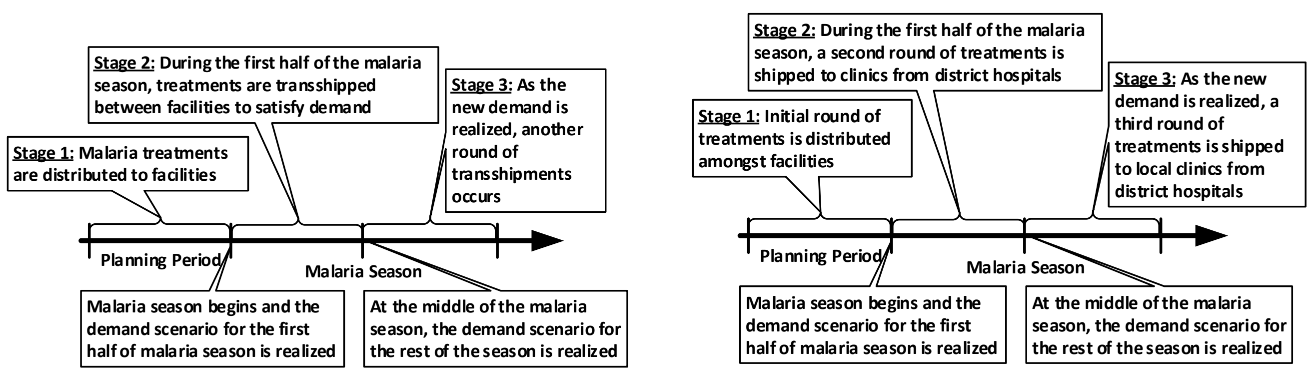
Online Appendices

A Alternative Stochastic Programming Model Formulations

A.1 Three-Stage Stochastic Formulation

In §3.2, we considered two-stage models in which the demand scenario for each clinic is realized at the beginning of the malaria season. Recourse actions address the disparity between the realized demand and the initial inventory of ACTs at each clinic. One drawback of the two-stage model is that the recourse actions are aggregate-level surrogates for the actual periodic decisions. This assumption allows for a tractable solution at the cost of ignoring the temporal (e.g., bi-monthly, monthly, weekly, etc.) fluctuations in demand. When temporal demand fluctuations are high, the two-stage models may underestimate the actual shortage in each period.

Model accuracy can be improved by increasing the granularity of the recourse actions. For instance, the transshipments or delayed shipments can be delivered periodically so the model can better estimate the actual shortage in each period. However, as the granularity of the model increases, the computation time increases dramatically. Moreover, collecting and processing the demand data at a very detailed level is often not feasible in a developing nation.



(a) Three-Stage Transshipment Model (b) Three-Stage Delayed Shipment Model

Figure 14: Event timelines for three-stage stochastic models.

In §A.2 we explore the benefits of increasing the granularity of the recourse actions by extending the former analysis to a three-stage stochastic program, using a revised timeline shown in Figure 14.

A.2 Three-Stage Transshipment Model and Delayed Shipment Models

The first stage of the three-stage problem (initial distribution of medications) is identical to that of the two-stage problem in §3.3. The second stage represents the initial round of transshipment; therefore it includes an additional term ($\mathcal{Q}'$) in the objective function (21) to represent the third-stage problem - the final round of transshipment. We also define a new auxiliary decision variable (l_i^s) to represent the number of ACTs left at clinic i under scenario s after the second stage and the term l_i^s is subtracted from the left-hand-side of the flow conservation constraints

(27). Note that in the third-stage problem, l_i^s is calculated by the flow conservation constraint of the second-stage problem (23) and thus is considered input data.

$$\mathcal{Q} = \min \sum_{s\in\mathcal{S}} p_s \left(\sum_{(i,j)\in\mathcal{A}^\mathcal{T}\cup\mathcal{A}^\mathcal{C}} c_{ij} y_{ij}^s + \sum_{i\in\mathcal{C}} \pi_i z_i^s \right) + \mathcal{Q}' \tag{21}$$

$$\text{s.t.} \quad \sum_{j:(i,j)\in\mathcal{A}^\mathcal{C}} y_{ij}^s \leq \sum_{j:(j,i)\in\mathcal{A}^\mathcal{D}} x_{ij} - \sum_{j:(i,j)\in\mathcal{A}^\mathcal{C}} x_{ij} \qquad \forall i \in \mathcal{D}, \forall s \in \mathcal{S} \tag{22}$$

$$\sum_{j:(j,i)\in\mathcal{A}^\mathcal{T}\cup\mathcal{A}^\mathcal{C}} y_{ji}^s - \sum_{j:(i,j)\in\mathcal{A}^\mathcal{T}\cup\mathcal{A}^\mathcal{C}} y_{ij}^s + z_i^s - l_i^s = - \sum_{j:(j,i)\in\mathcal{A}^\mathcal{C}} x_{ji} + d_i^s \qquad \forall i \in \mathcal{C}, \forall s \in \mathcal{S} \tag{23}$$

$$y_{ij}^s \geq 0 \qquad \forall (i,j) \in \mathcal{A}^\mathcal{T} \cup \mathcal{A}^\mathcal{C}, \forall s \in \mathcal{S} \tag{24}$$

$$z_i^s \geq 0, l_i^s \geq 0 \qquad \forall i \in \mathcal{C}, \forall s \in \mathcal{S}. \tag{25}$$

