DEFICIENCIES OF LATTICE SUBGROUPS OF LIE GROUPS

JOHN LOTT

Abstract

Let Γ be a lattice in a connected Lie group. We show that, besides a few exceptional cases, the deficiency of Γ is nonpositive.

1. *Introduction*

If Γ is a finitely presented group, then its deficiency $\mathrm{def}(\Gamma)$ is the maximum, over all finite presentations of Γ, of the number of generators minus the number of relations. If G is a connected Lie group, then a lattice in G is a discrete subgroup Γ such that G/Γ has finite volume. It is uniform if G/Γ is compact. Lubotzky proved the following result [**7**, Proposition 6.2].

Theorem 1 (Lubotzky). *Let Γ be a lattice in a simple Lie group G.*
(a) *If* $\mathbf{R}-\mathrm{rank}(G) \geqslant 2$ *or* $G = \mathrm{Sp}(n,1)$ *or* $G = F_4$, *then* $\mathrm{def}(\Gamma) \leqslant 0$.
(b) *If* $G = \mathrm{SO}(n,1)$ (*for* $n \geqslant 3$) *or* $G = \mathrm{SU}(n,1)$ (*for* $n \geqslant 2$), *then* $\mathrm{def}(\Gamma) \leqslant 1$.

We give an improvement of Lubotzky's result.

Theorem 2. *Let G be a connected Lie group. Let Γ be a lattice in G. If* $\mathrm{def}(\Gamma) > 0$, *then*
(1) Γ *has a finite normal subgroup F such that Γ/F is a lattice in* $\mathrm{PSL}_2(\mathbf{R})$,
or
(2) $\mathrm{def}(\Gamma) = 1$ *and either*
 (A) Γ *is isomorphic to a torsion-free nonuniform lattice in* $\mathbf{R}\times\mathrm{PSL}_2(\mathbf{R})$ *or* $\mathrm{PSL}_2(\mathbf{C})$, *or*
 (B) Γ *is* $\mathbf{Z}$, $\mathbf{Z}^2$ *or the fundamental group of a Klein bottle.*

The examples in case (2) do have deficiency one [**5**]. A free group on r generators, $r > 1$, has deficiency r and gives an example of case (1).
In some cases, we have sharper bounds on $\mathrm{def}(\Gamma)$.

Theorem 3. (1) *If Γ is a lattice in* $\mathrm{SO}(4,1)$, *then*

$$\mathrm{def}(\Gamma) \leqslant 1 - \frac{3}{4\pi^2}\,\mathrm{vol}(H^4/\Gamma). \tag{1.1}$$

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(2) *If* Γ *is a lattice in* $\mathrm{SU}(2,1)$, *then*

$$\operatorname{def}(\Gamma) \leqslant 1 - \frac{6}{\pi^2} \operatorname{vol}(\mathbf{C}H^2/\Gamma). \tag{1.2}$$

(*We normalize* $\mathbf{C}H^2$ *to have sectional curvatures between* -4 *and* -1.)

(3) *If* Γ *is a lattice in* $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$, *then*

$$\operatorname{def}(\Gamma) \leqslant 1 - \frac{1}{4\pi^2} \operatorname{vol}((H^2 \times H^2)/\Gamma). \tag{1.3}$$

2. *Proofs*

To prove Theorems 2 and 3, we use methods of L^2-homology. For a review of L^2-homology, see [**8**]. Let G and Γ be as in the hypotheses of Theorem 2. Let $b_i^{(2)}(\Gamma) \in \mathbf{R}$ denote the ith L^2-Betti number of Γ. Let Rad be the radical of G, let L be a Levi subgroup of G, and let K be the maximal compact connected normal subgroup of L. Put $G_1 = \mathrm{Rad} \cdot K$ and $G_2 = G/G_1$, a connected semisimple Lie group whose Lie algebra has no compact factors. Let $\beta : G \to G_2$ be the projection map. Put $\Gamma_1 = \Gamma \cap G_1$ and $\Gamma_2 = \beta(\Gamma)$. Then there is an exact sequence

$$1 \longrightarrow \Gamma_1 \longrightarrow \Gamma \xrightarrow{\beta} \Gamma_2 \longrightarrow 1, \tag{2.1}$$

where Γ_1 is a lattice in G_1 and Γ_2 is a lattice in G_2 [**1**].

