Meromorphic Dirichlet Series

by

Corey Everlove

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Doctoral Committee:

Professor Jeffrey C. Lagarias, Chair
Professor Deborah E. Goldberg
Associate Professor Sarah C. Koch
Professor Hugh L. Montgomery
Professor Karen E. Smith

Corey Everlove

everlove@umich.edu

ORCID id: 0000-0001-6301-3085

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Abstract

This thesis studies several problems concerning the meromorphic continuation of Dirichlet series to the complex plane.

We show that if a Dirichlet series $f(s) = \sum a_n n^{-s}$ has a meromorphic continuation to the complex plane and the power series generating function $\sum b_n z^n$ of a sequence b_n has a meromorphic continuation to $z = 1$, then the Dirichlet series

$$\sum_{n=1}^{\infty}\Big(\sum_{i+j=n} a_i b_j\Big)n^{-s}$$

has a meromorphic continuation to the complex plane. Using specific choices of the sequence b_n, we show that the Dirichlet series whose coefficients are a shift, forward or backward difference, or partial sum of the coefficients of $f(s)$ has a meromorphic continuation to the complex plane. We study several examples of such Dirichlet series involving important arithmetic functions, including the Dirichlet series whose coefficients are the Chebyshev function, the Mertens function, the partial sums of the divisor function, or the partial sums of a Dirichlet character.

We apply these results to study the Dirichlet series whose coefficients are the sum of the base-b digits of the integers. We also study the Dirichlet series whose coefficients are the cumulative sum of the base-b digits of the integers less than n. We show that these Dirichlet series have a meromorphic continuation to the complex plane, and we give the locations of the poles and the residue at each pole. We also consider an interpolation of the sum-of-digits and cumulative sum-of-digits functions from integer bases $b \geq 2$ to a real parameter $\beta > 1$, and show that Dirichlet series attached to these interpolated sum-of-digits functions meromorphically continue one unit left of their halfplanes of convergence.

Finally, we consider a one-parameter family of Dirichlet series related to Ramanujan sums. The classical Ramanujan sum $c_n(m)$ is a function of two integer variables; we replace the integer parameter m with a complex number and consider the Dirichlet series attached to such complex Ramanujan sums. We show that this Dirichlet series continues to a meromorphic function of two complex variables and locate its singularities.

Chapter 1

Introduction

1.1 Dirichlet series and meromorphic continuation

This thesis studies some topics related to the meromorphic continuation of Dirichlet series.

A ***Dirichlet series*** is a series of the form

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{1.1}$$

with coefficients $a_n \in \mathbb{C}$, considered as a function of a complex variable s. As we review in Chapter 2, the region of convergence of a Dirichlet series is a halfplane: each Dirichlet series has an abscissa of convergence σ_c (possibly ∞ or $-\infty$) such that the series converges to a holomorphic function if $\mathrm{Re}(s) > \sigma_c$ and diverges if $\mathrm{Re}(s) < \sigma_c$. The holomorphic function defined by a Dirichlet series in its halfplane of convergence may or may not analytically continue to a meromorphic function on a larger region of the complex plane.

1.1.1 The Riemann zeta function

The prototypical example of a Dirichlet series is the ***Riemann zeta function*** defined for $\mathrm{Re}(s) > 1$ by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.2}$$

In 1737, Euler studied $\zeta(s)$ as a function of a real variable. By writing the zeta function as a product over the prime numbers

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}} \tag{1.3}$$

and using the fact that $\zeta(s) \to \infty$ as $s \to 1+$, Euler deduced that the sum $\sum 1/p$ of the reciprocals of the primes diverges. Here we see the beginning of a major theme of analytic number theory:

information about the analytic behavior of a Dirichlet series (the growth of $\zeta(s)$ at $s = 1$) is used to deduce arithmetic information (a statement about the distribution of prime numbers).

In 1859, Riemann studied the function $\zeta(s)$ as a function of a complex variable (see the appendix to [18] for an English translation of Riemann's memoir). Riemann proved that $\zeta(s)$, originally defined for $\mathrm{Re}(s) > 1$, continues to a meromorphic function on $\mathbb{C}$ with only singularity a simple pole at $s = 1$ with residue 1. Riemann also proved the functional equation

$$\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \pi^{-(1-s)/2}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s) \tag{1.4}$$

relating the values of $\zeta(s)$ for $\mathrm{Re}(s) > 1/2$ and $\mathrm{Re}(s) < 1/2$. From the functional equation and the nonvanishing of $\zeta(s)$ for $\mathrm{Re}(s) > 1$, Riemann deduced that aside from the "trivial zeros" at $s = -2k$ for integers $k \geq 1$, the zeros of $\zeta(s)$ must lie in the strip $0 \leq \mathrm{Re}(s) \leq 1$. Riemann famously conjectured, based on some computational evidence, that all of the zeros in this strip must have real part $1/2$. Riemann made some further conjectures about $\zeta(s)$ and its relation to the distribution of the prime numbers; before stating a form of his conjectured "explicit formula", we return to more general Dirichlet series.

1.1.2 Perron's formula

The following formula provides one of the important links between the analytic behavior of a Dirichlet series and information about its coefficients.

Proposition 1.1.1 (Perron's formula, see [52, sec. 9.42])**.** *If*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{1.5}$$

is a Dirichlet series convergent in the halfplane $\mathrm{Re}(s) > \sigma_c$*, then for* $c > \max(\sigma_c, 0)$*,*

$$\sideset{}{'}\sum_{n\leq x} a_n = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s)\frac{x^s}{s}\, ds, \tag{1.6}$$

where if x is an integer, the term a_x in the sum on the left is counted with weight $1/2$.

More detailed versions of this relation bound the error in truncating the integral at height T instead of taking the limit or modify the factor x^s/s in the integral to introduce weights in the sum of the coefficents a_n.

If the Dirichlet series $f(s)$ has a meromorphic continuation to the complex plane, then one might hope to use information about the singularities of $f(s)$ to evaluate or estimate the

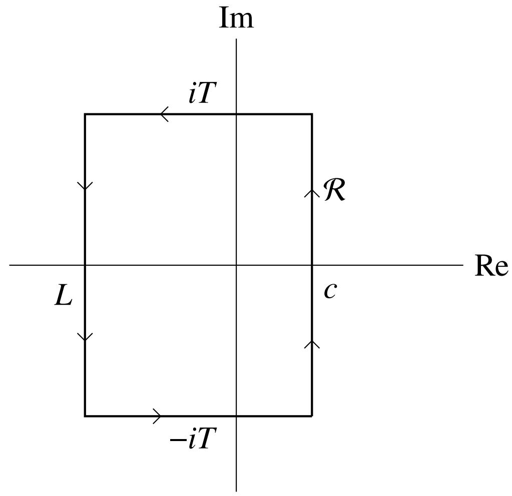

FIGURE 1.1: A rectangular contour for using Perron's formula.

integral on the right of (1.6) using the residue theorem. For example, one could integrate over a rectangular contour $\mathcal{R}$ as in Figure 1.1 by the residue theorem, then estimate the contributions of the integral over the top, left, and bottom sides. To apply this method to a Dirichlet series $f(s)$, one desires the following information about $f(s)$:

1. existence of a meromorphic continuation of $f(s)$ to $\mathbb{C}$,
2. information about the location of the poles of $f(s)$ and the residue at each pole,
3. information about the growth of $f(s)$, so that the contributions of the integral along the top, left, and bottom of the integral over $\mathcal{R}$ may be bounded.

In general, more precise information about the poles of the function $f(s)$ leads to better information about the behavior of the sums $\sum a_n$.

1.1.3 The explicit formula and the prime number theorem

Using the product formula (1.3), one finds that the logarithmic derivative of the zeta function is given by the Dirichlet series

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \tag{1.7}$$

where $\Lambda(n)$ is the von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p, \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

In 1895, von Mangoldt supplied the analytic details to apply Perron's formula to the Dirichlet series $\zeta'/\zeta(s)$ obtain a formula for $\sum_{n\le x}\Lambda(n)$. By proving that the integrals over the top, left, and bottom of the contour $\mathcal{R}$ in Figure 1.1 go to 0 for suitable sequences of $T\to\infty$ and $L\to-\infty$, von Mangoldt obtained

$$\sum_{n\le x}\Lambda(n) = x - \sum_{\substack{\zeta(\rho)=0\\ 0<\operatorname{Re}\rho<1}} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) + \sum_{k=1}^{\infty}\frac{x^{-2k}}{2k}. \tag{1.9}$$

Note the four terms on the right of (1.9) come from the poles of $\zeta'/\zeta(s)$ and the integrand x^s/s in Perron's formula: the pole of $\zeta(s)$ at $s=1$, the nontrivial zeros ρ in the strip $0<\operatorname{Re}(s)<1$, the pole of x^s/s at $s=0$, and the nontrivial zeros of $\zeta(s)$ at $s=-2k$. We therefore see in this example the importance of information about the poles of the Dirichlet series $\zeta'/\zeta(s)$, as each pole contributes to asymptotics of the summatory function $\sum\Lambda(n)$.

In 1896, Hadamard and de la Vallée-Poussin independently supplied the analytic details needed to complete the proof of the prime number theorem. By proving that $\zeta(s)$ has no zeros in a small region to the left of the line $\operatorname{Re}(s)=1$, the sum over nontrivial zeros in (1.9) may be estimated to be smaller than the main term x, so that

$$\sum_{n\le x}\Lambda(n)\sim x \tag{1.10}$$

as $x\to\infty$. By partial summation, we have the more familiar form of the prime number theorem

$$\pi(x)\sim\frac{x}{\log x}, \tag{1.11}$$

where $\pi(x)$ is the number of primes less than or equal to x.

1.1.4 Dirichlet series and arithmetic functions

Many important arithmetic functions are closely related to the Riemann zeta function; among the many well-known examples, we have

$$\zeta(s)^2 = \sum_{n=1}^{\infty}\frac{d(n)}{n^s} \tag{1.12}$$

where $d(n)$ is the number of divisors of the integer n and

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\phi(n)}{n^s} \tag{1.13}$$

where $\phi(n)$ is the Euler totient function (the number of reduced residues mod n). The meromorphic continuation and other analytic properties of Dirichlet series attached to such arithmetic functions thus follows from knowledge of $\zeta(s)$, and sums of such arithmetic functions can then be studied by Perron's formula.

Many other Dirichlet series have been used to study a wide variety of arithmetic problems. To give just one of the many possible interesting examples, the Dirichlet L-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{1.14}$$

attached to a Dirichlet character mod q (a character of the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^\times$) has a meromorphic continuation to $\mathbb{C}$; these L-functions are crucial to the study of the distribution of primes in arithmetic progressions.

1.2 Outline of the thesis and summary of results

We now give a brief overview of the contents and results of this thesis.

1.2.1 Rings of Dirichlet series

In Chapter 2, we present introductory material on the theory of Dirichlet series that will be needed in future chapters and collect some known results about the structure of some rings of Dirichlet series.

In Section 2.1, we begin by considering the ring of formal Dirichlet series—Dirichlet series considered as formal series without considering convergence. We survey some of the known algebraic properties of this ring.

We then consider the theory of convergence of Dirichlet series in Section 2.2, collecting the results we will need in the rest of the thesis. We also state some algebraic properties of the ring of Dirichlet series that converge in some halfplane.

In Section 2.3, we consider Dirichlet series with a meromorphic continuation to $\mathbb{C}$. We also give some examples of Dirichlet series that do not have a meromorphic continuation to $\mathbb{C}$.

Finally, in Section 2.4, we consider Dirichlet series with a continuation to $\mathbb{C}$ as a meromorphic function of finite order. Most of the well-known Dirichlet series of number theory are meromorphic functions of order 1. We briefly review the Selberg class, an important class of Dirichlet series all of order 1. We then give an example of a Dirichlet series of arbitary real order $\rho > 1$ and a Dirichlet series of infinite order.

1.2.2 Dirichlet series associated to additive convolutions

In Chapter 3, we study Dirichlet series associated to additive convolutions of sequences. Specifically, we study the problem of meromorphic continuation of the Dirichlet series

$$g(s) = \sum_{n=1}^{\infty}\Big(\sum_{i+j=n} a_i b_j\Big) n^{-s} \tag{1.15}$$

when the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{1.16}$$

has a meromorphic continuation. We prove that if the additive generating function $q(z) = \sum_{n=0}^{\infty} b_n z^n$ of the sequence b_n has a meromorphic continuation to $z = 1$, then the function $g(s)$ is meromorphic on $\mathbb{C}$ if $f(s)$ is meromorphic on $\mathbb{C}$. The main result is summarized by the following.

Theorem 1.2.1. *Let $B = (b_0, b_1, b_2, \ldots)$ be a sequence of complex numbers whose summatory function $B(x) = \sum_{n \le x} b_n$ satisfies $B(x) \ll x^r$ for some $r \ge 0$, and set*

$$q(z) = \sum_{n=0}^{\infty} b_n z^n. \tag{1.17}$$

Suppose that the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{1.18}$$

has abscissa of absolute convergence $\sigma_a(f)$ and has a meromorphic continuation to the halfplane $\mathrm{Re}(s) > \sigma_m$.

Then the Dirichlet series

$$g(s) = \sum_{n=1}^{\infty}\Big(\sum_{i+j=n} a_i b_j\Big) \frac{1}{n^s} \tag{1.19}$$

has the following properties.

(i) *The Dirichlet series $g(s)$ has abscissa of absolute convergence $\sigma_a(g) \le \max(\sigma_a(f) + r, r)$.*

(ii) *If the function $q(z)$ has a meromorphic continuation to $z = 1$, then $g(s)$ has a meromorphic continuation to the halfplane* $\mathrm{Re}(s) \ge \sigma_m - M$, *where M is the order of the zero (if $M > 0$) or pole (if $M < 0$) of $q(z)$ at $z = 1$.*

See Theorem 3.3.4 for a more precise version, with a formula for $g(s)$ in terms of $f(s)$.

Our motivation for studying such additive convolutions is to study the Dirichlet series obtained by shifting, differencing, or summing the coefficients of a Dirichlet series. We prove that such series have a meromorphic continuation if the original series have a meromorphic continuation.

Theorem 1.2.2. *If the Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{1.20}$$

converges in a right halfplane $\mathrm{Re}(s) > \sigma_c$ *for some* $\sigma_c < \infty$ *and has a meromorphic continuation to the halfplane* $\mathrm{Re}(s) > \sigma_m$*, then the Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{a_{n-1}}{n^s}, \quad \sum_{n=1}^{\infty} \frac{a_n - a_{n-1}}{n^s}, \quad \textit{and} \quad \sum_{n=1}^{\infty} \Big(\sum_{m \leq n} a_m \Big) \frac{1}{n^s} \tag{1.21}$$

have meromorphic continuation to the halfplanes $\mathrm{Re}(s) > \sigma_m$*,* $\mathrm{Re}(s) > \sigma_m - 1$*, and* $\mathrm{Re}(s) > \sigma_m + 1$*, respectively.*

See Theorems 3.4.3, 3.4.6, 3.4.11 for more detailed results. From these results, we deduce the meromorphic continuation of each of the following examples, explicitly computing the locations of the poles and the resudes at the poles.

- Let $H_n = \sum_{m \leq n} m^{-1}$ be the nth harmonic number. We study the Dirichlet series

  $$H(s) = \sum_{n=1}^{\infty} \frac{H_n}{n^s} \tag{1.22}$$

  and show that this function has a meromorphic continuation to $\mathbb{C}$, with a double pole at $s = 2$ and simple poles at $s = 0$ and $s = 1 - 2m$ for $m \geq 1$. We explicitly compute the residues at the poles and determine the value of $H(-2m)$ for $m \geq 1$. These results were obtained by a different method by Matsuoka [36]. See Theorem 3.5.1 for details.

- Let $d(n)$ be the number of divisors of the positive integer n. The summatory function $D(x)$ of $d(n)$ satisfies

  $$D(x) := \sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + \Delta(x) \tag{1.23}$$

  with error term $\Delta(x) = o(x)$, where γ is the Euler-Mascheroni constant. Determining the order of the error term $\Delta(x)$ is known as the Dirichlet divisor problem. We prove that the

Dirichlet series

$$\sum_{n=1}^{\infty} \frac{D(n)}{n^s} \tag{1.24}$$

has a meromorphic continuation to $\mathbb{C}$, with double poles at $s = 2$ and $s = 1$ and simple poles at $s = -2m$ for $m \geq 0$; we also compute the value at the negative odd integers. See Theorem 3.5.2 for details.

We also note that the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\Delta(n)}{n^s} \tag{1.25}$$

associated to the error term in the Dirichlet divisor problem has a meromorphic continuation to $\mathbb{C}$; see Corollary 3.5.3.

- Let $\Lambda(n)$ be the von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, \\ 0 & \text{otherwise,} \end{cases} \tag{1.26}$$

and let $\psi(x) = \sum_{n \leq x} \Lambda(n)$ be the Chebyshev function. We show that the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} \tag{1.27}$$

has a meromorphic continuation to $\mathbb{C}$ with poles at $s = m$ for integers $m \leq 2$ and at points $s = m + \rho$ with $m = 1, 0, -1, -3, -5, \ldots$ where ρ is a nontrivial zero of $\zeta(s)$, and we determine the residue at each pole. See Theorem 3.5.4 for details.

- Let $\mu(n)$ be the Möbius function

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is squarefree and has } k \text{ prime factors,} \\ 0 & \text{otherwise,} \end{cases} \tag{1.28}$$

and let $M(x) = \sum_{n \leq x} \mu(n)$ be the Mertens function. We show that the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{M(n)}{n^s} \tag{1.29}$$

has a meromorphic continuation to $\mathbb{C}$ with simple poles at $s = m$ for all integers $m \leq 1$ and poles at left translates of the nontrivial zeros of $\zeta(s)$ whose orders depend on the orders of the nontrivial zeros. See Theorem 3.5.6 for details.

- Let $\chi(n)$ be a nonprincipal Dirichlet character. We show that the Dirichlet series

$$\sum_{n=1}^{\infty}\Big(\sum_{m\le n}\chi(m)\Big)\frac{1}{n^s} \tag{1.30}$$

continues to $\mathbb{C}$ as an entire function if $\chi(-1)=-1$ or as a meromorphic function with at most a simple pole at $s=1$ if $\chi(-1)=1$. See Theorem 3.5.8 for details.

1.2.3 Dirichlet series associated to digital sums

In Chapter 4, we study Dirichlet series associated to the sum of the base-b digits of the integers. For an integer base $b\ge 2$, let $d_b(n)$ be the sum of the base-b digits of n, and let

$$S_b(n) := \sum_{m=1}^{n-1} d_b(m) \tag{1.31}$$

be the summatory function of $d_b(n)$. Delange [15] proved that the function $S_b(n)$ can be written as

$$S_b(n) = \frac{b-1}{\log b}n\log n + h_b\Big(\frac{\log n}{\log b}\Big)n \tag{1.32}$$

where $h_b(n)$ is a nowhere-differentiable function of period 1 whose Fourier coefficients involve values of the Riemann zeta function on the imaginary axis.

We associate to the functions $d_b(n)$ and $S_b(n)$ the Dirichlet series generating functions

$$F_b(s) = \sum_{n=1}^{\infty}\frac{d_b(n)}{n^s} \tag{1.33}$$

and

$$G_b(s) = \sum_{n=1}^{\infty}\frac{S_b(n)}{n^s}. \tag{1.34}$$

Using the methods of Chapter 3, we prove the meromorphic continuation of $F_b(s)$ and $G_b(s)$ to $\mathbb{C}$. For $F_b(s)$ we have the following result.

Theorem 1.2.3. *For each integer base $b\ge 2$, the function $F_b(s)=\sum_{n=1}^{\infty}d_b(n)n^{-s}$ has a meromorphic continuation to $\mathbb{C}$. The poles of $F_b(s)$ consist of a double pole at $s=1$ with Laurent expansion beginning*

$$F_b(s) = \frac{b-1}{2\log b}(s-1)^{-2} + \left(\frac{b-1}{2\log b}\log(2\pi) - \frac{b+1}{4}\right)(s-1)^{-1} + O(1), \tag{1.35}$$

simple poles at each other point $s = 1 + 2\pi im/\log b$ with $m \in \mathbb{Z}$ $(m \neq 0)$ with residue

$$\operatorname{Res}\left(F_b(s), s = 1 + \frac{2\pi im}{\log b}\right) = -\frac{b-1}{2\pi im}\zeta\left(\frac{2\pi im}{\log b}\right), \tag{1.36}$$

and simple poles at each point $s = 1 - k + 2\pi im/\log b$ with $k = 1$ or $k \geq 2$ an even integer and with $m \in \mathbb{Z}$, with residue

$$\operatorname{Res}\left(F_b(s), s = 1 - k + \frac{2\pi im}{\log b}\right) = (-1)^{k+1}\frac{b-1}{\log b}\zeta\left(\frac{2\pi im}{\log b}\right)\frac{B_k}{k!}\prod_{j=1}^{k-1}\left(\frac{2\pi im}{\log b} - j\right) \tag{1.37}$$

where B_k is the kth Bernoulli number.

Dumas proved in his 1993 thesis [17] that the function $F_b(s)$ has a meromorphic continuation to $\mathbb{C}$ but did not completely determine the locations of the poles or their residues.

Similary, for $G_b(s)$, we have the following result.

Theorem 1.2.4. *For each integer $b \geq 2$, the function $G_b(s) = \sum_{n=1}^{\infty} S_b(n)n^{-s}$ has a meromorphic continuation to $\mathbb{C}$. The poles of $G_b(s)$ consist of a double pole at $s = 2$ with Laurent expansion*

$$G_b(s) = \frac{b-1}{2\log b}(s-2)^{-2} + \left(\frac{b-1}{2\log b}\left(\log(2\pi) - 1\right) - \frac{b+1}{4}\right)(s-2)^{-1} + O(1), \tag{1.38}$$

a simple pole at $s = 1$ with residue

$$\operatorname{Res}(G_b(s), s = 1) = \frac{b+1}{12}, \tag{1.39}$$

simple poles at $s = 2 + 2\pi im/\log b$ with $m \in \mathbb{Z}$ $(m \neq 0)$ with residue

$$\operatorname{Res}\left(G_b(s), s = 2 + \frac{2\pi im}{\log b}\right) = -\frac{b-1}{2\pi im}\left(1 + \frac{2\pi im}{\log b}\right)^{-1}\zeta\left(\frac{2\pi im}{\log b}\right) \tag{1.40}$$

and simple poles at point $s = 2 - k + 2\pi im/\log b$ with $k \geq 2$ an even integer and $m \in \mathbb{Z}$ with residue

$$\operatorname{Res}\left(G_b(s), s = 2 - k + \frac{2\pi im}{\log b}\right) = \frac{b-1}{\log b}\zeta\left(\frac{2\pi im}{\log b}\right)\left(\frac{B_k}{k(k-2)!}\right)\prod_{j=1}^{k-2}\left(\frac{2\pi im}{\log b} - j\right). \tag{1.41}$$

Next, we use Delange's formula (1.32) to interpolate the function $S_b(n)$ from integer bases $b \geq 2$ to a real parameter $\beta \geq 1$. We then define an interpolated sum-of-digits function $d_\beta := S_\beta(n+1) - S_\beta(n)$. For the associated Dirichlet series

$$F_\beta(s) = \sum_{n=1}^{\infty} \frac{d_b(n)}{n^s} \tag{1.42}$$

and

$$G_\beta(s) = \sum_{n=1}^{\infty} \frac{S_b(n)}{n^s}, \tag{1.43}$$

we obtain a meromorphic continuation 1 unit left of the absicssa of convergence of each series.

For the function $F_\beta(s)$ we have the following theorem.

Theorem 1.2.5. *For each real $\beta > 1$, the function $F_\beta(s)$ has a meromorphic continuation to the halfplane $\mathrm{Re}(s) > 0$, with a double pole at $s = 1$ with Laurent expansion*

$$F_\beta(s) = \frac{\beta - 1}{2\log\beta}(s-1)^{-2} + \left(\frac{\beta-1}{2\log\beta}\big(\log(2\pi)\big) - \frac{\beta+1}{4}\right)(s-1)^{-1} + O(1) \tag{1.44}$$

and simple poles at $s = 1 + 2\pi i m/\log\beta$ for $m \in \mathbb{Z}$ with $m \neq 0$ with residue

$$\mathrm{Res}\left(F_\beta(s), s = 1 + \frac{2\pi i m}{\log\beta}\right) = -\frac{\beta-1}{2\pi i m}\zeta\left(\frac{2\pi i m}{\log\beta}\right). \tag{1.45}$$

For the function $G_\beta(s)$ we have the following theorem.

Theorem 1.2.6. *For each real $\beta > 1$, the function $G_\beta(s)$ is meromorphic in the region $\mathrm{Re}(s) > 1$ with a double pole at $s = 2$ with Laurent expansion*

$$G_\beta(s) = \frac{\beta - 1}{2\log\beta}(s-2)^{-2} + \left(\frac{\beta-1}{2\log\beta}\big(\log(2\pi) - 1\big) - \frac{\beta+1}{4}\right)(s-2)^{-1} + O(1) \tag{1.46}$$

and simple poles at $s = 2 + 2\pi i m/\log\beta$ for $m \in \mathbb{Z}$ with $m \neq 0$ with residue

$$\mathrm{Res}\left(G_b(s), s = 2 + \frac{2\pi i m}{\log\beta}\right) = -\frac{\beta-1}{2\pi i m}\left(1 + \frac{2\pi i m}{\log\beta}\right)^{-1}\zeta\left(\frac{2\pi i m}{\log\beta}\right). \tag{1.47}$$

1.2.4 Complex Ramanujan sums and interpolation of the sum-of-divisors function

For positive integers m and n, the Ramanujan sum $c_n(m)$ is

$$c_n(m) = \sum_{\substack{a=0 \\ (a,n)=1}}^{n-1} \exp\left(2\pi i \frac{am}{n}\right). \tag{1.48}$$

For fixed m, the Dirichlet series generating function of $c_n(m)$ is

$$\sum_{n=1}^{\infty} \frac{c_n(m)}{n^s} = \begin{cases} \zeta(s-1)/\zeta(s) & \text{if } m = 0, \\ \sigma_{1-s}(m)/\zeta(s) & \text{if } m \neq 0 \end{cases} \tag{1.49}$$

where $\sigma_\alpha(m)$ is the sum-of-divisors function

$$\sigma_\alpha(m) = \sum_{d|m} d^\alpha. \tag{1.50}$$

In Chapter 5, we consider Ramanujan sums with the integer m replaced by a complex parameter λ and consider the Dirichlet series

$$Z(s,\lambda) = \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{n^s}. \tag{1.51}$$

We prove that $Z(s,\lambda)$ has a meromorphic continuation to $\mathbb{C}^2$ as a function of two complex variables s and λ.