For the third-stage problem ($\mathcal{Q}'$) represented by (26)-(29), we define $y_{ij}^{\prime s}$ to be the decision on the number of ACT units transshipped from clinic i to clinic j under scenario s in the third stage. We allow the transshipment cost in the third stage, (c'_{ij}), to be different from the second-stage cost (c_{ij}).

$$\mathcal{Q}' = \min \sum_{s\in\mathcal{S}} p_s \left(\sum_{(i,j)\in\mathcal{A}^\mathcal{C}} c'_{ij} y_{ij}^{\prime s} + \sum_{i\in\mathcal{C}} \pi_i z_i^{\prime s} \right) \tag{26}$$

$$\text{s.t.} \quad \sum_{j:(i,j)\in\mathcal{A}^\mathcal{T}} y_{ij}^{\prime s} - \sum_{j:(j,i)\in\mathcal{A}^\mathcal{T}} y_{ji}^{\prime s} + z_i^{\prime s} \geq -l_i^s + d_i^{\prime s} \qquad \forall i \in \mathcal{C}, \forall s \in \mathcal{S} \tag{27}$$

$$y_{ij}^{\prime s} \geq 0 \qquad \forall (i,j) \in \mathcal{A}^\mathcal{T}, \forall s \in \mathcal{S} \tag{28}$$

$$z_i^{\prime s} \geq 0 \qquad \forall i \in \mathcal{C}, \forall s \in \mathcal{S}. \tag{29}$$

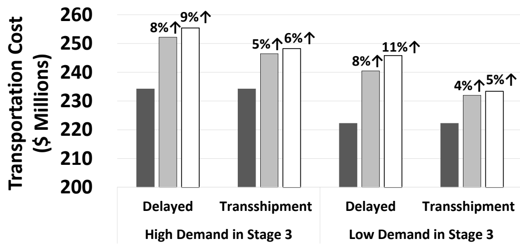
The third stage, Eq. 26-29 has the same structure and intuition as the second stage.

Delayed Shipment. The form of the three-stage delayed shipment model is analogous to the three-stage transshipment model with the necessary changes illustrated in §3.4 for the two-stage version. For brevity we do not repeat them here.

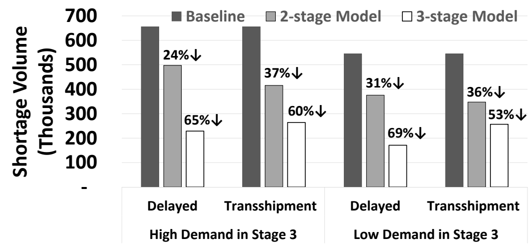
A.3 Two-Stage vs. Three-Stage Models

Through computational experiments, we also analyze the marginal benefit of adding another recourse stage to the stochastic model. To focus on this impact given different uncertainty profiles, we performed separate experiments focusing on two scenario subsets: (1) scenarios where the majority of demand is realized in the second stage (LOW2, MED2, HIGH1, CONS2, and VAR1); (2) scenarios where the majority of the demand is realized in the third stage (LOW1, MED1, HIGH2, CONS1, and VAR2). The three-stage model provides more opportunities to react to demand uncertainty, but adding more stages makes the problem harder to solve.

As observed in Figures 4, 6, and 15, three-stage models outperform two-stage and the difference is accentuated as supply increases. From Figure 15, the marginal benefit of moving from the baseline model to the two-stage model is higher in transshipment models. On the other hand, the three-stage delayed shipment model is more effective at reducing shortage compared to the three-stage transshipment model. Depending on parameters such as supply, shortage penalty, and demand uncertainty profile, the magnitude of these marginal benefits can vary.



(a) Expected Transportation Cost



(b) Expected Shortage Volume

Figure 15: Stochastic models compared to the baseline.

A.4 Supply Equity

When the total supply of malaria medication is less than the demand, shortage is inevitable. In such a scenario, it is possible that some clinics may face significantly higher shortages than others. An equitable policy, however, distributes ACTs in a manner that limits the shortage disparity between clinics. Minimizing the *sum of absolute differences* between the shortage of each clinic and the average shortage among all clinics can limit this disparity. Let $\bar{z}^s$ be the average shortage of ACTs in all the clinics in scenario s and $\tilde{z}_i^s$ be the absolute difference between the shortage in clinic i and the average shortage. The following constraints capture this concept:

$$\bar{z}^s = \frac{1}{|\mathcal{C}|} \sum_{i \in \mathcal{C}} z_i^s \qquad \forall s \in \mathcal{S} \tag{30}$$

$$\tilde{z}_i^s \geq z_i^s - \bar{z}^s \qquad \forall s \in \mathcal{S}, \forall i \in \mathcal{C} \tag{31}$$

$$\tilde{z}_i^s \geq -z_i^s + \bar{z}^s \qquad \forall s \in \mathcal{S}, \forall i \in \mathcal{C} \tag{32}$$

$$\tilde{z}_i^s \geq 0. \tag{33}$$

Equations (30) define the average shortage. Equations (31) - (33) linearize the absolute value function. Based on the above definition of equity, we can modify the objective function in each model by adding a new term $\sum_{s,i} \pi_i \tilde{z}_i^s$. Other approaches can maintain equity without violating linearity as well, such as minimizing the maximum shortage, or minimizing the difference between the minimum and the maximum shortage values.

