Online Appendix

Dynamic Pricing and Replenishment with Customer Upgrades

Online Appendix:

Glossary of Terms: Below, we provide a list of the main notation (in a rough order of appearance) used for the proof of the main results presented in the Appendix. For $i, j = \{1, 2\}$,

x_i^t	:	inventory position of product type-i at the beginning of period t where product type-1 refers to the higher quality product and product type-2 refers to the lower quality product
w_i^t	:	intermediate inventory position of product type-i in period t after demand realization
w^t	:	total intermediate inventory in period t after demand realization, i.e., $w^t = w_1^t + w_2^t$
u_i^t	:	ending inventory position for product type-i after upgrades
d_i^t	:	mean demand for product type-i
y_i^t	:	replenishment level for product type-i
z_i^t	:	target safety stock level for product type-i, i.e., $z_i^t = y_i^t - d_i^t$
$V^t(x_1^t, x_2^t)$	:	first-stage optimal value function starting at state (x_1^t, x_2^t) with t periods remaining
$G^t(w_1^t, w_2^t)$	:	second-stage optimal value function starting at state (w_1^t, w_2^t) with t periods remaining
V_i^t	:	partial derivative of $V^t(\cdot)$ with respect to its i^{th} argument (similarly defined for $G^t(\cdot)$)
V_{ij}^t	:	second partial derivative of $V^t(\cdot)$ with respect to its i^{th} and j^{th} arguments (similarly defined for $G^t(\cdot)$)
$\bar{G}^t(\cdot)$	:	second-stage profit as a function of the inventory position for product type-i after upgrades
$J^t(\cdot)$	:	first stage profit as a function of the decision variables z_i^t and d_i^t
$J_{d_i}^t$	:	partial derivative of $J^t(\cdot)$ with respect to d_i^t
$J_{z_i}^t$	:	partial derivative of $J^t(\cdot)$ with respect to z_i^t
J_{d_i,d_j}^t	:	second partial derivative of $J^t(\cdot)$ with respect to d_i^t and d_j^t
J_{z_i,z_j}^t	:	second partial derivative of $J^t(\cdot)$ with respect to z_i^t and z_j^t

In order to derive the optimal policy structure, we first make an inductional assumption on the properties of the value function $V^t(x_1^t, x_2^t)$. We will then show that these properties hold throughout the dynamic programming recursions. In the following analysis, we assume that $V^t(x_1^t, x_2^t)$ is twice-continuously differentiable. As part of the inductional step, assume that the value function in period $t-1$, $V^{t-1}(x_1^{t-1}, x_2^{t-1})$ satisfies the following properties:

Induction Assumption: $V^{t-1}(x_1^{t-1}, x_2^{t-1})$ is jointly concave, submodular, and its Hessian is diagonally dominant: $V_{ii}^{t-1}(x_1^{t-1}, x_2^{t-1}) \leq V_{ij}^{t-1}(x_1^{t-1}, x_2^{t-1}) \leq 0$ where V_{ij}^{t-1} represents $\frac{\partial^2 V^{t-1}}{\partial x_i^{t-1} \partial x_j^{t-1}}$ for $i, j = 1, 2$.

These properties enable us to derive the structure of the optimal upgrade policy in period t. After characterizing the optimal upgrade policy, and later the optimal production and pricing policies, we will subsequently show in the forthcoming Lemma 4 that these properties also hold for $V^t(x_1^t, x_2^t)$. Note that the induction assumption is trivially satisfied for $V^0(x_1^0, x_2^0)$.

Proof of Theorem 1 (Optimal Upgrade Policy):

The optimal upgrade policy is determined by solving the second-stage problem described in (4), which we analyze through a variable transformation. For any intermediate inventory position w_1^t and w_2^t, with a total intermediate inventory position $w^t = w_1^t + w_2^t$, let $\bar{G}^t(u_1^t)$ be defined such that $\bar{G}^t(u_1^t) = -h_1(u_1^t) - h_2(w^t - u_1^t) + \beta V^{t-1}(u_1^t, w^t - u_1^t)$ where u_1^t and $w^t - u_1^t$ represent, respectively, the period ending inventory positions for product type-1 and product type-2 after upgrades. In particular, when u^t units of upgrades are given, we have $u_1^t = w_1^t - u^t$. Thus, the choice of u_1^t will immediately determine the upgrade quantity u^t. The constraint $u_1^t \le w_1^t$, i.e., $u^t \ge 0$, guarantees that the upgrade quantity is nonnegative, implying unidirectional product substitutions for the demand for the lower quality product by a higher quality product and not vice versa.

The first derivative of $\bar{G}^t(u_1^t)$ with respect to u_1^t is given by

$$\begin{aligned}\frac{d\bar{G}^t(u_1^t)}{du_1^t} = -h_1^+ I_{(u_1^t>0)} + h_1^- I_{(u_1^t<0)} + h_2^+ I_{(w>u_1^t)} - h_2^- I_{(w<u_1^t)} \\ + \beta V_1^{t-1}(u_1^t, w - u_1^t) - \beta V_2^{t-1}(u_1^t, w - u_1^t)\end{aligned} \tag{8}$$

where $I_{(\cdot)}$ denotes the indicator function and $V_1^{t-1}(\cdot,\cdot)$ denotes the partial derivative of $V^{t-1}(\cdot,\cdot)$ with respect to its first argument and $V_2^{t-1}(\cdot,\cdot)$ denotes the partial derivative of $V^{t-1}(\cdot,\cdot)$ with respect to its second argument. For expositional clarity, when a function's arguments are evident, we suppress the notation and write for example, $\bar{G}^t$, V^t, or V_i^t and V_{ij}^t for $i,j = 1,2$. The second derivative of $\bar{G}^t(u_1^t)$ with respect to u_1^t is $\beta(V_{11}^{t-1} - V_{12}^{t-1}) + \beta(V_{22}^{t-1} - V_{21}^{t-1})$. By the induction assumption, V^{t-1} is concave, submodular and its Hessian has diagonal dominance property, i.e., $V_{11}^{t-1} \le V_{12}^{t-1} \le 0$ and $V_{22}^{t-1} \le V_{21}^{t-1} \le 0$. Hence, $\bar{G}^t$ is concave in u_1^t and its first derivative with respect to u_1^t is decreasing. Let $r^t(w^t)$ be defined such that, if for given w^t, $\frac{d\bar{G}^t(u_1^t)}{du_1^t} > 0$ for all u_1^t, then $r^t(w^t) = \infty$. Else, $r^t(w^t) = \min\{u_1^t \mid \frac{d\bar{G}^t(u_1^t)}{du_1^t} \le 0\}$. Then, given w^t, $r^t(w^t)$ is the optimal protection level for product type-1 and we can express the optimal upgrade quantity, u^{t*}, through this protection level. Specifically, $u^{t*} = (w_1^t - r^t(w^t))^+$.

To show that $r^t(w^t)$ is increasing with respect to w^t, consider $\overline{w}^t$ and $\underline{w}^t$ such that $\overline{w}^t > \underline{w}^t$. We would like to show that $r^t(\overline{w}^t) \ge r^t(\underline{w}^t)$. Let $g(u_1^t, w^t)$ represent the first derivative given in (8) as a function of u_1^t and w^t. We have $\frac{\partial g(u_1^t, w^t)}{\partial w^t} = \beta(V_{12}^{t-1} - V_{22}^{t-1})$. Since $V_{12}^{t-1} \ge V_{22}^{t-1}$ by the induction assumption, we have $g(u_1^t, \overline{w}^t) - g(u_1^t, \underline{w}^t) \ge 0$. The result then immediately follows from the definition of $r^t(w^t) = \min\{u_1^t \mid \frac{d\bar{G}^t(u_1^t)}{du_1^t} \le 0\}$. In order to show that $r^t(w^t) - w^t$ is decreasing with respect to w^t, we utilize a different variable transformation. Specifically, let $u_2^t = w_2^t + u^t$ denote the ending inventory for product type-2 after u^t units of upgrades. Hence, $w^t - u_2^t$ will denote the inventory level for product type-1 after upgrades. We can then rewrite $\bar{G}^t(u_2^t) = -h_1(w^t - u_2^t) - h_2(u_2^t) + \beta V^{t-1}(w^t - u_2^t, u_2^t)$. Define u_2^{t*} as the optimal inventory position for product type-2 after upgrades. Note that since the total inventory after upgrades equals total inventory prior to upgrades, we have $u_2^{t*} + u_1^{t*} = w^t$. The first and second derivatives of $\bar{G}^t(u_2^t)$ with respect to u_2^t are given by $h_1^+ I_{(u_2^t<w^t)} - h_1^- I_{(u_2^t>w^t)} - h_2^+ I_{(u_2^t>0)} + h_2^- I_{(u_2^t<0)} - \beta V_1^{t-1}(w^t - u_2^t, u_2^t) + \beta V_2^{t-1}(w^t - u_2^t, u_2^t)$ and $\beta(V_{11}^{t-1} - V_{12}^{t-1}) + \beta(V_{22}^{t-1} - V_{21}^{t-1})$, respectively. Due to the inductional assumption, $\bar{G}^t(u_2^t)$ is concave with respect to u_2^t. Since the first derivative of $\bar{G}^t(u_2^t)$ is increasing with w^t, we find u_2^{t*} is increasing with w^t. This in turn directly implies $w^t - u_1^{t*}$ is increasing in w^t, or equivalently $u_1^{t*} - w^t$ is decreasing in w^t. Note that when the first derivative of $\bar{G}^t(u_2^t) < 0$ for all u_2^t, we have $u_2^{t*} = w_2^t$ and thus $u_1^{t*} = w_1^t$, i.e., no upgrades are given and, as before, the protection threshold can be stated as $r^t(w^t) = \infty$. Otherwise, when the first derivative of $\bar{G}^t(u_2^t) \ge 0$, the protection level $r^t(w^t)$ equals u_1^{t*}. Therefore, $u_1^{t*} - w^t$ decreasing in w^t implies $r^t(w^t) - w^t$ is decreasing in w^t. □

Proof of Theorem 2 (Optimal Replenishment Policy):

The outline for the proof of Theorem 2 is as follows. We first present a reformulation of the first-stage problem with change of variables that facilitate the subsequent analysis. We then derive several structural properties on the first-stage profit function. Finally, we complete the characterization of the optimal replenishment policy structure.

In order to simplify the analysis, we define a new set of variables (z_1^t, z_2^t) such that $z_i^t = y_i^t - d_i^t$ for $i = 1,2$, where, as a reminder to the reader, y_i^t and d_i^t denote the replenish-up-to level and target mean demand for product type-i, respectively. An economic interpretation of z_i^t is that it represents the target safety-stock level for product type-i after its current inventory position is augmented by the replenishment quantity and depleted by the expected demand for the product. In addition, rather than the prices (p_1^t, p_2^t), we work with the decision variable pair (d_1^t, d_2^t) referring to mean demands. This choice simplifies the exposition of our results. The prices of both products can then be determined from the expected demands for each product. Using (z_1^t, z_2^t) and (d_1^t, d_2^t), as the new decision variables, we rewrite the first-stage problem equivalently as follows:

$$V^t(x_1^t, x_2^t) = c_1^t x_1^t + c_2^t x_2^t + \max_{\substack{z_i^t, d_i^t \\ x_i^t \le z_i^t + d_i^t \le x_i^t + K_i}} J^t(z_1^t, z_2^t, d_1^t, d_2^t) \tag{9}$$

where $J^t(z_1^t, z_2^t, d_1^t, d_2^t) = d_1^t p_1^t(d_1^t, d_2^t) + d_2^t p_2^t(d_1^t, d_2^t) - (c_1(z_1^t + d_1^t) + c_2(z_2^t + d_2^t)) + \mathrm{E}_{\epsilon_1^t, \epsilon_2^t}[(G^t(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t)]$. The following lemma provides several structural properties on the second partials of $J^t(z_1^t, z_2^t, d_1^t, d_2^t)$ that will be utilized to derive the optimal production and pricing policies. In the following analysis, we assume that $J^t(z_1^t, z_2^t, d_1^t, d_2^t)$ is twice-continuously differentiable. Let $J^t_{z_1,z_2}$ refer to the second partials of $J^t(z_1^t, z_2^t, d_1^t, d_2^t)$ with respect to the variables z_1^t and z_2^t and $J^t_{d_1,d_2}$ to refer to the second partials of $J^t(z_1^t, z_2^t, d_1^t, d_2^t)$ with respect to the variables d_1^t and d_2^t.

