Supplemental Materials: Rank Theory Approach to Ridge, LASSO, Preliminary Test and Stein-type Estimators: A Comparative Study

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The supplement is divided as follows. Section S1 contains the analysis of comparison of the proposed estimators. Section S2 show omitted information on the tables related to section 4 of the paper. In Section S3 the dataset and the codes related to the paper are expressed.

S1. COMPARISON BETWEEN ESTIMATORS

Now, we will prove the dominance characteristics of the estimators.
(i) RE vs. URE

$$\text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n\right) - \text{ADL}_2\text{-risk}\left(\tilde{\boldsymbol{\beta}}_n\right) = \eta^2(\Delta^2 - p_2).$$

Hence, URE outperforms the RE for $\Delta^2 > p_2$ and RE outperforms URE for $\Delta^2 < p_2$. Neither RE nor URE dominates the other uniformly.

(ii) PTE vs. URE

$$\begin{aligned}\text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{PT}}\right) - \text{ADL}_2\text{-risk}\left(\tilde{\boldsymbol{\beta}}_n\right) &= \eta^2[p_1 + p_2(1 - \mathcal{H}_{p_2+2}(\chi^2_{p_2}(\alpha);\Delta^2)) \\ &\quad +\Delta^2\{2\mathcal{H}_{p_2+2}(\chi^2_{p_2}(\alpha);\Delta^2) - \mathcal{H}_{p_2+4}(\chi^2_{p_2}(\alpha);\Delta^2)\} - (p_1+p_2)] \\ &= \eta^2\left[-p_2\mathcal{H}_{p_2+2}(\chi^2_{p_2}(\alpha);\Delta^2)\right. \\ &\quad \left.+\Delta^2\{2\mathcal{H}_{p_2+2}(\chi^2_{p_2}(\alpha);\Delta^2) - \mathcal{H}_{p_2+4}(\chi^2_{p_2}(\alpha);\Delta^2)\}\right]\end{aligned}$$

Hence, if

$$0 < \Delta^2 \leq \frac{p_2\mathcal{H}_{p_2+2}(\chi^2_{p_2}(\alpha);\Delta^2)}{2\mathcal{H}_{p_2+2}(\chi^2_{p_2}(\alpha);\Delta^2) - \mathcal{H}_{p_2+4}(\chi^2_{p_2}(\alpha);\Delta^2)}$$

PTE dominates URE. If

$$\Delta^2 > \frac{p_2\mathcal{H}_{p_2+2}(\chi^2_{p_2}(\alpha);\Delta^2)}{2\mathcal{H}_{p_2+2}(\chi^2_{p_2}(\alpha);\Delta^2) - \mathcal{H}_{p_2+4}(\chi^2_{p_2}(\alpha);\Delta^2)}$$

URE dominates PTE. Thus, neither PTE nor URE dominates the other, uniformly.

(iii) JSE vs. URE

$$\text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{JS}}\right) - \text{ADL}_2\text{-risk}\left(\tilde{\boldsymbol{\beta}}_n\right) = \eta^2\left[(p_2-2)^2\mathbb{E}[\chi^{-2}_{p_2}(\Delta^2)]\right] \geq 0$$

for all $\Delta^2 \geq 0$. Hence, JSE dominates URE uniformly.
(iii) JSE vs. PRSE

$$\begin{aligned}\text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{JS}}\right) - \text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{S+}}\right) = \\ \eta^2 p_2 \mathbb{E}[(1-(p_2-2)\chi^{-2}_{p_2+2}(\Delta^2))^2 I(\chi^2_{p_2+2}(\Delta^2) < p_2-2)] \\ +\Delta^2\{2\mathbb{E}[(1-(p_2-2)\chi^{-2}_{p_2+2}(\Delta^2))^2 I(\chi^2_{p_2+2}(\Delta^2) < p_2-2)] \\ +\mathbb{E}[((p_2-2)\chi^{-2}_{p_2+4}(\Delta^2)-1) I(\chi^2_{p_2+4}(\Delta^2) < p_2-2)]\}\end{aligned}$$

for all $\Delta^2 \geq 0$, since

$$0 < \chi^2_{p_2+2}(\Delta^2) < p_2 - 2 \Rightarrow \left((p_2-2)\chi^2_{p_2+2}(\Delta^2) - 1\right) \geq 0.$$

Hence, combining (iii) and (iv) we find that

$$\text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{S+}}\right) \leq \text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{JS}}\right) \leq \text{ADL}_2\text{-risk}\left(\tilde{\boldsymbol{\beta}}_n\right).$$

(v) LASSO vs. URE

$$\text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{LASSO}}\right) - \text{ADL}_2\text{-risk}\left(\tilde{\boldsymbol{\beta}}_n\right) = \eta^2(\Delta^2 - p_2).$$