Theorem 1.2.7. *The function $\widetilde{Z}(s,\lambda)$ defined for* $\mathrm{Re}(s) > 2$ *and* $\lambda \in \mathbb{C}$ *by*

$$\widetilde{Z}(s,\lambda) = \zeta(s) \cdot \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{n^s} \tag{1.52}$$

has a meromorphic continuation as a function of two complex variables to $\mathbb{C}^2$. The polar set of $\widetilde{Z}(s,\lambda)$ is the union of hyperplanes

$$H_1 \cup \Big(\bigcup_{k=0}^{\infty} H_{2-2k}\Big) \tag{1.53}$$

with $H_k = \{s = k\} \cup \{\lambda \in \mathbb{C}\}$. In particular:

(i) *For fixed $\lambda \in \mathbb{C}$, the function $\widetilde{Z}(s,\lambda)$ is a meromorphic function on $\mathbb{C}$ in the variable s. If $\lambda \notin \mathbb{Z}$, then $\widetilde{Z}(s,\lambda)$ has simple poles at $s = 2-k$ for $k = 1$ and for even integers $k \geq 0$ with residue*

$$\operatorname*{Res}_{s=2-k} \widetilde{Z}(s,\lambda) = (e(\lambda) - 1)\,\frac{B_k}{k!}(2\pi i\lambda)^{k-1} \tag{1.54}$$

where B_k is the kth Bernoulli number. If $\lambda = 0$ then $\widetilde{Z}(s,\lambda)$ has only a simple pole at $s = 1$ with residue 1*. If $\lambda \in \mathbb{Z} \setminus \{0\}$, then $\widetilde{Z}(s,\lambda)$ is an entire function of s.*

(ii) *For fixed $s \neq 2-k$ for $k = 1$ or for even integers $k \geq 0$, the function $\widetilde{Z}(s,\lambda)$ is an entire function of the variable λ.*

We prove the above result in the more general setting of Cohen's generalization $c_n^{\beta}(m)$ of the Ramanujan sum defined for positive integers n and β and integer m by

$$c_n^{(\beta)}(m) = \sum_{\substack{a=0 \\ (a,n^{\beta})_{\beta}=1}}^{n^{\beta}-1} e\left(\frac{am}{n^{\beta}}\right) \tag{1.55}$$

where $(a, n^{\beta})_{\beta}$ is the greatest common divisor of a and n^{β} that is a βth power. Note that $c_n^{(\beta)}(m)$ is equal to the classical Ramanujan sum $c_n(m)$ when $\beta = 1$. We prove the meromorphic continuation of the Dirichlet series

$$Z_{\beta}(s, \lambda) = \sum_{n=1}^{\infty} \frac{c_n^{(\beta)}(\lambda)}{n^s} \tag{1.56}$$

to $\mathbb{C}^2$ as a function of the complex variables s and λ. See Theorem 5.1.2 for details.

Chapter 2

Rings of Dirichlet series

In this chapter, we review the basics of the theory of ordinary Dirichlet series.

In Section 2.1, we consider the ring $\mathcal{A}$ of formal Dirichlet series, equivalent to the ring of arithmetic functions under addition and Dirichlet convolution. We state some basic algebraic properties of this ring.

In Section 2.2, we consider the ring $\mathcal{D}$ of convergent Dirichlet series—that is, Dirichlet series that converge in some halfplane of $\mathbb{C}$. We present the basic theory of convergence of Dirichlet series needed for the following chapters. We also review the known algebraic properties of this ring.

In Section 2.3, we consider the ring $\mathcal{M}$ of Dirichlet series with a meromorphic continuation to the complex plane. We give some examples of Dirichlet series that do or do not possess such a continuation.

In Section 2.4, we consider the subring $\mathcal{M}_{\text{fin}}$ of $\mathcal{M}$ of Dirichlet series that continue to $\mathbb{C}$ as a meromorphic function of finite order. Most of the important Dirichlet series of number theory belong to (or are conjectured to belong to) this ring. We review the basic definitions of the order of an entire function and a meromorphic function. We also present an example of a Dirichlet series of arbitrary real order $\rho > 1$.

Each of the rings of Dirichlet series presented is a subring of the previous ring: we have the proper inclusions

$$\mathcal{M}_{\text{fin}} \subset \mathcal{M} \subset \mathcal{D} \subset \mathcal{A}. \tag{2.1}$$

2.1 Formal Dirichlet series

First we consider the ring $\mathcal{A}$ of formal Dirichlet series.

Definition 2.1.1. A ***formal Dirichlet series*** is a series of the form

$$\sum_{n=1}^{\infty} a_n n^{-s} \tag{2.2}$$

with coefficients $a_n \in \mathbb{C}$.

Two Dirichlet series may be added termwise, giving

$$\sum_{n=1}^{\infty} a_n n^{-s} + \sum_{n=1}^{\infty} b_n n^{-s} = \sum_{n=1}^{\infty} (a_n + b_n) n^{-s}. \tag{2.3}$$

Formally multiplying two Dirichlet series and collecting terms gives the multiplication rule

$$\left(\sum_{n=1}^{\infty} a_n n^{-s}\right)\left(\sum_{n=1}^{\infty} b_n n^{-s}\right) = \sum_{n=1}^{\infty}\left(\sum_{d|n} a_d b_{n/d}\right) n^{-s}. \tag{2.4}$$

We also have the operation of scalar multiplication by $\lambda \in \mathbb{C}$, giving

$$\lambda \sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} \lambda a_n n^{-s}. \tag{2.5}$$

It is easily verified that the operations of addition and multiplication endow the set of formal Dirichlet series with a ring structure; with scalar multiplication, the formal Dirichlet series form a $\mathbb{C}$-algebra.

Definition 2.1.2. The ring $\mathcal{A}$ is the ring of formal Dirichlet series under the operations of addition and multiplication.

In practice, the coefficients of a Dirichlet series are often values of some interesting arithmetic function. To this end, we associate a Dirichlet series to each arithmetic function as follows. Recall that an ***arithmetic function*** is simply a function $f : \mathbb{N} \to \mathbb{C}$.

Definition 2.1.3. The ***Dirichlet series generating function*** of a function $f : \mathbb{N} \to \mathbb{C}$ is the Dirichlet series

$$D(f, s) = \sum_{n=1}^{\infty} f(n) n^{-s}. \tag{2.6}$$

Given arithmetic functions f and g, one can consider the ***Dirichlet convolution*** or ***multiplicative convolution*** $f * g$ defined by multiplication of the associated Dirichlet series generating functions

$$D(f * g, s) = D(f, s) D(g, s). \tag{2.7}$$

We see that

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d). \tag{2.8}$$

Then the ring $\mathcal{A}$ can also be identified with the ring of arithmetic functions $\mathbb{N} \to \mathbb{C}$ under the operations of pointwise addition and Dirichlet convolution. In this interpretation of the ring $\mathcal{A}$, the identity is the arithmetic function δ with $D(\delta, s) = 1$, or

$$\delta(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Remark. The ring of formal Dirichlet series should be compared to the ring of formal power series of the form

$$\sum_{n=0}^{\infty} a_n x^n. \tag{2.10}$$

The product of two power series is given by

$$\Big(\sum_{n=0}^{\infty} a_n x^n\Big)\Big(\sum_{n=0}^{\infty} b_n x^n\Big) = \sum_{n=0}^{\infty}\Big(\sum_{i+j=n} a_i b_j\Big)x^n. \tag{2.11}$$

To each arithmetic function $f: \mathbb{N} \to \mathbb{C}$, one can attach a power series generating function

$$P(f, x) = \sum_{n=1}^{\infty} f(n)x^n. \tag{2.12}$$

The corresponding product of arithmetic functions, called the ***Cauchy convolution*** or ***additive convolution*** is given by

$$(f *_{\text{add}} g)(n) = \sum_{i+j=n} f(i)g(j). \tag{2.13}$$

The operations of addition and Cauchy convolution give a different ring structure on the set of arithmetic functions.

2.1.1 Algebraic properties

The group $\mathcal{A}^{\times}$ of units of the ring $\mathcal{A}$ of formal Dirichlet series is exactly the set of Dirichlet series with nonzero constant term.

Proposition 2.1.4. *The formal Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ is invertible in the ring $\mathcal{A}$ if and only if $a_1 \neq 0$. If $f(s)^{-1} = \sum_{n=1}^{\infty} b_n n^{-s}$, then the coefficients b_n satisfy $b_1 = 1/a_1$ and*

$$b_n = -\frac{1}{a_1} \sum_{\substack{d|n \\ d\neq 1}} a_d b_{n/d}. \tag{2.14}$$

Proof. If two Dirichlet series $f(s) = \sum a_n n^{-s}$ and $g(s) = \sum b_n n^{-s}$ satisfy $f(s)g(s) = 1$, then

$$\sum_{n=1}^{\infty} \Big(\sum_{d|n} a_d b_{n/d} \Big) n^{-s} = 1. \tag{2.15}$$

We must have $a_1 b_1 = 1$ and

$$\sum_{d|n} a_d b_{n/d} = 0 \tag{2.16}$$

for $n \geq 2$, which gives the proposition. □

We define an additive valuation v on the ring $\mathcal{A}$. Let $\Omega(n)$ be the number of prime factors of the natural number n, counted with multiplicity, setting $\Omega(1) = 0$. For a formal Dirichlet series $f(s) = \sum a_n n^{-s}$, define

$$v(f) = \begin{cases} +\infty & \text{if } f(s) = 0, \\ \min\{m \mid a_n \neq 0 \text{ for some } n \text{ with } \Omega(n) = m\} & \text{otherwise.} \end{cases} \tag{2.17}$$

One easily verifies that

$$v(fg) = v(f) + v(g) \tag{2.18}$$

and

$$v(f+g) \geq \min(v(f), v(g)). \tag{2.19}$$

That is, v is an additive valuation on $\mathcal{A}$. From the valuation v, one can equip the ring $\mathcal{A}$ with a non-archmidean absolute value by setting

$$|f|_v := \exp(-v(f)). \tag{2.20}$$

We summarize the important algebraic properties of the ring $\mathcal{A}$ in the following theorem.

Theorem 2.1.5. *The ring $\mathcal{A}$ of formal Dirichlet series has the following algebraic properties.*

(i) $\mathcal{A} \cong \mathbb{C}[[x_1, x_2, \dots]]$ *as rings and as $\mathbb{C}$-algebras.*

(ii) $\mathcal{A}$ *is a local ring.*

(iii) *$\mathcal{A}$ is a non-noetherian ring.*

(iv) *$\mathcal{A}$ is a unique factorization domain.*

(v) *$\mathcal{A}$ is complete with respect to the non-archimedean absolute value $|\cdot|_v$.*

Proof.

(i) Writing the prime numbers as $p_1, p_2, \ldots$, each positive integer n has a unique factorization $n = p_1^{i_1} p_2^{i_2} \cdots$, with all but finitely many exponenents equal to 0. Consider the map $\phi\colon \mathcal{A} \to \mathbb{C}[[x_1, x_2, \ldots]]$ that takes the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{2.21}$$

to the power series

$$\phi(f(s)) = \sum_I b_I \cdot x_1^{i_1} x_2^{i_2} \cdots, \tag{2.22}$$

with the sum over sequences $I = (i_1, i_2, \ldots)$ of non-negative integers, only finitely many nonzero, with coefficient b_I equal to a_n, where $n = p_1^{i_1} p_2^{i_2} \cdots$. The map ϕ is a bijection by the unique factorization of the positive integers into primes. One easily verifies that $\phi(f(s) + g(s)) = \phi(f(s)) + \phi(g(s))$ and that $\phi(f(s)g(s)) = \phi(f(s))\phi(g(s))$.

(ii) The set of Dirichlet series $\sum a_n n^{-s}$ with constant term $a_1 = 0$ (the set of non-units) is the unique maximal ideal of $\mathcal{A}$.

(iii) The ascending chain of ideals

$$(2^{-s}) \subset (2^{-s}, 3^{-s}) \subset (2^{-s}, 3^{-s}) \subset (2^{-s}, 3^{-s}, 5^{-s}) \subset \cdots \tag{2.23}$$

does not terminate.

(iv) See Cashwell and Everett [9].

(v) Suppose that $\{f_i(s)\}$ is a Cauchy sequence with respect to $|\cdot|_v$ in $\mathcal{A}$ with

$$f_i(s) = \sum_{n=1}^{\infty} a_{i,n} n^{-s}. \tag{2.24}$$

For each $m \geq 0$, we have $v(f_i - f_j) \geq m$ for all sufficiently large i and j; this means that the Dirichlet series coefficients of f_i and f_j differ only for terms with $\Omega(n) \geq m$. Therefore, for each n, the sequence $\{a_{i,n}\}_{i=1}^{\infty}$ is eventually constant, so

$$\lim_{i\to\infty} f_i(s) = \sum_{n=1}^{\infty} \Big(\lim_{i\to\infty} a_{i,n}\Big) n^{-s} \tag{2.25}$$

exists as a formal Dirichlet series. □

2.2 Convergent Dirichlet series

We now consider Dirichlet series as functions of a complex variable s. We first in Section 2.2.1 review the basic theory of convergence of Dirichlet series: the region of convergence of a Dirichlet series is a right half-plane in $\mathbb{C}$ determined by the growth of the summatory function of the coefficients of $f(s)$. The product of two Dirichlet series that each converge in some halplane again converges in some halfplane, so the set of such convergent Dirichlet series is a ring.

Definition 2.2.1. Let $\mathcal{D}$ be the ring of Dirichlet series convergent in some halfplane of $\mathbb{C}$.

In Section 2.2.2 we survey the known results on the algebraic structure of this ring, including that $\mathcal{D}$ is a unique factorization domain, integrally closed in the ring $\mathcal{A}$ of formal Dirichlet series.

2.2.1 Convergence of Dirichlet series

In this section, we review the basic theory of convergence of Dirichlet series that we will need in later chapters. Proofs are omitted; we refer the reader to Titchmarsh [52, ch. 9] for further details.

The region of convergence of a Dirichlet series is a right halfplane in $\mathbb{C}$.

Proposition 2.2.2. *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series. Then there is an extended real number $\sigma_c(f)$ (possibly ∞ or $-\infty$) such that $f(s)$ converges and is holomorphic for* $\mathrm{Re}(s) > \sigma_c(f)$ *and diverges when* $\mathrm{Re}(s) < \sigma_c(f)$.

Proof. See Titchmarsh [52, sec. 9.11]. □

Accordingly, we have the following definition.

Definition 2.2.3. The ***abscissa of (conditional) convergence*** of a Dirichlet series $f(s) = \sum a_n n^{-s}$ is the extended real number $\sigma_c(f)$ such that the Dirichlet series for $f(s)$ converges if $\mathrm{Re}(s) > \sigma_c(f)$ and diverges if $\mathrm{Re}(s) < \sigma_c(f)$. If $\sigma_c(f) = \infty$ then $f(s)$ diverges for every $s \in \mathbb{C}$; if $\sigma_c(f) = -\infty$, then $f(s)$ converges for every $s \in \mathbb{C}$.

The abscissa of convergence of a Dirichlet series is determined by the growth of the partial sums of the coefficients of the series. Specifically, we have the following formulas for the abscissa of convergence analogous to the "root test" for power series.

Proposition 2.2.4. *Let $f(s)$ be the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. Then the abscissa of convergence $\sigma_c(f)$ is given by the following formulas.*

1. *If $\sum_{n=1}^{\infty} a_n$ diverges, then the abscissa of convergence is*

$$\sigma_c(f) = \limsup_{x\to\infty} \frac{\log |A(x)|}{\log x} \tag{2.26}$$

   where $A(x) = \sum_{n\le x} a_n$.

2. *If $\sum_{n=1}^{\infty} a_n$ converges, then the abscissa of convergence is*

$$\sigma_c(f) = \limsup_{x\to\infty} \frac{\log |R(x)|}{\log x} \tag{2.27}$$

   where $R(x) = \sum_{n>x} a_n$.

Proof. See Titchmarsh [52, sec. 9.14]. □

A Dirichlet series does not necessarily converge absolutely at each point of its halfplane of convergence $\mathrm{Re}(s) > \sigma_c(f)$. For example, the Dirichlet series

$$\sum_{n=1}^{\infty}(-1)^{n+1} n^{-s} = (1 - 2\cdot 2^{-s})\zeta(s) \tag{2.28}$$

converges for $\mathrm{Re}(s) > 0$ but only converges absolutely for $\mathrm{Re}(s) > 1$. However, since $|n^{-s}| = n^{-\mathrm{Re}(s)}$, we find that the halfplane of convergence of the Dirichlet series $\sum |a_n| n^{-s}$ is exactly the region in which the Dirichlet series $\sum a_n n^{-s}$ is absolutely convergent. Hence we can make the following definition.

Definition 2.2.5. The ***abscissa of absolute convergence*** of a Dirichlet series $f(s) = \sum a_n n^{-s}$ is the extended real number $\sigma_a(f)$ such that the Dirichlet series for $f(s)$ converges absolutely if $\mathrm{Re}(s) > \sigma_a(f)$ and does not converge absolutely if $\mathrm{Re}(s) < \sigma_a(f)$.

While the halfplane of conditional convergence may be strictly larger than the halfplane of absolute convergence, the strip of conditional convergence $\sigma_c < \mathrm{Re}(s) < \sigma_a$ can never be wider than in the example (2.28).

Proposition 2.2.6. *For any Dirichlet series $f(s) = \sum a_n n^{-s}$, we have $\sigma_c(f) \le \sigma_a(f) \le \sigma_c(f) + 1$.*

Proof. See Titchmarsh [52, sec. 9.13]. □

A power series with finite radius of convergence must have a singularity on the boundary of its disc of convergence. In contrast, a Dirichlet series $f(s)$ does not necessarily have a singularity on the line of convergence $\mathrm{Re}(s) = \sigma_c(f)$. For example, the Dirichlet series (2.28) has abscissa of convergence $\sigma_c = 0$ but continues to an entire function on $\mathbb{C}$. However, a Dirichlet series whose coefficients are eventually of one sign must have a singularity not just on the line $\mathrm{Re}(s) = \sigma_c = \sigma_a$, but on the real axis, as in the following theorem.

Proposition 2.2.7. *If the coefficients of a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ satisfy $a_n \geq 0$ for all n, then the function $f(s)$ has a singularity at the point $s = \sigma_c(f)$.*

Proof. See Titchmarsh [52, sec. 9.2]. □

If two Dirichlet series $f(s)$ and $g(s)$ are absolutely convergent at a point $s \in \mathbb{C}$, then the Dirichlet series $f(s)g(s)$ is also absolutely convergent at s; thus we have $\sigma_a(fg) \leq \max(\sigma_a(f), \sigma_a(g))$. This implies that the set $\mathcal{D}$ of convergent Dirichlet series is indeed a ring.

2.2.2 Algebraic properties

The ring $\mathcal{D}$ of convergent Dirichlet series can be identified with the ring of arithmetic functions of at most polynomial growth (i.e. arithmetic functions f satisfying $f(n) \ll n^p$ for some $p \in \mathbb{R}$), by Proposition 2.2.4.

The group $\mathcal{D}^\times$ of units in $\mathcal{D}$ is the set of Dirichlet series in $\mathcal{D}$ with nonzero constant term—that is, $\mathcal{D}^\times = \mathcal{A}^\times \cap \mathcal{D}$.

The ring of convergent Dirichlet series was studied by Bayart and Mouze in the papers [6], [5], [7]. The important algebraic properties of this ring are summarized in the following theorem.

Theorem 2.2.8. *The ring $\mathcal{D}$ of convergent Dirichlet series has the following properties*:

(i) *$\mathcal{D}$ is a local ring.*

(ii) *$\mathcal{D}$ is a non-noetherian ring.*

(iii) *$\mathcal{D}$ is a unique factorization domain.*

(iv) *$\mathcal{D}$ is integrally closed in $\mathcal{A}$.*

(v) *$f \in \mathcal{D}$ is irreducible if and only if f is irreducible in $\mathcal{A}$.*

Proof.

(i) The proof is the same as for $\mathcal{A}$: the set $\mathcal{D} \setminus \mathcal{D}^{\times}$ of nonunits (Dirichlet series with constant term $a_1 = 0$) is the unique maximal ideal of $\mathcal{D}$.

(ii) The example given in the proof of Theorem 2.1.5 works in $\mathcal{D}$ as well: the ascending chain of ideals

$$(2^{-s}) \subset (2^{-s}, 3^{-s}) \subset (2^{-s}, 3^{-s}) \subset (2^{-s}, 3^{-s}, 5^{-s}) \subset \cdots \tag{2.29}$$

does not terminate.

(iii) See Bayart and Mouze [6], [5].

(iv) See Bayart and Mouze [6], [5].

(v) See Bayart and Mouze [7]. □

2.2.3 Closure under other operations

We first consider when the composition $p(f(s))$ of a power series $p(z)$ and a convergent Dirichlet series $f(s)$ is a holomorphic function defined in some right halfplane by a convergent Dirichlet series. If $p(f(s))$ is to be a Dirichlet series, then since

$$\lim_{\mathrm{Re}(s)\to\infty} \sum_{n=1}^{\infty} a_n n^{-s} = a_1, \tag{2.30}$$

we will require that the constant term a_1 of $f(s)$ fall within the radius of convergence of $p(z)$. This condition is sufficient to guarantee that $p(f(s))$ is given by a Dirichlet series in a right halfplane, as in the following proposition.

Proposition 2.2.9. *Let $p(z) = \sum_{n=0}^{\infty} b_n z^n$ be an analytic function with radius of convergence $r > 0$, and let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with $|a_1| < r$. If $\sigma_c(f) < \infty$, then $p(f(s))$ is a Dirichlet series with $\sigma_c(p \circ f) < \infty$.*

Proof. Since $|a_1| < r$, there is some σ_0 such that

$$\sum_{n=1}^{\infty} |a_n| n^{-\sigma} < r \tag{2.31}$$

for all $\sigma > \sigma_0$. Then for $\mathrm{Re}(s) > \sigma_0$, we have $|f(s)| < r$, so the number $z = f(s)$ is within the radius of convergence of the power series $p(z)$. So for $\mathrm{Re}(s) > \sigma_0$, we have

$$p(f(s)) = \sum_{m=0}^{\infty} b_m \Big(\sum_{n=1}^{\infty} a_n n^{-s}\Big)^m = \sum_{m=0}^{\infty} b_m \sum_{n=1}^{\infty} \Big(\sum_{d_1 \cdots d_m = n} a_{d_1} \cdots a_{d_m} \Big) n^{-s}. \tag{2.32}$$

Since
$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}|b_m|\cdot\Big|\sum_{d_1\cdots d_m=n}a_{d_1}\cdots a_{d_m}\Big|n^{-\sigma}\leq\sum_{m=0}^{\infty}|b_m|\Big(\sum_{n=1}^{\infty}|a_n|n^{-\sigma}\Big)^m \tag{2.33}$$

converges, we may interchange the sums to get

$$p(f(s))=\sum_{n=1}^{\infty}\bigg(\sum_{m=0}^{\infty}b_m\Big(\sum_{d_1\cdots d_m=n}a_{d_1}\cdots a_{d_m}\Big)\bigg)n^{-s}. \tag{2.34}$$

We have therefore expressed $p(f(s))$ as a Dirichlet series, convergent in at least the halfplane $\mathrm{Re}(s)>\sigma_0$. □

In addition to the ring operations of addition and multiplication (Dirichlet convolution) and scalar multiplication, the ring $\mathcal{D}$ of convergent Dirichlet series is also closed under the following common operations.

Theorem 2.2.10. *The ring $\mathcal{D}$ of convergent Dirichlet series is closed under the following operations*:

(i) *If $f(s)\in\mathcal{D}$, then the derivative $f'(s)\in\mathcal{D}$.*

(ii) *If $f(s)\in\mathcal{D}\setminus\mathcal{D}^{\times}$, then any antiderivative of $f(s)$ is in $\mathcal{D}$.*

(iii) *If $f(s)\in\mathcal{D}$ and $p(z)$ is an entire function, then $p(f(s))\in\mathcal{D}$.*

Proof.

(i) Dirichlet series converge uniformly on compact subsets of their halfplane of convergence, so termwise differentiation is valid. The derivative of a Dirichlet series $f(s)=\sum a_n n^{-s}$ is $f'(s)=-\sum a_n(\log n)n^{-s}$.

(ii) If a convergent Dirichlet series $f(s)=\sum a_n n^{-s}$ has constant term $a_1=0$ (that is, if $f(s)$ is a nonunit in $\mathcal{D}$), then any antiderivative

$$\int f(s)\,ds=c-\sum_{n=2}^{\infty}\frac{a_n}{\log n}n^{-s} \tag{2.35}$$

(with $c\in\mathbb{C}$ a constant) is again a convergent Dirichlet series.

If the constant term a_1 is nonzero, then the antiderivative

$$\int f(s)\,ds=c+a_1 s-\sum_{n=2}^{\infty}\frac{a_n}{\log n}n^{-s} \tag{2.36}$$

cannot be given by a Dirichlet series in any right halfplane since it is unbounded as $\mathrm{Re}(s) \to \infty$.

(iii) See Proposition 2.2.9. Here the radius of convergence of $p(z)$ is ∞, so there is no restriction on the constant term of $f(s)$. □

2.3 Meromorphic Dirichlet series

A Dirichlet series that converges in a right halfplane defines a holomorphic function in that halfplane; this function may or may not analytically continue to a larger region. In this section we consider Dirichlet series with a meromorphic continuation to the complex plane $\mathbb{C}$.

As we noted in the previous section, the product of two Dirichlet series with finite abscissas of convergence has a finite abscissa of convergence. Additionally, the product of two meromorphic functions is meromorphic. Thus the set of Dirichlet series with a meromorphic continuation to $\mathbb{C}$ forms a ring.

Definition 2.3.1. The ring $\mathcal{M}$ is the ring of Dirichlet series with finite abscissa of convergence and with a meromorphic continuation to the complex plane $\mathbb{C}$.

2.3.1 Examples

The simplest examples of Dirichlet series with continuation to $\mathbb{C}$ are the finite Dirichlet series (or ***Dirichlet polynomials***)

$$\sum_{n=1}^{N} a_n n^{-s} \tag{2.37}$$

which are entire functions.

The Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ has abscissa of convergence $\sigma_c = 1$ and continues to a meromorphic function on $\mathbb{C}$, with only singularity a simple pole at $s = 1$ of residue 1, satisfying the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{2.38}$$

See Titchmarsh's book [53, ch. 2] on the zeta function for many different proofs of the analytic continuation and functional equation.

Similarly, for χ a Dirichlet character, the Dirichlet L-function $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ continues to $\mathbb{C}$ and satisfies a functional equation. If χ is the principal character mod q, then

$L(s, \chi)$ has a simple pole at $s = 1$; otherwise, $L(s, \chi)$ is an entire function. For a proof of the analytic continuation and functional equation for Dirichlet L-functions, see Davenport [14, ch. 9] or Montgomery and Vaughan [40, sec. 10.1].

We also note some examples of Dirichlet series that do not have a meromorphic continuation to $\mathbb{C}$.

Some Dirichlet series have a natural boundary beyond which the function defined by the Dirichlet series cannot be analytically continued. For a well-known simple example, consider the Dirichlet series

$$f(s) = \sum_{k=0}^{\infty} (2^{k!})^{-s}. \tag{2.39}$$

For any $a, b \in \mathbb{Z}$, we find

$$f\left(\sigma + \frac{a}{b} \cdot \frac{2\pi i}{\log 2}\right) \to \infty \tag{2.40}$$

as $\sigma \to 0+$ since the terms of the sum (2.39) are equal to 1 for all sufficiently large k. Hence the function $f(s)$ must have a singularity at the point $s = (a/b)(2\pi i/\log 2)$ for every $a, b \in \mathbb{Z}$. As such points are dense on the line $\mathrm{Re}(s) = 0$, this line must be a natural boundary of the function $f(s)$.