LEMMA 1. *$J^t(z_1^t, z_2^t, d_1^t, d_2^t)$ is strictly concave and (a) $J^t(z_1^t, z_2^t, d_1^t, d_2^t)$ is submodular in (d_1^t, d_2^t) and possesses the following strict diagonal dominance property: $J^t_{d_1,d_1} < J^t_{d_1,d_2} \le 0$ and $J^t_{d_2,d_2} < J^t_{d_2,d_1} \le 0$; and (b) $J^t(z_1^t, z_2^t, d_1^t, d_2^t)$ is submodular in (z_1^t, z_2^t) and possesses the following strict diagonal dominance property: $J^t_{z_1,z_1} < J^t_{z_1,z_2} \le 0$ and $J^t_{z_2,z_2} < J^t_{z_2,z_1} \le 0$.*

Proof: We first note that $J^t(z_1^t, z_2^t, d_1^t, d_2^t)$ is separable in (d_1^t, d_2^t) and (z_1^t, z_2^t). For part (a), first note that the inverse price-demand relationships corresponding to (1) and (2) are given by $p_1(d_1, d_2) = \frac{\bar{p}(q_1 - \underline{q}) + \underline{p}(\bar{q} - q_1)}{(\bar{q} - \underline{q})} - \frac{(\bar{q} - q_1)}{\delta(\bar{q} - \underline{q})}\Big[(q_1 - \underline{q})d_1 + (q_2 - \underline{q})d_2\Big]$ and $p_2(d_1, d_2) = \frac{\bar{p}(q_2 - \underline{q}) + \underline{p}(\bar{q} - q_2)}{(\bar{q} - \underline{q})} - \frac{(q_2 - \underline{q})}{\delta(\bar{q} - \underline{q})}\Big[(\bar{q} - q_1)d_1 + (\bar{q} - q_2)d_2\Big]$. We denote the partial derivative of $J^t(z_1^t, z_2^t, d_1^t, d_2^t)$ with respect to d_i^t by $J^t_{d_i}$, and have $J^t_{d_i} = \frac{\partial}{\partial d_i^t}\Big(d_1^t p_1^t(d_1^t, d_2^t) + d_2^t p_2^t(d_1^t, d_2^t) - c_1^t d_1^t - c_2^t d_2^t\Big)$ for $i = 1, 2$. Written explicitly, we have

$$\begin{aligned} J^t_{d_1} &= \frac{\bar{p}(q_1 - \underline{q}) + \underline{p}(\bar{q} - q_1)}{(\bar{q} - \underline{q})} - \frac{2(\bar{q} - q_1)}{\delta(\bar{q} - \underline{q})}\Big[(q_1 - \underline{q})d_1 + (q_2 - \underline{q})d_2\Big] - c_1^t \\ J^t_{d_2} &= \frac{\bar{p}(q_2 - \underline{q}) + \underline{p}(\bar{q} - q_2)}{(\bar{q} - \underline{q})} - \frac{2(q_2 - \underline{q})}{\delta(\bar{q} - \underline{q})}\Big[(\bar{q} - q_1)d_1 + (\bar{q} - q_2)d_2\Big] - c_2^t \end{aligned} \tag{10}$$

Further, the Hessian is given by $-\frac{2}{\delta(\bar{q} - \underline{q})}\begin{bmatrix} (\bar{q} - q_1)(q_1 - \underline{q}) & (\bar{q} - q_1)(q_2 - \underline{q}) \\ (\bar{q} - q_1)(q_2 - \underline{q}) & (\bar{q} - q_2)(q_2 - \underline{q}) \end{bmatrix}$. Since $\bar{q} > q_1 > q_2 > \underline{q}$, $J^t(z_1^t, z_2^t, d_1^t, d_2^t)$ is strictly concave and submodular in (d_1^t, d_2^t), and possesses strict diagonal dominance property.

For part (b), since $c_1 z_1^t + c_2 z_2^t$ is linear in (z_1^t, z_2^t), it suffices to show that the properties hold for $\mathrm{E}_{\epsilon_1^t, \epsilon_2^t}[(G^t(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t)]$. In order to show the strict concavity and diagonal dominance, consider a momentary partitioning of the function $G^t(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t)$ such that $G^t(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t) = H^t(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t) + \hat{G}^t(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t)$ where $H^t(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t) := -\big(h_1^{t+}(z_1^t - \epsilon_1^t)^+ + h_1^{t-}(z_1^t - \epsilon_1^t)^- + h_2^{t+}(z_2^t - \epsilon_2^t)^+ + h_2^{t-}(z_2^t - \epsilon_2^t)^-\big)$ and $\hat{G}^t(w_1^t, w_2^t) := \max_{u_1^t} \bar{G}'^t(w_1^t, w_2^t, u_1^t)$ s.t. $w_1^t - u_1^t \ge 0$ where $\bar{G}'^t(w_1^t, w_2^t) := H'^t(w_1^t, w_2^t, u_1^t) + \beta V^{t-1}(u_1^t, w_1^t + w_2^t - u_1^t)$ and $H'^t(w_1^t, w_2^t, u_1^t) := \big(h_1^{t+}(w_1^t - u_1^t)\big) I_{(w_1^t \ge u_1^t > 0)} + \big(h_1^{t+} w_1^t + h_1^{t-} u_1^t\big) I_{(w_1^t > 0 > u_1^t)} + \big(h_1^{t-}(w_1^t - u_1^t)\big) I_{(0 > w_1^t \ge u_1^t)} + \big(h_2^{t+}(w_1^t - u_1^t)\big) I_{(w_2^t > 0 > u_1^t - w_1^t)} + \big(-h_2^{t+}(w_1^t + w_2^t - u_1^t) - h_2^{t-} w_2^t\big) I_{(0 > w_2^t > u_1^t - w_1^t)} + \big(h_2^{t-}(w_1^t - u_1^t)\big) I_{(0 > u_1^t - w_1^t > w_2^t)}$ with $I_{(\cdot)}$ denoting the indicator function. In words, we are partitioning the expected profit-to-go function $\mathrm{E}_{\epsilon_1^t, \epsilon_2^t}[(G^t(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t)]$ into two components, where the first component $\mathrm{E}_{\epsilon_1^t, \epsilon_2^t}[H^t(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t)]$ refers to the expected holding and shortage cost if no upgrades were given and the second component $\mathrm{E}_{\epsilon_1^t, \epsilon_2^t}[\hat{G}^t(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t)]$ is the result of the maximization problem when the optimal upgrade quantity is selected, and reflects the discounted profit-to-go function for the next period as well as the adjustment to the holding and shortage costs due to upgrades. One can straightforwardly show that the Hessian of $\mathrm{E}_{\epsilon_1^t, \epsilon_2^t}[H^t(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t)]$ is a diagonal matrix with elements $-(h_i^{t+} + h_i^{t-}) f_{\epsilon_i}(z_i^t)$ for $i = 1, 2$ and therefore is strictly concave, submodular, and possesses the strict diagonal dominance property for positive unit holding and shortage costs and continuously defined probability density functions $f_{\epsilon_1}(\cdot)$ and $f_{\epsilon_2}(\cdot)$ for the demand uncertainty terms ϵ_1^t and

ϵ_2^t. Next we show $\mathrm{E}_{\epsilon_1^t,\epsilon_2^t}[\hat{G}^t(z_1^t-\epsilon_1^t, z_2^t-\epsilon_2^t)]$ is weakly concave, submodular and possesses a weak diagonal dominance property. Consider the Lagrangian function $\bar{G}'^t(w_1^t, w_2^t, u_1^t) - \mu_1^t(u_1^t - w_1^t)$ for the second-stage optimization problem given in (4) where $\mu_1^t \geq 0$ denotes the Lagrangian variable associated with constraint $u_1^t - w_1^t \leq 0$. For the case where $\mu_1^t = 0$, the envelope theorem yields $\hat{G}_{i,j}^t = \bar{G}_{i,j}'^t = \beta V_{22}^{t-1} \leq 0$ for $i,j=1,2$ where the inequality follows from the inductional assumption. For the case where $\mu_1^t > 0$, the envelope theorem results in $\hat{G}_{ij}^t = \bar{G}_{ij}'^t = \beta V_{ij}^{t-1}$ for $i,j=1,2$. Based on the inductional assumption, we then have $\hat{G}_{ii}^t - \hat{G}_{ij}^t \leq 0$ for $i \neq j$ and $i=1,2$. Hence, $\hat{G}^t(w_1^t, w_2^t)$ is concave, submodular, and possesses the weak diagonal dominance property. As $J^t(z_1^t, z_2^t, d_1^t, d_2^t)$ is an addition of three terms one of which, $\mathrm{E}_{\epsilon_1^t,\epsilon_2^t}[H^t(z_1^t-\epsilon_1^t, z_2^t-\epsilon_2^t)]$, is strictly concave and submodular in (z_1^t, z_2^t) with a Hessian possessing strict diagonal dominance, the other, $\mathrm{E}_{\epsilon_1^t,\epsilon_2^t}[\hat{G}^t(z_1^t-\epsilon_1^t, z_2^t-\epsilon_2^t)]$, which is concave and submodular in (z_1^t, z_2^t) with a Hessian possessing weak diagonal dominance, and the last $c_1 z_1^t + c_2 z_2^t$ being linear in (z_1^t, z_2^t), itself is strictly concave and submodular in (z_1^t, z_2^t) with a Hessian possessing strict diagonal dominance. □

To proceed with the characterization of the optimal policy, we first construct the first order conditions by introducing Lagrange multipliers $\lambda_{ij}^t \geq 0$ for $i,j=\{1,2\}$ where λ_{i1}^t and λ_{i2}^t are associated with constraints $-z_i^t - d_i^t \leq -x_i^t$ and $z_i^t + d_i^t \leq x_i^t + K_i$, respectively (i.e., $y_i^t \geq x_i^t$ and $y_i^t \leq x_i^t + K_i$ in the original formulation before the change of variables). We note that these constraints form 'box constraints' and some may not be simultaneously active for positive capacity parameters. We can exploit this special structure of constraints to represent the first-order optimality conditions in simpler notation by defining $\lambda_i^t := \lambda_{i1}^t - \lambda_{i2}^t$. Note that λ_i^t uniquely determines λ_{ij}^t for $j=1,2$ where (a) $\lambda_i^t < 0$ implies $\lambda_{i1}^t = 0$ and $\lambda_{i2}^t > 0$, (b) $\lambda_i^t > 0$ implies $\lambda_{i1}^t > 0$ and $\lambda_{i2}^t = 0$; and (c) $\lambda_i^t = 0$ implies $\lambda_{i1}^t = \lambda_{i2}^t = 0$.

The first order optimality conditions for the first-stage problem can then be expressed as: $J_{z_i}^t(z_1^t, z_2^t, d_1^t, d_2^t) + \lambda_i^t = 0$, and $J_{d_i}^t(z_1^t, z_2^t, d_1^t, d_2^t) + \lambda_i^t = 0$ for $i=1,2$ where we again use the notation $J_{z_i}^t$, $J_{d_i}^t$ to denote the first derivative of $J^t(z_1^t, z_2^t, d_1^t, d_2^t)$ with respect to z_i^t and d_i^t, respectively.

Using Lemma 1 and the first order conditions presented earlier, we can write the decision variables (z_1^t, z_2^t) and (d_1^t, d_2^t) in terms of λ_1^t and λ_2^t. The following lemma formally presents this result.

LEMMA 2. *There exists implicit functions $z_i'^t$ and $d_i'^t$ such that $z_i^t = z_i'^t(\lambda_1^t, \lambda_2^t)$ and $d_i^t = d_i'^t(\lambda_1^t, \lambda_2^t)$. Further, $\frac{\partial z_i'^t}{\partial \lambda_i^t} > 0$, $\frac{\partial z_i'^t}{\partial \lambda_j^t} < 0$, with $\frac{\partial z_i'^t}{\partial \lambda_i^t} > |\frac{\partial z_i'^t}{\partial \lambda_j^t}|$, and $\frac{\partial d_i'^t}{\partial \lambda_i^t} > 0$, $\frac{\partial d_i'^t}{\partial \lambda_j^t} < 0$, with $\frac{\partial d_i'^t}{\partial \lambda_i^t} > |\frac{\partial d_i'^t}{\partial \lambda_j^t}|$ for $i \neq j$ and $i,j=1,2$.*

Proof: The proof follows from the Implicit Function Theorem, the first order conditions and Lemma 1. We first present the proof for the existence and monotonicity of $z_i'^t(\lambda_1^t, \lambda_2^t)$. From the first order conditions, we have $J_{z_1}^t + \lambda_1^t = 0$ and $J_{z_2}^t + \lambda_2^t = 0$. Hence, $\frac{\partial \lambda_1^t}{\partial z_1^t} = -J_{z_1,z_1}^t$, $\frac{\partial \lambda_1^t}{\partial z_2^t} = -J_{z_1,z_2}^t$, $\frac{\partial \lambda_2^t}{\partial z_1^t} = -J_{z_2,z_1}^t$, and $\frac{\partial \lambda_2^t}{\partial z_2^t} = -J_{z_2,z_2}^t$. Therefore $\left(\frac{\partial \boldsymbol{\lambda}^t}{\partial \mathbf{z}^t}\right) = -\begin{bmatrix} J_{z_1,z_1}^t & J_{z_1,z_2}^t \\ J_{z_2,z_1}^t & J_{z_2,z_2}^t \end{bmatrix}$. Since J^t is strictly concave in (z_1^t, z_2^t), this matrix is invertible and $\left(\frac{\partial \mathbf{z}^t}{\partial \boldsymbol{\lambda}^t}\right) = -\begin{bmatrix} J_{z_1,z_1}^t & J_{z_1,z_2}^t \\ J_{z_2,z_1}^t & J_{z_2,z_2}^t \end{bmatrix}^{-1} = \frac{1}{\hat{J}_z^t}\begin{bmatrix} -J_{z_2,z_2}^t & J_{z_1,z_2}^t \\ J_{z_2,z_1}^t & -J_{z_2,z_2}^t \end{bmatrix}$, where $\hat{J}_z^t > 0$ denotes the determinant. Thus there exists implicit functions $z_i'^t(\lambda_1^t, \lambda_2^t)$. The properties outlined in Lemma 1 lead to $\frac{\partial z_i'^t}{\partial \lambda_i^t} > 0$, $\frac{\partial z_i'^t}{\partial \lambda_j^t} < 0$, and $\frac{\partial z_i'^t}{\partial \lambda_i^t} > |\frac{\partial z_i'^t}{\partial \lambda_j^t}|$ for $i \neq j$ and $i,j=1,2$.

