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CENTRALIZERS IN DOMAINS OF GELFAND–KIRILLOV DIMENSION 2

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Abstract

Given an affine domain of Gelfand–Kirillov dimension 2 over an algebraically closed field, it is shown that the centralizer of any non-scalar element of this domain is a commutative domain of Gelfand–Kirillov dimension 1 whenever the domain is not polynomial identity. It is shown that the maximal subfields of the quotient division ring of a finitely graded Goldie algebra of Gelfand–Kirillov dimension 2 over a field F all have transcendence degree 1 over F. Finally, centralizers of elements in a finitely graded Goldie domain of Gelfand–Kirillov dimension 2 over an algebraically closed field are considered. In this case, it is shown that the centralizer of a non-scalar element is an affine commutative domain of Gelfand–Kirillov dimension 1.

1. *Introduction*

In this paper we consider domains of Gelfand–Kirillov dimension 2. This dimension was first considered by Gelfand and Kirillov in 1966 [**6**]. It is defined as follows. Given a field F and an affine F-algebra A, we define the *Gelfand–Kirillov dimension* of A to be

$$\mathrm{GKdim}(A) := \limsup_{n\to\infty} \log(\dim V^n)/\log n,$$

where V is a finite-dimensional subspace of A that generates A as an F-algebra. We note that this definition is independent of choice of V. In the case that A is not affine, we define the Gelfand–Kirillov dimension of A to be the supremum of the Gelfand–Kirillov dimensions of all affine subalgebras of A. Basic facts about Gelfand–Kirillov dimension can be found in [**6**].

An affine domain of Gelfand–Kirillov dimension 0 is a division ring that is finite-dimensional over its center. An affine domain of Gelfand–Kirillov dimension 1 over a field F is a finite module over its center, and hence polynomial identity (see [**9**]); if, in addition, F is algebraically closed, a routine application of Tsen's theorem shows that this domain is in fact commutative. Domains of Gelfand–Kirillov dimension 2 are not well understood. The second author conjectures that such domains must either be primitive or polynomial identity. In the case that such a domain is a finitely generated ℕ-graded algebra with the property that the homogenous elements of any given degree form a finite-dimensional vector space, it has been shown by Artin and Stafford [**2**] that this conjecture holds. We consider the centralizers of elements of an affine domain of Gelfand–Kirillov dimension 2. Centralizers of the first Weyl algebra – a domain of Gelfand–Kirillov dimension 2 – were studied as long ago as 1922 by Burchnall and Chaundy [**4**], and also by Amitsur [**1**]. Using the work of Artin and Stafford, we are also able to obtain results about the centralizers of

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elements of the quotient division algebra of a finitely graded domain of Gelfand–Kirillov dimension 2. We state our main results.

THEOREM 1.1. *Let A be a non-polynomial identity affine domain of Gelfand–Kirillov dimension 2 over an algebraically closed field F. Then the centralizer of a non-scalar element of A is a commutative domain of Gelfand–Kirillov dimension 1.*

In the case where we are dealing with graded rings, we obtain somewhat stronger results. We first give the following definition.

DEFINITION 1.2. Let

$$A = \bigoplus_{n=0}^{\infty} A_n$$

be a finitely generated $\mathbb{N}$-graded F-algebra. We say that A is *finitely graded* if $\dim_F(A_n) < \infty$ for all $n \geqslant 0$.

THEOREM 1.3. *Let A be a finitely graded non-polynomial identity domain of Gelfand–Kirillov dimension 2 over a field F. Then any subfield of $Q(A)$ has transcendence degree at most 1 over F.*

THEOREM 1.4. *Let A be a finitely graded non-polynomial identity domain of Gelfand–Kirillov dimension 2 over an algebraically closed field F. Then the centralizer of a non-scalar element $a \in Q(A)$ is a finitely generated field extension of F of transcendence degree 1.*

2. *Algebras of Gelfand–Kirillov dimension 2*

LEMMA 2.1. *Let A be a non-polynomial identity affine domain of Gelfand–Kirillov dimension 2 over an algebraically closed field F, and let B be a polynomial identity subalgebra of A. Then B has Gelfand–Kirillov dimension at most 1.*

