Optimal Policies for Selling New and Remanufactured Products

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Abstract

Because of environmental and economic reasons, an increasing number of original equipment manufacturers (OEMs) nowadays sell both new and remanufactured products. When both products are available, customers will buy the one that gives them a higher (and nonnegative) utility. Thus, if the firm does not price the products properly, then product cannibalization may arise and its revenue may be adversely impacted. In this paper, we study the pricing problem of a firm that sells both new and remanufactured products over a finite planning horizon. Customer demand processes for both new and remanufactured products are random and price-sensitive, and product returns (also called cores) are random and remanufactured upon receipt. We characterize the optimal pricing and manufacturing policies that maximize the expected total discounted profit. If new products are made to order (MTO), we show that when the inventory level of remanufactured product increases, the optimal price of remanufactured product decreases while the price difference between new and remanufactured products increases; however, the optimal selling price of new product may increase *or* decrease. If new products are made to stock (MTS), then the optimal manufacturing policy is of a base-stock policy with the base-stock level decreasing in the remanufactured product inventory level. To understand the potential benefit in implementing an MTO system, we study the difference between the value functions of the MTO and MTS systems, and develop lower and upper bounds for it. Finally, we study several extensions of the base model and show that most of our results extend to those more general settings.

Key words: remanufactured product, new product, pricing, optimal policy
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1 Introduction

An increasing number of original equipment manufacturers (OEMs), such as those in machinery, automobile, and personal electronics, are producing and selling products in both new and remanufactured conditions to cater for demands in different market segments. Meanwhile, more stringent environmental regulations such as the WEEE directive in Europe also boost the growth of remanufacturing industry. When both new and remanufactured products are sold in the same market, the remanufactured product is often cheaper than the new one because most customers still prefer a new product to a remanufactured one. For example, Dell offers discounts to customers who are willing to buy remanufactured (or refurbished) products. Apple sells both new and refurbished products such as iPad; the price of a refurbished new generation iPad with 32GB and Wi-Fi is sold at $469, 14% cheaper than the new one.[1] The remanufactured product often attracts customers with low valuation, who originally would not buy the product. Thus, if the price of remanufactured product is set too low or the price of new product is too high, although the demand of the remanufactured product would increase, some customers who would have bought the new product may switch to the remanufactured product. For instance, a Xerox study shows that the presence of a remanufactured product decreases the consumer's wiliness to pay for the new product (Vietor 1993). This may hurt the profitability of the firm. Therefore, the firm needs to balance such trade-off when setting the prices for its new and remanufactured products.

This paper studies the optimal pricing and manufacturing policies for a firm selling new and remanufactured products. Customers choose which product to buy (or buy nothing) based on their product valuations and the selling prices. Both demand and product return are random while demand is price-sensitive. Returned products are remanufactured upon receipt and then used to satisfy demand for the remanufactured product. The manufacturing of new product follows either a make-to-order (MTO) or a make-to-stock (MTS) strategy. Unused inventory of both products at the end of each period is carried over to the next period, and unsatisfied demand is backlogged. The inventory incurs holding cost while the demand backlog incurs shortage cost. The objective is to maximize the expected total discounted profit over a finite planning horizon. For the MTO system under which the new product is produced after demand is realized, we find that when the remanufactured product inventory level increases, it is optimal for the firm to drop its selling price for the remanufactured product, but increase the price difference between the new and remanufactured products. However, the optimal selling price of the new product may go up *or* go down. For the MTS system under which the new product is produced before demand realization, the optimal manufacturing policy is of base-stock type with the base-stock level decreasing in the remanufactured product inventory level. Under stochastic demand, it is shown that MTO system results in a higher profit for the firm than the MTS system because the firm incurs either inventory

[1] http://store.apple.com/us/browse/home/specialdeals/ipad.

holding or demand backlogging cost under the MTS system but not the MTO system. However, even if shifting from the MTS system to the MTO system is feasible for the firm, it incurs cost and requires investment. Therefore, to help the firm assess the potential profit increment of changing from the MTS system to the MTO system, we analyze the difference between the value functions of the MTO and MTS systems. We derive lower and upper bounds for the profit difference and test its sensitivity to system primitives by a set of comprehensive numerical experiments.

We also study three extensions of the aforementioned base model. In the first extension, we consider positive manufacturing and remanufacturing lead times. We employ the concept of $L^\natural-$concavity and show that under the MTO system, the optimal price of the remanufactured product is decreasing and the price difference of the new and remanufactured product is increasing with respect to the remanufactured product inventory level and the work-in-process (WIP) core inventory level. Furthermore, the optimal price of the remanufactured product and the price difference of the two products are both more sensitive to changes in the WIP cores that are closer to finish than to changes in the WIP cores that take longer to finish. The optimal manufacturing policy for the MTS system remains a base-stock type. In the second extension, we consider effort-dependent product return, where the firm needs to decide how much effort/resource to spend on core acquisition. We find that the optimal effort level decreases in the inventory level of remanufactured product and WIP cores. In the third extension, we study sales-dependent product return, where the number of returned products in each period is modeled as a random proportion of the new product sold in the earlier period. Most results from our base model extend to the case of sales-dependent return. However, the optimal price for the remanufactured product may not decrease in the remanufactured product inventory level when product return depends on the previous sales. This is because, selling more new products in one period can lead to more product returns in the future period, which may be beneficial to the firm in the subsequent periods.

Literature review. The literature on remanufacturing operations is very extensive. Simpson (1978) studies a system with a single type of return and shows that the optimal policy is determined by three state-independent parameters. DeCroix and Zipkin (2005) and DeCroix (2006) extend Simpson's model to multi-echelon inventory systems. Zhou et al. (2011) generalize Simpson (1978) to multi-type of product returns that differ in remanufacturing costs, and characterize the optimal inventory policies. When pricing is considered, Ferrer and Swaminathan (2006) study the optimal pricing strategies in both a monopoly and a duopoly model with remanufacturing. In the monopoly setting, a proportion of new products sold in the previous period will be returned to the OEM, remanufactured, and sold. And in their duopoly setting, an independent operator intercepts some of the returns and competes with the OEM in remanufactured product market. Guide et al. (2003) show that the quantity and quality of product returns can be influenced by varying quality-dependent acquisition prices and develop a simple framework for profit maximization. Zhou and Yu (2011) incorporate product acquisition effort and pricing decisions into the model of Simpson

(1978) and characterize the structure of the optimal operational and pricing/effort strategies. Other related papers include Inderfurth (1997), Van der Laan et al. (1999), Savaskan et al. (2004), and Atasu (2008). All these papers assume that there is only one type of serviceable product for filling demand, i.e., the new and remanufactured products are indistinguishable.

When customers view remanufactured products as different from new ones, Debo et al. (2005) study the joint pricing and production technology selection problem in an infinite horizon deterministic model. Ferrer and Swaminathan (2010) extend the model of Ferrer and Swaminathan (2006) to new and differentiated remanufactured products. Debo et al. (2006) investigate the sequence of prices for new and remanufactured products to maximize the firm's total discounted profit. Akan et al. (2013) develop a continuous-time model where the price of a remanufactured product is assumed to be a fixed percentage of the new product's price. The firm sets price of the new product, production rates of new and remanufactured products, and disposal rate of the remanufactured product to maximize the total profit. Our paper differs from the preceding ones in the several aspects: We consider stochastic demand and both make-to-order and make-to-stock systems for manufacturing operation; we derive the structural properties of the optimal pricing and manufacturing policies; we also study positive manufacturing and remanufacturing lead times, effort-dependent product return, and sales-dependent return; and finally, different from Debo et al. (2006) and Akan et al. (2013), we do not model a detailed product diffusion process as we do not focus on product life-cycle dynamics.

Another stream of related research is dynamic pricing in multiproduct inventory systems (see e.g., Song and Xue, 2007; Zhu and Thonemann, 2009). Our model differs from these in the following aspects. First, customers value the new product higher than the remanufactured product. Second, the inventory level of remanufactured product is affected by random product return which can depend on the acquisition effort and past sales. Third, we allow positive production lead times for both new and remanufactured products.

Organization. The rest of the paper is organized as follows. In Section 2, we describe the base model in detail and analyze the optimal pricing and manufacturing strategies. In Section 3, we extend the base model to the case with positive lead times and derive additional results. In Section 4, we incorporate acquisition effort on product returns and examine the structural property of the optimal effort. In Section 5, we consider a scenario where product return depends on the new product sales of the previous period and show that most results in the base model extend to that setting. Section 6 concludes the paper. All of the technical proofs are provided in the Appendix.

Throughout the paper, we consider increasing and decreasing in a non-strict sense, i.e., they represent non-decreasing and non-increasing, respectively. In addition, we use notation $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$ for any real number x, and "$\triangleq$" stands for "defined as".

2 Model Formulation

Consider a firm selling both new and remanufactured products over a finite planning horizon with N periods, indexed by $n = 1, 2, \ldots, N$. The planning horizon represents a segment of the life cycle of the product, during which new and remanufactured products co-exist. The length of a period can be a week, two weeks, or a month, depending on how frequently the firm manufactures new product.

Customer preferences. As noted in Debo et al. (2005), customers typically value remanufactured products less than new products. To model the customer purchasing behavior, we extend a consumer choice model proposed by Debo et al. (2005) by including the no-purchase option. Specifically, we model the value of a new product by a random variable v with distribution $F(\cdot)$, while the value of a remanufactured product is $\eta(v)$ with $\eta(v) \in [0, v]$. We do not assume any specific form of $\eta(v)$ except that both $\eta(v)$ and $v - \eta(v)$ are strictly increasing in v (one example is $\eta(v) = av$ with $a \in [0, 1)$).[2] At the beginning of each period n, the firm sets the selling price p_1 for its new product and p_2 for the remanufactured product. A customer' utility of buying a new (resp., remanufactured) product is $v - p_1$ (resp., $\eta(v) - p_2$). A customer will choose to buy the product that gives her a higher non-negative utility. Hence, the probabilities for a customer to buy new and remanufactured products, denoted by $\lambda_1(p_1, p_2)$ and $\lambda_2(p_1, p_2)$, can be computed as

$$\lambda_1(p_1, p_2) = \mathsf{P}\big(v - p_1 \geq \eta(v) - p_2,\, v - p_1 \geq 0\big) = \mathsf{P}\big(v - \eta(v) \geq p_1 - p_2,\, v \geq p_1\big), \quad (1)$$

$$\lambda_2(p_1, p_2) = \mathsf{P}\big(\eta(v) - p_2 > v - p_1,\, \eta(v) - p_2 \geq 0\big) = \mathsf{P}\big(v - \eta(v) < p_1 - p_2,\, \eta(v) \geq p_2\big). \quad (2)$$

A customer does not buy any product if her utility of doing so is negative, thus

$$1 - (\lambda_1(p_1, p_2) + \lambda_2(p_1, p_2)) \geq 0$$

is the probability for a customer to not make a purchase.

The demand processes for new and remanufactured products are random and depend on both prices, and they are modeled by

$$\begin{aligned} D_{1n}(p_1, p_2) &= \lambda_1(p_1, p_2) d_n + \varepsilon_{1n}, \\ D_{2n}(p_1, p_2) &= \lambda_2(p_1, p_2) d_n + \varepsilon_{2n}, \end{aligned}$$

where $\varepsilon_{in} \in [\underline{\varepsilon}_i, \bar{\varepsilon}_i]$ is the random noise with $\mathsf{E}[\varepsilon_{in}] = 0$ $(i = 1, 2)$, d_n is a deterministic number representing the potential total demand for the firm's products in period n, and $\lambda_1(p_1, p_2)$ and $\lambda_2(p_1, p_2)$ are the fractions of the potential demand that purchase new and remanufactured products, respectively, described above.

[2] Although we assume a stationary v for notational conciseness, all the results in this paper hold for a dynamic v, i.e., v can change over time.

For ease of analysis, in the following lemma we present the price decisions in terms of fractions of customers who purchase new and remanufactured products λ_1 and λ_2 of (1) and (2).

Lemma 1. *The price decisions for new and remanufactured products, p_1 and p_2, can be written as functions of λ_1 and λ_2 as follows:*

$$p_1(\lambda_1,\lambda_2) = \eta\big(F^{-1}(1-\lambda_1-\lambda_2)\big) + F^{-1}(1-\lambda_1) - \eta\big(F^{-1}(1-\lambda_1)\big), \quad (3)$$

$$p_2(\lambda_1,\lambda_2) = \eta\big(F^{-1}(1-\lambda_1-\lambda_2)\big), \quad (4)$$

where $(\lambda_1,\lambda_2) \in \Omega = \big\{(\lambda_1,\lambda_2) : 0 \le \lambda_1 \le 1, 0 \le \lambda_2 \le 1, 0 \le \lambda_1+\lambda_2 \le 1\big\}$.

As there exists a one-to-one correspondence between (p_1, p_2) and (λ_1, λ_2), in what follows we shall take λ_1 and λ_2 as decision variables. By (3) and (4), the expected one-period revenue (when sufficient on-hand inventory is available) can be written as

$$\big[\lambda_1 p_1(\lambda_1,\lambda_2) + \lambda_2 p_2(\lambda_1,\lambda_2)\big] d_n = \big[G_1(\lambda_1) + G_2(\lambda_1+\lambda_2)\big] d_n,$$

where

$$G_1(x) = xF^{-1}(1-x) - x\eta\big(F^{-1}(1-x)\big), \quad G_2(x) = x\eta\big(F^{-1}(1-x)\big).$$

We can interpret $G_1(\lambda_1)$ as the additional revenue from selling a new product at a higher price than the remanufactured product while $G_2(\lambda_1+\lambda_2)$ the total revenue excluding the preceding part from the price markup of the new product.

To facilitate the analysis, we make the following assumption.

Assumption 1. *$G_1(x)$ and $G_2(x)$ are concave in x.*

This assumption is satisfied by many examples. For instance, when $\eta(v) = av$ for some constant $a \in (0,1)$, the assumption is valid if and only if $xF^{-1}(1-x)$ is concave, which is satisfied by many distributions of v, including the class of IFR (increasing failure rate) distributions. If $\eta(v) = a\ln(1+bv)$ with $ab \le 1$ and $ab(b+2) \le 2$ or $\eta(v) = \alpha(1-e^{-\beta v})$ with $\alpha\beta \le 1$ and $2\alpha + \alpha\beta \le 2$, then $G_1(x)$ and $G_2(x)$ are both strictly concave in x if v is uniform.

We consider random product returns, which is a key characteristic and a main challenge in managing a remanufacturing inventory system (Guide et al., 2003). In the base model, we assume product return R_n is random, uncontrollable, and is realized at the end of period n. In Section 4, we will study the case with effort-dependent product return. Upon receiving returned cores, they are inspected, preprocessed, and remanufactured, costing c_2 per unit. Each unit of the new product costs the firm c_1 to manufacture, with $c_1 \ge c_2$. The manufacturing and remanufacturing lead times are assumed to be zero. Positive lead times will be considered in Section 3.

In each period, demands for new and remanufactured products are satisfied by on-hand stocks to the maximum possible extent, and unsatisfied demand is backlogged with a unit shortage cost π_0 for remanufactured product and a unit shortage cost π for new product. Excess inventories of new and remanufactured products are carried over to the next period, at a unit holding cost h_0 for remanufactured product and unit holding cost h for new product. The objective of the firm is to maximize its expected total discounted profit by determining the selling prices of both new and remanufactured products as well as the production quantity of new product.

In this paper, we consider the scenario where the holding cost for a returned core is the same (or similar) as its holding cost after it is remanufactured. Thus, there is little incentive in postponing the remanufacturing of cores. Therefore, since we have no operations capacity constraint, we assume that returned cores are remanufactured as soon as they are received. A large portion of remanufacturing in the US is remanufacture-to-stock (e.g., Guide, et al., 2003; Hauser and Lund, 2003). For the new product, we assume that there are sufficient raw materials for manufacturing. Alternatively, new products can be modeled as being ordered from external suppliers (instead of being manufactured in-house).

In the following we will consider two production strategies for new product: make-to-order and make-to-stock. In the first one, manufacturing takes place after orders for the new product are received, while in the second, manufacturing of the new product takes place in anticipation of future demand.

2.1 Make-To-Order System

We first consider the case where the manufacturing of the new product follows make-to-order (MTO) strategy. At the beginning of each period, the firm first determines the selling prices of both the new and remanufactured products, then demands for both products are received. Then the firm uses on-hand inventory to satisfy demand of the remanufactured product to the maximum extent; while it produces new products to meet demand, i.e., MTO. Since manufacturing lead time is 0, in this special case the new product incurs neither holding nor shortage cost.

Let x_0 denote the starting inventory level of remanufactured product. After observing x_0, the firm decides (λ_1, λ_2) to maximize its expected profit. The dynamic program can be formulated as

$$\begin{aligned} V_n(x_0) \quad = \quad \max_{(\lambda_1,\lambda_2)\in\Omega} \Big\{ & \Gamma_n(\lambda_1, \lambda_2) - \mathsf{E}[L_0(x_0 - \lambda_2 d_n - \varepsilon_{2n})] - c_2 \mathsf{E}[R_n] \\ & + \gamma \mathsf{E}\left[V_{n+1}(x_0 - \lambda_2 d_n - \varepsilon_{2n} + R_n)\right] \Big\}, \end{aligned}$$

where $0 < \gamma < 1$ is the discount factor, and

$$\Gamma_n(\lambda_1, \lambda_2) = \big[G_1(\lambda_1) + G_2(\lambda_1 + \lambda_2) - c_1 \lambda_1\big] d_n \tag{5}$$

is the expected revenue from the sales of both the new and remanufactured products minus manufacturing cost, which is jointly concave and submodular in (λ_1, λ_2) by Assumption 1, and $L_0(x) = h_0 x^+ + \pi_0 x^-$ is the holding and shortage cost rate of the remanufactured product. The third term in the optimality equation $c_2\mathsf{E}[R_n]$ is the expected processing and remanufacturing cost of cores. Because it is a constant and in this section we are concerned with the structure of the optimal control policy, we omit it in the subsequent analysis of this section. For simplicity we assume that $V_{N+1}(x_0) \equiv 0$, but the results and analysis can be easily extended to more general boundary conditions.

When the inventory level of remanufactured product is x_0 at the beginning of period n, the optimal fractions of customers to purchase the two products, $(\lambda_{1n}^*(x_0), \lambda_{2n}^*(x_0))$, are determined by

$$\begin{aligned} \lambda_{2n}^*(x_0) &= \underset{0\le\lambda_2\le1}{\arg\max}\Big\{U_n(\lambda_2) - \mathsf{E}[L_0(x_0-\lambda_2 d_n - \varepsilon_{2n})] + \gamma\mathsf{E}\left[V_{n+1}(x_0 - \lambda_2 d_n - \varepsilon_{2n} + R_n)\right]\Big\}, \\ \lambda_{1n}^*(x_0) &= \lambda_1\big(\lambda_{2n}^*(x_0)\big), \end{aligned} \tag{6}$$

where

$$U_n(\lambda_2) = \max_{0\le\lambda_1\le1-\lambda_2}\big\{\Gamma_n(\lambda_1,\lambda_2)\big\}, \quad \lambda_1(\lambda_2) = \underset{0\le\lambda_1\le1-\lambda_2}{\arg\max}\big\{\Gamma_n(\lambda_1,\lambda_2)\big\}. \tag{7}$$

The following lemma presents the monotonicity result of $\lambda_1(\lambda_2)$ that will be used to derive the structural properties of the optimal policy.

Lemma 2. *$\lambda_1(\lambda_2)$ is decreasing in λ_2 and $\lambda_2 + \lambda_1(\lambda_2)$ is increasing in λ_2, where $\lambda_1(\lambda_2)$ is the optimal solution defined in (7).*

The next theorem presents the structural properties of the optimal fractions of customers who purchase new and remanufactured products.

Theorem 1. *Suppose the starting inventory level of remanufactured product at the beginning of period n is x_0. The optimal fractions of customers who purchase new and remanufactured products in period n, $(\lambda_{1n}^*(x_0), \lambda_{2n}^*(x_0))$, have the following properties:*

(i) $\lambda_{1n}^*(x_0)$ is decreasing in x_0 while both $\lambda_{2n}^*(x_0)$ and $\lambda_n^*(x_0) = \lambda_{1n}^*(x_0) + \lambda_{2n}^*(x_0)$ are increasing in x_0.

(ii) The expected ending inventory of remanufactured product in period n, $x_0 - \lambda_{2n}^*(x_0)d_n$, is increasing in x_0.

