^{m_k}}\left(\sum_{j=k+1}^{\infty}\left(1-\frac{1}{3^j}\right)\frac{1}{2^{(j-k-1)(2j)}}\right) < \frac{1}{2^{m_k}}, \end{aligned}$$

proving Claim 2.3. $\square$

CLAIM 2.4. *For any choice of* $\{d_j : 1 \leqslant j \leqslant k-1\}$ *that satisfy both* (P1) *and* (P2), *there are at least* $2^{r_k} - 2^{r_k - k}$ *choices of* $d_k$ *that satisfy* (P1) *and* (P2).

*Proof.* To prove this, first note that

$$\lambda_{k-1} 2^{m_k} = M_{k-1} 2^{m_k - m_{k-1}} = M_{k-1} 2^{l_k} = M_{k-1} 3^{r_k} + M_{k-1}(2^{l_k} - 3^{r_k}). \tag{2.19}$$

We assert that

$$0 \leqslant M_{k-1}(2^{l_k} - 3^{r_k}) \leqslant 3^{r_k - k}. \tag{2.20}$$

The left inequality is immediate, and using (2.18) we have $M_{k-1} \leqslant \lambda 2^{m_{k-1}} \leqslant 2^{m_{k-1}+1}$, while (2.12) gives

$$\begin{aligned} M_{k-1}(2^{l_k} - 3^{r_k}) &\leqslant 2^{m_{k-1}+1} \left( 3^{r_k} \frac{\ln 3}{2^{m_{k-1}+2k+4}} \right) \\ &\leqslant 3^{r_k} \frac{1}{2^{2k+3}} \leqslant 3^{r_k - k}, \end{aligned}$$

proving (2.20).

From (2.19) and (2.20) we see that the ternary expansion of $\lambda_{k-1} 2^{m_k}$ repeats that of $M_{k-1}$ shifted $r_k$ positions to the left, then has a block of at least $k$ zeros, and following this has the ternary expansion of the integer $M_{k-1}(2^{l_k} - 3^{r_k})$. It follows that, choosing from the range of values $0 \leqslant d_k \leqslant 3^{r_k} - 3^{r_k - k}$, and setting $\lambda_k := \sum_{j=0}^{k} d_j / 2^{m_j}$, the integers

$$M_k := \lambda_k 2^{m_k} = \lambda_{k-1} 2^{m_k} + d_k \tag{2.21}$$

can be selected to give all ternary integers that:

(i) have the ternary expansion matching $M_{k-1}$ to the left of the $r_k$th position;
(ii) omit the digit 1; and
(iii) have at least one 2 and at least one 0 in positions between $r_k$ and $r_k - k$;

we call these *allowable values*. In these $k+1$ positions the largest allowed value is $222\ldots20$ and the smallest is $000\ldots02$. These produce exactly $2^{r_k} - 2^{r_k - k}$ such ternary integers $M_k$, constructed by the choice of the same number of allowable values $d_k$. This proves Claim 2.4. □

CLAIM 2.5. *The set* $\tilde{\Sigma}$ *contains uncountably many admissible* $\lambda$, *and each of them has the property that every*

$$M_k = \lfloor \lambda 2^{m_k} \rfloor, \quad k \geqslant 1, \tag{2.22}$$

*has a ternary expansion* $(M_k)_3$ *that omits the digit 1.*

*Proof.* Indeed, Claim 2.4 implies that there are uncountably many such $\lambda$, since the construction has a Cantor set form that gives an infinite tree of values with branching at least two at every node at every level $k \geqslant 2$. The relation (2.22) holds by Claim 2.3, and these $M_k$ have ternary expansions omitting 2 by (P2). Thus Claim 2.5 follows. □

It remains to verify the upper and lower bounds (1.4) on the growth rate of the sequence $m_k$. The size of $m_k$ is determined by the Diophantine condition on $l_k$ given by equation (2.11). (The numbers $l_k$ grow so rapidly that the side condition $l_k \geqslant 2k$ is automatically satisfied for $k \geqslant 2$.) Note that we cannot directly use Dirichlet's box principle to get an upper bound for the size of the minimal $l_k$ satisfying (2.11) because this is a one-sided approximation condition. Instead we have that the minimal $l_k$ will be no larger than that even-numbered convergent $q_{2l}$ of the continued fraction expansion of $\alpha_0$ satisfying

$$q_{2l-2} \leqslant 2^{m_{k-1}+2k+4} < q_{2l}.$$

Lemma 2.2(ii) gives the bound

$$q_{2l} \leqslant \frac{1}{C_0^2}(q_{2l-2})^{2c_1} = (1200)^2(q_{2l-2})^{26.6} \leqslant 2^{27m_{k-1}+54k+132}. \tag{2.23}$$

Since $n_k = m_k - 1$, we obtain

$$n_k \leqslant m_k \leqslant m_{k-1} + q_{2l} \leqslant m_{k-1} + 2^{27m_{k-1}+54k+132} \leqslant 2^{27(n_{k-1}+2k+6)},$$

which is the upper bound in (1.4).

Lemma 2.2 implies a lower bound on how small $l_{k+1}$ can be to make (2.11) hold, namely we must have

$$(l_{k+1})^{c_0} \geqslant 2^{m_k+2j-7}, \tag{2.24}$$

with $c_0 = 13.3$, to avoid contradicting Lemma 2.2(i). This yields the lower bound in (1.4), which holds for $n_k = m_k - 1$ produced in this construction, and completes the proof of Theorem 1.2. □

2.3. *Hausdorff dimension of truncated exceptional set*

We recall basic facts on Hausdorff dimension; see [**7**, **8**, **19**]. It is a notion defined for any metric space $(X, \|\cdot\|)$; here we take $(\mathbb{R}, \|\cdot\|)$ with the Euclidean metric. Given a set $S$, its diameter is $\mathrm{diam}(S) = \sup\{\|x-y\| : x, y \in S\}$. We consider finite or countable coverings of $S$ with closed sets $\mathcal{C} := \{I_j; j \in A\}$, so $S \subset \bigcup_\alpha I_\alpha$. The diameter $\mathrm{diam}(\mathcal{C})$ of a covering $\mathcal{C}$ is

$$\mathrm{diam}(\mathcal{C}) := \sup_{I_j \in \mathcal{C}}[\mathrm{diam}(I_j)].$$

The Hausdorff dimension $\dim_H(S)$ of a set $S$ is the infimum of all $\alpha$ such that, for arbitrarily small $\epsilon > 0$, there exists a covering of $S$ with sets of diameter at most $\epsilon$, such that

$$V_\alpha(\mathcal{C}) := \sum_j (\mathrm{diam}(I_j))^\alpha < \epsilon.$$

Alternatively, for any value $\alpha$ one defines the $\alpha$-dimensional Hausdorff (outer) measure of $S$ to be

$$\mathcal{H}^\alpha(S) := \liminf_{\epsilon \to 0}\left(\inf_{\mathrm{diam}(\mathcal{C})<\epsilon} V_\alpha(\mathcal{C})\right),$$

where $\mathcal{C}$ runs over allowable covers of $S$. The Hausdorff dimension $\dim_H(S)$ is determined as the unique value of $\alpha$ such that $\mathcal{H}^{\alpha'}(S) = 0$ for $\alpha' > \alpha$ and $\mathcal{H}^{\alpha'}(S) = \infty$ for $0 \leqslant \alpha' < \alpha$. In the critical dimension $\alpha$, a set $S$ can have zero, positive, or infinite Hausdorff measure. In the case of $(\mathbb{R}, \|\cdot\|)$, we get the same notions on restricting coverings to consist of closed intervals. An important property of Hausdorff dimension is that if $S = \bigcup_{j=1}^\infty S_j$ then

$$\dim_H(S) = \sup_j[\dim_H(S_j)]. \tag{2.25}$$

*Proof of Theorem* 1.3. We consider the truncated exceptional set $\mathcal{E}_T(\mathbb{R}_+)$ . We first establish the upper bound $\dim_H(\mathcal{E}_T(\mathbb{R}_+)) \leqslant \alpha_0$. We have

$$\mathcal{E}_T(\mathbb{R}_+) = \bigcup_{M=2}^\infty \left(\mathcal{E}_T(\mathbb{R}_+) \cap \left[\frac{1}{M}, M\right]\right).$$