The proof for $d_i'^t(\lambda_1^t, \lambda_2^t)$ follows similarly. From the first order conditions, we have $J_{d_1}^t + \lambda_1^t = 0$ and $J_{d_2}^t + \lambda_2^t = 0$. Hence, $\frac{\partial \lambda_1^t}{\partial d_1^t} = -J_{d_1,d_1}^t$, $\frac{\partial \lambda_1^t}{\partial d_2^t} = -J_{d_1,d_2}^t$, $\frac{\partial \lambda_2^t}{\partial d_1^t} = -J_{d_2,d_1}^t$, and $\frac{\partial \lambda_2^t}{\partial d_2^t} = -J_{d_2,d_2}^t$. Therefore $\left(\frac{\partial \boldsymbol{\lambda}^t}{\partial \mathbf{d}^t}\right) = -\begin{bmatrix} J_{d_1,d_1}^t & J_{d_1,d_2}^t \\ J_{d_2,d_1}^t & J_{d_2,d_2}^t \end{bmatrix}$. Since J^t is strictly concave in (d_1^t, d_2^t), this matrix is invertible and $\left(\frac{\partial \mathbf{d}^t}{\partial \boldsymbol{\lambda}^t}\right) = -\begin{bmatrix} J_{d_1,d_1}^t & J_{d_1,d_2}^t \\ J_{d_2,d_1}^t & J_{d_2,d_2}^t \end{bmatrix}^{-1}$. Thus there exists implicit functions $d_i'^t(\lambda_1^t, \lambda_2^t)$. Substituting in the terms for J_{d_i,d_j}^t as given in the corresponding Hessian in the proof of Lemma 1, $-\begin{bmatrix} J_{d_1,d_1}^t & J_{d_1,d_2}^t \\ J_{d_2,d_1}^t & J_{d_2,d_2}^t \end{bmatrix}^{-1}$ reduces to $\frac{\delta}{2(q_1-q_2)}\begin{bmatrix} \frac{\bar{q}-q_2}{\bar{q}-q_1} & -1 \\ -1 & \frac{q_1-\underline{q}}{q_2-\underline{q}} \end{bmatrix}$. The results $\frac{\partial d_i'^t}{\partial \lambda_i^t} > 0$, $\frac{\partial d_i'^t}{\partial \lambda_j^t} < 0$, with $\frac{\partial d_i'^t}{\partial \lambda_i^t} > |\frac{\partial d_i'^t}{\partial \lambda_j^t}|$ for $i \neq j$ and $i,j=1,2$ directly follow as $\bar{q} > q_1 > q_2 > \underline{q}$.

Lemma 3. *$\frac{\partial\lambda_i^t}{\partial x_i^t} \geq \frac{\partial\lambda_i^t}{\partial x_j^t} \geq 0$ for $i \neq j$ and $i,j = 1,2$. Further, (a) for all (x_1^t, x_2^t) such that $\lambda_i^t \neq 0$ and $\lambda_j^t = 0$, we have $\frac{\partial\lambda_i^t}{\partial x_i^t} > 0$ and $\frac{\partial\lambda_i^t}{\partial x_j^t} = 0$, (b) for all (x_1^t, x_2^t) such that $\lambda_1^t \neq 0$ and $\lambda_2^t \neq 0$, we have $\frac{\partial\lambda_i^t}{\partial x_i^t} > \frac{\partial\lambda_i^t}{\partial x_j^t} > 0$.*

Proof: The signs of λ_1^t and λ_2^t segment the state space into nine different regions depending on whether each variable is negative, zero, or positive. For the region where $\lambda_1^t = \lambda_2^t = 0$, we have $\frac{\partial\lambda_i^t}{\partial x_j^t} = 0$ for $i,j = 1,2$ and the result holds. Consider the region for which $\lambda_1^t > 0$ and $\lambda_2^t = 0$, implying that the constraint $x_1^t - z_1'^{*t}(\lambda_1^t, 0) - d_1'^{*t}(\lambda_1^t, 0) \leq 0$ is active. Specifically, in this region we'd like to show that λ_1^t is strictly increasing in x_1^t and independent of x_2^t. Differentiating the active constraint with respect to x_1^t gives $1 - \frac{\partial z_1^{*t}}{\partial\lambda_1^t}\frac{\partial\lambda_1^t}{\partial x_1^t} - \frac{\partial d_1^{*t}}{\partial\lambda_1^t}\frac{\partial\lambda_1^t}{\partial x_1^t} = 0$, from which we find $\frac{\partial\lambda_1^t}{\partial x_1^t} = 1/(\frac{\partial z_1'^t}{\partial\lambda_1^t} + \frac{\partial d_1'^t}{\partial\lambda_1^t}) > 0$ where the last inequality follows from Lemma 2. Similarly, differentiating the active constraint with respect to x_2^t gives $-\frac{\partial z_1^{*t}}{\partial\lambda_1^t}\frac{\partial\lambda_1^t}{\partial x_2^t} - \frac{\partial d_1^{*t}}{\partial\lambda_1^t}\frac{\partial\lambda_1^t}{\partial x_2^t} = 0$. Through Lemma 2, we find $\frac{\partial\lambda_1^t}{\partial x_2^t} = 0$.

Next, consider the region where $\lambda_1^t > 0$ and $\lambda_2^t < 0$, which implies that the constraints $x_1^t - z_1'^{*t}(\lambda_1^t, \lambda_2^t) - d_1'^{*t}(\lambda_1^t, \lambda_1^t) \leq 0$ and $z_2'^{*t}(\lambda_1^t, \lambda_2^t) + d_2'^{*t}(\lambda_1^t, \lambda_1^t) - x_2^t - K_2 \leq 0$ are active. First, differentiating each of the active constraints with respect to x_1^t and solving for $\frac{\partial\lambda_i^t}{\partial x_1^t}$ for $i = 1,2$, we get $\frac{\partial\lambda_1^t}{\partial x_1^t} = \left(\frac{\partial z_2^{*t}}{\partial\lambda_2^t} + \frac{\partial d_2^{*t}}{\partial\lambda_2^t}\right)/Q$ where $Q := \left(\frac{\partial z_1'^{*t}}{\partial\lambda_1^t}\frac{\partial z_2'^{*t}}{\partial\lambda_2^t} - \frac{\partial z_1'^{*t}}{\partial\lambda_2^t}\frac{\partial z_2'^{*t}}{\partial\lambda_1^t}\right) + \left(\frac{\partial z_1'^{*t}}{\partial\lambda_1^t}\frac{\partial d_2'^{*t}}{\partial\lambda_2^t} - \frac{\partial z_1'^{*t}}{\partial\lambda_2^t}\frac{\partial d_2'^{*t}}{\partial\lambda_1^t}\right) + \left(\frac{\partial z_2'^{*t}}{\partial\lambda_2^t}\frac{\partial d_1'^{*t}}{\partial\lambda_1^t} - \frac{\partial z_2'^{*t}}{\partial\lambda_1^t}\frac{\partial d_1'^{*t}}{\partial\lambda_2^t}\right) + \left(\frac{\partial d_1'^{*t}}{\partial\lambda_1^t}\frac{\partial d_2'^{*t}}{\partial\lambda_2^t} - \frac{\partial d_1'^{*t}}{\partial\lambda_2^t}\frac{\partial d_2'^{*t}}{\partial\lambda_1^t}\right)$. The terms in each of the brackets in Q collectively can be shown to be positive through Lemma 2, hence $Q > 0$. Also by Lemma 2, the numerator is positive, thus $\frac{\partial\lambda_1^t}{\partial x_1^t} > 0$. Similarly, we find $\frac{\partial\lambda_2^t}{\partial x_1^t} = -\left(\frac{\partial z_2^{*t}}{\partial\lambda_1^t} + \frac{\partial d_2^{*t}}{\partial\lambda_1^t}\right)/Q$ and $\frac{\partial\lambda_2^t}{\partial x_1^t} > 0$ by Lemma 2. Next, differentiating the active constraints with respect to x_2^t this time and solving for $\frac{\partial\lambda_i^t}{\partial x_1^t}$ for $i = 1,2$, we find $\frac{\partial\lambda_1^t}{\partial x_2^t} = -\left(\frac{\partial z_1^{*t}}{\partial\lambda_2^t} + \frac{\partial d_1^{*t}}{\partial\lambda_2^t}\right)/Q > 0$ and $\frac{\partial\lambda_2^t}{\partial x_2^t} = \left(\frac{\partial z_1^{*t}}{\partial\lambda_1^t} + \frac{\partial d_1^{*t}}{\partial\lambda_1^t}\right)/Q > 0$ by Lemma 2. The properties in Lemma 2 then also immediately lead to $\frac{\partial\lambda_i^t}{\partial x_i^t} > \frac{\partial\lambda_i^t}{\partial x_j^t} > 0$ for $i \neq j$ and $i,j = 1,2$.

The analysis for the remaining cases are very similar to the ones considered and are omitted for brevity. □

To complete the proof of Theorem 2, we examine each of the nine state-space regions defined by the signs of λ_1^t and λ_2^t. First, consider the region corresponding to $\lambda_1^t = \lambda_2^t = 0$. In this region, we have $y_1^{*t} = z_1'^{*t}(0,0) + d_1'^{*t}(0,0)$ and $y_2^{*t} = z_2'^{*t}(0,0) + d_2'^{*t}(0,0)$. For future reference, we denote this base-stock level pair by $(x_1^{\circ t}, x_2^{\circ t})$, i.e., $x_1^{\circ t} := z_1'^{*t}(0,0) + d_1'^{*t}(0,0)$ and $x_2^{\circ t} := z_2'^{*t}(0,0) + d_2'^{*t}(0,0)$. Hence, anywhere in this region, the optimal replenishment policy brings the inventory level to the base-stock levels given by $y_i^{*t} = x_1^{\circ t}$ and the base-stock levels for each product is independent of the initial inventory level of the other product.

Next, consider the region for which $\lambda_1^t > 0$ and $\lambda_2^t = 0$. Then, from complementary slackness, there is no replenishment for item 1. The optimal base-stock level for item 2 is given by the expression $y_2^{*t} = z_2'^{*t}(\lambda_1^t, 0) + d_2'^{*t}(\lambda_1^t, 0)$. We are interested in how this optimal base-stock level changes with the initial inventory level of item 1 which is found by $\frac{\partial y_2^{*t}}{\partial x_1^t} = \left(\frac{\partial z_2'^{*t}(\lambda_1^t,0)}{\partial\lambda_1^t} + \frac{\partial d_2'^{*t}(\lambda_1^t,0)}{\partial\lambda_1^t}\right)\frac{\partial\lambda_1^t}{\partial x_1^t}$. By Lemma 2, we have $\frac{\partial z_2'^{*t}(\lambda_1^t,0)}{\partial\lambda_1^t} < 0$ and $\frac{\partial d_2'^{*t}(\lambda_1^t,0)}{\partial\lambda_1^t} < 0$. By Lemma 3, in this region we have $\frac{\partial\lambda_1^t}{\partial x_1^t} > 0$. Thus $\frac{\partial y_2^{*t}}{\partial x_1^t} < 0$.

Now, we consider the region for which $\lambda_1^t < 0$ and $\lambda_2^t = 0$. From complementary slackness, the available capacity for product type-1 is used in its entirety to replenish this item. The optimal base-stock level for item 2 is again given by the expression $y_2^{*t} = z_2'^{*t}(\lambda_1^t, 0) + d_2'^{*t}(\lambda_1^t, 0)$. For the monotonicity of this base-stock level with respect to the initial inventory of product type-1, we write $\frac{\partial y_2^{*t}}{\partial x_1^t} = \left(\frac{\partial z_2'^{*t}(\lambda_1^t,0)}{\partial\lambda_1^t} + \frac{\partial d_2'^{*t}(\lambda_1^t,0)}{\partial\lambda_1^t}\right)\frac{\partial\lambda_1^t}{\partial x_1^t}$. By Lemma 2, we have $\frac{\partial z_2'^{*t}(\lambda_1^t,0)}{\partial\lambda_1^t} < 0$ and $\frac{\partial d_2'^{*t}(\lambda_1^t,0)}{\partial\lambda_1^t} < 0$. By Lemma 3, $\frac{\partial\lambda_1^t}{\partial x_1^t} > 0$. Hence, $\frac{\partial y_2^{*t}}{\partial x_1^t} < 0$. Since in this region $\lambda_1^t < 0$, we also find $y_2^{*t} > x_2^{\circ t}$.