Hence, none of the estimators outperforms the other uniformly.
(vi) RR vs. URE

$$\text{ADL}_2\text{-risk}\left(\tilde{\boldsymbol{\beta}}_n\right) - \text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{RR}}(k_{\text{opt}})\right) = \eta^2 \frac{p_2^2}{\Delta^2 + p_2} > 0.$$

Hence, RR dominates uniformly over URE.
(vii) RE/LASSO vs. RR

$$\text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n\right) - \text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{RR}}(k_{\text{opt}})\right) = \eta^2 \frac{\Delta^4}{\Delta^2 + p_2} \geq 0.$$

Hence, RR uniformly dominates RE and LASSO.
(viii) PTE vs. RR

$$\begin{aligned}\text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{PT}}\right) - \text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{RR}}(k_{\text{opt}})\right) = \\ \eta^2 \Big[p_2(1-\mathcal{H}_{p_2+2}(\chi^2_{p_2}(\alpha);\Delta^2)) + \\ \Delta^2(2\mathcal{H}_{p_2+2}(\chi^2_{p_2}(\alpha);\Delta^2) - \mathcal{H}_{p_2+4}(\chi^2_{p_2}(\alpha);\Delta^2)) - \frac{p_2\Delta^2}{p_2+\Delta^2}\Big],\end{aligned}$$

Note that the risk of PTE is an increasing function of Δ^2, crossing the p_2-line to a maximum and then it drops monotonically towards p_2-line as $\Delta^2 \to \infty$. At $\Delta^2 = 0$, the value of this risk is $p_2[1 - \mathcal{H}_{p_2+2}(\chi^2_{p_2}(\alpha);0)] < p_2$. On the other hand, $p_2\Delta^2/(p_2+\Delta^2)$ is an increasing function of Δ^2 below the p_2-line with a minimum value 0 at $\Delta^2 = 0$ and it converges to p_2 as $\Delta^2 \to \infty$. Hence, the risk difference is non-negative for all $\Delta^2 \geq 0$ and RR outperforms PTE uniformly.

(ix) JSE vs. RR

$$\text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{JS}}\right) - \text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{RR}}(k_{\text{opt}})\right) = \eta^2\left\{p_2 - (p_2-2)^2\mathbb{E}[\chi_{p_2}^{-2}(\Delta^2)] - \frac{p_2\Delta^2}{p_2+\Delta^2}\right\}.$$

Note that the first function is increasing in Δ^2 with its minimum at $\Delta^2 = 0$ and as $\Delta^2 \to \infty$ it tends to p_2. The second function is also increasing in Δ^2 with a value 0 at Δ^2 and approaches the value p_2 as $\Delta^2 \to \infty$. Hence, the risk difference is non-negative for all $\Delta^2 \geq 0$. Hence, RR uniformly outperforms JSE.

(x) PRSE vs. RR

$$\text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{S+}}\right) - \text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{RR}}(k_{\text{opt}})\right) = \tag{S1}$$

$$\eta^2\left\{p_2 - (p_2-2)^2\mathbb{E}[\chi_{p_2}^{-2}(\Delta^2)] - \frac{p_2\Delta^2}{p_2+\Delta^2} - \text{R}^*\right\} \geq 0, \quad \forall\, \Delta^2 \geq 0, \tag{S2}$$

Consider the risk of PRSE. It is an increasing function of Δ^2. At $\Delta^2 = 0$, its value is

$$(p_1+2) - p_2\mathbb{E}[(1-(p_2-2)\chi_{p_2+2}^{-2}(0))I(\chi_{p_2+2}^2(0) < p_2-2)] \geq 0$$

and as $\Delta^2 \to \infty$, it tends to $p_1 + p_2$. For RR, at $\Delta^2 = 0$, the value is p_1 and as $\Delta^2 \to \infty$, it tends to $p_1 + p_2$. Hence, the risk difference is non-negative and therefore RR outperforms PRSE uniformly.

(xi) LASSO vs. PTE

$$\text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{PT}}\right) - \text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{LASSO}}\right) = \eta^2\Big[p_2(1-\mathcal{H}_{p_2+2}(\chi_{p_2}^2(\alpha);\Delta^2) - \Delta^2\{1-2\mathcal{H}_{p_2+2}(\chi_{p_2}^2(\alpha);\Delta^2)+\mathcal{H}_{p_2+4}(\chi_{p_2}^2(\alpha);\Delta^2)\})\Big].$$

Then, for

$$0 \leq \Delta^2 \leq \frac{p_2[1-\mathcal{H}_{p_2+2}(\chi_{p_2}^2(\alpha);\Delta^2)]}{1-2\mathcal{H}_{p_2+2}(\chi_{p_2}^2(\alpha);\Delta^2)+\mathcal{H}_{p_2+4}(\chi_{p_2}^2(\alpha);\Delta^2)}$$

LASSO outperforms PTE. On the other hand, outside this interval PTE outperforms LASSO. Hence, neither LASSO nor PTE dominates the other uniformly.

