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ON THE NUMBER OF SUBREPRESENTATIONS OF A GENERAL QUIVER REPRESENTATION

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Abstract

It is well known that the intersection multiplicities of Schubert classes in the Grassmannian are Littlewood–Richardson coefficients. We generalize this statement in the context of quiver representations. Here the intersection multiplicity of Schubert classes is replaced by the number of subrepresentations of a general quiver representation, and the Littlewood–Richardson coefficients are replaced by the dimension of a certain space of semi-invariants.

1. *Introduction*

It is well known that the intersection multiplicities of Schubert classes in the Grassmannian are Littlewood–Richardson coefficients. For a partition λ inside an $r \times (n-r)$ rectangle, let Y_λ be the Schubert variety inside the Grassmannian $\mathrm{Grass}(r,n)$ corresponding to λ and let $[Y_\lambda]$ be its cohomology class. Then we have

$$[Y_\lambda] \cdot [Y_\mu] = \sum_\nu c^\nu_{\lambda,\mu} [Y_\nu], \tag{1}$$

where $c^\nu_{\lambda,\mu}$ is a Littlewood–Richardson coefficient and the sum runs over all partitions ν inside an $r \times (n-r)$ rectangle. This result can be translated in terms of quiver representations as follows. Consider the triple flag quiver

$$\begin{tikzcd} x_1 \arrow[r] & x_2 \arrow[r] & \cdots \arrow[r] & x_n = y_n = z_n & \cdots \arrow[l] & y_2 \arrow[l] & y_1 \arrow[l] \\ & & & \vdots \arrow[u] \\ & & & z_2 \arrow[u] \\ & & & z_1. \arrow[u] \end{tikzcd}$$

Suppose that λ, μ and ν are partitions with Young diagrams that fit into an $r \times (n-r)$ rectangle such that $|\lambda| + |\mu| + |\nu| = r(n-r)$. The intersection of the classes $[Y_\lambda]$, $[Y_\mu]$ and $[Y_\nu]$ is zero-dimensional. Define dimension vectors α and β by

$$\alpha = \begin{array}{ccccccc} 1 & 2 & \cdots & n & \cdots & 2 & 1 \\ & & & \vdots & & & \\ & & & 2 & & & \\ & & & 1 & & & \end{array}$$

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and

$$\beta(x_{n-r-\lambda_i+i}) = \cdots = \beta(x_{n-r-\lambda_{i+1}+i}) = i,$$
$$\beta(y_{n-r-\mu_i+i}) = \cdots = \beta(y_{n-r-\mu_{i+1}+i}) = i,$$
$$\beta(z_{n-r-\nu_i+i}) = \cdots = \beta(z_{n-r-\nu_{i+1}+i}) = i$$

for $i = 1, 2, \dots, r$ with the *ad hoc* conventions $\lambda_{r+1} = \mu_{r+1} = \nu_{r+1} = 0$ and $\lambda_0 = \mu_0 = \nu_0 = n - r - 1$.

The number of β-dimensional subrepresentations of a general α-dimensional representation is finite and equal to the multiplicity of $[Y_{\overline{\nu}}]$ inside $[Y_\lambda] \cdot [Y_\mu]$. Here

$$\overline{\nu} = (n - r - \nu_r, n - r - \nu_{r-1}, \cdots, n - r - \nu_1)$$

is the complementary partition of ν inside the $r \times (n - r)$ rectangle.

Following the calculation in [3], the dimension of the space of semi-invariants of weight $\langle \beta, \cdot \rangle$ on the space $\mathrm{Rep}(Q, \alpha - \beta)$ of $(\alpha - \beta)$-dimensional representations is the Littlewood–Richardson coefficient $c^{\overline{\nu}}_{\lambda,\mu}$, where $\langle \cdot, \cdot \rangle$ is the Euler form defined in Section 2.

In this paper, we generalize the connection between Schubert calculus and Littlewood–Richardson coefficients to representations of quivers without oriented cycles and arbitrary dimension vectors. For a quiver Q without oriented cycles and two dimension vectors α and β, we define $N(\beta, \alpha)$ as the number of β-dimensional subrepresentations of a general α-dimensional representation and $M(\beta, \alpha)$ as the dimension of the space of semi-invariant polynomials of weight $\langle \beta, \cdot \rangle$ on the representation space of dimension $\gamma := \alpha - \beta$. If $\langle \beta, \gamma \rangle$ equals 0, then $N(\beta, \alpha)$ is finite.

THEOREM 1. *If $\langle \beta, \gamma \rangle = 0$, then we have $N(\beta, \alpha) = M(\beta, \alpha)$.*

The proof compares the calculation of $N(\beta, \alpha)$ with that of $M(\beta, \alpha)$. The calculation of $N(\beta, \alpha)$ comes from intersection theory and was done by Crawley-Boevey [1]. The calculation of $M(\beta, \alpha)$ comes from standard calculations in the coordinate ring of a representation space involving the Littlewood–Richardson rule. In this paper we explain why both the calculations are the same. For this, we will make use of (1).

Suppose that a general representation of dimension α has infinitely many β-dimensional subrepresentations. In Section 6, we will see that in that case, the cohomology class of the variety of β-dimensional subrepresentations of an α-dimensional representation in general position is given by a formula with coefficients that can be interpreted as multiplicities of isotypic components in the coordinate ring of $\mathrm{Rep}(Q, \gamma)$.

Suppose that $\langle \beta, \gamma \rangle$ equals 0. Given a general representation of dimension α, we will construct in Section 7 a basis of the semi-invariant polynomials on $\mathrm{Rep}(Q, \gamma)$ of weight $\langle \beta, \cdot \rangle$.

2. *Basic notation*

A *quiver* is a pair $Q = (Q_0, Q_1)$, where Q_0 is the set of vertices and Q_1 the set of arrows. Each arrow a has a head ha and a tail ta, both in Q_0:

$$ta \xrightarrow{a} ha.$$

An *oriented cycle* is a sequence of arrows $a_1, a_2, \dots, a_r \in Q_1$ such that $ta_i = ha_{i+1}$ for $i = 1, 2, \dots, r - 1$ and $ha_1 = ta_r$. We will assume that Q has no oriented cycles.