LEMMA 1. *If* $b_1^{(2)}(\Gamma) \neq 0$, *then* Γ *has a finite normal subgroup* F *such that* Γ/F *is a lattice in* $\mathrm{PSL}_2(\mathbf{R})$.

Proof. There are the following possibilities.

(A) Γ_1 is infinite. Then Γ has an infinite normal amenable subgroup. By a result of Cheeger and Gromov, the L^2-Betti numbers of Γ vanish [**8**, Theorem 10.12].

(B) Γ_1 is finite and Γ_2 is finite (that is, $\Gamma_2 = \{e\}$). Then Γ is finite and $b_1^{(2)}(\Gamma) = 0$.

(C) Γ_1 is finite and Γ_2 is infinite. By the Leray–Serre spectral sequence for L^2-homology, $b_1^{(2)}(\Gamma) = b_1^{(2)}(\Gamma_2)/|\Gamma_1|$. Suppose that $b_1^{(2)}(\Gamma_2) \neq 0$. If G_2 had an infinite centre, then Γ_2, being a lattice, would have to have an infinite centre. This would imply, by [**8**, Theorem 10.12], that $b_1^{(2)}(\Gamma_2)$ vanishes, so G_2 must have a finite centre $Z(G_2)$. Put $G_3 = G_2/Z(G_2)$, let $\gamma : G_2 \to G_3$ be the projection, and put $\Gamma_3 = \gamma(\Gamma_2)$, a lattice in G_3. Then there is the exact sequence

$$1 \longrightarrow \Gamma_2 \cap Z(G_2) \longrightarrow \Gamma_2 \xrightarrow{\gamma} \Gamma_3 \longrightarrow 1, \tag{2.2}$$

and so $b_1^{(2)}(\Gamma_2) = b_1^{(2)}(\Gamma_3)/|\Gamma_2 \cap Z(G_2)|$. Let K_3 be a maximal compact subgroup of G_3, and let $\mathscr{F}$ be a fundamental domain for the Γ_3-action on G_3/K_3. Let $\Pi(x,y)$ be the Schwartz kernel for the projection operator onto the L^2-harmonic 1-forms on G_3/K_3. By [**4**, Theorem 1.1],

$$b_1^{(2)}(\Gamma_3) = \int_{\mathscr{F}} \operatorname{tr}(\Pi(x,x)) d\mathrm{vol}(x).$$

Hence G_3/K_3 has nonzero L^2-harmonic 1-forms. By the Künneth formula for L^2-cohomology and [**2**, Section II.5], the only possibility is $G_3 = \mathrm{PSL}_2(\mathbf{R})$. Then there is the exact sequence

$$1 \longrightarrow \Gamma \cap \operatorname{Ker}(\gamma \circ \beta) \longrightarrow \Gamma \xrightarrow{\gamma\circ\beta} \Gamma_3 \longrightarrow 1, \tag{2.3}$$

with $\Gamma \cap \operatorname{Ker}(\gamma \circ \beta)$ finite.

Let $\operatorname{geom}\dim\Gamma$ be the minimal dimension of a $K(\Gamma,1)$-complex [**3**, p. 185]. We shall need the following result of Hillman [**6**, Theorem 2]. For completeness, we give the short proof.

LEMMA 2 (Hillman). *If Γ is a finitely-presented group, then* $\operatorname{def}(\Gamma) \leqslant 1 + b_1^{(2)}(\Gamma)$. *Equality implies that there is a finite $K(\Gamma,1)$-complex X with* $\dim(X) \leqslant 2$.

Proof. If Γ is finite, then $\operatorname{def}(\Gamma) \leqslant 0$, so we may assume that Γ is infinite. Given a presentation of Γ with g generators and r relations, let X be the corresponding 2-complex. As X is two-dimensional, its second L^2-homology group is the same as the space of square-integrable real cellular 2-cycles on the universal cover $\widetilde{X}$. This contains the ordinary integer cellular 2-cycles as a subgroup.

