SIMPLE CLOSED GEODESICS IN HYPERBOLIC 3-MANIFOLDS

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1. *Introduction*

The question of which Riemannian manifolds admit simple closed geodesics is still a mystery. It is not known whether all closed Riemannian manifolds contain simple closed geodesics. For closed manifolds with nontrivial fundamental group, a simple closed geodesic can always be found by taking the shortest homotopically nontrivial closed geodesic. When the manifold is closed but simply connected, the question is open for dimensions three and above. In dimension two, it is known by the theorem of Lusternik and Schnirelmann [**6**] (see also [**3**] and [**4**]) that the 2-sphere equipped with any smooth Riemannian metric contains at least three distinct simple closed geodesics.

Non-compact manifolds do not necessarily contain closed geodesics, Euclidean space being an obvious example. Even if the manifold is not simply connected, it may not contain any simple closed geodesics, as with the hyperbolic thrice-punctured sphere. However, among the orientable, finite area, complete hyperbolic 2-manifolds, the thrice-punctured sphere is the only example that contains no simple closed geodesic. In this paper, we shall determine which orientable hyperbolic 3-manifolds do and do not contain simple closed geodesics. We shall prove that the Fuchsian group corresponding to the thrice-punctured sphere generates the only example of a complete non-elementary orientable hyperbolic 3-manifold that does not contain a simple closed geodesic. We do not assume that the manifold is geometrically finite, or even that it has finitely generated fundamental group. The simple closed geodesic which we produce arises from an interesting class of elements of the fundamental group. It is the shortest closed geodesic corresponding to a screw motion induced by the action of the fundamental group on hyperbolic 3-space.

In proving our result, we use geometric methods to obtain results about isometries of hyperbolic 3-space. These results apply to elliptic, as well as to parabolic and loxodromic isometries, and we state them in full generality.

A related question is whether a hyperbolic 3-manifold always contains a non-simple closed geodesic. In [**2**], Alan Reid and Ted Chinburg utilized arithmetic hyperbolic 3-manifold theory to construct examples of closed hyperbolic 3-manifolds in which every closed geodesic is simple.

We shall be working with orientable 3-manifolds, so all of the isometries discussed will be orientation preserving. A *screw motion* in hyperbolic space is an isometry that has two fixed points on S^2_∞, and is the composition of a nontrivial translation and a

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nontrivial rotation along the geodesic with those fixed points as endpoints. In the language of Kleinian groups, it is a loxodromic isometry that is not hyperbolic. A *non-screw motion* will be any isometry of hyperbolic space that is not a screw motion. Note that an isometry is a non-screw motion if and only if it preserves an oriented plane in H^3. Moreover, if an orientation preserving isometry is represented by a matrix in $\mathrm{SL}(2, C)$, then it is a non-screw motion if and only if its trace is real.

A natural place to look for a simple closed geodesic is to take the shortest essential closed curve in a manifold, if such a curve exists. If it has self-intersections, one can perform a cut and paste at a singular point to create shorter curves. The problem with this line of argument is that these shorter curves may represent parabolic elements of the fundamental group, and thus not be homotopic to a closed geodesic. The basic method of our argument is to show that the shortest screw motion, if it exists, does in fact give a simple closed geodesic. In geometrically infinite manifolds, a shortest screw motion may not exist, but an approximation argument extends the proof to this case.

2. *Non-screw motions of H^3*

In this section we examine the subgroups of the isometry group of H^3 which are generated by elements that preserve some oriented plane.