One particularly interesting example of a Dirichlet series with natural boundary is the ***prime zeta function*** $P(s) = \sum_p p^{-s}$, with the sum over primes. The summatory function of the coefficients of $P(s)$ is the prime-counting function $\pi(x)$, so one might try to recover asymptotics for $\pi(x)$ by applying Perron's formula to $P(s)$ and shifting the contour to the left in the usual fashion. The contour cannot be pushed past $\sigma = 0$, however, as $\sigma = 0$ is a natural boundary for $P(s)$ (proved by Landau and Walfisz [33] in 1919).

As a last example of a Dirichlet series with natural boundary, consider the Dirichlet series

$$F_{k,p}(s) = \sum_{n=1}^{\infty} \frac{(d_k(n))^p}{n^s}, \tag{2.41}$$

where $d_k(n)$ is the number of ways of writing n as a product of k positive integers. If $p = 1$, then $F_{k,1}(s) = \zeta^k(s)$, which has a meromorphic continuation to $\mathbb{C}$. If $k = 2$ and $p = 2$, then $F_{2,2}(s) = \zeta^4(s)/\zeta(2s)$, which has a meromorphic continuation to $\mathbb{C}$. For all other values of k and p, the series has a natural boundary at $\sigma = 0$, as proved by Estermann [20].

Other Dirichlet series can be analytically continued to $\mathbb{C}$ but with isolated singularities other than poles. For example, the Dirichlet series $\exp(\zeta(s))$ is analytic on $\mathbb{C}$ apart from an essential singularity at $s = 1$.

2.4 Finite-order meromorphic Dirichlet series

In this section, we consider Dirichlet series with a continuation to the plane $\mathbb{C}$ as a meromorphic function of finite order. As the sum and product of two meromorphic functions of finite order are again meromorphic of finite order, such Dirichlet series form a ring.

Definition 2.4.1. The ring $\mathcal{M}_{\text{fin}}$ is the ring of Dirichlet series with continuation to the complex plane $\mathbb{C}$ as a meromorphic function of finite order.

Muñoz and Pérez Marco [42, thm. 4.1] show that to each Dirichlet series $f(s)$ in $\mathcal{M}_{\text{fin}}$ with nonzero constant term (i.e. to each unit of the ring $\mathcal{M}_{\text{fin}}$), one may associate a distributional explicit formula relating a sum over the zeros and poles of $f(s)$ with a sum over the coefficients of the logarithmic derivative of $f(s)$.

2.4.1 Meromorphic functions of finite order

In this section we review some notions from complex analysis. For a short overview of the theory of entire and meromorphic functions and the connections between their order and their value distribution, see Titichmarsh [52, ch. 8]; for a more thorough study, see for example Nevanlinna [44] or Rubel [49].

Definition 2.4.2. The ***order*** of an entire function $f(z)$ is the infimum of the positive real numbers ρ with

$$|f(z)| \leq \exp(|z|^{\rho+\epsilon}) \tag{2.42}$$

for all $z \in \mathbb{C}$; we define the order to be ∞ if $f(z)$ does not satisfy such a bound for any positive real ρ. Equivalently, the order ρ is given by

$$\rho = \limsup_{r\to\infty} \frac{\log\log\big(\max_{|z|=r}|f(z)|\big)}{\log r}. \tag{2.43}$$

If an entire function $f(z)$ has order $\rho < \infty$, we say that $f(z)$ is of ***finite order***.

The order of a meromorphic function is typically defined as the order of growth of its associated Nevanlinna characteristic function. To simplify matters, we will use the following equivalent definition.

Definition 2.4.3. The ***order*** of a meromorphic function $f(z)$ is the infimum of the positive real numbers ρ such that $f(z)$ can be written as

$$f(z) = \frac{g(z)}{h(z)} \tag{2.44}$$

for entire functions $g(z)$ and $h(z)$ of order $\leq \rho$. If a meromorphic functino $f(z)$ has order $\rho < \infty$, we say that $f(z)$ is of ***finite order***.

For the equivalence of this definition with the more common definition, see Rubel [49], Lemma 15.6 and Theorem on page 91.

The order of a meromorphic function is closely connected to its value distribution. We note only the following basic result.

Proposition 2.4.4. *Let $f(z)$ be a meromorphic function of finite order ρ on $\mathbb{C}$, and for $a \in \mathbb{C} \cup \{\infty\}$, let $n(r, a)$ be the number of points z in the disc $|z| \leq r$ for which $f(z) = a$. Then $n(r, a) \ll r^{\rho+\epsilon}$ for any $\epsilon > 0$.*

Proof. See Titchmarsh [52, sec. 8.91]. □

In particular, knowledge of the number of zeros or poles of a meromorphic function can give a lower bound on the function's order.

2.4.2 The Selberg Class

Most of the important Dirichlet series of number theory referred to as "L-functions" or "zeta functions" have similar properties, including an Euler product and a functional equation. Selberg [50], studying the value distribution of such functions near the critical line, considered a class of Dirichlet series having properties shared by known L-functions and made several conjectures about such Dirichlet series.

Definition 2.4.5. The ***Selberg class*** is the set $\mathcal{S}$ of Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{2.45}$$

with $a_1 = 1$ satisfying the following properties.

S1. The Dirichlet series $f(s)$ is absolutely convergent for $\sigma > 1$.

S2. For some $m \geq 0$, the function $(s-1)^m f(s)$ has a continuation to $\mathbb{C}$ as an entire function of finite order.

S3. There is a complex number ε with $|\varepsilon| = 1$, real numbers $Q > 0$ and $\lambda_i > 0$, and complex numbers μ_i with $\mathrm{Re}(\mu_i) \geq 0$ such that the function

$$\Phi(s) = \varepsilon Q^s \prod_{i=1}^{k} \Gamma(\lambda_i s + \mu_i) f(s) \tag{2.46}$$

satisfies the functional equation

$$\Phi(s) = \overline{\Phi}(1 - s). \tag{2.47}$$

(Here and in the sequel, $\overline{f}(s) \coloneqq \overline{f(\overline{s})}$.)

S4. The Dirichlet series

$$\log f(s) = \sum_{n=1}^{\infty} b_n n^{-s} \tag{2.48}$$

has $b_n = 0$ if n is not a prime power.

S5. The coefficients a_n satisfy $a_n = O(n^\delta)$ for any fixed $\delta > 0$.

Since each Dirichlet series in the Selberg class has a continuation to the plane as a meromorphic function of finite order, we have $\mathcal{S} \subset \mathcal{M}_{\text{fin}}$.

As Selberg states in [50], the functional equation implies that $f(s)$ is a meromorphic function of order at most 1. We provide a proof of this fact for completeness.

Proposition 2.4.6. *If a Dirichlet series $f(s)$ satisfies properties* S1–S3 *in the definition of the Selberg class, then $f(s)$ is a meromorphic function of order at most* 1.

Proof. We show that $(s-1)^m f(s)$ is an entire function of order at most 1, which implies that $f(s)$ is meromorphic of order at most 1.

The Dirichlet series $f(s) = \sum a_n n^{-s}$ is absolutely convergent for $\sigma > 1$, so $f(s)$ is bounded in the halfplane $\sigma \geq 2$.

By the functional equation (2.46), we have $f(s) = \frac{\overline{P}(1-s)}{P(s)} \overline{f}(1-s)$ where

$$P(s) = \varepsilon Q^s \prod_{i=1}^{k} \Gamma(\lambda_i s + \mu_i). \tag{2.49}$$

$P(s)$ is meromorphic of order at most 1 and $\overline{f}(1-s)$ is bounded if $\sigma \leq -1$; since $f(s)$ is holomorphic except at $s = 1$, we deduce that $f(s) = O\big(\exp(|s|^{1+\delta})\big)$ for $\sigma \leq -1$.

Finally, we show that $f(s)$ has at most polynomial growth in the remaining strip $-1 \le \sigma \le 2$. Consider the function

$$g(s) = \frac{(s-1)^m}{(s-3)^m} f(s). \tag{2.50}$$

The function $g(s)$ is bounded on $\sigma = 2$ because $f(s)$ is bounded on $\sigma = 2$. The functional equation and Stirling's formula or the crude bound

$$|\Gamma(s)| \le \int_0^\infty |e^{-x} x^{s-1}| \, dx = \Gamma(\sigma) \qquad \text{if } \sigma > 0 \tag{2.51}$$

combined with $\Gamma(s) = \Gamma(s+1)/s$ show that $g(s)$ is bounded on the line $\sigma = -1$. Finally, since $(s-1)^m f(s)$ is assumed to be entire of finite order, we have $g(s) = O\big(\exp(|s|^\rho)\big)$ in the strip $-1 \le \sigma \le 2$ for some ρ. Then, by the Phragmén-Lindelöf Theorem, $g(s)$ is bounded in the strip $-1 \le \sigma \le 2$, which implies that $f(s)$ has at most polynomial growth in this strip. □

Proposition 2.4.7. *The Selberg class $\mathcal{S}$ is closed under multiplication.*

Proof. All five axioms of $\mathcal{S}$ are easily verified for the product $f(s)g(s)$ of two Dirichlet series $f(s), g(s) \in \mathcal{S}$. □

A Dirichlet series $f(s) \in \mathcal{S}$ is ***primitive*** if $f(s)$ does not have a nontrivial factorization in $\mathcal{S}$; that is, $f(s) \in \mathcal{S}$ is primitive if whenever $f(s) = f_1(s) f_2(s)$ with $f_1(s), f_2(s) \in \mathcal{S}$, either $f_1(s) = 1$ or $f_2(s) = 1$. Whether every Dirichlet series in $\mathcal{S}$ has a unique factorization into primitives remains an open problem.

2.4.3 Dirichlet series of arbitrary order

Our goal in this section is to show that, despite the prevalence of examples of Dirichlet series meromorphic of order 1 in the complex plane, there exist Dirichlet series meromorphic on $\mathbb{C}$ of arbitrary order $\rho \ge 1$. We will do so by providing an explicit example of a Dirichlet series convergent on all of $\mathbb{C}$ that is entire of an arbitrary real order $\rho > 1$ and an explicit example of a different Dirichlet series of infinite order. By considering the product of these series with $\zeta(s)$, for example, this shows that the meromorphic continuation of a Dirichlet series with finite abscissa of convergence can be meromorphic of arbitrary order.

The Dirichlet series

$$f(s) = \sum_{k=0}^{\infty} \frac{1}{k!} (2^k)^{-s}, \tag{2.52}$$

which converges on all of $\mathbb{C}$, is just the series expansion of $f(s) = \exp(2^{-s})$, which is an entire function of infinite order.

As an example of a Dirichlet series of arbitrary finite order, we have the following result.

Theorem 2.4.8. *For real $p \geq 1$, the Dirichlet series*

$$F_p(s) = \sum_{k=0}^{\infty} e^{-k^p} (2^k)^{-s} \tag{2.53}$$

converges on $\mathbb{C}$ to an entire function of order $p/(p-1)$.

Proof. The series for $F_p(s)$ converges on all of $\mathbb{C}$, so $F_p(s)$ is an entire function.

The coefficients of $F_p(s)$ are positive, so for any s in $\mathbb{C}$, we have $|F_p(s)| \leq F_p(-|s|)$. We therefore consider the growth of $F_p(-\sigma)$ as $\sigma \to \infty$ in $\mathbb{R}$. Break the sum into two parts

$$F_p(-\sigma) = \underbrace{\sum_{k \leq \sigma^{1/(p-1)}} e^{-k^p} (2^k)^{\sigma}}_{\mathbf{I}} + \underbrace{\sum_{k > \sigma^{1/(p-1)}} e^{-k^p} (2^k)^{\sigma}}_{\mathbf{II}} . \tag{2.54}$$

In the first sum, since $e^{-k^p} < 1$, we have

$$\mathbf{I} = \sum_{k \leq \sigma^{1/(p-1)}} e^{-k^p} (2^k)^{\sigma} \leq \sum_{k \leq \sigma^{1/(p-1)}} (2^k)^{\sigma} \tag{2.55}$$

and we make the trivial estimate

$$\mathbf{I} \leq \sigma^{1/(p-1)} \cdot (2^{\sigma^{1/(p-1)}})^{\sigma} = \exp\Big(\sigma^{p/(p-1)} \log 2 + \frac{1}{p-1} \log \sigma\Big). \tag{2.56}$$

In the second sum,

$$\mathbf{II} = \sum_{k > \sigma^{1/(p-1)}} e^{-k^p} (2^k)^{\sigma} \leq \sum_{k > \sigma^{1/(p-1)}} e^{-\sigma k} (2^k)^{\sigma} \ll 1. \tag{2.57}$$

Thus we have $F_p(s) \ll \exp(|s|^{p/(p-1)+\varepsilon})$ on all of $\mathbb{C}$, so $F_p(s)$ is of order at most $p/(p-1)$.

To show a lower bound, we note that if $\sigma = pk^{p-1}/\log 2$ with $k \in \mathbb{N}$, then $F_p(-\sigma)$ is larger than the kth term of the sum, which is

$$F_p(-\sigma) > e^{-k^p} (2^k)^{\sigma} = \exp\Big(-(\sigma \log 2/p)^{p/(p-1)} + \sigma(\sigma \log 2/p)^{1/(p-1)} \log 2\Big) \tag{2.58}$$

$$= \exp\Big(\sigma^{p/(p-1)} (\log 2)^{p/(p-1)} \frac{p-1}{p^{p/(p-1)}}\Big) \tag{2.59}$$

so that $F_p(-\sigma) \geq \exp(c \cdot |\sigma|^{p/(p-1)})$ for a sequence with $\sigma \to \infty$. Thus $F_p(s)$ cannot be of order less than $p/(p-1)$. □

Chapter 3

Dirichlet series and additive convolutions

3.1 Introduction

Suppose that the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{3.1}$$

has a meromorphic continuation to $\mathbb{C}$. In this chapter, we consider the Dirichlet series

$$\sum_{n=1}^{\infty} \Big(\sum_{i+j=n} a_i b_j \Big) \frac{1}{n^s} \tag{3.2}$$

associated to the additive (Cauchy) convolution of the coefficients a_n with a sequence $(b_0, b_1, b_2, \dots)$. As special cases of this construction, we consider the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_{n-1}}{n^s}, \quad \sum_{n=1}^{\infty} \frac{a_n - a_{n-1}}{n^s}, \quad \text{and} \quad \sum_{n=1}^{\infty} \Big(\sum_{m \le n} a_m \Big) \frac{1}{n^s} \tag{3.3}$$

obtained by shifting, differencing, and summing the coefficients of the Dirichlet series $f(s)$. We also consider these operations applied to several important Dirichlet series in number theory.

3.1.1 Summary of results

In Section 3.3, we consider the general problem of deducing the meromorphic continuation of the series (3.2) from that of $f(s)$. The results of this section are summarized by the following theorem.

Theorem 3.1.1. *Let $B = (b_0, b_1, b_2, \dots)$ be a sequence of complex numbers whose summatory function $B(x) = \sum_{n \le x} b_n$ satisfies $B(x) \ll x^r$ for some $r \ge 0$, and set*

$$q(z) = \sum_{n=0}^{\infty} b_n z^n. \tag{3.4}$$

Suppose that the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{3.5}$$

has abscissa of absolute convergence $\sigma_a(f)$ and has a meromorphic continuation to the halfplane $\mathrm{Re}(s) > \sigma_m$.

Then the Dirichlet series

$$g(s) = \sum_{n=1}^{\infty} \Big(\sum_{i+j=n} a_i b_j \Big) \frac{1}{n^s} \tag{3.6}$$

has the following properties.

(i) *The Dirichlet series $g(s)$ has abscissa of absolute convergence $\sigma_a(g) \leq \max(\sigma_a(f) + r, r)$.*

(ii) *If the function $q(z)$ has a meromorphic continuation to $z = 1$, then $g(s)$ has a meromorphic continuation to the halfplane* $\mathrm{Re}(s) \geq \sigma_m - M$, *where M is the order of the zero (if $M > 0$) or pole (if $M < 0$) of $q(z)$ at $z = 1$.*

See Theorem 3.3.4 for a more precise version, with a formula for $g(s)$ in terms of $f(s)$.

In Section 3.4, we use specific choices of the sequence B to prove the following.

Theorem 3.1.2. *If the Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{3.7}$$

has a meromorphic continuation to the halfplane $\mathrm{Re}(s) > \sigma_m$, *then the Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{a_{n-1}}{n^s}, \quad \sum_{n=1}^{\infty} \frac{a_n - a_{n-1}}{n^s}, \quad \textit{and} \quad \sum_{n=1}^{\infty} \Big(\sum_{m \leq n} a_m \Big) \frac{1}{n^s} \tag{3.8}$$

have meromorphic continuation to the halfplanes $\mathrm{Re}(s) > \sigma_m$, $\mathrm{Re}(s) > \sigma_m - 1$, *and* $\mathrm{Re}(s) > \sigma_m + 1$, *respectively.*

See Section 3.4 for more precise versions of these results. The formulas obtained in this section for the Dirichlet series (3.8) allow the precise determination of the structure of the poles (locations and Laurent expansions) if the structure of the poles of $f(s)$ is known.

In the notation of the previous chapter, we have the following corollary.

Corollary 3.1.3. *The ring $\mathcal{M}$ of meromorphic Dirichlet series is closed under the operations of shifting coefficients, differencing coefficients, and partial summation of coefficients.*

In Section 3.5, we consider the Dirichlet series associated to the summatory functions of some important number-theoretic functions. We prove the meromorphic continuation of each of the following Dirichlet series, and explicitly compute the locations and residues of the poles.

Theorem 3.1.4. *The following Dirichlet series have a meromorphic continuation to* $\mathbb{C}$:

(i) *The Dirichlet series* $\sum_{n=1}^{\infty} H_n n^{-s}$ *associated to the sequence of harmonic numbers* $H_n = \sum_{m \leq n} m^{-1}$.

(ii) *The Dirichlet series* $\sum_{n=1}^{\infty} D(n) n^{-s}$ *associated to the summatory function of the divisor function* $d(n)$.

(iii) *The Dirichlet series* $\sum_{n=1}^{\infty} \Delta(n) n^{-s}$ *associated to the error term in the Dirichlet divisor problem.*

(iv) *The Dirichlet series* $\sum_{n=1}^{\infty} \psi(n) n^{-s}$ *associated to the Chebyshev function* $\psi(n) = \sum_{m \leq n} \Lambda(m)$.

(v) *The Dirichlet series* $\sum_{n=1}^{\infty} M(n) n^{-s}$ *associated to the Mertens function* $M(n) = \sum_{m \leq n} \mu(m)$.

(vi) *The Dirichlet series* $\sum_{n=1}^{\infty} \big(\sum_{m \leq n} \chi(n)\big) n^{-s}$ *associated to the partial sums of a Dirichlet character.*

3.1.2 Related work

Matsumoto and Tanigawa [35] consider the analytic continuation of multiple Dirichlet series, including double Dirichlet series of the form

$$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{a_1(m_1) a_2(m_2)}{m_1^{s_1} (m_1 + m_2)^{s_2}} \tag{3.9}$$

for two arithmetic functions $a_1(n)$ and $a_2(n)$, which for $s_1 = 0$ reduce to the additive convolution series (3.2). They obtain an analytic continuation in both variables s_1 and s_2 under the assumptions that $f_1(s) = \sum a_1(n) n^{-s}$ and $f_2(s) = \sum a_2(n) n^{-s}$ each have abscissa of convergence greater than 0 and that $f_1(s)$ and $f_2(s)$ can each be continued to the plane with only one simple pole and with polynomial growth in vertical strips.

Egami and Matsumoto [19] consider the series

$$\Phi_2(s) := \sum_{n=1}^{\infty} \Big(\sum_{i+j=n} \Lambda(i)\Lambda(j) \Big) \frac{1}{n^s}, \tag{3.10}$$

and show under the Riemann Hypothesis that $\Phi_2(s)$ has a meromorphic continuation to $\mathrm{Re}(s) > 1$. They conjecture that $\mathrm{Re}(s) = 1$ is a natural boundary for $\Phi_2(s)$ and provide evidence supporting this conjecture.

3.2 A relation between power series and Dirichlet series

Many different generating functions can be attached to a given arithmetic function $a\colon \mathbb{N} \to \mathbb{C}$. The power series generating function

$$\sum_{n=1}^{\infty} a(n) z^n \tag{3.11}$$

tends to be useful for investigating properties of the function $a(n)$ that involve the additive structure of $\mathbb{N}$, while the Dirichlet series generating function

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} \tag{3.12}$$

tends to be useful for investigating properties involving the multiplicative structure of $\mathbb{N}$.

If the Dirichlet series $\sum_{n=1}^{\infty} a(n) n^{-s}$ converges in some halfplane (that is, if the abscissa of convergence satisfies $\sigma_c(a) < \infty$), then the coefficients $a(n)$ have at most polynomial growth in n, and the power series generating function $\sum_{n=1}^{\infty} a(n) z^n$ converges in the unit disk $|z| < 1$.

The converse, however, is not true: the convergence of $\sum_{n=1}^{\infty} a(n) z^n$ for $|z| < 1$ does not guarantee that the associated Dirichlet series generating function converges in any halfplane. For example, the power series

$$\sum_{n=1}^{\infty} n^{\log n} z^n \tag{3.13}$$

has radius of convergence 1, while the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{n^{\log n}}{n^s} \tag{3.14}$$

has abscissa of convergence $\sigma_c = \infty$.

In this chapter, we will be considering Dirichlet series attached to transformations of an arithmetic function that involve the additive structure of $\mathbb{N}$. We will accomplish this by considering the transformation on the associated power series generating function, then translating this to the Dirichlet series generating function by way of the Mellin transform. The following classical relation between the power series and Dirichlet series generating functions of a sequence a_n is the key ingredient.

Proposition 3.2.1. *For any sequence of complex numbers a_n, the generating functions*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{3.15}$$

and

$$p(x) = \sum_{n=1}^{\infty} a_n x^n \tag{3.16}$$

are related by

$$\Gamma(s) f(s) = \int_0^{\infty} p(e^{-x}) x^s \, \frac{dx}{x} \tag{3.17}$$

for $\mathrm{Re}(s) > \max(\sigma_c(f), 0)$.

Proof. The gamma function has the absolutely convergent integral representation

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^s \, \frac{dx}{x}, \tag{3.18}$$

valid for $\mathrm{Re}(s) > 0$. Replacing x with nx, we have

$$\Gamma(s) n^{-s} = \int_0^{\infty} e^{-nx} x^s \, \frac{dx}{x}. \tag{3.19}$$

Multiply by a_n and sum over n; for $\mathrm{Re}(s) > \max(\sigma_c(f), 0)$, we have

$$\Gamma(s) \cdot \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} a_n \int_0^{\infty} e^{-nx} x^s \, \frac{dx}{x}. \tag{3.20}$$

If $\mathrm{Re}(s)$ is greater than the abscissa of absolute convergence $\sigma_a(f)$, then the interchange of the sum and integral is justified, giving the proposition.

To show that the proposition holds for $\mathrm{Re}(s) > \max(\sigma_c(f), 0)$, it suffices to show that the right side of (3.17) is analytic in this halfplane since the left side is analytic in this halfplane. To this end, we use the fact that

$$A(x) \coloneqq \sum_{n \le x} a_n \ll x^{\theta} \tag{3.21}$$

for any $\theta > \max(\sigma_c(f), 0)$ to estimate the integrand. Partial summation gives

$$\sum_{n=1}^{\infty} a_n e^{-nx} = x \int_1^{\infty} A(u) e^{-ux} \, du \tag{3.22}$$

and so

$$\sum_{n=1}^{\infty} a_n e^{-nx} \ll x \int_1^{\infty} u^{\theta} e^{-ux} \, du = x^{-\theta} \int_x^{\infty} y^{\theta} e^{-y} \, dy \ll x^{-\theta} \tag{3.23}$$

as $x \to 0+$. Then we find that for s in any compact set K in the halfplane $\mathrm{Re}(s) > \theta$ where $\theta > \max(\sigma_c(f), 0)$, the integrand is uniformly

$$\Big(\sum_{n=1}^{\infty} a_n e^{-nx}\Big) x^{s-1} \ll x^{\sigma-\theta-1} \tag{3.24}$$

so the integral on the right of (3.17) converges uniformly on compact subsets of $\mathrm{Re}(s) > \max(\sigma_c(f), 0)$ and is analytic in that halfplane. □

Remark. We note that the proposition says that the Mellin transform (the Fourier transform on the *multiplicative* group of positive reals) of the *additive* generating function of the sequence a_n is equal to the gamma function (the Mellin transform of the *additive* character e^{-x}) times the *multiplicative* generating function of the sequence.

3.3 Dirichlet series associated to additive convolutions

In this section, we consider the following operations on the set of Dirichlet series.

Definition 3.3.1. For a sequence $B = (b_0, b_1, b_2, \dots)$ of complex numbers, define the map C_B on the set of Dirichlet series by

$$C_B\Big(\sum_{n=1}^{\infty} a_n n^{-s}\Big) = \sum_{n=1}^{\infty}\Big(\sum_{i+j=n} a_i b_j\Big) n^{-s}. \tag{3.25}$$

The operations of shifting, differencing, and summation that we consider later in this chapter are special cases of this operation, corresponding to specific choices of the sequence B.

In this section, we give a condition on the sequence B that guarantees the meromorphic continuation of (3.25) for *any* Dirichlet series $f(s) = \sum a_n n^{-s}$ with a meromorphic continuation to the plane; we do not place restrictions on the number or location of the poles of $f(s)$ or on the growth of $f(s)$. In the notation of the previous chapter, we give a sufficient condition for the the ring $\mathcal{M}$ of meromorphic Dirichlet series to be closed under the operation C_B.

We begin by considering the convergence of the series (3.25).

Lemma 3.3.2. *Suppose that the Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{3.26}$$

has abscissa of absolute convergence $\sigma_a(f)$ *and that the summatory function* $B(n) = \sum_{m\le n} b_m$ *of the sequence* b_n *satisfies* $B(n) \ll n^r$ *for some* $r \ge 0$*. Then the abscissa of absolute convergence* $\sigma_a(g)$ *of the Dirichlet series*

$$g(s) = \sum_{n=1}^{\infty}\Big(\sum_{i+j=n} a_i b_j\Big)\frac{1}{n^s} \tag{3.27}$$

satisfies $\sigma_a(g) \le \max(r, \sigma_a(f) + r)$*.*

Proof. To determine the abscissa of convergence of $g(s)$, we estimate the summatory function

$$C(n) = \sum_{m\le n}\sum_{i+j=m} a_i b_j \tag{3.28}$$

of its coefficients. We have

$$C(n) = \sum_{m\le n} a_m B(n-m) \ll n^r \sum_{m\le n} |a_m|. \tag{3.29}$$

Since $f(s)$ has abscissa of absolute convergence $\sigma_a(f)$, the summatory function $A(n) = \sum_{m\le n} |a_m|$ satisfies

$$A(n) \ll \begin{cases} n^{\sigma_a(f)+\varepsilon} & \text{if } \sigma_a(f) \ge 0, \\ 1 & \text{if } \sigma_a(f) < 0. \end{cases} \tag{3.30}$$

Using this bound for $A(n)$ in (3.29) gives the claimed bound on $\sigma_c(g)$. □

Corollary 3.3.3. *If the sequence* B *has at most polynomial growth, then the ring* $\mathcal{D}$ *of convergent Dirichlet series is closed under the operation* C_B*.*

Now we consider the question of meromorphic continuation of the Dirichlet series $C_B f(s)$. We show that if the sequence B has at most polynomial growth and the additive generating function of the sequence B meromorphically continues to a neighborhood of 1, then meromorphicity of $C_B f(s)$ follows from that of $f(s)$.