B Clinic Clustering.

The purpose of the paper was not to employ formal clustering approaches from the literature or to provide proof for an optimal clustering method. Instead, we found a convenient clustering structure based on the optimal solution of the strategic level stochastic program. This facilitated the decomposition into clusters that allowed us to solve the operational problem. These clusters are heuristically defined, but make a lot of operational sense, because flows between clusters were found to be sufficiently small as to not significantly affect the optimal solutions, as demonstrated by our numerical studies of the optimal solution of the decomposition method versus the fully integrated model (§4.3). We believe that this clustering method should work for problems with a similar structure and that there is intuition from an epidemiological sense why this should be so. Fig. 16 depicts an example of clustering results as a histogram of cluster sizes.

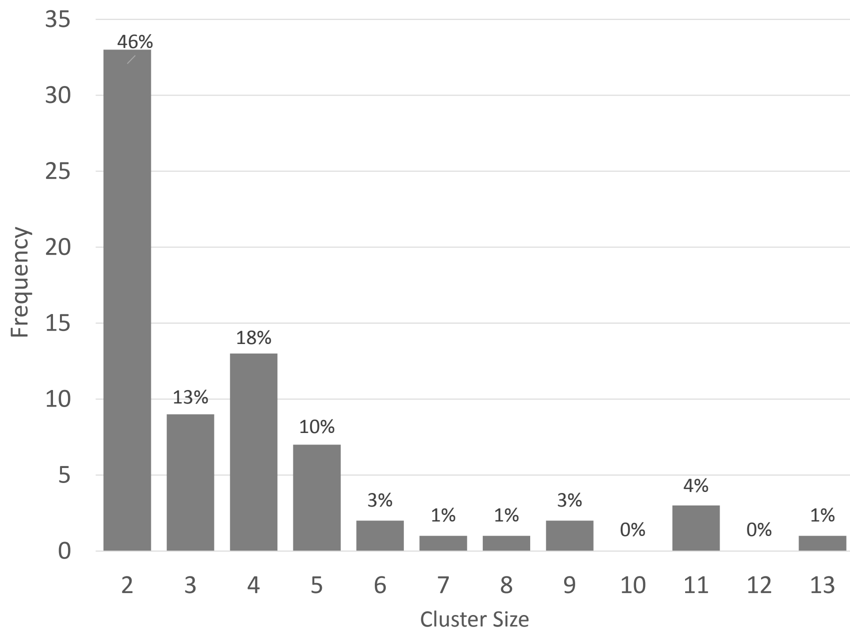


Figure 16: Clinic clusters histogram.

C Proofs.

Theorem 4.1. *Proof:* We prove this theorem by induction.

Base Case: $f_0(\mathbf{\Xi}) = 0$ for all $\mathbf{\Xi}$ and therefore is trivially non-increasing.

Induction Step: Assume$f_{n-1}(\mathbf{\Xi})$ is non-increasing in $\mathbf{\Xi}$.

$$\begin{aligned} f_n(\mathbf{\Xi}) - f_n(\mathbf{\Xi} - e_j) =& \Pi^{\mathrm{T}}(-\mathbf{\Xi})^+ + \min_{\mathbf{u}\in\mathcal{U}_{\Xi}}\{c\sum_{j\in\Phi}(u_j)^+ + \mathbb{E}\{f_{n-1}((\mathbf{\Xi})^+ + \mathbf{u} - \mathbf{d}_n)\}\}- \\ & \Pi^{\mathrm{T}}(-(\mathbf{\Xi} - e_j))^+ - \min_{\mathbf{u}\in\mathcal{U}_{\Xi-e_j}}\{c\sum_{j\in\Phi}(u_j)^+ + \mathbb{E}\{f_{n-1}((\mathbf{\Xi} - e_j)^+ + \mathbf{u} - \mathbf{d}_n)\}\}. \end{aligned}$$

e_j is the unit vector with 1 in the j^{th} dimension and 0's elsewhere. We compare the Equation term by term. First, the instantaneous cost is clearly greater in the system with less inventory:

$$\Pi^{\mathrm{T}}((-\mathbf{\Xi})^+ - (-(\mathbf{\Xi} - e_j))^+) \leq 0. \tag{34}$$

Next we compare the minimization term. If the optimal action, $\mathbf{u}^*$, is the same in both $f_n(\Xi)$ and $f_n(\Xi - e_j)$, it follows from the induction hypothesis that:

$$\mathbb{E}\{f_{n-1}((\Xi)^+ + \mathbf{u}^* - \mathbf{d}_n)\} - \mathbb{E}\{f_{n-1}((\Xi - e_j)^+ + \mathbf{u}^* - \mathbf{d}_n)\} \leq 0.$$