This result shows that when the inventory level of remanufactured product in a period goes up, the corresponding optimal selling prices of new and remanufactured products will make the optimal fraction of customers who purchase remanufactured products in this period increases while the fraction of customers who purchase new product decreases. This is intuitive as the firm wants to sell more remanufactured products when it has more inventory on hand and the new and remanufactured products compete for customers in the same market. However, the decrease in the number of customers who purchase new products is dominated by the increase in the number of customers who purchase remanufactured products, and as a result, the total sales of new and remanufactured products will still go up. Furthermore, when the starting inventory level of remanufactured product in a period goes up, we should expect to have more remanufactured products to be carried over to the following period.

Based on Theorem 1, we have the following monotonic properties of the optimal prices with respect to the starting inventory level of remanufactured product.

Theorem 2. *The optimal selling price of the remanufactured product, $p^*_{2n}(x_0)$, in period n is decreasing in the starting inventory level x_0 of remanufactured product at the beginning of that period. However, the price difference between new and remanufactured products, $\Delta p^*_n(x_0) \triangleq p^*_{1n}(x_0) - p^*_{2n}(x_0)$, is increasing in the starting inventory level x_0.*

Therefore, according to this result, when the inventory level of remanufactured product goes up, not only its selling price goes down, but price discount with respect to new product also goes up, i.e., a deeper discount is offered for the remanufactured product.

Intuitively, one would expect that the optimal price of new product is increasing in the inventory level of remanufactured product x_0 (so that more customers can buy remanufactured products). This intuition turns out to be incorrect. We consider the following numerical example with deterministic demand. Consider a single-period problem, and we study the optimal price p^*_{11} as a function of starting inventory level of remanufactured products x_0. Set $c_1 = 0.44$, $c_2 = 0$, $h_0 = 0$, $\pi_0 = 0.6$, $d_1 = 1$, $\eta(v) = av$ with $a = 0.6$ and the cumulative distribution of v is $F^{-1}(x) = 2x^3 - 3x^2 + 2x$. Then $\zeta(x) \triangleq xF^{-1}(1-x) = -2x^4 + 3x^3 - 2x^2 + x$ and $\zeta''(x) = -24x^2 + 18x - 4 < 0$ for any $x \in [0, 1]$. Hence, Assumption 1 is satisfied. It can be seen from Figure 1 that the optimal price of new product $p^*_{11}(x_0)$ is not even monotone in x_0.

We offer the following insight on this non-intuitive result. When setting the price of new product, the firm has to consider two conflicting factors: one is the fraction of customers who will purchase new products, which decreases with its own price, while the other is the fraction of customers who will purchase remanufactured products, which increases with the selling price of new product. Theorem 2 has shown that when the inventory level of remanufactured product x_0 increases, the price of remanufactured product decreases, which will reduce the fraction of customers who purchase new products. Therefore, when x_0 is low, the first factor outweighs the second because it is not urgent to sell out the remanufactured product, and the firm wants to maintain the fraction of customers who purchase new products, thus in this case, the firm lowers its price with a small increment in the inventory level of remanufactured product. However, when the starting inventory

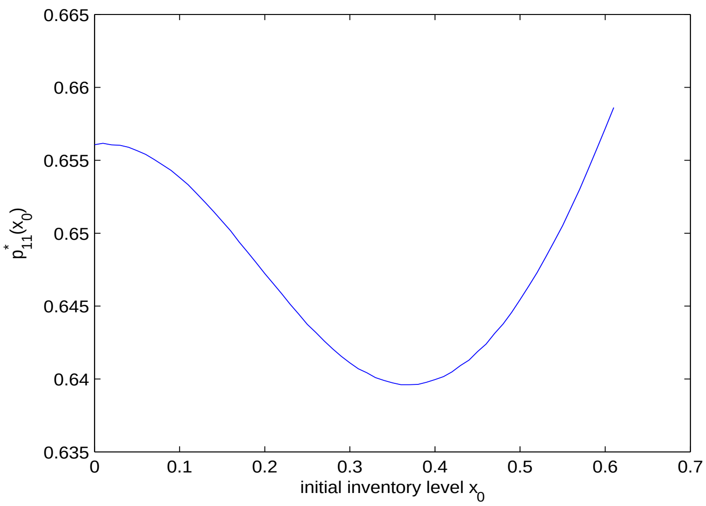

Figure 1: Non-monotonicity of selling price of new products $p^*_{11}(x_0)$

level of remanufactured product x_0 becomes high, the second factor outweighs the first as the firm wishes to firstly get rid of more remanufactured products, hence in this case the firm will raise the selling price of new product.

Comparative statics. To obtain additional insights into the optimal policy, we analyze the dependency of the optimal fractions of customers who purchase new and remanufactured products, and the optimal selling prices, on the inventory holding and shortage costs of the remanufactured product.

Theorem 3. *(i) The optimal fraction of customers who purchase new products, $\lambda^*_{1n}(x_0)$, decreases with the unit holding cost of remanufactured product h_0, while the optimal faction of customers who purchase remanufactured products, $\lambda^*_{2n}(x_0)$, and the fraction of customers who will purchase a product (new or remanufactured), $\lambda^*_n(x_0) = \lambda^*_{1n}(x_0) + \lambda^*_{2n}(x_0)$, increase with h_0. Conversely, $\lambda^*_{1n}(x_0)$ increases with the unit shortage cost of remanufactured product π_0, while $\lambda^*_{2n}(x_0)$ and $\lambda^*_n(x_0)$ decrease with π_0.*

(ii) The optimal price of remanufactured product, $p^*_{2n}(x_0)$, decreases with h_0, while the difference between the optimal prices of new and remanufactured products, $\Delta p^*_n(x_0) = p^*_{1n}(x_0) - p^*_{2n}(x_0)$, increases with h_0. Conversely, $p^*_{2n}(x_0)$ increases with π_0, while $\Delta p^*_n(x_0)$ decreases with π_0.

When h_0 increases or π_0 decreases, carrying inventory to the next period becomes more expensive. This drives the firm to sell more remanufactured products. As a result, the fraction of customers who buy new products decreases because the new and remanufactured products compete for customers in the market. However, the total sales of new and remanufactured products increase, which implies that the decrease of the sales of new product is less than the increase of the sales of remanufactured product. Theorem 3 also suggests that when h_0 increases or π_0 decreases, the optimal price of remanufactured product decreases and the price difference between new and remanufactured products increases, i.e., a deeper discount is offered for the remanufactured product, which is consistent with the optimal decisions on the fractions of customers who buy new and remanufactured products.

2.2 Make-To-Stock System

In this subsection, we consider the case where the manufacturing of new products follows a make-to-stock (MTS) strategy. In each period before demand is realized, the firm determines the selling prices of the new and remanufactured products as well as the production quantity of new products. Since the firm manufactures new products before seeing demand, product underage or overage will occur during the period.

The state of the system now becomes two-dimensional: The inventory levels of new and remanufactured products at the beginning of a period. As in the previous section, let x_0 be the inventory level of remanufactured product at the beginning of a period. Let u and z be the inventory levels of new product before and after manufacturing decision, respectively. Denote $V_n(u, x_0)$ as the maximum expected discount profit from period n onwards when the starting state is (u, x_0), then the optimality equation is

$$V_n(u, x_0) = \max_{(\lambda_1,\lambda_2)\in\Omega,\, z\geq u} \Big\{ \Gamma_n(\lambda_1, \lambda_2) + c_1\lambda_1 d_n - c_1(z-u) - \mathsf{E}[L(z-\lambda_1 d_n - \varepsilon_{1n})] \\ - \mathsf{E}[L_0(x_0 - \lambda_2 d_n - \varepsilon_{2n})] + \gamma \mathsf{E}\big[V_{n+1}(z - \lambda_1 d_n - \varepsilon_{1n}, x_0 - \lambda_2 d_n - \varepsilon_{2n} + R_n)\big] \Big\},$$

where $L(x) = hx^+ + \pi x^-$ is the holding and shortage cost rate of the new product. As before, we assume $V_{N+1}(u, x_0) \equiv 0$ though the results and analysis easily extend to more general boundary conditions.

For ease of exposition, we introduce notation

$$H_n(z, x_0) = \max_{(\lambda_1,\lambda_2)\in\Omega} \Big\{ \Gamma_n(\lambda_1, \lambda_2) - c_1(z - \lambda_1 d_n) - \mathsf{E}[L(z-\lambda_1 d_n - \varepsilon_{1n})] \\ - \mathsf{E}[L_0(x_0 - \lambda_2 d_n - \varepsilon_{2n})] + \gamma \mathsf{E}\big[V_{n+1}(z - \lambda_1 d_n - \varepsilon_{1n}, x_0 - \lambda_2 d_n - \varepsilon_{2n} + R_n)\big] \Big\},$$

then we have

$$V_n(u, x_0) = \max_{z\geq u}\big\{H_n(z, x_0)\big\} + c_1 u. \tag{8}$$

The optimal production policy for the new product is characterized in the following theorem.

Theorem 4. *Given the inventory level of remanufactured product x_0 at the beginning of a period, the optimal production policy for new product is of a base-stock type, i.e., there exists a base-stock level $z_n^0(x_0)$, such that if $u \leq z_n^0(x_0)$, then manufacture to raise the inventory level of new product to $z_n^0(x_0)$; otherwise, manufacture nothing. In addition, the optimal base-stock level $z_n^0(x_0)$ is decreasing in x_0.*

The optimality of a base-stock type of policy follows from the concavity of the objective function, which can be shown by induction. It is intuitive that the optimal base-stock level $z_n^0(x_0)$ is decreasing in x_0. When the inventory level of remanufactured product goes up, the firm will make effort to sell more remanufactured products, which negatively affects the sales of new product, leading to a lower stocking level for new product.

The optimal fractions of customers to purchase new and remanufactured products as well as the optimal selling prices for the two products can also be similarly studied. One question is whether these optimal decisions possess the monotonic properties in the starting inventory level of remanufactured product similar to those in the MTO system. The answer turns out to be negative. Here we offer some insights on why the monotonicity result breaks down for the MTS system. When the inventory level of remanufactured product goes up, if the fraction of customers to purchase remanufactured products increases, then the fraction of customers to purchase new products will tend to decrease because they are competing products. In the MTO system, the reduction in the fraction of customers to purchase new products in the current period will not affect the firm's profits in the subsequent periods because the new product has no inventory carryover or shortage. However, in the MTS system, the reduction in the fraction of customers to purchase new products in this period will change its inventory level in the next period, which then affects the firm's profit in the future periods. Because of this, the monotonic properties of the fractions of customers to purchase new and remanufactured products, as well as their optimal selling prices, will no longer hold. For the same reasoning, the results in Theorem 3 for the MTO system cannot be extended to the MTS system either.

2.3 Comparing Profits in MTO and MTS Systems

For the MTO system, the production of new products is determined after demand uncertainty is realized. Hence, because of the zero manufacturing lead time, the profit function for the MTO system is at least as much as that of the MTS system. However, switching from MTS to MTO, if feasible, involves other issues, some quantifiable and some not quantifiable. Hence, as a first step the firm would want to understand the potential profit increment of such a change in production strategy. This difference, in a sense, reflects the value of information (making decision after observing demand).

In the following, we analyze the difference in profits between the MTO and MTS systems, and establish lower and upper bounds for the difference. For ease of exposition, the notation with superscript s (o) corresponds to the MTS (MTO) system, and ε_{1n} are independent and identically distributed over periods. The inventory level of new product at the beginning of the planning horizon is assumed to be zero.

Proposition 1. *For any x_0, we have*

$$\gamma^{N-1}m_1 + \frac{(1-\gamma^{N-1})m_0}{1-\gamma} \le V_1^o(x_0) - V_1^s(0,x_0) \le \gamma^{N-1}m_2 + \frac{(1-\gamma^{N-1})m_0}{1-\gamma},$$

where

$$\begin{aligned} m_0 &= \min_t \big\{(1-\gamma)c_1 t + \mathsf{E}[L(t-\varepsilon_{1n})]\big\} \ge 0, \\ m_1 &= \min_t \big\{c_1 t + \mathsf{E}[L(t-\varepsilon_{1n})]\big\} \ge 0, \\ m_2 &= c_1 t_0 + \mathsf{E}[L(t_0-\varepsilon_{1n})], \end{aligned}$$

and

$$t_0 = \arg\min_t \big\{(1-\gamma)c_1 t + \mathsf{E}[L(t-\varepsilon_{1n})]\big\}.$$

It is worthy noting that the lower and upper bounds depend only on the cost parameters of the new product, viz., c_1, h and π, and they are independent of the costs related to the remanufactured product. In the special case that the demand for the new product is deterministic, i.e, $\varepsilon_{1n} \equiv 0$, these two systems result in the same profit. In that case, it can be verified that $m_0 = m_1 = m_2 = 0$. For the lower bound, it is the total discounted minimum one-period costs of the new product due to demand uncertainty over the planning horizon, which is the smallest possible additional cost that the MTS system will incur over the MTO system. For the upper bound, it is obtained by constructing a feasible policy for the MTS system that adopts the optimal pricing strategy of the MTO system and the corresponding optimal manufacturing policy of new product.

It is interesting to observe that, when the length of the planning horizon becomes long or $N \to \infty$, the lower and upper bounds both approach $m_0/(1-\gamma)$. In other words, the profit difference between these two systems converges to a constant that is independent of x_0, the initial inventory level of remanufactured product.

The above result provides analytical bounds on the profit increment when the firm switches from MTS to MTO. In the rest of this section, we conduct numerical experiments to demonstrate the benefit of MTO. In particular, how does the benefit change when remanufacturing is introduced to the system? How is it affected by the system parameters? We assume that the initial inventory level of new product is zero. The benefit of MTO is measured as

$$\frac{V_1^o(x_0) - V_1^s(0,x_0)}{V_1^s(0,x_0)} \times 100\%. \tag{9}$$

The setup of the experiments is as follows: ε_{1n} and ε_{2n} are independent and uniformly distributed over $[-\delta, \delta]$, R_n is uniformly distributed over $[0, \bar{R}]$, v is uniformly distributed over $[0, 1]$, and $\eta(v) = av$ with $a \in [0, 1]$. The benchmark setting is that $N = 4$, $x_0 = 0$, $\delta = 5$, $\gamma = 0.96$, $d = 50$, $\bar{R} = 30$, $a = 0.85$, $c_1 = 0.3$, $c_2 = 0.1$, $h = h_0 = 0.03$, $\pi = 0.09$, $\pi_0 = 0.06$, and the terminal condition is $V_5(u, x_0) = -0.4u^- - 0.3x_0^-$, i.e., the unit shortage costs of new and remanufactured products at the end of the planning horizon are 0.4 and 0.3, respectively.

Table 1: MTO over MTS: Remanufacturing vs. No-remanufacturing

δ	Holding cost of new product h				δ	Shortage cost of new product π			
	0.01	0.02	0.03	0.04		0.03	0.06	0.09	0.12
3	(2.08,2.24)	(2.31,2.53)	(2.49,2.74)	(2.66,2.94)	3	(1.51,1.69)	(2.12,2.35)	(2.49,2.74)	(2.79,3.07)
5	(3.45,3.64)	(3.82,4.08)	(4.13,4.45)	(4.38,4.76)	5	(2.49,2.71)	(3.48,3.77)	(4.13,4.45)	(4.59,4.95)
7	(4.81,5.04)	(5.34,5.65)	(5.82,6.18)	(6.25,6.65)	7	(3.53,3.74)	(4.92,5.23)	(5.82,6.18)	(6.50,6.88)
9	(6.31,6.49)	(6.99,7.28)	(7.61,7.96)	(8.15,8.55)	9	(4.50,4.80)	(6.37,6.73)	(7.61,7.96)	(8.53,8.89)
	Number of periods N					Initial level of remanufactured product x_0			
	2	4	6	8		0	10	20	30
3	(3.95,4.06)	(2.49,2.74)	(2.04,2.31)	(1.90,2.09)	3	(2.49,2.74)	(2.30,2.74)	(2.17,2.74)	(2.11,2.74)
5	(6.51,6.65)	(4.13,4.45)	(3.43,3.74)	(3.14,3.39)	5	(4.13,4.45)	(3.82,4.45)	(3.60,4.45)	(3.53,4.45)
7	(8.66,9.31)	(5.82,6.18)	(5.01,5.18)	(4.45,4.69)	7	(5.82,6.18)	(5.38,6.18)	(5.16,6.18)	(5.10,6.18)
9	(11.21,12.1)	(7.61,7.96)	(6.20,6.65)	(5.46,6.01)	9	(7.61,7.96)	(7.17,7.96)	(7.01,7.96)	(6.98,7.96)
	Value of remanufactured product a					Upper bound of product return $\bar{R}$			
	0.75	0.8	0.85	0.9		20	25	30	35
3	(2.70,2.74)	(2.60,2.74)	(2.49,2.74)	(2.38,2.74)	3	(2.50,2.74)	(2.49,2.74)	(2.49,2.74)	(2.47,2.74)
5	(4.40,4.45)	(4.31,4.45)	(4.13,4.45)	(3.97,4.45)	5	(4.21,4.45)	(4.15,4.45)	(4.13,4.45)	(4.10,4.45)
7	(6.14,6.18)	(6.04,6.18)	(5.82,6.18)	(5.60,6.18)	7	(6.13,6.18)	(5.88,6.18)	(5.82,6.18)	(5.61,6.18)
9	(7.92,7.96)	(7.90,7.96)	(7.61,7.96)	(7.33,7.96)	9	(7.79,7.96)	(7.67,7.96)	(7.61,7.96)	(7.47,7.96)

Table 1 summarizes the results for the benefit of MTO compared to MTS with and without remanufacturing. For example, when $\delta = 3$ and $h = 0.01$, the benefit of MTO with remanufacturing is 2.08% while that without remanufacturing is 2.24%. An observation from Table 1 is that introducing remanufacturing will lower the benefit of MTO, compare to the case without remanufacturing. This is mainly because the sales of remanufactured products will reduce the sales of new products, which decreases the proportion of profit incurred by selling new products. As a result, the benefit of implementing MTO strategy for the new product is reduced.

It is observed from Table 1 that the benefit of MTO increases with δ. Note that δ represents the variability of demand because the standard deviation of a uniform distribution over $[-\delta, \delta]$ is $\delta/\sqrt{3}$. When the variability of demand increases, the firm needs to pay more inventory holding and shortage costs in the MTS system because it is more difficult to match the demand. Therefore, the benefit of MTO increases. Similarly, when the inventory holding and shortage costs of the new product h and π increase, the benefit of MTO also increases because it can save more inventory holding and shortage costs. When the inventory holding and shortage costs of the remanufactured product h_0 and π_0 change, we find that their effects on the benefit of MTO are rather marginal, so they are not reported here.

Table 1 also shows that when the number of periods N increases, the benefit of MTO decreases. This is because, the benefit is measured by percentage increment as in (9). As the number of period N increases, the profit of the MTS system in the denominator increases faster than the profit difference between the MTO and MTS systems.

There are three other factors related to remanufacturing: the initial inventory level of remanufactured product x_0, the relative value of remanufactured product a, and the upper bound of product return $\bar{R}$. If any of these increases, the sales of remanufactured product tend to increase, which will decrease the sales of new product because the new and remanufactured products compete in the same market. This would reduce the proportion of the new product's profit in the system, which negatively impacts the benefit of MTO, as shown in Table 1.

3 Positive Lead Times

In this section, we extend the base model by including positive manufacturing and remanufacturing lead times. Specifically, suppose it takes $l > 0$ periods to produce a new product and $l_0 > 0$ periods to remanufacture a core. Let $\mathbf{u}_n = (u_{n0}, u_{n1}, \ldots, u_{n,l-1})$ and $\mathbf{x}_n = (x_{n0}, x_{n1}, \ldots, x_{nl_0})$ represent the inventory vectors of new product and work-in-process (WIP) of cores respectively at the beginning of period n, where u_{nj} is the amount of new products that will be ready to satisfy demand for the new product in period $n + j$, $j = 0, \ldots, l - 1$, and x_{ni} is the WIP of cores that will be finished in i periods, which can be used to satisfy demand for the remanufactured product in period $n + i$, $i = 0, 1, \ldots, l_0$. Note that x_{n0} is the on-hand inventory level of remanufactured product. Different from $\mathbf{x}_n$, $\mathbf{u}_n$ does not contain a term $u_{n,l}$ because manufacturing of new product starts at the beginning of each period while remanufacturing of cores starts at the end of each period after product returns in the period are received.

Make-To-Order System. When the manufacturing of new product follows a MTO strategy, new products are manufactured after demand is realized, thus demand in period t is satisfied in period $t + l$. In this case the state of the system is $\mathbf{x} = (x_0, x_1, \ldots, x_{l_0})$, the WIP of cores and on-hand inventory level of remanufactured product. The optimality equation of the dynamic program is

$$V_n(\mathbf{x}) = \max_{0 \leq \lambda_2 \leq 1} \Big\{ U_n(\lambda_2) - \mathsf{E}[L_0(x_0 - \lambda_2 d_n - \varepsilon_{2n})] + \gamma \mathsf{E}\left[V_{n+1}(\mathbf{x}_+)\right] \Big\}, \tag{10}$$

where $\mathbf{x}_+ = (x_0 - \lambda_2 d_n - \varepsilon_{2n} + x_1, \ldots, x_{l_0}, R_n)$, and $U_n(\lambda_2)$ is defined in (7).