(xii) LASSO vs. JSE

$$\text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{JS}}\right) - \text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{LASSO}}\right) = \eta^2\Big[p_2 - (p_2-2)^2\mathbb{E}[\chi_{p_2}^{-2}(\Delta^2)] \\ -\Delta^2\Big]$$

Hence, for $0 \leq \Delta^2 \leq p_2 - (p_2-2)^2\mathbb{E}[\chi_{p_2}^{-2}(\Delta^2)]$, LASSO outperforms JSE, otherwise JSE outperforms LASSO. Hence, neither LASSO nor JSE outperforms the other uniformly.

(xiii) LASSO vs. PRSE

$$\begin{aligned}\text{ADL}_2\text{-risk}\left(\widehat{\boldsymbol{\beta}}_n^{\text{S+}}\right) - \text{ADL}_2\text{-risk}\left(\widehat{\boldsymbol{\beta}}_n^{\text{LASSO}}\right) &= \eta^2\Big[\text{ADL}_2\text{-risk}\left(\hat{\boldsymbol{\beta}}_n^{\text{JS}}\right) - \text{R}^* \\ &\qquad -(p_1+\Delta^2)\Big] \\ &= \eta^2\left[p_2 - (p_2-2)^2\mathbb{E}[\chi_{p_2}^{-2}(\Delta^2)] \right. \\ &\qquad \left. -\text{R}^* - \Delta^2\right] \\ &= \eta^2(A - \Delta^2 B),\end{aligned}$$

where

$$\begin{aligned}A = p_2 &- (p_2-2)^2\mathbb{E}[\chi_{p_2}^{-2}(\Delta^2)] \\ &-p_2\mathbb{E}[(1-(p_2-2)\chi_{p_2+2}^{-2}(\Delta^2))^2 I(\chi_{p_2+2}^2(\Delta^2) < p_2-2)]\end{aligned}$$

and

$$\begin{aligned}B = 1 &+ 2\mathbb{E}\left[\left((p_2-2)\chi_{p_2+2}^{-2}(\Delta^2) - 1\right) I(\chi_{p_2+2}^2(\Delta^2) < p_2-2)\right] \\ &+\mathbb{E}\left[\left(1-(p_2-2)\chi_{p_2+4}^{-2}(\Delta^2)\right)^2 I(\chi_{p_2+4}^2(\Delta^2) < p_2-2)\right]. \qquad \text{(S3)}\end{aligned}$$

Again, for $0 \leq \Delta^2 \leq A/B$, LASSO outperforms PRSE, otherwise PRSE outperforms LASSO. Neither LASSO nor PRSE dominates the other.

S2. MORE DETAILS ON GRAPHICAL REPRESENTATIONS

In this section, more tables (Tables S1-S2) are presented to illustrate the properties of the proposed estimators, clearly.

Table S1: ADRRE for the estimators in case of $p_1 = 7$ and $p_2 = 33$.

Δ^2	URE	RE/LASSO	PTE $\alpha = 0.15$	PTE $\alpha = 0.2$	PTE $\alpha = 0.25$	JSE	PRSE	RRE
0	1.00	5.71	2.86	2.50	2.23	4.44	4.92	5.71
0.1	1.00	5.63	2.82	2.46	2.20	4.40	4.86	5.63
0.5	1.00	5.33	2.66	2.34	2.10	4.23	4.64	5.34
1	1.00	5.00	2.49	2.20	1.99	4.03	4.40	5.02
2	1.00	4.44	2.21	1.97	1.80	3.71	4.00	4.50
3	1.00	4.00	1.99	1.79	1.65	3.45	3.68	4.10
5	1.00	3.33	1.67	1.54	1.44	3.05	3.20	3.53
7	1.00	2.86	1.46	1.36	1.29	2.76	2.86	3.13
10	1.00	2.35	1.26	1.20	1.16	2.46	2.51	2.73
15	1.00	1.82	1.09	1.07	1.05	2.13	2.14	2.31
20	1.00	1.48	1.03	1.02	1.01	1.92	1.92	2.06
30	1.00	1.08	1.00	1.00	1.00	1.67	1.67	1.76
33	1.00	0.99	1.00	1.00	1.62	1.62	1.70	
50	1.00	0.70	1.00	1.00	1.00	1.43	1.43	1.49
100	1.00	0.37	1.00	1.00	1.00	1.12	1.12	1.26