We have

$$\chi(X) = 1 - g + r = b_0^{(2)}(X) - b_1^{(2)}(X) + b_2^{(2)}(X) = -b_1^{(2)}(\Gamma) + b_2^{(2)}(X). \tag{2.4}$$

Hence

$$g - r = 1 + b_1^{(2)}(\Gamma) - b_2^{(2)}(X) \leqslant 1 + b_1^{(2)}(\Gamma). \tag{2.5}$$

If $g - r = 1 + b_1^{(2)}(\Gamma)$, then $b_2^{(2)}(X) = 0$. Hence $\mathrm{H}_2(\widetilde{X};\mathbf{Z}) = 0$. From the Hurewicz theorem, $\widetilde{X}$ is contractible.

Proof of Theorem 2. Suppose that $\operatorname{def}(\Gamma) > 0$. Then, first, $|\Gamma| = \infty$. Suppose that Γ does not have a finite normal subgroup F such that G/F is a lattice in $\mathrm{PSL}_2(\mathbf{R})$. By Lemma 1, $b_1^{(2)}(\Gamma) = 0$. Then Lemma 2 implies that $\operatorname{def}(\Gamma) = 1$ and $\operatorname{geom}\dim\Gamma \leqslant 2$. In particular, Γ is torsion-free.

As Γ_1 is a lattice in $K \cdot \mathrm{Rad}$, it is a uniform lattice [**9**, Chapter III]. Furthermore, as Γ_1 is a subgroup of Γ, $\operatorname{geom}\dim\Gamma_1 \leqslant 2$, and so Γ_1 must be $\{e\}$, $\mathbf{Z}$, $\mathbf{Z}^2$ or the fundamental group of a Klein bottle. We go through the possibilities.

(i) $\Gamma_1 = \{e\}$. Then $\Gamma = \Gamma_2$ is a torsion-free lattice in the semisimple group G_2. Using a result of Borel and Serre [**3**, p. 218], the fact that $\operatorname{geom}\dim\Gamma \leqslant 2$ implies that the Lie algebra of G_2 is $\mathrm{sl}_2(\mathbf{R})$, $\mathrm{sl}_2(\mathbf{R}) \oplus \mathrm{sl}_2(\mathbf{R})$ or $\mathrm{sl}_2(\mathbf{C})$. One possibility is $G_2 = \widetilde{\mathrm{PSL}_2(\mathbf{R})}$. Using the embedding $\widetilde{\mathrm{PSL}_2(\mathbf{R})} \cong \mathbf{Z} \times_{\mathbf{Z}} \widetilde{\mathrm{PSL}_2(\mathbf{R})} \to \mathbf{R} \times_{\mathbf{Z}} \widetilde{\mathrm{PSL}_2(\mathbf{R})}$, in this case we can say that Γ is isomorphic to a lattice in $\mathbf{R} \times_{\mathbf{Z}} \widetilde{\mathrm{PSL}_2(\mathbf{R})}$. On the other hand, if G_2 is a finite covering of $\mathrm{PSL}_2(\mathbf{R})$, then $b_1^{(2)}(\Gamma) \neq 0$, contrary to assumption. If G_2 is an infinite covering of $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$, then the Leray–Serre spectral sequence implies that Γ_2 has cohomological dimension greater than two, contrary to assumption. If G_2 is a finite covering of $\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$, then Lemma 3 below will show that $\operatorname{def}(\Gamma) \leqslant 0$, contrary to assumption. If $G_2 = \mathrm{SL}_2(\mathbf{C})$, let $p : \mathrm{SL}_2(\mathbf{C}) \to \mathrm{PSL}_2(\mathbf{C})$ be the projection map. Then there is the exact sequence

$$1 \longrightarrow \Gamma \cap \operatorname{Ker}(p) \longrightarrow \Gamma \stackrel{p}{\longrightarrow} p(\Gamma) \longrightarrow 1. \tag{2.6}$$

As Γ is torsion-free, $\Gamma \cap \operatorname{Ker}(p) = \{e\}$, and so Γ is isomorphic to $p(\Gamma)$, a lattice in $\mathrm{PSL}_2(\mathbf{C})$. Thus in any case, Γ is isomorphic to a torsion-free lattice in $\mathbf{R} \times_{\mathbf{Z}} \widetilde{\mathrm{PSL}_2(\mathbf{R})}$ or $\mathrm{PSL}_2(\mathbf{C})$. If Γ is uniform, then $\operatorname{geom}\dim\Gamma = 3$. Thus Γ must be nonuniform. The torsion-free nonuniform lattices in $\mathbf{R} \times_{\mathbf{Z}} \widetilde{\mathrm{PSL}_2(\mathbf{R})}$ and $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$ are isomorphic, as they both correspond to the Seifert fibre spaces whose base is a hyperbolic orbifold with boundary [**10**]. We conclude that Γ is isomorphic to a torsion-free nonuniform lattice in $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$ or $\mathrm{PSL}_2(\mathbf{C})$.