To facilitate the analysis, we conduct the following change of variables. Let

$$\begin{aligned} x_i^c &= \sum_{j=0}^{i} x_j, \qquad 0 \le i \le l_0, \\ \mathbf{x}^c &= (x_0, x_1^c, \ldots, x_{l_0}^c). \end{aligned}$$

And define

$$V_n^c(\mathbf{x}^c) = V_n(x_0, x_1^c - x_0, \ldots, x_{l_0}^c - x_{l_0-1}^c),$$

then the dynamic program (10) can be rewritten as

$$V_n^c(\mathbf{x}^c) = \max_{0 \le \lambda_2 \le 1} \Big\{ U_n(\lambda_2) - \mathsf{E}[L_0(x_0 - \lambda_2 d_n - \varepsilon_{2n})] + \gamma \mathsf{E}\big[V_{n+1}^c(\mathbf{x}_+^c)\big] \Big\}, \tag{11}$$

where

$$\mathbf{x}_+^c = (x_1^c, \ldots, x_{l_0}^c, x_{l_0}^c + R_n) - (\lambda_2 d_n + \varepsilon_{2n}) \cdot \mathbf{1},$$

and $\mathbf{1}$ is the row vector of 1s.

To characterize the structure of the optimal policy, we use $L^\natural$-concavity. Recall that a function $f\colon \Lambda \to R$ is $L^\sharp-$concave on a sublattice Λ of R^n if the function $\varphi(\mathbf{w}, \xi) = f(\mathbf{w} - \xi\mathbf{1})$, $\xi \ge 0$, is supermodular on $\big\{(\mathbf{w}, \xi) | \mathbf{w} \in \Lambda, \xi \ge 0, \mathbf{w} - \xi\mathbf{1} \in \Lambda\big\}$. The concept of $L^\sharp-$concavity has been applied to analyze various inventory models, e.g., Zipkin (2008), Huh and Janakiraman (2011), and Pang et al. (2012). Lemma 3 presents some important properties of our problem under the transformed state $\mathbf{x}^c$.

Lemma 3. *(1) The value function $V_n^c(\boldsymbol{x}^c)$ is $L^\sharp-$concave in $\boldsymbol{x}^c$.*

(2) The optimal fraction of customers to purchase remanufactured products is increasing in $\boldsymbol{x}^c$ with bounded sensitivity. That is, the optimal solution of (11), denoted by $\lambda_{2n}(\boldsymbol{x}^c)$, is an increasing function, and it satisfies $\lambda_{2n}(\boldsymbol{x}^c + \xi\boldsymbol{1}) \le \lambda_{2n}(\boldsymbol{x}^c) + \xi/d_n$ for $\xi \ge 0$.

This result allows us to establish the following theorem that characterizes the optimal fractions of customers who purchase new and remanufactured products.

Theorem 5. *For each period $n = 1, 2, \ldots, N$, given the starting state $\boldsymbol{x}$, the optimal fractions of customers who purchase new and remanufactured products, $(\lambda_{1n}^*(\boldsymbol{x}), \lambda_{2n}^*(\boldsymbol{x}))$, have the following properties:*

(i) $\lambda_{1n}^*(\boldsymbol{x})$ is decreasing in $\boldsymbol{x}$ while both $\lambda_{2n}^*(\boldsymbol{x})$ and $\lambda_n^*(\boldsymbol{x}) = \lambda_{1n}^*(\boldsymbol{x}) + \lambda_{2n}^*(\boldsymbol{x})$ are increasing in $\boldsymbol{x}$. Furthermore, the expected ending inventory level of remanufactured product, $x_0 - \lambda_{2n}^*(\boldsymbol{x})d_n$, is increasing in x_0 but decreasing in x_i ($1 \le i \le l_0$).

(ii) $\lambda_{2n}^*(x_i + \xi, \boldsymbol{x}_{-i}) - \lambda_{2n}^*(\boldsymbol{x}) \ge \lambda_{2n}^*(x_{i+1} + \xi, \boldsymbol{x}_{-(i+1)}) - \lambda_{2n}^*(\boldsymbol{x})$, $\lambda_{1n}^*(\boldsymbol{x}) - \lambda_{1n}^*(x_i + \xi, \boldsymbol{x}_{-i}) \ge \lambda_{1n}^*(\boldsymbol{x}) - \lambda_{1n}^*(x_{i+1} + \xi, \boldsymbol{x}_{-(i+1)})$ and $\lambda_n^*(x_i + \xi, \boldsymbol{x}_{-i}) - \lambda_n^*(\boldsymbol{x}) \ge \lambda_n^*(x_{i+1} + \xi, \boldsymbol{x}_{-(i+1)}) - \lambda_n^*(\boldsymbol{x})$ for any $\xi \ge 0$, where $\boldsymbol{x}_{-i} = (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{l_0})$.

By Theorem 5(i), when the WIP of cores increase, the fraction of customers to purchase remanufactured product increases, but the fraction of customers to purchase new product decreases, while the total sales of new and remanufactured products increase. This is because, when the WIP of cores go up, the firm faces the pressure to sell more remanufactured products, and this will negatively affect the sales of new products, but total sales of new and remanufactured products still go up. In addition, when the inventory level of remanufactured product x_0 increases, more remanufactured products are expected to be carried over to the next period. Part (ii) of Theorem 5 further shows that the optimal market segmentations are more sensitive to the inventory level of remanufactured product than to the WIP of cores; and among the WIP of cores, they are more sensitive to those that are closer toward the end of the remanufacturing process.

Theorem 6. *The optimal selling price of remanufactured product, $p_{2n}^*(\boldsymbol{x})$, is decreasing in $\boldsymbol{x}$ and the difference between the selling prices of two products, $\Delta p_n^*(\boldsymbol{x}) = p_{1n}^*(\boldsymbol{x}) - p_{2n}^*(\boldsymbol{x})$, is increasing in $\boldsymbol{x}$. Furthermore, $p_{2n}^*(\boldsymbol{x}) - p_{2n}^*(x_i + \xi, \boldsymbol{x}_{-i}) \geq p_{2n}^*(\boldsymbol{x}) - p_{2n}^*(x_{i+1} + \xi, \boldsymbol{x}_{-(i+1)})$ and $\Delta p_n^*(x_i + \xi, \boldsymbol{x}_{-i}) - \Delta p_n^*(\boldsymbol{x}) \geq \Delta p_n^*(x_{i+1} + \xi, \boldsymbol{x}_{-(i+1)}) - \Delta p_n^*(\boldsymbol{x})$ for any $\xi \geq 0$.*

Therefore, the structure of the optimal selling prices resembles that of the optimal fractions of customers who buy new and remanufactured products: When the WIP of cores increase, the firm will cut its selling price of remanufactured product but to the extent that the price difference between new and remanufactured products still increases. Furthermore, this result shows that the optimal selling prices are more sensitive to the inventory level of remanufactured product than to the WIP of cores; and among the WIP of cores, they are more sensitive to those that are closer to finish remanufacturing.

Make-To-Stock System. We next consider the case where the firm employs MTS for manufacturing new product. In such a system, the manufacturing lead time of the new product affects the pricing and production decisions, and the firm needs to keep track of the pipeline inventories of new product. The dynamic program can be written as

$$\begin{aligned} V_n(\mathbf{u}, \mathbf{x}) \quad = \quad & \max_{(\lambda_1,\lambda_2)\in\Omega,\, q\geq 0} \Big\{ \Gamma_n(\lambda_1, \lambda_2) - \mathsf{E}[L(u_0 - \lambda_1 d_n - \varepsilon_{1n})] - \mathsf{E}[L_0(x_0 - \lambda_2 d_n - \varepsilon_{2n})] \\ & \qquad - c_1 q + \gamma \mathsf{E}\big[V_{n+1}(\mathbf{u}_+, \mathbf{x}_+)\big] \Big\}, \end{aligned}$$

where $\mathbf{u} = (u_0, u_1, \ldots, u_{l-1})$ includes the inventory level and pipeline inventories of new product, $\mathbf{u}_+ = (u_0 - \lambda_1 d_n - \varepsilon_{1n} + u_1, u_2, \ldots, u_{l-1}, q)$, and q is the production quantity in period n.

Define

$$u_i^c = \sum_{j=0}^{i} u_j \text{ for } 1 \leq i \leq l-1, \quad \mathbf{u}^c = (u_0, u_1^c, \ldots, u_{l-1}^c),$$

and $z = u_{l-1}^c + q$, where u_{l-1}^c is the inventory position of new product at the beginning of the period and z is the inventory position of new product after production decision. We also define

$$V_n^c(\mathbf{u}^c, \mathbf{x}) = V_n(u_0, u_1^c - u_0, \ldots, u_l^c - u_{l-1}^c, \mathbf{x}).$$

Making use of these, we can write the dynamic program as

$$V_n^c(\mathbf{u}^c, \mathbf{x}) = \max_{(\lambda_1,\lambda_2)\in\Omega,\, z\geq u_{l-1}^c} \Big\{\Gamma_n(\lambda_1,\lambda_2) - \mathsf{E}[L(u_0 - \lambda_1 d_n - \varepsilon_{1n})] - \mathsf{E}[L_0(x_0 - \lambda_2 d_n - \varepsilon_{2n})]$$
$$-c_1(z - u_{l-1}^c) + \gamma \mathsf{E}\big[V_{n+1}^c(\mathbf{u}_+^c, \mathbf{x}_+)\big]\Big\},$$

where

$$\mathbf{u}_+^c = (u_1^c - \lambda_1 d_n - \varepsilon_{1n}, \ldots, u_{l-1}^c - \lambda_1 d_n - \varepsilon_{1n}, z - \lambda_1 d_n - \varepsilon_{1n}).$$

The following result on the optimal policy for manufacturing new product extends Theorem 4 from zero lead time to positive manufacturing and remanufacturing lead times. We remark that the concavity of the objective function, the structure of optimal fractions of customers who purchase new and remanufactured products, as well as the optimal selling prices, can also be similarly determined.

Theorem 7. *For any given $(\boldsymbol{u}^c, \boldsymbol{x})$, the optimal manufacturing policy of the new product is of the base-stock type, i.e., there exists a base-stock level $z_n^0(\boldsymbol{u}^c, \boldsymbol{x})$ such that, if $u_{l-1}^c \leq z_n^0(\boldsymbol{u}^c, \boldsymbol{x})$, then manufacture to raise the inventory position of new product to $z_n^0(\boldsymbol{u}^c, \boldsymbol{x})$; otherwise, manufacture nothing.*

4 Effort-Dependent Product Return

In this section, we present another extension of the base model in which the product return is affected by the firm's acquisition effort $\widetilde{e}$. As uncertain and often insufficient product return is a major concern of many remanufacturers, they have tried to actively manage the process of core acquisition by providing incentives for customers to return their used products. For examples, Apple gives its customers gift cards if they trade in their used Apple products, e.g., iPhone. Here we assume the product return R_n in period n follows

$$R_n = \delta(\widetilde{e}) + \epsilon_n,$$

where $\delta(\widetilde{e}) \geq 0$ is an increasing, concave function of acquisition effort level $\widetilde{e}$, and ϵ_n, $n = 1, \ldots, N$, are independent nonnegative random variables across different periods. The effort exerted by the firm on core acquisition results in a cost of $\widetilde{g}(\widetilde{e})$, which is assumed to be an increasing, convex function of the effort. This effort-dependent product return model has been used by Zhou and Yu (2011).

For ease of exposition, we assume that the manufacturing lead time of the new product is zero and the remanufacturing lead time of the returned product is one. Note that all the results of this section can be extended to the model with general positive lead times. We first consider the case where the production of new product follows the MTO strategy.

At the beginning of period n, after observing the state $\mathbf{x} = (x_0, x_1)$, the firm decides (λ_1, λ_2) and $\widetilde{e}$ to maximize its expected profit. Since $\delta(\widetilde{e})$ is an increasing function, we denote $e = \delta(\widetilde{e})$ and $\widetilde{e} = \delta^{-1}(e)$, where δ^{-1} is the inverse function. It is easy to show that $\delta^{-1}(e)$ is an increasing and convex function of e because $\delta(\widetilde{e})$ is increasing and concave. Moreover, by defining $\mathbf{x}^c = (x_0, x_1^c)$ with $x_1^c = x_0 + x_1$ and $y = x_1^c + e$, the dynamic program can be equivalently written as

$$V_n^c(\mathbf{x}^c) = \max_{(\lambda_1,\lambda_2)\in\Omega, e\geq 0} \Big\{ \Gamma_n(\lambda_1, \lambda_2) - \mathsf{E}[L_0(x_0 - \lambda_2 d_n - \varepsilon_{2n})] - g(e) + \gamma \mathsf{E}\big[V_{n+1}^c(\mathbf{y}_+^c)\big] \Big\},$$

where $g(e) = c_2(e + \mathsf{E}[\epsilon_n]) + \widetilde{g}(\delta^{-1}(e))$ and

$$\mathbf{y}_+^c = (x_1^c - \lambda_2 d_n - \varepsilon_{2n}, y - \lambda_2 d_n - \varepsilon_{2n} + \epsilon_n).$$

Let

$$J_n(\mathbf{x}^c, y) = \max_{0\leq\lambda_2\leq 1} \Big\{ U_n(\lambda_2) - \mathsf{E}[L_0(x_0 - \lambda_2 d_n - \varepsilon_{2n})] + \gamma \mathsf{E}\big[V_{n+1}^c(\mathbf{y}_+^c)\big] \Big\},$$

where $U_n(\lambda_2)$ is defined in (7). We denote the optimal fraction of customers who purchase remanufactured products and optimal y by

$$\begin{aligned} \lambda_{2n}(\mathbf{x}^c, y) &= \arg\max_{0\leq\lambda_2\leq 1} \Big\{ U_n(\lambda_2) - \mathsf{E}[L_0(x_0 - \lambda_2 d_n - \varepsilon_{2n})] + \gamma \mathsf{E}\big[V_{n+1}^c(\mathbf{y}_+^c)\big] \Big\}, \\ y_n(\mathbf{x}^c) &= \arg\max_{y\geq x_1^c} \Big\{ J_n(\mathbf{x}^c, y) - g(y - x_1^c) \Big\}. \end{aligned} \tag{12}$$

We can show that $V_n^c(\mathbf{x}^c)$ is $L^\natural$-concave and the optimal solutions are monotone functions of the initial inventory states, which is presented in the following lemma.

Lemma 4. *(1) $V_n^c(\boldsymbol{x}^c)$ is $L^\natural-$concave in $\boldsymbol{x}^c$.*

(2) Both $\lambda_{2n}(\boldsymbol{x}^c, y)$ and $y_n(\boldsymbol{x}^c)$ are increasing functions, and $\lambda_{2n}(\boldsymbol{x}^c + \xi\boldsymbol{1}, y + \xi) \leq \lambda_{2n}(\boldsymbol{x}^c, y) + \xi/d_n$ and $y_n(\boldsymbol{x}^c + \xi\boldsymbol{1}) \leq y_n(\boldsymbol{x}^c) + \xi$ for $\xi \geq 0$.

Based on the results in Lemma 4, we can establish structural properties of the optimal fractions of customers to purchase new and remanufactured products and the optimal selling prices similar to those in Theorem 5 and Theorem 6. Details are omitted.

The following proposition shows that the optimal acquisition effort decreases with the WIP of cores, and the optimal effort is more sensitive to the WIP of cores than to the inventory level of remanufactured product. In contrast, Theorem 5 and Theorem 6 posit that the fractions of customers to purchase new and remanufactured products and the optimal selling prices are more sensitive to the inventory level of remanufactured product than to the WIP of cores. This is because, the optimal fractions to purchase the two products and optimal prices are set to consume remanufactured products while the acquisition effort is made to attract more returned cores.

Proposition 2. *For $n = 1, 2, \ldots, N$, given the starting state $\boldsymbol{x}$ of the system in period n, the optimal acquisition effort, $e_n^*(\boldsymbol{x})$, is decreasing in $\boldsymbol{x}$. Furthermore, $e_n^*(\boldsymbol{x}) - e_n^*(x_0 + \xi, x_1) \leq e_n^*(\boldsymbol{x}) - e_n^*(x_0, x_1 + \xi)$ for any $\xi \geq 0$.*

We next consider the scenario where the production of new product follows the MTS strategy. Except the acquisition effort, other notation remains the same as before, and the dynamic program can be written as

$$V_n(u, \mathbf{x}) = \max_{z \geq u,\, e \geq 0} \Big\{ H_n(z, \mathbf{x}, e) - c_1(z - u) - g(e) \Big\},$$

where

$$\begin{aligned} H_n(z, \mathbf{x}, e) \quad = \quad & \max_{(\lambda_1, \lambda_2) \in \Omega} \Big\{ \Gamma_n(\lambda_1, \lambda_2) - \mathsf{E}[L(z - \lambda_1 d_n - \varepsilon_{1n})] - \mathsf{E}[L_0(x_0 - \lambda_2 d_n - \varepsilon_{2n})] \\ & \qquad + \gamma \mathsf{E}\big[V_{n+1}(z - \lambda_1 d_n - \varepsilon_{1n}, x_0 + x_1 - \lambda_2 d_n - \varepsilon_{2n}, e + \epsilon_n)\big] \Big\}. \end{aligned}$$

The optimal production policy and acquisition effort for the MTS case are described in the following theorem. It shows that, the optimal production policy for new product is of the base-stock type. That is, if the new product inventory level is below the base-stock level, the firm should produce to raise the inventory level of new product to the base-stock level; otherwise, it should not manufacture any new product. Depending on whether the firm produces any new product, there exists a corresponding critical level that specifies the firm's optimal acquisition effort.

Theorem 8. *For any given $(u, \boldsymbol{x})$, the optimal manufacturing policy and acquisition effort level are determined by functions $(z_n^0(\boldsymbol{x}), e_n^0(\boldsymbol{x}))$, and $e_n(u, \boldsymbol{x})$, such that if $u \leq z_n^0(\boldsymbol{x})$, then produce new products to level $z_n^* = z_n^0(\boldsymbol{x})$ and set effort level $e_n^* = e_n^0(\boldsymbol{x})$; otherwise, $z_n^* = u$ and $e_n^* = e_n(u, \boldsymbol{x})$.*

5 Sales-Dependent Product Return

In this section, we extend the base model to a case with sales-dependent product returns, where the number of returned cores in each period is a random proportion of new products sold in the immediate previous period. This happens when the remanufactured product is made from the firm's own products only and the lifetime of the product is relatively short. This stylized sales-dependent return model will enable us to analyze the effect of sales-dependent returns on the optimal production and pricing strategies of the firm. This product return model has been adopted in Debo et al. (2005) and Ferrer and Swaminathan (2006, 2010), except that they assume a deterministic proportion of the sales in the immediately previous period will be returned.

We first consider the MTO system. Recall that the decision variables of the firm in each period are the fractions of customers to purchase new and remanufactured products (λ_1, λ_2). Denote the proportion of the previous demand that will be returned in period n by α_n, which is a random variable with support $[0,1]$ and $\mathsf{E}[\alpha_n]=\theta$. With a slight abuse of notation, the problem of the firm can be formulated as

$$V_n(x_0) = \max_{(\lambda_1,\lambda_2)\in\Omega}\Big\{\Gamma_n(\lambda_1,\lambda_2) - \mathsf{E}[L_0(x_0-\lambda_2 d_n-\varepsilon_{2n})] - c_2\mathsf{E}[\alpha_n(\lambda_1 d_n+\varepsilon_{1n})] + \gamma\mathsf{E}\big[V_{n+1}(x_0-\lambda_2 d_n-\varepsilon_{2n}+\alpha_n(\lambda_1 d_n+\varepsilon_{1n}))\big]\Big\}. \quad (13)$$

Note that the starting inventory level at period $n+1$ is $x_0-\lambda_2 d_n-\varepsilon_{2n}+\alpha_n(\lambda_1 d_n+\varepsilon_{1n})$ because the total amount of new products sold in period n is $\lambda_1 d_n+\varepsilon_{1n}$ and hence the amount of cores returned at the end of period n is $\alpha_n(\lambda_1 d_n+\varepsilon_{1n})$, i.e., $R_n=\alpha_n(\lambda_1 d_n+\varepsilon_{1n})$. For simplicity, we assume that $V_{N+1}(x_0)\equiv 0$, but the results and analysis can be easily extended to more general boundary conditions.

Similar to Theorems 1 and 2, the following result presents the monotonic properties of the optimal fractions of customers to purchase new and remanufactured products and the corresponding optimal prices for the two products.