In the region corresponding to $\lambda_1^t = 0$ and $\lambda_2^t > 0$, complementary slackness yields no replenishment for item 2. The optimal base-stock level for product type-1 is given by the expression $y_1^{*t} = z_1'^{*t}(0, \lambda_2^t) + d_1'^{*t}(0, \lambda_2^t)$. In terms of the change of this base-stock level with respect to an increase in the initial inventory level of product type-2, we find $\frac{\partial y_1^{*t}}{\partial x_2^t} = \left(\frac{\partial z_1'^{*t}(0,\lambda_2^t)}{\partial\lambda_2^t} + \frac{\partial d_1'^{*t}(0,\lambda_2^t)}{\partial\lambda_2^t}\right)\frac{\partial\lambda_2^t}{\partial x_2^t}$. By Lemma 2, we have $\frac{\partial z_1'^{*t}(0,\lambda_2^t)}{\partial\lambda_2^t} < 0$ and $\frac{\partial d_1'^{*t}(0,\lambda_2^t)}{\partial\lambda_2^t} < 0$. By Lemma 3, $\frac{\partial\lambda_2^t}{\partial x_2^t} > 0$, hence we find $\frac{\partial y_1^{*t}}{\partial x_2^t} < 0$. Therefore, in this region we also have $y_1^{*t} < x_1^{\circ t}$. In the region where $\lambda_1^t > 0$ and $\lambda_2^t > 0$, no replenishment takes place for either item. The analysis for the remaining four regions are similar. □

Proof of Theorem 3 (Optimal Pricing Policy):

Solving for d_1^t and d_2^t in the first order conditions corresponding to the demand selection decisions, $J_{d_1}^t + \lambda_1^t = 0$ and $J_{d_2}^t + \lambda_2^t = 0$, where $J_{d_1}^t$ and $J_{d_2}^t$ are as given in (10), and substituting the inverse demand-price relationships derived in the proof of Lemma 1, we get:

$$\begin{aligned} p_1^t =& \frac{\bar{p}(q_1 - \underline{q}) + \underline{p}(\bar{q} - q_1)}{2(\bar{q} - \underline{q})} + \frac{c_1^t}{2} - \frac{\lambda_1^t}{2} \\ p_2^t =& \frac{\bar{p}(q_2 - \underline{q}) + \underline{p}(\bar{q} - q_2)}{2(\bar{q} - \underline{q})} + \frac{c_2^t}{2} - \frac{\lambda_2^t}{2} \end{aligned} \tag{11}$$

First, consider the region for which $\lambda_1^t = \lambda_2^t = 0$. The expressions in (11) become $p_1^t = \frac{\bar{p}(q_1 - \underline{q}) + \underline{p}(\bar{q} - q_1)}{2(\bar{q} - \underline{q})} + \frac{c_1^t}{2}$, and $p_2^t = \frac{\bar{p}(q_2 - \underline{q}) + \underline{p}(\bar{q} - q_2)}{2(\bar{q} - \underline{q})} + \frac{c_2^t}{2}$, where both prices are independent of the starting inventory levels within the region. We refer to these prices as the "list prices" and let $p_1^{0\,t} = \frac{\bar{p}(q_1 - \underline{q}) + \underline{p}(\bar{q} - q_1)}{2(\bar{q} - \underline{q})} + \frac{c_1^t}{2}$ and $p_2^{0\,t} = \frac{\bar{p}(q_2 - \underline{q}) + \underline{p}(\bar{q} - q_2)}{2(\bar{q} - \underline{q})} + \frac{c_2^t}{2}$ denote the list price for product 1 and product 2, respectively.

Next, consider the region defined by $\lambda_1^t > 0$ and $\lambda_2^t = 0$. We have $p_1^t = p_1^{0\,t} - \frac{\lambda_1^t}{2}$ and $p_2^t = p_2^{0\,t}$. Thus, $p_1^t < p_1^{0\,t}$. Further, as $\frac{\partial p_1^t}{\partial x_1^t} = -\frac{1}{2}\frac{\partial \lambda_1^t}{\partial x_1^t} < 0$, and $\frac{\partial p_1^t}{\partial x_2^t} = -\frac{1}{2}\frac{\partial \lambda_1^t}{\partial x_2^t} = 0$ (where the inequality and the equality follow from Lemma 3), p_1^t is decreasing with x_1^t and is independent of x_2^t. Since $p_2^t = p_2^{0\,t}$, we also have p_2^t independent of x_1^t and x_2^t. The case for which $\lambda_1^t < 0$ and $\lambda_2^t = 0$ is very similar and results in $p_1^t > p_1^{0\,t}$, and $p_2^t = p_2^{0\,t}$ with the same monotonicity properties as in the previous case.

When $\lambda_1^t = 0$ and $\lambda_2^t > 0$, we similarly have $p_1^t = p_1^{0\,t}$ and $p_2^t = p_2^{0\,t} - \frac{\lambda_2^t}{2}$. Since $\lambda_2^t > 0$, we have $p_2^t > p_2^{0\,t}$. We also find p_1^t independent of x_1^t and x_2^t, while p_2^t independent of x_1^t and decreasing in x_2^t. Again, the case with $\lambda_1^t = 0$ and $\lambda_2^t < 0$ is very similar and leads to $p_1^t = p_1^{0\,t}$ and $p_2^t < p_2^{0\,t}$ with the same monotonicity properties p_1^t independent of x_1^t and x_2^t, and p_2^t independent of x_1^t and decreasing in x_2^t.

For $\lambda_1^t > 0$ and $\lambda_2^t > 0$, we have $p_1^t = p_1^{0\,t} - \frac{\lambda_1^t}{2}$ and $p_2^t = p_2^{0\,t} - \frac{\lambda_2^t}{2}$. As both $\lambda_1^t > 0$ and $\lambda_2^t > 0$, this leads to $p_1^t < p_1^{0\,t}$ and $p_2^t < p_2^{0\,t}$. Regarding price monotonicities, recall from Lemma 3 that $\frac{\partial \lambda_i^t}{\partial x_i^t} > \frac{\partial \lambda_i^t}{\partial x_j^t} > 0$ for $i,j = 1,2$ and $i \neq j$. Hence, we find that $\frac{\partial p_i^t}{\partial x_j^t} < 0$ for $i,j = \{1,2\}$, i.e., both p_1^t and p_2^t are decreasing in x_1^t and x_2^t. These monotonicity results also similarly carry to all other cases where $\lambda_1^t \neq 0$ and $\lambda_2^t \neq 0$. □

Preservation of the Structural Properties of the Value Function:

To complete the analysis of the optimal policy characterization, lastly we show that the value function $V^t(x_1^t, x_2^t)$ retains the properties of $V^{t-1}(x_1^{t-1}, x_2^{t-1})$ that were assumed in the induction step as outlined in the Induction Assumption.

LEMMA 4 **(Completing the Induction)**. *$V^t(x_1^t, x_2^t)$ is jointly concave, submodular, and its Hessian possesses weak diagonal dominance property: $V_{11}^t(x_1^t, x_2^t) \leq V_{12}^t(x_1^t, x_2^t) \leq 0$ and $V_{22}^t(x_1^t, x_2^t) \leq V_{21}^t(x_1^t, x_2^t) \leq 0$.*

Proof: The proof follows from Lemma 3. By (9) and the envelope theorem, we have $\frac{\partial V^t(x_1^t, x_2^t)}{\partial x_i^t} = c_i - \lambda_i^t$ for $i = 1,2$. Hence, $V_{ij}^t(x_1^t, x_2^t) = -\frac{\partial \lambda_i^t}{\partial x_j^t}$ for $i,j = 1,2$. By Lemma 3, $\frac{\partial \lambda_i^t}{\partial x_j^t} \geq 0$. Hence, $V_{ij}^t(x_1^t, x_2^t) \leq 0$ and therefore $V^t(x_1^t, x_2^t)$ is submodular. For diagonal dominance, $V_{ii}^t(x_1^t, x_2^t) - V_{ij}^t(x_1^t, x_2^t) = -\frac{\partial \lambda_i^t}{\partial x_i^t} + \frac{\partial \lambda_i^t}{\partial x_j^t} \leq 0$, where the inequality follows from Lemma 3. $V^t(x_1^t, x_2^t)$ being concave follows immediately from $V_{ij}^t(x_1^t, x_2^t) \leq 0$ and $V_{ii}^t(x_1^t, x_2^t) - V_{ij}^t(x_1^t, x_2^t) \leq 0$. □

Proof of Theorem 4 (Impact of Upgrades on Pricing and Replenishment):

As we discuss in Section 5.1, our preceding analysis to characterize the structure of the optimal policy can straightforwardly be extended to incorporate an upgrade limit $\bar{u}^t$ in the second-stage upgrade problem. For brevity, we omit the replication of the derivation but would like to highlight that the optimal upgrade policy can now be stated as $u^{t*} = \min((w_1^t - r^t(w^t))^+, \bar{u}^t)$, while the structure of the optimal pricing and replenishment policies continue to hold as stated in Theorems 2 and 3. The first order conditions outlined in the proof of Theorem 2 for the modified problem where we incorporate an upgrade limit $\bar{u}^t$ can be written explicitly by substituting in the structure of the optimal second-stage upgrade decision as stated below:

$$
\begin{aligned}
&-c_1^t + \mathrm{E}_{\epsilon_1^t,\epsilon_2^t}[\frac{\partial}{\partial z_1^t} G^t(z_1^t-\epsilon_1^t, z_2^t-\epsilon_2^t)] + \lambda_1^t = 0 \\
&-c_2^t + \mathrm{E}_{\epsilon_1^t,\epsilon_2^t}[\frac{\partial}{\partial z_2^t} G^t(z_1^t-\epsilon_1^t, z_2^t-\epsilon_2^t)] + \lambda_2^t = 0 \\
&\frac{\bar{p}(q_1-\underline{q})+\underline{p}(\bar{q}-q_1)}{(\bar{q}-\underline{q})} - \frac{2(\bar{q}-q_1)}{\delta(\bar{q}-\underline{q})}\Big[(q_1-\underline{q})d_1+(q_2-\underline{q})d_2\Big] - c_1^t + \lambda_1^t = 0 \\
&\frac{\bar{p}(q_2-\underline{q})+\underline{p}(\bar{q}-q_2)}{(\bar{q}-\underline{q})} - \frac{2(q_2-\underline{q})}{\delta(\bar{q}-\underline{q})}\Big[(\bar{q}-q_1)d_1+(\bar{q}-q_2)d_2\Big] - c_2^t + \lambda_2^t = 0
\end{aligned}
\tag{12}
$$

where $G^t(z_1^t-\epsilon_1^t, z_2^t-\epsilon_2^t) =$

$$
\begin{aligned}
&[-h_1(z_1^t-\epsilon_1^t) - h_2(z_2^t-\epsilon_2^t) + \beta V^{t-1}(z_1^t-\epsilon_1^t, z_2^t-\epsilon_2^t)] \cdot I_{(z_1^t-\epsilon_1^t < r^t(z_1^t+z_2^t-\epsilon_1^t-\epsilon_2^t))} \\
&\quad + [-h_1(r^t(z_1^t+z_2^t-\epsilon_1^t-\epsilon_2^t)) - h_2(z_1^t+z_2^t-\epsilon_1^t-\epsilon_2-r^t(z_1^t+z_2^t-\epsilon_1^t-\epsilon_2^t)) \\
&\quad + \beta V^{t-1}(r^t(z_1^t+z_2^t-\epsilon_1^t-\epsilon_2^t), z_1^t+z_2^t-\epsilon_1^t-\epsilon_2^t-r^t(z_1^t+z_2^t-\epsilon_1^t-\epsilon_2^t))] \cdot I_{(z_1^t-\epsilon_1^t-\bar{u}^t < r^t(z_1^t+z_2^t-\epsilon_1^t-\epsilon_2^t) < z_1^t-\epsilon_1^t)} \\
&\quad + [-h_1(z_1^t-\epsilon_1^t-\bar{u}^t) - h_2(z_2^t-\epsilon_2^t+\bar{u}) + \beta V^{t-1}(z_1^t-\epsilon_1^t-\bar{u}^t, z_2^t-\epsilon_2^t+\bar{u}^t)] \cdot I_{(r^t(z_1^t+z_2^t-\epsilon_1^t-\epsilon_2^t) < z_1^t-\epsilon_1^t-\bar{u}^t)}
\end{aligned}
$$

and $I_{(\cdot)}$ is the indicator function.