LEMMA 1. *Let α, β and $\alpha\beta$ be non-screw motions of H^3. If α and β do not share a fixed point in $H^3 \cup S^2_\infty$, then they preserve a common oriented plane, and the group $\langle \alpha, \beta \rangle$ consists of non-screw motions which preserve this plane.*

Proof. We shall give a geometric proof of this result. Every orientation preserving isometry of H^3 can be represented as the product of 180° rotations about two axes in hyperbolic space. Call such a rotation a *half-turn*. If the isometry is parabolic, then both half-turn axes pass through the parabolic fixed point. The second endpoint of one of the half-turn axes can be chosen arbitrarily. If the isometry is hyperbolic or elliptic, then both half-turn axes pass perpendicularly through the axis of the isometry. One of the half-turn axes can be chosen arbitrarily, subject to this property. If the isometry is elliptic, then the two half-turn axes intersect the axis of the given isometry in the same point. Let r_l denote the half-turn with axis l. Our hypothesis that α and β have no common fixed points implies that, in any combination of these cases, we can choose half-turns r_{α_1}, r_{α_2} for α and r_{β_1}, r_{β_2} for β such that $r_{\alpha_2} = r_{\beta_1}$. This choice determines the half-turns r_{α_1} and r_{β_2}. Then

$$\alpha\beta = r_{\alpha_1} r_{\alpha_2} r_{\beta_1} r_{\beta_2} = r_{\alpha_1} r_{\beta_2}.$$

Note that if a hyperbolic isometry α is expressed as a product of two half-turns r_{α_1}, r_{α_2} as above, then α is non-screw if and only if α_1 and α_2 lie in a common plane.

Suppose that all three of α, β and $\alpha\beta$ are either parabolic or elliptic. In order that $\alpha\beta$ be parabolic or elliptic, it must be that α_1 and β_2 have a point in common, possibly on the sphere at infinity. Thus the three axes α_1, α_2 and β_2 create a triangle in H^3, and the product of any two of the half-turns about these axes preserves the oriented plane containing that triangle.

Suppose next that exactly two of α, β and $\alpha\beta$ are parabolic or elliptic. Since the inverse of a parabolic or elliptic element is parabolic or elliptic, we can rename the elements so that α and β are parabolic or elliptic. Then α_1 and α_2 share a common point, as do $\beta_1(=\alpha_2)$ and β_2. Since $\alpha\beta$ is non-screw, α_1 and β_2 lie in a common plane. As two distinct points on α_2 also lie in this plane, so does α_2 itself. As before, the product of any two of the half-turns about α_1, α_2 or β_2 preserve this oriented plane.

In the case where exactly one of α, β and $\alpha\beta$ is parabolic or elliptic, we can assume that it is α. Then α_1 and α_2 share a common point, while $\beta_1(=\alpha_2)$ and β_2 lie in a common plane P. Moreover, α_1 and β_2 also lie in a common plane Q. If P and Q are distinct, then they intersect in a line passing through $\alpha_1 \cap \alpha_2$, and this line must equal β_2. Thus α_1, α_2 and β_2 have a common point, which must be fixed by both α and β, contradicting our hypothesis. If $P = Q$, then α_1, α_2 and β_2 all lie in this plane, and the product of any two of the half-turns about them will preserve this oriented plane as required.

The final case occurs when none of α, β and $\alpha\beta$ is parabolic or elliptic. In this case, each pair of axes from α_1, α_2 and β_2 are disjoint, even at infinity, but share a common plane. Let P denote the plane containing α_1 and α_2, let Q denote the plane containing α_2 and β_2, and let R denote the plane containing α_1 and β_2. Let λ denote the common perpendicular line segment between α_1 and α_2. Let Π denote the plane which contains λ and is orthogonal to P. Then Π is also orthogonal to each of Q and R. If $Q \neq R$, then it follows that Π is orthogonal to $Q \cap R = \beta_2$. The product of any two of the half-turns about α_1, α_2 and β_2 will preserve this oriented plane as required. If $Q = R$, then α_1, α_2 and β_2 all lie in this plane, and, as before, the product of any two of the half-turns about these axes will preserve this oriented plane. This completes the proof of Lemma 1.