Theorem 9. *For $n=1,2,\ldots,N$, given the starting state x_0 of period n, the optimal fractions of customers to purchase new and remanufactured products, $(\lambda^*_{1n}(x_0),\lambda^*_{2n}(x_0))$, and the optimal selling prices of the two products, $(p^*_{1n}(x_0),p^*_{2n}(x_0))$, have the following properties:*

(i) $\lambda^*_{1n}(x_0)$ is decreasing in x_0 while $\lambda^*_{2n}(x_0)$ is increasing in x_0. Furthermore, the expected carryover inventory of remanufactured product, $x_0-\lambda^*_{2n}(x_0)d_n$, to the next period is increasing in x_0.

(ii) The difference between the optimal prices of new and remanufactured products, $p^*_{1n}(x_0)-p^*_{2n}(x_0)$, is increasing in x_0.

The results in Theorem 9 are similar to those in Theorems 1 and 2. That is, under the optimal policy, when the inventory level of remanufactured product x_0 increases, the fraction of customers to purchase remanufactured product increases while the fraction of customers to purchase new product decreases, and the price difference between new and remanufactured products increases. It is interesting to point out that, in the case of sales-dependent product return, the optimal price of remanufactured product $p^*_{2n}(x_0)$ is not always decreasing in its initial inventory level x_0. This can be explained as follows. When the price of remanufactured product decreases, sales of new product decrease, which reduces the number of product returns in the next period and hence may not be beneficial to the firm. As the remanufactured product's price may not decrease, the total sales of new and remanufactured products $\lambda^*_{1n}(x_0)+\lambda^*_{2n}(x_0)$ may not increase in the inventory level x_0 of remanufactured product.

For the MTS system, similar to what we did in Section 2.2, we can show that the optimal value function is jointly concave. Therefore, the optimal production policy of new product is of base-stock type and the optimal fractions of customers to purchase new and remanufactured products (and their corresponding prices) can be determined by recursively solving the dynamic program. However, the base-stock level of new product does not possess monotonic property with respect to the initial inventory level of remanufactured product. This is because, in the sales-dependent product return model, selling more new products in the current period can increase product returns in the next period, which may be beneficial to the firm in the subsequent periods.

6 Conclusion

In this paper, we study the optimal pricing and manufacturing policies for a firm selling both new and remanufactured products over a finite planning horizon. Demand and product returns are random. The firm either employs a make-to-order or a make-to-stock strategy for its new product. When new products are made to order, we establish certain monotonicity properties of the optimal prices with respect to the inventory level of remanufactured product, and present insights why some other optimal prices fail to have any monotonicity property. When new products are made to stock, we establish the optimality of the base-stock type production policy. We further investigate the difference in profit values between the make-to-order and the make-to-stock strategies, and derive upper and lower bounds for that difference. Additional results are derived when the base model is extended to positive manufacturing and remanufacturing lead times, effort-dependent product return, and sales-dependent product return.

There are several directions for further research. First, in this paper we have focused our discussions on an additive demand model (i.e., the price only affects the location parameter of the demand distribution). This model applies to products whose demand uncertainties come mainly from forecast errors (Agrawal and Seshadri 2000). For some other cases, a multiplicative demand model (i.e., the price affects the scale parameter of the demand distribution) or a more general demand model may be more suitable. The optimal policy may become complicated and exhibit nonintuitive structure, and it will be interesting to develop simple but near optimal heuristic policy for the problem. Second, in this paper we provide qualitative insights into the effects of the remanufacturing decision on the firm's dynamic pricing and inventory strategies. We assume that the firm knows how customers value a remanufactured product relative to a new product. Estimating the demand distributions from real sales data or consumer surveys is another important research direction. Third, in this paper the new and remanufactured products are produced and sold by a monopoly firm. Allowing multiple firms to complete in the same market, and analyzing how the competition affects firms' optimal price and inventory decisions may also result in interesting insights.

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Appendix

In this Appendix, we provide the mathematical proofs of all the results.

Proof of Lemma 1.

When $p_2 \geq \eta(p_1)$, $\eta(v) \geq p_2$ implies $\eta(v) \geq \eta(p_1)$ and hence $v \geq p_1$ because $\eta(\cdot)$ is a strictly increasing function. At the same time, $v - \eta(v) < p_1 - p_2 \leq p_1 - \eta(p_1)$, which implies that $v < p_1$ because $v - \eta(v)$ is a strictly increasing function. These arguments show that if $p_2 \geq \eta(p_1)$ then $\lambda_2(p_1, p_2) = 0$, i.e., no customer will buy the remanufactured product in this case. Moreover, when $p_2 \geq \eta(p_1)$, $v \geq p_1$ implies that $v - \eta(v) \geq p_1 - \eta(p_1) \geq p_1 - p_2$, hence it follows from (1) that $\lambda_1(p_1, p_2) = \mathsf{P}(v \geq p_1) = 1 - F(p_1)$. If $p_2 < \eta(p_1)$, then $v - \eta(v) > p_1 - p_2 \geq p_1 - \eta(p_1)$, which implies that $v \geq p_1$. Therefore, (1) is reduced to $\lambda_1(p_1, p_2) = \mathsf{P}\big(v - \eta(v) \geq p_1 - p_2\big)$. The analysis above shows that to find the optimal prices, it is sufficient to focus on the range $p_2 \leq \eta(p_1)$, as $p_2 > \eta(p_1)$ is captured by $p_2 = \eta(p_1)$.

Since $\eta(v)$ and $v - \eta(v)$ are both strictly increasing in v, we define v_h and v_l such that $v_h - \eta(v_h) = p_1 - p_2$ and $\eta(v_l) = p_2$. Then $p_2 \leq \eta(p_1)$ implies that $v_l \leq p_1$ and $v_h - \eta(v_h) \geq p_1 - \eta(p_1)$. Therefore $v_h \geq p_1 \geq v_l$ and (1) and (2) can be simplified to

$$\lambda_1(p_1, p_2) = 1 - F(v_h), \tag{14}$$
$$\lambda_2(p_1, p_2) = F(v_h) - F(v_l), \tag{15}$$

whenever $p_2 \leq \eta(p_1)$. From (14) and (15), the price decisions can be written as the functions of the fractions of customers that purchase new and remanufactured products, given by

$$\begin{aligned} p_1(\lambda_1, \lambda_2) &= \eta\big(F^{-1}(1 - \lambda_1 - \lambda_2)\big) + F^{-1}(1 - \lambda_1) - \eta\big(F^{-1}(1 - \lambda_1)\big), \\ p_2(\lambda_1, \lambda_2) &= \eta\big(F^{-1}(1 - \lambda_1 - \lambda_2)\big), \end{aligned}$$

where $(\lambda_1, \lambda_2) \in \Omega = \big\{(\lambda_1, \lambda_2) : 0 \leq \lambda_1 \leq 1, 0 \leq \lambda_2 \leq 1, 0 \leq \lambda_1 + \lambda_2 \leq 1\big\}$. □

Proof of Lemma 2.

Note that the concavity of $G_2(x)$ implies that $G_2(\lambda_2 - u)$ is supermodular in (u, λ_2), which indicates that $-\lambda_1(\lambda_2)$ is increasing in λ_2 because

$$-\lambda_1(\lambda_2) = \arg\max_{\lambda_2 - 1 \leq u \leq 0} \Big\{G_2(\lambda_2 - u) + c_1 u + G_1(-u)\Big\}$$

and $\big\{(u, \lambda_2) : \lambda_2 - 1 \leq u \leq 0\big\}$ is a lattice. Hence, $\lambda_1(\lambda_2)$ is decreasing in λ_2.

Because $\lambda_1(\lambda_2)$ can be written as

$$\lambda_1(\lambda_2) = \arg\max_{\lambda_2 \leq \lambda_1 + \lambda_2 \leq 1} \Big\{G_2(\lambda_1 + \lambda_2) - c_1(\lambda_1 + \lambda_2) + G_1(\lambda_1 + \lambda_2 - \lambda_2) + c_1 \lambda_2\Big\},$$

we have

$$\lambda_2 + \lambda_1(\lambda_2) = \arg\max_{\lambda_2 \leq z \leq 1} \Big\{G_2(z) - c_1 z + G_1(z - \lambda_2) + c_1 \lambda_2\Big\}.$$

Note that $G_1(z - \lambda_2)$ is supermodular in (z, λ_2) and $\big\{(z, \lambda_2) : \lambda_2 \leq z \leq 1\big\}$ is a lattice. Therefore, $\lambda_2 + \lambda_1(\lambda_2)$ is increasing in λ_2. □

Proofs of Theorems 1 and 2.

The proofs are special cases of Theorem 5 and Theorem 6 respectively.

Proof of Theorem 3.

(i) In this proof, we denote the profit-to-go function by $V_n(x_0, h_0)$, and the optimal fractions of customers who buy new and remanufactured products by $\lambda^*_{1n}(x_0, h_0)$ and $\lambda^*_{2n}(x_0, h_0)$, where

$$\begin{aligned} V_n(x_0, h_0) &= \max_{0\le\lambda_2\le 1}\Big\{U_n(\lambda_2) - h_0\mathsf{E}[(x_0-\lambda_2 d_n-\varepsilon_{2n})^+] - \pi_0\mathsf{E}[(x_0-\lambda_2 d_n-\varepsilon_{2n})^-] \\ &\qquad +\gamma\mathsf{E}\left[V_{n+1}(x_0-\lambda_2 d_n-\varepsilon_{2n}+R_n, h_0)\right]\Big\}, \qquad (16) \\ \lambda^*_{2n}(x_0, h_0) &= \arg\max_{0\le\lambda_2\le 1}\Big\{U_n(\lambda_2) - h_0\mathsf{E}[(x_0-\lambda_2 d_n-\varepsilon_{2n})^+] - \pi_0\mathsf{E}[(x_0-\lambda_2 d_n-\varepsilon_{2n})^-] \\ &\qquad +\gamma\mathsf{E}\left[V_{n+1}(x_0-\lambda_2 d_n-\varepsilon_{2n}+R_n, h_0)\right]\Big\}. \qquad (17) \end{aligned}$$

Assumption 1 implies that $\Gamma_n(\lambda_1, \lambda_2)$ is jointly concave in (λ_1, λ_2), which implies that $U_n(\lambda_2)$ is concave in λ_2, where $\Gamma_n(\lambda_1, \lambda_2)$ and $U_n(\lambda_2)$ are defined in (5) and (7), respectively. Therefore, by induction, it is straightforward to show that $V_n(x_0, h_0)$ is concave in x_0 for all given h_0 and n. Let $\varsigma = x_0 - \lambda_2 D_n$. The dynamic program (16) can be written as

$$\begin{aligned} V_n(x_0, h_0) &= \max_{x_0-d_n\le\varsigma\le x_0}\Big\{U_n\Big(\frac{x_0-\varsigma}{d_n}\Big) - h_0\mathsf{E}[(\varsigma-\varepsilon_{2n})^+] - \pi_0\mathsf{E}[(\varsigma-\varepsilon_{2n})^-] \\ &\qquad +\gamma\mathsf{E}\left[V_{n+1}(\varsigma-\varepsilon_{2n}+R_n, h_0)\right]\Big\}. \qquad (18) \end{aligned}$$

We first show that if $V_{n+1}(x_0, h_0)$ is submodular in (x_0, h_0), then $V_n(x_0, h_0)$ is also submodular in (x_0, h_0). For ease of analysis, we denote $\hat{h}_0 = -h_0$ and $\hat{V}_n(x_0, \hat{h}_0) = V_n(x_0, -\hat{h}_0)$ for all n. As $V_{n+1}(x_0, h_0)$ is submodular in (x_0, h_0), $\hat{V}_{n+1}(x_0, \hat{h}_0)$ is supermodular in $(x_0, \hat{h}_0)$. Note that the dynamic program (18) can be rewritten as

$$\begin{aligned} \hat{V}_n(x_0, \hat{h}_0) &= \max_{x_0-d_n\le\varsigma\le x_0}\Big\{U_n\Big(\frac{x_0-\varsigma}{d_n}\Big) + \hat{h}_0\mathsf{E}[(\varsigma-\varepsilon_{2n})^+] - \pi_0\mathsf{E}[(\varsigma-\varepsilon_{2n})^-] \\ &\qquad +\gamma\mathsf{E}\big[\hat{V}_{n+1}(\varsigma-\varepsilon_{2n}+R_n, \hat{h}_0)\big]\Big\}. \end{aligned}$$

As $U_n(\lambda_2)$ is concave in λ_2, $\mathsf{E}[(\varsigma-\varepsilon_{2n})^+]$ is increasing in ς and $\hat{V}_{n+1}(x_0, \hat{h}_0)$ is supermodular in $(x_0, \hat{h}_0)$, it is derived that $U_n\big(\frac{x_0-\varsigma}{d_n}\big)$ is supermodular in (x_0, ς) (Theorem 2.3.6 (b) in Simchi-Levi et al., 2005), $\hat{h}_0\mathsf{E}[(\varsigma-\varepsilon_{2n})^+]$ is supermodular in $(\hat{h}_0, \varsigma)$, and $\mathsf{E}[\hat{V}_{n+1}(\varsigma-\varepsilon_{2n}+R_n, \hat{h}_0)]$ is supermodular in $(\hat{h}_0, \varsigma)$ (Proposition 2.3.5 (d) in Simchi-Levi et al., 2005). Therefore,

$$U_n\Big(\frac{x_0-\varsigma}{d_n}\Big) + \hat{h}_0\mathsf{E}[(\varsigma-\varepsilon_{2n})^+] - \pi_0\mathsf{E}[(\varsigma-\varepsilon_{2n})^-] + \gamma\mathsf{E}\big[\hat{V}_{n+1}(\varsigma-\varepsilon_{2n}+R_n, \hat{h}_0)\big]$$

is supermodular in $(x_0, \varsigma, \hat{h}_0)$ (Proposition 2.3.5 (a) in Simchi-Levi et al., 2005). Note that $\{(x_0, \varsigma) : x_0 - d_n \le \varsigma \le x_0\}$ is a lattice. Hence, $\hat{V}_n(x_0, \hat{h}_0)$ is supermodular in $(x_0, \hat{h}_0)$ (Proposition 2.3.5 (e) in Simchi-Levi et al., 2005) and $\varsigma_n(\hat{h}_0)$ is increasing in $\hat{h}_0$ (Theorem 2.3.7 in Simchi-Levi et al., 2005), where

$$\varsigma_n(\hat{h}_0) = \arg\max_{x_0-d_n\le\varsigma\le x_0}\Big\{U_n\Big(\frac{x_0-\varsigma}{d_n}\Big) + \hat{h}_0\mathsf{E}[(\varsigma-\varepsilon_{2n})^+] - \pi_0\mathsf{E}[(\varsigma-\varepsilon_{2n})^-]$$

$$+\gamma \mathsf{E}\big[\hat{V}_{n+1}(\varsigma - \varepsilon_{2n} + R_n, \hat{h}_0)\big]\Big\}. \tag{19}$$

Recall that $V_n(x_0, h_0) = \hat{V}_n(x_0, \hat{h}_0) = \hat{V}_n(x_0, -h_0)$. Thus, $V_n(x_0, h_0)$ is submodular in (x_0, h_0).

We next prove that for given x_0, $\lambda^*_{1n}(x_0, h_0)$ decreases with h_0, while $\lambda^*_{2n}(x_0, h_0)$ and $\lambda^*_n(x_0, h_0) = \lambda^*_{1n}(x_0, h_0) + \lambda^*_{2n}(x_0, h_0)$ increase with h_0, where $\lambda^*_{1n}(x_0, h_0) = \lambda_1(\lambda^*_{2n}(x_0, h_0))$ and $\lambda_1(\lambda_2)$ is defined in (7). As $\varsigma = x_0 - \lambda_2 d_n$, Equations (17) and (19) imply that $\lambda^*_{2n}(x_0, h_0) = \frac{x_0 - \varsigma_n(\hat{h}_0)}{d_n}$ is increasing in h_0 because $\varsigma_n(\hat{h}_0)$ is increasing in $\hat{h}_0$. Lemma 2 has shown that $\lambda_1(\lambda_2)$ is decreasing in λ_2 and $\lambda_2 + \lambda_1(\lambda_2)$ is increasing in λ_2. Therefore, we obtain from (6) that $\lambda^*_{1n}(x_0, h_0) = \lambda_1(\lambda^*_{2n}(x_0, h_0))$ is decreasing in h_0 and $\lambda^*_n(x_0, h_0) = \lambda^*_{1n}(x_0, h_0) + \lambda^*_{2n}(x_0, h_0)$ is increasing in h_0.

Similarly, we can prove the monotonicity of the optimal fractions of customers to purchase new and remanufactured products with respect to the unit shortage cost of remanufactured product π_0. We denote the profit-to-go function by $V_n(x_0, \pi_0)$, and the optimal fractions of customers that purchase new and remanufactured products by $\lambda^*_{1n}(x_0, \pi_0)$ and $\lambda^*_{2n}(x_0, \pi_0)$, where

$$\begin{aligned} V_n(x_0, \pi_0) &= \max_{0 \le \lambda_2 \le 1} \Big\{ U_n(\lambda_2) - h_0 \mathsf{E}[(x_0 - \lambda_2 d_n - \varepsilon_{2n})^+] - \pi_0 \mathsf{E}[(x_0 - \lambda_2 d_n - \varepsilon_{2n})^-] \\ &\qquad + \gamma \mathsf{E}\left[V_{n+1}(x_0 - \lambda_2 d_n - \varepsilon_{2n} + R_n, \pi_0)\right] \Big\}, \\ \lambda^*_{2n}(x_0, \pi_0) &= \arg\max_{0 \le \lambda_2 \le 1} \Big\{ U_n(\lambda_2) - h_0 \mathsf{E}[(x_0 - \lambda_2 d_n - \varepsilon_{2n})^+] - \pi_0 \mathsf{E}[(x_0 - \lambda_2 d_n - \varepsilon_{2n})^-] \\ &\qquad + \gamma \mathsf{E}\left[V_{n+1}(x_0 - \lambda_2 d_n - \varepsilon_{2n} + R_n, \pi_0)\right] \Big\}. \end{aligned}$$

Let $\varsigma = x_0 - \lambda_2 d_n$. By the same argument, we can prove that $V_n(x_0, \pi_0)$ is supermodular in (x_0, π_0) for all n, and for given x_0, $\lambda^*_{1n}(x_0, \pi_0) = \lambda_1(\lambda^*_{2n}(x_0, \pi_0))$ increases with π_0, while $\lambda^*_{2n}(x_0, \pi_0)$ and $\lambda^*_n(x_0, \pi_0) = \lambda^*_{1n}(x_0, \pi_0) + \lambda^*_{2n}(x_0, \pi_0)$ decrease with π_0. We omit the proof because it is the same as above.

(ii) Equations (3) and (4) imply that

$$\begin{aligned} p^*_{1n}(x_0) &= \eta\big(F^{-1}(1 - \lambda^*_{1n}(x_0) - \lambda^*_{2n}(x_0))\big) + F^{-1}(1 - \lambda^*_{1n}(x_0)) - \eta\big(F^{-1}(1 - \lambda^*_{1n}(x_0))\big), \\ p^*_{2n}(x_0) &= \eta\big(F^{-1}(1 - \lambda^*_{1n}(x_0) - \lambda^*_{2n}(x_0))\big). \end{aligned}$$

In part (i), we have shown that $\lambda^*_{1n}(x_0)$ decreases with h_0, while $\lambda^*_n(x_0)$ increases with h_0. Therefore, $p^*_{2n}(x_0)$ decreases with h_0. As $\Delta p^*_n(x_0) = p^*_{1n}(x_0) - p^*_{2n}(x_0) = F^{-1}(1 - \lambda^*_{1n}(x_0)) - \eta(F^{-1}(1 - \lambda^*_{1n}(x_0)))$ and $x - \eta(x)$ is increasing in x, we derive that $\Delta p^*_n(x_0)$ increases with h_0.

We have also shown in part (i) that $\lambda^*_{1n}(x_0)$ increases with π_0, while $\lambda^*_n(x_0)$ decreases with π_0. Hence, $p^*_{2n}(x_0)$ increases with π_0 and $\Delta p^*_n(x_0)$ decreases with π_0. □

Proof of Theorem 4.