Consider first the case for which $\lambda_1^t = \lambda_2^t = 0$. Differentiating the first two expressions in (12) with respect to $\bar{u}^t$, we get

$$
\begin{aligned}
&\mathrm{E}_{\epsilon_1^t,\epsilon_2^t}\left[\left(V_{11}^{t-1}(\cdot,\cdot)\left(\frac{\partial z_1^t}{\partial \bar{u}^t}-1\right) + V_{12}^{t-1}(\cdot,\cdot)\left(\frac{\partial z_2^t}{\partial \bar{u}^t}+1\right)\right) \cdot I_{(r^t(\cdot)<z_1^t-\epsilon_1^t-\bar{u}^t)} + 0 \cdot I_{(r^t(\cdot)>z_1^t-\epsilon_1^t-\bar{u}^t)}\right] = 0, \\
&\mathrm{E}_{\epsilon_1^t,\epsilon_2^t}\left[\left(V_{21}^{t-1}(\cdot,\cdot)\left(\frac{\partial z_1^t}{\partial \bar{u}^t}-1\right) + V_{22}^{t-1}(\cdot,\cdot)\left(\frac{\partial z_2^t}{\partial \bar{u}^t}+1\right)\right) \cdot I_{(r^t(\cdot)<z_1^t-\epsilon_1^t-\bar{u}^t)} + 0 \cdot I_{(r^t(\cdot)>z_1^t-\epsilon_1^t-\bar{u}^t)}\right] = 0.
\end{aligned}
$$

Solving for $\frac{\partial z_1^t}{\partial \bar{u}^t}$ and $\frac{\partial z_2^t}{\partial \bar{u}^t}$ and using the concavity, submodularity and diagonal dominance properties of V^{t-1}, we find $\frac{\partial z_1^t}{\partial \bar{u}^t} \geq 0$ and $\frac{\partial z_1^t}{\partial \bar{u}^t} \leq 0$.

Next, we differentiate the last two expressions in (12) with respect to $\bar{u}^t$ and have $-\frac{2(\bar{q}-q_1)(q_1-\underline{q})}{\delta(\bar{q}-\underline{q})}\frac{\partial d_1^t}{\partial \bar{u}^t} - \frac{2(\bar{q}-q_1)(q_2-\underline{q})}{\delta(\bar{q}-\underline{q})}\frac{\partial d_2^t}{\partial \bar{u}^t} = 0$ and $-\frac{2(q_2-\underline{q})(\bar{q}-q_1)}{\delta(\bar{q}-\underline{q})}\frac{\partial d_1^t}{\partial \bar{u}^t} - \frac{2(q_2-\underline{q})(\bar{q}-q_2)}{\delta(\bar{q}-\underline{q})}\frac{\partial d_2^t}{\partial \bar{u}^t} = 0$. Solving for $\frac{\partial d_1^t}{\partial \bar{u}^t}$ and $\frac{\partial d_2^t}{\partial \bar{u}^t}$ results in $\frac{\partial d_1^t}{\partial \bar{u}^t} = \frac{\partial d_2^t}{\partial \bar{u}^t} = 0$. Consequently, $\frac{\partial p_1^t}{\partial \bar{u}^t} = \frac{\partial p_2^t}{\partial \bar{u}^t} = 0$. Since $y_i^t = z_i^t + d_i^t$, we also have $\frac{\partial y_1^t}{\partial \bar{u}^t} \geq 0$ and $\frac{\partial y_2^t}{\partial \bar{u}^t} \leq 0$.

For the case where $\lambda_1^t > 0$ and $\lambda_2^t = 0$, we have the active constraint $y_1^t = x_1^t$. Differentiating the expressions in (12) and $z_1^t + d_1^t = x_1^t$ with respect to $\bar{u}^t$ and solving for $\frac{\partial d_i^t}{\partial \bar{u}^t}$ and $\frac{\partial z_i^t}{\partial \bar{u}^t}$ for $i = 1,2$ results in $\frac{\partial d_1^t}{\partial \bar{u}^t} = -\frac{\frac{\delta(\bar{q}-q_2)}{(\bar{q}-q_1)(q_1-q_2)}\left(\tilde{V}_{11}^t\tilde{V}_{22}^t - \tilde{V}_{12}^t\tilde{V}_{21}^t\right)}{\frac{\delta(\bar{q}-q_2)}{(\bar{q}-q_1)(q_1-q_2)}\left(\tilde{V}_{11}^t\tilde{V}_{22}^t - \tilde{V}_{12}^t\tilde{V}_{21}^t\right) - 2\tilde{V}_{22}^t} < 0$ where $\tilde{V}_{ij}^t$ stands for $\mathrm{E}_{\epsilon_1^t,\epsilon_2^t}\left[V_{ij}^{t-1}(\cdot,\cdot)\cdot I_{(r^t(\cdot)<z_1^t-\epsilon_1^t-\bar{u})} + 0 \cdot I_{(r^t(\cdot)>z_1^t-\epsilon_1^t-\bar{u})}\right]$. The inequality follows due to $\frac{\delta(\bar{q}-q_2)}{(\bar{q}-q_1)(q_1-q_2)} > 0$, and the inductional assumption. Similarly, we find $\frac{\partial d_2^t}{\partial \bar{u}^t} = \frac{\frac{\delta}{(q_1-q_2)}\left(\tilde{V}_{11}^t\tilde{V}_{22}^t - \tilde{V}_{12}^t\tilde{V}_{21}^t\right)}{\frac{\delta(\bar{q}-q_2)}{(\bar{q}-q_1)(q_1-q_2)}\left(\tilde{V}_{11}^t\tilde{V}_{22}^t - \tilde{V}_{12}^t\tilde{V}_{21}^t\right) - 2\tilde{V}_{22}^t} > 0$, $\frac{\partial z_1^t}{\partial \bar{u}^t} = -\frac{\partial d_1^t}{\partial \bar{u}^t} > 0$, and $\frac{\partial z_2^t}{\partial \bar{u}^t} = -\frac{\frac{\delta(\bar{q}-q_2)}{(\bar{q}-q_1)(q_1-q_2)}\left(\tilde{V}_{11}^t\tilde{V}_{22}^t - \tilde{V}_{12}^t\tilde{V}_{21}^t\right) - 2\left(\tilde{V}_{22}^t - \tilde{V}_{21}^t\right)}{\frac{\delta(\bar{q}-q_2)}{(\bar{q}-q_1)(q_1-q_2)}\left(\tilde{V}_{11}^t\tilde{V}_{22}^t - \tilde{V}_{12}^t\tilde{V}_{21}^t\right) - 2\tilde{V}_{22}^t} > 0$. Consequently, we get $\frac{\partial y_1^t}{\partial \bar{u}^t} = 0$ and $\frac{\partial y_2^t}{\partial \bar{u}^t} < 0$. Substituting the corresponding demand parameter expressions, we also find $\frac{\partial p_1^t}{\partial \bar{u}^t} > 0$ and $\frac{\partial p_2^t}{\partial \bar{u}^t} = 0$.

Next, consider the case for which $\lambda_1^t > 0$ and $\lambda_2^t > 0$, where we have two active constraints, $y_1^t = x_1^t$ and $y_2^t = x_2^t$. Differentiating the expressions in (12) and $z_1^t + d_1^t = x_1^t$, $z_2^t + d_2^t = x_2^t$ with respect to $\bar{u}^t$, and solving for $\frac{\partial d_i^t}{\partial \bar{u}^t}$, $\frac{\partial z_i^t}{\partial \bar{u}^t}$, and $\frac{\partial \lambda_i^t}{\partial \bar{u}^t}$ for $i = 1,2$, and through similar steps we find $\frac{\partial p_1^t}{\partial \bar{u}^t} > 0$ and $\frac{\partial p_2^t}{\partial \bar{u}^t} < 0$. The analysis for the remaining cases are similar and hence omitted for brevity. □

Proof of Theorem 5 (Sensitivity to Quality Differential):

Recall from Theorem 3 that in this region we have $p_1^{0\,t} = \frac{\bar{p}(q_1-\underline{q})+\underline{p}(\bar{q}-q_1)}{2(\bar{q}-\underline{q})} + \frac{c_1^t}{2}$ and $p_2^{0\,t} = \frac{\bar{p}(q_2-\underline{q})+\underline{p}(\bar{q}-q_2)}{2(\bar{q}-\underline{q})} + \frac{c_2^t}{2}$. Thus, regarding the monotonicity of the list prices with respect to q_1, we find $\frac{\partial p_1^{0\,t}}{\partial q_1} = \frac{(\bar{p}-\underline{p})}{2(\bar{q}-\underline{q})} > 0$ and $\frac{\partial p_2^{0\,t}}{\partial q_1} = 0$, i.e. an increase in the quality level of the higher quality product results in an increase in the list price of the higher quality product but does not impact the optimal list price charged for the lower quality product. Similarly, with respect to an increase in the quality level of the lower quality product, we find $\frac{\partial p_1^{0\,t}}{\partial q_1} = 0$ and $\frac{\partial p_2^{0\,t}}{\partial q_1} = \frac{(\bar{p}-\underline{p})}{2(\bar{q}-\underline{q})} > 0$.

Regarding the monotonicities of the base-stock levels, we first need to consider the monotonicities of the mean demand selections, d_1^{*t} and d_2^{*t}, as an intermediate step. Solving for d_1^t and d_2^t in the first order conditions corresponding to the demand selection decisions, $J_{d_1}^t + \lambda_1^t = 0$ and $J_{d_2}^t + \lambda_2^t = 0$, where $J_{d_1}^t$ and $J_{d_2}^t$ are as given in (10), and recalling that in this region, we have $\lambda_1^t = 0$ and $\lambda_2^t = 0$, we get $d_1^{*t} = \frac{\delta}{2}(\frac{-c_1^t + c_2^t}{q_1 - q_2} + \frac{\bar{p} - c_1^t}{\bar{q} - q_1})$ and $d_2^{*t} = \frac{\delta}{2}(\frac{c_1^t - c_2^t}{q_1 - q_2} + \frac{\underline{p} - c_2^t}{q_2 - \underline{q}})$. We first consider the monotonicities with respect to q_1. Differentiating, we find $\frac{\partial d_1^{*t}}{\partial q_1} = \frac{\delta}{2}(\frac{c_1^t - c_2^t}{(q_1 - q_2)^2} + \frac{\bar{p} - c_1^t}{(\bar{q} - q_1)^2}) > 0$ where the inequality follows since $c_1^t > c_2^t$ and $\bar{p} > c_1^t$ as $\bar{p} > p_1^t > c_1^t$. Similarly, we find $\frac{\partial d_2^{*t}}{\partial q_1} = -\frac{\delta}{2}(\frac{c_1^t - c_2^t}{(q_1 - q_2)^2}) < 0$. Thus, an increase in the quality level of the higher quality product leads the firm to select a higher mean demand value for the higher quality product and a lower mean demand value for the lower quality product. Now, considering the first order conditions with respect to z_1^t and z_2^t, i.e., $-c_1^t + \mathrm{E}_{\epsilon_1^t, \epsilon_2^t}[\frac{\partial}{\partial z_1^t} G^t(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t)] = 0$ and $-c_2^t + \mathrm{E}_{\epsilon_1^t, \epsilon_2^t}[\frac{\partial}{\partial z_2^t} G^t(z_1^t - \epsilon_1^t, z_2^t - \epsilon_2^t)] = 0$ with respect to q_1, and solving for $\frac{\partial z_1^{*t}}{\partial q_1}$ and $\frac{\partial z_2^{*t}}{\partial q_1}$, we get $\frac{\partial z_1^{*t}}{\partial q_1} = \frac{\partial z_2^{*t}}{\partial q_1} = 0$. Thus, the expected number of customers subsequently receiving an upgrade does not change with a change in only the current-period value of q_1. Lastly, since $y_1^{*t} = z_1^{*t} + d_1^{*t}$ and $y_2^{*t} = z_2^{*t} + d_2^{*t}$ by definition, we find $\frac{\partial y_1^{*t}}{\partial q_1} = \frac{\partial d_1^{*t}}{\partial q_1} > 0$ and $\frac{\partial y_2^{*t}}{\partial q_1} = \frac{\partial d_2^{*t}}{\partial q_1} < 0$. Lastly, we consider the monotonicities with respect to q_2^t. Through similar steps, we find that $\frac{\partial y_1^{*t}}{\partial q_2} = \frac{\partial d_1^{*t}}{\partial q_2} \le 0$ and $\frac{\partial y_2^{*t}}{\partial q_2} = \frac{\partial d_2^{*t}}{\partial q_2} < 0$ when $\underline{p} \le \left(\frac{q_2 - \underline{q}}{q_1 - q_2}\right)^2 (c_1^t - c_2^t) + c_2^t$. □

Proof of Theorem 6 (Sensitivity to Cost Parameters):