We fix an algebraically closed base field K. A representation V of Q is a family of finite-dimensional K-vector spaces

$$\{V(x) \mid x \in Q_0\}$$

together with a family of K-linear maps

$$\{V(a) : V(ta) \to V(ha) \mid a \in Q_1\}.$$

The dimension vector of a representation V is the function $\underline{d}(V) : Q_0 \to \mathbb{Z}$ defined by $\underline{d}(V)(x) := \dim V(x)$. The dimension vectors lie in the space $\Gamma = \mathbb{Z}^{Q_0}$ of integer-valued functions on Q_0. A morphism $\phi : V \to W$ of two representations is a collection of K-linear maps, that is,

$$\{\phi(x) : V(x) \to W(x) \mid x \in Q_0\}$$

such that for each $a \in Q_1$, we have $W(a)\phi(ta) = \phi(ha)V(a)$, that is, the diagram

$$\begin{tikzcd} V(ta) \arrow[r, "V(a)"] \arrow[d, "\phi(ta)"'] & V(ha) \arrow[d, "\phi(ha)"] \\ W(ta) \arrow[r, "W(a)"'] & W(ha) \end{tikzcd}$$

commutes. We denote the vector space of morphisms from V to W by $\mathrm{Hom}_Q(V, W)$.

The category of representations of Q is hereditary, that is, a subobject of a projective object is also projective. This implies that if V and W are representations, then $\mathrm{Ext}^i_Q(V, W)$ equals 0 for all $i \geqslant 2$. We shall write $\mathrm{Ext}_Q(V, W)$ instead of $\mathrm{Ext}^1_Q(V, W)$.

The spaces $\mathrm{Hom}_Q(V, W)$ and $\mathrm{Ext}_Q(V, W)$ can be calculated, respectively, as the kernel and cokernel of the following linear map:

$$d^V_W : \bigoplus_{x \in Q_0} \mathrm{Hom}(V(x), W(x)) \longrightarrow \bigoplus_{a \in Q_1} \mathrm{Hom}(V(ta), W(ha)), \tag{2}$$

where the map d^V_W restricted to $\mathrm{Hom}(V(x), W(x))$ is equal to

$$\sum_{\substack{a \\ ta=x}} \mathrm{Hom}(\mathrm{id}_{V(x)}, W(a)) - \sum_{\substack{a \\ ha=x}} \mathrm{Hom}(V(a), \mathrm{id}_{W(x)}).$$

In other words,

$$d^V_W\big(\{\phi(x) \mid x \in Q_0\}\big) = \{W(a)\phi(ta) - \phi(ha)V(a) \mid a \in Q_1\}.$$

For a dimension vector β, we denote by

$$\mathrm{Rep}(Q, \beta) = \bigoplus_{a \in Q_1} \mathrm{Hom}(K^{\beta(ta)}, K^{\beta(ha)})$$

the vector space of representations of Q of dimension vector β. The group

$$\mathrm{GL}(Q, \beta) := \prod_{x \in Q_0} \mathrm{GL}(\beta(x))$$

and its subgroup

$$\mathrm{SL}(Q, \beta) := \prod_{x \in Q_0} \mathrm{SL}(\beta(x))$$

act on $\mathrm{Rep}(Q, \beta)$ as follows. If

$$A = \{A(x) \mid x \in Q_0\} \in \mathrm{GL}(Q, \beta)$$

and

$$V = \{V(a) \mid a \in Q_1\} \in \mathrm{Rep}(Q, \beta),$$

then we define

$$A \cdot V := \{A(ha)V(a)A(ta)^{-1} \mid a \in Q_1\}.$$

The group $\mathrm{GL}(Q,\beta)$ acts on the coordinate ring $K[\mathrm{Rep}(Q,\beta)]$ as follows. If $f \in K[\mathrm{Rep}(Q,\beta)]$ and $A \in \mathrm{GL}(Q,\beta)$, then

$$(A \cdot f)(V) = f(A^{-1} \cdot V), \quad V \in \mathrm{Rep}(Q,\beta).$$

We are interested in the ring of semi-invariants

$$\mathrm{SI}(Q,\beta) = K[\mathrm{Rep}(Q,\beta)]^{\mathrm{SL}(Q,\beta)}.$$

To each $\sigma \in \Gamma = \mathbb{Z}^{Q_0}$, we can associate a character of $\mathrm{GL}(\beta)$ defined by

$$A = \{A(x) \mid x \in Q_0\} \in \mathrm{GL}(\beta) \mapsto \prod_{x \in Q_0} \det(A(x))^{\sigma(x)}.$$

By abuse of notation this character is also denoted by σ. The one-dimensional representation corresponding to this multiplicative character will be denoted by $\det^{\sigma}$. For any two characters $\sigma, \tau \in \mathbb{Z}^{Q_0}$, we have $\det^{\sigma+\tau} = \det^{\sigma} \otimes \det^{\tau}$. We can write

$$\det^{\sigma} = \bigotimes_{x \in Q_0} \det_x^{\sigma(x)},$$

where $\det_x^k$ is the one-dimensional representation of $\mathrm{GL}(\beta(x))$ corresponding to the multiplicative character $A \mapsto \det(A)^k$.

The ring $\mathrm{SI}(Q,\beta)$ has a weight space decomposition

$$\mathrm{SI}(Q,\beta) = \bigoplus_{\sigma} \mathrm{SI}(Q,\beta)_{\sigma},$$

where σ runs through the characters of $\mathrm{GL}(Q,\beta)$ and

$$\mathrm{SI}(Q,\beta)_{\sigma} = \{f \in K[\mathrm{Rep}(Q,\beta)] \mid g(f) = \sigma(g) f \ \forall g \in \mathrm{GL}(Q,\beta)\}.$$

Let α and β be the two elements of Γ. We define the Euler inner product

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha). \tag{3}$$

It follows from (2) and (3) that

$$\langle \underline{d}(V), \underline{d}(W) \rangle = \dim_K \mathrm{Hom}_Q(V,W) - \dim_K \mathrm{Ext}_Q(V,W).$$

3. *Computation of* $N(\beta,\alpha)$

If r and n are non-negative integers with $r \leqslant n$, then $\mathrm{Grass}(r,n)$ denotes the Grassmannian of r-dimensional subspaces of K^n. Let α, β and γ be dimension vectors such that $\alpha = \beta + \gamma$. Then we define

$$\mathrm{Grass}(\beta,\alpha) = \prod_{x \in Q_0} \mathrm{Grass}(\beta(x), \alpha(x)).$$

A point $W = \{W(x) \mid x \in Q_0\} \in \mathrm{Grass}(\beta,\alpha)$ is a collection of subspaces with $\dim W(x) = \beta(x)$ for all $x \in Q_0$. Let us consider the incidence variety

$$Z(Q,\beta,\alpha) = \{(V,W) \in \mathrm{Rep}(Q,\alpha) \times \mathrm{Grass}(\beta,\alpha) \mid \forall a \in Q_1 \ V(a)(W(ta)) \subseteq W(ha)\}.$$

There are two projections as we can see in the diagram below.

$$\begin{tikzcd} & Z(Q,\beta,\alpha) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{Rep}(Q,\alpha) & & \mathrm{Grass}(\beta,\alpha) \end{tikzcd}$$

The following proposition was proved in [7].