Theorem 3.3.4. *Let* b_n *be a sequence of complex numbers with* $\sum_{m\le n} b_m \ll n^r$ *for some* $r \ge 0$ *and such that the function* $q(z)$ *defined in* $|z| < 1$ *by*

$$q(z) = \sum_{n=0}^{\infty} b_n z^n \tag{3.31}$$

has a meromorphic continuation to a neighborhood of $z = 1$*. Let* $M = \operatorname{ord}(q(z), 1)$ *be the order of the pole (if* $M < 0$*) or zero (if* $M > 0$*) of* $q(z)$ *at* $z = 1$*.*

Let

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{3.32}$$

be a Dirichlet series with abscissa of absolute convergence $\sigma_a(f) < \infty$ *and meromorphic continuation to a halfplane* $\mathrm{Re}(s) > \sigma_m$ *(with* $\sigma_m = -\infty$ *if* $f(s)$ *continues to* $\mathbb{C}$*).*

Then the following are true of the Dirichlet series

$$g(s) = \sum_{n=1}^{\infty} \Big(\sum_{i+j=n} a_i b_j \Big) \frac{1}{n^s}. \tag{3.33}$$

(i) *The Dirichlet series* $g(s)$ *converges absolutely for* $\mathrm{Re}(s) > \max(r, \sigma_a(f) + r)$ *and has a meromorphic continuation to* $\mathrm{Re}(s) > \sigma_m - M$.

(ii) *For each integer* $K \geq 0$*, the function* $g(s)$ *satisfies*

$$g(s) = \sum_{k=M}^{K} c_k \frac{\Gamma(s+k)}{\Gamma(s)} f(s+k) + R_K(s) \tag{3.34}$$

in the halfplane $\mathrm{Re}(s) > \max(\sigma_c(f) - K - 1, -K - 1, \sigma_m - M)$*, where* $R_K(s)$ *is a holomorphic function in* $\mathrm{Re}(s) > \max(\sigma_c(f) - K - 1, -K - 1)$ *with* $R_K(-m) = 0$ *for integers* $m \geq 0$ *and the numbers* c_k *are the coefficients of the series expansion*

$$q(e^{-z}) = \sum_{k=M}^{\infty} c_k z^k. \tag{3.35}$$

Proof. If $\sum_{m \leq n} b_m \ll n^r$ for some $r \geq 0$, then the power series (3.31) converges and defines a holomorphic function in $|z| < 1$.

By Lemma 3.3.2, the Dirichlet series $g(s)$ converges absolutely in the halfplane $\mathrm{Re}(s) > \max(r, \sigma_a(f) + r)$.

By Proposition 3.2.1, in the halfplane $\mathrm{Re}(s) > \max(\sigma_c(g), 0)$, we have

$$\Gamma(s) g(s) = \int_0^{\infty} q(e^{-x}) p(e^{-x}) x^s \, \frac{dx}{x}. \tag{3.36}$$

Since $q(z)$ is meromorphic at $z = 1$, the function $q(e^{-z})$ is meromorphic at $z = 0$. If $q(z)$ has a zero or pole at $z = 1$, then $q(e^{-z})$ has a zero or pole of the same order at $z = 0$. Thus $q(e^{-z})$ has a series expansion (3.35).

For a fixed $K \geq M$, by adding and subtracting the first terms of the series expansion of $q(e^{-z})$, we have

$$\Gamma(s)g(s) = \int_0^\infty \Big(\sum_{k=M}^{K} c_k x^k\Big) p(e^{-x}) x^s \, \frac{dx}{x} + \int_0^\infty \Big(q(e^{-x}) - \sum_{k=M}^{K} c_k x^k\Big) p(e^{-x}) x^s \, \frac{dx}{x}. \tag{3.37}$$

Interchanging the first (finite) sum and the integral, we have

$$\Gamma(s)g(s) = \sum_{k=M}^{K} c_k \int_0^\infty p(e^{-x}) x^{s+k} \, \frac{dx}{x} + \int_0^\infty \Big(q(e^{-x}) - \sum_{k=M}^{K} c_k x^k\Big) p(e^{-x}) x^s \, \frac{dx}{x}. \tag{3.38}$$

Then, using Proposition 3.2.1 again and dividing by $\Gamma(s)$, we have

$$g(s) = \sum_{k=M}^{K} c_k \frac{\Gamma(s+k)}{\Gamma(s)} f(s+k) + R_K(s) \tag{3.39}$$

where the remainder term $R_K(s)$ is

$$R_K(s) = \frac{1}{\Gamma(s)} \int_0^\infty \Big(q(e^{-x}) - \sum_{k=M}^{K} c_k x^k\Big) p(e^{-x}) x^s \, \frac{dx}{x}. \tag{3.40}$$

If $f(s)$ is meromorphic in the halfplane $\mathrm{Re}(s) > \sigma_m$, then the sum

$$\sum_{k=M}^{K} c_k \frac{\Gamma(s+k)}{\Gamma(s)} f(s+k) \tag{3.41}$$

is meromorphic in $\mathrm{Re}(s) > \sigma_m - M$ since each term $f(s+k)$ is meromophic in $\mathrm{Re}(s) > \sigma_m - k$.

We now show that the remainder term $R_K(s)$ is holomorphic in the halfplane $\mathrm{Re}(s) > \max(\sigma_c(f) - K - 1, -K - 1)$ by showing that the integral in (3.40) is uniformly convergent on compact subsets of this halfplane. We first have the estimate

$$q(e^{-x}) - \sum_{k=M}^{K} c_k x^k \ll x^{K+1} \tag{3.42}$$

as $x \to 0+$. In (3.23), we showed that $p(e^{-x}) \ll x^{-\theta}$ as $x \to 0+$ for any $\theta > \max(\sigma_c(f), 0)$. Therefore, for s in any compact set Ω in the halfplane $\mathrm{Re}(s) > \theta - K - 1$ where $\theta > \max(\sigma_c(f), 0)$, the integrand is uniformly

$$\Big(q(e^{-x}) - \sum_{k=M}^{K} c_k x^k\Big) p(e^{-x}) x^{s-1} \ll x^{\sigma+K-\theta} \tag{3.43}$$

as $x \to 0+$, so the integral converges unformly on Ω if $\sigma + K - \theta > -1$. We therefore have that the integral in (3.40) converges uniformly on compact subsets of $\mathrm{Re}(s) > \max(\sigma_c(f) - K - 1, -K - 1)$ and is analytic in that halfplane.

Finally, since $R_K(s)$ is holomorphic in $\mathrm{Re}(s) > \max(\sigma_c(f) - K - 1, -K - 1)$ and $1/\Gamma(-m) = 0$ for all $m \geq 0$, we have that $R_K(-m) = 0$. □

Corollary 3.3.5. *If the sequence B has at most polynomial growth and the additive generating function*

$$q(z) = \sum_{n=0}^{\infty} b_n z^n \tag{3.44}$$

meromorphically continues to $z = 1$, then the ring $\mathcal{M}$ of meromorphic Dirichlet series is closed under the operation C_B.

3.4 Dirichlet series and the calculus of finite differences

In this section, we consider applying the basic operations of the calculus of finite differences—right and left shifts, forward and backward differences, and summation—to the coefficients of a Dirichlet series.

In this section, it will be convenient to use the generating function notation, writing the coefficients of a Dirichlet series as values of an arithmetic function $a\colon \mathbb{N} \to \mathbb{C}$ instead of as a sequence of numbers a_n.

3.4.1 Shifts

The first operations we will consider are the right and left shifts of the sequence of coefficients.

Definition 3.4.1. For an arithmetic function $a\colon \mathbb{N} \to \mathbb{C}$, the ***right shift*** $\mathcal{R}a$ and the ***left shift*** $\mathcal{L}a$ are the arithmetic functions defined by

$$(\mathcal{R}a)(n) = a(n-1) \quad \text{for all } n \in \mathbb{N} \tag{3.45}$$

with $a(0) = 0$ and

$$(\mathcal{L}a)(n) = a(n+1) \quad \text{for all } n \in \mathbb{N}. \tag{3.46}$$

These operations are almost inverses; we have

$$(\mathcal{L}\mathcal{R}a)(n) = a(n) \quad \text{for all } n \in \mathbb{N} \tag{3.47}$$

but

$$(\mathcal{R}\mathcal{L}a)(n) = \begin{cases} a(n) & \text{if } n \geq 2, \\ 0 & \text{if } n = 1. \end{cases} \tag{3.48}$$

We note also that the shift operations can be iterated; we have

$$(\mathcal{R}^k a)(n) = a(n-k) \quad \text{and} \quad (\mathcal{L}^k a)(n) = a(n+k). \tag{3.49}$$

In this section, we will mainly consider the right shift $\mathcal{R}$. Our aim is to study the Dirichlet series

$$D(\mathcal{R}a, s) = \sum_{n=1}^{\infty} \frac{(\mathcal{R}a)(n)}{n^s} = \sum_{n=1}^{\infty} \frac{a(n-1)}{n^s} \tag{3.50}$$

obtained by shifting the coefficients of a Dirichlet series. Note that the Dirichlet series $D(a, s)$ and $D(\mathcal{R}a, s)$ have the same abscissa of convergence $\sigma_c(a)$ and the same abscissa of absolute convergence $\sigma_a(a)$.

The right shift operation $\mathcal{R}$ on an arithmetic function a is just the additive convolution of a with the sequence $(0, 1, 0, 0, \ldots)$, so we have the following lemma.

Lemma 3.4.2. *If $a\colon \mathbb{N} \to \mathbb{C}$ is an arithmetic function, then*

$$D(\mathcal{R}a, s) = C_B D(a, s) \tag{3.51}$$

for the sequence B with $b_0 = 0$, $b_1 = 1$, and $b_n = 0$ for $n \geq 2$.

Using the results of the previous section, the Dirichlet series $D(\mathcal{R}a, s)$ can be expressed as an infinite sum involving the original Dirichlet series $D(a, s)$ evaluated at points to the right of s.

Theorem 3.4.3. *Let $\sigma_c(a)$ be the abscissa of convergence of the Dirichlet series $D(a, s)$ and suppose that $D(a, s)$ has a meromorphic continuation to the halfplane* $\mathrm{Re}(s) > \sigma_m$*. Then the Dirichlet series $D(\mathcal{R}a, s)$ has a meromorphic continuation to* $\mathrm{Re}(s) > \sigma_m$*. For each $K \geq 0$, the Dirichlet series $D(\mathcal{R}a, s)$ satisfies*

$$D(\mathcal{R}a, s) = \sum_{k=0}^{K} \frac{(-1)^k}{k!} \Big(\prod_{i=0}^{k-1} (s+i) \Big) D(a, s+k) + R_K(s) \tag{3.52}$$

where $R_K(s)$ is holomorphic in the halfplane $\mathrm{Re}(s) > \max(\sigma_c(a) - K, -K)$ *and $R_K(-m) = 0$ for integers $m \geq 0$.*

Proof. We apply Theorem 3.3.4. For the sequence $B = (0, 1, 0, 0, \dots)$, we have in the notation of the Theorem that $q(z) = z$ and

$$q(e^{-z}) = e^{-z} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} z^k. \tag{3.53}$$

Then Theorem 3.3.4 gives

$$D(\mathcal{R}a, s) = \sum_{k=0}^{K} \frac{(-1)^k}{k!} \frac{\Gamma(s+k)}{\Gamma(s)} D(a, s+k) + R_K(s), \tag{3.54}$$

and reducing the product of gamma functions gives the theorem. □

Corollary 3.4.4. *The ring $\mathcal{M}$ of meromorphic Dirichlet series is closed under the right shift operation $\mathcal{R}$.*

The above theorem not only provides the meromorphic continuation of $D(\mathcal{R}a, s)$, but also allows us to determine the structure (locations and Laurent expansions) of the singularities of $D(\mathcal{R}a, s)$ in terms of the singularities of $D(a, s)$. Since $D(a, s)$ is holomorphic in the halfplane $\sigma > \sigma_c(a)$, the terms $D(a, s + k)$ in (3.52) are all holomorphic for k large enough.

3.4.2 Differences

Definition 3.4.5. For an arithmetic function $a \colon \mathbb{N} \to \mathbb{C}$, the ***forward difference*** Δa is

$$(\Delta a)(n) = a(n+1) - a(n) \quad \text{for all } n \in \mathbb{N} \tag{3.55}$$

and the ***backward difference*** ∇a is

$$(\nabla a)(n) = a(n) - a(n-1) \quad \text{for all } n \in \mathbb{N}. \tag{3.56}$$

We have the following theorem on the meromorphic continuation of the Dirichlet series

$$D(\nabla a, s) = \sum_{n=1}^{\infty} \frac{\nabla a(n)}{n^s} = \sum_{n=1}^{\infty} \frac{a(n) - a(n-1)}{n^s}. \tag{3.57}$$

Theorem 3.4.6. *Let $\sigma_c(a)$ be the abscissa of convergence of the Dirichlet series $D(a, s)$ and suppose that $D(a, s)$ has a meromorphic continuation to the halfplane* $\mathrm{Re}(s) > \sigma_m$. *Then the*

Dirichlet series $D(\nabla a, s)$ has a meromorphic continuation to $\mathrm{Re}(s) > \sigma_m - 1$. *For each $K \geq 0$, the Dirichlet series $D(\nabla a, s)$ satisfies*

$$D(\nabla a, s) = \sum_{k=0}^{K} \frac{(-1)^{k+1}}{k!} \Big(\prod_{i=0}^{k-1} (s+i) \Big) D(a, s+k) + R_K(s) \tag{3.58}$$

where $R_K(s)$ is holomorphic in the halfplane $\mathrm{Re}(s) > \max(\sigma_c(a) - K, -K)$ *and $R_K(-m) = 0$ for integers $m \geq 0$.*

Proof. Since $(\nabla a)(n) = a(n) - (\mathcal{R}a)(n)$, the theorem follows immediately from the results of the previous section. Equivalently, one could apply Theorem 3.3.4 to the sequence $B = (1, -1, 0, 0, \ldots)$. Note here that the power series generating function $q(z) = 1 - z$ has a simple zero at $z = 1$. □

3.4.3 Summation

Definition 3.4.7. For an arithmetic function $a \colon \mathbb{N} \to \mathbb{C}$, the ***summatory function*** $\mathcal{S}a$ is

$$(\mathcal{S}a)(n) = \sum_{m=1}^{n} a(m) \quad \text{for all } n \in \mathbb{N}. \tag{3.59}$$

The operations of summation and (backward) differencing are inverses in that

$$(\nabla \mathcal{S}a)(n) = a(n) \quad \text{and} \quad (\mathcal{S}\nabla a)(n) = a(n) \tag{3.60}$$

for all $n \in \mathbb{N}$.

Our goal is to study the Dirichlet series

$$D(\mathcal{S}a, s) = \sum_{n=1}^{\infty} \frac{\mathcal{S}a(n)}{n^s} = \sum_{n=1}^{\infty} \Big(\sum_{m=1}^{n} a(m) \Big) \frac{1}{n^s} \tag{3.61}$$

whose coefficients are the partial sums of the coefficients of a Dirichlet series $D(a, s)$.

We begin by considering the convergence of such a Dirichlet series.

Lemma 3.4.8. *If the Dirichlet series $D(a, s)$ has abscissa of convergence $\sigma_c(a)$, then the abscissa of convergence of $D(\mathcal{S}a, s)$ satisfies $\sigma_c(\mathcal{S}a) \leq \max(\sigma_c(a) + 1, 1)$.*

Proof. If $\sum_{n=1}^{\infty} a(n)$ converges, then $(\mathcal{S}a)(n)$ is bounded, so

$$\sum_{m=1}^{n} (\mathcal{S}a)(m) \ll n, \tag{3.62}$$

and $\sigma_c(\mathcal{S}a) \leq 1$.

If $\sum_{n=1}^{\infty} a(n)$ diverges, then $(\mathcal{S}a)(n) \ll n^{\sigma_c(a)+\varepsilon}$. Then

$$\sum_{m=1}^{n} (\mathcal{S}a)(m) \ll n^{\sigma_c(a)+1+\varepsilon}, \tag{3.63}$$

and so $\sigma_c(\mathcal{S}a) \leq \sigma_c(a) + 1$. □

The bound for $\sigma_c(\mathcal{S}a)$ provided by the lemma is sharp in general, as shown by the following two examples.

Example 3.4.9. For the constant Dirichlet series $D(\delta, s) = 1$ associated to the arithmetic function

$$\delta(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{3.64}$$

we have $D(\mathcal{S}\delta, s) = \zeta(s)$. The series $D(\delta, s)$ has absicssa of convergence $\sigma_c(\delta) = -\infty$ while the series $D(\mathcal{S}\delta, s)$ has abscissa of convergence $\sigma_c(\mathcal{S}\delta) = 1$.

Example 3.4.10. For the Dirichlet series $\zeta(s) = D(\mathbf{1}, s)$ with abscissa of convergence $\sigma_c(\mathbf{1}) = 1$, we have $D(\mathcal{S}\mathbf{1}, s) = \sum n \cdot n^{-s} = \zeta(s-1)$, which has abscissa of convergence $\sigma_c(\mathcal{S}\mathbf{1}) = 2$.

By Theorem 3.3.4, in any right halfplane, the function $D(\mathcal{S}a, s)$ can be expressed as a sum of finitely many shifts of the original Dirichlet series $D(a, s)$ and a holomorphic remainder term.

Theorem 3.4.11. *Let $\sigma_c(a)$ be the absicssa of convergence of the Dirichlet series $D(a, s)$ and suppose that $D(a, s)$ has a meromorphic continuation to* $\mathrm{Re}(s) > \sigma_m$. *Then the Dirichlet series $D(\mathcal{S}a, s)$ has a meromorphic continuation to* $\mathrm{Re}(s) > \sigma_m + 1$. *For each $K \geq 1$, the Dirichlet series $D(\mathcal{S}a, s)$ satisfies*

$$D(\mathcal{S}a, s) = \frac{D(a, s-1)}{s-1} + \frac{1}{2} D(a, s) + \sum_{k=1}^{K} \frac{B_{2k}}{(2k)!} \Big(\prod_{i=0}^{2k-2} (s+i) \Big) D(a, s+2k-1) + R_K(s) \tag{3.65}$$

where B_k is the kth Bernoulli number and $R_K(s)$ is holomorphic in the halfplane $\mathrm{Re}(s) > \max(\sigma_c(a) - 2K, -2K)$ *with $R_K(-m) = 0$ for integers $m \geq 0$.*

Proof. We apply Theorem 3.3.4. In the notation of the Theorem, the sequence $B = (1, 1, 1, \dots)$ has

$$q(z) = \frac{1}{1-z} \tag{3.66}$$

and

$$q(e^{-z}) = \frac{1}{1-e^{-z}} = \sum_{k=-1}^{\infty} \frac{(-1)^{k+1} B_{k+1}}{(k+1)!} z^k \tag{3.67}$$

(see Proposition A.0.3 in the Appendix). Then note that $B_0 = 1$, $B_1 = -1/2$, and $B_{2k+1} = 0$ for $k \geq 1$. □

Corollary 3.4.12. *The ring $\mathcal{M}$ of meromorphic Dirichlet series is closed under the operation $\mathcal{S}$ of partial summation of coefficients.*

Remark. If the abscissa of convergence $\sigma_c(a)$ of $D(a, s)$ is less than 0, then as noted earlier, the abscissa of convergence of $D(\mathcal{S}a, s)$ could be as large as 1. However, the theorem shows that while the Dirichlet series $D(\mathcal{S}a, s)$ may not converge to the left of $\sigma = 1$, the function $D(\mathcal{S}a, s)$ does have a meromorphic continuation to $\sigma_c(a) + 1$.

3.5 Examples

We now consider many examples of the above operations, especially the summation operation, applied to important Dirichlet series of analytic number theory.

3.5.1 Harmonic numbers

The ***harmonic numbers*** H_n are

$$H_n = \sum_{k=1}^{n} \frac{1}{k}. \tag{3.68}$$

Matsuoka [36] studied the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{H_{n-1}}{n^s}, \tag{3.69}$$

showing that this series has a meromorphic continuation to $\mathbb{C}$, locating its poles, and determining its values at negative integers. Since the harmonic numbers are the partial sums of the coefficients of the Dirichlet series $\zeta(s+1) = \sum (1/n) n^{-s}$, we can recover these results by simply applying Theorem 3.4.11 to $\zeta(s+1)$. We work with the series $\sum H_n n^{-s}$ instead of $\sum H_{n-1} n^{-s}$; since

$$\sum_{n=1}^{\infty} \frac{H_{n-1}}{n^s} = \sum_{n=1}^{\infty} \frac{H_n}{n^s} - \zeta(s+1), \tag{3.70}$$

there is no difficulty in translating the results to the series considered by Matsuoka.

Theorem 3.5.1 (Matsuoka [36])**.** *The Dirichlet series*

$$H(s) := \sum_{n=1}^{\infty} \frac{H_n}{n^s} \tag{3.71}$$

has a meromorphic continuation to the complex plane. For each $K \geq 1$, $H(s)$ satisfies

$$H(s) = \frac{\zeta(s)}{s-1} + \frac{1}{2}\zeta(s+1) + \sum_{k=1}^{K} \frac{B_{2k}}{(2k)!}\Big(\prod_{i=0}^{2k-2}(s+i)\Big)\zeta(s+2k) + R_K(s) \tag{3.72}$$

for a function $R_K(s)$ holomorphic in the halfplane $\sigma > 1-2K$. The function $H(s)$ has a double pole at $s = 1$ with

$$H(s) = (s-1)^{-2} + \gamma(s-1)^{-1} + O(1) \tag{3.73}$$

and simple poles at $s = 0$ and $s = 1-2k$ for $k \geq 1$ with residue $-B_{2k}/2k$. At negative even integers, we have

$$H(-2m) = \frac{1}{2}\Big(1 - \frac{1}{2m}\Big)B_{2m}. \tag{3.74}$$

Proof. The zeta function has a pole at $s = 1$ with

$$\zeta(s) = (s-1)^{-1} + \gamma + O(s-1). \tag{3.75}$$

The claims about the poles of $H(s)$ then follow by considering the pole of each term on the right side of (3.72).

To evaluate $H(-2m)$, we note that $\zeta(-2k) = 0$ for $k \geq 1$, that

$$\prod_{i=0}^{2k-2}(-2m+i) = 0 \tag{3.76}$$

if $k \geq m+1$, and that $R_K(-2m) = 0$ for all K and all $m \geq 0$. All that remains on the right side is

$$H(-2m) = \frac{1}{2}\zeta(-2m+1) + \frac{B_{2m}}{(2m)!}\Big(\prod_{i=0}^{2m-2}(-2m+i)\Big)\zeta(0). \tag{3.77}$$

The claim then follows from $\zeta(-2m+1) = -B_{2m}/2m$ and $\zeta(0) = -1/2$. □

3.5.2 The Dirichlet divisor problem

The function $d(n)$ equal to the number of divisors of n has the Dirichlet series generating function $D(d, s) = \zeta(s)^2$. The summatory function of $d(n)$ satisfies

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + \Delta(x) \tag{3.78}$$

with $\Delta(x) = o(x)$; here, $\gamma \approx 0.5772$ is the Euler-Mascheroni constant. Determining the order of growth of $\Delta(x)$ has become known as the ***Dirichlet divisor problem*** (see Titchmarsh [53, Ch. XII] for an overview). Dirichlet proved that $\Delta(x) \ll x^{1/2}$, and Voronoi improved this bound to $\Delta(x) \ll x^{1/3+\varepsilon}$. The best bound on $\Delta(x)$ is currently due to Huxley [26], who proved $\Delta(x) \ll x^{131/416+\varepsilon}$ (note that $131/416 \approx 0.3149$). It is conjectured that $\Delta(x) \ll x^{1/4+\varepsilon}$.

We now consider the Dirichlet series

$$D(\mathcal{S}d, s) = \sum_{n=1}^{\infty} \Big(\sum_{m \leq n} d(m) \Big) \frac{1}{n^s} \tag{3.79}$$

attached to the summatory function considered in the Dirichlet divisor problem.

Theorem 3.5.2. *The Dirichlet series $D(\mathcal{S}d, s)$ has a meromorphic continuation to $\mathbb{C}$. In the halfplane $\sigma > 1 - 2K$, $D(\mathcal{S}d, s)$ satisfies*

$$D(\mathcal{S}d, s) = \frac{1}{s-1}\zeta^2(s-1) + \frac{1}{2}\zeta^2(s) + \sum_{k=1}^{K} \frac{B_{2k}}{(2k)!} \Big(\prod_{i=0}^{2k-2} (s+i) \Big) \zeta^2(s + 2k - 1) + R_K(s) \tag{3.80}$$

where $R_K(s)$ is a holomorphic function. The function $D(\mathcal{S}d, s)$ has the following poles:

(i) *a double pole at $s = 2$ with*

$$D(\mathcal{S}d, s) = (s-2)^{-2} + (2\gamma - 1)(s-2)^{-1} + O(1) \tag{3.81}$$

(ii) *a double pole at $s = 1$ with*

$$D(\mathcal{S}d, s) = \frac{1}{2}(s-1)^{-2} + \Big(\gamma + \frac{1}{4}\Big)(s-1)^{-1} + O(1) \tag{3.82}$$

(iii) *a simple pole at $s = -2m$ for $m \geq 0$ with residue* $\dfrac{B_{2m+2}}{(2m+2)(2m+1)}$.

Further, for integers $m \geq 0$, we have

$$D(\mathcal{S}d, -2m-1) = \frac{1}{2}\Big(\frac{B_{2m}}{2m}\Big)^2 - \frac{1}{4}\frac{B_{2m+2}}{2m+2}. \tag{3.83}$$

Proof. Applying Theorem 3.4.11 to $\zeta^2(s)$ gives (3.80).

At $s = 2$, the first term is

$$\frac{1}{s-1}\zeta^2(s-1) = \Big(1-(s-2)+O((s-2)^2)\Big)\Big((s-2)^{-1}+\gamma+O(s-2)\Big)^2, \tag{3.84}$$

giving claim (i). At $s = 1$, the first term is

$$\frac{1}{s-1}\zeta^2(s-1) = \Big(-\frac{1}{2}\Big)^2(s-1)^{-1}+O(1) \tag{3.85}$$

while the second term is

$$\frac{1}{2}\zeta^2(s) = \frac{1}{2}\Big((s-1)^{-1}+\gamma+O(s-1)\Big)^2. \tag{3.86}$$

Adding the contributions from these two terms gives claim (ii). In each term

$$\frac{B_{2k}}{(2k)!}\Big(\prod_{i=0}^{2k-2}(s+i)\Big)\zeta^2(s+2k-1), \tag{3.87}$$

$\zeta^2(s+2k-1)$ has a double pole at $s = -2k+2$, while the factor of $(s+2k-2)$ has a zero; this contributes a simple pole as in claim (iii).