Table S2: ADRRE of the estimators for $p = 10$ and different Δ^2-value for varying p_1

Estimators	$p_1 = 2$	$p_1 = 3$	$p_1 = 5$	$p_1 = 7$	$p_1 = 2$	$p_1 = 3$	$p_1 = 5$	$p_1 = 7$
	$\Delta^2 = 0$				$\Delta^2 = 1$			
URE	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
RE/LASSO	5.00	3.33	2.00	1.43	3.33	2.50	1.66	1.25
PTE ($\alpha = 0.15$)	2.34	1.98	1.51	1.23	1.75	1.55	1.27	1.09
PTE ($\alpha = 0.2$)	2.07	1.80	1.43	1.19	1.60	1.45	1.22	1.07
PTE ($\alpha = 0.25$)	1.86	1.66	1.36	1.16	1.49	1.37	1.18	1.06
JSE	2.50	2.00	1.43	1.11	2.14	1.77	1.33	1.08
PRSE	3.04	2.31	1.56	1.16	2.46	1.98	1.42	1.11
RRE	5.00	3.33	2.00	1.43	3.46	2.58	1.71	1.29
	$\Delta^2 = 5$				$\Delta^2 = 10$			
URE	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
RE/LASSO	1.43	1.25	1.00	0.83	0.83	0.77	0.67	0.59
PTE ($\alpha = 0.15$)	1.05	1.01	0.95	0.92	0.92	0.92	0.92	0.94
PTE ($\alpha = 0.2$)	1.04	1.00	0.95	0.93	0.94	0.93	0.94	0.95
PTE ($\alpha = 0.25$)	1.03	1.00	0.96	0.94	0.95	0.95	0.95	0.97
JSE	1.55	1.38	1.15	1.03	1.32	1.232	1.09	1.01
PRSE	1.62	1.43	1.18	1.03	1.34	1.23	1.09	1.01
Ridge	1.97	1.69	1.33	1.13	1.55	1.40	1.20	1.07

Table S3: ADRRE of the estimators for $p = 20$ and different Δ^2-value for varying p_1

Estimators	$p_1 = 2$	$p_1 = 3$	$p_1 = 5$	$p_1 = 7$	$p_1 = 2$	$p_1 = 3$	$p_1 = 5$	$p_1 = 7$
		$\Delta^2 = 0$				$\Delta^2 = 1$		
URE	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
RE/LASSO	10.00	6.67	4.00	2.86	6.67	5.00	3.33	2.50
PTE ($\alpha = 0.15$)	3.20	2.84	2.31	1.95	2.50	2.27	1.93	1.68
PTE ($\alpha = 0.2$)	2.70	2.45	2.07	1.79	2.17	2.01	1.76	1.56
PTE ($\alpha = 0.25$)	2.35	2.17	1.89	1.67	1.94	1.82	1.63	1.47
JSE	5.00	4.00	2.86	2.22	4.13	3.43	2.56	2.04
PRSE	6.28	4.77	3.22	2.43	4.89	3.9 3	2.82	2.20
RRE	10.00	6.66	4.00	2.86	6.79	5.07	3.37	2.52
		$\Delta^2 = 5$			$\Delta^2 = 10$			
URE	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
RE/LASSO	2.86	2.50	2.00	1.67	1.67	1.54	1.33	1.18
PTE ($\alpha = 0.15$)	1.42	1.36	1.26	1.17	1.08	1.06	1.03	1.00
PTE ($\alpha = 0.2$)	1.33	1.29	1.21	1.14	1.06	1.04	1.02	1.00
PTE ($\alpha = 0.25$)	1.27	1.23	1.17	1.11	1.04	1.03	1.01	1.00
JSE	2.65	2.36	1.94	1.65	2.03	1.87	1.62	1.43
PRSE	2.83	2.49	2.02	1.70	2.07	1.90	1.64	1.45
RRE	3.38	2.91	2.29	1.88	2.37	2.15	1.82	1.58

Table S4: ADRRE of the estimators for $p = 40$ and different Δ^2-value for varying p_1