If, on the other hand, the optimal actions for $f_n(\Xi)$ and $f_n(\Xi - e_j)$ are different, without loss of generality we assume that optimal action in state $(\Xi - e_i)$ is $\mathbf{u}^0$. We then have:

$$\min_{\mathbf{u}\in\mathcal{U}_\Xi}\{c\sum_{j\in\Phi}(u_j)^+ + \mathbb{E}\{f_{n-1}(\Xi + \mathbf{u} - \mathbf{d}_n)\}\} - c\sum_{j\in\Phi}(u_j^0)^+ - \mathbb{E}\{f_{n-1}((\Xi - e_j)^+ + \mathbf{u}^0 - \mathbf{d}_n)\} \leq$$
$$c\sum_{j\in\Phi}(u_j^0)^+ + \mathbb{E}\{f_{n-1}((\Xi)^+ + \mathbf{u}^0 - \mathbf{d}_n) - c\sum_{j\in\Phi}(u_j^0)^+ - \mathbb{E}\{f_{n-1}((\Xi - e_j)^+ + \mathbf{u}^0 - \mathbf{d}_n)\} \leq 0. \quad (35)$$

Inequality (34) follows because the minimizing action at Ξ is clearly at least as small as action $\mathbf{u}^0$. Inequality (35) follows directly from the induction hypothesis. This completes the proof. □

Lemma 4.1. *Proof:* Without loss of generality let $\mathbf{d}_n$ be distributed as $q_{i,j}$ for $i, j = 1, \ldots, n$ where $q_{i,j} = q_{j,i}$ is the probability of observing i units of demand in clinic 1 and j units of demand in clinic 2 and vice versa. For notational convenience, for any state $\Xi = (\xi_1, \xi_2)$, define $\Delta\Xi = |\xi_1 - \xi_2|$ as the absolute difference between the inventory at the two clinics. Consider two different states, Ξ and Ξ', such that $\xi_1 + \xi_2 = \xi_1' + \xi_2'$ and $\Delta\Xi \leq \Delta\Xi'$. We now show that $\mathbb{E}[f_n(\Xi' - \mathbf{d}_n)] - \mathbb{E}[f_n(\Xi - \mathbf{d}_n)] \geq 0$.

$$\mathbb{E}[f_n(\Xi' - \mathbf{d}_n)] - \mathbb{E}[f_n(\Xi - \mathbf{d}_n)] = \sum_{i=1}^{n}\sum_{j=1}^{n} q_{i,j} f_n(\Xi' - i\mathbf{e}_1 - j\mathbf{e}_2) - \sum_{i=1}^{n}\sum_{j=1}^{n} q_{i,j} f_n(\Xi - i\mathbf{e}_1 - j\mathbf{e}_2)$$
$$= \sum_{i=1}^{n}\sum_{j=i}^{n}\Big(q_{i,j}[f_n(\Xi' - i\mathbf{e}_1 - j\mathbf{e}_2) - f_n(\Xi - i\mathbf{e}_1 - j\mathbf{e}_2)] +$$
$$q_{j,i}[f_n(\Xi' - j\mathbf{e}_1 - i\mathbf{e}_2) - f_n(\Xi - j\mathbf{e}_1 - i\mathbf{e}_2)]\Big). \quad (36)$$

We now perform a term by term comparison of (36). Without loss of generality assume that $\xi_1 < \xi_2$ and $\xi_1' < \xi_2'$. First note that if $j - i \leq \Delta\Xi$ then both terms within the sum are positive. Otherwise it is possible that $f_n(\xi_1' - i, \xi_2' - j) - f_n(\xi_1 - i, \xi_2 - j)$ is negative, while $f_n(\xi_1' - j, \xi_2' - i) - f_n(\xi_1 - j, \xi_2 - i)$ remains positive. What we show is that the magnitude of the negative portion is smaller than the magnitude of the positive portion, which implies that the sum of the negative and positive portions will be non-negative. To do so we consider two cases:

Case 1: $\Delta\Xi < j - i < \Delta\Xi'$.

First, when j is not too much larger than i we show that $f_n(\xi_1' - i, \xi_2' - j) - f_n(\xi_1 - i, \xi_2 - j) \geq 0$ so the sum of all 4 terms will be positive. In the cases where $f_n(\xi_1' - i, \xi_2' - j) - f_n(\xi_1 - i, \xi_2 - j) < 0$, we show that the imbalance between states $(\xi_1' - i, \xi_2' - j)$ and $(\xi_1 - i, \xi_2 - j)$ is smaller than the imbalance between the states of the positive terms: $(\xi_1' - j, \xi_2' - i)$ and $(\xi_1 - j, \xi_2 - i)$. This directly implies, by the fact that f_n is balanced, that $|f_n(\Xi' - i\mathbf{e}_1 - j\mathbf{e}_2) - f_n(\Xi - i\mathbf{e}_1 - j\mathbf{e}_2)| <$