Finally, to evaluate $D(\mathcal{S}d, -2m-1)$, we note that in the sum

$$\sum_{k=1}^{K}\frac{B_{2k}}{(2k)!}\Big(\prod_{i=0}^{2k-2}(-2m-1+i)\Big)\zeta^2(-2m-1+2k-1), \tag{3.88}$$

the product is 0 if $k \geq m+2$, while $\zeta(-2m-1+2k-1) = 0$ if $k \leq m$. Then, since $\zeta(-2m-2) = 0$ and $R_K(-2m-1) = 0$, we have

$$\begin{aligned} D(\mathcal{S}d, -2m-1) &= \frac{1}{2}\zeta^2(-2m-1) + \frac{B_{2m+2}}{(2m+2)!}\Big(\prod_{i=0}^{2m}(-2m-1+i)\Big)\zeta^2(0) \\ &= \frac{1}{2}\Big(\frac{B_{2m}}{2m}\Big)^2 - \frac{1}{4}\frac{B_{2m+2}}{2m+2}. \end{aligned}$$

□

Since

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{\Delta(n)}{n^s} &= \sum_{n=1}^{\infty}\frac{(\mathcal{S}d)(n)}{n^s} - \sum_{n=1}^{\infty}\frac{n\log n}{n^s} - (2\gamma-1)\sum_{n=1}^{\infty}\frac{n}{n^s} && (3.89) \\ &= D(\mathcal{S}d, s) + \zeta'(s-1) - (2\gamma-1)\zeta(s-1), && (3.90) \end{aligned}$$

we have the following corollary.

Corollary 3.5.3. *The Dirichlet series $D(\Delta, s)$ attached to the error term in the Dirichlet divisor problem has a meromorphic continuation to the complex plane.*

3.5.3 The Chebyshev and Mertens functions

Recall the von Mangoldt function, defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, \\ 0 & \text{otherwise,} \end{cases} \tag{3.91}$$

with Dirichlet series generating function

$$D(\Lambda, s) = -\frac{\zeta'}{\zeta}(s). \tag{3.92}$$

The Chebyshev function $\psi(x)$ is the summatory function of the von Mangoldt function $\Lambda(n)$; that is,

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{p^k \leq x} \log p. \tag{3.93}$$

We consider the Dirichlet series generating function

$$D(\psi, s) = \sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} \tag{3.94}$$

attached to the Chebyshev function. Since $\psi(x) \sim x$ by the Prime Number Theorem, we have

$$\sum_{n \leq x} \psi(n) \sim \frac{1}{2} x^2 \tag{3.95}$$

and $D(\psi, s)$ has abscissa of convergence $\sigma_c(\psi) = 2$.

Theorem 3.5.4. *The Dirichlet series $D(\psi, s)$ has a meromorphic continuation to $\mathbb{C}$. In the halfplane $\sigma > 1 - 2K$, $D(\psi, s)$ satisfies*

$$D(\psi, s) = \frac{-1}{s-1} \frac{\zeta'}{\zeta}(s-1) - \frac{1}{2} \frac{\zeta'}{\zeta}(s) - \sum_{k=1}^{K} \frac{B_{2k}}{(2k)!} \Big(\prod_{i=0}^{2k-2} (s+i) \Big) \frac{\zeta'}{\zeta}(s + 2k - 1) + R_K(s) \tag{3.96}$$

where $R_K(s)$ is a holomorphic function. On the real axis, the function $D(\psi, s)$ has the following poles:

(i) *a simple pole at $s = 2$ with residue* 1,

(ii) *a simple pole at $s = 1$ with residue* $-\log 2\pi + 1/2$,

(iii) *a simple pole at* $s = -2m$ *for* $m \geq 1$ *with residue* $-1/2$*,*

(iv) *a simple pole at* $s = -2m - 1$ *for* $m \geq 0$ *with residue* $1/2$*.*

Off the real axis, the function $D(\psi, s)$ *has the following poles for each nontrival zero* ρ *of* $\zeta(s)$ *with multiplicity* n_ρ*:*

(v) *a simple pole at* $s = 1 + \rho$ *with residue* $n_\rho/(1 - \rho)$*,*

(vi) *a simple pole at* $s = \rho$ *with residue* $-n_\rho/2$*,*

(vii) *a simple pole at* $s = -2m + 1 + \rho$ *with residue* $n_\rho \frac{B_{2m}}{(2m)!} \prod_{i=1}^{2m-1} (\rho - i)$ *for each* $m \geq 1$*.*

Proof. The function $\zeta'/\zeta(s)$ has a simple pole at $s = 1$ with residue -1, a simple pole at each trivial zero $s = -2k$ with residue 1, and a simple pole at each nontrivial zero ρ of zeta with residue equal to the multiplicity n_ρ of the zero.

The poles at $s = 2$ and $s = 1$ follow from considering the first two terms of (3.96). To determine the residue at $s = 1$, we have used that $\zeta'/\zeta(0) = \log 2\pi$.

At $s = -2m$, only the second term $-\frac{1}{2}\frac{\zeta'}{\zeta}(s)$ has a pole.

At $s = -2m - 1$, only finitely many terms contribute to the pole since

$$\prod_{i=0}^{2k-2} (-2m - 1 + i) = 0 \tag{3.97}$$

for $k \geq m + 1$. We find then that at $s = -2m - 1$, there is a simple pole of residue

$$\frac{1}{2m + 2} - \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} \prod_{i=0}^{2k-2} (-2m - 1 + i) = \frac{1}{2m + 2} + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} \frac{(2m + 1)!}{(2m + 2 - 2k)!} \tag{3.98}$$

which we can write as

$$\frac{1}{2m + 2} \sum_{k=0}^{m} B_{2k} \binom{2m + 2}{2k} = \frac{1}{2m + 2} \left(\sum_{k=0}^{2m} B_k \binom{2m + 2}{k} + \frac{1}{2} \binom{2m + 2}{1} \right). \tag{3.99}$$

Finally, we appeal to identity (A.9) to show that the inner sum is 0, giving claim (iv). □

The summatory function of the Möbius function $\mu(n)$ is the Mertens function

$$M(x) = \sum_{n \leq x} \mu(n). \tag{3.100}$$

Since $D(\mu, s) = 1/\zeta(s)$, the growth of $M(x)$ is closely related to the zeros of the zeta function; if θ is the supremum of the real parts of the zeros of $\zeta(s)$, then $M(x) \ll x^{\theta+\varepsilon}$.

By applying Theorem 3.4.11 to the Dirichlet series $1/\zeta(s)$, one can obtain the meromorphic continuation of the series

$$D(M, s) = \sum_{n=1}^{\infty} \frac{M(n)}{n^s} \tag{3.101}$$

and determine the location of its poles in terms of the zeros of zeta. The residues involve the derivative of $\zeta(s)$ at the negative even integers; we remind the reader of the value of these derivatives in terms of more familiar zeta values.

Lemma 3.5.5. *The derivative of the zeta function at negative even integers is*

$$\zeta'(-2m) = \frac{(-1)^m (2m)!}{2^{2m+1}\pi^{2m}} \zeta(2m+1) \tag{3.102}$$

for $m \geq 1$.

Proof. The asymmetric functional equation for the zeta function is

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s). \tag{3.103}$$

Since $\sin(\pi s/2) = 0$ when $s = -2m$, differentiating by the product rule gives

$$\begin{aligned} \zeta'(-2m) &= 2^{-2m}\pi^{-2m-1}(\pi/2)\cos(\pi(-2m)/2)\Gamma(1+2m)\zeta(1+2m) \\ &= 2^{-2m-1}\pi^{-2m}(-1)^m(2m)!\zeta(2m+1). \end{aligned}$$

□

Theorem 3.5.6. *The Dirichlet series $D(M, s)$ has a meromorphic continuation to $\mathbb{C}$. In the halfplane $\sigma > 1 - 2K$, $D(M, s)$ satisfies*

$$D(M, s) = \frac{\zeta(s-1)^{-1}}{s-1} + \frac{1}{2}\zeta(s)^{-1} + \sum_{k=1}^{K} \frac{B_{2k}}{(2k)!} \Big(\prod_{i=0}^{2k-2} (s+i) \Big) \zeta(s+2k-1)^{-1} + R_K(s) \tag{3.104}$$

where $R_K(s)$ is holomorphic. On the real axis, the function $D(M, s)$ has the following poles:

(i) *a simple pole at $s = 1$ with residue -2,*

(ii) *a simple pole at $s = -2m$ for $m \geq 1$ with residue $(2\zeta'(-2m))^{-1}$,*

(iii) *a simple pole at $s = -2m-1$ for $m \geq 0$ with residue*

$$-\frac{1}{2m+2} \sum_{k=0}^{m} B_{2k} \binom{2m+2}{2k} \frac{1}{\zeta'(-2m-2+2k)}. \tag{3.105}$$

If ρ is a nontrivial zero of $\zeta(s)$ of order n_ρ, then $D(M,s)$ has a pole of order n_ρ at the points $s = 1+\rho$, $s = \rho$, and $s = \rho - 2k - 1$ for integers $k \geq 0$.

Proof. We apply Theorem 3.4.11 to the series $\zeta(s)^{-1}$ to obtain (3.104). As in the previous examples, the poles of $D(M,s)$ are found by considering the poles of each term of (3.104). $\square$

3.5.4 Dirichlet L-functions

Recall that a Dirichlet character mod q is a homomorphism

$$\chi \colon (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times \tag{3.106}$$

extended to a q-periodic function on $\mathbb{Z}$ by setting $\chi(n) = 0$ if $(n,q) \neq 1$. If $\chi(n) = 1$ for all n in $(\mathbb{Z}/q\mathbb{Z})^\times$, we say that χ is the principal character mod q; otherwise, we say that χ is nonprincipal.

To each Dirichlet character, we associate the Dirichlet L-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{3.107}$$

When χ is a nonprincipal Dirichlet character, the function $L(s,\chi)$ has a continuation to $\mathbb{C}$ as an entire function; if χ is a principal character, then $L(s,\chi)$ has a simple pole at $s = 1$.

We will make use of the following evaluation of the Dirichlet L-function at $s = 0$.

Lemma 3.5.7. *If χ is a non-principal Dirichlet character mod q, then*

$$L(0,\chi) = \frac{-1}{q} \sum_{m=1}^{q} \chi(m) m. \tag{3.108}$$

If $\chi(-1) = 1$, then $L(0,\chi) = 0$.

Proof. By periodicity of χ, if we shift the coefficients of $L(s,\chi)$ right by q, we get

$$\sum_{n=1}^{\infty} \chi(n)(n+q)^{-s} = L(s,\chi) - \sum_{n=1}^{q} \chi(n) n^{-s}. \tag{3.109}$$

We apply Theorem 3.3.4 with $q(z) = z^q$. We have the series expansion

$$e^{-qz} = \sum_{k=0}^{\infty} \frac{(-q)^k}{k!} z^k. \tag{3.110}$$

Then the theorem gives

$$L(s,\chi)-\sum_{n=1}^{q}\chi(n)n^{-s}=\sum_{k=0}^{K}\frac{(-q)^k}{k!}\frac{\Gamma(s+k)}{\Gamma(s)}L(s+k,\chi)+R_K(s) \tag{3.111}$$

where $R_K(s)$ is holomorphic in the halfplane $\mathrm{Re}(s)>-K-1$ and $R_K(-m)=0$ for integer $m\geq 0$.

Take $K\geq 1$ and $s=-1$. Then we have

$$L(-1,\chi)-\sum_{n=1}^{q}\chi(n)n=L(-1,\chi)+qL(0,\chi) \tag{3.112}$$

which gives (3.108).

If $\chi(-1)=1$, then we use the identity

$$\sum_{m=1}^{q}\chi(m)m=\sum_{m=1}^{q}\chi(q-m)(q-m)+\sum_{m=1}^{q}\chi(q-m)m \tag{3.113}$$

to deduce that

$$\sum_{m=1}^{q}\chi(m)m=0. \tag{3.114}$$

□

We can now analyze the Dirichlet series

$$D(\mathcal{S}\chi,s)=\sum_{n=1}^{\infty}\Big(\sum_{m\leq n}\chi(m)\Big)\frac{1}{n^s}, \tag{3.115}$$

using Theorem 3.4.11.

Theorem 3.5.8. *Let χ be a non-principal Dirichlet character mod q. Then the Dirichlet series $D(\mathcal{S}\chi,s)$ has a meromorphic continuation to $\mathbb{C}$, satisfying*

$$D(\mathcal{S}\chi,s)=\frac{L(s-1,\chi)}{s-1}+\frac{1}{2}L(s,\chi)+\sum_{k=1}^{K}\frac{B_{2k}}{(2k)!}\Big(\prod_{i=0}^{2k-2}(s+i)\Big)L(s+2k-1,\chi)+R_K(s) \tag{3.116}$$

in the halfplane $\sigma>1-2K$. If $\chi(-1)=1$, then $D(\mathcal{S}\chi,s)$ is an entire function. If $\chi(-1)=-1$, then $D(\mathcal{S}\chi,s)$ has at most a simple pole at $s=1$ with residue $L(0,\chi)$.

Proof. We apply Theorem 3.4.11. Since $L(s,\chi)$ is entire, the only possible pole on the right side of (3.116) comes from the first term $L(s-1,\chi)/(s-1)$. If $\chi(-1)=1$, then this term is analytic at $s=1$; if $\chi(-1)=-1$, then there is at most a simple pole of residue $L(0,\chi)$. □

Chapter 4

Dirichlet series associated to digital sums

4.1 Introduction

There has been a great deal of study of properties of the radix expansions to an integer base $b \geq 2$ of integers n. For each integer base $b \geq 2$, every positive integer n has a unique base-b expansion

$$n = \sum_{i \geq 0} \delta_{b,i}(n) b^i \tag{4.1}$$

with digits $\delta_{b,i} \in \{0, 1, \ldots, b-1\}$ given by

$$\delta_{b,i} = \left\lfloor \frac{n}{b^i} \right\rfloor - b \left\lfloor \frac{n}{b^{i+1}} \right\rfloor. \tag{4.2}$$

This paper considers two summatory functions of base b digits of n:

1. The *base-b sum-of-digits function $d_b(n)$* is

$$d_b(n) = \sum_{i \geq 0} \delta_{b,i}(n). \tag{4.3}$$

2. The *(base b) cumulative sum-of-digits function $S_b(n)$* is

$$S_b(n) = \sum_{m=1}^{n-1} d_b(m). \tag{4.4}$$

   We follow here the convention of previous authors (including [15] and [21]), with the sum defining $S_b(n)$ running to $n-1$ instead of n.

We associate to the functions $d_b(n)$ and $S_b(n)$ the Dirichlet series generating functions

$$F_b(s) = \sum_{n=1}^{\infty} \frac{d_b(n)}{n^s} \tag{4.5}$$

and

$$G_b(s) = \sum_{n=1}^{\infty} \frac{S_b(n)}{n^s}. \tag{4.6}$$

These Dirichlet series have abscissa of convergence $\mathrm{Re}(s) = 1$ and $\mathrm{Re}(s) = 2$, respectively.

This chapter studies the problem of the meromorphic continuation to $\mathbb{C}$ of Dirichlet series associated to the base-b digit sums $d_b(n)$ and $S_b(n)$ using the methods of the previous chapter. We obtain the meromorphic continuation and determine the exact pole and residue structure. The pole structure contains half of a two-dimensional lattice and the residues involve Bernoulli numbers and values of the Riemann zeta function on the line $\mathrm{Re}(s) = 0$. A meromorphic continuation of these functions was previously obtained in the thesis of Dumas [17] by a different method, which specified a half-lattice containing all the poles but did not determine the residues; in fact infinitely many of the residues on his possible pole set vanish.

The asymptotics of $S_b(n)$ have been extensively studied, see Section 4.1.2. We mention particularly work of Delange [15], given below as Theorem 4.1.6, which gives an exact formula for $S_b(n)$ in terms of a continuous nondifferentiable function with Fourier coefficients involving values of the Riemann zeta function on the imaginary axis. Using an interpolation of Delange's formula we formulate a continuous interpolation of $S_b(n)$ in the base parameter b, permitting definitions of $d_\beta(n)$ and $S_\beta(n)$ for a real parameter $\beta > 1$. We obtain a meromorphic continuation of the associated Dirichlet series $F_\beta(s)$ and $G_\beta(s)$ to the half-planes $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(s) > 1$, respectively. We note apparent fractal properties of $d_\beta(n)$ as β is varied.

4.1.1 Results

Our first results concern the meromorphic continuation of the functions $F_b(s)$ and $G_b(s)$ to the entire complex plane $\mathbb{C}$.

Theorem 4.1.1. *For each integer base $b \geq 2$, the function $F_b(s) = \sum_{n=1}^{\infty} d_b(n) n^{-s}$ has a meromorphic continuation to $\mathbb{C}$. The poles of $F_b(s)$ consist of a double pole at $s = 1$ with Laurent expansion beginning*

$$F_b(s) = \frac{b-1}{2\log b}(s-1)^{-2} + \left(\frac{b-1}{2\log b}\log(2\pi) - \frac{b+1}{4}\right)(s-1)^{-1} + O(1), \tag{4.7}$$

simple poles at each other point $s = 1 + 2\pi i m/\log b$ with $m \in \mathbb{Z}$ $(m \neq 0)$ with residue

$$\mathrm{Res}\left(F_b(s), s = 1 + \frac{2\pi i m}{\log b}\right) = -\frac{b-1}{2\pi i m}\zeta\left(\frac{2\pi i m}{\log b}\right), \tag{4.8}$$

and simple poles at each point $s = 1 - k + 2\pi im/\log b$ with $k = 1$ or $k \geq 2$ an even integer and with $m \in \mathbb{Z}$, with residue

$$\operatorname{Res}\left(F_b(s), s = 1 - k + \frac{2\pi im}{\log b}\right) = (-1)^{k+1} \frac{b-1}{\log b} \zeta\left(\frac{2\pi im}{\log b}\right) \frac{B_k}{k!} \prod_{j=1}^{k-1} \left(\frac{2\pi im}{\log b} - j\right) \tag{4.9}$$

where B_k is the kth Bernoulli number.

Theorem 4.1.1 is proved by first considering the Dirichlet series $\sum \big(d_b(n) - d_b(n-1)\big) n^{-s}$ and then exploiting a relation between power series and Dirichlet series to recover $F_b(s)$. The proof is presented in Section 4.3.

The meromorphic continuation of Dirichlet series attached to b-regular sequences, of which our Dirichlet series $F_b(s)$ is a particular example, was studied by Dumas in his 1993 thesis [17]; this work also showed that the poles of $F_b(s)$ must be contained in a certain half-lattice, strictly larger than the half-lattice here.

A similar method allows us to meromorphically continue the series $G_b(s)$ to the complex plane.

Theorem 4.1.2. *For each integer $b \geq 2$, the function $G_b(s) = \sum_{n=1}^{\infty} S_b(n) n^{-s}$ has a meromorphic continuation to $\mathbb{C}$. The poles of $G_b(s)$ consist of a double pole at $s = 2$ with Laurent expansion*

$$G_b(s) = \frac{b-1}{2\log b}(s-2)^{-2} + \left(\frac{b-1}{2\log b}\big(\log(2\pi) - 1\big) - \frac{b+1}{4}\right)(s-2)^{-1} + O(1), \tag{4.10}$$

a simple pole at $s = 1$ with residue

$$\operatorname{Res}(G_b(s), s = 1) = \frac{b+1}{12}, \tag{4.11}$$

simple poles at $s = 2 + 2\pi im/\log b$ with $m \in \mathbb{Z}$ $(m \neq 0)$ with residue

$$\operatorname{Res}\left(G_b(s), s = 2 + \frac{2\pi im}{\log b}\right) = -\frac{b-1}{2\pi im}\left(1 + \frac{2\pi im}{\log b}\right)^{-1} \zeta\left(\frac{2\pi im}{\log b}\right) \tag{4.12}$$

and simple poles at point $s = 2 - k + 2\pi im/\log b$ with $k \geq 2$ an even integer and $m \in \mathbb{Z}$ with residue

$$\operatorname{Res}\left(G_b(s), s = 2 - k + \frac{2\pi im}{\log b}\right) = \frac{b-1}{\log b} \zeta\left(\frac{2\pi im}{\log b}\right) \left(\frac{B_k}{k(k-2)!}\right) \prod_{j=1}^{k-2} \left(\frac{2\pi im}{\log b} - j\right). \tag{4.13}$$

An interesting feature of the above theorems is the abundance of poles. Since each function $F_b(s)$ and $G_b(s)$ has $\asymp r^2$ poles in the disk $|s| < r$, we have the following corollary, which we discuss further in Section 4.3.3.

Corollary 4.1.3. *The functions $F_b(s)$ and $G_b(s)$ are meromorphic functions of order at least* 2 *on* $\mathbb{C}$.

The Riemann zeta function, the Dirichlet L-functions, and the Dirichlet series generating functions of many important arithmetic functions (such as the Möbius function $\mu(n)$, the von Mangoldt function $\Lambda(n)$, the Euler totient function $\phi(n)$, and the sum-of-diviors functions $\sigma_\alpha(n)$) analytically continue as meromorphic functions of order 1 on the complex plane. The Dirichlet series $F_b(s)$ and $G_b(s)$ thus have a different analytic character than many other Dirichlet series considered in number theory.

In Section 4.4, we use a formula of Delange [15] for $S_b(n)$ to define continuous real-valued interpolations of the functions $d_b(n)$ and $S_b(n)$ from integer bases $b \geq 2$ to a real parameter $\beta > 1$. As before, we associate to these interpolated sum-of-digits functions the Dirichlet series

$$F_\beta(s) = \sum_{n=1}^{\infty} \frac{d_\beta(n)}{n^s} \tag{4.14}$$

and

$$G_\beta(s) = \sum_{n=1}^{\infty} \frac{S_\beta(n)}{n^s}. \tag{4.15}$$

We prove that these Dirichlet series each have a meromorphic continuation one unit to the left of their halfplane of absolute convergence. For the function $F_\beta(s)$ we have the following theorem.

Theorem 4.1.4. *For each real $\beta > 1$, the function $F_\beta(s)$ has a meromorphic continuation to the halfplane* $\mathrm{Re}(s) > 0$*, with a double pole at $s = 1$ with Laurent expansion*

$$F_\beta(s) = \frac{\beta - 1}{2\log\beta}(s-1)^{-2} + \left(\frac{\beta-1}{2\log\beta}\big(\log(2\pi)\big) - \frac{\beta+1}{4}\right)(s-1)^{-1} + O(1) \tag{4.16}$$

and simple poles at $s = 1 + 2\pi i m/\log\beta$ for $m \in \mathbb{Z}$ with $m \neq 0$ with residue

$$\mathrm{Res}\left(F_\beta(s), s = 1 + \frac{2\pi i m}{\log\beta}\right) = -\frac{\beta-1}{2\pi i m}\zeta\left(\frac{2\pi i m}{\log\beta}\right). \tag{4.17}$$

For the function $G_\beta(s)$ we have the following theorem.

Theorem 4.1.5. *For each real $\beta > 1$, the function $G_\beta(s)$ is meromorphic in the region* $\mathrm{Re}(s) > 1$ *with a double pole at $s = 2$ with Laurent expansion*

$$G_\beta(s) = \frac{\beta - 1}{2\log\beta}(s-2)^{-2} + \left(\frac{\beta-1}{2\log\beta}\big(\log(2\pi) - 1\big) - \frac{\beta+1}{4}\right)(s-2)^{-1} + O(1) \tag{4.18}$$

and simple poles at $s = 2 + 2\pi i m/\log\beta$ *for* $m \in \mathbb{Z}$ *with* $m \neq 0$ *with residue*

$$\operatorname{Res}\left(G_b(s), s = 2 + \frac{2\pi i m}{\log\beta}\right) = -\frac{\beta - 1}{2\pi i m}\left(1 + \frac{2\pi i m}{\log\beta}\right)^{-1}\zeta\left(\frac{2\pi i m}{\log\beta}\right). \tag{4.19}$$

To prove these theorems, we start by obtaining the continuation of the series $G_\beta(s)$ by working directly with its Dirichlet series and then obtain the continuation of $F_\beta(s)$ by studying the relation between these two Dirichlet series.

4.1.2 Previous work

There has been much previous work studying the functions $d_b(n)$ and $S_b(n)$. The function $d_b(n)$ exhibits significant fluctuations as n changes to $n + 1$. It can only increase slowly, having $d_b(n+1) \leq d_b(n) + 1$ but it can decrease by an arbitrarily large amount. The sequence $d_b(n)$ is a b-regular sequence in the sense of Allouche and Shallit [3, Ex. 2, Sec. 7] and is a member of the b-th arithmetic fractal group $\Gamma_b(\mathbb{Z})$ of Morton and Mourant [41, p. 256]. Chen et al. [10] survey results on the sum-of-digits function of random integers, and give many references.

Concerning the cumulative sum-of-digits function, Mirsky [39] proved in 1949 that for any integer base $b \geq 2$, the function $S_b(n)$ has the asymptotic

$$S_b(n) = \frac{b-1}{2\log b} n \log n + O(n). \tag{4.20}$$

In 1968 Trollope [54] expressed the error term for the base-2 cumulative digit sum $S_2(n)$ in terms of a continuous everywhere nondifferentiable function, the Takagi function—see [32] for a survey of the properties of this function. In 1975 Delange [15] proved the following formula for $S_b(n)$, expressing the error term as a Fourier series with coefficients involving values of the Riemann zeta function on the imaginary axis.

Theorem 4.1.6 (Delange [15])**.** *The cumulative sum-of-digits function* $S_b(n)$ *satisfies*

$$S_b(n) = \frac{b-1}{2\log b} n \log n + h_b\left(\frac{\log n}{\log b}\right) n \tag{4.21}$$

where h_b *is a nowhere-differentiable function of period* 1*. The function* h_b *has a Fourier series*

$$h_b(x) = \sum_{k=-\infty}^{\infty} c_b(k) e^{2\pi i k x} \tag{4.22}$$

with coefficients

$$c_b(k) = -\frac{b-1}{2\pi i k}\left(1 + \frac{2\pi i k}{\log b}\right)^{-1}\zeta\left(\frac{2\pi i k}{\log b}\right) \tag{4.23}$$

for $k \neq 0$ and

$$c_b(0) = \frac{b-1}{2\log b}\big(\log(2\pi) - 1\big) - \frac{b+1}{4}. \tag{4.24}$$

A complex-analytic proof of a summation formula for general q-additive functions, of which the base-q sum-of-digits function is an example, was given by Mauclaire and Murata in 1983 [37, 38, 43]. A shorter complex-analytic proof of (4.21) in the specific case of $S_2(n)$ was given by Flajolet, Grabner, Kirschenhofer, Prodinger, and Tichy in 1994 [21]. The method of Flajolet et al. is based on applying a variant of Perron's formula to the Dirichlet series

$$\sum_{n=1}^{\infty}\big(d_2(n) - d_2(n-1)\big)n^{-s}. \tag{4.25}$$

Grabner and Hwang [23] study higher moments of the sum-of-digits function by similar complex-analytic methods.

Our formulas for the residues of $F_b(s)$ and $G_b(s)$ involve the Bernoulli numbers. Kellner [28] and Kellner and Sondow [29] investigate another relation between sums of digits and Bernoulli numbers, proving that the least common multiple of the denominators of the coefficients of the polynomial $\sum_{i=0}^{n} n^k$, which can be written in terms of a Bernoulli polynomial, can be expressed as a certain product of primes satisfying $d_p(n+1) \geq p$.