Estimators	$p_1 = 2$	$p_1 = 3$	$p_1 = 5$	$p_1 = 7$	$p_1 = 2$	$p_1 = 3$	$p_1 = 5$	$p_1 = 7$
	$\Delta^2 = 0$				$\Delta^2 = 1$			
URE	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
RE/LASSO	20.00	13.33	8.00	5.71	13.33	10.00	6.67	5.00
PTE ($\alpha = 0.15$)	4.05	3.74	3.24	2.86	3.33	3.12	2.77	2.49
PTE ($\alpha = 0.2$)	3.28	3.09	2.76	2.50	2.77	2.63	2.40	2.20
PTE ($\alpha = 0.25$)	2.78	2.65	2.42	2.23	2.40	2.30	2.13	1.99
JSE	10.00	8.00	5.71	4.44	8.12	6.75	5.05	4.03
PRSE	12.80	9.69	6.53	4.92	9.81	7.87	5.65	4.40
RRE	20.00	13.33	8.00	5.71	13.45	10.07	6.70	5.02
	$\Delta^2 = 5$				$\Delta^2 = 10$			
URE	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
RE/LASSO	5.71	5.00	4.00	3.33	3.33	3.08	2.67	2.35
PTE ($\alpha = 0.15$)	1.96	1.90	1.78	1.67	1.38	1.35	1.30	1.26
PTE ($\alpha = 0.2$)	1.75	1.70	1.61	1.54	1.29	1.27	1.24	1.20
PTE ($\alpha = 0.25$)	1.60	1.57	1.50	1.44	1.23	1.22	1.19	1.16
JSE	4.87	4.35	3.59	3.05	3.46	3.20	2.78	2.46
PRSE	5.29	4.68	3.80	3.20	3.57	3.29	2.85	2.51
RRE	6.23	5.40	4.27	3.53	4.03	3.68	3.13	2.73

Table S5: ADRRE of the estimators for $p = 60$ and different Δ^2-value for varying p_1

Estimators	$p_1 = 2$	$p_1 = 3$	$p_1 = 5$	$p_1 = 7$	$p_1 = 2$	$p_1 = 3$	$p_1 = 5$	$p_1 = 7$
	$\Delta^2 = 0$				$\Delta^2 = 1$			
URE	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
RE/LASSO	30.00	20.00	12.00	8.57	20.00	15.00	10.00	7.50
PTE ($\alpha = 0.15$)	4.49	4.23	3.79	3.43	3.80	3.62	3.29	3.02
PTE ($\alpha = 0.2$)	3.58	3.42	3.14	2.91	3.10	2.99	2.78	2.59
PTE ($\alpha = 0.25$)	2.99	2.89	2.70	2.54	2.64	2.56	2.42	2.29
JSE	15.00	12.00	8.57	6.67	12.12	10.09	7.55	6.03
PRSE	19.35	14.63	9.83	7.40	14.74	11.83	8.48	6.61
RRE	30.00	20.00	12.00	8.57	20.11	15.06	10.03	7.52
	$\Delta^2 = 5$				$\Delta^2 = 10$			
URE	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
RE/LASSO	8.57	7.50	6.00	5.00	5.00	4.62	4.00	3.53
PTE ($\alpha = 0.15$)	2.35	2.28	2.16	2.05	1.63	1.60	1.55	1.50
PTE ($\alpha = 0.2$)	2.04	2.00	1.91	1.83	1.49	1.47	1.43	1.39
PTE ($\alpha = 0.25$)	1.83	1.79	1.73	1.67	1.39	1.37	1.34	1.31
JSE	7.10	6.35	5.25	4.47	4.89	4.53	3.95	3.50
PRSE	7.76	6.87	5.60	4.72	5.09	4.70	4.07	3.60
RRE	9.09	7.90	6.26	5.19	5.70	5.21	4.46	3.89

Table S6: ADRRE values of estimators for $p_1 = 5$, different values of p_2 and Δ^2

Δ^2	p_2	URE	RE/LASSO	PTE $\alpha = 0.15$	PTE $\alpha = 0.2$	PTE $\alpha = 0.25$	JSE	PRSE	RRE
0	5	1.00	2.00	1.51	1.43	1.36	1.43	1.56	2.00
	15	1.00	4.00	2.31	2.07	1.89	2.86	3.22	4.00
	25	1.00	6.00	2.85	2.48	2.20	4.29	4.87	6.00
	35	1.00	8.00	3.24	2.76	2.42	5.71	6.53	8.00
	55	1.00	12.00	3.79	3.14	2.70	8.57	9.83	12.00
0.5	5	1.00	1.82	1.38	1.31	1.26	1.37	1.48	1.83
	15	1.00	3.64	2.10	1.90	1.75	2.70	3.01	3.65
	25	1.00	5.45	2.61	2.29	2.05	4.03	4.53	5.46
	35	1.00	7.27	2.99	2.57	2.27	5.36	6.05	7.28
	55	1.00	10.91	3.52	2.95	2.55	8.02	9.10	10.92
5	5	1.00	1.00	0.95	0.95	0.96	1.15	1.18	1.33
	15	1.00	2.00	1.26	1.21	1.17	1.94	2.02	2.29
	25	1.00	3.00	1.54	1.43	1.35	2.76	2.90	3.27
	35	1.00	4.00	1.78	1.61	1.50	3.59	3.80	4.27
	55	1.00	6.00	2.16	1.91	1.73	5.25	5.60	6.26