PROPOSITION 2.
(a) *The first projection* $q : Z(Q, \beta, \alpha) \to \mathrm{Rep}(Q, \alpha)$ *is proper.*
(b) *The second projection* $p : Z(Q, \beta, \alpha) \to \mathrm{Grass}(\beta, \alpha)$ *gives* $Z(Q, \beta, \alpha)$ *the structure of a vector bundle over* $\mathrm{Grass}(\beta, \alpha)$*.*
(c) $\dim Z(Q, \beta, \alpha) - \dim \mathrm{Rep}(Q, \alpha) = \langle \beta, \gamma \rangle$.

We will use the result of Crawley-Boevey [1] who proved the formula for the cohomology class of the general fiber of q in terms of intersection theory.

In order to formulate this result, we need some notation. For a variety X, its associated cohomology ring will be denoted by $\mathcal{A}^*(X)$ (the Chow ring or the singular cohomology ring). We recall that for the Grassmannian $\mathrm{Grass}(r, n)$, the ring $\mathcal{A}^*(\mathrm{Grass}(r, n))$ is spanned by classes of Schubert varieties Y_λ, where λ denotes a partition contained in the rectangle $((n-r)^r)$, that is, $\lambda = (\lambda_1, \dots, \lambda_r)$ with $n - r \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_1 \geqslant 0$. The sum $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_r$ of the parts of λ is equal to the codimension of Y_λ. In our setup, we have

$$\mathcal{A}^*(\mathrm{Grass}(\beta, \alpha)) = \bigotimes_{x \in Q_0} \mathcal{A}^*(\mathrm{Grass}(\beta(x), \alpha(x)).$$

We denote by $[\lambda]_x$ the cohomology class of the Schubert variety Y_λ in the factor $\mathcal{A}^*(\mathrm{Grass}(\beta(x), \alpha(x)))$. We use the convention $[\lambda]_x = 0$ if λ is not contained in the rectangle $(\gamma(x)^{\beta(x)})$.

PROPOSITION 3 [1]. *For general* $V \in \mathrm{Rep}(Q, \alpha)$*, the cycle of* $q^{-1}(V)$ *in* $\mathcal{A}^*(\mathrm{Grass}(\beta, \alpha))$ *is equal to*

$$[q^{-1}(V)] = \prod_{a \in Q_1} \left(\sum_\lambda [\lambda]_{ta} [\overline{\lambda}]_{ha} \right), \tag{4}$$

where in $\sum_\lambda [\lambda]_{ta} [\overline{\lambda}]_{ha}$*,* λ *runs over all the partitions which fit inside a* $\beta(ta) \times \gamma(ha)$ *rectangle. In* (4)*,* $\overline{\lambda}$ *denotes the complement of* λ *inside a* $\beta(ta) \times \gamma(ha)$ *rectangle.*

If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\beta(ta)})$ with $\gamma(ha) \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{\beta(ta)} \geqslant 0$ is a partition inside a $\beta(ta) \times \gamma(ha)$ rectangle, then its complement is given by

$$\overline{\lambda} = (\gamma(ha) - \lambda_{\beta(ta)}, \dots, \gamma(ha) - \lambda_2, \gamma(ha) - \lambda_1).$$

From this, it is clear that (4) is in agreement with the formula in [1, Theorem].

Now we exchange the sum and the product in (4). Clearly the summands in the formula correspond to functions $\underline{\lambda} : Q_1 \to \mathcal{P}$, where $\mathcal{P}$ is a set of partitions and $\lambda(a)$ is contained in a $\beta(ta) \times \gamma(ha)$ rectangle for all arrows a. Now (4) can be rewritten as

$$[q^{-1}(V)] = \sum_{\underline{\lambda} : Q_1 \to \mathcal{P}} \prod_{x \in Q_0} \prod_{\substack{a \in Q_1 \\ ta = x}} [\underline{\lambda}(a)]_x \prod_{\substack{a \in Q_1 \\ ha = x}} [\overline{\underline{\lambda}}(a)]_x. \tag{5}$$

Suppose that $\langle \beta, \gamma \rangle$ equals 0. Then we have $\dim Z(Q, \beta, \alpha) = \dim \mathrm{Rep}(Q, \alpha)$ by Proposition 2(c). This means that the general fiber of q is finite. Such a general fiber is reduced, even in positive characteristic (see [1, Corollary 3]). Let $N(\beta, \alpha)$ be the cardinality of a general fiber. Therefore, a general representation V of dimension α has $N(\beta, \alpha)$ subrepresentations of dimension β. We can rewrite equation (5) in terms of Schur functors as

$$N(\beta, \alpha) = |q^{-1}(V)| = \sum_{\underline{\lambda} : Q_1 \to \mathcal{P}} \prod_{x \in Q_0} \mathrm{mult} \left(S^{\gamma(x)^{\beta(x)}}, \bigotimes_{\substack{a \in Q_1 \\ ta = x}} S^{\underline{\lambda}(a)} \otimes \bigotimes_{\substack{a \in Q_1 \\ ha = x}} S^{\overline{\underline{\lambda}}(a)} \right). \tag{6}$$

Here $\mathrm{mult}(S^\lambda; T)$ denotes the multiplicity of S^λ in T.