LEMMA 2. *If Γ is a non-elementary Kleinian group, and if every element of Γ is a non-screw motion, then Γ is Fuchsian.*

Proof. A proof appears in [**7**, p. 108]. We give here a geometric proof. Since Γ is not elementary, there exist infinitely many distinct axes for hyperbolic (that is, non-screw loxodromic) elements. Choose two hyperbolic elements with distinct axes. Then Lemma 1 shows that the two axes lie in a plane P that is preserved by the subgroup that they generate. Suppose that there exists a limit point of Γ that is not contained in the circle that bounds P. There is a hyperbolic axis with an endpoint that is arbitrarily close to this point. Let γ be a hyperbolic element with axis A that has an endpoint off the circle. The set of all geodesics on P that are coplanar with A forms a foliation of the plane P. However, there exists a hyperbolic axis in P that is not one of the geodesics in this foliation, since we can take any two of the axes in the foliation and let the hyperbolic element of the first act a number of times on the second. The second axis will be carried close to one endpoint of the first, so it cannot be a geodesic of the foliation. It follows that the whole limit set is in the boundary of P, and Γ is Fuchsian.

3. *The main result*

In this section we prove the following result.

THEOREM 1. *Let M be an orientable hyperbolic 3-manifold. Then exactly one of the following three cases occurs.*

(1) *There exists a simple closed geodesic in M.*

(2) *M is the quotient of H^3 by a Fuchsian group corresponding to a thrice-punctured sphere.*

(3) *The fundamental group of M is elementary with zero or one limit point.*

Proof. Let Γ be a discrete group of isometries such that $M = H^3/\Gamma$. Note that if Γ is elementary with two limit points, then there exists a hyperbolic isometry such that the endpoints of its axis are these two limit points. In order that these be the only limit points, it must be that this is the only axis of an isometry of Γ in H^3. This means that this axis projects to a simple closed geodesic. We now assume that Γ is non-elementary. If Γ is Fuchsian, then either Γ is the fundamental group of a thrice-punctured sphere, or there is a totally geodesic surface in M that contains a simple closed geodesic. Hence the theorem holds in this case. We now assume that Γ is a non-elementary non-Fuchsian discrete group. By Lemma 2, we can find an element in Γ that is a screw motion. If γ is a screw motion, we let $l(\gamma)$ denote the translation length of γ, and define $I = \inf\{l(\gamma)\colon \gamma \text{ is a screw motion in } \pi_1(M)\}$. There are now two cases, depending on whether or not I is realized as the translation length of a screw motion.

Case 1. There is a screw motion γ with $l(\gamma) = I$. We shall denote the corresponding closed geodesic by c. We claim that c must be embedded. If c is not embedded, then c intersects itself transversely at some point. We can cut and paste at this point, to produce curves a, b and ab, each shorter than c, with c homotopic to ab^{-1}. Then, a, b and ab must correspond to non-screw motions, as otherwise we could find closed geodesics homotopic to them that are shorter than c, contradicting the minimizing property of c. Lemma 1 now implies that the isometry γ corresponding to c must also be a non-screw motion. This contradiction shows that c must be embedded, as claimed, so M admits a simple closed geodesic.

Case 2. There is no screw motion γ with $l(\gamma) = I$. Note that in the manifolds which we are considering, the fundamental group may not be finitely generated, and there need not exist a shortest closed geodesic or a shortest screw motion. Explicit examples of hyperbolic 3-manifolds containing no shortest closed geodesic have been constructed by Bonahon and Otal [**1**].