$|f_n(\Xi' - j\mathbf{e}_1 - i\mathbf{e}_2) - f_n(\Xi - j\mathbf{e}_1 - i\mathbf{e}_2)|$ and therefore $f_n(\Xi' - i\mathbf{e}_1 - j\mathbf{e}_2) - f_n(\Xi - i\mathbf{e}_1 - j\mathbf{e}_2) + f_n(\Xi' - j\mathbf{e}_1 - i\mathbf{e}_2) - f_n(\Xi - j\mathbf{e}_1 - i\mathbf{e}_2) \geq 0$. The imbalance for each term of the sum in (36) is given below.

$$\Delta(\Xi' - i\mathbf{e}_1 - j\mathbf{e}_2) = \xi_2' - j - \xi_1' + i = \Delta\Xi' - (j - i), \tag{37}$$

$$\Delta(\Xi - i\mathbf{e}_1 - j\mathbf{e}_2) = \xi_1 - i - \xi_2 + j = -\Delta\Xi + (j - i), \tag{38}$$

$$\Delta(\Xi' - j\mathbf{e}_1 - i\mathbf{e}_2) = \xi_2' - i - \xi_1' + j = \Delta\Xi' + (j - i), \tag{39}$$

$$\Delta(\Xi - j\mathbf{e}_1 - i\mathbf{e}_2) = \xi_2 - i - \xi_1 + j = \Delta\Xi + (j - i). \tag{40}$$

In Equations (37) and (38), if $-\Delta\Xi+(j-i) \leq \Delta\Xi'-(j-i)$, then $|\xi_2-j-(\xi_1-i)| < |\xi_2'-j-(\xi_1'-i)|$ and since f_n is balanced we have that $f_n(\xi_1' - i, \xi_2' - j) - f_n(\xi_1 - i, \xi_2 - j) \geq 0$, so that term of the sum in (36) will be positive. If, however, the opposite is true, then the amount of imbalance for the negative term – which directly correlates with the magnitude – is given by subtracting (37) from (38). In this situation, the state $(\xi_1' - i, \xi_2' - j)$ actually becomes more balanced than the state $(\xi_1 - i, \xi_2 - j)$. Therefore the difference in the amount of imbalance of the negative term is given by:

$$0 \leq -\Delta\Xi + (j - i) - (\Delta\Xi' - (j - i)) < -\Delta\Xi + (j - i) < \Delta\Xi' - \Delta\Xi. \tag{41}$$

The first inequality holds by the assumption that $-\Delta\Xi + (j - i) \geq (\Delta\Xi' - (j - i))$. Then second inequality holds because we have $j - i < \Delta\Xi' \Rightarrow \Delta\Xi' - (j - i) > 0$. The final inequality follows from the fact that $(j - i) < \Xi'$.

Likewise we know that the difference in imbalance for the positive term, $f_n(\xi_1' - j, \xi_2' - i) - f_n(\xi_1 - j, \xi_2 - i)$, is at least as large as the difference in imbalance for the negative term by subtracting (40) from (39).

$$0 \leq \Delta\Xi' + (j - i) - (\Delta\Xi + (j - i)) = \Delta\Xi' - \Delta\Xi. \tag{42}$$

Where the inequality follows from the fact that the Ξ' term is more imbalanced than the Ξ term and the equality follows directly. Clearly the negative term has less difference in imbalance between its components than the positive term, and therefore $|f_n(\Xi' - i\mathbf{e}_1 - j\mathbf{e}_2) - f_n(\Xi - i\mathbf{e}_1 - j\mathbf{e}_2)| < |f_n(\Xi' - j\mathbf{e}_1 - i\mathbf{e}_2) - f_n(\Xi - j\mathbf{e}_1 - i\mathbf{e}_2)|$, which implies that $f_n(\Xi' - i\mathbf{e}_1 - j\mathbf{e}_2) - f_n(\Xi - i\mathbf{e}_1 - j\mathbf{e}_2) + f_n(\Xi' - j\mathbf{e}_1 - i\mathbf{e}_2) - f_n(\Xi - j\mathbf{e}_1 - i\mathbf{e}_2) \geq 0$.

Case 2: $\Delta\Xi' \leq j - i$.

This case is straightforward, because we now have that for the pair of terms for the negative term, $f_n(\xi_1' - i, \xi_2' - j) - f_n(\xi_1 - i, \xi_2 - j)$, both $\xi_2 - j < \xi_1 - i$ and $\xi_2' - j < \xi_1' - i$. Therefore the imbalance for each component is now given by:

$$\Delta(\Xi' - i\mathbf{e}_1 - j\mathbf{e}_2) = \xi_1' - i - \xi_2' + j = -\Delta\Xi' + (j - i) \tag{43}$$

$$\Delta(\Xi - i\mathbf{e}_1 - j\mathbf{e}_2) = \xi_1 - i - \xi_2 + j = -\Delta\Xi + (j - i) \tag{44}$$

$$\Delta(\Xi' - j\mathbf{e}_1 - i\mathbf{e}_2) = \xi_2' - i - \xi_1' + j = \Delta\Xi' + (j - i) \tag{45}$$

$$\Delta(\Xi - j\mathbf{e}_1 - i\mathbf{e}_2) = \xi_2 - i - \xi_1 + j = \Delta\Xi + (j - i) \tag{46}$$