4. *Computation of $M(\beta,\alpha)$*

Let us calculate the dimension of a weight space $\mathrm{SI}(Q,\gamma)_{\langle\beta,\cdot\rangle}$. Now, we assume that the base field has characteristic 0. This is sufficient because we will show in Section 5 that the number $M(\beta,\alpha)$ does not depend on the (algebraically closed) base field. The space $\mathrm{Rep}(Q,\gamma)$ can be identified with

$$\prod_{a\in Q_1}\mathrm{Hom}_K(W(ta),W(ha)),$$

where $W(x)$ is a vector space of dimension $\gamma(x)$ for all $x\in Q_0$. The coordinate ring can now be identified with the symmetric algebra on the dual space

$$K[\mathrm{Rep}(Q,\gamma)]=\bigotimes_{a\in Q_1}\mathrm{Sym}\left(W(ta)\otimes W(ha)^*\right).$$

Here Sym denotes the symmetric algebra on a vector space. By Cauchy's formula, we can rewrite this in terms of Schur functors. We use the exterior power notation for Schur functors, that is, for a partition μ, we write $\bigwedge^{\mu}W:=S^{\mu'}W$, where μ' is the conjugate partition of μ. In particular, $\bigwedge^{(m)}$ is the mth exterior power. We have

$$K[\mathrm{Rep}(Q,\gamma)]=\bigoplus_{\underline{\lambda}\,:\,Q_1\to\mathcal{P}}\ \bigotimes_{a\in Q_1}\left(\bigwedge\nolimits^{\underline{\lambda}(a)}W(ta)\otimes\bigwedge\nolimits^{\underline{\lambda}(a)}W(ha)^*\right).\tag{7}$$

This can be rewritten as

$$K[\mathrm{Rep}(Q,\gamma)]=\bigoplus_{\underline{\lambda}\,:\,Q_1\to\mathcal{P}}\ \bigotimes_{x\in Q_0}\left(\bigotimes_{\substack{a\in Q_1\\ ta=x}}\bigwedge\nolimits^{\underline{\lambda}(a)}W(x)\right)\otimes\left(\bigotimes_{\substack{a\in Q_1\\ ha=x}}\bigwedge\nolimits^{\underline{\lambda}(a)}W(x)^*\right).\tag{8}$$

Let us calculate the dimension of the space of semi-invariants of weight $\langle\beta,\cdot\rangle$.

The partition $\underline{\lambda}(a)$ has parts at most $\gamma(ha)$, because $\dim W(ha)=\gamma(ha)$. We need additional restriction to match the one in Proposition 3. This is provided by the following lemma.

LEMMA 4. *If a summand in (8) corresponding to the function $\underline{\lambda}:Q_1\to\mathcal{P}$ contains a non-zero semi-invariant of weight $\langle\beta,\cdot\rangle$, then for each $a\in Q_1$ the partition $\underline{\lambda}(a)$ is contained inside a $\beta(ta)\times\gamma(ha)$ rectangle.*

Proof. Let us look at the space of semi-invariants $\mathrm{SI}(Q,\gamma)_{\langle\beta,\cdot\rangle}$. By [**3**, Theorem 1] (see also [**8**]), this space is spanned by the semi-invariants c^V defined by the formula $c^V(W):=\det d^V_W$, where $V\in\mathrm{Rep}(Q,\beta)$, and d^V_W is the differential in (2). Similarly, $\mathrm{SI}(Q,\beta)_{-\langle\cdot,\gamma\rangle}$ is generated by semi-invariants c_W, where $c_W(V):=\det d^V_W=c^V(W)$. Let us investigate the contribution of the coefficients of the matrix $W(a)$ to such a semi-invariant. The only block of d^V_W where this matrix $W(a)$ occurs is the block $\mathrm{Hom}(\mathrm{id}_{V(x)},W(a))$

$$\begin{pmatrix}W(a)&0&\cdots&0\\0&W(a)&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&W(a)\end{pmatrix}$$

with $\beta(ta)$ blocks $W(a)$. Now any multihomogeneous semi-invariant will arise from exhibiting a deteminant $c^V(W)$ as a polynomial in coefficients of matrices from V and taking a coefficient of some monomial. Such a semi-invariant has to be a linear combination of minors of the above block, multiplied by polynomials depending on other matrices $W(b)$. However, the minors of the above block matrix are products of minors of $W(a)$ with at most $\beta(ta)$ factors in each summand. By the straightening law (compare, for example, [**2**]), we know that the products of

$\beta(ta)$ minors of $W(a)$ are contained in the space

$$\sum_{\nu} \bigwedge^{\nu} W(ta) \otimes \bigwedge^{\nu} W(ha)^*$$

in $\mathrm{Sym}(W(ta) \otimes W(ha)^*)$, where ν runs over all partitions with at most $\beta(ta)$ parts. $\square$

Let us calculate the dimension of the space of semi-invariants of weight $\sigma := \langle \beta, \cdot \rangle$. We have

$$\sigma(x) = \beta(x) - \sum_{\substack{a \in Q_1 \\ ha = x}} \beta(ta) \tag{9}$$

for all $x \in Q_0$. By duality, we have the following $\mathrm{GL}(W(ha))$ isomorphism:

$$\bigwedge^{\underline{\mu}(a)} W(ha)^* = \bigwedge^{\overline{\underline{\mu}}(a)} W(ha) \otimes \det_{ha}^{-\beta(ta)}, \tag{10}$$

where $\overline{\underline{\mu}}(a)$ is the complement of $\underline{\mu}(a)$ inside a $\beta(ta) \times \gamma(ha)$ rectangle.

From (8) and (10), it follows that the dimension $M(\beta, \alpha)$ of $\mathrm{SI}(Q, \gamma)_{\langle \beta, \cdot \rangle}$ is equal to

$$\sum_{\underline{\lambda} : Q_1 \to \mathcal{P}} \prod_{x \in Q_0} \mathrm{mult} \left(\det_x^{\sigma(x)}; \bigotimes_{\substack{a \\ ta = x}} \bigwedge^{\underline{\lambda}(a)} W(x) \otimes \bigotimes_{\substack{a \\ ha = x}} \left(\bigwedge^{\overline{\underline{\lambda}}(a)} W(x) \otimes \det_x^{-\beta(ta)} \right) \right). \tag{11}$$

Equations (9) and (11) imply that

$$M(\beta, \alpha) = \sum_{\underline{\lambda} : Q_1 \to \mathcal{P}} \prod_{x \in Q_0} \mathrm{mult} \left(\det_x^{\beta(x)}; \bigotimes_{\substack{a \\ ta = x}} \bigwedge^{\underline{\lambda}(a)} W(x) \otimes \bigotimes_{\substack{a \\ ha = x}} \bigwedge^{\overline{\underline{\lambda}}(a)} W(x) \right). \tag{12}$$

In view of Lemma 4, we only need to sum over those functions $\underline{\lambda} : Q_1 \to \mathcal{P}$ for which $\underline{\lambda}(a)$ lies in a $\beta(ta) \times \gamma(ha)$ rectangle. We rewrite (12) in terms of Schur functors as

$$M(\beta, \alpha) = \sum_{\underline{\lambda}} \prod_{x \in Q_0} \mathrm{mult} \left(\bigwedge^{(\gamma(x)^{\beta(x)})}; \bigotimes_{\substack{a \in Q_1 \\ ta = x}} \bigwedge^{\underline{\lambda}(a)} \otimes \bigotimes_{\substack{a \in Q_1 \\ ha = x}} \bigwedge^{\overline{\underline{\lambda}}(a)} \right). \tag{13}$$