Since the Hausdorff dimension of a countable union of sets is the supremum of the Hausdorff dimensions of the separate sets, it suffices to show that

$$\dim_H\left(\mathcal{E}_T(\mathbb{R}_+) \cap \left[\frac{1}{M}, M\right]\right) \leqslant \alpha_0 = \log_3 2. \tag{2.26}$$

To show this we find suitable coverings of these sets. For each $n \geqslant 1$, we have

$$\mathcal{E}_T(\mathbb{R}_+) \cap \left[\frac{1}{M}, M\right] \subset S_n(M) := \bigcup_{j=N}^{\infty} \Sigma_j\left(\left[\frac{1}{M}, M\right]\right), \tag{2.27}$$

with

$$\Sigma_j\left(\left[\frac{1}{M}, M\right]\right) := \left\{\lambda : -\frac{1}{M} \leqslant \lambda \leqslant M \text{ and } (\lfloor \lambda 2^j \rfloor)_3 \text{ omits the digit } 2\right\}.$$

The set $S_n(M)$ thus encodes a 'tail event' that there are arbitrarily large $j$ for which $(\lfloor \lambda 2^j \rfloor)_3$ omit the digit 2. We will eventually let $n \to \infty$, so we suppose that $n \geqslant \log_3 M + 2$, so that $\lambda 2^j \geqslant 1$ for any $j \geqslant n$. Now consider such $j$ as fixed, and note that $\lfloor \lambda 2^j \rfloor$ takes a fixed integer value on an interval of length $1/2^j$. Letting $\mathbf{b} = (\lfloor \lambda 2^j \rfloor)_3$, we see that allowable values of $\mathbf{b}$ satisfy $1 \leqslant \mathbf{b} \leqslant M2^j$. As $\lambda$ varies over $[1/M, M]$ these integers vary over a subset of $[1, M2^j]$ and, of these, the number of such ternary expansions $\mathbf{b}$ that omit the digit 2 is at most (counting integers over successive blocks $[3^{k-1}, 3^k)$)

$$\begin{aligned} 1 + 2 + \ldots + 2^{\lceil \log_3(2^j M) \rceil} &\leqslant 2^{\log_2(2^j M)+2} \\ &\leqslant 2^{j\alpha_0 + \log_3 M + 2} \leqslant 4M2^{j\alpha_0}. \end{aligned}$$

Thus we obtain a collection

$$\mathcal{I}_j(M) := \left\{ I_j(\mathbf{b}) : \mathbf{b} \text{ gives an admissible interval for } \lfloor \lambda 2^j \rfloor, \frac{1}{M} \leqslant \lambda \leqslant M \right\}$$

of at most $4M2^{j\alpha_0}$ intervals of length $1/3^j$, and these intervals cover the set $\Sigma_j([1/M, M])$. Summing over all $j \geqslant n$, we obtain an infinite collection of intervals

$$\mathcal{I}(n, M) := \bigcup_{j=n}^{\infty} \mathcal{I}_j(M),$$

which cover the set $\mathcal{E}_T(\mathbb{R}_+) \cap [1/M, M])$ by (2.27), and every interval included has length at most $1/2^n$. Now fix $\epsilon > 0$ and observe that

$$\begin{aligned} \sum_{I \in \mathcal{I}(n,M)} |I|^{\alpha_0 + \epsilon} &= \sum_{j=n}^{\infty} \left( \sum_{I \in \mathcal{I}_j(M)} \left(\frac{1}{2^j}\right)^{\alpha_0+\epsilon} \right) \\ &\leqslant \sum_{j=n}^{\infty} 4M2^{j\alpha_0} \left(\frac{1}{2^j}\right)^{\alpha_0+\epsilon} \\ &= 4M \left(\sum_{j=n}^{\infty} 2^{-j\epsilon}\right) = \left(\frac{4M}{1-2^{-\epsilon}}\right) 2^{-n\epsilon}. \end{aligned}$$

Letting $n \to \infty$, the diameter of the covering $\mathcal{I}(n, M)$ goes to zero, and the scaled length goes to zero as well, which establishes

$$\dim_H\left(\mathcal{E}_T(\mathbb{R}_+) \cap \left[\frac{1}{M}, M\right]\right) \leqslant \alpha_0 + \epsilon.$$

Now we can let $\epsilon \to 0$ to obtain (2.6), and the upper bound $\dim_H(\mathcal{E}_T(\mathbb{R}_+)) \leqslant \alpha_0$ follows.

To establish the lower bound $\dim_H(\mathcal{E}_T(\mathbb{R})) \geqslant \alpha_0$ is more difficult, as it requires controlling all coverings of the set. We will actually establish the stronger result that

$$\text{meas}_{\alpha_0}(\tilde{\Sigma}) > \frac{1}{16}, \tag{2.28}$$

where $\tilde{\Sigma} \subset [1, 2]$ is the set constructed in (2.13) in Theorem 1.2. The set $\tilde{\Sigma}$ had a construction resembling a Cantor set, with two differences. The first difference is that the dissection at each layer $k$ depended on the previous layers, and the second difference is that the layer at level $k$ involved denominators $2^{m_k}$ with

$$m_k = l_0 + l_1 + \ldots + l_k,$$

with the $l_k$ growing extremely rapidly. We can, however, adapt an argument given in Falconer [**8**, Example 2.7, p. 31] for the Cantor set to show (2.28).

To begin, we claim that $\tilde{\Sigma}$ has a representation as

$$\tilde{\Sigma} = \bigcap_{s=1}^{\infty} X_s, \tag{2.29}$$

in which $X_s$ consists of a union of a collection $\mathcal{J}_s$ of disjoint intervals of size proportional to $3^{-s}$, and the sets are nested:

$$\ldots X_3 \subset X_2 \subset X_1.$$

Here the intervals in $\mathcal{J}_s$ will play the role of the Cantor set dissection into intervals at level $s$, for each power of $3^s$.

We first define the collection $\mathcal{J}_s$ for those levels $s = s_k$ with

$$s_j := \lfloor m_j \alpha_0 \rfloor, \tag{2.30}$$

which are directly given in the construction of Theorem 1.2. Then we show that one can fill in all of the intermediate layers $s_k \leqslant s < s_{k+1}$.

We have $3^{s_k} < 2^{m_k} < 3^{s_k+1}$, and the set $\mathcal{J}_{s_k}$ is the union of all closed intervals:

$$\mathcal{J}_{s_k} := \left\{ \left[ \frac{M}{2^{m_k}}, \frac{M+1}{2^{m_k}} \right] : M = \lambda_k 2^{m_k} \text{ with } \lambda_k = \sum_{j=0}^{k} \frac{d_j}{2^{m_j}} \text{ admissible} \right\},$$

with admissibility in the construction in Theorem 1.2. Here we have

$$2^{m_k} = 2^{l_1+\ldots+l_k} = 3^{l_1\alpha_0+\ldots+l_k\alpha_0} = 3^{r_1+r_2+\ldots+r_k} \cdot 3^{\{\{l_1\alpha_0\}\}+\ldots+\{\{l_k\alpha_0\}\}} \leqslant 2 \cdot 3^{r_1+\ldots+r_k},$$

using the fact that

$$\sum_{k=1}^{\infty} \{\{l_k\alpha_0\}\} \leqslant \sum_{k=1}^{\infty} 2^{-m_{k-1}-2k-2} \leqslant \frac{1}{2},$$

using (2.11). This also establishes that

$$s_k = r_1 + r_2 + \ldots + r_k. \tag{2.31}$$

Inside each interval at level $s = s_{k-1}$ there fit exactly $2^{r_k} - 2^{r_k-k}$ subintervals at ternary level $s = s_k$, each of length $2^{-m_k}$, and we now know that $\frac{1}{2}3^{-s_k} \leqslant 2^{-m_k} \leqslant 3^{-s_k}$. This dissection of an interval at ternary level $s_{k-1}$ into subintervals at ternary level $s_k$ is exactly that of the Cantor set, except that the two ends of the interval are trimmed off by a small amount, to a relative distance $3^{-k}$ from each end of the interval.