The proof is similar to the proof of Theorem 5. Recall that our focus is limited to the region corresponding to $\lambda_1^t = \lambda_2^t = 0$. Differentiating the first order conditions with respect to c_1^t, we obtain (i) $\frac{\partial d_1^{*t}}{\partial c_1^t} = -\frac{\delta}{2}(\frac{1}{q_1 - q_2} + \frac{1}{\bar{q} - q_1})$, (ii) $\frac{\partial d_2^{*t}}{\partial c_1^t} = \frac{\delta}{2}(\frac{1}{q_1 - q_2})$, (iii) $\frac{\partial z_1^{*t}}{\partial c_1^t} = \frac{\mathrm{E}[G_{22}^t]}{\mathrm{E}[G_{11}^t]\mathrm{E}[G_{22}^t] - \mathrm{E}[G_{12}^t]\mathrm{E}[G_{21}^t]}$, and (iv) $\frac{\partial z_2^{*t}}{\partial c_1^t} = -\frac{\mathrm{E}[G_{21}^t]}{\mathrm{E}[G_{11}^t]\mathrm{E}[G_{22}^t] - \mathrm{E}[G_{12}^t]\mathrm{E}[G_{21}^t]}$. Thus, $\frac{\partial y_1^{*t}}{\partial c_1^t} = \frac{\partial d_1^{*t}}{\partial c_1^t} + \frac{\partial z_1^{*t}}{\partial c_1^t} = -\frac{\delta}{2}(\frac{1}{q_1 - q_2} + \frac{1}{\bar{q} - q_1}) + \frac{\mathrm{E}[G_{22}^t]}{\mathrm{E}[G_{11}^t]\mathrm{E}[G_{22}^t] - \mathrm{E}[G_{12}^t]\mathrm{E}[G_{21}^t]} < 0$ (recall that $\mathrm{E}[G_{22}^t] < 0$, and $\mathrm{E}[G_{22}^t]\mathrm{E}[G_{11}^t] - \mathrm{E}[G_{12}^t]\mathrm{E}[G_{21}^t] > 0$) , hence the base-stock level for product type-1 decreases, and $\frac{\partial y_2^{*t}}{\partial c_1^t} = \frac{\partial d_2^{*t}}{\partial c_1^t} + \frac{\partial z_2^{*t}}{\partial c_1^t} = \frac{\delta}{2}(\frac{1}{q_1 - q_2}) - \frac{\mathrm{E}[G_{21}^t]}{\mathrm{E}[G_{11}^t]\mathrm{E}[G_{22}^t] - \mathrm{E}[G_{12}^t]\mathrm{E}[G_{21}^t]} > 0$, i.e., the base-stock level for product type-2 increases. Further, differentiating the expressions for the list prices, we obtain $\frac{\partial p_1^{*t}}{\partial c_1^t} = \frac{1}{2}$ and $\frac{\partial p_2^{*t}}{\partial c_1^t} = 0$, thus in this region, the list price of product type-1 increases while the list price for product type-2 remains same. In addition, by (iii) and (iv), we have $\frac{\partial z_1^{*t}}{\partial c_1^t} + \frac{\partial z_2^{*t}}{\partial c_1^t} = \frac{\mathrm{E}[G_{22}^t] - \mathrm{E}[G_{21}^t]}{\mathrm{E}[G_{11}^t]\mathrm{E}[G_{22}^t] - \mathrm{E}[G_{12}^t]\mathrm{E}[G_{21}^t]} < 0$, which indicates (by Theorem 1) that the expected protection level decreases. Further, by (iv), $\frac{\partial z_2^t}{\partial c_1^t} > 0$, thus $\frac{\partial z_1^t}{\partial c_1^t} < \frac{\partial z_1^t}{\partial c_1^t} + \frac{\partial z_2^t}{\partial c_1^t}$, i.e., the decrease in the expected protection level is accompanied by an even greater decrease in the expected intermediate inventory level of product type-1. Therefore, again by Theorem 1, the expected number of subsequent upgrades decreases. The analysis for the sensitivity with respect to c_2^t is similar.

For part (b), differentiating the list price expressions with respect to the underlying cost parameter c^t gives $\frac{\partial p_1^{0\,t}}{\partial c^t} = \frac{\gamma}{2} > 0$ and $\frac{\partial p_2^{0\,t}}{\partial c^t} = \frac{1}{2} > 0$, thus the list price for both products increase. Note that since $\gamma > 1$, the increase in the list price for product type-1 is larger than that for product type-2. Next, differentiation of the first order conditions with respect to the underlying cost parameter c^t leads to: (i) $\frac{\partial d_1^{*t}}{\partial c^t} = -\gamma\frac{\delta}{2}(\frac{1}{q_1 - q_2} + \frac{1}{\bar{q} - q_1}) + \frac{\delta}{2}(\frac{1}{q_1 - q_2})$, (ii) $\frac{\partial d_2^{*t}}{\partial c^t} = \gamma\frac{\delta}{2}(\frac{1}{q_1 - q_2}) - \frac{\delta}{2}(\frac{1}{q_1 - q_2} + \frac{1}{q_2 - \underline{q}})$, (iii) $\frac{\partial z_1^{*t}}{\partial c^t} = \frac{\gamma\mathrm{E}[G_{22}^t] - \mathrm{E}[G_{12}^t]}{\mathrm{E}[G_{11}^t]\mathrm{E}[G_{22}^t] - \mathrm{E}[G_{12}^t]\mathrm{E}[G_{21}^t]}$, and (iv) $\frac{\partial z_2^{*t}}{\partial c^t} = \frac{-\gamma\mathrm{E}[G_{21}^t] + \mathrm{E}[G_{11}^t]}{\mathrm{E}[G_{11}^t]\mathrm{E}[G_{22}^t] - \mathrm{E}[G_{12}^t]\mathrm{E}[G_{21}^t]}$. We have $\frac{\partial y_1^{*t}}{\partial c^t} = \frac{\partial d_1^{*t}}{\partial c^t} + \frac{\partial z_1^{*t}}{\partial c^t} < 0$ (as both $\frac{\partial d_1^{*t}}{\partial c^t} < 0$ and $\frac{\partial z_1^{*t}}{\partial c^t} < 0$), thus the base-stock level for product type-1 decreases. The sign of $\frac{\partial y_2^t}{\partial c^t}$ can be either positive or negative depending on the magnitude of γ. Explicitly, $\frac{\partial y_2^t}{\partial c^t} = \gamma\frac{\delta}{2}(\frac{1}{q_1 - q_2}) - \frac{\delta}{2}(\frac{1}{q_1 - q_2} + \frac{1}{q_2 - \underline{q}}) + \frac{-\gamma\mathrm{E}[G_{21}^t] + \mathrm{E}[G_{11}^t]}{\mathrm{E}[G_{11}^t]\mathrm{E}[G_{22}^t] - \mathrm{E}[G_{12}^t]\mathrm{E}[G_{21}^t]}$. Introduce and let $\gamma_r^t := \frac{\delta(\frac{1}{q_1 - q_2} + \frac{1}{q_2 - \underline{q}})(\mathrm{E}[G_{11}^t]\mathrm{E}[G_{22}^t] - \mathrm{E}[G_{12}^t]\mathrm{E}[G_{21}^t]) - 2\mathrm{E}[G_{11}^t]}{(\frac{\delta}{q_1 - q_2})(\mathrm{E}[G_{11}^t]\mathrm{E}[G_{22}^t] - \mathrm{E}[G_{12}^t]\mathrm{E}[G_{21}^t]) - 2\mathrm{E}[G_{21}^t]}$. Then, $\frac{\partial y_2^{*t}}{\partial c^t} < 0$ if $\gamma < \gamma_r^t$, and $\frac{\partial y_2^{*t}}{\partial c^t} \ge 0$ if $\gamma \ge \gamma_r^t$. Regarding the upgrade policy, from (iii) and (iv), we find that $\frac{\partial z_1^t}{\partial c^t} + \frac{\partial z_2^t}{\partial c^t} < 0$, thus the expected protection level decreases. Introduce and let $\gamma_u^t := \frac{\tilde{G}_{11}}{\tilde{G}_{12}}$. Then for $\gamma > \gamma_u^t$, we have $\frac{\partial z_1^t}{\partial c^t} < \frac{\partial z_1^t}{\partial c^t} + \frac{\partial z_2^t}{\partial c^t} < 0$, i.e., the decrease in the expected protection level is accompanied by an even greater decrease in the expected intermediate inventory level of product type-1, and thus the expected number of subsequent upgrades decreases by Theorem 1. □

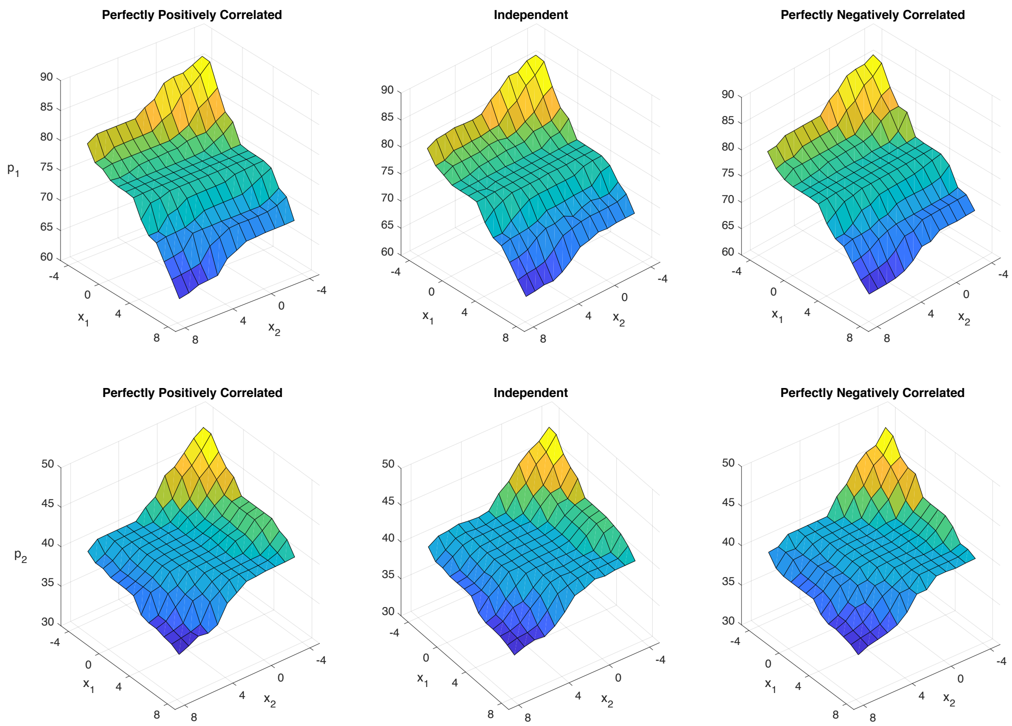

Figure 3 **Price of Product 1 (top) and Price of Product 2 (bottom) in period 5 across perfectly positively correlated demand (left), independent demand (center), and perfectly negatively correlated demand (right)**

Supplement to Numerical Study (Section 6):

Section 6.2 Impact of Demand Correlation: As described in the main text, our numerical studies indicate that the optimal policy structure in the presence of demand correlation is similar to the optimal policy structure shown for the independent demand setting. As an example, Figure 3 displays the similarity in the pricing policy for the higher and lower quality products in period 5 for different correlations, namely, perfectly positively correlated demand, independent demand, and perfectly positively correlated demand.

