A CHARACTER-THEORETIC COMPLEMENTATION THEOREM FOR CARTER SUBGROUPS

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Let G be a finite group and H a subgroup with the property that any two elements in H which are conjugate in G must be conjugate in H. Frobenius **[2]** proved in 1901 that if H is an Abelian Hall subgroup of G (i.e. $|H|$ and $|G:H|$ are coprime), then H has a normal complement (i.e. there is a normal subgroup N of G with $HN = G$ and $N \cap H = 1$). In fact, this conclusion is valid if "Abelian" is replaced by "nilpotent" (e.g. see Sah **[4]**).

There is a stronger hypothesis on the subgroup H than the conjugacy hypothesis. Suppose that every complex irreducible character of H extends to an irreducible character of G. Then since the irreducible characters of H are a basis for the space of class functions on H, these characters must separate classes. Because the irreducible characters of H extend to irreducible characters of G (which are class functions of G), distinct H-conjugacy classes must belong to distinct G-conjugacy classes. Consequently, the character extension property on H implies the conjugacy property.

Sah **[4]** has also proved that if H is a solvable Hall subgroup and if H has the character extension property, then H has a normal complement. The purpose of this note is to prove a normal complementation theorem along these lines for solvable groups, retaining the character extension property on H while replacing the Hall condition on H by a condition which controls the imbedding of H in G.

The notation and concepts are standard. Recall that a finite solvable group contains a nilpotent self-normalizing subgroup called a Carter subgroup and that any two Carter subgroups are conjugate **[1]**. If G is a finite solvable group, the unique smallest normal subgroup N of G such that G/N is nilpotent is called the nilpotent residual of G. From these facts and definitions it is easy to see that if C is a Carter subgroup of G then $NC = G$, and if $L \lhd G$ then LC/L and NL/L are a Carter subgroup and the nilpotent residual of G/L respectively. $Z(H)$ denotes the centre of a group H. The kernel of a character ζ on H is the kernel of any representation affording ζ.

THEOREM *Let G be a finite solvable group, C a Carter subgroup and N the nilpotent residual. If every complex irreducible character of C extends to an irreducible character of G, then N is a normal complement to C.*

Proof. It suffices to prove that $N \cap C = 1$. If the theorem is false let G be a counter-example of the smallest order. If ζ is an irreducible character of C let ζ^1 denote some fixed character of G such that $\zeta^1 \mid C = \zeta$. Let $K = \bigcap \ker \zeta^1$ where the intersection is over all irreducible characters ζ of C. Then $K \lhd G$ and $K \cap C = 1$ since $K \cap C = \bigcap(\ker \zeta^1 \cap C) = \bigcap \ker \zeta$.

Suppose $K \neq 1$. Then G/K has smaller order than G, and CK/K is a Carter subgroup of G. Moreover, every character CK/K extends to a character of G/K.

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To see this let χ be an irreducible character of CK/K. The isomorphism $C \cong C/(C \cap K) \cong CK/K$ gives that the function ζ defined by $\zeta(c) = \chi(cK)$ is an irreducible character of C. The extension ζ^1 of ζ to G has K in its kernel by the definition of K. Thus χ^1 defined by $\chi^1(gK) = \zeta^1(g)$ is an irreducible character of G/K extending χ. Minimality of the counter-example G implies that $NK \cap CK = K$. Thus $N \cap C \leqslant K$ and so $N \cap C = N \cap C \cap K = 1$ contrary to G being a counter-example.

Thus $K = 1$. Let T_ζ be a matrix representation of G affording the character ζ^1, and let T be the representation of G which is the direct sum of the T_ζ's as ζ ranges over all irreducible characters of C. Suppose $x \in Z(C)$. Then by Schur's lemma $T_\zeta(x)$ is a scalar matrix and so $T_\zeta(x)$ commutes with $T_\zeta(g)$ for all $g \in G$. Thus $T(x)$ commutes with $T(g)$ for all $g \in G$. But the kernel of T, K, is 1 and so T is a monomorphism. Hence x commutes with all $g \in G$, and $Z(C) \leqslant Z(G)$. But $Z(G) \leqslant Z(C)$ also, since C is self-normalizing. It now follows that $Z(G)$ is not trivial since C is nilpotent.

Let A be a minimal normal subgroup of G in $Z(G) \leqslant C$. Then as before we can see that G/A satisfies the hypotheses of the theorem. By the choice of G we have $NA \cap CA = A$. But this gives $N \cap C \leqslant A$ for any minimal normal subgroup of G which is contained in $Z(G)$. Thus $Z(G)$ contains a unique minimal normal subgroup of G, and as a result it is a p-group for some prime p. Thus $Z(C)$ is a p-group, and since C is nilpotent C is a p-group.

Now C is a self-normalizing p-subgroup of G, whereupon C is a p-Sylow subgroup of G. Since C has the character extension property, whenever x and y in C are G-conjugate they are also C-conjugate. But then in this case a standard argument with the transfer shows (or see [3; p. 432, 4.9]) that G has a normal p-complement L. Since $C \cong G/L$ is nilpotent it follows that $N \leqslant L$. Thus $N \cap C \leqslant L \cap C = 1$, and G is no counter-example. This contradiction concludes the proof.

Note that the converse of the theorem is trivial.

The same proof proves an apparently more general result. Recall that a subgroup H of a finite group G is called abnormal if $g \in \langle H, H^g \rangle$ for all $g \in G$. (See [1] for basic properties of abnormal subgroups.) The following is true: if G is a finite group with a nilpotent, abnormal subgroup H and if H has the character extension property, then H has a normal complement. However, we have no example which shows this to be more general than the theorem.

Note that the example of G cyclic of order 4 and C the subgroup of index 2 shows that some assumption on C beyond the character-extension assumption is necessary to achieve the conclusion. The example of $SL_2(3)$ where a Carter subgroup C is cyclic of order 6 and the nilpotent residual N is the quaternion subgroup and where $N \cap C$ is the centre of $SL_2(3)$ shows that the character extension property of $C \leqslant G$ is really stronger than the conjugacy property of $C \leqslant G$. Moreover it shows that the theorem is false with the conjugacy hypothesis on $C \leqslant G$ replacing the character extension hypothesis.

References

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