In the following, we suppose that M does not contain a simple closed geodesic, and shall obtain a contradiction. Let c_n be a sequence of closed geodesics corresponding to screw motions whose lengths approach I as $n \to \infty$. Each of these geodesics must itself be singular. Therefore each geodesic can be subdivided at a singular point into two sub-loops, which we call a_n and b_n. Let α_n, β_n and γ_n be the closed geodesics homotopic to a_n, b_n and $a_n b_n^{-1}$, if they exist. Let ε denote the Margulis constant in dimension three, so that any closed geodesic of length less than ε must be embedded. It follows that $I \geqslant \varepsilon$. We also claim that there exists a number $K > 0$ such that $l(a_n) > K$ and $l(b_n) > K$ for all n. If not, then by passing to a subsequence we can assume that $\lim_{n\to\infty} l(a_n) = 0$. For any value of n with $l(a_n) < \varepsilon$, the isometry corresponding to a_n must be parabolic. However, a very short parabolic loop must occur far out in a cusp, forcing c_n to be very long since it is not contained completely in a cusp. More explicitly, a shortest curve in a maximal horosurface (which must be an annulus or torus) cutting off a cusp has length at least 1. The shortest nontrivial curve that comes to within a distance d of the horosurface can be computed by standard hyperbolic geometry. Inverting to obtain the distance in terms of the length of the curve shows that a homotopically nontrivial curve in a cusp whose length is less than $\delta < 1$ has distance at least $\ln(\operatorname{csch}(\delta/2)/2)$ from the maximal horosurface bounding the cusp. In particular, if a_n represents a parabolic and $\lim_{n\to\infty} l(a_n) = 0$, then $\lim_{n\to\infty} l(c_n) = \infty$, contradicting our choice of c_n.

Now let θ_n be the angle between the two strands of b_n meeting at the singular point on c_n. We shall show that $\lim_{n\to\infty} \theta_n = 0$. Suppose that $\theta_n > \theta_0 > 0$ for all n. Since $l(a_n)$ and $l(b_n)$ are both greater than K, we can choose a segment of a_n of length greater than K, and a segment of b_n of length greater than K, each with one end at the singular point of c_n, and consider the triangle determined by their endpoints. This triangle has angle $\pi - \theta_n$ at the singular point of c_n. The assumption that $\theta_n > \theta_0 > 0$ for all n implies that the third edge of this triangle is shorter than the corresponding path along a_n and b_n by a constant $L > 0$, depending only on K and θ_0. In particular, L is independent of n. If there is a closed geodesic γ_n homotopic to the loop $a_n b_n^{-1}$, then it follows that $l(\gamma_n) < l(c_n) - L$, as $a_n b_n^{-1}$ and the original loop c_n have the same length. If $l(c_n) < I + L$, then it follows that either $a_n b_n^{-1}$ represents a parabolic, or $l(\gamma_n) < I$. In either case, $a_n b_n^{-1}$ represents a non-screw motion. Now if $l(c_n) < I + K$, then the fact that $l(a_n) > K$ and $l(b_n) > K$ for all n implies that $l(a_n)$ and $l(b_n)$ are each less than I, so that a_n and b_n each represent non-screw motions. Thus if n is large enough to ensure that $l(c_n) < I + L$ and $l(c_n) < I + K$, then a_n, b_n and $a_n b_n^{-1}$ will all represent non-screw motions. But now Lemma 1 implies that c_n represents a non-screw motion—a contradiction.

Finally, consider the case where the angle θ_n at the singular point of the closed geodesic c_n satisfies $\lim_{n\to\infty} \theta_n = 0$. Recall that there exists a number $K > 0$ such that $l(a_n) > K$. As the sequence of closed geodesics c_n has length bounded above, and it is trivial that $l(a_n) < l(c_n)$, it follows that we can bound the length of a_n between two constants, $0 < K < l(a_n) < M$. Lifting to the universal cover, we have two geodesic lifts of c_n crossing at a pre-image Q of the singular point. Along each geodesic at a distance $l(a_n) < M$ from the crossing point, we have two points which also project down to the singular point. If P is one of these points, then $a_n^2 P$ is the other, where a_n denotes the element of Γ which sends P to Q. Recall that we are assuming that as n approaches ∞, θ_n approaches 0. Thus $\lim_{n\to\infty} d(P, a_n^2 P) = 0$. Since sufficiently short closed geodesics are simple, it must be that a_n^2, and therefore a_n itself, are parabolic for large n. For a parabolic element, $d(P, a_n^2 P) \to 0$ implies $d(P, a_n P) \to 0$, which contradicts the fact that $l(a_n) > K$. This contradiction completes the proof that M must contain a simple closed geodesic. Note that, in fact, our argument shows more than this. It shows that all sufficiently short closed geodesics in M coming from screw motions must be simple.

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