Table S7: ADRRE values of estimators for $p_1 = 7$, different values of p_2 and Δ^2

Δ^2	p_2	URE	RE/LASSO	PTE $\alpha = 0.15$	PTE $\alpha = 0.2$	PTE $\alpha = 0.25$	JSE	PRSE	RRE
0	5	1.00	1.71	1.39	1.33	1.28	1.33	1.43	1.71
	15	1.00	3.14	2.06	1.89	1.75	2.44	2.68	3.14
	25	1.00	4.57	2.55	2.27	2.05	3.56	3.92	4.57
	35	1.00	6.00	2.93	2.55	2.27	4.67	5.17	6.00
	55	1.00	8.86	3.47	2.94	2.56	6.89	7.65	8.86
0.5	5	1.00	1.60	1.29	1.25	1.21	1.29	1.37	1.61
	15	1.00	2.93	1.91	1.76	1.63	2.34	2.54	2.94
	25	1.00	4.27	2.37	2.12	1.92	3.39	3.71	4.27
	35	1.00	5.60	2.73	2.39	2.14	4.44	4.88	5.61
	55	1.00	8.27	3.26	2.77	2.43	6.54	7.22	8.27
5	5	1.00	1.00	0.96	0.96	0.97	1.13	1.14	1.26
	15	1.00	1.83	1.23	1.18	1.15	1.78	1.85	2.05
	25	1.00	2.67	1.49	1.39	1.32	2.49	2.60	2.87
	35	1.00	3.50	1.71	1.57	1.46	3.19	3.35	3.69
	55	1.00	5.17	2.08	1.85	1.69	4.62	4.88	5.35