4.2 Sum-of-digits Dirichlet series

First we consider basic properties of the Dirichlet series

$$F_b(s) = \sum_{n=1}^{\infty} \frac{d_b(n)}{n^s} \tag{4.26}$$

attached to the base-b digit sum of n and the Dirichlet series

$$G_b(s) = \sum_{n=1}^{\infty} \frac{S_b(n)}{n^s} \tag{4.27}$$

attached to the cumulative base-b digit sum.

Proposition 4.2.1. *For each integer $b \geq 2$, the Dirichlet series*

$$F_b(s) = \sum_{n=1}^{\infty} \frac{d_b(n)}{n^s} \tag{4.28}$$

converges and defines a holomorphic function for $\mathrm{Re}(s) > 1$.

Proof. A positive integer n has $[\log n/\log b] + 1$ digits when written in base b, each of which is at most $b - 1$, so

$$d_b(n) \le (b-1)\left(\left\lfloor \frac{\log n}{\log b} \right\rfloor + 1\right). \tag{4.29}$$

We then obtain the estimate

$$S_b(n) \ll n \log n \tag{4.30}$$

with an implied constant depending on b. This implies that the Dirichlet series (4.28) has abscissa of absolute convergence at most 1 and therefore defines a holomorphic function for $\mathrm{Re}(s) > 1$. □

Proposition 4.2.2. *For each integer $b \ge 2$, the Dirichlet series*

$$G_b(s) = \sum_{n=1}^{\infty} \frac{S_b(n)}{n^s} \tag{4.31}$$

converges and defines a holomorphic function for $\mathrm{Re}(s) > 2$.

Proof. The estimate (4.30) gives

$$\sum_{m=1}^{n} S_b(m) \ll n^2 \log n, \tag{4.32}$$

which shows that the Dirichlet series (4.31) converges for $\mathrm{Re}(s) > 2$. □

It follows from Delange's formula (4.21) that $F_b(s)$ and $G_b(s)$ have abscissa of absolute convergence $\mathrm{Re}(s) = 1$ and $\mathrm{Re}(s) = 2$, respectively, and this can be proven directly using a more careful estimate of the function $d_b(n)$. We can also obtain the exact values of the abscissas of convergence as a corollary of our theorems on the meromorphic continuation of $F_b(s)$ and $G_b(s)$, since $F_b(s)$ has a pole at $s = 1$ and $G_b(s)$ has a pole at $s = 2$.

As in previous work on Dirichlet series associated to q-additive sequences, it is advantageous to consider the Dirichlet series

$$Z_b(s) = \sum_{n=1}^{\infty} \big(d_b(n) - d_b(n-1)\big) n^{-s} \tag{4.33}$$

obtained by differencing the coefficients of the series $F_b(s)$, setting $d_b(0) = 0$. Identity (4.34) in the following proposition appears in a more general form (for q-additive functions) in the work of Mauclaire and Murata [37, 38, 43] and is stated and proved explicitly for the sum-of-digits series by Allouche and Shallit [2]. We give a more direct proof of this result.

Proposition 4.2.3. *For each integer $b \geq 2$, the Dirichlet series $Z_b(s)$ has abscissa of absolute convergence $\sigma_a = 1$, abscissa of conditional convergence $\sigma_c = 0$, and has a meromorphic continuation to $\mathbb{C}$, satisfying*

$$Z_b(s) = \frac{b^s - b}{b^s - 1}\zeta(s). \tag{4.34}$$

Proof. For bases $b \geq 3$, we have $|d_b(n) - d_b(n-1)| \geq 1$ for all n; if $b = 2$, we have $|d_b(n)-d_b(n-1)| \geq 1$ for at least all odd n. Hence $\sigma_a \geq 1$. We also have $d_b(n)-d_b(n-1) \ll \log n$, so $\sigma_a \leq 1$. The abscissa of conditional convergence $\sigma_c = 0$ follows from the bound

$$\sum_{m \leq n} \big(d_b(m) - d_b(m-1)\big) = d_b(n) \ll \log n. \tag{4.35}$$

The effect of adding 1 on the digit sum in base-b arithmetic depends on the divisibility of n by b; in particular, we have

$$d_b(n) - d_b(n-1) = 1 - k(b-1) \tag{4.36}$$

where k is the largest integer such that $b^k \mid n$. We may also express this as

$$d_b(n) - d_b(n-1) = \sum_{m \mid n} \alpha(m)\beta(m/n) \tag{4.37}$$

where

$$\alpha(n) = \begin{cases} 1 & \text{if } n = b^k \text{ for some } k \\ 0 & \text{otherwise} \end{cases} \qquad \beta(n) = \begin{cases} 1-b & \text{if } b \mid n \\ 1 & \text{if } b \nmid n \end{cases}. \tag{4.38}$$

Then we have, for $\mathrm{Re}(s) > 1$,

$$Z_b(s) = \sum_{n=1}^{\infty} \big(d_b(n) - d_b(n-1)\big) n^{-s} = \sum_{n=1}^{\infty} \Big(\sum_{m \mid n} \alpha(m)\beta(m/n) \Big) n^{-s} \tag{4.39}$$

Writing the right side as a product of two Dirichlet series, we have

$$Z_b(s) = \sum_{n=1}^{\infty} \alpha(n) n^{-s} \sum_{n=1}^{\infty} \beta(n) n^{-s} = \sum_{n=1}^{\infty} b^{-ns} \sum_{n=1}^{\infty} (1-b)(bn)^{-s}. \tag{4.40}$$

Summing the geometric series, we obtain

$$Z_b(s) = \frac{1}{1 - b^{-s}} \big(\zeta(s) - b b^{-s} \zeta(s)\big) = \frac{b^s - b}{b^s - 1}\zeta(s) \tag{4.41}$$

as claimed. Equation (4.34) then provides a meromorphic continuation of $Z_b(s)$ since the right side is meromorphic on $\mathbb{C}$. □

We will obtain information about the meromorphic continuation of $F_b(s)$ and $G_b(s)$ by considering the relation between these series and the series $Z_b(s)$. For future use, we list the poles of the function $Z_b(s)$.

Lemma 4.2.4. *The function $Z_b(s)$ is meromorphic on $\mathbb{C}$, with simple poles at $s = 2\pi i m/\log b$ for $m \in \mathbb{Z}$. The residue at each pole is*

$$\operatorname{Res}\left(Z_b(s), s = \frac{2\pi i m}{\log b}\right) = -\frac{b-1}{\log b}\zeta\left(\frac{2\pi i m}{\log b}\right). \tag{4.42}$$

In particular, at $s = 0$, the function $Z_b(s)$ has a Laurent expansion beginning

$$Z_b(s) = \left(\frac{b-1}{2\log b}\right)s^{-1} + \left(-\frac{b+1}{4} + \frac{b-1}{2\log b}\log(2\pi)\right) + O(s). \tag{4.43}$$

Proof. The function $(b^s - b)/(b^s - 1)$ has simple poles at $s = 2\pi i m/\log b$ for each $m \in \mathbb{Z}$, with residue

$$\operatorname{Res}\left(\frac{b^s - b}{b^s - 1}, s = \frac{2\pi i m}{\log b}\right) = -\frac{b-1}{\log b}. \tag{4.44}$$

The Laurent expansion at $s = 0$ follows from multiplying the expansions

$$\frac{b^s - b}{b^s - 1} = -\frac{b-1}{\log b}s^{-1} + \frac{b+1}{2} + O(s) \tag{4.45}$$

and

$$\zeta(s) = -\frac{1}{2} - \frac{1}{2}\log(2\pi)s + O(s^2). \tag{4.46}$$

The function $\zeta(s)$ has only a simple pole at $s = 1$, cancelled by a zero of $(b^s - b)$. $\square$

4.3 Meromorphic continuation of $F_b(s)$ and $G_b(s)$

In this section, we show that for integers $b \geq 2$, the Dirichlet series $F_b(s)$ and $G_b(s)$ have a meromorphic continuation to $\mathbb{C}$ and determine the structure of the poles, proving Theorems 4.1.1 and 4.1.2.

4.3.1 Meromorphic continuation of $F_b(s)$

We now prove the meromorphic continuation of the Dirichlet series $F_b(s)$ using Theorem 3.4.11 of the previous chapter.

Proof of Theorem 4.1.1. Apply Theorem 3.4.11 to the Dirichlet series $Z_b(s)$ to obtain the meromorphic continuation of $F_b(s)$ to $\mathbb{C}$. We have the formula

$$F_b(s) = \frac{Z_b(s-1)}{s-1} + \frac{1}{2}Z_b(s) + \sum_{k=1}^{K} \frac{B_{2k}}{(2k)!}\Big(\prod_{j=0}^{2k-2}(s+j)\Big)Z_b(s+2k-1) + R_K(s) \tag{4.47}$$

for $R_K(s)$ holomorphic in the halfplane $\mathrm{Re}(s) > 1-2K$. From Lemma 4.2.4, we know $Z_b(s)$ has simple poles at $s = 2\pi i k/\log b$ for $k \in \mathbb{Z}$. The first term

$$\frac{Z_b(s-1)}{s-1} \tag{4.48}$$

has a double pole at $s = 1$ with Laurent expansion beginning

$$\frac{1}{s-1}Z_b(s-1) = \frac{b-1}{2\log b}(s-1)^{-2} + \left(\frac{b-1}{2\log b}\log(2\pi) - \frac{b+1}{4}\right)(s-1)^{-1} + O(1), \tag{4.49}$$

and simple poles at each other point $s = 1 + 2\pi i m/\log b$. Each term

$$\frac{B_{2k}}{(2k)!}\prod_{j=0}^{2k-2}(s+j)\cdot Z_b(s+2k-1) \tag{4.50}$$

has a simple pole at $s = 1-2k+2\pi i m/\log b$ for $m \in \mathbb{Z}$ with residue

$$\frac{B_{2k}}{(2k)!}\prod_{j=1}^{2k-1}\left(\frac{2\pi i m}{\log b} - j\right)\cdot\left(-\frac{b-1}{\log b}\right)\zeta\left(\frac{2\pi i m}{\log b}\right). \tag{4.51}$$

□

4.3.2 Meromorphic continuation of $G_b(s)$

We continue the function $G_b(s)$ to the plane in a similar fashion, using the fact that $S_b(n)$ is a double sum of the differences $d_b(n) - d_b(n-1)$ appearing in the series $Z_b(n)$.

Proof of Theorem 4.1.2. Since

$$\sum_{i+j=n} i\cdot\big(d_b(j) - d_b(j-1)\big) = S_b(n), \tag{4.52}$$

we apply Theorem 3.3.4 using the sequence $B = (0,1,2,3,\ldots)$ with $b_n = n$ for all $n \geq 0$. In the notation of Theorem 3.3.4, we have the generating function

$$q(z) = \sum_{n=0}^{\infty} b_n z^n = \frac{z}{(1-z)^2}. \tag{4.53}$$

In Proposition A.0.3 of the Appendix, we have the series expansion

$$q(e^{-z}) = \frac{1}{z^2}\Big(1 - \sum_{k=2}^{\infty} \frac{B_k}{k(k-2)!} z^k\Big). \tag{4.54}$$

Then, applying Theorem 3.3.4, we find that $G_b(s)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies the formula

$$G_b(s) = \frac{Z_b(s-2)}{(s-1)(s-2)} - \sum_{k=2}^{K} \frac{B_k}{k(k-2)!} \prod_{j=0}^{k-3} (s+j) Z_b(s-2+k) + R_K(s) \tag{4.55}$$

with remainder R_K holomorphic for $\mathrm{Re}(s) > 2 - K$.

As before, we consider the poles of each term of (4.55). The first term

$$\frac{Z_b(s-2)}{(s-1)(s-2)} \tag{4.56}$$

has a double pole at $s = 2$ with Laurent expansion

$$\frac{b-1}{2\log b}(s-2)^{-2} + \left(\frac{b-1}{2\log b}\big(\log(2\pi) - 1\big) - \frac{b+1}{4}\right)(s-2)^{-1} + \cdots, \tag{4.57}$$

a simple pole at each point $s = 2 + 2\pi i m/\log b$ with $m \neq 0$, and a simple pole at $s = 1$ with residue $(b+1)/12$. Each other term

$$\frac{B_k}{k(k-2)!} \prod_{j=0}^{k-3} (s+j) \cdot Z_b(s-2+k). \tag{4.58}$$

has simple poles at $s = 2 - k + 2\pi i m/\log b$ for all m. □

4.3.3 Order of $F_b(s)$ and $G_b(s)$ as meromorphic functions

The functions $F_b(s)$ and $G_b(s)$ are meromorphic functions on the complex plane with infinitely many poles on a left half-lattice. We now raise a further question about the analytic properties of these functions.

By Theorems 4.1.1 and 4.1.2, the functions $F_b(s)$ and $G_b(s)$ each have $\gg r^2$ poles in the disc $|z| < r$. Hence we have the following corollary.

Corollary 4.3.1. *The functions $F_b(s)$ and $G_b(s)$ are meromorphic functions of order at least* 2.

A meromorphic function of order greater than 2 could still have only $O(r^2)$ poles in $|z| < r$, so without further information, we cannot deduce that $F_b(s)$ and $G_b(s)$ have order 2.

Question 4.3.2. Are the functions $F_b(s)$ and $G_b(s)$ mermorphic functions of order exactly 2?

Such a question has been answered positively in the related setting of strongly q-multiplicative functions: the Dirichlet series attached to such functions are entire of order exactly 2 (see Alkauskas [1]).

4.4 Meromorphic continuation of Dirichlet series for non-integer bases

In this section, we consider the problem of extending the digit sums $d_b(n)$ and $S_b(n)$ from integer bases b to real parameters $\beta > 1$. There are a number of possible ways to do this. One natural approach concerns the notion of β-expansions introduced by Renyi [48] and studied at length by Parry [45]. However, for non-integer values of β, the β-expansion of an integer generally has infinitely many digits, so the sum of the digits will generally be infinite. Digit sums related to a different digit expansion with respect to an irrational base were considered by Grabner and Tichy [24].

The approach which we consider in this section is to use the formula of Delange to define the cumulative digit sum $S_\beta(n)$ for real parameters $\beta > 1$, from which we can define a digit sum $d_\beta(n)$ by differencing. The resulting functions are continuous in the β-parameter.

4.4.1 Extension to non-integer bases by Delange's formula

We begin by replacing the integer variable b in Theorem 4.1.6, which gives a formula for $S_b(n)$ for integer bases $b \geq 2$, by a real parameter $\beta > 1$.

Definition 4.4.1. For $\beta \in \mathbb{R}$ with $\beta > 1$, define a generalized cumulative sum-of-digits function $S_\beta(n)$ for integer $n \geq 0$ by

$$S_\beta(n) := \frac{\beta - 1}{2\log\beta} n \log n + h_\beta\left(\frac{\log n}{\log\beta}\right) n, \tag{4.59}$$

where the function $h_\beta(x)$ is defined by the Fourier series

$$h_\beta(x) = \sum_{k=-\infty}^{\infty} c_\beta(k) e^{2\pi i k x} \tag{4.60}$$

with coefficients

$$c_\beta(k) = -\frac{\beta - 1}{2\pi i k}\left(1 + \frac{2\pi i k}{\log\beta}\right)^{-1} \zeta\left(\frac{2\pi i k}{\log\beta}\right) \tag{4.61}$$

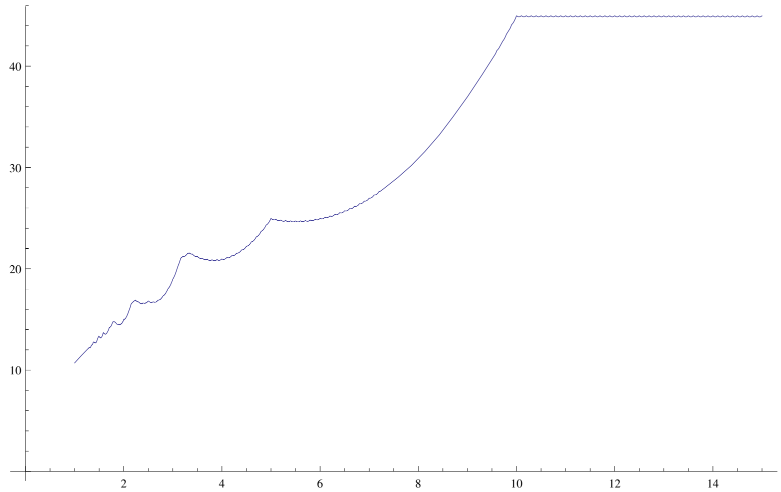

FIGURE 4.1: A plot of $S_\beta(10)$ for $1 \leq \beta \leq 15$, using terms with $|k| \leq 1000$ in the Fourier series for $h_\beta(x)$.

for $k \neq 0$ and

$$c_\beta(0) = \frac{\beta - 1}{2\log\beta}(\log 2\pi - 1) - \frac{\beta + 1}{4}. \tag{4.62}$$

Definition 4.4.2. Define the generalized sum-of-digits function $d_\beta(n)$ for real $\beta > 1$ by

$$d_\beta(n) := S_\beta(n+1) - S_\beta(n). \tag{4.63}$$

A plot of $S_\beta(10)$ as a function of β for $1 \leq \beta \leq 15$ is shown in Figure 4.4.1. Note that $S_\beta(n)$ is approximately constant for $\beta \geq 10$.

4.4.2 The function $h_\beta(x)$

In this section, we study properties of the function $h_\beta(x)$ appearing in Definition 4.4.1 as a function of the variable $\beta > 1$ and as a function of the variable x. When $\beta = b \in \mathbb{N}$, Delange showed that $h_b(x)$ is a continuous but everywhere non-differentiable real-valued function of x with period 1.

Lemma 4.4.3. *For each fixed $\beta > 1$, the function $h_\beta(x)$ is a real-valued continuous function of x on $\mathbb{R}$.*

Proof. The zeta function satisfies the bound

$$|\zeta(it)| \ll t^{1/2+\varepsilon} \tag{4.64}$$

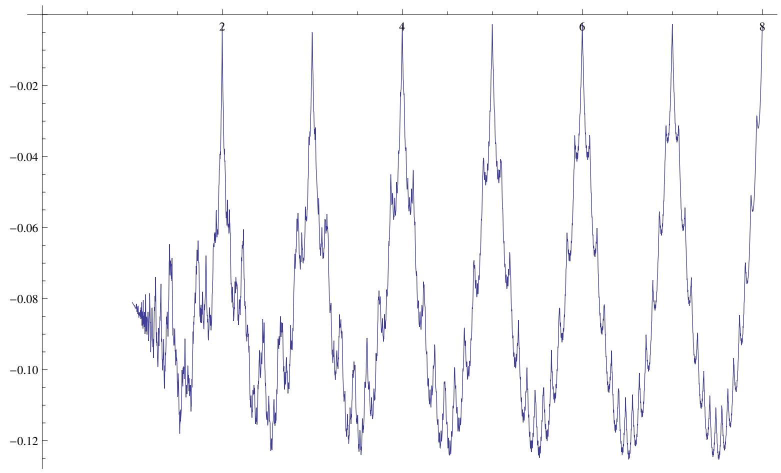

FIGURE 4.2: A plot of $h_\beta(2)$ for $1 \le \beta \le 8$, using terms with $|k| \le 1000$ in the Fourier series for $h_\beta(x)$.

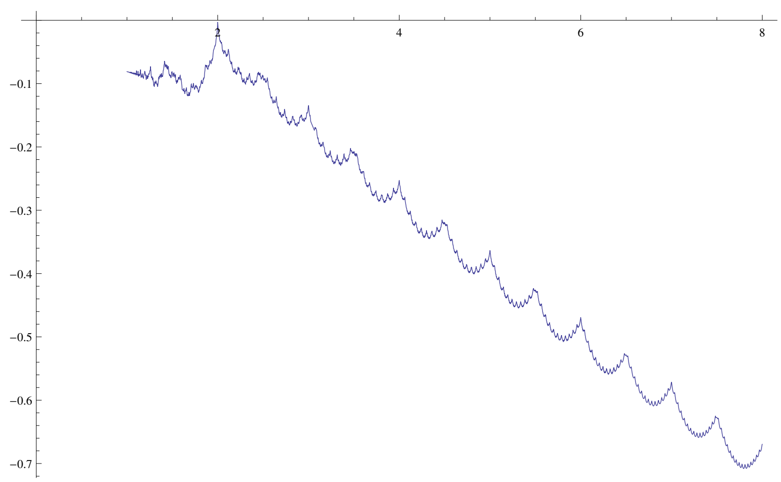

FIGURE 4.3: A plot of $h_\beta(\log 2/\log\beta)$ for $1 \le \beta \le 8$, using terms with $|k| \le 1000$ in the Fourier series for $h_\beta(x)$.

for $t \in \mathbb{R}$ (see for example [53, eq. 5.1.3], so the Fourier coefficients of h_β satisfy

$$c_\beta(k) = -\frac{\beta - 1}{2\pi i k}\left(1 + \frac{2\pi i k}{\log\beta}\right)^{-1} \zeta\left(\frac{2\pi i k}{\log\beta}\right) \ll_\beta k^{-3/2+\varepsilon}. \tag{4.65}$$

This estimate shows that the Fourier series (4.60) is absolutely and uniformly convergent for $x \in \mathbb{R}$, so gives a continuous function of x.

The function $h_\beta(x)$ is real-valued for $x \in \mathbb{R}$ since the Fourier coefficients $c_\beta(k)$ satisfy $\overline{c_\beta(k)} = c_\beta(-k)$. □

A plot of $h_\beta(2)$ as a function of the real parameter β for $1 \le \beta \le 8$ is shown in Figure 4.4.2. From the plot, it also appears that h_β might be non-differentiable as a function of the real parameter β.

Question 4.4.4. For fixed $x \in \mathbb{R}$, is the function $h_\beta(x)$ everywhere non-differentiable as a function of the real variable β?

4.4.3 Meromorphic continuation of $G_\beta(s)$

Our proofs of the meromorphic continuation of $F_b(s)$ and $G_b(s)$ for integer bases relied on the identity

$$Z_b(s) = \sum_{n=1}^{\infty} \big(d_b(n) - d_b(n-1)\big) n^{-s} = \frac{b^s - b}{b^s - 1}\zeta(s). \tag{4.66}$$

If for non-integer $\beta > 1$ we define

$$Z_\beta(s) := \sum_{n=1}^{\infty} \big(d_\beta(n) - d_\beta(n-1)\big) n^{-s}, \tag{4.67}$$

then $Z_\beta(s)$ is *not* equal to

$$\frac{\beta^s - \beta}{\beta^s - 1}\zeta(s) \tag{4.68}$$

as (4.68) is not an ordinary Dirichlet series. We must therefore take a different approach.

We first consider the Dirichlet series generating function

$$G_\beta(s) := \sum_{n=1}^{\infty} \frac{S_\beta(n)}{n^s} \tag{4.69}$$

for $\beta \in \mathbb{R}$ with $\beta > 1$. Since the coefficients satisfy

$$S_\beta(n) \asymp n \log n, \tag{4.70}$$

the Dirichlet series $G_\beta(s)$ has abscissa of absolute convergence $\sigma_a = 2$. We show that the function $G_\beta(s)$ can be analytically continued to a larger halfplane.

Theorem 4.4.5. *For each real $\beta > 1$, the function $G_\beta(s)$ is meromorphic in the region* $\mathrm{Re}(s) > 1$ *with a double pole at $s = 2$ with Laurent expansion*

$$G_\beta(s) = \frac{\beta - 1}{2\log\beta}(s-2)^{-2} + c_\beta(0)(s-2)^{-1} + O(1) \tag{4.71}$$

and simple poles at $s = 2 + 2\pi i k/\log\beta$ for $k \in \mathbb{Z}$ with $k \neq 0$ with residue

$$\mathrm{Res}\left(G_b(s), s = 2 + \frac{2\pi i k}{\log\beta}\right) = c_\beta(k), \tag{4.72}$$

where the numbers $c_\beta(k)$ are those in Definition 4.4.1.

Proof. Using the definition (4.59) of S_β, we have

$$G_\beta(s) = \sum_{n=1}^{\infty} \left(\frac{\beta - 1}{2 \log \beta} n \log n + h_\beta \left(\frac{\log n}{\log \beta} \right) n \right) n^{-s} \tag{4.73}$$

$$= -\frac{\beta - 1}{2 \log \beta} \zeta'(s-1) + \sum_{n=1}^{\infty} h_\beta \left(\frac{\log n}{\log \beta} \right) n^{-(s-1)}. \tag{4.74}$$

The function $\zeta'(s-1)$ is meromorphic on $\mathbb{C}$ with only singularity a double pole at $s = 2$ with Laurent expansion $\zeta'(s-1) = -(s-1)^{-2} + O(1)$. Using the Fourier series (4.60) for h_β, we have

$$\sum_{n=1}^{\infty} h_\beta \left(\frac{\log n}{\log \beta} \right) n^{-(s-1)} = \sum_{n=1}^{\infty} \sum_{k=-\infty}^{\infty} c_\beta(k) \exp\left(2\pi i k \frac{\log n}{\log \beta} \right) n^{-(s-1)} \tag{4.75}$$

$$= \sum_{n=1}^{\infty} \sum_{k=-\infty}^{\infty} c_\beta(k) n^{-(s-1-2\pi i k/\log \beta)}. \tag{4.76}$$

This double sum is absolutely convergent, so we may exchange the sums, giving

$$\sum_{n=1}^{\infty} h_\beta \left(\frac{\log n}{\log \beta} \right) n^{-(s-1)} = \sum_{k=-\infty}^{\infty} c_\beta(k) \zeta \left(s - 1 - \frac{2\pi i k}{\log \beta} \right). \tag{4.77}$$

If $\mathrm{Re}(s) > 1$, then

$$\zeta \left(s - 1 - \frac{2\pi i k}{\log \beta} \right) \ll k^{1/2+\varepsilon} \tag{4.78}$$

for any $\varepsilon > 0$. On any compact set K in the halfplane $\mathrm{Re}(s) > 1$ not containing a point $s = 2 + 2\pi i k/\log \beta)$ for any $k \in \mathbb{Z}$, the sum

$$\sum_{k=-\infty}^{\infty} c_\beta(k) \zeta \left(s - 1 - \frac{2\pi i k}{\log \beta} \right) \tag{4.79}$$

is uniformly convergent on K; if the compact set K contains a point of the form $s = 2 + 2\pi i k_0/\log \beta$, then one term of the sum has a simple pole with residue $c_\beta(k_0)$ while the remaining sum is uniformly convergent. □

When $\beta \geq 2$ is an integer, we know that the function $G_\beta(s)$ has a meromorphic continuation to the entire complex plane.

Question 4.4.6. For noninteger $\beta > 1$, does the Dirichlet series $G_\beta(s)$ have a meromorphic continuation beyond $\mathrm{Re}(s) > 1$?

4.4.4 Meromorphic continuation of $F_\beta(s)$

We now consider the Dirichlet series

$$F_\beta(s) = \sum_{n=1}^{\infty} \frac{d_\beta(n)}{n^s} \tag{4.80}$$

for real $\beta > 1$. We already know that this series has a meromorphic continuation to $\mathbb{C}$ when $\beta \geq 2$ is an integer. We show that for each real $\beta > 1$, the Dirichlet series $F_\beta(s)$ has a meromorphic continuation to the halfplane $\mathrm{Re}(s) > 0$.