Proof. Suppose that B has Gelfand–Kirillov dimension greater than 1. Then $\text{GKdim}(B) > \text{GKdim}(A) - 1$. By a theorem of Borho and Kraft (see [**3**]), $Q(A)$ is a finite-dimensional vector space over $Q(B)$. In particular, $Q(A)$ is polynomial identity, since $Q(B)$ is polynomial identity, a contradiction. It follows that B must have Gelfand–Kirillov dimension 1. □

THEOREM 2.2. *Let A be a non-polynomial identity affine domain of Gelfand–Kirillov dimension 2 over an algebraically closed field F. Then the centralizer of a non-scalar element of A is a commutative domain of Gelfand–Kirillov dimension 1.*

Proof. Let a be a non-scalar element of A. Let $C(a)$ denote the *centralizer* of a in A, and let B be an affine subalgebra of $C(a)$ containing a. Notice that the Gelfand–Kirillov dimension of B is at most 2, as it is a subalgebra of A. On the other hand, the center of B has Gelfand–Kirillov dimension at least 1, since it contains $F[a]$. By [**10**, Corollary 2], B is polynomial identity. By Lemma 2.1, B has Gelfand–Kirillov dimension 1. By [**9**], B is commutative. Since B is an arbitrary affine subalgebra of $C(a)$, we see that $C(a)$ is a commutative domain of Gelfand–Kirillov dimension 1. □

3. *Graded algebras of Gelfand–Kirillov dimension* 2

In this section we prove our main results for graded domains of Gelfand–Kirillov dimension 2. Throughout this section we use arguments involving Krull dimension.

NOTATION 3.1. Given an algebra A and a right A-module M, we denote the Krull dimension of M by $\mathcal{K}(M)$.

For basic facts about Krull dimension we refer the reader to [**7**].

LEMMA 3.2. *Let A be a prime Noetherian polynomial identity ring that is not primitive. Suppose that σ is an automorphism of A such that $A[x, x^{-1}; \sigma]$ is simple. Then*

$$\mathcal{K}(A) = \mathcal{K}(A[x, x^{-1}; \sigma]).$$

Proof. The proof is similar to that of [**7**, Corollary 6.6.7]. We show that there are no nonzero right $A[x, x^{-1}; \sigma]$-modules of finite length over A. Suppose that M is a nonzero right $A[x, x^{-1}; \sigma]$-module of finite length over A. Since A is not primitive, $\mathrm{ann}_A(M)$ must contain a finite product of primitive ideals of A. Thus $A/\mathrm{ann}_A(M)$ is a subdirect product of semisimple Artinian rings. It follows that $\mathcal{K}(A/\mathrm{ann}_A(M)) = 0$. Notice that $\mathrm{ann}_A(M)$ is stable under σ, and hence generates a proper nonzero two-sided ideal of $A[x, x^{-1}; \sigma]$, contradicting the fact that $A[x, x^{-1}; \sigma]$ is simple. The result now follows from [**7**, Theorem 6.6.10]. □

We now prove a useful lemma about subfields of quotient division algebras.

LEMMA 3.3. *Let A be a finitely graded Goldie domain of Gelfand–Kirillov dimension* 2. *Then the subfields of $Q(A)$ are finitely generated.*

Proof. By [**2**, Theorem 0.1] we know that A has a graded quotient ring

$$Q_{\mathrm{gr}}(A) \cong D[x, x^{-1}; \sigma]$$

for some division ring D which is a finite module over its center and some automorphism σ of D. By [**8**, Theorem 3] the subfields of $Q(D[x, x^{-1}; \sigma])$ are finitely generated. The result follows. □

PROPOSITION 3.4. *Let A be a finitely graded non-polynomial identity domain of Gelfand–Kirillov dimension* 2 *over a field F. Then $Q_{\mathrm{gr}}(A)$ is a simple F-algebra and $Z(Q_{\mathrm{gr}}(A))$ is a finite extension of F consisting of homogeneous elements of degree* 0.