We now fill in the intermediate levels $X_s$ for $s_{k-1} < s < s_k$ by gluing together all intervals in $\mathcal{J}_{s_k}$ that have matching initial ternary expansions $[M]_3$ of $M = \lambda_k 2^{m_k}$, disregarding the last $s_k - s$ ternary digits of $[M]_3$, and filling in the space between them. The resulting intervals of $\mathcal{J}_s$ all have size exactly $3^{s_k-s}2^{-m_k}$ (except possibly for two subintervals adjacent to the truncated ends); their size lies between $\frac{1}{2}3^{-s}$ and $3^{-s}$. Also, the gaps between any two adjacent intervals at ternary level $s$ are of size at least as large as

$$G_s = 3^{s_k-s}2^{-m_k} \geqslant \frac{1}{2}3^{-s}. \tag{2.32}$$

This fact holds because this construction uses ternary integers omitting the digit 1; the set of ternary integers omitting the digit 2 has some intervals of this kind that are adjacent, so the gap size would be zero in that case.

The above construction defines the intervals in $\mathcal{J}_s$ at level $s$ for all $s \geqslant 1$. This dissection imitates the Cantor set in that each interval at level $s$ contains at most $2^{s'-s}$ subintervals at any deeper ternary level $s' \geqslant s$. It may contain fewer subintervals, due to the trimming at ends of the subinterval, but it always contains at least $2^{s'-s-1}$ such subintervals.

The set $\tilde{\Sigma}$ is a compact set contained in the interval $[1, 2]$. To bound its $\alpha_0$-dimensional Hausdorff measure from below, we must show that, in *every* covering $\{U_i\}$ by closed intervals, there holds

$$\sum_i |U_i|^{\alpha_0} \geqslant \frac{1}{16}. \tag{2.33}$$

By enlarging the intervals slightly (by $1 + \epsilon$) and observing that their interiors give an open cover of $\tilde{\Sigma}$, we can extract a finite subcover. Since we can extract a finite subcover for any $\epsilon > 0$, it suffices to verify that (2.33) holds for every finite cover $\{U_i\}$ of $\tilde{\Sigma}$ by intervals.

Given an interval $U_i$ in a covering, define $s$ by

$$3^{-s} \leqslant |U_i| < 3^{-s+1}. \tag{2.34}$$

Then $U_i$ can touch at most two subintervals at level $s$ because all subintervals in $\mathcal{J}_s$ are separated by gaps of size at least $\frac{1}{2}3^{-s}$. If $s' \geqslant s$ then $U_i$ intersects at most $2 \cdot 2^{s'-s}$ subintervals at level $s' - s$; by (2.34) this number is bounded above by

$$2 \cdot 2^{s'-s} \leqslant 2^{s'} 3^{-\alpha_0 s} \leqslant 2 \cdot 2^{s'} (3^{\alpha_0} |U_i|^{\alpha_0}) = 4 \cdot 2^{s'} |U_i|^{\alpha_0}. \tag{2.35}$$

Given a finite cover, choose $s' = s_k$ large enough so that $|U_i| \geqslant 3^{-s'}$ for all $i$. Then the collection $\{U_i\}$ necessarily covers all subintervals at level $s' = s_k$. By construction, $\mathcal{I}_{s_k}$ contains at least

$$\prod_{i=1}^{k} (2^{r_i} - 2^{r_i - i}) = 2^{r_1 + \ldots + r_k} \prod_{i=1}^{n} (1 - 2^{-i}) \geqslant \frac{1}{4} 2^{s_k} \tag{2.36}$$

intervals, since $\prod_{i=1}^{k}(1 - 2^{-i}) \geqslant \prod_{i=1}^{\infty}(1 - 2^{-i}) \geqslant \frac{1}{4}$. Now we count how many intervals at level $s_k$ are covered. Since $U_i$ intersects at most $4 \cdot 2^{s_k} |U_i|^{\alpha_0}$ such intervals, we must have

$$\sum_i 4 \cdot 2^{s_k} |U_i|^{\alpha_0} \geqslant |\mathcal{J}_{s_k}| \geqslant \frac{1}{4} 2^{-s_k}.$$

This yields

$$\sum_i |U_i|^{\alpha_0} \geqslant \frac{1}{16},$$

which establishes (2.28). □

Remark 2.6. More generally, we may consider the real dynamical system $y \to \beta y$, where $\beta > 1$, and consider the truncated ternary expansions $\{(\lfloor \lambda \beta^n \rfloor)_3 : n \geqslant 0\}$. The methods above should extend to those $\beta$ such that $\alpha := \log_3 \beta$ satisfies a Diophantine condition

$$\left| \alpha - \frac{p}{q} \right| \geqslant c_2 \frac{1}{q^{c_1+1}} \text{ for all } p, q \text{ with } q \geqslant 1, \tag{2.37}$$

for constants $c_1 > 1$ and $c_2 > 0$. The conclusions of the results require appropriate modification, with constants depending on the Diophantine condition.

## 3. 3-*adic integer dynamical systems: proofs*

We consider the 3-adic integers $\mathbb{Z}_3$ and write the 3-adic expansion of $\lambda \in \mathbb{Z}_3$ as

$$\lambda = \sum_{j=0}^{\infty} d_j 3^j \quad \text{with each } d_j \in \{0, 1, 2\}. \tag{3.1}$$

We write the 3-adic digit expansion as $(\lambda)_3 = (\dots d_2 d_1 d_0)_3$.

For this dynamical system we consider the sequence of 3-adic integers $y_n = \lambda 2^n$, where $\lambda$ is a given nonzero 3-adic integer. For $n \geqslant 0$ these form the forward orbit of the first-order linear recurrence $y_n = 2y_{n-1}$, with initial condition $y_0 = \lambda$. The map $T : x \to 2x$ is an automorphism of the 3-adic integers $\mathbb{Z}_3$ that leaves each of the sets $\Sigma_j := 3^j \mathbb{Z}_3^*$, for $j \geqslant 0$, invariant. (Here $\mathbb{Z}_3^*$ are the 3-adic units.) These sets partition $\mathbb{Z}_3$ and this map acts ergodically on each component $\Sigma_j$.

We are interested in the possible ways that the orbit $\{y_n : n \geqslant 0\}$ can intersect the set $\Sigma_{3,\bar{2}} := \{w : w = \sum_{j=0}^{\infty} a_j 3^j \in \mathbb{Z}_3, \text{ with each } a_j = 0 \text{ or } 1\}$.

### 3.1. *Quantitative upper bound*

We upper bound the number of $n \leqslant X$ such that $(\lambda 2^n)_3$ falls in the set $\Sigma_{3,\bar{2}}$.

*Proof of Theorem* 1.5. Let $\lambda \in \mathbb{Z}_3$ with $\lambda \neq 0$. We study the set

$$\tilde{N}_\lambda(X) := \#\{1 \leqslant n \leqslant X : (\lambda 2^n)_3 \text{ omits the digit } 2\}. \tag{3.2}$$

Write $\lambda = 3^j \lambda^*$, with $\lambda^* \in \mathbb{Z}_3^\times := \{\lambda \in \mathbb{Z}_3 : \lambda \not\equiv 0 \pmod 3\}$. Then we have $\tilde{N}_\lambda(X) = \tilde{N}_{\lambda^*}(X)$, since multiplication by $3^j$ simply shifts 3-adic digits to the left. Thus to prove the desired inequality there is no loss of generality in requiring $\lambda \not\equiv 0 \pmod 3$, by replacing $\lambda$ with $\lambda^*$.