Note that Ω is a convex set, and $\Gamma_n(\lambda_1, \lambda_2)$ is jointly concave in (λ_1, λ_2). Then by mathematical induction on n, we can easily show that $H_n(z, x_0)$ and $V_n(u, x_0)$ are both concave functions. Hence, $H_n(z, x_0) - c_1 z$ is concave in z for any given x_0. Define

$$z^0_n(x_0) = \arg\max_z \big\{H_n(z, x_0)\big\}, \tag{20}$$

which is the global maximizer of the objective function. Therefore, if $u \le z^0_n(x_0)$, then $z^*_n = z^0_n(x_0)$; and if $u > z^0_n(x_0)$, then $z^*_n = u$. Moreover, the optimal segmentations of customers, $(\lambda^*_{1n}, \lambda^*_{2n})$, are determined by

$$\begin{aligned}(\lambda_{1n}^*, \lambda_{2n}^*) &= \arg\max_{(\lambda_1,\lambda_2)\in\Omega} \Big\{ \Gamma_n(\lambda_1,\lambda_2) - c_1(z_n^* - \lambda_1 d_n) - \mathsf{E}[L(z_n^* - \lambda_1 d_n - \varepsilon_{1n})] \\ &\quad - \mathsf{E}[L_0(x_0 - \lambda_2 d_n - \varepsilon_{2n})] + \gamma \mathsf{E}\big[V_{n+1}(z_n^* - \lambda_1 d_n - \varepsilon_{1n}, x_0 - \lambda_2 d_n - \varepsilon_{2n} + R_n)\big] \Big\}.\end{aligned}$$

We now show that $z_n^*(x_0)$ is decreasing in x_0.

Note that the dynamic program is described as

$$\begin{aligned}V_n(u,x_0) &= \max_{(\lambda_1,\lambda_2)\in\Omega,\, z\geq u} \Big\{ \Gamma_n(\lambda_1,\lambda_2) - c_1(z-u) - \mathsf{E}[L(z-\lambda_1 d_n - \varepsilon_{1n})] - \mathsf{E}[L_0(x_0 - \lambda_2 d_n - \varepsilon_{2n})] \\ &\quad + \gamma \mathsf{E}\big[V_{n+1}(z - \lambda_1 d_n - \varepsilon_{1n}, x_0 - \lambda_2 d_n - \varepsilon_{2n} + R_n)\big] \Big\}.\end{aligned}$$

Let $\overline{x}_0 = -x_0$, $\overline{\lambda}_2 = -\lambda_2$ and $\overline{V}_n(u,\overline{x}_0) = V_n(u, -\overline{x}_0)$. The dynamic program can be written as

$$\begin{aligned}\overline{V}_n(u,\overline{x}_0) &= \max_{(\lambda_1,\overline{\lambda}_2)\in\overline{\Omega},\, z\geq u} \Big\{ \Gamma_n(\lambda_1,-\overline{\lambda}_2) - c_1(z-u) - \mathsf{E}[L(z-\lambda_1 d_n - \varepsilon_{1n})] - \mathsf{E}[\overline{L}_0(\overline{x}_0 - \overline{\lambda}_2 d_n + \varepsilon_{2n})] \\ &\quad + \gamma \mathsf{E}\big[\overline{V}_{n+1}(z - \lambda_1 d_n - \varepsilon_{1n}, \overline{x}_0 - \overline{\lambda}_2 d_n + \varepsilon_{2n} - R_n)\big] \Big\},\end{aligned}$$

where $\overline{\Omega} = \{(\lambda_1,\overline{\lambda}_2) : (\lambda_1,\lambda_2)\in\Omega\}$ and $\overline{L}_0(x) = L_0(-x)$. Denote

$$\begin{aligned}\overline{H}_n(z,\overline{x}_0) &= \max_{(\lambda_1,\overline{\lambda}_2)\in\overline{\Omega}} \Big\{ \Gamma_n(\lambda_1,-\overline{\lambda}_2) - \mathsf{E}[L(z-\lambda_1 d_n - \varepsilon_{1n})] - \mathsf{E}[\overline{L}_0(\overline{x}_0 - \overline{\lambda}_2 d_n + \varepsilon_{2n})] \\ &\quad + \gamma \mathsf{E}\big[\overline{V}_{n+1}(z - \lambda_1 d_n - \varepsilon_{1n}, \overline{x}_0 - \overline{\lambda}_2 d_n + \varepsilon_{2n} - R_n)\big] \Big\}. \qquad (21)\end{aligned}$$

First, we show that if $\overline{V}_{n+1}(u,\overline{x}_0)$ is supermodular in $(u,\overline{x}_0)$, then $\overline{H}_n(z,\overline{x}_0)$ is supermodular in $(z,\overline{x}_0)$. Let $\widehat{t}_1 = z - \lambda_1 d_n$ and $\widehat{t}_2 = \overline{x}_0 - \overline{\lambda}_2 d_n$. Equation (21) is written as

$$\begin{aligned}\overline{H}_n(z,\overline{x}_0) &= \max_{(\lambda_1,\overline{\lambda}_2)\in\overline{\Omega},\, \widehat{t}_1 = z-\lambda_1 d_n,\, \widehat{t}_2 = \overline{x}_0 - \overline{\lambda}_2 d_n} \Big\{ \Gamma_n(\lambda_1,-\overline{\lambda}_2) - \mathsf{E}[L(\widehat{t}_1 - \varepsilon_{1n})] - \mathsf{E}[\overline{L}_0(\widehat{t}_2 + \varepsilon_{2n})] \\ &\quad + \gamma \mathsf{E}\big[\overline{V}_{n+1}(\widehat{t}_1 - \varepsilon_{1n}, \widehat{t}_2 + \varepsilon_{2n} - R_n)\big] \Big\}.\end{aligned}$$

Note that $\overline{\Omega}$ is a lattice, and $\Gamma_n(\lambda_1,-\overline{\lambda}_2)$ is supermodular in $(\lambda_1,\overline{\lambda}_2)$. Moreover, $\widehat{t}_1 = z - \lambda_1 d_n$ and $\widehat{t}_2 = \overline{x}_0 - \overline{\lambda}_2 d_n$ can be written as

$$\begin{pmatrix} d_n & 1 & 0 & 0 \\ 0 & 0 & d_n & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \widehat{t}_1 \\ \overline{\lambda}_2 \\ \widehat{t}_2 \end{pmatrix} = \begin{pmatrix} z \\ \overline{x}_0 \end{pmatrix}.$$

Therefore, $\overline{H}_n(z,\overline{x}_0)$ is supermodular in $(z,\overline{x}_0)$ by Theorem 1 in Chen et al. (2013). Recall that

$$\overline{V}_n(u,\overline{x}_0) = \max_{z\geq u} \big\{ \overline{H}_n(z,\overline{x}_0) - c_1(z-u) \big\}.$$

Therefore, $\overline{V}_n(u,\overline{x}_0)$ is supermodular in $(u,\overline{x}_0)$, and $z_n^0(x_0)$ is decreasing in x_0. □

Proof of Proposition 1.

Define $W_n^s(u,x) = V_n^s(u,x) - c_1 u$. We first show that for any given x, $-(1-\gamma)c_1 u - hu^+ - \pi u^- + \gamma W_{n+1}^s(u,x)$ is increasing in u when $u \le 0$ and decreasing in u when $u > 0$. We prove this by induction. When $n = N$,

$$\begin{aligned} & -(1-\gamma)c_1 u - hu^+ - \pi u^- + \gamma W_{N+1}^s(u,x) \\ = & -(1-\gamma)c_1 u - hu^+ - \pi u^- - \gamma c_1 u \\ = & -(c_1+h)u^+ - (\pi - c_1)u^-. \end{aligned}$$

Because $\pi > c_1$, $-(1-\gamma)c_1 u - hu^+ - \pi u^- + \gamma W_{N+1}^s(u,x)$ is increasing in u when $u \le 0$ and decreasing in u when $u > 0$.

Suppose that the result holds for $n+1$, i.e., for any given x, $-(1-\gamma)c_1 u - hu^+ - \pi u^- + \gamma W_{n+1}^s(u,x)$ is increasing in u when $u \le 0$ and decreasing in u when $u > 0$. We next show that the result holds for n. To this end, we first need to show that $z_n^0(x_0) \ge 0$. Recall that $z_n^0(x_0)$ is defined as (equivalent to (20))

$$\begin{aligned} & \big(z_n^0(x_0), \lambda_{1n}^0(x_0), \lambda_{2n}^0(x_0)\big) \\ = & \operatorname*{arg\,max}_{z,\,(\lambda_1,\lambda_2)\in\Omega} \Big\{ \Gamma_n(\lambda_1,\lambda_2) - (1-\gamma)c_1(z - \lambda_1 d_n) - \mathsf{E}[L(z - \lambda_1 d_n - \varepsilon_{1n})] - \mathsf{E}[L_0(x_0 - \lambda_2 d_n - \varepsilon_{2n})] \\ & \qquad + \gamma \mathsf{E}\big[W_{n+1}^s(z - \lambda_1 d_n - \varepsilon_{1n}, x_0 - \lambda_2 d_n - \varepsilon_{2n} + R_n)\big] \Big\}. \end{aligned} \tag{22}$$

We prove this by contradiction. Suppose that $z_n^0(x_0) < 0$ for some x_0. We next show that $\big(0, \lambda_{1n}^0(x_0), \lambda_{2n}^0(x_0)\big)$ is better than $\big(z_n^0(x_0), \lambda_{1n}^0(x_0), \lambda_{2n}^0(x_0)\big)$, i.e.,

$$\begin{aligned} & \Gamma_n(\lambda_{1n}^0(x_0), \lambda_{2n}^0(x_0)) - (1-\gamma)c_1(-\lambda_{1n}^0(x_0)d_n) - \mathsf{E}[L(-\lambda_{1n}^0(x_0)d_n - \varepsilon_{1n})] \\ & -\mathsf{E}[L_0(x_0 - \lambda_{2n}^0(x_0)d_n - \varepsilon_{2n})] + \gamma\mathsf{E}\big[W_{n+1}^s(0 - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}, x_0 - \lambda_{2n}^0(x_0)d_n - \varepsilon_{2n} + R_n)\big] \\ \ge & \Gamma_n(\lambda_{1n}^0(x_0), \lambda_{2n}^0(x_0)) - (1-\gamma)c_1(z_n^0(x_0) - \lambda_{1n}^0(x_0)d_n) - \mathsf{E}[L(z_n^0(x_0) - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n})] \\ & -\mathsf{E}[L_0(x_0 - \lambda_{2n}^0(x_0)d_n - \varepsilon_{2n})] + \gamma\mathsf{E}\big[W_{n+1}^s(z_n^0(x_0) - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}, x_0 - \lambda_{2n}^0(x_0)d_n - \varepsilon_{2n} + R_n)\big]. \end{aligned}$$

Recall that $L(x) = hx^+ + \pi x^-$ and $\mathsf{E}[\varepsilon_{1n}] = 0$. After some simplifications, the above expression is equivalent to

$$\begin{aligned} & -(1-\gamma)c_1\mathsf{E}[0 - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}] - h\mathsf{E}[0 - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}]^+ - \pi\mathsf{E}[0 - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}]^- \\ & +\gamma\mathsf{E}\big[W_{n+1}^s(0 - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}, x_0 - \lambda_{2n}^0(x_0)d_n - \varepsilon_{2n} + R_n)\big] \\ \ge & -(1-\gamma)c_1\mathsf{E}[z_n^0(x_0) - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}] - h\mathsf{E}[z_n^0(x_0) - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}]^+ \\ & -\pi\mathsf{E}[z_n^0(x_0) - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}]^- \\ & +\gamma\mathsf{E}\big[W_{n+1}^s(z_n^0(x_0) - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}, x_0 - \lambda_{2n}^0(x_0)d_n - \varepsilon_{2n} + R_n)\big]. \end{aligned} \tag{23}$$

As $\lambda_{1n}^0(x_0)d_n + \varepsilon_{1n} \ge 0$ and $z_n^0(x_0) < 0$, we have $z_n^0(x_0) - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n} < 0$. Recall that for any given x, $-(1-\gamma)c_1 u - hu^+ - \pi u^- + \gamma W_{n+1}^s(u,x)$ is increasing in u when $u \le 0$ and decreasing in u when $u > 0$. We obtain

$$\begin{aligned} & -(1-\gamma)c_1[0 - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}] - h[0 - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}]^+ - \pi[0 - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}]^- \\ & +\gamma W_{n+1}^s(0 - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}, x_0 - \lambda_{2n}^0(x_0)d_n - \varepsilon_{2n} + R_n) \\ \ge & -(1-\gamma)c_1[z_n^0(x_0) - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}] - h[z_n^0(x_0) - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}]^+ \end{aligned}$$

$$-\pi[z_n^0(x_0) - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}]^-$$
$$+\gamma W_{n+1}^s(z_n^0(x_0) - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}, x_0 - \lambda_{2n}^0(x_0)d_n - \varepsilon_{2n} + R_n).$$

Taking expectation on both sides of above expression, we have

$$\begin{aligned}
& -(1-\gamma)c_1\mathsf{E}[0 - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}] - h\mathsf{E}[0 - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}]^+ - \pi\mathsf{E}[0 - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}]^- \\
& +\gamma\mathsf{E}\big[W_{n+1}^s(0 - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}, x_0 - \lambda_{2n}^0(x_0)d_n - \varepsilon_{2n} + R_n)\big] \\
\geq\ & -(1-\gamma)c_1\mathsf{E}[z_n^0(x_0) - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}] - h\mathsf{E}[z_n^0(x_0) - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}]^+ \\
& -\pi\mathsf{E}[z_n^0(x_0) - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}]^- \\
& +\gamma\mathsf{E}\big[W_{n+1}^s(z_n^0(x_0) - \lambda_{1n}^0(x_0)d_n - \varepsilon_{1n}, x_0 - \lambda_{2n}^0(x_0)d_n - \varepsilon_{2n} + R_n)\big],
\end{aligned}$$

which is (23). Hence, $\big(0, \lambda_{1n}^0(x_0), \lambda_{2n}^0(x_0)\big)$ is better than $\big(z_n^0(x_0), \lambda_{1n}^0(x_0), \lambda_{2n}^0(x_0)\big)$, which contradicts to the optimality of $\big(z_n^0(x_0), \lambda_{1n}^0(x_0), \lambda_{2n}^0(x_0)\big)$. Therefore, $z_n^0(x_0) \geq 0$ because the objective function of (22) is concave.

Equation (8) implies that the dynamic program of the make-to-stock system can be equivalently written as $W_n^s(u, x_0) = \max_{z\geq u}\big\{H_n^s(z, x_0)\big\}$, where

$$\begin{aligned}
& H_n^s(z, x_0) \\
=\ & \max_{(\lambda_1,\lambda_2)\in\Omega}\Big\{\Gamma_n(\lambda_1,\lambda_2) - (1-\gamma)c_1(z - \lambda_1 d_n) - \mathsf{E}[L(z - \lambda_1 d_n - \varepsilon_{1n})] - \mathsf{E}[L_0(x_0 - \lambda_2 d_n - \varepsilon_{2n})] \\
& +\gamma\mathsf{E}\big[W_{n+1}^s(z - \lambda_1 d_n - \varepsilon_{1n}, x_0 - \lambda_2 d_n - \varepsilon_{2n} + R_n)\big]\Big\}. \qquad (24)
\end{aligned}$$

Note that $H_n^s(z, x_0)$ is concave in z and $z_n^0(x_0) \geq 0$. Therefore, for any given x_0, $W_n^s(u, x_0) = W_n^s(0, x_0)$ for $u \leq 0$, and $W_n^s(u, x_0)$ is decreasing in u when $u > 0$. Moreover, $\pi - (1-\gamma)c_1 > 0$ implies that for any given x_0, $-(1-\gamma)c_1 u - hu^+ - \pi u^- + \gamma W_n^s(u, x_0)$ is increasing in u when $u \leq 0$ and decreasing in u when $u > 0$.

Now, we are ready to show the lower bound, i.e., $V_n^o(x_0) - V_n^s(0, x_0) \geq \gamma^{N-n}m_1 + \frac{(1-\gamma^{N-n})m_0}{1-\gamma}$ by induction. As $V_{N+1}(u, x_0) = 0$, we have

$$\begin{aligned}
& V_N^s(0, x_0) \\
=\ & \max_{z\geq 0,\,(\lambda_1,\lambda_2)\in\Omega}\Big\{\Gamma_N(\lambda_1,\lambda_2) - c_1(z - \lambda_1 d_N) - \mathsf{E}[L(z - \lambda_1 d_N - \varepsilon_{1N})] - \mathsf{E}[L_0(x_0 - \lambda_2 d_N - \varepsilon_{2N})]\Big\} \\
\leq\ & \max_{(\lambda_1,\lambda_2)\in\Omega}\Big\{\Gamma_N(\lambda_1,\lambda_2) - \mathsf{E}[L_0(x_0 - \lambda_2 d_N - \varepsilon_{2N})]\Big\} \\
& - \min_{z\geq 0,\,(\lambda_1,\lambda_2)\in\Omega}\Big\{c_1(z - \lambda_1 d_N) + \mathsf{E}[L(z - \lambda_1 d_N - \varepsilon_{1N})]\Big\} \\
\leq\ & \max_{(\lambda_1,\lambda_2)\in\Omega}\Big\{\Gamma_N(\lambda_1,\lambda_2) - \mathsf{E}[L_0(x_0 - \lambda_2 d_N - \varepsilon_{2N})]\Big\} - \min_t\Big\{c_1 t + \mathsf{E}[L(t - \varepsilon_{1N})]\Big\} \\
=\ & V_N^o(x_0) - m_1.
\end{aligned}$$

Hence, the result holds for $n = N$. Suppose that the result holds for $n+1$, i.e.,

$$V_{n+1}^o(x_0) - V_{n+1}^s(0, x_0) \geq \gamma^{N-(n+1)}m_1 + \frac{(1-\gamma^{N-(n+1)})m_0}{1-\gamma}. \qquad (25)$$

We prove that

$$V_n^o(x_0) - V_n^s(0, x_0) \geq \gamma^{N-n}m_1 + \frac{(1-\gamma^{N-n})m_0}{1-\gamma}.$$

As the dynamic program of the make-to-stock system can be written as $W_n^s(u,x_0) = \max_{z\ge u}\big\{H_n^s(z,x_0)\big\}$, where $H_n^s(z,x_0)$ is defined in (24), we have

$$\begin{aligned}
W_n^s(0,x_0) &= \max_{z\ge 0,\,(\lambda_1,\lambda_2)\in\Omega}\Big\{\Gamma_n(\lambda_1,\lambda_2) - (1-\gamma)c_1(z-\lambda_1 d_n) - \mathsf{E}[L(z-\lambda_1 d_n - \varepsilon_{1n})] \\
&\quad -\mathsf{E}[L_0(x_0-\lambda_2 d_n-\varepsilon_{2n})] + \gamma\mathsf{E}\big[W_{n+1}^s(z-\lambda_1 d_n-\varepsilon_{1n}, x_0-\lambda_2 d_n-\varepsilon_{2n}+R_n)\big]\Big\} \\
&\le \max_{z\ge 0,\,(\lambda_1,\lambda_2)\in\Omega}\Big\{\Gamma_n(\lambda_1,\lambda_2) - \mathsf{E}[L_0(x_0-\lambda_2 d_n-\varepsilon_{2n}) \\
&\qquad +\gamma\mathsf{E}\big[W_{n+1}^s(z-\lambda_1 d_n-\varepsilon_{1n}, x_0-\lambda_2 d_n-\varepsilon_{2n}+R_n)\big]\Big\} \\
&\quad - \min_{z\ge 0,\,(\lambda_1,\lambda_2)\in\Omega}\Big\{(1-\gamma)c_1(z-\lambda_1 d_n) + \mathsf{E}[L(z-\lambda_1 d_n-\varepsilon_{1n})]\Big\} \\
&\le \max_{z\ge 0,\,(\lambda_1,\lambda_2)\in\Omega}\Big\{\Gamma_n(\lambda_1,\lambda_2) - \mathsf{E}[L_0(x_0-\lambda_2 d_n-\varepsilon_{2n})] + \gamma\mathsf{E}\big[W_{n+1}^s(0,x_0-\lambda_2 d_n-\varepsilon_{2n}+R_n)\big]\Big\} \\
&\quad - \min_t\big\{(1-\gamma)c_1 t + \mathsf{E}[L(t-\varepsilon_{1n})]\big\} \\
&\le \max_{z\ge 0,\,(\lambda_1,\lambda_2)\in\Omega}\Big\{\Gamma_n(\lambda_1,\lambda_2) - \mathsf{E}[L_0(x_0-\lambda_2 d_n-\varepsilon_{2n})] + \gamma\mathsf{E}\big[V_{n+1}^o(x_0-\lambda_2 d_n-\varepsilon_{2n}+R_n)\big] \\
&\quad -\gamma\cdot\gamma^{N-(n+1)}m_1 - \gamma\frac{(1-\gamma^{N-(n+1)})m_0}{1-\gamma}\Big\} - m_0 \\
&= V_n^o(x_0) - \gamma^{N-n}m_1 - \frac{(1-\gamma^{N-n})m_0}{1-\gamma}, \qquad (26)
\end{aligned}$$

where the second inequality follows from the fact that for any given x_0, $W_n^s(u,x_0) = W_n^s(0,x_0) = V_n^s(0,x_0)$ for $u\le 0$, and $W_n^s(u,x_0)$ is decreasing in u when $u>0$. The third inequality follows from (25). As $W_n^s(0,x_0) = V_n^s(0,x_0)$, (26) implies that

$$V_n^o(x_0) - V_n^s(0,x_0) \ge \gamma^{N-n}m_1 + \frac{(1-\gamma^{N-n})m_0}{1-\gamma}.$$

Next, we prove the upper bound, i.e., for any x_0, $V_1^o(x_0)-V_1^s(0,x_0) \le \gamma^{N-1}m_2+\frac{(1-\gamma^{N-1})m_0}{1-\gamma}$. We choose an inventory and pricing policy for the make-to-stock system $(z_n^s(u,x_0), \lambda_{1n}^s(u,x_0), \lambda_{2n}^s(u,x_0))$ with $\lambda_{1n}^s(u,x_0) = \lambda_{1n}^*(x_0)$ and $\lambda_{2n}^s(u,x_0) = \lambda_{2n}^*(x_0)$, where $(\lambda_{1n}^*(x_0), \lambda_{2n}^*(x_0))$ is determined by Theorem 1 and $z_n^s(u,x_0)$ is the optimal inventory level for the make-to-stock system when the sales policy, $(\lambda_{1n}^s(u,x_0), \lambda_{2n}^s(u,x_0))$, is given. Let $V_n^h(u,x_0)$ be the profit generated by this policy. We have $V_1^h(0,x_0)\le V_1^s(0,x_0)$ because $V_1^s(0,x_0)$ is the maximal profit generated by the optimal policy. To prove the upper bound, we show $W_n^h(u,x_0) \ge V_n^o(x_0) - \gamma^{N-n}m_2 - \frac{(1-\gamma^{N-n})m_0}{1-\gamma}$ by induction, where $W_n^h(u,x_0) = V_n^h(u,x_0) - c_1 u$.

