MATHEMATICS WORKOUT:

ARITHMETIC, ADDITION AND MULTIPLICATION THROUGH FRACTIONS

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"Musical form is close to mathematics -- not perhaps to mathematics itself,
but certainly to something like mathematical thinking and relationship."

Igor Stravinsky

Introduction to the Workout Series

The world of technology makes life easier, and in many instances, more fun and more efficient. There are, however, situations in which such is not the case. As digital calculators appear to have become ubiquitous, some of us may forget to use our best calculators: our brains. Do we 'forget' or do we become 'lazy' or do we not remember how much fun mathematics can be? There is a problem, though. Most of mathematics is based on the Law of the Excluded Middle: things are either 'right' or they are 'wrong'. Getting things 'right' feels good. Getting them 'wrong' may not: it depends on the attitude of the person evaluating answers. When students are made to feel stupid or shamed for getting a wrong answer, they understandably lose interest in moving forward with the subject. When students see wrong answers as an exercise in error detection and correction, with an opportunity from an expert to guide them in a positive manner along a forward path, then, and perhaps only then, they become responsive and even enthusiastic about continuing to study mathematics. So, in the technological world, there is no human out there belittling wrong answers…that is clearly a good thing. There is also, however, no human out there to share the joy of learning and offer constructive advice on moving forward.

Recently, we had the opportunity to meet a young adult man, Danny Rushing, who had been studying symphonic tuba playing and was doing extremely well at it in his studies at Delta State University where he was studying to become a music teacher. An accident to his lip ended the tuba career suddenly and left him depressed and floundering a bit in regard to his future. Danny can play all sorts of musical instruments and can sing. But his first love was the tuba. One day he was telling us how easy music is for him—of the transformations that carried musical scores for him across the sheaves of notation. He told us how he explained these transformations to his student friends who were always astounded by the clarity of his thought and explanations. Danny said he knew that it was really all a bunch of math—that he knew how it worked but not why it worked.

With that, we offered to go back to the roots of math with Danny and show him how to think clearly about things he already knew, or thought he knew, and to show him why they worked. Those conversations resulted in a series of math workouts. The first set is devoted to arithmetic. It draws on the fact that students who think they are no good at math as adults, but claim that they were as young children, often went astray when it came to fractions. The reason is simple. The curriculum teaches addition of whole numbers first and then teaches multiplication of whole numbers. That's fine and makes sense because then one can see multiplication as repetition of addition. When it comes to fractions, however, the order should be reversed for the process to make sense. In some curricula, that order is not reversed. Students must learn to multiply fractions before learning to add them; otherwise the quest for a common denominator becomes a guessing game, frustration sets in, and former good students become confused, weak students. It was not clear to us that Danny had experienced this issue; nonetheless, it seemed prudent to back up to that point and let him see what a difference clarity of thought can make in math, as well as in music.

Thus the first set of workouts focuses on arithmetic involving addition and multiplication of non-negative integers and fractions. In all cases, these workouts will work best for students who have already been exposed, in a formal classroom setting, to the topics being covered. The workouts are not a replacement for a steady curricular diet: they are a healthy supplement!

Workout Materials

Root Concepts, Part 1

Concept: The Distributive Law.

$$a*(b+c) = a*b + a*c$$

Remember another Concept: Order of Operations…Consider:

$$2*(3 + 4)$$

Using Order of Operations, the steps are: $2*(3 + 4) = 2*7 = 14$.

That doesn't expose the Distributive Law (DL)…of course, using DL for steps should give the same answer: $2*(3 + 4) = 2*3 + 2*4 = 6 + 8$.

Simple problems like this don't expose any power to DL…so, why bother with it? Let's see…

Book Pricing Example

One book costs $7.99. How much do 8 books cost? Use DL:

$8*(800 – 1) = 8*800 – 8*1 = 6400 – 8 = 6392$ or $63.92

Here, using the DL enables solving the problem quickly in your head; it might be hard to solve 8*799 in your head otherwise.

Notice that DL applies when there is a minus sign because $(800 – 1)$ is the same as $(800 + (-1))$.

Rapid Multiplication: Difference of Two Squares

Find the product: $(x – y)*(x + y)$

a in the DL is (x-y); (b+c) is (x+y).

So, using DL:

$(x-y)*(x+y)= (x-y)*x + (x-y)*y$ One use of DL

$= x*x - y*x + x*y - y*y$ Another use of DL

$= x^2 + 0 - y^2$ Notice that y*x is the same as x*y and that adding zero causes no change

$= x^2 - y^2$

Rapid calculation using the Difference of Two Squares:

29 * 31 = (30 – 1)*(30 + 1) = 30 * 30 – 1 * 1 = 900 – 1 = 899.

67 * 73 = (70 – 3)*(70 + 3) = 70 * 70 – 3 * 3 = 4900 – 9 = 4891.

Squaring a number ending in 5

Try 25 * 25. Use the DL (more than once):

(20 + 5)(20 + 5)= (20+5)*20 + (20+5)*5 = 20*20+5*20+20*5+5*5 = 20(20+5+5)+25 = 20*30 +25 = 2*10*3*10 + 25 = 2*3 * 100 +25

Try 75 * 75.