The proof is based on the fact that 2 is a primitive root $(\mathrm{mod}\ 3^k)$ for each $k \geqslant 1$. Thus, for each $k \geqslant 1$, we have that

$$\{\lambda 2^n \pmod 3 : 1 \leqslant n \leqslant \phi(3^k) = 2 \cdot 3^{k-1}\} \tag{3.3}$$

runs over all $2 \cdot 3^{k-1}$ invertible residue classes $(\mathrm{mod}\ 3^k)$. Of these, exactly $2^{k-1}$ residue classes have a 3-adic expansion that omits the digit 2. Now, given $X$, choose $k$ such that

$$2 \cdot 3^{k-2} < X \leqslant 2 \cdot 3^{k-1}.$$

Then, applying (3.3) over $1 \leqslant n \leqslant 2 \cdot 3^{k-1}$, we have exactly $2^{k-1}$ values of $n$ with $(\lambda 2^n)_3$ omitting the digit 2 in its first $k$ 3-adic digits $(d_{k-1} \dots d_1 d_0)_3$. Thus

$$\begin{aligned} \tilde{N}_\lambda(X) &\leqslant 2^{k-1} = 2 \cdot 2^{k-2} = 2 \cdot 3^{\alpha_0(k-2)} \\ &= 2^{1-\alpha_0}(2 \cdot 3^{k-2})^{\alpha_0} \leqslant 2X^{\alpha_0}, \end{aligned}$$

which is the desired upper bound. □

### 3.2. *Hausdorff dimension bounds*

The objective of Theorem 1.6 is to establish upper bounds on the Hausdorff dimension of the 3-adic exceptional set $\mathcal{E}(\mathbb{Z}_3)$ through upper bounds on various sets $\mathcal{E}^{(j)}(\mathbb{Z}_3)$ that contain it.

The notion of Hausdorff dimension for subsets $S$ of 3-adic integers using the 3-adic metric is quite similar to Hausdorff dimension for real numbers on the interval $[0, 1]$ (cf. Abercrombie [1]). In fact, we have a continuous (and almost one-to-one) mapping $\iota : \mathbb{Z}_3 \to [0, 1]$ that sends a 3-adic number $\lambda = (\dots d_2 d_1 d_0)_3$ to the real number with ternary expansion $.d_0 d_1 d_2 \dots$. We can show that this mapping preserves the Hausdorff dimension of sets, that is, a 3-adic set $X$ and its image $\iota(X)$ have the same Hausdorff dimension. This holds because we can expand each set in a 3-adic covering of a set $X$ to a closed-open disk

$$B(m, 3^j) = \{x \in \mathbb{Z}_3 : x \equiv m \ (\mathrm{mod}\ 3^j)\}$$

(which has diameter $1/3^j$), with at most a factor of 3 increase in diameter, and similarly we can inflate any real covering to a covering with ternary intervals $[m/3^j, (m+1)/3^j]$ with at most a factor of 3 increase in diameter. However, these special intervals are assigned the same diameter under their respective metrics, and this can be used to show that the Hausdorff dimensions of $X$ and $\iota(X)$ coincide. In particular, the standard 3-adic Cantor set $\Sigma_{3,\bar{1}}$ maps under $\iota$ to the usual Cantor set in $[0,1]$, and hence it has Hausdorff dimension $d_H(\Sigma_{3,\bar{1}}) = \log_3(2) \approx 0.63092$. Now $\Sigma_{3,\bar{1}} = 2\Sigma_{3,\bar{2}}$, and hence $\dim_H(\Sigma_{3,\bar{2}}) = \log_3(2)$ as well.

*Proof of Theorem* 1.6. This proof assumes that Theorem 1.8 is proved in order to deduce the upper bound in (ii).

(i) We have

$$\mathcal{E}^{(1)}(\mathbb{Z}_3) = \bigcup_{m=0}^{\infty} \mathcal{C}(2^m),$$

with $\mathcal{C}(2^m) := \{\lambda : (\lambda 2^n)_3 \text{ omits the digit } 2\}$. Then

$$\mathcal{C}(2^m) = \frac{1}{2^m}\mathcal{C}(1) = \frac{1}{2^m}(\Sigma_{3,\bar{2}}) = \frac{1}{2^{m+1}}(\Sigma_{3,\bar{1}}).$$

Each $\mathcal{C}(2^m)$ is a linearly rescaled version of the Cantor set $\Sigma_{3,\bar{1}}$ and so has Hausdorff dimension $\log_3 2$. Thus

$$\log_3 2 = \dim_H(\mathcal{C}(1)) \leqslant \dim_H(\mathcal{E}^{(1)}(\mathbb{Z}_3)) \leqslant \sup_{m \geqslant 0} \dim_H(\mathcal{C}(2^m)) = \log_3 2,$$

as required.

(ii) We have

$$\mathcal{E}^{(2)}(\mathbb{Z}_3) = \bigcup_{0 \leqslant m_1 < m_2} \mathcal{C}(2^{m_1}, 2^{m_2}),$$

with $\mathcal{C}(2^{m_1}, 2^{m_2}) := \{\lambda : (\lambda 2^{m_i})_3 \text{ omits the digit } 2\}$. Now

$$\mathcal{C}(2^{m_1}, 2^{m_2}) = \frac{1}{2^{m_1}}\mathcal{C}(1, 2^{m_2 - m_1}),$$

which gives $\dim_H(\mathcal{C}(2^{m_1}, 2^{m_2})) = \dim_H(\mathcal{C}(1, 2^{m_2 - m_1}))$. Since $m_2 - m_1 \geqslant 1$, Theorem 1.8 applies to give

$$\dim_H(\mathcal{C}(1, 2^{m_2 - m_1})) \leqslant \frac{1}{2} \quad \text{for all } m_2 > m_1 \geqslant 0.$$

This yields the upper bound

$$\dim_H(\mathcal{E}^{(2)}(\mathbb{Z}_3)) = \sup_{0 \leqslant m_1 < m_2} \dim_H(\mathcal{C}(2^{m_1}, 2^{m_2})) \leqslant \frac{1}{2}.$$

To establish the lower bound, we use the fact that $4 = (11)_3$. Then the set

$$\Sigma_A := \{\lambda = (\dots d_2 d_1 d_0)_3 : \text{all blocks } d_{2n+1} d_{2n} \in \{00, 01\}\} \subset \Sigma_{3,\bar{2}}$$

satisfies

$$4\Sigma_A = \{\lambda = (\dots d_2 d_1 d_0)_3 : \text{all blocks } d_{2n+1} d_{2n} \in \{00, 11\}\} \subset \Sigma_{3,\bar{2}},$$

which shows that $\Sigma_A \subset \mathcal{C}(1, 4)$. Now $\Sigma_A$ is given by a Cantor set construction, which permits its Hausdorff dimension to be computed in a standard way. We obtain

$$\dim_H(\mathcal{E}^{(2)}(\mathbb{Z}_3)) \geqslant \dim_H(\mathcal{C}(1, 2^2)) \geqslant \dim_H(\Sigma_A) = \frac{\log_3(2)}{\log_3(9)} = \frac{1}{2}\log_3(2) \approx 0.31596.$$

(iii) We have

$$\mathcal{E}^{(2)}(\mathbb{Z}_3) = \bigcup_{0 \leqslant m_1 < m_2 < m_3} \mathcal{C}(2^{m_1}, 2^{m_2}, 2^{m_3}).$$