Proof of Theorem 1. The number $N(\beta, \alpha)$ does not depend on the base field, and neither does $M(\beta, \alpha)$, by Proposition 8 in Section 5. We may assume that the base field K is algebraically closed and has characteristic 0. Note that $S^{\lambda} = \bigwedge^{\lambda'}$, where λ' is the conjugate partition. Also, the Littlewood–Richardson coefficients $c^{\nu}_{\lambda,\mu}$ and $c^{\nu'}_{\lambda',\mu'}$ are the same. From these observations, it follows that the right sides of (6) and (11) are identical. $\square$

5. *Good filtrations*

We work over an algebraically closed field K of arbitrary characteristic. Suppose that V is an n-dimensional vector space. We use the convention that a Schur module $L_{\lambda'} V$ is denoted by the partition of its highest weight, that is, $\bigwedge^i V = L_{1^i} V$, $S^i V = L_i V$.

Let us recall that a *good filtration* for a rational finite-dimensional $\mathrm{GL}(n) := \mathrm{GL}(V)$ module W is a filtration, that is,

$$0 = W_0 \subset W_1 \subset \cdots \subset W_{s-1} \subset W_s = W$$

for which each factor W_{i+1}/W_i is isomorphic to some Schur module $L_{\lambda(i)} V$. Such filtration, if it exists, may not be unique. The number of factors $L_{\lambda} V$ in any good filtration of W does not depend on the choice of a good filtration and will be denoted by $n_{\lambda}(W)$.

PROPOSITION 5. *If the modules W_1 and W_2 have good filtrations, then the tensor product $W_1 \otimes W_2$ has a good filtration.*

Proof. See [9, Theorem 4.2] or [4, Theorem 4.3.1]. □

PROPOSITION 6. *Let W be a module with a good filtration. Then there exists a good filtration*

$$0 = W'_0 \subset W'_1 \subset \cdots \subset W'_{s-1} \subset W'_s = W$$

such that the submodule of the $\mathrm{SL}(V)$*-invariants $W^{\mathrm{SL}(V)}$ in W is equal to W'_t for some t. In particular, the dimension of $W^{\mathrm{SL}(V)}$ is equal to the number of factors W_i/W_{i-1} isomorphic to the trivial representation.*

Proof. We use the results of [4] freely. Let us order the highest weights λ by saying that $\lambda < \mu$ if $\lambda - \mu$ is a sum of positive roots for $\mathrm{SL}(V)$.

Let W be a module with a good filtration. We can assume without loss of generality that W is a polynomial homogeneous representation of degree d. Then by [4, Proposition 3.2.6] there exists a filtration

$$0 = W'_0 \subset W'_1 \subset \cdots \subset W'_{s-1} \subset W'_s = W$$

with $W'_i/W'_{i-1} = L_{\lambda(i)}V$ for which $\lambda(1) \leqslant \lambda(2) \leqslant \cdots \leqslant \lambda(s)$. We notice that for the existence of $\mathrm{SL}(V)$-invariants, it is nessesary that $d = ne$ for some e. Now among all the possible highest weights λ that correspond to partitions of d with at most n parts, the smallest one is $\lambda = (e^n)$. Let t be maximal such that $\lambda(1) = \cdots = \lambda(t) = (e^n)$. All composition factors of W'_t are isomorphic to the trivial $\mathrm{SL}(V)$-representation $L_{(e^n)}V$. Hence $\mathrm{SL}(V)$ acts trivially on W'_t. This proves the first part of the proposition.

It remains to show that the submodule $W^{\mathrm{SL}(V)}$ is equal to W'_t. Let us assume that the opposite is true. Then the module W/W'_t is a polynomial homogeneous representation of $\mathrm{GL}(V)$ which has a good filtration with no factors isomorphic to $L_{(e^n)}V$ and with a non-trivial submodule of $\mathrm{SL}(V)$-invariants. Let u be the smallest number for which the factor module W'_{t+u}/W'_t has a non-zero module of $\mathrm{SL}(V)$-invariants. Then the factor $W'_{t+u}/W'_{t+u-1} = L_{\lambda(t+u)}$ contains a non-zero $\mathrm{SL}(V)$-invariant. This is impossible because the unique irreducible submodule of $L_{\lambda(t+u)}$ has highest weight $\lambda(t+u)$ and is not one-dimensional. □

COROLLARY 7. *Let W be a polynomial homogeneous $\mathrm{GL}(V)$ module of degree $d = en$ with a good filtration. Then the dimension of the $W^{\mathrm{SL}(V)}$-invariants in W is equal to $n_{(e^n)}(W)$.*

We continue with the application to quiver representations. Assume that Q is a quiver without oriented cycles. The above reasoning generalizes directly to products of general linear groups, as the Schur modules for products of general linear groups are just tensor products of the Schur modules for the factors. In dealing with the coordinate rings $K[\mathrm{Rep}(Q, \alpha)]$ and their good filtrations, notice that we are really dealing with their homogeneous components which are finite-dimensional representations.

PROPOSITION 8. *Let Q be a quiver with no oriented cycles and let α be a dimension vector. The coordinate ring $K[\mathrm{Rep}(Q, \alpha)]$ has a good filtration as a $\mathrm{GL}(Q, \alpha)$ module. Thus, the dimension of the spaces of semi-invariants $\mathrm{SI}(Q, \alpha)_\sigma$ can be calculated as a multiplicity of the corresponding tensor products of Schur functors in the coordinate ring $K[\mathrm{Rep}(Q, \alpha)]$. In particular, this dimension does not depend on the characteristic of K.*

Proof. The coordinate ring $K[\mathrm{Rep}(Q,\alpha)]$ has the following decomposition:

$$K[\mathrm{Rep}(Q,\alpha)] = \bigotimes_{a\in Q_1} \mathrm{Sym}(V(ta)\otimes V(ha)^*).$$

Now, using the straightening law (compare [**2**] or [**10**, Theorem 2.3.2]), we get that $\mathrm{Sym}(V(ta)\otimes V(ha)^*)$ has a characteristic-free filtration with associated graded object $\bigoplus_{\lambda(a)} L_{\lambda(a)}V(ta)\otimes L_{\lambda(a)}V(ha)^*$. This is a good filtration. Applying Proposition 5, we get that $K[\mathrm{Rep}(Q,\alpha)]$ has a good filtration as a $\mathrm{GL}(Q,\alpha)$ module. Now Proposition 6 (for a product of general linear groups) gives the result. □

6. *Generalization to covariants*

Theorem 1 generalizes from semi-invariants to covariants. Let us assume that $\langle\beta,\gamma\rangle \geqslant 0$. Assume that the cycle $[q^{-1}(V)]$ for generic $V\in\mathrm{Rep}(Q,\alpha)$ decomposes as follows:

$$[q^{-1}(V)] = \sum_{\underline{\lambda}\,:\,Q_0\to\mathcal{P}} N(\beta,\alpha,\underline{\lambda}) \prod_{x\in Q_0} [\overline{\underline{\lambda}}(x)]_x. \tag{14}$$

Here $\overline{\underline{\lambda}}(x)$ is the complement of $\underline{\lambda}(x)$ in a $\beta(x)\times\gamma(x)$ rectangle.