Proof. Note that since A is a finitely generated domain of Gelfand–Kirillov dimension 2, we have

$$\mathrm{GKdim}(A/I) \leqslant 1$$

for any nonzero ideal I of A. In particular, A/I is polynomial identity, by [**9**]. It follows that A/I satisfies a multilinear homogeneous polynomial identity, say $p(x_1,\dots,x_d)$. Now since p is multilinear and A is not polynomial identity, there exist homogeneous elements $a_1,\dots,a_d \in A$ such that $u := p(a_1,\dots,a_d) \neq 0$. Thus u is a nonzero homogeneous element of A which is in I. It follows that every nonzero ideal of I contains a homogeneous regular element u. Thus $Q_{\mathrm{gr}}(A)$ is simple. By [**2**, Theorem 0.1],

$$Q_{\mathrm{gr}}(A) \cong D[x,x^{-1};\sigma]$$

with D finite-dimensional over its center Z and Z a finitely generated field extension of transcendence degree 1 over F. Note that if $a \in Q_{\mathrm{gr}}(A)$ is central, then each of the homogeneous parts of a must also be central. Suppose that $b = \alpha x^i$ with $\alpha \in D$ and $i \neq 0$ is central. Then since D is a finite module over its center,

$$D[\alpha x^i,(\alpha x^i)^{-1}] = D[x^i,x^{-i};\sigma^i]$$

is polynomial identity. Since $D[x,x^{-1};\sigma]$ is a finite module over $D[x^i,x^{-i};\sigma^i]$, we conclude that $Q_{\mathrm{gr}}(A)$ is polynomial identity, a contradiction. Thus any homogeneous central element of $Q_{\mathrm{gr}}(A)$ must have degree 0. Consequently, $Z(Q_{\mathrm{gr}}(A)) \subseteq D$. Next, suppose that $\alpha \in D$ is in $Z(Q_{\mathrm{gr}}(A))$. Then $F(\alpha)[x,x^{-1}]$ is commutative. We claim that α is algebraic over F. To see this, suppose that this is not the case. Then $[Z:F(\alpha)] < \infty$, and hence $Q_{\mathrm{gr}}(A)$ is a finite module over $F(\alpha)[x,x^{-1}]$. Since $F(\alpha)[x,x^{-1}]$ is commutative, we conclude that $Q_{\mathrm{gr}}(A)$ is polynomial identity, which contradicts the fact that A is not polynomial identity. Thus α is algebraic over F. It follows that $Z(Q_{\mathrm{gr}}(A))$ is an algebraic extension of F. By Lemma 3.3, it is also a finitely generated extension. The result follows. □

THEOREM 3.5. *Let A be a finitely graded non-polynomial identity domain of Gelfand–Kirillov dimension 2 over a field F. Then any subfield of $Q(A)$ has transcendence degree at most 1 over F.*

Proof. By [**2**, Theorem 0.1], A has a graded quotient

$$Q_{\mathrm{gr}}(A) = D[x,x^{-1};\sigma],$$

where D is a division algebra which is finite-dimensional over its center Z, with Z a finitely generated extension of F of transcendence degree 1. Let K be a subfield of $Q_{\mathrm{gr}}(A)$ that is a purely transcendental extension of $F' := Z(Q_{\mathrm{gr}}(A)) \subseteq D$. Observe that $K \otimes_{F'} Q_{\mathrm{gr}}(A) \cong (D \otimes_{F'} K)[x,x^{-1};\sigma]$. Now $D \otimes_{F'} K$ is a prime Noetherian algebra. Since it is polynomial identity and not simple, it is not primitive. Note that

$$Q_{\mathrm{gr}}(A) \otimes_{F'} K \cong (D \otimes_{F'} K)[x,x^{-1};\sigma]$$

is simple, because by Proposition 3.4, $Q_{\mathrm{gr}}(A)$ is a central simple F'-algebra and K is a simple F'-algebra. By Lemma 3.2,

$$\mathcal{K}(Q_{\mathrm{gr}}(A) \otimes_{F'} K) = \mathcal{K}(D \otimes_{F'} K).$$

Since D is a finite Z-module, we have

$$\mathcal{K}(D \otimes_{F'} K) = \mathcal{K}(Z \otimes_{F'} K) = \min\{\mathrm{trdeg}_{F'}(Z), \mathrm{trdeg}_{F'}(K)\},$$

and so

$$\mathcal{K}(Q_{\mathrm{gr}}(A) \otimes_{F'} K) = \min\{\mathrm{trdeg}_{F'}(Z), \mathrm{trdeg}_{F'}(K)\}. \tag{3.1}$$