We first need to show that under the policy $(z_n^s(u,x_0), \lambda_{1n}^s(u,x_0), \lambda_{2n}^s(u,x_0))$, $u_n \le t_0 - \underline{\varepsilon}_1$ where u_n is the initial inventory level of new product at the beginning of period n and $\underline{\varepsilon}_1$ is the lower bound of the support of ε_{1n}. As $u_1 = 0$, the result holds for $n=1$ because $t_0\ge\underline{\varepsilon}_1$ by its definition in (1) and $\pi-(1-\gamma)c_1>0$. Suppose that $u_n\le t_0-\underline{\varepsilon}_1$ for some $n\ge 1$. We prove that $u_{n+1}\le t_0-\underline{\varepsilon}_1$ in two cases: (i) it is optimal not to order in period n, and (ii) it is optimal to order in period n. If it is optimal not to order in period n, then $u_{n+1} = u_n - \lambda_{1n}^*(x_0)d_n - \varepsilon_{1n} \le u_n \le t_0 - \underline{\varepsilon}_1$ because $\lambda_{1n}^*(x_0)d_n + \varepsilon_{1n}$ is nonnegative. If it is optimal to order in period n, then we want to prove that the optimal order-up-to level is less than $t_0 - \underline{\varepsilon}_1$. Note that $W_n^h(u,x_0) = V_n^h(u,x_0) - c_1 u$ and so we have

$$W_n^h(u_n,x_0)$$

$$
\begin{aligned}
&= \max_{z\geq u_n}\Big\{\Gamma_n(\lambda^*_{1n}(x_0),\lambda^*_{2n}(x_0))-(1-\gamma)c_1(z-\lambda^*_{1n}(x_0)d_n)-\mathsf{E}[L(z-\lambda^*_{1n}(x_0)d_n-\varepsilon_{1n})]\\
&\quad -\mathsf{E}[L_0(x_0-\lambda^*_{2n}(x_0)d_n-\varepsilon_{2n})]+\gamma\mathsf{E}\big[W^h_{n+1}(z-\lambda^*_{1n}(x_0)d_n-\varepsilon_{1n},x_0-\lambda^*_{2n}(x_0)d_n-\varepsilon_{2n}+R_n)\big]\Big\}\\
&= \Gamma_n(\lambda^*_{1n}(x_0),\lambda^*_{2n}(x_0))-\mathsf{E}[L_0(x_0-\lambda^*_{2n}(x_0)d_n-\varepsilon_{2n})]\\
&\quad +\max_{z\geq u_n}\Big\{-(1-\gamma)c_1(z-\lambda^*_{1n}(x_0)d_n)-\mathsf{E}[L(z-\lambda^*_{1n}(x_0)d_n-\varepsilon_{1n})]\\
&\quad +\gamma\mathsf{E}\big[W^h_{n+1}(z-\lambda^*_{1n}(x_0)d_n-\varepsilon_{1n},x_0-\lambda^*_{2n}(x_0)d_n-\varepsilon_{2n}+R_n)\big]\Big\}. \qquad (27)
\end{aligned}
$$

Equation (27) implies that for any n and x_0, $W^h_n(u_n,x_0)$ is decreasing in u_n. Moreover, it is straightforward to prove by induction that $W^h_n(u_n,x_0)$ is concave in u_n. Because $-(1-\gamma)c_1t-\mathsf{E}[L(t-\varepsilon_{1n})]+\gamma\mathsf{E}[W^h_{n+1}(t-\varepsilon_{1n},x_0-\lambda^*_{2n}(x_0)d_n-\varepsilon_{2n}+R_n)]$ is concave in t and for any given x, $W^h_{n+1}(u_{n+1},x)$ is decreasing in u_{n+1}, we have that

$$
\begin{aligned}
&\arg\max_t\Big\{-(1-\gamma)c_1t-\mathsf{E}[L(t-\varepsilon_{1n})]+\gamma\mathsf{E}[W^h_{n+1}(t-\varepsilon_{1n},x_0-\lambda^*_{2n}(x_0)d_n-\varepsilon_{2n}+R_n)]\Big\}\\
\leq\ &\arg\max_t\Big\{-(1-\gamma)c_1t-\mathsf{E}[L(t-\varepsilon_{1n})]\Big\}=t_0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\arg\max_z\Big\{-(1-\gamma)c_1(z-\lambda^*_{1n}(x_0)d_n)-\mathsf{E}[L(z-\lambda^*_{1n}(x_0)d_n-\varepsilon_{1n})]\\
&+\gamma\mathsf{E}\big[W^h_{n+1}(z-\lambda^*_{1n}(x_0)d_n-\varepsilon_{1n},x_0-\lambda^*_{2n}(x_0)d_n-\varepsilon_{2n}+R_n)\big]\Big\}\leq t_0+\lambda^*_{1n}(x_0)d_n. \qquad (28)
\end{aligned}
$$

Combining (27) and (28), we obtain $u_n\leq z^s_n(u_n,x_0)\leq t_0+\lambda^*_{1n}(x_0)d_n$. Hence, $u_{n+1}=z^s_n(u_n,x_0)-\lambda^*_{1n}(x_0)d_n-\varepsilon_{1n}\leq t_0+\lambda^*_{1n}(x_0)d_n-\lambda^*_{1n}(x_0)d_n-\varepsilon_{1n}=t_0-\varepsilon_{1n}\leq t_0-\underline{\varepsilon}_1$. The last inequality is due to $\varepsilon_{1n}\geq\underline{\varepsilon}_1$.

As $V^h_{N+1}(u,x_0)\equiv 0$,

$$
\begin{aligned}
V^h_N(u_N,x_0) &= \max_{z\geq u_N}\Big\{\Gamma_N(\lambda^*_{1N}(x_0),\lambda^*_{2N}(x_0))+c_1\lambda^*_{1N}(x_0)d_N-c_1(z-u_N)\\
&\quad -\mathsf{E}[L(z-\lambda^*_{1N}(x_0)d_N-\varepsilon_{1N})]-\mathsf{E}[L_0(x_0-\lambda^*_{2N}(x_0)d_N-\varepsilon_{2N})]\Big\}\\
&= \max_{z\geq u_N}\Big\{\Gamma_N(\lambda^*_{1N}(x_0),\lambda^*_{2N}(x_0))+c_1\lambda^*_{1N}(x_0)d_N-c_1(z-u_N)\\
&\quad -\mathsf{E}[L(z-\lambda^*_{1N}(x_0)d_N-\varepsilon_{1N})]-\mathsf{E}[L_0(x_0-\lambda^*_{2N}(x_0)d_N-\varepsilon_{2N})]\Big\}\\
&= V^o_N(x_0)-\min_{z\geq u_N}\Big\{c_1(z-\lambda^*_{1N}(x_0)d_N)+\mathsf{E}[L(z-\lambda^*_{1N}(x_0)d_N-\varepsilon_{1N})]\Big\}+c_1u_N, \qquad (29)
\end{aligned}
$$

where the last equality follows from the definition of $V^o_N(x_0)$. Because $\underline{\varepsilon}_1$ is the lower bound of the support of ε_{1n}, $\lambda^*_{1n}(x_0)d_n+\varepsilon_{1n}\geq 0$ for any realization of ε_{1n} implies that $\lambda^*_{1n}(x_0)d_n\geq-\underline{\varepsilon}_1$. Therefore, we obtain that $u_n\leq t_0-\underline{\varepsilon}_1\leq t_0+\lambda^*_{1n}(x_0)d_n$. We have

$$
\begin{aligned}
&\min_{z\geq u_N}\Big\{c_1(z-\lambda^*_{1N}(x_0)d_N)+\mathsf{E}[L(z-\lambda^*_{1N}(x_0)d_N-\varepsilon_{1N})]\Big\}\\
\leq\ & c_1(t_0+\lambda^*_{1N}(x_0)d_N-\lambda^*_{1N}(x_0)d_N)+\mathsf{E}[L(t_0+\lambda^*_{1N}(x_0)d_N-\lambda^*_{1N}(x_0)d_N-\varepsilon_{1N})]\\
=\ & c_1t_0+\mathsf{E}[L(t_0-\varepsilon_{1N})]=m_2. \qquad (30)
\end{aligned}
$$

Combining (29) and (30), we get that $W_N^h(u_N, x_0) = V_N^h(u_N, x_0) - c_1 u_N \geq V_N^o(x_0) - m_2$.

Suppose that

$$W_{n+1}^h(u, x_0) \geq V_{n+1}^o(x_0) - \gamma^{N-(n+1)} m_2 - \frac{(1-\gamma^{N-(n+1)})m_0}{1-\gamma}. \quad (31)$$

We need to prove that $W_n^h(u, x_0) \geq V_n^o(x_0) - \gamma^{N-n} m_2 - \frac{(1-\gamma^{N-n})m_0}{1-\gamma}$. By (27), we have

$$\begin{aligned}
& W_n^h(u_n, x_0) \\
= \;& \max_{z \geq u_n} \Big\{ \Gamma_n(\lambda_{1n}^*(x_0), \lambda_{2n}^*(x_0)) - (1-\gamma)c_1(z - \lambda_{1n}^*(x_0)d_n) - \mathsf{E}[L(z - \lambda_{1n}^*(x_0)d_n - \varepsilon_{1n})] \\
& -\mathsf{E}[L_0(x_0 - \lambda_{2n}^*(x_0)d_n - \varepsilon_{2n})] + \gamma \mathsf{E}\big[W_{n+1}^h(z - \lambda_{1n}^*(x_0)d_n - \varepsilon_{1n}, x_0 - \lambda_{2n}^*(x_0)d_n - \varepsilon_{2n} + R_n)\big] \Big\} \\
\geq \;& \max_{z \geq u_n} \Big\{ \Gamma_n(\lambda_{1n}^*(x_0), \lambda_{2n}^*(x_0)) - (1-\gamma)c_1(z - \lambda_{1n}^*(x_0)d_n) - \mathsf{E}[L(z - \lambda_{1n}^*(x_0)d_n - \varepsilon_{1n})] \\
& -\mathsf{E}[L_0(x_0 - \lambda_{2n}^*(x_0)d_n - \varepsilon_{2n})] + \gamma \mathsf{E}\Big[V_{n+1}^o(x_0 - \lambda_{2n}^*(x_0)d_n - \varepsilon_{2n} + R_n) \\
& -\gamma^{N-(n+1)} m_2 - \frac{(1-\gamma^{N-(n+1)})m_0}{1-\gamma}\Big] \Big\} \\
= \;& \Gamma_n(\lambda_{1n}^*(x_0), \lambda_{2n}^*(x_0)) - \mathsf{E}[L_0(x_0 - \lambda_{2n}^*(x_0)d_n - \varepsilon_{2n})] + \gamma \mathsf{E}\big[V_{n+1}^o(x_0 - \lambda_{2n}^*(x_0)d_n - \varepsilon_{2n} + R_n)\big] \\
& + \max_{z \geq u_n} \Big\{ -(1-\gamma)c_1(z - \lambda_{1n}^*(x_0)d_n) - \mathsf{E}[L(z - \lambda_{1n}^*(x_0)d_n - \varepsilon_{1n})] \Big\} \\
& -\gamma^{N-n} m_2 - \gamma \cdot \frac{(1-\gamma^{N-(n+1)})m_0}{1-\gamma} \\
= \;& V_n^o(x_0) - \min_{z \geq u_n} \Big\{ (1-\gamma)c_1(z - \lambda_{1n}^*(x_0)d_n) + \mathsf{E}[L(z - \lambda_{1n}^*(x_0)d_n - \varepsilon_{1n})] \Big\} \\
& -\gamma^{N-n} m_2 - \frac{(\gamma - \gamma^{N-n})m_0}{1-\gamma},
\end{aligned} \quad (32)$$

where the first inequality follows from (31), and the last equality follows from the definition of $V_n^o(x_0)$. Recall that $u_n \leq t_0 - \underline{\varepsilon}_1 \leq t_0 + \lambda_{1n}^*(x_0)d_n$. We have

$$\begin{aligned}
& \min_{z \geq u_n} \Big\{ (1-\gamma)c_1(z - \lambda_{1n}^*(x_0)d_n) + \mathsf{E}[L(z - \lambda_{1n}^*(x_0)d_n - \varepsilon_{1n})] \Big\} \\
\leq \;& (1-\gamma)c_1(t_0 + \lambda_{1n}^*(x_0)d_n - \lambda_{1n}^*(x_0)d_n) + \mathsf{E}[L(t_0 + \lambda_{1n}^*(x_0)d_n - \lambda_{1n}^*(x_0)d_n - \varepsilon_{1n})] \\
= \;& (1-\gamma)c_1 t_0 + \mathsf{E}[L(t_0 - \varepsilon_1)] = m_0.
\end{aligned} \quad (33)$$

Combining (32) and (33), we get

$$\begin{aligned}
W_n^h(u_n, x_0) \;&\geq\; V_n^o(x_0) - m_0 - \gamma^{N-n} m_2 - \frac{(\gamma - \gamma^{N-n})m_0}{1-\gamma} \\
&=\; V_n^o(x_0) - \gamma^{N-n} m_2 - \frac{(1-\gamma^{N-n})m_0}{1-\gamma}.
\end{aligned}$$

Therefore, $W_1^h(u, x_0) \geq V_1^o(x_0) - \gamma^{N-1} m_2 - \frac{(1-\gamma^{N-1})m_0}{1-\gamma}$. Note that $W_1^h(0, x_0) = V_1^h(0, x_0) \leq V_1^s(0, x_0)$. Hence, $V_1^o(x_0) - \gamma^{N-1} m_2 - \frac{(1-\gamma^{N-1})m_0}{1-\gamma} \leq V_1^s(0, x_0)$, i.e.,

$$V_1^o(x_0) - V_1^s(0, x_0) \leq \gamma^{N-1} m_2 + \frac{(1-\gamma^{N-1})m_0}{1-\gamma}.$$

Finally, we prove that $m_0 \geq 0$ and $m_1 \geq 0$.

Note that $\pi - (1-\gamma)c_1 > 0$ implies that $(1-\gamma)c_1 x + hx^+ + \pi x^-$ is nonnegative with minimum at $x = 0$. Thus, $(1-\gamma)c_1(t-\varepsilon_{1n}) + h(t-\varepsilon_{1n})^+ + \pi(t-\varepsilon_{1n})^- \geq 0$, which implies $(1-\gamma)c_1\mathsf{E}(t-\varepsilon_{1n}) + h\mathsf{E}(t-\varepsilon_{1n})^+ + \pi\mathsf{E}(t-\varepsilon_{1n})^- \geq 0$. Hence, $(1-\gamma)c_1 t + \mathsf{E}[L(t-\varepsilon_{1n})] \geq 0$ and

$$m_0 = \min_t \Big\{(1-\gamma)c_1 t + \mathsf{E}[L(t-\varepsilon_{1n})]\Big\} \geq 0.$$

$\pi \geq c_1$ implies that $c_1 x + hx^+ + \pi x^- = (c_1+h)x^+ + (\pi - c_1)x^-$ is nonnegative with minimum at $x = 0$. Therefore, $c_1(t-\varepsilon_{1n}) + h(t-\varepsilon_{1n})^+ + \pi(t-\varepsilon_{1n})^- \geq 0$, which implies $c_1\mathsf{E}(t-\varepsilon_{1n}) + h\mathsf{E}(t-\varepsilon_{1n})^+ + \pi\mathsf{E}(t-\varepsilon_{1n})^- \geq 0$. So we have $c_1 t + \mathsf{E}[L(t-\varepsilon_{1n})] \geq 0$. Therefore,

$$m_1 = \min_t \Big\{c_1 t + \mathsf{E}[L(t-\varepsilon_{1n})]\Big\} \geq 0.$$

□

Proof of Lemma 3.

(1) We prove this result by induction. Note that $V^c_{N+1}(\mathbf{x}^c) \equiv 0$ for any $\mathbf{x}^c$. The result holds for $N+1$. Suppose that the result holds for n. That is, $V^c_n(\mathbf{x}^c)$ is a $L^\natural-$concave function. We want to prove that the result also holds for $n-1$.

We first show that $W_n(w_1, \ldots, w_{l_0}) = \mathsf{E}\big[V^c_n(w_1 - \varepsilon_{2n}, \ldots, w_{l_0} - \varepsilon_{2n}, w_{l_0} - \varepsilon_{2n} + R_n)\big]$ is a $L^\natural-$concave function. That is, for any $\xi \geq 0$, $W_n(w_1 - \xi, \ldots, w_{l_0} - \xi)$ is supermodular in $(w_1, \ldots, w_{l_0}, \xi)$. For any $(w^1_1, \ldots, w^1_{l_0})$, $(w^2_1, \ldots, w^2_{l_0})$, $\xi^1 \geq 0$ and $\xi^2 \geq 0$, we have

$$\begin{aligned}
& W_n(w^1_1 \wedge w^2_1 - \xi^1 \wedge \xi^2, \ldots, w^1_{l_0} \wedge w^2_{l_0} - \xi^1 \wedge \xi^2) + W_n(w^1_1 \vee w^2_1 - \xi^1 \vee \xi^2, \ldots, w^1_{l_0} \vee w^2_{l_0} - \xi^1 \vee \xi^2) \\
= \;& \mathsf{E}\big[V^c_n(w^1_1 \wedge w^2_1 - \xi^1 \wedge \xi^2 - \varepsilon_{2n}, \ldots, w^1_{l_0} \wedge w^2_{l_0} - \xi^1 \wedge \xi^2 - \varepsilon_{2n}, w^1_{l_0} \wedge w^2_{l_0} - \xi^1 \wedge \xi^2 - \varepsilon_{2n} + R_n)\big] \\
& + \mathsf{E}\big[V^c_n(w^1_1 \vee w^2_1 - \xi^1 \vee \xi^2 - \varepsilon_{2n}, \ldots, w^1_{l_0} \vee w^2_{l_0} - \xi^1 \vee \xi^2 - \varepsilon_{2n}, w^1_{l_0} \vee w^2_{l_0} - \xi^1 \vee \xi^2 - \varepsilon_{2n} + R_n)\big] \\
= \;& \mathsf{E}\Big[V^c_n\big((w^1_1 - \varepsilon_{2n}) \wedge (w^2_1 - \varepsilon_{2n}) - \xi^1 \wedge \xi^2, \ldots, (w^1_{l_0} - \varepsilon_{2n}) \wedge (w^2_{l_0} - \varepsilon_{2n}) - \xi^1 \wedge \xi^2, \\
& \quad (w^1_{l_0} - \varepsilon_{2n} + R_n) \wedge (w^2_{l_0} - \varepsilon_{2n} + R_n) - \xi^1 \wedge \xi^2\big) \\
& + V^c_n\big((w^1_1 - \varepsilon_{2n}) \vee (w^2_1 - \varepsilon_{2n}) - \xi^1 \vee \xi^2, \ldots, (w^1_{l_0} - \varepsilon_{2n}) \vee (w^2_{l_0} - \varepsilon_{2n}) - \xi^1 \vee \xi^2, \\
& \quad (w^1_{l_0} - \varepsilon_{2n} + R_n) \vee (w^2_{l_0} - \varepsilon_{2n} + R_n) - \xi^1 \vee \xi^2\big)\Big] \\
\geq \;& \mathsf{E}\Big[V^c_n(w^1_1 - \varepsilon_{2n} - \xi^1, \ldots, w^1_{l_0} - \varepsilon_{2n} - \xi^1, w^1_{l_0} - \varepsilon_{2n} + R_n - \xi^1) \\
& + V^c_n(w^2_1 - \varepsilon_{2n} - \xi^2, \ldots, w^2_{l_0} - \varepsilon_{2n} - \xi^2, w^2_{l_0} - \varepsilon_{2n} + R_n - \xi^2)\Big] \\
= \;& W_n(w^1_1 - \xi^1, \ldots, w^1_{l_0} - \xi^1) + W_n(w^2_1 - \xi^2, \ldots, w^2_{l_0} - \xi^2),
\end{aligned}$$

where the inequality follows from the $L^\natural-$concavity of function $V^c_n(\mathbf{x}^c)$.