Let $W(x)$ be a $\gamma(x)$-dimensional K-vector space for all $x\in Q_0$. We can identify $\mathrm{Rep}(Q,\gamma)$ with $\bigoplus_{a\in Q_0}\mathrm{Hom}(W(ta),W(ha))$ and $\mathrm{GL}(Q,\gamma)$ with $\prod_{x\in Q_0}\mathrm{GL}(W(x))$.

We define $M(\beta,\alpha,\underline{\lambda})$ as the multiplicity of $\det^\sigma$ in

$$K[\mathrm{Rep}(Q,\gamma)]\otimes\bigotimes_{x\in Q_0}\bigwedge^{\underline{\mu}(x)}W(x).$$

Proposition 9. *Assume that $\sum_{x\in Q_0}|\underline{\mu}(x)| = \langle\beta,\gamma\rangle$. Then*

$$N(\beta,\alpha,\underline{\mu}) = M(\beta,\alpha,\underline{\mu}).$$

We will reduce Proposition 9 to Theorem 1. Let us write

$$\overline{\underline{\mu}}(x) = (\gamma(x)^{b_1(x)}, (\gamma(x)-1)^{b_2(x)},\dots,1^{b_{\gamma(x)}(x)})$$

for all x, where $\overline{\underline{\mu}}(x)$ is the complement of $\underline{\mu}(x)$ inside a $\beta(x)\times\gamma(x)$ rectangle. We introduce the quiver $\widehat{Q}$ with

$$\widehat{Q}_0 = Q_0\cup\bigcup_{x\in Q_0}\{y_{1,x},y_{2,x},\dots,y_{\gamma(x),x}\}$$

and

$$\widehat{Q}_1 = Q_1\cup\bigcup_{x\in Q_1}\{a_{1,x},a_{2,x},\dots,a_{\gamma(x),x}\},$$

where

$$a_{i,x} : y_{i-1,x}\to y_{i,x}$$

for all i and x. We use the convention $y_{0,x}=x$.

We define dimension vectors $\widehat{\beta}$ and $\widehat{\gamma}$ by

$$\widehat{\beta}(x) = \beta(x),\quad x\in Q_0,$$
$$\widehat{\beta}(y_{i,x}) = b_1(x)+b_2(x)+\cdots+b_{\gamma(x)-i+1},\quad i=1,2,\dots,\gamma(x),$$

and

$$\widehat{\gamma}(x) = \gamma(x),\quad x\in Q_0,$$
$$\widehat{\gamma}(y_{i,x}) = \gamma(x)-i+1,\quad i=1,2,\dots,\gamma(x).$$

LEMMA 10.

$$N(\beta, \alpha, \underline{\mu}) = N(\widehat{\beta}, \widehat{\alpha}).$$

Proof. From (5), it follows that

$$N(\widehat{\beta}, \widehat{\alpha}) = \sum_{\underline{\lambda}\,:\,\widehat{Q}_1 \to \mathcal{P}} \prod_{x \in \widehat{Q}_0} \prod_{\substack{a \in \widehat{Q}_1 \\ ta = x}} [\underline{\lambda}(a)]_x \prod_{\substack{a \in \widehat{Q}_1 \\ ha = x}} [\overline{\underline{\lambda}}(a)]_x.$$

To get the class of a point at vertex $y_{\gamma(x),x}$ we must have

$$\underline{\lambda}(a_{\gamma(x),x}) = (\gamma(y_{\gamma(x),x})^{\beta(y_{\gamma(x),x})}) = (1^{b_1}).$$

Now $\overline{\underline{\lambda}}(a_{\gamma(x),x})$ and $\underline{\lambda}(a_{\gamma(x),x})$ fit together into a $\beta(y_{\gamma(x)-1,x}) \times \gamma(y_{\gamma(x),x}) = (b_1 + b_2) \times 1$ rectangle. It follows that

$$\overline{\underline{\lambda}}(a_{\gamma(x),x}) = (1^{b_2}).$$

To get a non-zero summand, $\overline{\underline{\lambda}}(a_{\gamma(x),x})$ and $\underline{\lambda}(a_{\gamma(x)-1,x})$ have to fit together into a $\beta(y_{\gamma(x)-1,x}) \times \gamma(y_{\gamma(x)-1,x}) = (b_1 + b_2) \times 2$ rectangle. We see that

$$\underline{\lambda}(a_{\gamma(x)-1,x}) = (2^{b_1}, 1^{b_2}).$$

Continuing by induction, we see that

$$\underline{\lambda}(a_{1,x}) = (\gamma(x)^{b_1(x)}, \dots, 1^{b_{\gamma(x)}(x)}) = \underline{\mu}(x).$$

$$N(\widehat{\beta}, \widehat{\alpha}) = \prod_{a \in \widehat{Q}_1} \left(\sum_{\lambda} [\lambda]_{ta} [\overline{\lambda}]_{ha} \right) = \left(\prod_{a \in Q_1} \left(\sum_{\lambda} [\lambda]_{ta} [\overline{\lambda}]_{ha} \right) \right) \left(\prod_{a \in \widehat{Q}_1 \setminus Q_1} \left(\sum_{\lambda} [\lambda]_{ta} [\overline{\lambda}]_{ha} \right) \right).$$

Using our calculations for $\underline{\lambda}(a_{i,x})$ above, we see that this is equal to

$$\left(\prod_{a \in Q_1} \left(\sum_{\lambda} [\lambda]_{ta} [\overline{\lambda}]_{ha} \right) \right) \prod_{x \in Q_0} [\underline{\mu}(x)]_x.$$