Section 6.3 A Heuristic Policy: We provide below the explicit representation of the single-period reduced problem in which the firm with no capacity restrictions and no initial inventory determines optimal base-stock levels for the two products.$\hat{y}_1^\circ$ and $\hat{y}_2^\circ$, with expected demands d_1° and d_2° taking into account possible upgrades. The single-period expected cost function $C(y_1, y_2)$ consists of replenishment costs c_i per unit of product type-i, holding and shortage costs h_i^+ and h_i^- after demand realization and any subsequent upgrades, and a discounted cost βc_i for any negative inventory (imitating the replacement cost to return to the original zero inventory position) or a reward $-\beta c_i$ for any remaining positive inventory for product type-i, $i = \{1, 2\}$:

$$\min_{y_1, y_2} \quad C(y_1, y_2) \tag{13}$$

where

$$
\begin{aligned}
C(y_1,y_2) = {} & c_1 y_1 + c_2 y_2 + \iint_{S_1} \Big((h_1^+ - \beta c_1)(y_1 - d_1^\circ - \epsilon_1) + (h_2^+ - \beta c_2)(y_2 - d_2^\circ - \epsilon_2)\Big) f_1(\epsilon_1)\, f_2(\epsilon_2)\, d\epsilon_2\, d\epsilon_1 \\
& + \iint_{S_2} \Big((h_1^- + \beta c_1)(d_1^\circ - y_1 + \epsilon_1) + (h_2^+ - \beta c_2)(y_2 - d_2^\circ - \epsilon_2)\Big) f_1(\epsilon_1)\, f_2(\epsilon_2)\, d\epsilon_2\, d\epsilon_1 \\
& + \iint_{S_3} \Big((h_1^- + \beta c_1)(d_1^\circ - y_1 + \epsilon_1) + (h_2^- + \beta c_2)(d_2^\circ - y_2 + \epsilon_2)\Big) f_1(\epsilon_1)\, f_2(\epsilon_2)\, d\epsilon_2\, d\epsilon_1 \\
& + \iint_{S_4} \Big((h_1^+ - \beta c_1)(y_1 + y_2 - d_1^\circ - d_2^\circ - \epsilon_1 - \epsilon_2)\Big) f_1(\epsilon_1)\, f_2(\epsilon_2)\, d\epsilon_2\, d\epsilon_1 \\
& + \iint_{S_5} \Big((h_2^- + \beta c_2)(d_1^\circ + d_2^\circ - y_1 - y_2 + \epsilon_1 + \epsilon_2)\Big) f_1(\epsilon_1)\, f_2(\epsilon_2)\, d\epsilon_2\, d\epsilon_1
\end{aligned}
$$

with $S_1 = \{(\epsilon_1,\epsilon_2) : \epsilon_1 \le y_1 - d_1^\circ \text{ and } \epsilon_2 \le y_2 - d_2^\circ\}$, $S_2 = \{(\epsilon_1,\epsilon_2) : \epsilon_1 \ge y_1 - d_1^\circ \text{ and } \epsilon_2 \le y_2 - d_2^\circ\}$, $S_3 = \{(\epsilon_1,\epsilon_2) : \epsilon_1 \ge y_1 - d_1^\circ \text{ and } \epsilon_2 \ge y_2 - d_2^\circ\}$, $S_4 = \{(\epsilon_1,\epsilon_2) : \epsilon_1 \le y_1 - d_1^\circ \text{ and } y_2 - d_2^\circ \le \epsilon_2 \le y_1 + y_2 - d_1^\circ - d_2^\circ - \epsilon_1\}$, and $S_5 = \{(\epsilon_1,\epsilon_2) : \epsilon_1 \le y_1 - d_1^\circ \text{ and } \epsilon_2 \ge y_1 + y_2 - d_1^\circ - d_2^\circ - \epsilon_1\}$.

In words, S_1, S_2, and S_3 correspond to, respectively, demand uncertainty realizations that result in excess inventory in both products, a shortage of the higher quality product and excess in the lower quality product, and shortages in both types of products. Collectively, these three areas do not lead to upgrades. The next two areas, S_4 and S_5, correspond to uncertainty realizations where there is excess in the higher quality product and a shortage in the lower quality product, where in the former all demand for the lower quality product is upgraded, and in the latter, the upgrade quantity is limited by the availability of the higher quality product.

It can be easily verified that $C(y_1,y_2)$ is jointly convex in y_1 and y_2 when $h_1^- > h_2^-$, $c_1 > c_2$, and $h_1^+ + h_2^- \ge \beta(c_1 - c_2)$, and the optimal base-stock levels for this reduced problem, denoted by $\hat{y}_1^\circ$ and $\hat{y}_2^\circ$, simultaneously satisfy the following (as also presented in (5)):

$$
\begin{aligned}
F_1(\hat{y}_1^\circ - d_1^\circ) &= \frac{h_1^- - (1-\beta)c_1 + \big(h_1^+ + h_2^- - \beta(c_1 - c_2)\big) \displaystyle\int_{\underline{\epsilon}_1}^{\hat{y}_1^\circ - d_1^\circ} \Big(1 - F_2(\hat{y}_1^\circ + \hat{y}_2^\circ - d_1^\circ - d_2^\circ - \epsilon_1)\Big) f_1(\epsilon_1)\, d\epsilon_1}{h_1^+ + h_1^-} \\
F_2(\hat{y}_2^\circ - d_2^\circ) &= \frac{h_2^- - (1-\beta)c_2 + \big(h_1^+ + h_2^- - \beta(c_1 - c_2)\big) \displaystyle\int_{\underline{\epsilon}_1}^{\hat{y}_1^\circ - d_1^\circ} \Big(F_2(\hat{y}_1^\circ + \hat{y}_2^\circ - d_1^\circ - d_2^\circ - \epsilon_1) - F_2(\hat{y}_2^\circ - d_2^\circ)\Big) f_1(\epsilon_1)\, d\epsilon_1}{h_2^+ + h_2^-}
\end{aligned}
\tag{14}
$$

Proof of Theorem 7 (Optimal Upgrade Fee):

As in the proof of the earlier main results, we start with the inductional assumption that the value function in period $t-1$, $V^{t-1}(x_1^{t-1}, x_2^{t-1})$ is jointly concave, submodular, and its Hessian possesses the diagonally dominance property. The preservation of these properties will however require a new additional condition that we will establish within the subsequent proof of Theorem 8.

The second stage problem given by (7) is

$$
G^t(w_1^t, w_2^t, D_2^t) = \max_{\substack{u^t \\ 0 \le u^t \le D_2^t}} \bar{G}^t(w_1^t, w_2^t, D_2^t, u^t) \tag{15}
$$

where $\bar{G}^t(w_1^t, w_2^t, D_2^t, u^t) = \mathrm{E}_{\zeta^t}\Big[\big(\bar{p}_u - \frac{u^t}{D_2^t}(\bar{p}_u - \underline{p}_u)\big)(u^t + \zeta^t) - h_1(w_1^t - u^t - \zeta^t) - h_2(w_2^t + u^t + \zeta^t) + \beta V^{t-1}(w_1^t - u^t - \zeta^t, w_2^t + u^t + \zeta^t)\Big]$. Differentiating $\bar{G}^t(w_1^t, w_2^t, D_2^t, u^t)$ with respect to u^t yields:

$$
\begin{aligned}
\frac{\partial \bar{G}^t(\cdot)}{\partial u^t} = \bar{p}_u - \frac{2(\bar{p}_u - \underline{p}_u)}{D_2^t} u^t + h_1^+ I_{(w_1^t \ge D_2^t)} - h_1^- I_{(w_1^t \le 0)} + \big((h_1^+ + h_1^-)F(w_1^t - u^t) - h_1^-\big) I_{(0 < w_1^t < D_2^t)} \\
- h_2^+ I_{(w_2^t \ge 0)} + h_2^- I_{(w_2^t \le -D_2^t)} + \big((h_2^+ + h_2^-)F(-w_2^t - u^t) - h_2^+\big) I_{(-D_2^t < w_2^t < 0)} \\
- \beta \mathrm{E}_{\zeta^t}\Big[V_1^{t-1}(w_1^t - u^t - \zeta^t, w_2^t + u^t + \zeta^t) + V_2^{t-1}(w_1^t - u^t - \zeta^t, w_2^t + u^t + \zeta^t)\Big]
\end{aligned}
\tag{16}
$$

where $F(\cdot)$ is the cumulative distribution function for ζ^t and and $V_j^{t-1}(\cdot,\cdot)$ denotes the partial derivative of $V^{t-1}(\cdot,\cdot)$ with respect to its j^{th} argument. (Note: The assumption that $f(\cdot)$ has zero density at the boundaries $-u^t$ and $D_2^t - u^t$ is utilized in the derivation of the above expression, and together with the vanishing variance at $u^t = 0$ or $u^t = D_2^t$, also guarantees continuity of the objective function at the boundaries.) As before, for expositional clarity, when a function's arguments are evident, we suppress the notation and write for example, $\bar{G}^t$, V^t, or V_i^t and V_{ij}^t for $i,j = 1,2$. We further have:

$$\begin{aligned}\frac{\partial^2 \bar{G}^t}{\partial {u^t}^2} = -\frac{2(\bar{p}_u - \underline{p}_u)}{D_2^t} - (h_1^+ + h_1^-) f(w_1^t - u^t) I_{(0<w_1^t<D_2^t)} - (h_2^+ + h_2^-) f(-w_2^t - u^t) - h_2^+ I_{(-D_2^t<w_2^t<0)} \\ +\beta \mathrm{E}_{\zeta^t}\left[V_{11}^{t-1} - V_{12}^{t-1} - V_{21}^{t-1} + V_{22}^{t-1}\right] < 0\end{aligned} \tag{17}$$

where the strict inequality follows as the first three terms are strictly negative and the remaining term is nonpositive due to the inductional assumptions of submodularity and diagonal dominance for $V^{t-1}(x_1^{t-1}, x_2^{t-1})$. Thus, $\bar{G}^t$ is strictly concave in u^t. Let $\mu_1^t \geq 0$ and $\mu_2^t \geq 0$ be the Lagrangian variables associated with the constraints $u^t \geq 0$ and $u^t \leq D_2^t$, respectively. Note that since we assume $D_2^t > 0$, μ_1^t and μ_2^t cannot be simultaneously nonzero. Thus we need to consider three cases.

First, consider the case $\mu_1^{t*} = 0$ and $\mu_2^{t*} = 0$. Then, the optimal u^{t*} is the solution to $\frac{\partial \bar{G}^t}{\partial u^t} = 0$. To show how u^{t*} changes with w_1^t, we differentiate the first order condition with respect to w_1^t and solve for $\frac{\partial u^{t*}}{\partial w_1^t}$. We find: $\frac{\partial u^{t*}}{\partial w_1^t} = \frac{-(h_1^+ + h_1^-) f(w_1^t - u^{t*}) I_{(0<w_1^t<D_2^t)} + \beta \mathrm{E}_{\zeta^t}\left[V_{11}^{t-1} - V_{12}^{t-1}\right]}{-\frac{2(\bar{p}_u - \underline{p}_u)}{D_2^t} - (h_1^+ + h_1^-) f(w_1^t - u^{t*}) I_{(0<w_1^t<D_2^t)} - (h_2^+ + h_2^-) f(-w_2^t - u^{t*}) I_{(-D_2^t<w_2^t<0)} + \beta \mathrm{E}_{\zeta^t}\left[V_{11}^{t-1} - V_{12}^{t-1} - V_{21}^{t-1} + V_{22}^{t-1}\right]}$. As both the numerator and denominator are negative due to the inductional assumptions, we have $\frac{\partial u^{t*}}{\partial w_1^t} > 0$. Therefore, the optimal target upgrade quantity u^{t*} is increasing in w_1^t. Further, we have $\frac{\partial p_u^{t*}}{\partial w_1^t} = -\frac{(\bar{p}_u - \underline{p}_u)}{D_2^t} \frac{\partial u^{t*}}{\partial w_1^t} < 0$. Thus, the optimal upgrade fee p_u^{t*} is decreasing in w_1^t. A similar analysis results in: $\frac{\partial u^{t*}}{\partial w_2^t} = \frac{(h_2^+ + h_2^-) f(-w_2^t - u^{t*}) I_{(-D_2^t<w_2^t<0)} + \beta \mathrm{E}_{\zeta^t}\left[V_{12}^{t-1} - V_{22}^{t-1}\right]}{-\frac{2(\bar{p}_u - \underline{p}_u)}{D_2^t} - (h_1^+ + h_1^-) f(w_1^t - u^{t*}) I_{(0<w_1^t<D_2^t)} - (h_2^+ + h_2^-) f(-w_2^t - u^{t*}) I_{(-D_2^t<w_2^t<0)} + \beta \mathrm{E}_{\zeta^t}\left[V_{11}^{t-1} - V_{12}^{t-1} - V_{21}^{t-1} + V_{22}^{t-1}\right]} < 0$ where the inequality follows as the numerator is strictly positive and the denominator is strictly negative due to the induction assumptions. Further, we have $\frac{\partial p_u^{t*}}{\partial w_2^t} = -\frac{(\bar{p}_u - \underline{p}_u)}{D_2^t} \frac{\partial u^{t*}}{\partial w_2^t} > 0$. Hence, the optimal upgrade fee p_u^{t*} is increasing in w_2^t. Through a similar analysis, we also find $\frac{\partial u^{t*}}{\partial D_2^t} = \frac{2(\bar{p}_u - \underline{p}_u) u^{t*}/(D_2^t)^2}{-\frac{2(\bar{p}_u - \underline{p}_u)}{D_2^t} - (h_1^+ + h_1^-) f(w_1^t - u^{t*}) I_{(0<w_1^t<D_2^t)} - (h_2^+ + h_2^-) f(-w_2^t - u^{t*}) I_{(-D_2^t<w_2^t<0)} + \beta \mathrm{E}_{\zeta^t}\left[V_{11}^{t-1} - V_{12}^{t-1} - V_{21}^{t-1} + V_{22}^{t-1}\right]} < 0$, and consequently that $\frac{\partial p_u^{t*}}{\partial D_2^t} > 0$. Thus, the upgrade fee is increasing with the demand pool D_2^t.