For the negative term from (36), $f_n(\Xi' - i\mathbf{e}_1 - j\mathbf{e}_2) - f_n(\Xi - i\mathbf{e}_1 - j\mathbf{e}_2)$, the Ξ' component is more balanced, (43), than the Ξ component, (44). The difference in imbalance between the two terms is given by:

$$-\Delta\Xi + (j - i) - (-\Delta\Xi' + (j - i) = \Delta\Xi' - \Delta\Xi. \tag{47}$$

For the positive term, the difference in imbalance remains the same:

$$\Delta\Xi' + (j - i) - (\Delta\Xi + (j - i)) = \Delta\Xi' - \Delta\Xi. \tag{48}$$

Therefore in Case 2, the difference in imbalance between the components of the negative term and the difference in the imbalance between the components of the positive term are equal and thus the subtraction will be 0. □

Lemma 4.2. *Proof:* To prove this, we show that the action of "do nothing" will result in less cost than shipping from the clinic with a lower inventory level to the one with a higher stock of medication. Let $\xi_1 \leq \xi_2$. If the action is to do nothing, the cost function will be $J_n^0 = \Pi^{\mathrm{T}}(-\Xi)^+ + 0 + \mathbb{E}\{f_{n-1}((\Xi)^+ - \mathbf{d}_n)\}$, otherwise the amount of $\widehat{u}$ medication is moved from clinic 1 to clinic 2 ($\widehat{\mathbf{u}} = (-\widehat{u}, \widehat{u})$), we have $J_n^{\widehat{\mathbf{u}}} = \Pi^{\mathrm{T}}(-\Xi)^+ + c\widehat{u} + \mathbb{E}\{f_{n-1}((\Xi)^+ - \widehat{\mathbf{u}} - \mathbf{d}_n)\}$. Comparing the two term by term:

$$c\widehat{u} \geq 0, \tag{49}$$

$$\mathbb{E}\{f_{n-1}((\Xi - \widehat{u}\mathbf{e}_1 + \widehat{u}\mathbf{e}_2)^+ - \mathbf{d}_n)\} \geq \mathbb{E}\{f_{n-1}((\Xi)^+ - \mathbf{d}_n)\} \tag{50}$$

Equation (50) follows from the fact that $|\xi_2 - \xi_1| \leq |\xi_2 + \widehat{u} - \xi_1 + \widehat{u}|$ and that f_n is balanced, which carries through to the expectation via Lemma 4.1. From (49) and (50), $f_n^{\widehat{u}}(\Xi) \geq f_n^0(\Xi)$ follows directly. □

Lemma 4.3. *Proof:* Consider Ξ^* where $\xi_1^* = \xi_2^*$ versus Ξ where $\xi_1 \neq \xi_2$.

$$f_n(\Xi) = \Pi^{\mathrm{T}}(-\Xi)^+ + \min_{u \in \mathcal{U}_\Xi}\{c\sum_{j\in\Phi}(u_j)^+ + \mathbb{E}[f_{n-1}((\Xi)^+ + \mathbf{u} - \mathbf{d}_n)]\},$$

$$f_n(\Xi^*) = \Pi^{\mathrm{T}}(-\Xi^*)^+ + \min_{u \in \mathcal{U}_{\Xi^*}}\{c\sum_{j\in\Phi}(u_j)^+ + \mathbb{E}[f_{n-1}((\Xi^*)^+ + \mathbf{u} - \mathbf{d}_n)]\}.$$

We know that $f_n(\cdot)$ is balanced, thus by Lemma 4.1 for all n, $\mathbb{E}\{f_{n-1}((\Xi)^+ + \mathbf{u} - \mathbf{d}_n)\}$ is also balanced. Therefore the optimal action at Ξ^* is $\mathbf{u}^* = 0$, which achieves the lowest possible value for the expectation. Thus we have:

$$\mathbb{E}\{f_{n-1}((\Xi^*)^+ - \mathbf{d}_n)\} \leq \mathbb{E}\{f_{n-1}((\Xi)^+ + \mathbf{u} - \mathbf{d}_n)\}, \tag{51}$$

and because the optimal policy at Ξ^* has no penalty because $\mathbf{u}^* = 0$ it is clear that:

$$\min_{u \in \mathcal{U}_{\Xi^*}}\{c\sum_{j\in\Phi}(u_j)^+ + \mathbb{E}[f_{n-1}((\Xi^*)^+ + \mathbf{u} - \mathbf{d}_n)]\} \leq \min_{u \in \mathcal{U}_\Xi}\{c\sum_{j\in\Phi}(u_j)^+ + \mathbb{E}[f_{n-1}((\Xi)^+ + \mathbf{u} - \mathbf{d}_n)]\}. \tag{52}$$

Finally, it can quickly be verified that the instantaneous cost is lower for the balanced inventory (Ξ^*):

$$\Pi^{\mathrm{T}}(-\Xi^*)^+ \leq \Pi^{\mathrm{T}}(-\Xi)^+. \tag{53}$$

Thus we have shown that $f_n(\Xi^*) \leq f_n(\Xi)$. □

Theorem 4.2. *Proof:*

Proof: Without loss of generality, we assume $\xi_1 \leq \xi_2$ and $\xi_1' \leq \xi_2'$. This ordering along with our assumption that $\xi_1 + \xi_2 = \xi_1' + \xi_2'$ and $\Delta\Xi = |\xi_1 - \xi_2| \leq |\xi_1' - \xi_2'| = \Delta\Xi'$, implies that $\xi_1' \leq \xi_1 \leq \xi_2 \leq \xi_2'$. We proceed to prove this theorem by using induction.