(70+5)(70+5)=(70+5)*70+(70+5)*5=70*70+5*70+70*5+5*5=70(70+5+5)+25=70*80+25= 7*8*100+25=5625

Generally:

(10x+5)(10x+5)=(10x+5)*10x+(10x+5)*5=10x*10x+5*10x+10x*5+5*5=10x(10x+5+5)+25=10x(10x+10)+25=10*10x(x+1)+25=x(x+1)*100+25

So, a quick way to express it: 45*45 ends in 25 and the front part is 4*5, so 2025. Or, 65*65 ends in 25 and the front part is 6*7 so 4225.

Use the quick way; but make sure you understand WHY it works, that is, PROVE it…the answer for all of these is the Distributive Law. When you know WHY things work, you remember them and can be creative with them.

Putting things together…

What is 24 * 26?

(25-1)(25+1)=25*25-1*1=625-1=624

What is 32 * 38?

(35 – 3)(35 + 3)=35*35-3*3=1225-9=1216

KEEP LOOKING FOR THE DISTRIBUTIVE LAW…SEE IF YOU CAN MAKE UP OTHER INTERESTING USES OF IT…THE DISTRIBUTIVE LAW WILL COME UP IN ARITHMETIC, ALGEBRA, GEOMETRY, TRIGONOMETRY, CALCULUS AND MANY OTHER PLACES…IT IS TRULY A ROOT CONCEPT! ARE THERE OTHER ROOT CONCEPTS EMBEDDED IN THIS DOCUMENT? STAY TUNED…

Root Concepts, Part 2

Concept: The Distributive Law.

a*(b+c) = a*b + a*c

Concept: Multiplicative Identity.

a * 1 = 1 * a = a

Concept: Additive Identity.

a + 0 = 0 + a = a

Concept: Associative.

a + (b + c) = (a + b) +c

a*(b*c) = (a*b)*c

Concept: Commutative.

a*b = b*a

a+b = b+a

Concept: Symmetry.

a = b is the same as b = a

Where have we already assumed we are using any of these (look at the first part above that is focused on the Distributive Law).

Terminology

The parts of an expression linked by + signs are called 'terms'.

The parts of an expression linked by * signs are called 'factors'.

In a fraction, the top part is the "numerator" and the lower part, under, is called the denominator.

Multiply fractions:

a/b * c/d = (ac)/(bd)

Samples: 2/3 * 4/5 = (2*4)/(3*5)

6/15 * 45/48 = (6*45)/(15*48), or more efficiently, use the concept that a*1=a:

6/15 = 3*2/5*3 = 2/5 * 3/3 = 2/5 *1 = 2/5. Cancel first!

45/48 = 15*3/16*3 = 15/16 * 3/3 = 15/16 * 1 = 15/16. Cancel first!

So, now the problem has become:

2/5 * 15/16

Which is 2/5 * (3*5)/16; look to cancel again…cancel the 5s.

So, now the problem is:

2/1 * 3/16 = 2/1 * 3/(2*8) cancel the 2s.

So now it is:

1/1 * (3/8) = (3/8).

You can do this problem in your head when you use cancellation to make things more efficient! Look to cancel first before performing the multiplication. Could you have done this problem even more efficiently? Answer is YES…figure it out! See why it is good to be quick with the multiplication table, both in multiplying two numbers to get an answer and in doing it the other way around, expressing a single number as a product of factors.

Adding fractions.

Like denominators: a/c + b/c = (a+b)/c

Different denominators: a/b + c/d = (ad + bc)/bd

Why does this work? If we can reduce the case with different denominators to the first case with like denominators, then we are done. To do so, use a*1 = a, or written equivalently, a/a = 1.

So, a/b + c/d = a/b *d/d + c/d * b/b (multiplying by 1, and choosing values to yield the common denominator that we want)

=ad/bd +cb/db (multiplying fractions…see why you learn to do that first!)

= (ad + bc)/bd (using simple adding of fractions and commutative).

When teaching, it is CRITICAL therefore that students first understand well how to multiply fractions, then how to add simple fractions with the same denominator, and finally learn to add fractions with different denominators.

Samples: 1/5 + 2/3 = (1*3 + 2*5)/3*5 = (3+10)/15 = 13/15.

6/48 + 10/15 =

Look for cancellation possibilities…

GENERAL IDEA: THINK ABOUT WHAT YOU ARE DOING BEFORE YOU APPLY SOME MEMORIZED CUTE RULE…BE EFFICIENT AND DO AS MUCH AS POSSIBLE IN YOUR HEAD TO MAKE LIFE EASY! MATH IS FUN…NOT TEDIOUS!

Cool-down from the Workout

Make up some multiplication problems. Do not solve them, but instead, say what is the most efficient strategy for solving them. Make up some for which standard multiplication is the simplest and make up others for which some of the quick, in-the-head, strategies in this Workout are the most efficient. Justify your answer.

Then, go ahead and solve the problems. Was the method you chose the most efficient one? Did you discover, in the process of solving, that an alternate strategy might have been more efficient?

Repeat the two steps above for multiplying and adding fractions.

Circle back as many times as needed so that you become quick and comfortable at calculations.

Postscript

Danny is off and running; as he travels around Mississippi in his various part time jobs as a stage show host/personality, as a book salesman (he has read and remembers an amazing number of literary works, both classical and contemporary), and as a music store consultant on instrument selection for students of music, he does his math workout in the car. In fact, he is now working at creating a musical interpretation for the Distributive Law! From there, he became motivated on his own to discover the role that Pythagoras had in music theory. He does all of this because he loves it—and, because his ambition for a steady permanent job is as a teacher of mathematics—a platform from which he can continue to explore abstract connections between the teaching of mathematics and music and share his love for each with his future students!