We also have

$$\mathrm{trdeg}_{F'}(K) = \mathcal{K}(K \otimes_{F'} K).$$

By [**7**, Corollary 6.5.3], we have $\mathcal{K}(K \otimes_{F'} K) \leqslant \mathcal{K}(Q(A) \otimes_{F'} K)$. Additionally, by [**7**, Lemma 6.5.3.ii], we have

$$\mathcal{K}(Q(A) \otimes_{F'} K) \leqslant \mathcal{K}(Q_{\mathrm{gr}}(A) \otimes_{F'} K) = \min(\mathrm{trdeg}_{F'}(K), \mathrm{trdeg}_{F'}(Z)).$$

Since Z has transcendence degree at most 1 over F', we see that $\mathrm{trdeg}_{F'}(K) \leqslant 1$. Since F' is a finite extension of F, we obtain the desired result. □

We are now ready to prove our main result for graded domains of Gelfand–Kirillov dimension 2.

THEOREM 3.6. *Let A be a finitely graded non-polynomial identity domain of Gelfand–Kirillov dimension* 2 *over an algebraically closed field F. Then the centralizer of a non-scalar element $a \in Q(A)$ is a finitely generated field extension of F of transcendence degree* 1.

Proof. Observe that $C(a)$ is a division algebra over F. Notice also that for any $b \in C(a)$, a and b commute and hence $F(a, b)$ is a field. By Theorem 3.5, $F(a, b)$ has trancendence degree at most 1 over F. Thus b is algebraic over $F(a)$, as $F(a)$ has transcendence degree 1 over F. It follows that $C(a)$ is a division algebra that is algebraic over $F(a)$. Let Z denote the center of $C(a)$. Then $Z \supseteq F(a)$, and hence Z has transcendence degree at least 1 over F. On the other hand, Z is a subfield of $Q(A)$, and hence has transcendence degree at most 1 over F. By Lemma 3.3, Z is a finitely generated field extension of F. Let K denote a maximal subfield of $C(a)$. Then K has transcendence degree 1 over F and is a finitely generated extension of F by the same lemma. It follows that K is a finitely generated algebraic extension of Z, and so $[K : Z] < \infty$. It follows that $C(a)$ is finite-dimensional over Z using the results from [**5**, p. 165]; in fact, $[C(a) : Z] = [K : Z]^2$. Since Z is a finitely generated extension of transcendence degree 1 of an algebraically closed field, Z is necessarily a C_1 field. By Tsen's theorem, $C(a)$ is commutative, and hence is a field of transcendence degree 1 over F. □

PROPOSITION 3.7. *Let B be a commutative graded domain of Gelfand–Kirillov dimension* 1 *over an algebraically closed field F. Then there exist positive integers $m_1, \ldots, m_\ell$ and there exists a homogeneous element $t \in Q_{\mathrm{gr}}(B)$ such that $B = F[t^{m_1}, \ldots, t^{m_\ell}]$.*

Proof. Let

$$\mathcal{S} = \{\deg(u) \mid u \in B,\ u \text{ homogeneous}\}.$$

There exists some positive integer N such that every element of $\mathcal{S}$ can be expressed as a nonnegative integer linear combination of elements of $\mathcal{S} \cap \{1, 2, \ldots, N\}$. We can choose homogeneous elements $u_1, \ldots, u_\ell \in B$ such that

$$\{\deg(u_1), \ldots, \deg(u_\ell)\} = \mathcal{S} \cap \{1, 2, \ldots, N\}.$$

Let $u \in B$ be homogeneous. Now $R := F[u_1, \ldots, u_\ell, u]$ is a finitely graded commutative domain of Gelfand–Kirillov dimension 1. Since F is algebraically closed, it follows that $Q_{\mathrm{gr}}(R) \cong F[t,t^{-1}]$ where t is homogeneous of degree $\gcd(\deg(u_1),\dots,\deg(u_\ell))$. Since u_i is homogeneous and in $F[t,t^{-1}]$, it follows that $u_i = \alpha_i t^{m_i}$ with $m_i = \deg(u_i)/\deg(t)$ and $\alpha_i \in F$. Similarly, $u = \alpha t^m$ with $m = \deg(u)/\deg(t)$. By construction, m is a nonnegative integer linear combination of $m_1,\dots,m_\ell$. It follows that $u \in F[u_1,\dots,u_\ell] = F[t^{m_1},\dots,t^{m_\ell}]$. Thus