From (4) and (14) it follows that

$$N(\widehat{\beta}, \widehat{\alpha}) = \left(\sum_{\underline{\lambda}\,:\,Q_0 \to \mathcal{P}} N(\beta, \alpha, \underline{\lambda}) \prod_{x \in Q_0} [\overline{\underline{\lambda}}(x)]_x \right) \prod_{x \in Q_0} [\underline{\mu}(x)]_x = N(\beta, \alpha, \underline{\mu}). \qquad \square$$

LEMMA 11.

$$M(\beta, \alpha, \underline{\mu}) = M(\widehat{\beta}, \widehat{\alpha}).$$

Proof. This is similar to the proof of the previous lemma. From (13), it follows that

$$M(\widehat{\beta}, \widehat{\alpha}) = \sum_{\underline{\lambda}\,:\,\widehat{Q}_1 \to \mathcal{P}} \prod_{x \in \widehat{Q}_0} \operatorname{mult} \left(\bigwedge^{(\gamma(x)^{\beta(x)})} W(x); \left(\bigotimes_{\substack{a \in \widehat{Q}_1 \\ ta = x}} \bigwedge^{\underline{\lambda}(a)} W(x) \right) \otimes \left(\bigotimes_{\substack{a \in \widehat{Q}_1 \\ ha = x}} \bigwedge^{\overline{\underline{\lambda}}(a)} W(x) \right) \right).$$

To get a non-zero summand, we get the same conditions for $\underline{\lambda}(a_{i,x})$ as in the previous lemma. We obtain

$$
\begin{aligned}
&M(\widehat{\beta},\widehat{\alpha})\\
&\quad=\sum_{\underline{\lambda}\,:\,Q_1\to\mathcal{P}}\prod_{x\in Q_0}\operatorname{mult}\left(\det\nolimits_x^{\beta(x)};\left(\bigotimes_{\substack{a\in Q_1\\ ta=x}}\bigwedge^{\underline{\lambda}(a)}W(x)\right)\otimes\left(\bigotimes_{\substack{a\in Q_1\\ ha=x}}\bigwedge^{\overline{\lambda}(a)}W(x)\right)\otimes\bigwedge^{\underline{\mu}(x)}W(x)\right)\\
&\quad=\sum_{\underline{\lambda}\,:\,Q_1\to\mathcal{P}}\prod_{x\in Q_0}\operatorname{mult}\left(\det\nolimits_x^{\beta(x)};\left(\bigotimes_{\substack{a\in Q_1\\ ta=x}}\bigwedge^{\underline{\lambda}(a)}W(x)\right)\otimes\left(\bigotimes_{\substack{a\in Q_1\\ ha=x}}\bigwedge^{\underline{\lambda}(a)}W^\star(x)\otimes\det\nolimits_x^{\beta(ta)}\right)\right.\\
&\qquad\left.\otimes\bigwedge^{\underline{\mu}(x)}W(x)\right)\\
&\quad=\operatorname{mult}\left(\det{}^\sigma;\bigoplus_{\underline{\lambda}\,:\,Q_1\to\mathcal{P}}\bigotimes_{a\in Q_1}\left(\bigwedge^{\underline{\lambda}(a)}W(ta)\otimes\bigwedge^{\underline{\lambda}(a)}W^\star(ha)\right)\otimes\bigotimes_{x\in Q_0}\bigwedge^{\underline{\mu}(x)}W(x)\right)\\
&\quad=\operatorname{mult}\left(\det{}^\sigma;\operatorname{Rep}(Q,\gamma)\otimes\bigotimes_{x\in Q_0}\bigwedge^{\underline{\mu}(x)}W(x)\right)=M(\beta,\alpha,\underline{\mu}).\qquad\square
\end{aligned}
$$

7. *Applications*

Theorem 1 also allows us to exhibit an explicit basis of the weight space $\mathrm{SI}(Q,\gamma)_{\langle\beta,\cdot\rangle}$.

COROLLARY 12. *Let Q,α,β and γ be as in Theorem 1. Assume that the general representation V of dimension α has k subrepresentations of dimension β. There exists a non-empty Zariski open set U in $\operatorname{Rep}_K(Q,\alpha)$ such that for $V\in U$, the semi-invariants $c^{V_1},\dots,c^{V_k}$ form a basis in $\mathrm{SI}(Q,\gamma)_{\langle\beta,\cdot\rangle}$, where $V_1,V_2,\dots,V_k$ are the subrepresentations of V of dimension β.*

Proof. Let us choose $V\in\operatorname{Rep}_K(Q,\alpha)$ such that $q^{-1}(V)$ consists of k points. Let $V_1,\dots,V_k$ be the corresponding subrepresentations of V of dimension β. It is enough to prove that $c^{V_1},\dots,c^{V_k}$ are linearly independent in $\mathrm{SI}(Q,\gamma)_{\langle\beta,\cdot\rangle}$. Let us consider the exact sequences

$$0\to V_i\to V\to W_i\to 0$$

for $i=1,\dots,k$. It is clear that for $i\neq j$, we have $\operatorname{Hom}_Q(V_i,W_j)\neq 0$. Therefore, $c^{V_i}(W_j)=0$ for $i\neq j$. It is enough to show that $\operatorname{Hom}_Q(V_i,W_i)=0$ for $i=1,\dots,k$. In [7], it was proved that $\operatorname{Hom}_Q(V_i,W_i)$ is the tangent space to the fiber of the map $q:Z(Q,\beta,\alpha)\to\operatorname{Rep}_K(Q,\alpha)$ at the point (V,V_i). The map q is dominant and generically it is $k:1$. Moreover, it is shown in [1] that the differential Dq is generically surjective. Therefore, it is generically an isomorphism because $Z(Q,\beta,\alpha)$ is smooth. So for V in some non-empty Zariski open set, we have $\operatorname{Hom}_Q(V_i,W_i)=0$ for $i=1,\dots,k$. $\square$

PROPOSITION 13. *Let Q,α,β and γ be as in Theorem 1. Let $U_\beta\subseteq\operatorname{Rep}(Q,\beta)$ and $U_\gamma\subseteq\operatorname{Rep}(Q,\gamma)$ be non-empty Zariski open subsets that are stable under $\mathrm{GL}(\beta)$ and $\mathrm{GL}(\gamma)$, respectively. Then there exists a nonempty Zariski open subset U of $\operatorname{Rep}(Q,\alpha)$ such that for all $V\in U$ we have every β-dimensional subrepresentation of V lies in U_β and every γ-dimensional factor representation of V lies in U_γ.*