Base Case: Since $f_0(\Xi) = 0$ for all Ξ. As a result, the induction hypothesis holds.

Induction Step: We assume that the induction hypothesis holds for stage $n-1$. In order to show $f_n(\Xi)$ is less than or equal to $f_n(\Xi')$, we first write the expressions for both cases:

$$f_n(\Xi) = \Pi^{\mathrm{T}}(-\Xi)^+ + \min_{u \in \mathcal{U}_\Xi}\{c\sum_{j\in\Phi}(u_j)^+ + \mathbb{E}[f_{n-1}((\Xi)^+ + \mathbf{u} - \mathbf{d}_n)]\},$$
$$f_n(\Xi') = \Pi^{\mathrm{T}}(-\Xi')^+ + \min_{u \in \mathcal{U}_{\Xi'}}\{c\sum_{j\in\Phi}(u_j)^+ + \mathbb{E}[f_{n-1}((\Xi')^+ + \mathbf{u} - \mathbf{d}_n)]\}.$$

Let $\bar{\xi} = \lfloor \frac{\xi_1+\xi_2}{2} \rfloor$. Applying the inductive hypothesis that f_{n-1} is a balanced function, and Lemma 4.1 and Lemma 4.3, at stage $n-1$ the state $[\bar{\xi}, \xi_1 + \xi_2 - \bar{\xi}] = [\bar{\xi}', \bar{\xi}']$ achieves the minimum value for the expectation of the function $f_{n-1}(\cdot)$. By the induction hypothesis we have $f_{n-1}(\Xi') \geq f_{n-1}(\Xi)$. Since $\xi_1 + \xi_2 = \xi_1' + \xi_2'$, we can conjuncture that $f_{n-1}(\bar{\Xi})$ is the state which reaches the minimum possible cost.

$$\begin{aligned} f_n(\Xi') - f_n(\Xi) =& \Pi^{\mathrm{T}}(-\Xi')^+ + \min_{\mathbf{u} \in \mathcal{U}_{\Xi'}}\{c\sum_{j\in\Phi}(u_j)^+ + \mathbb{E}\{f_{n-1}((\Xi')^+ + \mathbf{u} - \mathbf{d}_n)\}\} - \\ & \Pi^{\mathrm{T}}(-\Xi)^+ - \min_{\mathbf{u} \in \mathcal{U}_{\Xi}}\{c\sum_{j\in\Phi}(u_j)^+ + \mathbb{E}\{f_{n-1}((\Xi)^+ + \mathbf{u} - \mathbf{d}_n)\}\}. \end{aligned}$$

First we compare the instant cost associated with the shortage penalties. As we mentioned before, without loss of generality, we consider the cases where $\xi_1 \leq \xi_2$ and $\xi_1' \leq \xi_2'$. Other cases can be investigated similarly. There are the following cases:

1. $\xi_1', \xi_2', \xi_1, \xi_2 \geq 0$: In this case both $f_n(\Xi)$ and $f_n(\Xi')$ incur zero shortage penalties. As a result $\Pi^{\mathrm{T}}(-\Xi')^+ - \Pi^{\mathrm{T}}(-\Xi)^+ = 0$.

2. $\xi_1' \leq 0$ and $\xi_2, \xi_1, \xi_2' \geq 0$: In this case $f_n(\Xi')$ has a positive shortage cost while $f_n(\Xi)$ incurs zero shortage penalty. Therefore $\Pi^{\mathrm{T}}(-\Xi')^+ - \Pi^{\mathrm{T}}(-\Xi)^+ \geq 0$.

3. $\xi_1', \xi_1 \leq 0$ and $\xi_2, \xi_2' \geq 0$: In this case $f_n(\Xi')$ and $f_n(\Xi)$ both have positive shortage cost but since $\xi_1' \leq \xi_1$, $f_n(\Xi)$ has a greater shortage penalty. As a result $\Pi^{\mathrm{T}}(-\Xi')^+ - \Pi^{\mathrm{T}}(-\Xi)^+ \geq 0$.

4. $\xi_1', \xi_1, \xi_2 \leq 0$ and $\xi_2' \geq 0$: We conclude that having $\xi_2' \geq 0$, will result in $\xi_1' \leq \xi_1 + \xi_2$ as a direct result of the assumption, since $\xi_1 + \xi_2 = \xi_1' + \xi_2'$. Therefore $\Pi^{\mathrm{T}}(-\boldsymbol{\Xi}')^+ - \Pi^{\mathrm{T}}(-\boldsymbol{\Xi})^+ \geq 0$.