The upper bound $\dim_H(\mathcal{E}^{(3)}(\mathbb{Z}_3)) \leqslant \dim_H(\mathcal{E}^{(2)}(\mathbb{Z}_3))$ is immediate. To establish the lower bound, we use the facts that $4 = (11)_3$ and $256 = (100111)_3$. Then

$$\Sigma_B := \{\lambda = (\dots d_2 d_1 d_0)_3 : \text{all } d_{6n+5}d_{6n+4}d_{6n+3}d_{6n+2}d_{6n+1}d_{6n} \in \{000000, 000001\}\} \subset \Sigma_{3,\bar{2}}$$

has

$$4\Sigma_B = \{\lambda = (\dots d_2 d_1 d_0)_3 : \text{all } d_{6n+5}d_{6n+4}d_{6n+3}d_{6n+2}d_{6n+1}d_{6n} \in \{000000, 000011\}\} \subset \Sigma_{3,\bar{2}},$$
$$256\Sigma_B = \{\lambda = (\dots d_2 d_1 d_0)_3 : \text{all } d_{6n+5}d_{6n+4}d_{6n+3}d_{6n+2}d_{6n+1}d_{6n} \in \{000000, 100111\}\} \subset \Sigma_{3,\bar{2}}.$$

Thus $\Sigma_B \subset \mathcal{C}(1, 4, 256) \subset \mathcal{E}^{(3)}(\mathbb{Z}_3)$. Now $\Sigma_B$ has a Cantor set construction, showing that

$$\dim_H(\Sigma_B) = \frac{\log_3(2)}{\log_3(3^6)} = \frac{1}{6}\log_3(2) \approx 0.10515,$$

which gives the asserted lower bound. □

REMARK 3.1. The proof of Theorem 1.6 exploited the known solutions to Erdős's problem. Consequently, this approach does not extend to give a nonzero lower bound for $\dim_H(\mathcal{E}^{(k)}(\mathbb{Z}_3))$, for any $k \geqslant 4$. Theorem 1.9 offers more flexibility in finding ternary expansion identities for integers that could potentially yield nonzero lower bounds in these cases.

## 4. *Intersections of multiplicative translates of the 3-adic Cantor set: proofs*

We study the 3-adic Cantor set $\Sigma_{3,\bar{2}}$, defined by

$$\Sigma_{3,\bar{2}} := \{\lambda \in \mathbb{Z}_3 : \text{the 3-adic digit expansion } (\lambda)_3 \text{ omits the digit } 2\}. \tag{4.1}$$

For integers $1 \leqslant M_1 < M_2 < \ldots < M_k$, we define the *intersection set*

$$\mathcal{C}(M_1, M_2, \dots, M_k) := \{\lambda \in \mathbb{Z}_3 : (M_i\lambda)_3 \text{ omits the digit } 2\} \tag{4.2}$$
$$= \bigcap_{i=1}^{k} \frac{1}{M_i}\Sigma_{3,\bar{2}}. \tag{4.3}$$

In Section 3 we used integers $M_i = 2^{m_i}$, but here we allow arbitrary positive integers $M_i$. We study $\mathcal{C}(1, M)$ for general $M$ and note first that $\mathcal{C}(1, 3^j M) = \mathcal{C}(1, M)$. Thus, without loss of generality, we may reduce to the case $\gcd(M, 3) = 1$. Another simple fact is the following.

LEMMA 4.1. *Let $M$ be a positive integer.*
(i) *If $M \equiv 2 \pmod 3$ then $\mathcal{C}(1, M) = \{0\}$.*
(ii) *If $M \equiv 1 \pmod 3$ then $\mathcal{C}(1, M)$ is an infinite set.*

*Proof.* (i) Suppose $M \equiv 2 \pmod 3$. If $\mathcal{C}(1, M) \neq \{0\}$, then it necessarily contains some $\lambda$ with $\lambda \not\equiv 0 \pmod 3$, since we may divide out any powers of 3, and multiplication by $3^j$ simply shifts digits to the left. Then $\lambda \in \Sigma_{3,\bar{2}}$ implies that $\lambda \equiv 1 \pmod 3$. Then $M\lambda \equiv 2 (\bmod 3)$, so $M\lambda \notin \Sigma_{3,\bar{2}}$, contradicting membership in $(1, M)$. Hence no such $\lambda$ exists, and $\mathcal{C}(1, M) = \{0\}$.

(ii) Suppose that $M \equiv 1 \pmod 3$. To show that $\mathcal{C}(1, M)$ is an infinite set it suffices to exhibit one nonzero element $\lambda \in \mathcal{C}^*(1, M)$, because $3^j\lambda \in \mathcal{C}^*(1, M)$ for all $j \geqslant 0$. We may construct such an element $\lambda = (\dots d_2 d_1 d_0)_3$ recursively, starting with the choice $d_0 = 1$. Write $M = \sum_{j=0}^{n} a_j 3^j$, with $a_0 = 1$. Let $M\lambda = \sum_{j=0}^{\infty} c_j 3^j$. Then the $k$th digit satisfies

$$c_k \equiv d_k + \left(\sum_{j=1}^{n} a_j d_{n-j}\right) + e_{k-1} \pmod 3$$

(with the convention $d_{-1} = d_{-2} = \ldots = d_{-n} = 0$), and with $e_{k-1}$ encoding the 'carry digit' information, from the previous terms, which is completely determined by $(d_0, d_1, \ldots, d_{k-1})$. Since we have two choices 0 or 1 for $d_k$, at least one of them will force $c_k \not\equiv 2 \pmod 3$. Thus we can recursively construct an admissible $\lambda$ by induction on $k \geqslant 1$. □

It is possible to make a detailed analysis of the structure of $\mathcal{C}(1, M)$ with $M \equiv 1 \pmod 3$, and to determine their Hausdorff dimensions, which we will consider elsewhere. We can show that the infinite set $\mathcal{C}(1, M)$ can be either countable or uncountable; for example, $\mathcal{C}(1, 49)$ is countably infinite, whereas $\mathcal{C}(1, 7)$ is uncountable.

Now we upper bound the Hausdorff dimension of $\mathcal{C}(1, M)$. For $M = 3^j$ $(j \geqslant 0)$, we have $\mathcal{C}(1, 3^j) = \Sigma_{3,\bar{2}}$, where $\dim_H(\mathcal{C}(1, 3^j)) = \log_3(2) \approx 0.63$. The following result treats all other $M \geqslant 1$.

*Proof of Theorem* 1.8. We suppose that $M > 1$ is an integer that is not a power of 3, that is, its ternary expansion $(M)_3$ contains at least two nonzero ternary digits. Our objective is to upper bound the Hausdorff dimension of

$$\mathcal{C}(1, M) := \Sigma_{3,\bar{2}} \cap M\Sigma_{3,\bar{2}}$$

by $\frac{1}{2}$. By the discussion above, we may reduce to the case that $\gcd(M, 3) = 1$, and by Lemma 4.1 we may suppose that $M \equiv 1 \pmod 3$, since the Hausdorff dimension is 0 if $M \equiv 2 \pmod 3$. Thus we can write

$$(M)_3 = b_0 + b_m 3^m + \sum_{j=m+1}^{n} b_j 3^j, \quad b_j \in \{0, 1, 2\}, \text{ with } b_0 b_m \neq 0, \tag{4.4}$$

and $b_0 = 1$, where the $m$th digit is the first nonzero ternary digit after the 0th digit.

We will study the minimal covers of $\mathcal{C}(1, M)$ with 3-adic open sets of measure $3^{-r-1}$ that specify the first $r+1$ digits of the 3-adic expansion of a number $\lambda \in \mathcal{C}(1, M)$. These sets are congruence classes $(\bmod\, 3^{r+1})$ and they have diameter $3^{-(r+1)}$. We call a congruence class $\lambda \pmod{3^{r+1}}$ *admissible* if $\mathcal{C}^*(1, M)$ contains at least one element in this congruence class. Our objective is to bound above the number of admissible congruence classes $\lambda \pmod{3^{r+1}}$.