Note that when the firms decides to upgrade u^{t*} customers, it is in effect, also selecting a target protection level $w_1^t - u^{t*}$ on the higher level product. We also provide several monotonicty results on this protection level. Following the above analysis, one can also straightforwardly establish that (a) $\frac{\partial u^{t*}}{\partial w_1^t} < 1$ and (b) $\frac{\partial u^{t*}}{\partial w_1^t} - \frac{\partial u^{t*}}{\partial w_2^t} < 1$. Through (a), we immediately find that the protection level $w_1^t - u^{t*}$ is increasing in w_1^t. Through the previous result $\frac{\partial u^{t*}}{\partial w_2^t} < 0$, we see that the protection level $w_1^t - u^{t*}$ is also increasing in w_2^t, and through (b) we find that the increase in the protection level with respect to w_1^t is stronger than the increase in the protection level with respect to w_2^t. Thus, the protection level is a function of w_1^t and w_2^t only through their sum, but is a function of w_1^t and w_2^t individually.

Finally, consider the case where $\mu_1^{t*} > 0$ and $\mu_2^{t*} = 0$. This indicates that the constraint $u^t \geq 0$ is active and we immediately have $p_u^{t*}(u_1^{t*}) = \bar{p}_u - \frac{(u_1^{t*} - u_1^{t*})}{D_2^t}(\bar{p}_u - \underline{p}_u) = \bar{p}_u$. Similarly, the case for which $\mu_1^{t*} = 0$ and $\mu_2^{t*} > 0$ implies $u^t = D_2^t$ and leads to $p_u^{t*}(u_1^{t*}) = \bar{p}_u - \frac{D_2^t}{D_2^t}(\bar{p}_u - \underline{p}_u) = \bar{p}_u$. □

Supplement to Pricing and Replenishment with Upgrade Fees: As a supplement to the manuscript, the below results summarize our findings regarding the optimal replenishment and pricing decisions when the firm selects and charges an upgrade fee.

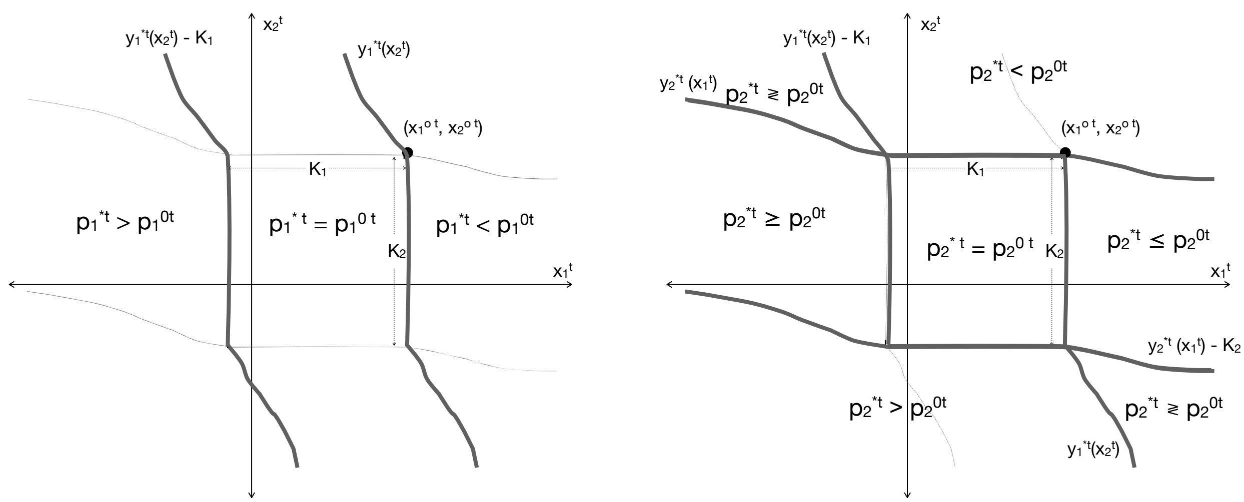

Figure 4 **Optimal pricing policy structure for the higher quality product (left) and the lower quality product (right) when the firm sets upgrade fees**

THEOREM 8. *(a) The optimal replenishment for both products follow the partially decoupled state-dependent base-stock policy characterized by $y_i^{*t}(x_j^t)$ with $x_1^{\circ t} = y_1^{*t}(x_2^{\circ t})$ and $x_2^{\circ t} = y_2^{*t}(x_1^{\circ t})$ as described in* Theorem 2 *with all monotonicity results preserved.*
(b) Let $p_1^{\circ t}$ and $p_2^{\circ t}$ denote list prices in period t for products type-1 and type-2, respectively. The optimal price for the higher quality product type-1 follows the structure of the pricing policy described in Theorem 3.
For the lower quality product type-2, it is optimal to apply its list price $p_2^{\circ t}$ if $y_1^{*t}(x_2^t) - K_1^t \le x_1^t \le y_1^{*t}(x_2^t)$ and $y_2^{*t}(x_1^t) - K_2^t \le x_2^t \le y_2^{*t}(x_1^t)$. A price discount is given if $x_1^t \ge y_1^{*t}(x_2^t) - K_1^t$ and $x_2^t \ge y_2^{*t}(x_1^t)$, and a price surcharge is given if $x_1^t \le y_1^{*t}(x_2^t)$ and $x_2^t \le y_2^{*t}(x_1^t) - K_2^t$. When $y_2^{*t}(x_1^t) - K_2^t \le x_2^t \le y_2^{*t}(x_1^t)$, either the list price or a price discount may be optimal if $x_1^t \ge y_1^{*t}(x_2^t)$, and either the list price or a price surcharge may be optimal if $x_1^t \le y_1^{*t}(x_2^t) - K_1^t$. In the two remaining regions corresponding to either $x_1^t < y_1^{*t}(x_2^t) - K_1$ and $x_2^t > y_2^{*t}(x_1^t)$, or $x_1^t > y_1^{*t}(x_2^t)$ and $x_2^t < y_2^{*t}(x_1^t) - K_2$, a price discount or a price surcharge may be optimal for product type-2.
Furthermore, the price of either product is decreasing with respect to the inventory level of either product.

Proof of Theorem 8 (Pricing and Replenishment with Upgrade Fees):
As the proof methodology is similar to the proofs of Theorem 2 and Theorem 3, for brevity, we only highlight the main arguments here and refer to earlier results where applicable. We start by introducing several properties of the second stage profit-to-go function, G^t.

LEMMA 5. *The second-partials of $G^t(w_1^t, w_2^t, D_2^t)$ satisfy the following: (i) $G_{11}^t(w_1^t, w_2^t, D_2^t) \le G_{12}^t(w_1^t, w_2^t, D_2^t) \le 0$, (ii) $G_{22}^t(w_1^t, w_2^t, D_2^t) \le G_{21}^t(w_1^t, w_2^t, D_2^t) \le 0$, and (iii) $G_{33}^t(w_1^t, w_2^t, D_2^t) \le 0$.*

Proof: For brevity, we only present the proof for property (*iii*). The properties (*i*) and (*ii*) are derived in a similar manner and are analogous to their earlier versions established in Lemma 1. Consider the Lagrangian for the second-stage problem, $\bar{G}^t(w_1^t, w_2^t, D_2^t, u^t) + \mu_1^t(u^t) - \mu_2^t(u^t - D_2^t)$, where, as before, $\mu_1^t \ge 0$ and $\mu_2^t \ge 0$ are the Lagrangian variables associated with the constraints $u^t \ge 0$ and $u^t \le D_2^t$, respectively. We first consider the case corresponding to $\mu_1^{t*} = 0$ and $\mu_1^{t*} = 0$. Through the envelope theorem, we have $G_{33}^t = \frac{2(\bar{p}_u - \underline{p}_u)u^{t*}}{(D_2^t)^2}\left(\frac{\partial u^{t*}}{\partial D_2^t} - \frac{u^{t*}}{D_2^t}\right) < 0$, where the inequality follows from $\frac{\partial u^{t*}}{\partial D_2^t} < 0$ established in the proof of Theorem 7. Next, consider the case where $\mu_1^{t*} > 0$ and $\mu_2^{t*} = 0$ corresponding to $u^{t*} = 0$ due to the active constraint. Through the envelope theorem, we have $G_{33}^t = 0$. Similarly, the case $\mu_1^{t*}=0$ and $\mu_2^{t*}>0$ corresponding to $u^{t*}=D_2^t$ leads to $G_{33}^t=\beta(V_{11}^{t-1}-V_{12}^{t-1}-V_{21}^{t-1}+V_{22}^{t-1})\leq 0$ due to the inductional assumption.

We make the same variable transformation introduced in the proof of Theorem 2 and rewrite the first-stage problem equivalently as follows:

$$V^t(x_1^t,x_2^t)=c_1^tx_1^t+c_2^tx_2^t+\max_{\substack{z_i^t,d_i^t \\ x_i^t\leq z_i^t+d_i^t\leq x_i^t+K_i}} J^t(z_1^t,z_2^t,d_1^t,d_2^t)$$

where $J^t(z_1^t,z_2^t,d_1^t,d_2^t)=d_1^tp_1^t(d_1^t,d_2^t)+d_2^tp_2^t(d_1^t,d_2^t)-(c_1(z_1^t+d_1^t)+c_2(z_2^t+d_2^t))+\mathrm{E}_{\epsilon_1^t,\epsilon_2^t}[(G^t(z_1^t-\epsilon_1^t,z_2^t-\epsilon_2^t,d_2^t+\epsilon_2^t)]$. The properties we establish for $G^t(w_1^t,w_2^t,D_2^t)$ in Lemma 5 suffice to preserve the properties of $J^t(z_1^t,z_2^t,d_1^t,d_2^t)$ introduced in Lemma 1. (Note that $J^t(z_1^t,z_2^t,d_1^t,d_2^t)$ is now separable in (d_1^t,d_2^t) and (z_1^t,z_2^t,d_2^t) and similar arguments as those presented in the proof of Lemma 1 verify its strict concavity.) For example, we now have $J_{d_2,d_1}^t=-\frac{2(q_2-\underline{q})(\bar{q}-q_1)}{\delta(\bar{q}-\underline{q})}<0$, and $J_{d_2,d_2}^t=-\frac{2(q_2-\underline{q})(\bar{q}-q_2)}{\delta(\bar{q}-\underline{q})}+\mathrm{E}_{\epsilon_1^t,\epsilon_2^t}[G_{33}]$, which yields $J_{d_2,d_2}^t-J_{d_2,d_1}^t=-\frac{2(q_2-\underline{q})(q_1-q_2)}{\delta(\bar{q}-\underline{q})}+\mathrm{E}_{\epsilon_1^t,\epsilon_2^t}[G_{33}]<0$ as the first term is strictly negative and $G_{33}\leq 0$. Hence Lemma 2, Lemma 3, and the remaining arguments in the proof of Theorem 2 also follow, resulting in the optimality of the partially decoupled state-dependent base-stock policy.

The analysis of the optimal pricing decisions are similar to the proof of Theorem 3. We first note that as $J_{d_2}^t=\frac{\bar{p}(q_2-\underline{q})+\underline{p}(\bar{q}-q_2)}{(\bar{q}-\underline{q})}-\frac{2(q_2-\underline{q})}{\delta(\bar{q}-\underline{q})}\Big[(\bar{q}-q_1)d_1+(\bar{q}-q_2)d_2\Big]-c_2^t+\mathrm{E}_{\epsilon_1^t,\epsilon_2^t}[G_3(\cdot)]$ now includes the term $\mathrm{E}_{\epsilon_1^t,\epsilon_2^t}[G_3(\cdot)]$, the derivation of the list prices through solving $J_{d_1}^t=0$ and $J_{d_2}^t=0$ no longer leads to a closed form solution of the problem parameters. As the optimal pricing policy for the higher quality product otherwise follows an identical structure to the one we derived for the original problem, for brevity, we limit our attention to only the pricing policy for the lower quality product and to the regions where its structure deviates from the optimal policy for the original problem. Specifically, consider the region where $\lambda_1^{t*}>0$ and $\lambda_1^{t*}=0$. Through similar steps as in the proof of Theorem 2, we get $\frac{\partial p_2^{t*}}{\partial\lambda_1^t}=\left(\frac{(q_2-\underline{q})(\bar{q}-q_1)}{\delta(\bar{q}-\underline{q})(J_{d_1,d_1}^tJ_{d_2,d_2}^t-(J_{d_1,d_2}^t)^2)}\right)\mathrm{E}_{\epsilon_1^t,\epsilon_2^t}[G_{33}]\leq 0$. Thus, $p_2^{t*}\leq p_2^{0\,t}$. Further, we also have $\frac{\partial p_2^{t*}}{\partial x_1^t}=\frac{\partial p_2^{t*}}{\partial\lambda_1^t}\frac{\partial\lambda_1^t}{\partial x_1^t}\leq 0$ and $\frac{\partial p_2^{t*}}{\partial x_2^t}=\frac{\partial p_2^{t*}}{\partial\lambda_1^t}\frac{\partial\lambda_1^t}{\partial x_2^t}=0$ by Lemma 3, establishing the monotonicity of p_2^{t*} in this region. For the region with $\lambda_1^{t*}<0$ and $\lambda_1^{t*}>0$, a similar analysis shows that p_2^{t*} may be greater than or smaller than $p_2^{0\,t}$, and that p_2^{t*} is decreasing with x_1^t and x_2^t. $\square$