Proof. Let

$$D\subset Z(Q,\beta,\alpha)\subset\operatorname{Rep}(Q,\alpha)\times\operatorname{Grass}(\beta,\alpha)$$

be the subset of pairs (V, W) such that W does not lie in U_β or V/W does not lie in U_γ. We claim that D is a Zariski closed strict subset of $Z(Q, \beta, \alpha)$. For every $z \in \mathrm{Grass}(\beta, \alpha)$, we can choose local sections $e_1, e_2, \ldots, e_{\alpha(x)}$ and an open neighborhood X of z such that $e_1(W), \ldots, e_{\beta(x)}(W)$ are a basis of W and $e_1(W), \ldots, e_{\alpha(x)}(W)$ are a basis of $K^{\alpha(x)}$ for all $W \in X$. With respect to this basis, V has the form

$$V(a) = \begin{pmatrix} V'(a) & \star \\ 0 & V''(a) \end{pmatrix},$$

where $V' \in \mathrm{Rep}(Q, \beta)$, $V'' \in \mathrm{Rep}(Q, \gamma)$ and $\star$ is an arbitrary matrix.

We define

$$r_X : p^{-1}(X) \to \mathrm{Rep}(Q, \beta) \times \mathrm{Rep}(Q, \gamma) \times X.$$

By

$$(V, W) \mapsto (V', V'', W),$$

it is clear that

$$D \cap p^{-1}(X) = r_X^{-1}((D_\beta \times \mathrm{Rep}(Q, \gamma) \times X) \cup (\mathrm{Rep}(Q, \beta) \times D_\gamma \times X)),$$

where D_β and D_γ are the complements of U_β and U_γ, respectively. Therefore, $D \cap p^{-1}(X)$ is a Zariski closed proper subset of $p^{-1}(X)$. Since such $p^{-1}(X)$ cover $Z(Q, \beta, \alpha)$, we conclude that D is a Zariski closed proper subset of $Z(Q, \beta, \alpha)$. For some $U' \subset \mathrm{Rep}(Q, \alpha)$ open nonempty, $q^{-1}(V)$ is finite for all $V \in U'$. Take $U = U' \setminus q(D)$. The map q is proper, so $q(D)$ is closed and U is therefore open. We also claim that U is nonempty. Indeed, the restriction $q : q^{-1}(U') \to U'$ is quasi-finite, so $q(q^{-1}(U') \cap D) \subseteq U' \cap q(D) \neq U'$ because the dimension of $q^{-1}(U') \cap D$ is strictly smaller than $\dim q^{-1}(U') = \dim U'$. If $V \in U$, then $q^{-1}(V)$ is finite and for every $(V, W) \in q^{-1}(V)$ we have W and V/W lie in U_β and U_γ, respectively. □

The proposition can be roughly reformulated as follows. If V is a general representation of dimension $\alpha = \beta + \gamma$ with $\langle \beta, \gamma \rangle = 0$, then all subrepresentations of dimension β and all factor representation of dimension γ are in general position as well.

COROLLARY 14. *Suppose that β, γ and δ are the dimension vectors such that $\langle \beta, \gamma \rangle = \langle \beta, \delta \rangle = \langle \gamma, \delta \rangle = 0$. Then we have the following equality:*

$$N(\beta, \beta + \gamma) N(\beta + \gamma, \beta + \gamma + \delta) = N(\beta, \beta + \gamma + \delta) N(\gamma, \gamma + \delta).$$

Proof. With the previous proposition, this is now a simple counting argument. Choose $V \in \mathrm{Rep}(Q, \beta + \gamma + \delta)$ is in general position. We count the number of pairs (V_1, V_2) such that V_1 is a β-dimensional subrepresentation of V_2, and V_2 is a $(\beta + \gamma)$-dimensional subrepresentation of V. On the one hand, V has $N(\beta + \gamma, \beta + \gamma + \delta)$ $(\beta + \gamma)$-dimensional subrepresentations V_2 and each such subrepresentation (since it is again in general position) has exactly $N(\beta, \beta + \gamma)$ subrepresentations V_1 of dimension β.

On the other hand, V has $N(\beta, \beta + \gamma + \delta)$ β-dimensional subrepresentations V_1. For each V_1, V/V_1 is again in general position and V/V_1 has exactly $N(\gamma, \gamma + \delta)$ subrepresentations of dimension γ. Note also that there is a one-to-one correspondence between γ-dimensional subrepresentations of V/V_1 and $(\beta + \gamma)$-dimensional subrepresentations V_2 of V containing V_1. Comparison of the two computations completes the proof. □

EXAMPLE 15. Let $Q = \theta(m)$ be the quiver with two vertices x and y and m arrows $a_1, \ldots, a_m$, with $ta_i = x, ha_i = y$ for $i = 1, \ldots, m$. Assume that $m = 2r$ is even. Let α be the dimension vector $\alpha(x) = \alpha(y) = r + 1$. Consider the subdimension vector β with $\beta(x) = 1$, $\beta(y) = r$. Then $\langle \beta, \alpha - \beta \rangle = 0$. The Littlewood–Richardson calculation shows that $N(\beta, \alpha) = \binom{2r}{r}$.

In particular, for $r = 2$, we have $Q = \theta(4)$ is the quiver with two vertices x and y and four arrows a, b, c and d from x to y. Consider the dimension vector α with $\alpha(x) = \alpha(y) = 3$. Consider the map

$$C : \mathrm{Rep}(Q, \alpha) \to \mathbb{A}^{20},$$

where we identify $\mathbb{A}^{20}$ with the space of quaternary cubics. The map C sends a representation V to the point $\det(X_0V(a) + X_1V(b) + X_2V(c) + X_3V(d))$.

Consider the subrepresentations of dimension β, where $\beta(x) = 1$ and $\beta(y) = 2$. Our calculation tells us that a generic representation of dimension α has six subrepresentations of dimension β. They correspond to six lines on the cubic surface defined by $C(V)$. These lines represent the cubic surface as a projective plane with six points blown up. Given a subrepresentation W of dimension β, the corresponding line is constructed as follows. Choose bases of $V(x)$ and $V(y)$ such that $W(x)$ is spanned by the first basis vector and $W(y)$ is spanned by the first two basis vectors. Then the cubic surface is defined by

$$\det \begin{pmatrix} l_{1,1} & l_{1,2} & l_{1,3} \\ l_{2,1} & l_{2,2} & l_{2,3} \\ 0 & l_{3,2} & l_{3,3} \end{pmatrix} = 0,$$

where $l_{i,j}$ are the linear functions. Now $l_{1,1} = l_{2,1} = 0$ defines a line on the surface.

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