Set $\lambda = \sum_{j=0}^{\infty} d_j 3^j \in \Sigma_{3,\bar{2}}$, so that each $d_j = 0$ or 1. Now define the digits $a_j$ by

$$M\lambda = \sum_{j=0}^{\infty} a_j 3^j, \quad a_j \in \{0, 1, 2\}.$$

The condition that $M\lambda \in \Sigma_{3,\bar{2}}$ means that each $a_j = 0$ or 1, which imposes extra constraints on the $d_j$s.

Claim 4.2. *Suppose that $(d_0, d_1, \ldots, d_{2lm+k-1})$, with $0 \leqslant k < m$, of $\lambda \in \mathcal{C}(1, M)$ are fixed. Then at least one of the following conditions holds.*

(i) *There is at most one admissible value for $d_{2lm+k}$ in $\lambda \pmod{3^{2lm+k+1}}$.*

(ii) *There are two admissible values for $d_{2lm+k}$ for $\lambda \pmod{3^{2lm+k+1}}$ and for any fixed choices of $(d_{2lm+k+1}, d_{2lm+k+2}, \ldots, d_{(2l+1)m+k-1})$ at most three of the four possible values of $(d_{2lm+k}, d_{(2l+1)m+k})$ give admissible sequences for $\lambda \pmod{3^{(2l+1)m+k+1}}$.*

*Proof.* To prove the claim, suppose that condition (i) does not hold. We then examine the digit $a_{(2l+1)m+k}$ using (4.4) to obtain

$$M\lambda \equiv b_0 d_{(2l+1)m+k} 3^{(2l+1)m+k} + b_m d_{2lm+k} 3^{(2l+1)m+k} + M\left(\sum_{j=0}^{2lm+k-1} d_j 3^j\right) + b_0\left(\sum_{j=2lm+k+1}^{(2l+1)m+k-1} d_j 3^j\right) + b_0 d_{2lm+k} 3^{2lm+k} \left(\bmod 3^{(2l+1)m+k+1}\right). \tag{4.5}$$

Define the digits $r_j$ by

$$M\left(\sum_{j=0}^{2lm+k-1} d_j 3^j\right) + b_0\left(\sum_{j=2lm+k+1}^{(2l+1)m+k-1} d_j 3^j\right) = \sum_{j=0}^{(2l+1)m+k+n} r_j 3^j, \quad r_j \in \{0,1,2\}.$$

We assert that (4.5) then gives the congruence

$$a_{(2l+1)m+k} \equiv b_0 d_{(2l+1)m+k} + b_m d_{2lm+k} + r_{(2l+1)m+k} \pmod 3; \tag{4.6}$$

that is, we assert that there cannot be any extra 'carry digit' from lower-order terms that affects the $((2l+1)m+k)$-th 3-adic digit, coming from the addition of $b_0 d_{2lm+k} 3^{2lm+k}$ in (4.5). By our assumption that (i) does not hold, both values $d_{2lm+k} = 0$ and 1 occur for admissible $\lambda \pmod{3^{2lm+k+1}}$ for these digits. Since $b_0 = 1$ and the 3-adic digit $a_{2lm+k}$ of $M\lambda$ is 0 or 1, this digit must have been 0 when $d_{2lm+k} = 0$, and 1 when $d_{2lm+k} = 1$, so there can be no 'carry digit' in the addition of $b_0 d_{2lm+k} 3^k$, as asserted.

Now consider the pairs $(d_{2lm+k}, d_{(2l+1)m+k})$. Of the four values (00), (01), (10), and (11) that these may take, the quantities $b_0 d_{(2l+1)m+k} + b_m d_{2lm+k}$, with $b_0 = 1$ and $b_m = 1$ or 2, will cover all residue classes $(\bmod 3)$. In particular, at least one choice will result in $a_{(2l+1)m+k} \equiv 2 \pmod 3$ in (4.6), and so give a non-admissible set of digits $(\bmod 3^{(2l+1)m+k})$. This proves (ii), and the claim. □

CLAIM 4.3. *For $M$ having the ternary expansion (4.4) and a given $r = 2lm$ for some $l \geqslant 1$, there are at most $3^{r/2}$ admissible congruence classes in $\mathcal{C}(1, M) \pmod{3^r}$.*

*Proof.* To prove the claim, we suppose $j \geqslant 0$ and that the initial block $(d_0, d_1, \ldots, d_{2jm-1})$ of $2jm$ digits of $\lambda$ is fixed, such that $M\lambda$ is admissible to level $2jm$ in the sense of having its first $2jm$ ternary digits 0 and 1 only. We then make the sub-claim that the number of possible admissible extensions $(d_{2jm}, d_{2jm+1}, \ldots, d_{(2j+2)m-1})$ of the initial block to the next $2m$ digits of this sequence is at most $3^m$. Assuming the sub-claim is shown for all $j \geqslant 0$, we may conclude by induction on $j$ that for $r = 2lm$, there are at most $3^{lm} = 3^{r/2}$ admissible sequences of the first $r$ digits, proving the claim.

To prove the sub-claim, we first define the digits $s_i$ by

$$M\left(\sum_{i=0}^{2jm-1} d_i 3^i\right) = \sum_{i=0}^{2jm+n} s_i 3^i, \quad s_i \in \{0,1,2\}.$$

The admissible extensions are then described by paths in a rooted unary–binary tree, whose branch (or branches) at the first level from the root node gives the admissible choices of $d_{2jm}$, the next level gives the allowed extensions $d_{2jm+1}$, and so on down to depth $2m$, where the allowed values of $d_{(2j+2)m-1}$ occur. We label the nodes of the tree at the first level by the value of $d_{2jm}$; those at the second level $d_{2jm+1}$, and so on to the leaf node labels $d_{(2j+2)m-1}$. The number of nodes $N_i$ of this tree at level $i$ counts the number of admissible extensions

$(d_{2jm}, d_{2jm+1}, \ldots, d_{2jm+i-1})$ of length $i$ of the initial block. The sub-claim asserts that for any initial block, the number of leaves $N_{2m}$ of this tree satisfies $N_{2m} \leqslant 3^m$.

We now assert that the branching at the first $m$ levels is *uniform* at level $i$ $(1 \leqslant i \leqslant m)$ with all branches being in case (i), or else all branches being in case (ii) of Claim 4.2, that is, either there is no branching at this level or else all nodes branch to two descending edges. We prove this assertion by induction on $i$. It is trivially true at the first level, the base case. For the induction step from level $i$ to level $i+1$, we note that the digit $a_{2jm+i}$ of $M\lambda$ at level $i+1$ satisfies

$$a_{2jm+i} \equiv b_0 d_{2jm+i} + s_{2jm+i} + c_{2jm+i} (\text{mod } 3),$$

in which $c_{2jm+1}$ is a possible 'carry digit' from the previous level $i$ which takes the value 0 or 1, and *a priori* depends on the node at the previous level. Here $b_0 = 1$ and the allowed choices of $d_{2m+i} = 0, 1$ are those giving $a_{2mj+i} \equiv 0$ or $1 \,(\text{mod } 3)$. The assertion will follow by showing that the 'carry digit' must be the same for every node at the previous level. We add this as an additional induction hypothesis on $i$, noting that the base case $i = 1$ holds because there is no 'carry digit' at the root node. Now at level $i$ we have (using the fact $b_0 = 1$)

$$a_{2jm+i-1} \equiv d_{2jm+i-1} + s_{2jm+i-1} + c_{2jm+i-1} (\text{mod } 3).$$

Here $s_{2jm+i-1}$ takes a constant value which may be $0, 1, 2$, and, by the carry digit induction hypothesis, $c_{2jm+i-1}$ takes a (node-independent) constant value 0 or 1, and $d_{2jm+i-1}$ may only take values 0 or 1. There is a carry digit $c_{2jm+i} = 1$ to the next level $i+1$ at this node if and only if the sum of these three numbers is 3 or higher. Now we know $S := s_{2jm+i-1} + c_{2jm+i-1}$ takes a node-independent value, which can be $0, 1, 2$, or 3. The restriction that $a_{2jm+i-1}$ can take only values 0 or 1 implies that if $S = 0$, we must be in case (ii) on level $i$ with no carry digit, if $S = 1$ we are in case (i) with no carry digit, if $S = 2$ we are in case (i) with a carry digit 1 and if $S = 3$ we are in case (ii) with a carry digit 1. So the carry digit $c_{2jm+i}$ is node-independent. We conclude that at the level $i+1$, both $s_{2jm+i}$ and $c_{2jm+i}$ are node-independent values, and hence the equation for $a_{2jm+i}$ above (mod 3) is the same for all nodes at this level. Now set $S' = s_{2jm+i} + c_{2jm+i}$, and we find that all nodes on level $i+1$ fall in case (i) uniformly if $S' = 1$ or 2 and in case (ii) uniformly if $S' = 0$ or 3. This verifies both hypotheses of the induction step, and the assertion follows.