Theorem 4.4.7. *The function $F_\beta(s)$ has a meromorphic continuation to the halfplane* $\mathrm{Re}(s) > 0$, *with a double pole at $s = 1$ with Laurent expansion*

$$F_\beta(s) = \frac{\beta-1}{2\log\beta}(s-1)^{-2} + \left(c_\beta(0) + \frac{\beta-1}{2\log\beta}\right)(s-1)^{-1} + O(1) \tag{4.81}$$

and simple poles at $s = 1 + 2\pi i k/\log\beta$ for $k \in \mathbb{Z}$ with $k \neq 0$ with residue

$$\mathrm{Res}\left(F_\beta(s), s = 1 + \frac{2\pi i k}{\log\beta}\right) = \left(1 + \frac{2\pi i k}{\log\beta}\right) c_\beta(k). \tag{4.82}$$

Proof. We follow the method of proof of Theorem 3.3.4. Let

$$p(x) = \sum_{n=2}^{\infty} S_\beta(n) x^n, \tag{4.83}$$

so that

$$\Gamma(s)\big(G_\beta(s) - S_\beta(1)\big) = \int_0^\infty p(e^{-x}) x^{s-1}\, dx. \tag{4.84}$$

By our definition of $d_\beta(n)$, we have

$$\sum_{n=1}^{\infty} d_\beta(n) x^n + S_\beta(1) = (x^{-1} - 1) p(x). \tag{4.85}$$

Hence by Proposition 3.2.1 we have

$$\Gamma(s)\big(F_\beta(s) + S_\beta(1)\big) = \int_0^\infty (e^x - 1) p(e^{-x}) x^{s-1}\, dx \tag{4.86}$$

for $\mathrm{Re}(s) > 1$. Using the power series expansion Then we write

$$\Gamma(s)\big(F_\beta(s) + S_\beta(1)\big) = \Gamma(s+1)\big(G_\beta(s+1) - S_\beta(1)\big) + \int_0^\infty (e^x - 1 - x) p(e^{-x}) x^{s-1}\, dx. \tag{4.87}$$

Dividing by $\Gamma(s)$ and rearranging, we obtain

$$F_\beta(s) = -S_\beta(1)(s+1) + sG_\beta(s+1) + R(s) \tag{4.88}$$

where the remainder term

$$R(s) = \frac{1}{\Gamma(s)} \int_0^\infty (e^x - 1 - x) p(e^{-x}) x^{s-1}\, dx \tag{4.89}$$

is holomorphic in $\mathrm{Re}(s) > 0$ since $e^x - 1 - x \ll x^2$ as $x \to 0^+$. Since $G_\beta(s+1)$ is meromorphic in $\mathrm{Re}(s) > 0$, we find that $F_\beta(s)$ is meromorphic in $\mathrm{Re}(s) > 0$, with poles coming from the poles of $G_\beta(s+1)$. Since

$$sG_\beta(s+1) = (s-1)G_\beta(s+1) + G_\beta(s+1), \tag{4.90}$$

we find that $F_\beta(s)$ has a double pole at $s = 1$ with Laurent expansion as given in the theorem. At each other point $s = 1 + 2\pi i m/\log\beta$, $F_\beta(s)$ has a simple pole. $\square$

Meromorphic continuation of $F_\beta(s)$ to a larger halfplane would follow from continuation of $G_\beta(s)$ to a larger halfplane; in particular, by using more terms of the power series for e^x in formula (4.87), we find that if $G_\beta(s)$ is meromorphic in $\mathrm{Re}(s) > c$ for some c, then $F_\beta(s)$ is meromorphic in $\mathrm{Re}(s) > c - 1$.

Chapter 5

Complex Ramanujan sums and interpolation of the sum-of-divisors function

5.1 Introduction

For an integer m and a positive integer n, the Ramanujan sum $c_n(m)$ is the sum of the mth powers of the nth primitive roots of unity; writing $e(x) = \exp(2\pi i x)$, we have

$$c_n(m) = \sum_{\substack{a=0 \\ (a,n)=1}}^{n-1} e\left(\frac{am}{n}\right). \tag{5.1}$$

Ramanujan sums have been widely studied for over a century; we will survey the relevant history and literature in 5.1.2.

5.1.1 Summary of results

In this chapter, we consider Ramanujan sums with the integer variable m above replaced by a complex number λ, which we call complex Ramanujan sums.

It is well-known that for a fixed integer m, the Dirichlet series generating function of the Ramanujan sums $c_n(m)$ can be expressed in terms of the sum-of-divisors function

$$\sigma_\alpha(m) = \sum_{d|m} d^\alpha \tag{5.2}$$

and the Riemann zeta function:

$$\sum_{n=1}^{\infty} \frac{c_n(m)}{n^s} = \begin{cases} \zeta(s-1)/\zeta(s) & \text{if } m = 0, \\ \sigma_{1-s}(m)/\zeta(s) & \text{if } m \neq 0. \end{cases} \tag{5.3}$$

In Section 5.4 we study the Dirichlet series generating function

$$Z(s,\lambda) = \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{n^s} \tag{5.4}$$

for complex values of the parameter λ and show that as a function of two complex variables s and λ, the function $Z(s,\lambda)$ has a meromorphic continuation to $\mathbb{C}^2$. It is clear from (5.3) that $Z(s,\lambda)$ has a meromorphic continuation to $\mathbb{C}$ in the s variable for fixed $\lambda = m \in \mathbb{Z}$. Our results for general λ are summarized by the following theorem.

Theorem 5.1.1. *The function $\widetilde{Z}(s,\lambda)$ defined for $\mathrm{Re}(s) > 2$ and $\lambda \in \mathbb{C}$ by*

$$\widetilde{Z}(s,\lambda) = \zeta(s) \cdot \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{n^s} \tag{5.5}$$

has a meromorphic continuation as a function of two complex variables to $\mathbb{C}^2$. The polar set of $\widetilde{Z}(s,\lambda)$ is the union of hyperplanes

$$H_1 \cup \left(\bigcup_{k=0}^{\infty} H_{2-2k} \right) \tag{5.6}$$

with $H_k = \{s = k\} \cup \{\lambda \in \mathbb{C}\}$. In particular:

(i) *For fixed $\lambda \in \mathbb{C}$, the function $\widetilde{Z}(s,\lambda)$ is a meromorphic function on $\mathbb{C}$ in the variable s. If $\lambda \notin \mathbb{Z}$, then $\widetilde{Z}(s,\lambda)$ has simple poles at $s = 2 - k$ for $k = 1$ and for even integers $k \geq 0$ with residue*

$$\operatorname*{Res}_{s=2-k} \widetilde{Z}(s,\lambda) = (e(\lambda) - 1)\, \frac{B_k}{k!} (2\pi i \lambda)^{k-1} \tag{5.7}$$

where B_k is the kth Bernoulli number. If $\lambda = 0$ then $\widetilde{Z}(s,\lambda)$ has only a simple pole at $s = 1$ with residue 1. If $\lambda \in \mathbb{Z} \setminus \{0\}$, then $\widetilde{Z}(s,\lambda)$ is an entire function of s.

(ii) *For fixed $s \neq 2 - k$ for $k = 1$ or for even integers $k \geq 0$, the function $\widetilde{Z}(s,\lambda)$ is an entire function of the variable λ.*

The function $\widetilde{Z}(s,\lambda)$ thus gives an analytic interpolation between the functions $\widetilde{Z}(s,0) = \zeta(s-1)$, $\widetilde{Z}(s,1) = 1$, and $\widetilde{Z}(s,m) = \sigma_{1-s}(m)$ ($m \neq 0$ an integer). The functions $\zeta(s-1)$ and $\sigma_{1-s}(m)$ are important functions in the theory of modular forms; specifically, they arise in the Fourier coefficients of both the holomorphic and non-holomorphic Eisenstein series for $\mathrm{SL}(2,\mathbb{Z})$.

We also consider the extension of these results to Cohen's generalization of the Ramanujan sum first introduced in [11], defined for positive integers n and β and integer m by

$$c_n^{(\beta)}(m) = \sum_{\substack{a=0 \\ (a,n^\beta)_\beta=1}}^{n^\beta-1} e\Big(\frac{am}{n^\beta}\Big) \tag{5.8}$$

where $(a, n^\beta)_\beta$ is the greatest common divisor of a and n^β that is a βth power. Note that $c_n^{(\beta)}(m)$ is equal to the classical Ramanujan sum $c_n(m)$ when $\beta = 1$. The Dirichlet series generating function of $c_n^{(\beta)}(m)$ satisfies

$$\sum_{n=1}^{\infty} \frac{c_n^{(\beta)}(m)}{n^s} = \begin{cases} \zeta(s-\beta)/\zeta(s) & \text{if } m = 0 \\ \sigma^\beta_{1-s/\beta}(m)/\zeta(s) & \text{if } m \neq 0 \end{cases} \tag{5.9}$$

where

$$\sigma_\alpha^\beta(n) = \sum_{d^\beta | n} d^{\alpha\beta}. \tag{5.10}$$

We consider the sums $c_n^{(\beta)}(m)$ with the integer variable m replaced by a complex parameter λ and prove the following theorem, which generalizes Theorem 5.1.1.

Theorem 5.1.2. *Let $\beta \geq 1$ be an integer. Then the function $\widetilde{Z}_\beta(s, \lambda)$ defined for* $\mathrm{Re}(s) > \beta$ *and $\lambda \in \mathbb{C}$ by*

$$\widetilde{Z}_\beta(s, \lambda) = \zeta(s) \cdot \sum_{n=1}^{\infty} \frac{c_n^{(\beta)}(\lambda)}{n^s} \tag{5.11}$$

has a meromorphic continuation as a function of two complex variables to $\mathbb{C}^2$. The polar set of $\widetilde{Z}(s, \lambda)$ is the union of hyperplanes

$$H_1 \cup \Big(\bigcup_{k=0}^{\infty} H_{1+\beta-2k\beta}\Big) \tag{5.12}$$

with $H_k = \{s = k\} \cup \{\lambda \in \mathbb{C}\}$. In particular:

(i) *For fixed $\lambda \in \mathbb{C}$, the function $\widetilde{Z}_\beta(s, \lambda)$ is a meromorphic function on $\mathbb{C}$ in the variable s. If $\lambda \notin \mathbb{Z}$, then $\widetilde{Z}_\beta(s, \lambda)$ has simple poles at $s = 1 + \beta - k\beta$ for $k = 1$ and for even integers $k \geq 0$ with residue*

$$\operatorname*{Res}_{s=1+\beta-k\beta} \widetilde{Z}_\beta(s, \lambda) = (e(\lambda) - 1)\, \frac{B_k}{k!} (2\pi i \lambda)^{k-1} \tag{5.13}$$

where B_k is the kth Bernoulli number. If $\lambda = 0$ then $\widetilde{Z}_\beta(s, \lambda)$ has only a simple pole at $s = 1 + \beta$ with residue 1*. If $\lambda \in \mathbb{Z} \setminus \{0\}$, then $\widetilde{Z}_\beta(s, \lambda)$ is an entire function of s.*

(ii) *For fixed $s \neq 1 + \beta - k\beta$ for $k = 1$ or for even integers $k \geq 0$, the function $\widetilde{Z}_\beta(s, \lambda)$ is an entire function of the variable λ.*

5.1.2 History and related work

Although the sums $c_n(m)$ now bear Ramanujan's name, they appeared in earlier works of Dirichlet and Dedekind [16], von Sterneck, Kluyver [30], Landau [34], and Jensen [27].

The sums $c_n(m)$ were studied by Ramanujan in a paper of 1918 [47], in which he obtained expansions of many important arithmetic functions as sums of the form $\sum_n a_n c_n(m)$, now known as Ramanujan expansions or Ramanujan-Fourier expansions, or of the form $\sum_m a_m c_n(m)$. Among the many interesting expansions listed by Ramanujan, one finds

$$d(m) = -\sum_{n=1}^{\infty} \frac{\log n}{n} c_n(m) \tag{5.14}$$

where $d(m)$ is the number of divisors of m and

$$\sigma(m) = \frac{\pi^2}{6} m \sum_{n=1}^{\infty} \frac{c_n(m)}{n^2} \tag{5.15}$$

where $\sigma(m)$ is the sum of the divisors of m. The expansion

$$0 = \sum_{n=1}^{\infty} \frac{c_n(m)}{n} \tag{5.16}$$

holds for all $m > 0$ and is equivalent to the prime number theorem. As an example of an expansion in the other variable, one has

$$\Lambda(n) = -\sum_{m=1}^{\infty} \frac{c_n(m)}{m}. \tag{5.17}$$

Hardy [25] continued this study, and found other expansions including

$$\frac{\phi(m)}{m} \Lambda(m) = \sum_{n=1}^{\infty} \frac{\mu(n)}{\phi(n)} c_n(m). \tag{5.18}$$

In the century since Ramanujan introduced these expansions, many authors have studied Ramanujan-Fourier expansions more systematically, establishing a theory similar in some ways to the classical theory of Fourier series. See [46] for a recent survey of the subject.

Ramanujan sums make other appearances in analytic number theory, often in the study of important questions concerning the distribution of prime numbers. The sums play a crucial role in Vinogradov's proof that every sufficiently large odd integer is a sum of three primes. Gadiyar and Padma [22] showed that if Hardy's expansion (5.18) were absolutely convergent, then an analogue Weiner-Khinchin theorem of Fourier analysis for absolutely convergent expansions of the form $\sum a_n c_n(m)$ would prove the twin prime conjecture.

Finally, we note that Ramanujan sums are a special case of more general sums appearing in number theory, including Gauss sums and Kloosterman sums. Given a Dirichlet character χ mod n and an additive character ψ mod n, the ***Gauss sum*** $\tau(\chi, \psi)$ is defined by

$$\tau(\chi, \psi) = \sum_{a=1}^{n} \chi(a)\psi(a). \tag{5.19}$$

Taking χ to be the principal character χ_0 mod n and ψ to be the additive character $\psi_m(a) = e(am/n)$, we recover the Ramanujan sum

$$\tau\big(\chi_0\,,\, e(\,\cdot\, m/n)\big) = c_n(m). \tag{5.20}$$

For integers a and b mod n, the ***Kloosterman sum*** $S(a, b; n)$ is defined by

$$S(a, b; n) = \sum_{\substack{c=1\\(c,n)=1}}^{n} e\Big(\frac{ac + bc^{-1}}{n}\Big) \tag{5.21}$$

where c^{-1} is the multiplicative inverse of c mod n. Thus if $b = 0$, we have the Ramanujan sum

$$S(a, 0; n) = c_n(a). \tag{5.22}$$

Cohen introduced the generalization $c_n^{(\beta)}(m)$ in [11] and studied further properties in [12, 13]. Kühn and Robles [31] study analytic properties of the Dirichlet series (5.9); among their results is an explicit formula for $\sum_{n\le x} c_n^{(\beta)}(m)$ in terms of zeros of the zeta function.

Singer [51] considers (in a different notation) the series

$$\sum e\Big(-\lambda\frac{a}{b}\Big)b^{-s} \tag{5.23}$$

where the sum ranges over all rationals a/b in the interval $[0, 1)$ and $\lambda \in [0, 1]$ is a real parameter. We note that this series (5.23) is equal to $Z(s, -\lambda)$, where Z is the function defined above in (5.4). Singer proves that (5.23) is convergent (pointwise) for $\lambda \in [0, 1]$, and notes that this gives an interpolation between $1/\zeta(s)$ and $\zeta(s-1)/\zeta(s)$. In this chapter, we extend Singer's work by considering arbitrary complex values of λ and by considering the analytic character of the function $Z(s, \lambda)$ in each variable.

5.2 Classical Ramanujan sums

In this section, we review some properties of the classical Ramanujan sums.

As above, for positive integers m and n, the Ramanujan sum $c_n(m)$ is the sum of the mth powers of the nth primitive roots of unity:

$$c_n(m) = \sum_{\substack{a=0 \\ (a,n)=1}}^{n-1} e\left(\frac{am}{n}\right). \tag{5.24}$$

Note that $c_n(m)$ is unchanged if the sum in (5.24) is replaced by the sum over any other set of reduced residues mod n. Letting ζ_n denote a primitive nth root of unity, we note that $c_n(m)$ is the trace from $\mathbb{Q}(\zeta_n)$ to $\mathbb{Q}$ of the algebraic integer ζ_n^m; this shows that $c_n(m)$ is always a rational integer.

We also note that $c_n(m)$ is the sum of a character of the additive group $\mathbb{Z}/m\mathbb{Z}$ over the multiplicative group $(\mathbb{Z}/m\mathbb{Z})^\times$.

The classical Ramanujan sums $c_n(m)$ are multiplicative in the variable n; that is, they satisfy

$$c_{n_1 n_2}(m) = c_{n_1}(m) c_{n_2}(m) \quad \text{if } (n_1, n_2) = 1. \tag{5.25}$$

The sums satisfy the relation

$$\sum_{d|n} c_d(m) = \begin{cases} n & \text{if } n \mid m, \\ 0 & \text{otherwise,} \end{cases} \tag{5.26}$$

which by Möbius inversion gives

$$c_n(m) = \sum_{d|(n,m)} d\mu(n/d). \tag{5.27}$$

By explictly computing $c_n(m)$ for $n = p^k$ a prime power, one finds that

$$c_n(m) = \frac{\mu\left(n/(n,m)\right)}{\phi\left(n/(n,m)\right)} \phi(n), \tag{5.28}$$

from which one notes in particular that $c_n(1) = \mu(n)$ for all $n \in \mathbb{N}$. We note also that $c_n(0) = \phi(n)$ for all $n \in \mathbb{N}$.

The divisor-sum relation (5.26) and the fact that $c_n(0) = \phi(n)$ imply the following formulas for the Dirichlet series generating function:

$$\sum_{n=1}^{\infty} \frac{c_n(m)}{n^s} = \begin{cases} \dfrac{\sigma_{1-s}(m)}{\zeta(s)} & \text{if } m \neq 0 \\ \dfrac{\zeta(s-1)}{\zeta(s)} & \text{if } m = 0. \end{cases} \tag{5.29}$$

We also have the related interesting formulas

$$\sum_{m=1}^{\infty} \frac{c_n(m)}{m^s} = \zeta(s) \sum_{d|n} \mu\left(\frac{n}{d}\right) d^{1-s} \tag{5.30}$$

and

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{c_n(m)}{m^r} n^s = \frac{\zeta(s)\zeta(r+s-1)}{\zeta(r)}. \tag{5.31}$$

5.3 Complex Ramanujan sums

We now consider extending the Ramanujan sum $c_n(m)$ to complex values of the parameter m.

Definition 5.3.1. For $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, the ***complex Ramanujan sum*** $c_n(\lambda)$ is

$$c_n(\lambda) = \sum_{\substack{a=0 \\ (a,n)=1}}^{n-1} e\Big(\frac{a\lambda}{n}\Big). \tag{5.32}$$

Definition 5.3.2. For $n \in \mathbb{N}$, $\beta \in \mathbb{N}$, and $\lambda \in \mathbb{C}$, the ***generalized complex Ramanujan sum*** $c_n^{(\beta)}(\lambda)$ is

$$c_n^{(\beta)}(\lambda) = \sum_{\substack{a=0 \\ (a,n^\beta)_\beta=1}}^{n^\beta-1} e\Big(\frac{a\lambda}{n^\beta}\Big) \tag{5.33}$$

where $(a, n^\beta)_\beta$ is the greatest common divisor of a and n^β that is the βth power of an integer.

When $\lambda \in \mathbb{Z}$, (5.32) reduces to the classical Ramanujan sum. The generalized sum $c_n^{(\beta)}(\lambda)$ reduces to $c_n(\lambda)$ when $\beta = 1$.

The sums $c_n(\lambda)$ and $c_n^{(\beta)}(\lambda)$ are periodic holomorphic functions of the variable λ, satisfying

$$c_n(\lambda + n) = c_n(\lambda), \qquad c_n^{(\beta)}(\lambda + n^\beta) = c_n^{(\beta)}(\lambda). \tag{5.34}$$

Remark. Recall that in the (5.24), the sum $c_n(m)$ can be thought of as a sum over $(\mathbb{Z}/n\mathbb{Z})^\times$, as the value of $c_n(m)$ is independent of the choice of representatives $a \in \mathbb{Z}$ used in the definition. In contrast, in (5.32), we have fixed a choice of reduced residues mod n, and the value of $c_n(\lambda)$ does depend on our choice.

5.3.1 Basic properties of complex Ramanujan sums

We begin by showing some basic properties of our complex Ramanujan sums $c_n(\lambda)$ and $c_n^{(\beta)}(\lambda)$. Since $c_n^{(1)}(\lambda) = c_n(\lambda)$, we will generally consider $c_n^{(\beta)}(\lambda)$.

Lemma 5.3.3. *For $\beta \in \mathbb{N}$, if $\lambda \notin \mathbb{Z}$, then*

$$\sum_{d|n} c_d^{(\beta)}(\lambda) = \frac{1 - e(\lambda)}{1 - e(\lambda/n^\beta)}. \tag{5.35}$$

If $\lambda \in \mathbb{Z}$, then

$$\sum_{d|n} c_d^{(\beta)}(\lambda) = \begin{cases} n^\beta & \text{if } n^\beta \mid \lambda, \\ 0 & \text{if } n^\beta \nmid \lambda. \end{cases} \tag{5.36}$$

Note that (5.35) holds for all $\lambda \in \mathbb{C}$ if the expression on the right is extended by continuity to $\lambda \in \mathbb{Z}$.

Proof. For any $\lambda \in \mathbb{C}$ we have

$$\sum_{d|n} c_d^{(\beta)}(n) = \sum_{d|n} \sum_{\substack{a=0 \\ (a,d^\beta)_\beta=1}}^{d^\beta-1} e\left(\frac{a\lambda}{d^\beta}\right) = \sum_{a=0}^{n^\beta-1} e\left(\frac{a\lambda}{n^\beta}\right). \tag{5.37}$$

If $\lambda \in \mathbb{Z}$ and $n^\beta \mid \lambda$, then $e(a\lambda/n^\beta) = 1$ for all a, so

$$\sum_{d|n} c_d^{(\beta)}(n) = n^\beta. \tag{5.38}$$

Otherwise, the formula for a finite geometric sum gives (5.35). □

5.4 The Dirichlet series generating function of the complex Ramanujan sums

The Dirichlet series generating function of the complex Ramanujan sums $c_n(\lambda)$ is

$$Z(s, \lambda) = \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{n^s}. \tag{5.39}$$

We also consider the Dirichlet series generating function of the sums $c_n^{(\beta)}(\lambda)$

$$Z_\beta(s,\lambda) = \sum_{n=1}^{\infty} \frac{c_n^{(\beta)}(\lambda)}{n^s}. \tag{5.40}$$

Note that $Z_1(s,\lambda) = Z(s,\lambda)$.

In this section, we study $Z_\beta(s,\lambda)$ as a function of two complex variables s and λ.

Lemma 5.4.1. *For $\beta \in \mathbb{N}$, the function $Z_\beta(s,\lambda)$ is a holomorphic function for $(s,\lambda) \in \{\mathrm{Re}(s) > \beta+1\} \times \mathbb{C}$.*

Proof. The trivial estimate

$$\left|c_n^{(\beta)}(\lambda)\right| \le \sum_{\substack{a=0\\(a,n^\beta)_\beta=1}}^{n^\beta-1} \exp\Big(\frac{-2\pi a\,\mathrm{Im}(\lambda)}{n^\beta}\Big) \le \exp(-2\pi a\,\mathrm{Im}(\lambda))n^\beta \tag{5.41}$$

shows that for each fixed $\lambda \in \mathbb{C}$, (5.40) is absolutely convergent in the halfplane $\mathrm{Re}(s) > \beta+1$ and thus defines a holomorphic function of the variable s in that halfplane.

Furthermore, for fixed s with $\mathrm{Re}(s) > \beta+1$, the series (5.40) is uniformly convergent for λ varying in any compact subset of $\mathbb{C}$ (by our estimate (5.41)), so $Z_\beta(s,\lambda)$ is an entire function of λ for each fixed s with $\mathrm{Re}(s) > \beta+1$. Thus the Dirichlet series definition (5.40) defines $Z_\beta(s,\lambda)$ as a holomorphic function for $(s,\lambda) \in \{\mathrm{Re}(s) > \beta+1\} \times \mathbb{C}$. □

Remark. For certain values of $\lambda \in \mathbb{C}$, the series (5.39) is convergent in a wider halfplane; for example, at $\lambda = 1$, the series

$$Z(s,1) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \tag{5.42}$$

has abscissa of absolute convergence $\sigma_a = 1$, and the Riemann Hypothesis is equivalent to $\sigma_c = 1/2$.

5.4.1 Meromorphic continuation

We showed above that for each fixed $\lambda \in \mathbb{C}$, the function $Z_\beta(s,\lambda)$ is holomorphic in the halfplane $\mathrm{Re}(s) > \beta+1$ and that for each fixed s in that halfplane, $Z_\beta(s,\lambda)$ is an entire function of λ. Our goal in this section is to continue $Z_\beta(s,\lambda)$ to a meromorphic function on all of $\mathbb{C}^2$.

It will be more convenient to work with the functions

$$\widetilde{Z}(s,\lambda) = \zeta(s)Z(s,\lambda). \tag{5.43}$$

and

$$\widetilde{Z}_\beta(s,\lambda) = \zeta(s) Z_\beta(s,\lambda). \tag{5.44}$$

Since $\zeta(s)$ is meromorphic on $\mathbb{C}$, it will be sufficient to obtain the continuation of $\widetilde{Z}_\beta(s,\lambda)$ to $\mathbb{C}^2$. We note that in this notation, the formulas (5.3) become

$$\widetilde{Z}(s,0) = \zeta(s-1) \tag{5.45}$$

and

$$\widetilde{Z}(s,m) = \sigma_{1-s}(m) \tag{5.46}$$

for $m \in \mathbb{Z}$ with $m \neq 0$.

By formula (5.35), we may write the function $\widetilde{Z}_\beta(s,\lambda)$ as the Dirichlet series

$$\widetilde{Z}_\beta(s,\lambda) = \sum_{n=1}^{\infty} \frac{1-e(\lambda)}{1-e(\lambda/n^\beta)} n^{-s}. \tag{5.47}$$

Since the Dirichlet series for $\zeta(s)$ and $Z(s,\lambda)$ are absolutely convergent for $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(s) > \beta + 1$, respectively, we know that (5.47) is absolutely convergent in the halfplane $\mathrm{Re}(s) > \beta + 1$.

The meromorphic continuation of $\widetilde{Z}_\beta(s,\lambda)$ will be deduced from the following proposition.