$$B = F[t^{m_1},\dots,t^{m_\ell}]. \qquad \square$$

THEOREM 3.8. *Let A be a finitely graded Goldie F-algebra with the property that for any homogeneous element of positive degree, $x \in A$, there exist positive integers $m_1,\dots,m_\ell$ and a homogeneous element $t \in Q_{\mathrm{gr}}(A)$ of positive degree d such that*

$$C(x) = F[t^{m_1},\dots,t^{m_\ell}].$$

Then the centralizer of any element of A is an affine F-algebra.

Proof. Let $a \in A$. Write $a = a_0 + a_1 + \dots + a_n$, where a_i is homogeneous of degree i and $a_n \neq 0$. Without loss of generality, $n > 0$. Observe that if $b = b_0 + \dots + b_p \in C(a)$ with b_i homogeneous of degree i, then $b_p \in C(a_n)$. There exist a homogeneous element $t \in Q_{\mathrm{gr}}(A)$ of degree d and positive integers $m_1,\dots,m_\ell$ such that $C(a_n) = F[t^{m_1},\dots,t^{m_\ell}]$. It follows that if $b = b_0 + \dots + b_p \in C(a)$, then $b_p = \alpha t^m$, with $\alpha \in F$ and $m \in \mathbb{N}m_1 + \dots + \mathbb{N}m_\ell$. Let

$$\mathcal{S} = \{\deg(b) \mid b \in C(a),\ b \neq 0\}.$$

Let d' be the greatest common divisor of $\mathcal{S}$. Then there exists an integer N such that all integers larger than N can be expressed as a *nonnegative* integer linear combination of elements of $\mathcal{S}$. Choose elements $r_1,\dots,r_\ell \in C(a)$ such that

$$\{\deg(r_1),\dots,\deg(r_\ell)\} = \mathcal{S} \cap \{0,1,\dots,N\}.$$

Since the leading homogeneous part of r_i is in $C(a_n)$, we can multiply by appropriate scalars so that the leading homogeneous part of r_i is t^{m_i} for some positive integer m_i. We claim that $C(a) = F[r_1,\dots,r_\ell]$. To see this, suppose this is not the case. Choose $b = b_0 + \dots + b_p \in C(a) \setminus F[r_1,\dots,r_\ell]$ with p minimal. Again, we may assume that $b_p = t^{p/d}$. By assumption, $p/d = i_1 m_1 + \dots + i_\ell m_\ell$ for some nonnegative integers $i_1,\dots,i_\ell$. Observe that both b and $b' := r_1^{i_1}\dots r_\ell^{i_\ell}$ have degree p and both have the same homogeneous part of degree p, namely $t^{p/d}$. Thus $b - b' \in C(a)$ has degree at most $p-1$. By the minimality of $\deg(b)$, we see that $b - b' \in F[r_1,\dots,r_\ell]$, contradicting the fact that $b \notin F[r_1,\dots,r_\ell]$. $\square$

COROLLARY 3.9. *Let A be a finitely graded non-polynomial identity Goldie domain of Gelfand–Kirillov dimension 2 over an algebraically closed field F. Then the centralizer of a non-scalar element is an affine commutative domain of Gelfand–Kirillov dimension 1.*

Proof. By Theorem 2.2, the centralizer of a non-scalar element $b \in A$ is a commutative domain of Gelfand–Kirillov dimension 1. Hence by Proposition 3.7 and Theorem 3.8, the centralizer of a is affine. $\square$

We make the conjecture that the same result holds in the ungraded case.

CONJECTURE 3.10. Let A be an affine Noetherian non-polynomial identity domain of Gelfand–Kirillov dimension 2 over an algebraically closed field. Then the centralizer of a non-scalar element is an affine domain.

COROLLARY 3.11. *Let A be an affine domain of Gelfand–Kirillov dimension* 2 *with a non-polynomial identity domain for an associated graded ring. Then the centralizer of a non-scalar element is an affine commutative domain of Gelfand–Kirillov dimension* 1.

Proof. The same argument used in Theorem 3.8 and Corollary 3.9 gives the result. □

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