5. $\xi_1', \xi_1, \xi_2, \xi_2' \leq 0$: this implies that $\Pi^{\mathrm{T}}(-\boldsymbol{\Xi}')^+ = \Pi^{\mathrm{T}}(-\boldsymbol{\Xi})^+ \geq 0$.

The next step is to investigate the possible actions and compare the cost-to-go terms for both $f_n(\boldsymbol{\Xi})$ and $f_n(\boldsymbol{\Xi}')$. Let assume the optimal action in state $\boldsymbol{\Xi}'$ is $u^{'*}$. Having $\xi_1' \leq \xi_2'$ and based on the result of Lemma 4.2, $u^{'*} \in= \{0, ..., u_{\bar{\xi}'}\} = \mathcal{U}_{\Xi'}$ and $\mathbf{u}^{'*} = (u^{'*}, -u^{'*})$, the optimal action either will be to ship from clinic 2 to clinic 1 or do nothing (result of Lemma 4.2). We also know that $\xi_2' - \xi_1' \geq \xi_2 - \xi_1$, as a result, the optimal action u^* of the state $\boldsymbol{\Xi}'$ will be a member of $\{0, ..., u_{\bar{\xi}}\}$. We have:

$$\mathcal{U}_{\Xi} \subset \mathcal{U}_{\Xi'}$$

There are two possible scenarios, either $\mathbf{u}^{'*} \in \mathcal{U}_{\Xi}$ or $\mathbf{u}^{'*} \in \mathcal{U}_{\Xi'} \setminus \mathcal{U}_{\Xi'}$.

1. Scenario 1: $u^{'*} \in \mathcal{U}_{\Xi}$:

$$\begin{aligned}
&f_n(\boldsymbol{\Xi}') - f_n(\boldsymbol{\Xi}) \geq \overbrace{\Pi^{\mathrm{T}}(-\boldsymbol{\Xi}')^+ - \Pi^{\mathrm{T}}(-\boldsymbol{\Xi})^+}^{=Q^1 \geq 0} + \\
&cu^{'*} + \mathbb{E}\{f_{n-1}((\boldsymbol{\Xi}')^+ + \mathbf{u}^{'*} - \mathbf{d}_n)\} - \min_{\mathbf{u} \in \mathcal{U}_{\Xi}} \{c \sum_{j \in \Phi} (u_j)^+ + \mathbb{E}\{f_{n-1}((\boldsymbol{\Xi})^+ + \mathbf{u} - \mathbf{d}_n)\}\} \geq \\
&Q^1 + cu^{'*} + \mathbb{E}\{f_{n-1}((\boldsymbol{\Xi}')^+ + \mathbf{u}^{'*} - \mathbf{d}_n)\} - cu^{'*} + +E\{f_{n-1}((\boldsymbol{\Xi})^+ + \mathbf{u}^{'*} - \mathbf{d}_n)\} = \\
&\mathbb{E}\{f_{n-1}((\boldsymbol{\Xi}')^+ + \mathbf{u}^{'*} - \mathbf{d}_n)\} - E\{f_{n-1}((\boldsymbol{\Xi})^+ + \mathbf{u}^{'*} - \mathbf{d}_n)\} \geq 0. \quad \text{by induction hypothesis}
\end{aligned}$$

2. Scenario 2: $\mathbf{u}^{'*} \in \mathcal{U}_{\Xi'} \setminus \mathcal{U}_{\Xi}$:

$$\begin{aligned}
&f_n(\boldsymbol{\Xi}') - f_n(\boldsymbol{\Xi}) \geq \overbrace{\Pi^{\mathrm{T}}(-\boldsymbol{\Xi}')^+ - \Pi^{\mathrm{T}}(-\boldsymbol{\Xi})^+}^{Q^2 \geq 0} + \\
&cu^{'*} + \mathbb{E}\{f_{n-1}((\boldsymbol{\Xi}')^+ + \mathbf{u}^{'*} - \mathbf{d}_n)\} - \min_{\mathbf{u} \in \mathcal{U}_{\Xi}} \{c \sum_{j \in \Phi} (u_j)^+ + \mathbb{E}\{f_{n-1}((\boldsymbol{\Xi})^+ + \mathbf{u} - \mathbf{d}_n)\}\} \geq \\
&Q^2 + \overbrace{cu^{'*} - cu_{\bar{\xi}}}^{B \geq 0} + \overbrace{E\{f_{n-1}((\boldsymbol{\Xi}')^+ + \mathbf{u}^*_{\xi'} - \mathbf{d}_n)\} - E\{f_{n-1}((\boldsymbol{\Xi})^+ + \mathbf{u}_{\bar{\xi}} - \mathbf{d}_n)\}}^{C \geq 0} \geq 0. \\
&\text{by induction hypothesis}
\end{aligned}$$

We should note that $B \geq 0$ since the total units shipped from clinic 2 to clinic 1 in scenario 2 is more than $u_{\bar{\xi}}$. Also after transshipping $u_{\bar{\xi}}$, $f_{n-1}(\boldsymbol{\Xi})$ is reaching its minimum (as a direct result of Lemma 4.3), therefore $C \geq 0$. This ends the proof. □