Proposition 5.4.2. *Fix $M \in \mathbb{N}$ and $\beta \in \mathbb{N}$. Then for $\lambda \in \mathbb{C}$ with $|\lambda| < M^\beta$ and for* $\mathrm{Re}(s) > \beta + 1$*, the function $\widetilde{Z}_\beta(s,\lambda)$ satisfies*

$$\widetilde{Z}_\beta(s,\lambda) = \sum_{n=1}^{M-1} \frac{1-e(\lambda)}{1-e(\lambda/n^\beta)} n^{-s} + \frac{e(\lambda)-1}{2\pi i\lambda} \sum_{k=0}^{\infty} \frac{B_k}{k!} (2\pi i\lambda)^k \sum_{n=M}^{\infty} n^{-s+\beta(1-k)} \tag{5.48}$$

where B_k is the kth Bernoulli number. In particular, for $|\lambda| < 1$, we have

$$\widetilde{Z}_\beta(s,\lambda) = \frac{e(\lambda)-1}{2\pi i\lambda} \sum_{k=0}^{\infty} \frac{B_k}{k!} (2\pi i\lambda)^k \zeta(s+\beta(k-1)). \tag{5.49}$$

Proof. Suppose that $\lambda < M^\beta$ and that $\sigma > \beta + 1$. We write $\widetilde{Z}_\beta(s,\lambda)$ as

$$\widetilde{Z}_\beta(s,\lambda) = \sum_{n=1}^{M-1} \frac{1-e(\lambda)}{1-e(\lambda/n^\beta)} n^{-s} + \sum_{n=M}^{\infty} \frac{1-e(\lambda)}{1-e(\lambda/n^\beta)} n^{-s}. \tag{5.50}$$

We rewrite the second sum to get

$$\widetilde{Z}_\beta(s,\lambda) = \sum_{n=1}^{M-1} \frac{1-e(\lambda)}{1-e(\lambda/n^\beta)} n^{-s} + \frac{e(\lambda)-1}{2\pi i\lambda} \sum_{n=M}^{\infty} \frac{2\pi i \frac{\lambda}{n^\beta}}{e(\lambda/n^\beta)-1} n^{-s+\beta}. \tag{5.51}$$

Since $|\lambda/n^\beta| < 1$ for every term in the second sum, we can apply the series expansion

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k \qquad \text{for } |z| < 2\pi \tag{5.52}$$

to get

$$\widetilde{Z}_\beta(s,\lambda) = \sum_{n=1}^{M-1} \frac{1 - e(\lambda)}{1 - e(\lambda/n^\beta)} n^{-s} + \frac{e(\lambda) - 1}{2\pi i \lambda} \sum_{n=M}^{\infty} \sum_{k=0}^{\infty} \frac{B_k}{k!} \Big(2\pi i \frac{\lambda}{n^\beta}\Big)^k n^{-s+\beta}. \tag{5.53}$$

The double sum is absolutely convergent so the sums may be interchanged, giving

$$\widetilde{Z}_\beta(s,\lambda) = \sum_{n=1}^{M-1} \frac{1 - e(\lambda)}{1 - e(\lambda/n^\beta)} n^{-s} + \frac{e(\lambda) - 1}{2\pi i \lambda} \sum_{k=0}^{\infty} \frac{B_k}{k!} (2\pi i \lambda)^k \sum_{n=M}^{\infty} n^{-s+\beta(1-k)}, \tag{5.54}$$

as stated in the proposition. □

We already know that $\widetilde{Z}_\beta(s,0) = \zeta(s-\beta)$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = \beta + 1$, and that for $\lambda \in \mathbb{Z} \setminus \{0\}$, $\widetilde{Z}_\beta(s,\lambda) = \sigma^{(\beta)}_{1-s/\beta}(\lambda)$ is entire in s. We now show that even when $\lambda \notin \mathbb{Z}$, the function $\widetilde{Z}_\beta(s,\lambda)$ has a meromorphic continuation to $\mathbb{C}$ in s.

Proposition 5.4.3. *For each $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ and $\beta \in \mathbb{N}$, the function $\widetilde{Z}_\beta(s,\lambda)$ has a meromorphic continuation to $\mathbb{C}$ in the s variable with a simple pole at $s = 1 + \beta - k\beta$ for $k = 1$ or $k \geq 0$ an even integer. At the point $s = 1 + \beta - k\beta$, the function $\widetilde{Z}_\beta(s,\lambda)$ has residue*

$$\operatorname*{Res}_{s=1+\beta-k\beta} \widetilde{Z}_\beta(s,\lambda) = \big(e(\lambda) - 1\big) \frac{B_k}{k!} (2\pi i \lambda)^{k-1} \tag{5.55}$$

where B_k is the kth Bernoulli number.

Proof. Fix λ, and fix an arbitrary $M \in \mathbb{N}$ with $M^\beta > |\lambda|$. Let $\sigma_0 < 0$ be an integer. We will show that $\widetilde{Z}_\beta(s,\lambda)$ is meromorphic in the region $\mathrm{Re}(s) > \sigma_0$; since σ_0 is arbitrary, this shows that for our fixed λ, the function $\widetilde{Z}(s,\lambda)$ is meromorphic on all of $\mathbb{C}$.

Set $k_0 = (1 - \sigma_0)/\beta + 1$. We begin by writing (5.48) as

$$\begin{aligned}\widetilde{Z}_\beta(s,\lambda) = \sum_{n=1}^{M-1} \frac{1 - e(\lambda)}{1 - e(\lambda/n^\beta)} n^{-s} &+ \frac{e(\lambda) - 1}{2\pi i \lambda} \sum_{k=0}^{k_0} \frac{B_k}{k!} (2\pi i \lambda)^k \sum_{n=M}^{\infty} n^{-s+(1-k)\beta} \\ &+ \frac{e(\lambda) - 1}{2\pi i \lambda} \sum_{k=k_0}^{\infty} \frac{B_k}{k!} (2\pi i \lambda)^k \sum_{n=M}^{\infty} n^{-s+(1-k)\beta}, \end{aligned} \tag{5.56}$$

initially valid for $\mathrm{Re}(s) > 1 + \beta$ as stated in Proposition 5.4.2. The first sum

$$\sum_{n=1}^{M-1} \frac{1 - e(\lambda)}{1 - e(\lambda/n^{\beta})} n^{-s} \tag{5.57}$$

in (5.56) is an entire function of s. In the second sum,

$$\sum_{n=M}^{\infty} n^{-s+(1-k)\beta} = \zeta(s + (k-1)\beta) - \sum_{n=1}^{M-1} n^{-s+(1-k)\beta} \tag{5.58}$$

continues to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1 - (k-1)\beta$ with residue 1. Recalling that $B_k = 0$ when $k \geq 3$ is odd, we find that

$$\frac{e(\lambda) - 1}{2\pi i \lambda} \sum_{k=0}^{2-\sigma_0} \frac{B_k}{k!} (2\pi i \lambda)^k \sum_{n=M}^{\infty} n^{-s+(1-k)\beta} \tag{5.59}$$

continues to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1 + \beta - k\beta$ for $k \geq 1$ or $k \geq 0$ an even integer with residue given by (5.55).

In the third sum

$$\frac{e(\lambda) - 1}{2\pi i \lambda} \sum_{k=k_0}^{\infty} \frac{B_k}{k!} (2\pi i \lambda)^k \sum_{n=M}^{\infty} n^{-s+(1-k)\beta}, \tag{5.60}$$

we have that

$$\sum_{n=M}^{\infty} n^{-s+(1-k)\beta} \tag{5.61}$$

converges on $\mathrm{Re}(s) > \sigma_0$ by our choice of k_0. Further, we have

$$\left| \sum_{n=M}^{\infty} n^{-s+(1-k)\beta} \right| \ll M^{-k\beta} \tag{5.62}$$

as $k \to \infty$ with constant independent of s for s varying in a compact subset of $\mathrm{Re}(s) > \sigma_0$. Therefore, since (5.52) converges for $|z| < 2\pi$ and since $|\lambda| < M^{\beta}$, the sum (5.60) is uniformly convergent on any compact subset of $\mathrm{Re}(s) > \sigma_0$ and defines a holomorphic function in that region. □

We have already shown that for fixed s with $\mathrm{Re}(s) > \beta + 1$, the function $\widetilde{Z}_{\beta}(s, \lambda)$ is entire in the variable λ. We now show that $\widetilde{Z}_{\beta}(s, \lambda)$ is entire in λ for any fixed s away from the poles located in the previous proposition.

Proposition 5.4.4. *For $\beta \in \mathbb{N}$ and for fixed s with $s \neq 1 + (1-k)\beta$ for $k = 1$ or $k \geq 0$ an even integer, the function $\widetilde{Z}_{\beta}(s, \lambda)$ is an entire function of λ.*

Proof. Let s be as stated in the proposition. We fix an arbitrary integer $M > 0$ and show that $\widetilde{Z}(s, \lambda)$ is holomorphic as a function of λ in the disc $|\lambda| < M^{1/\beta}$.

Write $\widetilde{Z}_\beta(s, \lambda)$ as in Proposition 5.4.2:

$$\widetilde{Z}_\beta(s, \lambda) = \sum_{n=1}^{M-1} \frac{1 - e(\lambda)}{1 - e(\lambda/n^\beta)} n^{-s} + \frac{e(\lambda) - 1}{2\pi i \lambda} \sum_{k=0}^{\infty} \frac{B_k}{k!} (2\pi i \lambda)^k \sum_{n=M}^{\infty} n^{-s+\beta(1-k)}. \tag{5.63}$$

The first sum

$$\sum_{n=1}^{M-1} \frac{1 - e(\lambda)}{1 - e(\lambda/n^\beta)} n^{-s} \tag{5.64}$$

is easily seen to be an entire function of the variable λ; note that $1 - e(\lambda)$ vanishes when $\lambda \in \mathbb{Z}$ so the factor of $(1 - e(\lambda))/(1 - e(\lambda/n^\beta))$ does not contribute any singularities. In the second sum

$$\frac{e(\lambda) - 1}{2\pi i \lambda} \sum_{k=0}^{\infty} \frac{B_k}{k!} (2\pi i \lambda)^k \sum_{n=M}^{\infty} n^{-s+\beta(1-k)}, \tag{5.65}$$

we have

$$\sum_{n=M}^{\infty} n^{-s+\beta(1-k)} \sim M^{-\beta k} \quad \text{as } k \to \infty. \tag{5.66}$$

so that the power series (5.65) converges and defines a holomorphic function in the disc $\lambda < M^\beta$. Since $M > |\lambda|^{1/\beta}$ was arbitrary, this shows that $\widetilde{Z}_\beta(s, \lambda)$ is entire in the variable λ. □

Since $\widetilde{Z}_\beta(s, \lambda)$ is entire in the variable λ for fixed s avoiding the poles, it has a power series expansion in λ which is convergent on all of $\mathbb{C}$.

Proposition 5.4.5. *For $\beta \in \mathbb{N}$ and for fixed s with $s \neq 1 + (1 - k)\beta$ for $k = 1$ or $k \geq 0$ an even integer, the function $\widetilde{Z}_\beta(s, \lambda)$ has the power series expansion*

$$\widetilde{Z}_\beta(s, \lambda) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{B_k}{k!(n - k + 1)!} \zeta(s + \beta(k - 1)) \right) (2\pi i)^n \lambda^n \tag{5.67}$$

convergent for all $\lambda \in \mathbb{C}$.

Proof. We start with formula (5.49)

$$\widetilde{Z}_\beta(s, \lambda) = \frac{e(\lambda) - 1}{2\pi i \lambda} \sum_{k=0}^{\infty} \frac{B_k}{k!} (2\pi i \lambda)^k \zeta(s + \beta(k - 1)) \tag{5.68}$$

which is valid for $|\lambda| < 1$. Using the power series expansion for the exponential function, we have

$$\frac{e(\lambda) - 1}{2\pi i\lambda} = \sum_{k=0}^{\infty} \frac{(2\pi i\lambda)^k}{(k+1)!}. \tag{5.69}$$

Then equation (5.68) gives

$$\widetilde{Z}_\beta(s, \lambda) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{B_k}{k!(n-k+1)!} \zeta(s + \beta(k-1)) \right) (2\pi i)^n \lambda^n. \tag{5.70}$$

This power series expansion must be valid for all $\lambda \in \mathbb{C}$ since $\widetilde{Z}_\beta(s, \lambda)$ is entire in λ. □

5.4.2 Formulas in terms of zeta values

In the previous section, we wrote Proposition 5.4.2, the residue of $\widetilde{Z}_\beta(s, \lambda)$ at each pole, and the power series expansion (5.67) in terms of the Bernoulli numbers. We now restate some of these results in terms of values of the zeta function at nonnegative even integers. Recall that

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!} \quad \text{for } n \geq 0, \tag{5.71}$$

that $B_1 = -1/2$, and that $B_k = 0$ for odd $k \geq 3$.

We begin be writing the expansion (5.49) for $\widetilde{Z}_\beta(s, \lambda)$ with $|\lambda| < 1$ in terms of zeta values. Formula (5.48) may be written similarly.

Proposition 5.4.6. *If $\beta \in \mathbb{N}$ and $|\lambda| < 1$, then*

$$\widetilde{Z}_\beta(s, \lambda) = \frac{1 - e(\lambda)}{2} \zeta(s) + \frac{1 - e(\lambda)}{\pi i\lambda} \sum_{k=0}^{\infty} \zeta(2k)\zeta(s + (2k-1)\beta)\lambda^{2k}. \tag{5.72}$$

Proof. Writing the $k = 1$ term of (5.49) separately, we obtain

$$\widetilde{Z}_\beta(s, \lambda) = \frac{1 - e(\lambda)}{2} \zeta(s) + \frac{e(\lambda) - 1}{2\pi i\lambda} \sum_{\substack{k=0 \\ k \neq 1}}^{\infty} \frac{B_k}{k!} (2\pi i\lambda)^k \zeta(s + \beta(1-k)). \tag{5.73}$$

Since $B_k = 0$ for odd $k \geq 3$, we can reindex the sum to get

$$\widetilde{Z}_\beta(s, \lambda) = \frac{1 - e(\lambda)}{2} \zeta(s) + \frac{e(\lambda) - 1}{2\pi i\lambda} \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} (2\pi i\lambda)^{2k} \zeta(s + \beta(1-2k)). \tag{5.74}$$

Formula (5.72) then follows by comparing (5.74) with the formula (5.71) for $\zeta(2k)$. □

5.5 Interpolation of Fourier coefficients of Eisenstein series

The nonconstant terms of the Fourier expansions of the holomorphic Eisenstein series and nonholomorphic Eisenstein series for $\mathrm{SL}(2,\mathbb{Z})$ include the sum-of-divisors function $\sigma_\alpha(n)$, which we have interpolated in the n variable by our function $\widetilde{Z}(s,\lambda)$. We thus obtain an interpolation of the Fourier coefficients of such Eisenstein series.

5.5.1 Holomorphic Eisenstein series

For even integer $k \geq 4$, define the ***(holomorphic) Eisenstein series*** as a function of the variable z in the upper halfplane $\mathcal{H} \subset \mathbb{C}$ by

$$E_k(z) = \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)\neq(0,0)}} \frac{1}{(cz+d)^k}. \tag{5.75}$$

It is well-known that $E_k(z)$ is a modular form of weight k for $\mathrm{SL}(2,\mathbb{Z})$ and has the Fourier expansion

$$E_k(z) = \sum_{n=0}^{\infty} a_n(k) e^{2\pi i n z} \tag{5.76}$$

with constant term

$$a_0(k) = \zeta(k) \tag{5.77}$$

and higher coefficients

$$a_n(k) = \frac{(2\pi i)^k}{(k-1)!} \sigma_{k-1}(n). \tag{5.78}$$

See Bump [8, Sec. 1.3] for a proof of the modularity and the Fourier expansion of $E_k(z)$.

We show now that if the Fourier coefficients $a_n(k)$ for integer $n > 0$ are interpolated to complex n using the function $Z(s,\lambda)$, then the resulting interpolation at $n = 0$ differs from the correct constant term $a_0(k)$.

Theorem 5.5.1. *Fix an even integer $k \geq 4$. Define b_λ for complex λ by*

$$b_\lambda(k) = \frac{(2\pi i)^k}{(k-1)!} \widetilde{Z}(k,\lambda) \lambda^{k-1} \tag{5.79}$$

and let $a_n(k)$ be the nth Fourier coefficient of the holomorphic Eisenstein series $E_k(z)$. Then we have

$$a_n(k) = \begin{cases} b_n(k) & \text{if } n > 0, \\ b_0(k) + \zeta(k) & \text{if } n = 0. \end{cases} \tag{5.80}$$

Proof. For integer $n \neq 0$, we have

$$\widetilde{Z}(s,n) = \sigma_{1-s}(n). \tag{5.81}$$

Then for $n > 0$, using the relation

$$\sigma_\alpha(n) = n^\alpha \sigma_{-\alpha}(n), \tag{5.82}$$

we have

$$b_n(k) = \frac{(2\pi i)^k}{(k-1)!}\sigma_{1-k}(n)n^{k-1} = a_n(k). \tag{5.83}$$

At $\lambda = 0$, we have

$$\widetilde{Z}(s,0) = \zeta(s-1). \tag{5.84}$$

Thus we have

$$b_0(k) = \frac{(2\pi i)^k}{(k-1)!}\zeta(k-1)\cdot 0^{k-1} = 0, \tag{5.85}$$

so that $a_0(k) = b_0(k) + \zeta(k)$. □

Remark. Because of the relation

$$\sigma_\alpha(n) = n^\alpha \sigma_{-\alpha}(n), \tag{5.86}$$

we have for integers $n \neq 0$ that

$$\sigma_\alpha(n) = \widetilde{Z}(1-\alpha, n) = n^\alpha \cdot \widetilde{Z}(1+\alpha, n). \tag{5.87}$$

For a general complex λ, however, it is *not* true that

$$\widetilde{Z}(1-\alpha,\lambda) = \lambda^\alpha \cdot \widetilde{Z}(1+\alpha,\lambda). \tag{5.88}$$

Hence we have essentially two different ways of extending the sum-of-divisors function $\sigma_\alpha(n)$ to non-integer arguments using $\widetilde{Z}(s,\lambda)$.

5.5.2 Nonholomorphic Eisenstein series

Define the ***(completed) nonholomorphic Eisenstein series*** by

$$E(z,s) = \pi^{-s}\Gamma(s)\frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)\neq(0,0)}} \frac{y^s}{|cz+d|^{2s}} \tag{5.89}$$

where $z = x + iy \in \mathcal{H}$ and $\mathrm{Re}(s) > 1$. The function $E(z,s)$ has a Fourier expansion

$$E(z,s) = \sum_{n=-\infty}^{\infty} a_n(y,s)\exp(2\pi i n z). \tag{5.90}$$

For $n \neq 0$, the Fourier coefficients are

$$a_n(y,s) = 2\,|n|^{s-1/2}\,\sigma_{1-2s}\,(|n|)\,\sqrt{y}K_{s-1/2}\,(2\pi\,|n|\,y) \tag{5.91}$$

where $K_\nu(s)$ is the K-Bessel function, defined for $\mathrm{Re}(\nu) > 0$ by the Mellin integral

$$K_\nu(s) = \frac{1}{2}\int_0^\infty \exp\Big(-\frac{\nu}{2}\big(t + t^{-1}\big)\Big)t^{\nu-1}\,dt. \tag{5.92}$$

The constant term is given by

$$a_0(y,s) = \zeta^*(2s)y^s + \zeta^*(2-2s)y^{1-s} \tag{5.93}$$

where $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed zeta function. See Bump [8, Sec. 1.6] for a proof of the Fourier expansion.

As we did in the previous section for the holomorphic Eisenstein series, we interpolate the Fourier coefficients $a_n(y,s)$ for $n \neq 0$ using the function $Z(s,\lambda)$, and we note that this interpolation at $n = 0$ gives exactly one of the two parts of the constant term $a_0(y,s)$.

Theorem 5.5.2. *Fix s with $\mathrm{Re}(s) > 1$ and $z = x + iy \in \mathcal{H}$. Define $b_\lambda(y,s)$ for $\lambda \in \mathbb{R}$ with $\lambda \neq 0$ by*

$$b_\lambda(y,s) = 2|\lambda|^{s-1/2}Z(2s,\lambda)\zeta(2s)\sqrt{y}K_{s-1/2}(2\pi|\lambda|y), \tag{5.94}$$

and let $a_n(y,s)$ be the nth Fourier coefficient of the nonholomorphic Eisenstein series $E(z,s)$. Then $b_\lambda(y,s)$ extends continuously to $\lambda = 0$, and

$$a_n(y,s) = \begin{cases} b_n(y,s) & \text{if } n \neq 0, \\ b_0(y,s) + \zeta^*(2s)y^s & \text{if } n = 0. \end{cases} \tag{5.95}$$

Proof. Using the formula

$$Z(s,n) = \frac{\sigma_{1-s}(n)}{\zeta(s)}, \tag{5.96}$$

we find that $b_n(y,s) = a_n(y,s)$ for integer $n \neq 0$.

For $n = 0$, we make use of the asymptotic

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\Big(\frac{1}{2}z\Big)^{-\nu} \tag{5.97}$$

as $z \to 0^+$, valid for $\mathrm{Re}(\nu) > 0$, which gives us

$$K_{s-1/2}\,(2\pi|\lambda|y) \sim \frac{1}{2}\Gamma\Big(s - \frac{1}{2}\Big)(\pi|\lambda|y)^{-s+1/2} \tag{5.98}$$

as $\lambda \to 0$. Then as $\lambda \to 0$ we have

$$|\lambda|^{s-1/2}K_{s-1/2}(2\pi|\lambda|y) \to \frac{1}{2}\Gamma\Big(s - \frac{1}{2}\Big)(\pi y)^{-s+1/2}. \tag{5.99}$$

Since $Z(2s,0)\zeta(2s) = \zeta(2s-1)$, we can take $\lambda \to 0$ to get

$$b_0(y,s) = \zeta(2s-1)\Gamma\Big(s - \frac{1}{2}\Big)(\pi y)^{-s+1/2}\sqrt{y} = \zeta(2s-1)\Gamma\Big(\frac{2s-1}{2}\Big)\pi^{-(2s-1)/2}y^{1-s} \tag{5.100}$$

$$= \zeta^*(2s-1)y^{1-s}. \tag{5.101}$$

By the functional equation $\zeta^*(1-s) = \zeta^*(s)$, this is

$$b_0(y,s) = \zeta^*(2-2s)y^{1-s} = a_0(y,s) - \zeta^*(2s)y^s. \tag{5.102}$$

□

Appendix

Bernoulli numbers

In this appendix, we collect some properties of the Bernoulli numbers that are either used explicitly in the thesis or are relevant to the results of the thesis.

For a thorough treatment of Bernoulli numbers, their history, and connections to zeta and L-functions, see Arakawa, Ibukiyama, and Kaneko [4].

Definition A.0.1. The ***Bernoulli numbers*** are the sequence of rational numbers B_k defined by the power series expansion

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k. \tag{A.1}$$

One can easily show from this definition that the odd-index Bernoulli numbers other than B_1 all vanish.

Proposition A.0.2. $B_{2k+1} = 0$ *for all* $k \geq 1$.

Proof. We have that $B_1 = -1/2$ and the function

$$\frac{x}{e^x - 1} + \frac{1}{2}x \tag{A.2}$$

is an even function, so its odd-index power series coefficients must be 0. □

We will make use of two power series expansions closely related to (A.1).

Proposition A.0.3. *In the disc* $|x| < 2\pi$, *we have the series expansions*

$$\frac{x}{1 - e^{-x}} = \sum_{k=0}^{\infty} \frac{(-1)^k B_k}{k!} x^k \tag{A.3}$$

and

$$\frac{x^2 e^{-x}}{(1 - e^{-x})^2} = 1 - \sum_{k=2}^{\infty} \frac{B_k}{k(k-2)!} x^k. \tag{A.4}$$

Proof. We have

$$\frac{x}{1-e^{-x}} - x = \frac{x}{e^x - 1}, \tag{A.5}$$

so using (A.1), we have

$$\frac{x}{1-e^{-x}} = x + \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k. \tag{A.6}$$

The expansion (A.3) then follows from noting that $B_1 = -1/2$ and that $B_{2k+1} = 0$ for $k \geq 1$.

From (A.1), we have

$$\frac{1}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^{k-1}. \tag{A.7}$$

Taking the derivative with respect to x gives

$$\frac{-e^x}{(e^x-1)^2} = -x^{-2} + \sum_{k=2}^{\infty} \frac{B_k}{k(k-2)!} x^{k-2}, \tag{A.8}$$

which gives (A.3). □

Bernoulli numbers can be computed recursively from the following formula. We make use of this formula to compute values of some Dirichlet series at negative integers in Chapter 3.

Proposition A.0.4. *For all $n \geq 1$, the Bernoulli numbers satisfy the recursion*

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0. \tag{A.9}$$

Proof. Multiply both sides of (A.1) by $(e^x - 1)$. Using the power series expansion of $(e^x - 1)$, we have

$$x = \Big(\sum_{j=1}^{\infty} \frac{1}{j!} x^j\Big)\Big(\sum_{k=0}^{\infty} \frac{B_k}{k!} x^k\Big). \tag{A.10}$$

Multiplying the power series gives

$$x = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n-1} \frac{1}{(n-k)!k!} B_k\Big) x^n. \tag{A.11}$$

Comparing coefficients gives the proposition. □

Bernoulli numbers are closely related to the values of the Riemann zeta function at the integers. At the negative integers, we have the following result.

Proposition A.0.5. *For $n \geq 0$,*

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}. \tag{A.12}$$

In particular, we have $\zeta(0) = -1/2$, $\zeta(-2n) = 0$ for $n \geq 1$, and $\zeta(-2n+1) = -B_{2n}/2n$.

Proof. There are many proofs of this result; we outline one proof related to the methods of this thesis.

In the notation of Chapter 3, we have that $\zeta(s) = D(\mathcal{S}\delta, s)$, where $\delta(1) = 1$ and $\delta(n) = 0$ for all $n > 1$. Then Theorem 3.4.11 gives

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{k=1}^{K} \frac{B_{2k}}{(2k)!} \Big(\prod_{i=0}^{2k-2} (s+i) \Big) + R_K(s) \tag{A.13}$$

in the halfplane $\mathrm{Re}(s) > -2K$, where $R_K(s)$ is a holomorphic function in this halfplane with $R_K(-m) = 0$ for integers $m \geq 0$. The value of $\zeta(-n)$ stated in the proposition follows from evaluating $\zeta(-n)$ using (A.13) and using the relation (A.9). □

At the positive even integers, we have the following formula.

Proposition A.0.6. *For $n \geq 1$,*

$$\zeta(2n) = \frac{(-1)^{n+1} 2^{2n-1} \pi^{2n} B_{2n}}{(2n)!}. \tag{A.14}$$

Proof. The zeta function satisfies the functional equation

$$\pi^{-s/2} \Gamma\Big(\frac{s}{2}\Big) \zeta(s) = \pi^{-(1-s)/2} \Gamma\Big(\frac{1-s}{2}\Big) \zeta(1-s). \tag{A.15}$$

Combining the functional equation with our formula for $\zeta(-n)$ gives the proposition. □

Finally, we note an asymptotic for the non-zero Bernoulli numbers.

Proposition A.0.7. *The Bernoulli numbers satisfy the asymptotic*

$$|B_{2n}| \sim \frac{2(2n)!}{(2\pi)^{2n}} \tag{A.16}$$

as $n \to \infty$.

Proof. Equation (A.14) gives a formula for B_{2n} in terms of $\zeta(2n)$. Since $\zeta(2n) \to 1$ as $n \to \infty$, we have the asymptotic in the proposition. □

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