(ii) $\Gamma_1 = \mathbf{Z}$. Let Γ_2' be a finite-index torsion-free subgroup of Γ_2 which acts trivially on $\mathbf{Z}$, and put $\Gamma' = \beta^{-1}(\Gamma_2')$, a finite-index subgroup of Γ. Then there is the exact sequence

$$1 \longrightarrow \Gamma_1 \longrightarrow \Gamma' \xrightarrow{\beta} \Gamma_2' \longrightarrow 1. \tag{2.7}$$

Let M be a Γ_2'-module, and let β^*M be the corresponding Γ'-module. If $\mathrm{H}^*(\Gamma_2'; M) \neq 0$, let k be the largest integer such that $\mathrm{H}^k(\Gamma_2'; M) \neq 0$. Then by the Leray–Serre spectral sequence, $\mathrm{H}^{k+1}(\Gamma'; \beta^*M) \neq 0$. As $\operatorname{geom\,dim} \Gamma' \leqslant 2$, we must have $k \leqslant 1$. Thus the cohomological dimension of Γ_2' is at most one, and the Stallings–Swan theorem implies that Γ_2' must be trivial or a free group [**3**, p. 185]. If $\Gamma_2' = \{e\}$, then $G_2 = \{e\}$ and $\Gamma = \mathbf{Z}$. If Γ_2' is a free group, then G_2 is a finite covering of $\mathrm{PSL}_2(\mathbf{R})$. Let $\sigma : G_2 \to \mathrm{PSL}_2(\mathbf{R})$ be the projection map, and put $L = (\sigma \circ \beta)(\Gamma)$. Then there is the exact sequence

$$1 \longrightarrow \Gamma \cap \operatorname{Ker}(\sigma \circ \beta) \longrightarrow \Gamma \xrightarrow{\sigma\circ\beta} L \longrightarrow 1, \tag{2.8}$$

where L is a lattice in $\mathrm{PSL}_2(\mathbf{R})$ and $\Gamma \cap \operatorname{Ker}(\sigma\circ\beta)$ is virtually cyclic. As $\Gamma\cap\operatorname{Ker}(\sigma\circ\beta)$ is torsion-free, it must equal $\mathbf{Z}$. It follows that Γ is isomorphic to a lattice in $\mathbf{R}\times\mathrm{PSL}_2(\mathbf{R})$ or $\mathbf{R} \times_{\mathbf{Z}} \widetilde{\mathrm{PSL}_2(\mathbf{R})}$. If Γ is uniform, then $\operatorname{geom\,dim} \Gamma = 3$. Thus Γ is nonuniform and is isomorphic to a lattice in $\mathbf{R} \times \mathrm{PSL}_2(\mathbf{R})$.

(iii) $\Gamma_1 = \mathbf{Z}^2$. Let Γ_2' be a finite-index torsion-free subgroup of Γ_2 which acts on $\mathbf{Z}^2$ with determinant 1, and put $\Gamma' = \beta^{-1}(\Gamma_2')$, a finite-index subgroup of Γ. Let M be a Γ_2'-module, and let β^*M be the corresponding Γ'-module. If $\mathrm{H}^*(\Gamma_2'; M) \neq 0$, let k be the largest integer such that $\mathrm{H}^k(\Gamma_2'; M) \neq 0$. Then by the Leray–Serre spectral sequence, $\mathrm{H}^{k+2}(\Gamma'; \beta^*M) \neq 0$. As $\operatorname{geom\,dim} \Gamma' \leqslant 2$, we must have $k = 0$. Thus the cohomological dimension of Γ_2' is zero, so $\Gamma_2' = \{e\}$ and $G_2 = \{e\}$. Then $\Gamma = \mathbf{Z}^2$.