We now prove that $V^c_{n-1}(\mathbf{x}^c)$ is $L^\natural-$concave in $\mathbf{x}^c$.

$$V^c_{n-1}(\mathbf{x}^c) = \max_{0 \leq \lambda_2 \leq 1} \Big\{U_{n-1}(\lambda_2) - \mathsf{E}[L(x_0 - \lambda_2 d_{n-1} - \varepsilon_{2n})] + \gamma W_n\big(\mathbf{x}^c_{-0} - (\lambda_2 d_{n-1})\mathbf{1}\big)\Big\},$$

where $\mathbf{x}^c_{-0} = (x^c_1, \ldots, x^c_{l_0})$. Note that the joint concavity of $\Gamma_{n-1}(\lambda_1, \lambda_2)$ implies that $U_{n-1}(\lambda_2)$ is concave in λ_2. For any $\xi \geq 0$, we have

$$V^c_{n-1}(\mathbf{x}^c - \xi\mathbf{1})$$

$$
\begin{aligned}
&= \max_{0\le\lambda_2\le 1}\Big\{U_{n-1}(\lambda_2) - \mathsf{E}[L(x_0-\xi-\lambda_2 d_{n-1}-\varepsilon_{2n-1})] + \gamma W_n(\mathbf{x}^c_{-0}-\xi\mathbf{1}-(\lambda_2 d_{n-1})\mathbf{1})\Big\} \\
&= \max_{\xi\le\widetilde{\xi}\le d_{n-1}+\xi}\Big\{U_{n-1}\Big(\frac{\widetilde{\xi}-\xi}{d_{n-1}}\Big) - \mathsf{E}[L(x_0-\widetilde{\xi}-\varepsilon_{2n})] + \gamma W_n(\mathbf{x}^c_{-0}-\widetilde{\xi}\mathbf{1})\Big\},
\end{aligned}
$$

where $\widetilde{\xi} = \xi + \lambda_2 d_{n-1}$. As $\{(\xi,\widetilde{\xi})|\xi \le \widetilde{\xi} \le d_{n-1}+\xi\}$ is a lattice, the $L^\natural-$concavity of $W_n(\cdot)$ implies that $W_n(\mathbf{x}^c_{-0}-\widetilde{\xi}\mathbf{1})$ is supermodular in $(x^c_1,\ldots,x^c_{l_0},\widetilde{\xi})$. The concavity of $U_{n-1}(\cdot)$ implies that $U_{n-1}\big(\frac{\widetilde{\xi}-\xi}{d_{n-1}}\big)$ is supermodular in $(\widetilde{\xi},\xi)$. Therefore, $V^c_{n-1}(\mathbf{x}^c-\xi\mathbf{1})$ is supermodular in $(\mathbf{x}^c,\xi)$ (Proposition 2.3.5 (e) in Simchi-Levi et al., 2005). That is, $V^c_{n-1}(\mathbf{x}^c)$ is also $L^\natural-$concave in $\mathbf{x}^c$.

(2) Note that $W_{n+1}(\mathbf{x}^c_{-0})$ is a $L^\natural-$concave function and $W_{n+1}(\mathbf{x}^c_{-0}-(\lambda_2 d_n)\mathbf{1})$ is supermodular in $(\mathbf{x}^c_{-0},\lambda_2)$. Therefore,

$$
\lambda_{2n}(\mathbf{x}^c) = \arg\max_{0\le\lambda_2\le 1}\Big\{U_n(\lambda_2) - \mathsf{E}[L(x_0-\lambda_2 d_n-\varepsilon_{2n})] + \gamma W_{n+1}\big(\mathbf{x}^c_{-0}-(\lambda_2 d_n)\mathbf{1}\big)\Big\}, \tag{34}
$$

$\lambda_{2n}(\mathbf{x}^c)$ is increasing in $\mathbf{x}^c$. By Lemma 3 in Zipkin (2008), it is implied that $\lambda_{2n}(\mathbf{x}^c+\xi\mathbf{1}) \le \lambda_{2n}(\mathbf{x}^c)+\xi/d_n$ for $\xi \ge 0$. □

Proof of Theorem 5.

(i) Because $V^c_{n+1}(\mathbf{x}^c) = V_{n+1}(\mathbf{x})$, Equation (34) implies that $\lambda^*_{2n}(\mathbf{x}) = \lambda_{2n}(\mathbf{x}^c)$. Recall that $x^c_i = \sum_{j=0}^i x_j$ for $0 \le i \le l_0$. As $\lambda_{2n}(\mathbf{x}^c)$ is increasing in $\mathbf{x}^c$ and $\mathbf{x}^c$ is increasing in $\mathbf{x}$, $\lambda^*_{2n}(\mathbf{x})$ is increasing in $\mathbf{x}$. Lemma 2 has shown that $\lambda_1(\lambda_2)$ is decreasing in λ_2 and $\lambda_2+\lambda_1(\lambda_2)$ is increasing in λ_2. Therefore, $\lambda^*_{1n}(\mathbf{x})$ is decreasing in $\mathbf{x}$ and $\lambda^*_n(\mathbf{x}) = \lambda^*_{1n}(\mathbf{x})+\lambda^*_{2n}(\mathbf{x})$ is increasing in $\mathbf{x}$.

Now we prove that $x_0-\lambda^*_{2n}(\mathbf{x})d_n$ is increasing in x_0 while decreasing in $\mathbf{x}_{-0}$. Recall that $\lambda_{2n}(\mathbf{x}^c)$ is increasing in $\mathbf{x}^c$, $\lambda^*_{2n}(\mathbf{x}) = \lambda_{2n}(\mathbf{x}^c)$ is increasing in $\mathbf{x}$. Hence, $x_0-\lambda^*_{2n}(\mathbf{x})d_n$ is decreasing in $\mathbf{x}_{-0}$. As $x_0-\lambda^*_{2n}(\mathbf{x})d_n = x_0-\lambda_{2n}(\mathbf{x}^c)d_n$, and for any $\xi \ge 0$, we have

$$
\begin{aligned}
x_0+\xi-\lambda^*_{2n}(x_0+\xi,\mathbf{x}_{-0})d_n &= x_0+\xi-\lambda_{2n}(\mathbf{x}^c+\xi\mathbf{1})d_n \\
&\ge x_0+\xi-\lambda_{2n}(\mathbf{x}^c)d_n-\xi \\
&= x_0-\lambda_{2n}(\mathbf{x}^c)d_n = x_0-\lambda^*_{2n}(\mathbf{x})d_n.
\end{aligned}
$$

The inequalities follows from $\lambda_{2n}(\mathbf{x}^c+\xi\mathbf{1}) \le \lambda_{2n}(\mathbf{x}^c)+\xi/d_n$ by Lemma 3. Hence, $x_0-\lambda^*_{2n}(\mathbf{x})d_n$, is increasing in x_0.

(ii) Recall that $\lambda^*_{2n}(\mathbf{x}) = \lambda_{2n}(\mathbf{x}^c)$ and $\lambda_{2n}(\mathbf{x}^c)$ is increasing in $\mathbf{x}^c$. Then, for any $\xi \ge 0$,

$$
\begin{aligned}
\lambda^*_{2n}(x_i+\xi,\mathbf{x}_{-i}) &= \lambda_{2n}(x_0,\ldots,x^c_{i-1},x^c_i+\xi,\ldots,x^c_{l_0}+\xi) \\
&\ge \lambda_{2n}(x_0,\ldots,x^c_i,x^c_{i+1}+\xi,\ldots,x^c_{l_0}+\xi) = \lambda^*_{2n}(x_{i+1}+\xi,\mathbf{x}_{-(i+1)}),
\end{aligned} \tag{35}
$$

i.e., $\lambda^*_{2n}(x_i+\xi,\mathbf{x}_{-i})-\lambda^*_{2n}(\mathbf{x}) \ge \lambda^*_{2n}(x_{i+1}+\xi,\mathbf{x}_{-(i+1)})-\lambda^*_{2n}(\mathbf{x})$.

Lemma 2 has shown that $\lambda_1(\lambda_2)$ is decreasing in λ_2 and $\lambda(\lambda_2) \triangleq \lambda_2+\lambda_1(\lambda_2)$ is increasing in λ_2. Therefore, Inequality (35) implies

$$
\begin{aligned}
\lambda_1\big(\lambda^*_{2n}(x_i+\xi,\mathbf{x}_{-i})\big) &\le \lambda_1\big(\lambda^*_{2n}(x_{i+1}+\xi,\mathbf{x}_{-(i+1)})\big), \\
\lambda\big(\lambda^*_{2n}(x_i+\xi,\mathbf{x}_{-i})\big) &\ge \lambda\big(\lambda^*_{2n}(x_{i+1}+\xi,\mathbf{x}_{-(i+1)})\big),
\end{aligned}
$$

i.e., $\lambda_{1n}^*(\mathbf{x}) - \lambda_{1n}^*(x_i+\xi,\mathbf{x}_{-i}) \geq \lambda_{1n}^*(\mathbf{x}) - \lambda_{1n}^*(x_{i+1}+\xi,\mathbf{x}_{-(i+1)})$ and $\lambda_n^*(x_i+\xi,\mathbf{x}_{-i}) - \lambda_n^*(\mathbf{x}) \geq \lambda_n^*(x_{i+1}+\xi,\mathbf{x}_{-(i+1)}) - \lambda_n^*(\mathbf{x})$. □

Proof of Theorem 6.

From Equations (3) and (4),

$$\begin{aligned} p_{1n}^*(\mathbf{x}) &= \eta\big(F^{-1}(1-\lambda_{1n}^*(\mathbf{x})-\lambda_{2n}^*(\mathbf{x}))\big) + F^{-1}(1-\lambda_{1n}^*(\mathbf{x})) - \eta\big(F^{-1}(1-\lambda_{1n}^*(\mathbf{x}))\big), \\ p_{2n}^*(\mathbf{x}) &= \eta\big(F^{-1}(1-\lambda_{1n}^*(\mathbf{x})-\lambda_{2n}^*(\mathbf{x}))\big). \end{aligned}$$

Recall that $\lambda_{1n}^*(\mathbf{x})+\lambda_{2n}^*(\mathbf{x})$ increases in $\mathbf{x}$. Therefore, $p_{2n}^*(\mathbf{x})$ decreases in $\mathbf{x}$. As $x-\eta(x)$ is increasing in x and $\lambda_{1n}^*(\mathbf{x}) = \lambda_1(\lambda_{2n}^*(\mathbf{x}))$ is decreasing in $\mathbf{x}$, $\Delta p_n^*(\mathbf{x}) = p_{1n}^*(\mathbf{x}) - p_{2n}^*(\mathbf{x}) = F^{-1}(1-\lambda_{1n}^*(\mathbf{x})) - \eta(F^{-1}(1-\lambda_{1n}^*(\mathbf{x})))$ is increasing in $\mathbf{x}$.

Because $\eta(F^{-1}(x))$ is increasing in x and $\lambda_n^*(x_i+\xi,\mathbf{x}_{-i}) \geq \lambda_n^*(x_{i+1}+\xi,\mathbf{x}_{-(i+1)})$, we must have $p_{2n}^*(x_i+\xi,\mathbf{x}_{-i}) \leq p_{2n}^*(x_{i+1}+\xi,\mathbf{x}_{-(i+1)})$, i.e, $p_{2n}^*(\mathbf{x})-p_{2n}^*(x_i+\xi,\mathbf{x}_{-i}) \geq p_{2n}^*(\mathbf{x})-p_{2n}^*(x_{i+1}+\xi,\mathbf{x}_{-(i+1)})$.

Recall that $\Delta p_n^*(\mathbf{x}) = F^{-1}(1-\lambda_{1n}^*(\mathbf{x})) - \eta(F^{-1}(1-\lambda_{1n}^*(\mathbf{x})))$, $x-\eta(x)$ is increasing in x, and $\lambda_{1n}^*(x_i+\xi,\mathbf{x}_{-i}) \leq \lambda_{1n}^*(x_{i+1}+\xi,\mathbf{x}_{-(i+1)})$. These imply that $\Delta p_n^*(x_i+\xi,\mathbf{x}_{-i}) \geq \Delta p_n^*(x_{i+1}+\xi,\mathbf{x}_{-(i+1)})$, i.e, $\Delta p_n^*(x_i+\xi,\mathbf{x}_{-i}) - \Delta p_n^*(\mathbf{x}) \geq \Delta p_n^*(x_{i+1}+\xi,\mathbf{x}_{-(i+1)}) - \Delta p_n^*(\mathbf{x})$. □

Proof of Theorem 7.

Note that Ω is a convex set, and $\Gamma_n(\lambda_1,\lambda_2)$ is jointly concave in (λ_1,λ_2). Then we can show that $H_n(\mathbf{u}^c,z,\mathbf{x})$ and $V_n^c(\mathbf{u}^c,\mathbf{x})$ are both concave functions by induction. Hence, $H_n(\mathbf{u}^c,z,\mathbf{x}) - c_1 z$ is concave in z for any given $(\mathbf{u}^c,\mathbf{x})$. Define

$$z_n^0(\mathbf{u}^c,\mathbf{x}) = \arg\max_z \Big\{ H_n(\mathbf{u}^c,z,\mathbf{x}) - c_1 z \Big\}.$$

Therefore, if $u_{l-1}^c \leq z_n^0(\mathbf{u}^c,\mathbf{x})$, then $z_n^* = z_n^0(\mathbf{u}^c,\mathbf{x})$; and if $u_{l-1}^c > z_n^0(\mathbf{u}^c,\mathbf{x})$, then $z_n^* = u_{l-1}^c$.

Moreover, the optimal segmentation of customers, $(\lambda_{1n}^*,\lambda_{2n}^*)$, are determined by

$$\begin{aligned} (\lambda_{1n}^*,\lambda_{2n}^*) &= \arg\max_{(\lambda_1,\lambda_2)\in\Omega} \Big\{ \Gamma_n(\lambda_1,\lambda_2) - \mathsf{E}[L(u_0-\lambda_1 d_n - \varepsilon_{1n})] - \mathsf{E}[L_0(x_0-\lambda_2 d_n-\varepsilon_{2n})] \\ &\quad + \gamma \mathsf{E}\big[V_{n+1}^c(u_1^c-\lambda_1 d_n-\varepsilon_{1n},\ldots,u_{l-1}^c-\lambda_1 d_n-\varepsilon_{1n}, z_n^*-\lambda_1 d_n-\varepsilon_{1n},\mathbf{x}_+)\big] \Big\}. \end{aligned}$$

□

Proof of Lemma 4.

(1) We prove this result by induction. Note that $V_{N+1}^c(\mathbf{x}^c) \equiv 0$ for any $\mathbf{x}^c$. The result holds for $N+1$. Suppose that the result holds for n. That is, $V_n^c(\mathbf{x}^c)$ is a $L^\natural$−concave function. We need to show that the result also holds for $n-1$. We first prove that $W_n(w_1,y) = \mathsf{E}\big[V_n^c(w_1-\varepsilon_{2n}, y-\varepsilon_{2n}+\epsilon_n)\big]$ is $L^\natural$−concave. That is for any $\xi \geq 0$, $W_n(w_1-\xi,y-\xi)$ is supermodular in (w_1,y,ξ). For any (w_1^1,y^1), (w_1^2,y^2), $\xi^1 \geq 0$ and $\xi^2 \geq 0$, we have

$$\begin{aligned} & W_n(w_1^1\wedge w_1^2 - \xi^1\wedge\xi^2, y^1\wedge y^2 - \xi^1\wedge\xi^2) + W_n(w_1^1\vee w_1^2 - \xi^1\vee\xi^2, y^1\vee y^2-\xi^1\vee\xi^2) \\ = \;& \mathsf{E}\big[V_n^c(w_1^1\wedge w_1^2 - \xi^1\wedge\xi^2 - \varepsilon_{2n}, y^1\wedge y^2 - \xi^1\wedge\xi^2 - \varepsilon_{2n}+\epsilon_n)\big] \\ & + \mathsf{E}\big[V_n^c(w_1^1\vee w_1^2 - \xi^1\vee\xi^2 - \varepsilon_{2n}, y^1\vee y^2 - \xi^1\vee\xi^2 - \varepsilon_{2n}+\epsilon_n)\big] \end{aligned}$$

$$
\begin{aligned}
&= \mathsf{E}\big[V_n^c\big((w_1^1-\varepsilon_{2n})\wedge(w_1^2-\varepsilon_{2n})-\xi^1\wedge\xi^2,(y^1-\varepsilon_{2n}+\epsilon_n)\wedge(y^2-\varepsilon_{2n}+\epsilon_n)-\xi^1\wedge\xi^2\big)\\
&\quad +V_n^c\big((w_1^1-\varepsilon_{2n})\vee(w_1^2-\varepsilon_{2n})-\xi^1\vee\xi^2,(y^1-\varepsilon_{2n}+\epsilon_n)\vee(y^2-\varepsilon_{2n}+\epsilon_n)-\xi^1\vee\xi^2\big)\big]\\
&\geq \mathsf{E}\big[V_n^c(w_1^1-\varepsilon_{2n}-\xi^1,y^1-\varepsilon_{2n}+\epsilon_n-\xi^1)+V_n^c(w_1^2-\varepsilon_{2n}-\xi^2,y^2-\varepsilon_{2n}+\epsilon_n-\xi^2)\big]\\
&= W_n(w_1^1-\xi^1,y^1-\xi^1)+W_n(w_1^2-\xi^2,y^2-\xi^2).
\end{aligned}
$$

The inequality follows from the $L^\natural-$concavity of function $V_n^c(\mathbf{x}^c)$.

Next, we prove that $\widetilde{G}_n(\mathbf{x}^c,y)$ is $L^\natural-$concave, where

$$
\widetilde{G}_n(\mathbf{x}^c,y)=\max_{0\leq\lambda_2\leq 1}\Big\{U_n(\lambda_2)-\mathsf{E}[L_0(x_0-\lambda_2 d_n-\varepsilon_{2n})]+\gamma W_n(x_1^c-\lambda_2 d_n,y-\lambda_2 d_n)\Big\}. \tag{36}
$$

For any $\xi\geq 0$, we have

$$
\begin{aligned}
&\widetilde{G}_n(\mathbf{x}^c-\xi\mathbf{1},y-\xi)\\
&= \max_{0\leq\lambda_2\leq 1}\Big\{U_n(\lambda_2)-\mathsf{E}[L_0(x_0-\xi-\lambda_2 d_n-\varepsilon_{2n})]+\gamma W_n(x_1^c-\xi-\lambda_2 d_n,y-\xi-\lambda_2 d_n)\Big\}\\
&= \max_{\xi\leq\widetilde{\xi}\leq d_n+\xi}\Big\{U_n\Big(\frac{\widetilde{\xi}-\xi}{d_n}\Big)-\mathsf{E}[L_0(x_0-\widetilde{\xi}-\varepsilon_{2n})]+\gamma W_n(x_1^c-\widetilde{\xi},y-\widetilde{\xi})\Big\},
\end{aligned}
$$

where $\widetilde{\xi}=\xi+\lambda_2 d_n$. Note that $\big\{(\xi,\widetilde{\xi})|\xi\leq\widetilde{\xi}\leq d_n+\xi\big\}$ is a lattice, the $L^\natural-$concavity of $W_n(\cdot)$ implies that $W_n(x_1^c-\widetilde{\xi},y-\widetilde{\xi})$ is supermodular in $(x_1^c,y,\widetilde{\xi})$ and the concavity of $U_n(\cdot)$ implies that $U_n\big(\frac{\widetilde{\xi}-\xi}{d_n}\big)$ is supermodular in $(\widetilde{\xi},\xi)$. Therefore, $\widetilde{G}_n(\mathbf{x}^c-\xi\mathbf{1},y-\xi)$ is supermodular in $(\mathbf{x}^c,y,\xi)$ (Proposition 2.3.5 (e) in Simchi-Levi et al., 2005). That is, $\widetilde{G}_n(\mathbf{x}^c,y)$ is also $L^\natural-$concave.