S3. THE CODES RELATED TO? TABLES, FIGURES, AND APPLICATION EXAMPLE

Table 1:

```
p = 10 # p can be one of 10, 20, 30, 40, 60, and 128
p1 = 2  # p1 can be one 2, 3, 4, and 5
ADRRE <- 1+ (p-p1)/p1 #ADRRE of RE or LASSO or
ADRRE

rm(list=ls())
set.seed(2231346)

n=100;  #Number of Observasions
# p=20;  p1 = 5      #Case 1 (Table 2)
p=40;  p1 = 7        #Case 2 (Table S1)
p2=p-p1
##########################################
# Note that in the following, D means \Delta^2 #
##########################################
#Fucntion E[\chi_{p+2}^{-2} (\Delta^2)]
Echi2.2<-function(p,D){
r=seq(0,50,1)
exp(-(D/2))*sum((1/factorial(r) )*((D/2)^r)*(1/(p+2*r)))}
Echi2.2<-Vectorize(Echi2.2)
#Fucntion E[\chi_{p+2}^{-4} (\Delta^2)]
Echi4.2<-function(p,D){
r=seq(0,50,1)
exp(-(D/2))*sum((1/factorial(r) )
*((D/2)^r)*(1/(p+2*r))*(1/(p+2*r-2)))}
Echi4.2<-Vectorize(Echi4.2)
#Fucntion E[\chi_{p+4}^{-2} (\Delta^2)]
Echi2.4<-function(p,D){
r=seq(0,50,1)
exp(-(D/2))*sum((1/factorial(r))*((D/2)^r)*(1/(p+2*r-2)))}
Echi2.4<-Vectorize(Echi2.4)
#Fucntion E[\chi_{p+4}^{-4} (\Delta^2)]
Echi4.4<-function(p,D){
r=seq(0,50,1)
exp(-(D/2))*sum((1/factorial(r) )*((D/2)^r)
*(1/(p+2*r))*(1/(p+2*r-2)))}
Echi4.4<-Vectorize(Echi4.4)
#Fucntion
#E[\chi_{p+4}^{-2}(\Delta^2)I(\chi_{p+4}^2(\Delta^2)<c)]
```

```
EchiI2.4<-function(p,D){
c=p-2
r=seq(0,50,1)
exp(-(D/2))*sum((1/factorial(r) )*((D/2)^r)*(1/(p+2*r+2))
*pchisq(c,df=p+2*r+2))}
EchiI2.4<-Vectorize(EchiI2.4)
#Fucntion
# E[\chi_{p+4}^{-4} (\Delta^2) I(\chi_{p+4}^2(\Delta^2)<c)]
EchiI4.4<-function(p,D){
c=p-2
r=seq(0,50,1)
exp(-(D/2))*sum((1/factorial(r) )*((D/2)^r)*(1/(p+2*r+2))
*(1/(p+2*r))*pchisq(c,df=p+2*r))}
EchiI4.4<-Vectorize(EchiI4.4)
#Fucntion
#E[\chi_{p+2}^{-2}(\Delta^2)I(\chi_{p+2}^2(\Delta^2)<c)]
EchiI2.2<-function(p,D){
c=p-2
r=seq(0,50,1)
exp(-(D/2))*sum((1/factorial(r))*((D/2)^r)*(1/(p+2*r))
*pchisq(c,df=p+2*r))}
EchiI2.2<-Vectorize(EchiI2.2)
#Fucntion
#E[\chi_{p+2}^{-4}(\Delta^2)I(\chi_{p+2}^2(\Delta^2)<c)]
EchiI4.2<-function(p,D){
c=p-2
r=seq(0,50,1)
exp(-(D/2))*sum((1/factorial(r) )*((D/2)^r)*(1/(p+2*r))
*(1/(p-2+2*r))*pchisq(c,df=p-2+2*r))}
EchiI4.2<-Vectorize(EchiI4.2)
###
#ADRRE function of R estimator
ADRRE.UR<-function(D) { 1 }
#ADRRE function of Restricted/LASSO R estimator
ADRRE.RE <-function(D){ (1+p2/p1)/(1 +(D/p1))}
#ADRRE function of preliminary test R estimator
ADRRE.PT <-function(D,alpha){
c.alpha=qchisq(alpha,df=p2,lower.tail = FALSE)
(1 + (p2/p1)) / (1 +(p2/p1)
*(1- pchisq(c.alpha,df=p2+2,ncp = D))
+ (D/p1)*(2*pchisq(c.alpha,df=p2+2,ncp = D)
-pchisq(c.alpha,df=p2+4,ncp = D)))}
```

```
#ADRRE function of James-Stein R estimator
ADRRE.JS <-function(D){
(1 + (p2/p1)) /( 1 + (p2/p1) - (((p2-2)^2)/p1)
* Echi2.2(p2-2,D) )}
#ADRRE function of Positive-Rule Stein-type R estimator
ADRRE.PR <- function(D){
a = pchisq(p2-2,df=p2+2,ncp = D)-2*(p2-2)*EchiI2.2(p2,D)
     +((p2-2)^2)*EchiI4.2(p2,D)
b = 2*pchisq(p2-2,df=p2+2,ncp = D)-2*(p2-2)*EchiI2.2(p2,D)
   -pchisq(p2-2,p2+4,D)+2*(p2-2)*EchiI2.4(p2,D)
   -((p2-2)^2)*EchiI4.4(p2,D);
result=(1+(p2/p1))/(1+(p2/p1)-(((p2-2)^2)/p1)
    *Echi2.2(p2-2,D)-((p2/p1)*a)+((D/p1)*b))
return(result) }
#ADRRE function of Ridge estimator
ADRRE.RR <- function(D){
(1 + (p2/p1)) /(1 + (p2*D)/(p1*(p2+D)))}
####################################
#Vectorice the above functions
ADRRE.UR<-Vectorize(ADRRE.UR)
ADRRE.RE<-Vectorize(ADRRE.RE)
ADRRE.PT<-Vectorize(ADRRE.PT)
ADRRE.JS<-Vectorize(ADRRE.JS)
ADRRE.PR<-Vectorize(ADRRE.PR)
ADRRE.RR<-Vectorize(ADRRE.RR)
######################################
#Delta Vector
Delta<-c(0,0.1,0.5,1,2,3,5,7,10,15,20,30,50,100)
#result matrix (Table)
result <- cbind(Delta, ADRRE.UR(Delta),
ADRRE.RE(Delta),ADRRE.PT(Delta,0.15),
ADRRE.PT(Delta,0.2),ADRRE.PT(Delta,0.25),
ADRRE.JS(Delta),ADRRE.PR(Delta),ADRRE.RR(Delta))
round(result,digits = 2)
```