Using the assertion, we conclude that in the first $m$ levels, we get no branching at case (i) level and total binary branching at case (ii) level. We conclude that the number of nodes $N_m$ of the tree at depth $m$ has $N_m = 2^t$, where $t$ is the number of case (ii) levels that occurred in the first $m$ levels.

Next we consider the final $m$ levels of the tree, where the branching may be non uniform, with branching at level $m + i (1 \leqslant i \leqslant m)$ of a node being controlled by the value of $d_{2jm+i-1}$ at level $i$ via equation (4.6), and we distinguish whether the branching at level $i$ was case (i) or case (ii). When case (i) occurs at level $i$, we allow full branching at level $m+i$. When case (ii) occurs at level $i$, each node at level $i-1$ has two branches leading to two nodes labelled $d_{(2jm+i-1)} = 0, 1$ at level $i$. Claim 4.2(ii) now implies that the two subtrees formed using these two nodes as root nodes, going from level $i+1$ to level $m+i$, necessarily branch identically down to level $m+i-1$ and then at least one of them is completely unbranched at level $m+i$, while the other subtree may fully branch at level $m+i$, as dictated by equation (4.6). (The identical branching to level $m+i-1$ holds because the value of $d_{2jm+i-1} = 0, 1$ does not affect any subsequent carry digits down to level $m+i-1$.) It follows that if case (i) occurred at level $i$, then $N_{m+i} \leqslant 2N_{m+i-1}$, while if case (ii) occurred at level $i$, then $N_{m+i} \leqslant \frac{3}{2} N_{m+i-1}$.

Putting these facts together, we conclude that

$$N_{2m} = \left((3/2)^t 2^{m-t}\right) 2^t \leqslant 3^t 2^{m-t} \leqslant 3^m.$$

This completes the proof of the sub-claim. □

To conclude the proof of Theorem 1.8, Claim 4.3 implies that we have a covering $\mathcal{I}_r$ of $\mathcal{C}(1,M)$ with a set of at most $3^{r/2}$ sets, each of diameter $3^{-(r+1)}$. For each $\epsilon > 0$, this covering satisfies

$$\sum_{I \in \mathcal{I}_r} |I|^{1/2+\epsilon} \leqslant 3^{r/2}(3^{-(r+1)})^{1/2+\epsilon} \leqslant 3^{-(r+1)\epsilon}.$$

Letting $r \to \infty$, this bound implies that $\dim_H(\mathcal{C}(1,M)) \leqslant \frac{1}{2} + \epsilon$. Letting $\epsilon \to 0$ gives the result. □

We do not know whether the bound in Theorem 1.6 is sharp. However, it is possible to show that $\mathcal{C}(1,7)$ has $\dim_H \mathcal{C}(1,7) = \log_3(\frac{1+\sqrt{5}}{2}) \approx 0.43$.

*Proof of Theorem* 1.9. Suppose that we are given a positive integer $N$ with $N \in \Sigma_{3,\bar{2}}$ and $1 \leqslant M_1 < M_2 < \ldots < M_k$ with all $NM_i \in \Sigma_{3,\bar{2}}$. Our objective is to obtain an explicit nonzero lower bound on the Hausdorff dimension $\dim_H(\mathcal{C}(M_1, M_2, \ldots, M_k))$. We set $n$ equal to the number of ternary digits in $NM_k$, so that $n = \lceil \log_3 NM_k \rceil$. Now we consider the set

$$\Sigma_C := \{\lambda = (\ldots d_2 d_1 d_0)_3 : \text{all blocks } d_{(k+1)n-1} \ldots d_{kn+1} d_{kn} \in \{0^n, (N)_3\}\} \subset \Sigma_{3,\bar{2}}.$$

Since each $NM_j \in \Sigma_{3,\bar{2}}$ is an integer with at most $n$ ternary digits, we have

$$M_j \Sigma_C := \{\lambda = (\ldots d_2 d_1 d_0)_3 : \text{all blocks } d_{(k+1)n-1} \ldots d_{kn+1} d_{kn} \in \{0^n, (NM_j)_3\}\} \subset \Sigma_{3,\bar{2}}.$$

Thus $\Sigma_C \subset \mathcal{C}(M_1, M_2, \ldots, M_k)$. By inspection, $\Sigma_C$ is a Cantor set that has Hausdorff dimension

$$\dim_H \Sigma_C = \frac{\log_3(2)}{\log_3(3^n)} = \frac{\log_3(2)}{\lceil \log_3(NM_k) \rceil},$$

and the result follows. □

## 5. *Furstenberg conjecture and transversality of semigroup actions*

In 1970 Furstenberg [**11**, p. 43] formulated the following conjecture, which is in the same direction as Erdős's question.

CONJECTURE 5.1 (Furstenberg). Suppose that $p$ and $q$ are not powers of the same integer. Then the expansions to the base $B = pq$ of the powers $\{(p^n)_{pq} : n \geqslant 1\}$ have the property that any given finite pattern of consecutive base $B$ digits occurs in $(p^n)_{pq}$ for all sufficiently large $n$.

For example, for $p = 2$ and $q = 3$, this conjecture asserts that any given pattern of base $B = 6$ digits will occur as consecutive digits in the base 6 expansion of $(2^n)_6$, for all sufficiently large $n$. The restriction to products $B = pq$ of two (or more) multiplicatively independent elements was motivated by results in Furstenberg's seminal work [**10**]. There he showed that, for any irrational number $\theta$, the set $\{p^m q^n \theta \pmod 1 : m, n \geqslant 0\}$ is dense on the torus $\mathbb{R}/\mathbb{Z}$. However, it is well known that there is an uncountable set of irrational numbers $\theta$ for which $\{p^m \theta : m \geqslant 0\}$ is not dense on the torus.

Conjecture 1.12 proposes, nevertheless, that Furstenberg's conjecture continues to hold even when the base $B = q$. More generally, one can ask whether Furstenberg's conjecture might be valid more generally for base $B$ expansions $(p^n)_B$ for arbitrary $B$ with $\gcd(B, p) = 1$.

A main objective of Furstenberg [**11**] was to introduce a notion of transversality of two semigroups of transformations $\mathcal{S}_1$ and $\mathcal{S}_2$ acting on a compact metric space $X$ with respect to a (suitable) dimension function $\dim(A)$ defined on all closed sets $A$.

DEFINITION 5.2. Two closed sets $A$ and $B$ in a compact metric space $X$ are *transverse* (for a given dimension function) if

$$\dim(A \cap B) \leqslant \max(\dim(A) + \dim(B) - \dim(X), 0).$$

DEFINITION 5.3. Two semigroups $\mathcal{S}_1$ and $\mathcal{S}_2$ acting on a compact metric space $X$ are *transverse* (for a given dimension function) if any closed $\mathcal{S}_1$-invariant set $A$ and any closed $\mathcal{S}_2$-invariant set $B$ are themselves transverse, for that dimension function.

He obtained as an immediate consequence of this definition the following result concerning simultaneous invariant sets [**11**, p. 42], which draws on earlier work [**10**].