Finally, we prove that $V_{n-1}^c(\mathbf{x}^c)$ is $L^\natural-$concave, where

$$
V_{n-1}^c(\mathbf{x}^c)=\max_{y\geq x_1^c}\Big\{\widetilde{G}_n(\mathbf{x}^c,y)-g(y-x_1^c)\Big\}.
$$

For any $\xi\geq 0$, we have

$$
\begin{aligned}
V_{n-1}^c(\mathbf{x}^c-\xi\mathbf{1}) &= \max_{y\geq x_1^c-\xi}\Big\{\widetilde{G}_n(\mathbf{x}^c-\xi\mathbf{1},y)-g(y-x_1^c+\xi)\Big\}\\
&= \max_{\widetilde{y}\geq x_1^c}\Big\{\widetilde{G}_n(\mathbf{x}^c-\xi\mathbf{1},\widetilde{y}-\xi)-g(\widetilde{y}-x_1^c)\Big\},
\end{aligned}
$$

where $\widetilde{y}=y+\xi$. Note that $\big\{(\mathbf{x}^c,\xi,\widetilde{y})|\widetilde{y}\geq x_1^c\big\}$ is a lattice, the $L^\natural-$concavity of $\widetilde{G}_n(\cdot)$ implies that $\widetilde{G}_n(\mathbf{x}^c-\xi\mathbf{1},\widetilde{y}-\xi)$ is supermodular in $(\mathbf{x}^c,\xi,\widetilde{y})$ and the convexity of $g(\cdot)$ implies that $g(\widetilde{y}-x_1^c)$ is submodular in $(\widetilde{y},x_1^c)$. Therefore, $V_{n-1}^c(\mathbf{x}^c-\xi\mathbf{1})$ is supermodular in $(\mathbf{x}^c,\xi)$ (Proposition 2.3.5 (e) in Simchi-Levi et al., 2005). That is, $V_{n-1}^c(\mathbf{x}^c)$ is also $L^\natural-$concave.

(2) As $W_{n+1}(x_1^c,y)$ is $L^\natural-$concave, $W_{n+1}(x_1^c-\lambda_2 d_n,y-\lambda_2 d_n)$ is supermodular in (x_1^c,y,λ_2). Therefore,

$$
\lambda_{2n}(\mathbf{x}^c,y)=\arg\max_{0\leq\lambda_2\leq 1}\Big\{U_n(\lambda_2)-\mathsf{E}[L_0(x_0-\lambda_2 d_n-\varepsilon_{2n})]+\gamma W_{n+1}(x_1^c-\lambda_2 d_n,y-\lambda_2 d_n)\Big\},
$$

is increasing in $(\mathbf{x}^c,y)$. Because $\widetilde{G}_n(\mathbf{x}^c,y)$ defined in (36) is a $L^\natural-$concave function and $g(e)$ is convex in e, $\widetilde{G}_n(\mathbf{x}^c,y)$ is supermodular in $(\mathbf{x}^c,y)$ and $g(y-x_1^c)$ is submodular in (y,x_1^c). Hence,

$$
y_n(\mathbf{x}^c)=\arg\max_{y\geq x_1^c}\Big\{\widetilde{G}_n(\mathbf{x}^c,y)-g(y-x_1^c)\Big\}
$$

is increasing in $\mathbf{x}^c$ because $\{(\mathbf{x}^c, y)|y \geq x_l^c\}$ is a lattice. It follows from Lemma 3 in Zipkin (2008) that $\lambda_{2n}(\mathbf{x}^c + \xi\mathbf{1}, y + \xi) \leq \lambda_{2n}(\mathbf{x}^c, y) + \xi/d_n$ and $y_n(\mathbf{x}^c + \xi\mathbf{1}) \leq y_n(\mathbf{x}^c) + \xi$ for $\xi \geq 0$. □

Proof of Proposition 2.

Define

$$\begin{aligned}
\big(\lambda_{2n}^*(\mathbf{x}), e_n^*(\mathbf{x})\big) &= \underset{0\leq\lambda_2\leq 1,\, e\geq 0}{\arg\max} \Big\{U_n(\lambda_2) - \mathsf{E}[L_0(x_0 - \lambda_2 d_n - \varepsilon_{2n})] - g(e) + \gamma\mathsf{E}\left[V_{n+1}(\mathbf{x}_+)\right]\Big\},\\
\lambda_{1n}^*(\mathbf{x}) &= \lambda_1\big(\lambda_{2n}^*(\mathbf{x})\big).
\end{aligned}$$

As $y_n(\mathbf{x}^c)$ is defined in (12), we have $e_n^*(\mathbf{x}) = y_n(\mathbf{x}^c) - x_1^c$, where the one-to-one correspondence between $\mathbf{x}$ and $\mathbf{x}^c$ is given by $x_0^c = x_0$ and $x_1^c = x_0 + x_1$. Because it is shown in Lemma 4 that $y_n(\mathbf{x}^c + \xi\mathbf{1}) \leq y_n(\mathbf{x}^c) + \xi$ for $\xi \geq 0$, we obtain that

$$\begin{aligned}
e_n^*(x_0 + \xi, x_1) - e_n^*(\mathbf{x}) &= y_n(\mathbf{x}^c + \xi\mathbf{1}) - (x_1^c + \xi) - [y_n(\mathbf{x}^c) - x_1^c]\\
&= y_n(\mathbf{x}^c + \xi\mathbf{1}) - \xi - y_n(\mathbf{x}^c) \leq 0. \qquad (37)
\end{aligned}$$

As it follows from Lemma 4 that $y_n(\mathbf{x}^c)$ is increasing in $\mathbf{x}^c$, we have $y_n(x_0, x_1^c + \xi) \leq y_n(\mathbf{x}^c + \xi\mathbf{1})$. Hence, $e_n^*(x_0, x_1 + \xi) = y_n(x_0, x_1^c + \xi) - (x_1^c + \xi) \leq y_n(\mathbf{x}^c + \xi\mathbf{1}) - (x_1^c + \xi)$, which implies that

$$\begin{aligned}
e_n^*(x_0, x_1 + \xi) - e_n^*(\mathbf{x}) &\leq y_n(\mathbf{x}^c + \xi\mathbf{1}) - (x_1^c + \xi) - [y_n(\mathbf{x}^c) - x_1^c]\\
&= y_n(\mathbf{x}^c + \xi\mathbf{1}) - \xi - y_n(\mathbf{x}^c) \leq 0. \qquad (38)
\end{aligned}$$

Equations (37) and (38) imply that the optimal effort $e_n^*(\mathbf{x})$ is decreasing in $\mathbf{x}$.

Furthermore, for any $\xi \geq 0$, we have

$$\begin{aligned}
e_n^*(x_0 + \xi, x_1) - e_n^*(x_0, x_1 + \xi) &= y_n(\mathbf{x}^c + \xi\mathbf{1}) + x_1^c + \xi - (y_n(x_0, x_1^c + \xi) + x_1^c + \xi)\\
&= y_n(\mathbf{x}^c + \xi\mathbf{1}) - y_n(x_0, x_1^c + \xi) \geq 0.
\end{aligned}$$

Hence, $e_n^*(\mathbf{x}) - e_n^*(x_0 + \xi, x_1) \leq e_n^*(\mathbf{x}) - e_n^*(x_0, x_1 + \xi)$ for any $\xi \geq 0$. □

Proof of Theorem 8.

Note that Ω is a convex set, $g(e)$ is convex in e and $\Gamma_n(\lambda_1, \lambda_2)$ is jointly concave in (λ_1, λ_2). Then we can show $H_n(z, x_0, x_1, e)$ and $V_n(u, x_0, x_1)$ are both concave functions by induction on n.

Define

$$\begin{aligned}
(z_n^0(\mathbf{x}), e_n^0(\mathbf{x})) &= \underset{z,\, e\geq 0}{\arg\max} \Big\{H_n(z, \mathbf{x}, e) - g(e) - c_1 z\Big\},\\
e_n(u, \mathbf{x}) &= \underset{e\geq 0}{\arg\max} \Big\{H_n(u, \mathbf{x}, e) - g(e)\Big\}.
\end{aligned}$$

Moreover, the optimal segmentation of customers, $(\lambda_{1n}^*, \lambda_{2n}^*)$, are determined by

$$\begin{aligned}
(\lambda_{1n}^*, \lambda_{2n}^*) = \underset{(\lambda_1,\lambda_2)\in\Omega}{\arg\max} \Big\{&\Gamma_n(\lambda_1, \lambda_2) - \mathsf{E}[L(z_n^* - \lambda_1 d_n - \varepsilon_{1n})] - \mathsf{E}[L_0(x_0 - \lambda_2 d_n - \varepsilon_{2n})]\\
&+\gamma\mathsf{E}\big[V_{n+1}^c(z_n^* - \lambda_1 d_n - \varepsilon_{1n}, x_0 + x_1 - \lambda_2 d_n - \varepsilon_{2n}, e_n^* + \epsilon_n)\big]\Big\}.
\end{aligned}$$

Therefore, it is clear that if $u \le z_n^0(\mathbf{x})$, $z_n^* = z_n^0(\mathbf{x})$ and $e_n^* = e_n^0(\mathbf{x})$; otherwise, $z_n^* = u$ and $e_n^* = e_n(u, \mathbf{x})$. □

Proof of Theorem 9.

(i) We let $\varsigma = x_0 - \lambda_2 d_n$, which is the expected number of cores carried over to the next period. Consequently, the dynamic programming (13) can be rewritten as

$$\begin{aligned} V_n(x_0) &= \max_{\lambda_1,\varsigma}\Bigg\{\Big[G_1(\lambda_1) + G_2\Big(\frac{\lambda_1 d_n + x_0 - \varsigma}{d_n}\Big) - (c_1+\theta c_2)\lambda_1\Big]d_n - L_0(\varsigma) \\ &\qquad + \gamma \mathsf{E}\big[V_{n+1}(\varsigma + \alpha_n \lambda_1 d_n - \varepsilon_{2n} + \alpha_n \varepsilon_{1n})\big]\Bigg\}, \end{aligned}$$

subject to $x_0 - d_n \le \varsigma \le x_0$ and $0 \le \lambda_1 \le 1 - \frac{x_0-\varsigma}{d_n}$. Note that $\big\{(\lambda_1,\varsigma) : x_0 - d_n \le \varsigma \le x_0,\ 0 \le \lambda_1 \le 1 - \frac{x_0-\varsigma}{d_n}\big\}$ is a convex set. Then it is straightforward to show, by induction, that $V_n(x)$ is concave in x for all n. For notational convenience in the subsequent proofs, we define $\widehat{\lambda}_1 = -\lambda_1$, $t = x_0 - \varsigma$ and

$$\begin{aligned} \Lambda_{1n}(t,\varsigma) &= \max_{\frac{t}{d_n}-1\le\widehat{\lambda}_1\le 0}\Bigg\{\Big[G_1(-\widehat{\lambda}_1) + G_2\Big(\frac{t - \widehat{\lambda}_1 d_n}{d_n}\Big) + (c_1+\theta c_2)\widehat{\lambda}_1\Big]d_n \\ &\qquad + \gamma \mathsf{E}\big[V_{n+1}(\varsigma - \alpha_n \widehat{\lambda}_1 d_n - \varepsilon_{2n} + \alpha_n \varepsilon_{1n})\big]\Bigg\}, \\ \widehat{\lambda}_{1n}(t,\varsigma) &= \arg\max_{\frac{t}{d_n}-1\le\widehat{\lambda}_1\le 0}\Bigg\{\Big[G_1(-\widehat{\lambda}_1) + G_2\Big(\frac{t - \widehat{\lambda}_1 d_n}{d_n}\Big) + (c_1+\theta c_2)\widehat{\lambda}_1\Big]d_n \\ &\qquad + \gamma \mathsf{E}\big[V_{n+1}(\varsigma - \alpha_n \widehat{\lambda}_1 d_n - \varepsilon_{2n} + \alpha_n \varepsilon_{1n})\big]\Bigg\}, \qquad (39) \\ \varsigma_n(x_0) &= \max_{x_0 - d_n \le \varsigma \le x_0}\Big\{\Lambda_{1n}(x_0-\varsigma,\varsigma) - L_0(\varsigma)\Big\}. \qquad (40) \end{aligned}$$

By above notations, we write

$$V_n(x_0) = \max_{x_0 - d_n \le \varsigma \le x_0}\Big\{\Lambda_{1n}(x_0-\varsigma,\varsigma) - L_0(\varsigma)\Big\}. \qquad (41)$$

Note that $G_2(x)$ is concave in x (by Assumption 1) and $V_{n+1}(x)$ is concave in x. Hence, $G_2\big(\frac{t-\widehat{\lambda}_1 d_n}{d_n}\big)$ is concave and supermodular in $(t, \widehat{\lambda}_1)$, and $\mathsf{E}[V_{n+1}(\varsigma - \alpha_n\widehat{\lambda}_1 d_n - \varepsilon_{2n} + \alpha_n\varepsilon_{1n})]$ is concave and supermodular in $(\varsigma, \widehat{\lambda}_1)$ (Theorem 2.3.6 (b) in Simchi-Levi et al., 2005). Therefore,

$$\Big[G_1(-\widehat{\lambda}_1) + G_2\Big(\frac{t - \widehat{\lambda}_1 d_n}{d_n}\Big) + (c_1+\theta c_2)\widehat{\lambda}_1\Big]d_n + \gamma \mathsf{E}\big[V_{n+1}(\varsigma - \alpha_n \widehat{\lambda}_1 d_n - \varepsilon_{2n} + \alpha_n \varepsilon_{1n})\big]$$

is supermodular in $(t, \widehat{\lambda}_1, \varsigma)$ (Proposition 2.3.5 (a) in Simchi-Levi et al., 2005). Note that $\big\{(t, \widehat{\lambda}_1) : \frac{t}{d_n} - 1 \le \widehat{\lambda}_1 \le 0\big\}$ is a lattice. Hence, $\widehat{\lambda}_{1n}(t,\varsigma)$ is an increasing function of t and ς (Theorem 2.3.7 in Simchi-Levi et al., 2005), and $\Lambda_{1n}(t,\varsigma)$ is supermodular and concave in (t,ς) (Proposition 2.3.5 (e) in Simchi-Levi et al., 2005). Now we want to prove that $\Lambda_{1n}(x_0 - \varsigma, \varsigma)$ is supermodular in (x_0, ς), that is, for any (x_0^1, ς^1) and (x_0^2, ς^2),

$$\begin{aligned} & \Lambda_{1n}(x_0^1 \wedge x_0^2 - \varsigma^1 \wedge \varsigma^2, \varsigma^1 \wedge \varsigma^2) + \Lambda_{1n}(x_0^1 \vee x_0^2 - \varsigma^1 \vee \varsigma^2, \varsigma^1 \vee \varsigma^2) \\ \geq \;& \Lambda_{1n}(x_0^1 - \varsigma^1, \varsigma^1) + \Lambda_{1n}(x_0^2 - \varsigma^2, \varsigma^2). \end{aligned} \tag{42}$$

Without loss of generality, we assume $x_0^1 > x_0^2$ and $\varsigma^1 < \varsigma^2$. Therefore,

$$\begin{aligned} & \Lambda_{1n}(x_0^1 \wedge x_0^2 - \varsigma^1 \wedge \varsigma^2, \varsigma^1 \wedge \varsigma^2) + \Lambda_{1n}(x_0^1 \vee x_0^2 - \varsigma^1 \vee \varsigma^2, \varsigma^1 \vee \varsigma^2) \\ = \;& \Lambda_{1n}(x_0^2 - \varsigma^1, \varsigma^1) + \Lambda_{1n}(x_0^1 - \varsigma^2, \varsigma^2). \end{aligned}$$

Note that $x_0^1 - \varsigma^1 > x_0^1 - \varsigma^2$ and $x_0^2 - \varsigma^1 > x_0^2 - \varsigma^2$. We have

$$\begin{aligned} \Lambda_{1n}(x_0^1 - \varsigma^1, \varsigma^1) - \Lambda_{1n}(x_0^2 - \varsigma^1, \varsigma^1) &\leq \Lambda_{1n}(x_0^1 - \varsigma^2, \varsigma^1) - \Lambda_{1n}(x_0^2 - \varsigma^2, \varsigma^1) \\ &\leq \Lambda_{1n}(x_0^1 - \varsigma^2, \varsigma^2) - \Lambda_{1n}(x_0^2 - \varsigma^2, \varsigma^2), \end{aligned} \tag{43}$$

where the first inequality holds because $\Lambda_{1n}(t, \varsigma)$ is concave in t, and the second inequality holds because $\Lambda_{1n}(t, \varsigma)$ is supermodular in (t, ς). Inequality (43) implies that

$$\Lambda_{1n}(x_0^2 - \varsigma^1, \varsigma^1) + \Lambda_{1n}(x_0^1 - \varsigma^2, \varsigma^2) \geq \Lambda_{1n}(x_0^1 - \varsigma^1, \varsigma^1) + \Lambda_{1n}(x_0^2 - \varsigma^2, \varsigma^2),$$

which is (42) because $x_0^1 > x_0^2$ and $\varsigma^1 < \varsigma^2$. Therefore, $\Lambda_{1n}(x_0 - \varsigma, \varsigma)$ is supermodular in (x_0, ς). As $\{(x_0, \varsigma) : x_0 - d_n \leq \varsigma \leq x_0\}$ is a lattice, it is derived from (40) that $\varsigma_n(x_0)$ is increasing in x_0.

Recall that $t = x_0 - \varsigma$. The dynamic program, defined by (41), can be written as

$$V_n(x_0) = \max_{0 \leq t \leq d_n} \Big\{ \Lambda_{1n}(t, x_0 - t) - L_0(x_0 - t) \Big\}.$$

Let

$$t_n(x_0) = \arg\max_{0 \leq t \leq d_n} \Big\{ \Lambda_{1n}(t, x_0 - t) - L_0(x_0 - t) \Big\}. \tag{44}$$

As $\Lambda_{1n}(t, \varsigma)$ is supermodular and concave in (t, ς) and $L_0(z)$ is concave in z, both $\Lambda_{1n}(t, x_0 - t)$ and $L_0(x_0 - t)$ are supermodular in (x_0, t) (Theorem 2.3.6 (b) in Simchi-Levi et al., 2005). Therefore, $t_n(x_0)$ is increasing in x_0, i.e., $x_0 - \varsigma_n(x_0)$ is increasing in x_0 because $x_0 - \varsigma_n(x_0) = t_n(x_0)$. As $\varsigma = x_0 - \lambda_2 d_n$ and $t = x_0 - \varsigma$, we have $\lambda_2 = t/d_n$. By the definition of $t_n(x_0)$ in (44), we obtain $\lambda_{2n}^*(x_0) = t_n(x_0)/d_n$, which implies that $\lambda_{2n}^*(x_0)$ is increasing in x_0. By (39), we have $\lambda_{1n}^*(x_0) = -\widehat{\lambda}_{1n}(x_0 - \varsigma_n(x_0), \varsigma_n(x_0))$. As both $\varsigma_n(x_0)$ and $x_0 - \varsigma_n(x_0)$ are increasing in x_0 and $\widehat{\lambda}_{1n}(t, \varsigma)$ is an increasing function of t and ς, it is derived that $\lambda_{1n}^*(x_0)$ is decreasing in x_0. Furthermore, the expected leftover of the remanufactured product, $x_0 - \lambda_{2n}^*(x_0) d_n$, is increasing in x_0 because $\varsigma_n(x_0) = x_0 - \lambda_{2n}^*(x_0) d_n$ is increasing in x_0.

(ii) Equations (3) and (4) imply that

$$\begin{aligned} p_{1n}^*(x_0) &= \eta\big(F^{-1}(1 - \lambda_{1n}^*(x_0) - \lambda_{2n}^*(x_0))\big) + F^{-1}(1 - \lambda_{1n}^*(x_0)) - \eta\big(F^{-1}(1 - \lambda_{1n}^*(x_0))\big), \\ p_{2n}^*(x_0) &= \eta\big(F^{-1}(1 - \lambda_{1n}^*(x_0) - \lambda_{2n}^*(x_0))\big). \end{aligned}$$

As $\Delta p_n^*(x_0) = F^{-1}(1 - \lambda_{1n}^*(x_0)) - \eta(F^{-1}(1 - \lambda_{1n}^*(x_0)))$, $x - \eta(x)$ is increasing in x and $\lambda_{1n}^*(x_0)$ is decreasing in x_0, it is derived that $\Delta p_n^*(x_0)$ increases with x_0. □