(iv) Γ_1 is the fundamental group of a Klein bottle. Let $\mathbf{Z}^2$ be the unique maximal abelian subgroup of Γ_1. Any automorphism of Γ_1 acts as an automorphism of $\mathbf{Z}^2$. Thus we obtain a homomorphism $\phi : \operatorname{Aut}(\Gamma_1) \to \mathrm{GL}_2(\mathbf{Z})$. Let $\rho : \Gamma \to \operatorname{Aut}(\Gamma_1)$ be given by $(\rho(\gamma))(\gamma_1) = \gamma\gamma_1\gamma^{-1}$. Put $\widetilde{\Gamma} = \operatorname{Ker}(\det \circ \phi \circ \rho)$, an index-2 subgroup of Γ, and put $\widetilde{\Gamma_2} = \beta(\widetilde{\Gamma})$. Then there is an exact sequence

$$1 \longrightarrow \mathbf{Z}^2 \longrightarrow \widetilde{\Gamma} \xrightarrow{\beta} \widetilde{\Gamma_2} \longrightarrow 1. \tag{2.9}$$

As in case (iii), it follows that $G_2 = \{e\}$ and $\Gamma = \Gamma_1$ is the fundamental group of a Klein bottle.

This proves Theorem 2.

Proof of Theorem 3. Let X be as in the proof of Lemma 2. As the classifying map $X \to B\Gamma$ is 2-connected, $b_2^{(2)}(X) \geqslant b_2^{(2)}(\Gamma)$. Then from (2.5),

$$\operatorname{def}(\Gamma) \leqslant 1 + b_1^{(2)}(\Gamma) - b_2^{(2)}(\Gamma). \tag{2.10}$$

For the lattices in question, let G be the Lie group, let K now be a maximal compact subgroup of G, and put $M = \Gamma\backslash G/K$, an orbifold. As G/K has no L^2-harmonic 1-forms [**2**, Section II.5], it follows from [**4**, Theorem 1.1] that $b_1^{(2)}(\Gamma) = b_3^{(2)}(\Gamma) = 0$. As $|\Gamma| = \infty$, we have $b_0^{(2)}(\Gamma) = b_4^{(2)}(\Gamma) = 0$. If $\chi(\Gamma)$ is the rational-valued group Euler characteristic of Γ [**3**, p. 249], then

$$\chi(\Gamma) = b_0^{(2)}(\Gamma) - b_1^{(2)}(\Gamma) + b_2^{(2)}(\Gamma) - b_3^{(2)}(\Gamma) + b_4^{(2)}(\Gamma) = b_2^{(2)}(\Gamma). \tag{2.11}$$

From (2.10) and (2.11), we obtain

$$\operatorname{def}(\Gamma) \leqslant 1 - \chi(\Gamma). \tag{2.12}$$

Furthermore, letting $e(M,g) \in \Omega^4(M)$ denote the Euler density, it follows from [**4**, Theorem 1.1] that

$$\chi(\Gamma) = \int_M e(M,g). \tag{2.13}$$

Let G^d/K be the compact dual symmetric space to G/K. By the Hirzebruch proportionality principle,

$$\frac{\int_M e(M,g)}{\chi(G^d/K)} = \frac{\mathrm{vol}(M)}{\mathrm{vol}(G^d/K)}. \tag{2.14}$$

We have the following table.

G	G^d/K	$\chi(G^d/K)$	$\mathrm{vol}(G^d/K)$
$\mathrm{SO}(4,1)$	S^4	2	$8\pi^2/3$
$\mathrm{SU}(2,1)$	$\mathbf{C}P^2$	3	$\pi^2/2$
$\mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$	$S^2 \times S^2$	4	$16\pi^2$

This proves Theorem 3.

Lemma 3. *Let G be a connected Lie group with a surjective homomorphism $\rho : G \to \mathrm{PSL}_2(\mathbf{R}) \times \mathrm{PSL}_2(\mathbf{R})$ such that* $\mathrm{Ker}(\rho)$ *is central in G and finite. If Γ is a lattice in G, then* $\mathrm{def}(\Gamma) \leqslant 0$.

Proof. Equation (2.12) is still valid for Γ. We have $\chi(\Gamma) = \chi(\rho(\Gamma))/|\Gamma \cap \mathrm{Ker}(\rho)|$. Applying (2.13) to $\rho(\Gamma)$, the proof of Theorem 3 gives $\chi(\rho(\Gamma)) > 0$. Hence $\chi(\Gamma) > 0$ and $\mathrm{def}(\Gamma) \leqslant 0$.

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Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1109
USA