PROPOSITION 5.4 (Furstenberg). *Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are transverse semigroups acting on a compact metric space $X$, and that $\mathcal{S}_1$ has the following additional property:*
($\bullet$) *if $A$ is a closed $\mathcal{S}_1$-invariant set with $\dim(A) = \dim(X)$, then $A = X$.*
*Then any proper closed subset of $X$ that is invariant under both $\mathcal{S}_1$ and $\mathcal{S}_2$ has $\dim(A) = 0$.*

Furstenberg did not construct any transverse semigroups, but as evidence for their existence showed for the following pair of transformation semigroups that their (nontrivial) simultaneously invariant closed sets satisfy this property (see [**11**, Theorem 3]).

PROPOSITION 5.5 (Furstenberg). *Let $\mathbb{Z}_r$ be the ring of $r$-adic integers, and suppose that $r = pq$, with $p > 1$ and $q > 1$ not both powers of the same integer. Define transformations $D_s(x) = \lfloor x/s \rfloor$, for $s = p, q$, and $pq$, and note that $D_{pq} = D_p D_q = D_q D_p$. Let $\mathcal{S}_p$ and $\mathcal{S}_q$ denote the semigroups generated by $D_p$ and $D_q$, respectively. If $A$ is a simultaneously $\mathcal{S}_p$- and $\mathcal{S}_q$-invariant proper closed subset of $\mathbb{Z}_r$, then $A$ has Hausdorff dimension zero.*

The proof of this result draws on his earlier work [**10**]. Furstenberg [**11**, p. 45] went on to conjecture that $\mathcal{S}_p$ and $\mathcal{S}_q$ are transverse semigroups acting on $\mathbb{Z}_r$.

Conjectures 1.4 and 1.7 are partially motivated by Furstenberg's framework but fall outside it. One could approach Conjecture 1.4 by considering only the ternary expansions of fractional parts $\{\{\lambda 2^n\}\}$, and thus iterating $x \to 2x$ on the compact space $X = \mathbb{R}/\mathbb{Z}$. This defines a larger exceptional set $\mathcal{E}(\mathbb{R}/\mathbb{Z})$ that contains $\mathcal{E}(\mathbb{R})$. Does $\mathcal{E}(\mathbb{R}/\mathbb{Z})$ have Hausdorff dimension zero? This set includes all dyadic rationals (thus $\lambda = 1$), which is a dense set in $\mathbb{R}/\mathbb{Z}$, so its closure is the whole space $X$ and is not covered by Furstenberg's results.

Furstenberg's formulation does not apply to semigroups of transformations on the real numbers because $\mathbb{R}$ is not compact. One may ask: Can Furstenberg's framework be generalized to apply to semigroups of operators acting on the real numbers, or the integers?

## 6. *Concluding remarks*

We conclude by reviewing the history of Erdős's question. Erdős [**5**] raised his question on ternary expansions of $2^n$ in connection with his conjecture that the binomial coefficient $\binom{2n}{n}$ is not squarefree for all $n \geqslant 5$. This binomial coefficient is divisible by 4 except for $n = 2^k$, so it is natural to examine when larger primes divide $\binom{2^{k+1}}{2^k}$. Here one has

$$3 \text{ does not divide } \binom{2^{k+1}}{2^k} \iff \text{the ternary expansion of } 2^n \text{ omits the digit 2,}$$

as follows from Lucas's theorem (Lucas [**16**]; see Graham *et al.* [**12**, Exercise 5.61]). This led Erdős to raise his ternary expansion question, since a positive answer to it would establish his binomial coefficient conjecture.

Erdős's binomial coefficient conjecture was later resolved affirmatively, without answering the ternary expansion question. In 1985 Sārközy [**20**] proved that $\binom{2n}{n}$ is not squarefree for all sufficiently large $n$. In about 1995 Granville and Ramaré [**13**] and, independently, Velammal [**23**] proved it for all $n \geqslant 5$.

The theme of this paper is that Erdős's unconventional question retains interest for its own sake, although the problem that originally motivated its study is now solved.

## *References*

**1.** A. G. Abercrombie, 'The Hausdorff dimension of some exceptional sets of $p$-adic integer matrices', *J. Number Theory* 53 (1995) 311–341.
**2.** A. Baker, *Transcendental number theory* (Cambridge University Press, Cambridge, 1975).
**3.** A. Dubickas, 'Arithmetical properties of powers of algebraic integers', *Bull. London Math. Soc.* 38 (2006) 70–80.
**4.** A. Dubickas and A. Novikas, 'Integer parts of powers of rational numbers', *Math. Z.* 251 (2005) 635–648.
**5.** P. Erdős, 'Some unconventional problems in number theory', *Math. Mag.* 52 (1979) 67–70.
**6.** P. Erdős and R. L. Graham, *Old and new problems and results in combinatorial number theory*, Monograph 28 de L'Enseignement Mathematique (University of Geneva, Geneva, 1980).
**7.** K. Falconer, *The geometry of fractal sets*, Cambridge Tracts in Mathematics 85 (Cambridge University Press, Cambridge, 1985).
**8.** K. Falconer, *Fractal geometry: mathematical foundations and applications* (Wiley, Chichester, 1990).
**9.** L. Flatto, J. C. Lagarias and A. Pollington, 'On the range of fractional parts $\{\xi(p/q)^n\}$', *Acta Arith.* 70 (1995) 125–147.
**10.** H. Furstenberg, 'Disjointness in ergodic theory, minimal sets, and a problem in diophantine approximation', *Math. Systems Theory* 1 (1967) 1–49.
**11.** H. Furstenberg, 'Intersections of Cantor sets and transversality of semigroups', *Problems in analysis: a symposium in honor of Salomon Bochner, Princeton University*, 1969 (ed. R. C. Gunning; Princeton University Press, Princeton, NJ, 1970) 41–59.
**12.** R. L. Graham, D. Knuth and O. Patashnik, *Concrete mathematics*, 2nd edn (Addison-Wesley, Reading, MA, 1994).
**13.** A. Granville and O. Ramaré, 'Explicit bounds on exponential sums and the scarcity of squarefree binomial coefficients', *Mathematika* 43 (1996) 73–107.
**14.** H. Gupta, 'Powers of 2 and sums of distinct powers of 3', *Univ. Beograd Publ. Elecktrotehn. Fak. Ser. Mat. Fiz.* 602–633 (1978) 151–158 (MR 0580438).
**15.** R. K. Guy, *Unsolved problems in number theory*, 2nd edn (Springer, New York, 1994).
**16.** E. Lucas, 'Sur les congruences des nombres eulériens et des coefficients différentials des fonctions trigonométriques, suivant un module premier', *Bull. Soc. Math. France* 6 (1878) 49–54.
**17.** W. Narkiewicz, 'A note on a paper of H. Gupta concerning powers of 2 and 3', *Univ. Beograd Publ. Elecktrotehn. Fak. Ser. Mat. Fiz.* 678–715 (1980) 173–174 (MR 0623247).
**18.** G. Rhin, 'Approximants de Padé et mesures effectives d'irrationalité', *Séminaire de théorie des nombres, Paris 1985–86*, Progress in Math. 71 (ed. C. Goldstein; Birkhäuser, Boston, MA, 1987).
**19.** C. A. Rogers, *Hausdorff measures* (Cambridge University Press, Cambridge, 1970). (Reprint 1998.)
**20.** A. Sárközy, 'On divisors of binomial coefficients I', *J. Number Theory* 20 (1985) 70–80.
**21.** J. Simons and B. M. M. de Weger, 'Theoretical and computational bounds for $m$-cycles of the $3n+1$ problem', *Acta Arith.* 117 (2005) 51–70.
**22.** N. E. Slater, 'Gaps and steps for the sequence $n\theta \pmod 1$', *Math. Proc. Camb. Phil. Soc.* 63 (1967) 1115–1123.
**23.** G. Velammal, 'Is the binomial coefficient $\binom{2n}{n}$ squarefree?', *Hardy–Ramanujan J.* 18 (1995) 23–45.

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