Figure 1:

```
#Delta Vector
Delta<-c(0,0.1,0.5,1,2,3,5,7,10,15,20,30,50,100)

p = 10; p1 = 5 ; p2 = p - p1
plot(Delta,ADRRE.RE(Delta),ylim=c(0,2.5),
xlab = expression(Delta^2),ylab="ADRRE",col="black",
```

```
lwd=2,type = "o")
title(main=bquote( paste(p[1]== .(p1), ' , ', p[2]== .(p2))))
lines(Delta,ADRRE.PT(Delta,0.25),col="darkturquoise",
lwd=2,type = "o")
lines(Delta,ADRRE.JS(Delta),col="darkblue",lwd=2,type = "o")
lines(Delta,ADRRE.PR(Delta),col="green",lwd=2,type = "o")
lines(Delta,ADRRE.RR(Delta),col="deeppink",lwd=2,type = "o")
abline(h=1,lty=2)
legend("topright" ,c("RE/LASSO","PTE(0.25)","JSE","PRE",
"Ridge"),lwd=2,lty=1,cex=0.8,col=c("black","darkturquoise",
"darkblue","green","deeppink"))
zm(type = "s")

p = 20; p1 = 5 ; p2 = p - p1
plot(Delta,ADRRE.RE(Delta),ylim=c(0,4),
xlab = expression(Delta^2),ylab="ADRRE",col="black",lwd=2,
type = "o")
title(main=bquote( paste(p[1]== .(p1), ' , ', p[2]== .(p2))))
lines(Delta,ADRRE.PT(Delta,0.25),col="darkturquoise",lwd=2,
type = "o")
lines(Delta,ADRRE.JS(Delta),col="darkblue",lwd=2,type = "o")
lines(Delta,ADRRE.PR(Delta),col="green",lwd=2,type = "o")
lines(Delta,ADRRE.RR(Delta),col="deeppink",lwd=2,type = "o")
abline(h=1,lty=2)
legend("topright" ,c("RE/LASSO","PTE(0.25)","JSE","PRE",
"Ridge"),lwd=2,lty=1,cex=1,col=c("black","darkturquoise",
"darkblue","green","deeppink"))

p = 30; p1 = 5 ; p2 = p - p1
plot(Delta,ADRRE.RE(Delta),ylim=c(0,6),
xlab = expression(Delta^2),ylab="ADRRE",col="black",lwd=2,
type = "o")
title(main=bquote( paste(p[1]== .(p1), ' , ', p[2]== .(p2))))
lines(Delta,ADRRE.PT(Delta,0.25),col="darkturquoise",lwd=2,
type = "o")
lines(Delta,ADRRE.JS(Delta),col="darkblue",lwd=2,type = "o")
lines(Delta,ADRRE.PR(Delta),col="green",lwd=2,type = "o")
lines(Delta,ADRRE.RR(Delta),col="deeppink",lwd=2,type = "o")
abline(h=1,lty=2)
legend("topright" ,c("RE/LASSO","PTE(0.25)","JSE","PRE",
"Ridge"), lwd=2,lty=1,cex=1,col=c("black","darkturquoise",
"darkblue", "green","deeppink"))
```

```
p = 40; p1 = 5 ; p2 = p - p1
plot(Delta,ADRRE.RE(Delta),ylim=c(0,8),
xlab = expression(Delta^2), ylab="ADRRE",col="black",lwd=2,
type = "o")
title(main=bquote( paste(p[1]== .(p1), ' , ', p[2]== .(p2))))
lines(Delta,ADRRE.PT(Delta,0.25),col="darkturquoise",lwd=2,
type = "o")
lines(Delta,ADRRE.JS(Delta),col="darkblue",lwd=2,type = "o")
lines(Delta,ADRRE.PR(Delta),col="green",lwd=2,type = "o")
lines(Delta,ADRRE.RR(Delta),col="deeppink",lwd=2,type = "o")
abline(h=1,lty=2)
legend("topright" ,c("RE/LASSO","PTE(0.25)","JSE","PRE",
"Ridge"), lwd=2,lty=1,cex=1,col=c("black",
"darkturquoise","darkblue", "green","deeppink"))

p = 50; p1 = 5 ; p2 = p - p1
plot(Delta,ADRRE.RE(Delta),ylim=c(0,10),
xlab = expression(Delta^2), ylab="ADRRE",col="black",
lwd=2,type = "o")
title(main=bquote( paste(p[1]== .(p1), ' , ', p[2]== .(p2))))
lines(Delta,ADRRE.PT(Delta,0.25),col="darkturquoise",lwd=2,
type = "o")
lines(Delta,ADRRE.JS(Delta),col="darkblue",lwd=2,type = "o")
lines(Delta,ADRRE.PR(Delta),col="green",lwd=2,type = "o")
lines(Delta,ADRRE.RR(Delta),col="deeppink",lwd=2,type = "o")
abline(h=1,lty=2)
legend("topright" ,c("RE/LASSO","PTE(0.25)","JSE","PRE",
"Ridge") ,lwd=2,lty=1,cex=1,col=c("black","darkturquoise",
"darkblue", "green","deeppink"))
```

Tables S2-S5:

```
Delta.p1 <- c(0,1,5,10)
p = 60 # p1 can be 10 , 20 , 40 , 60
p1 = c(2,3,5,7)
p2 = p - p1

result.p1.1.1 <- rbind(
ADRRE.UR(Delta.p1[1],p1,p2),
ADRRE.RE(Delta.p1[1],p1,p2),
ADRRE.PT(Delta.p1[1],p1,p2,0.15),
ADRRE.PT(Delta.p1[1],p1,p2,0.2),
```

```
ADRRE.PT(Delta.p1[1],p1,p2,0.25),
ADRRE.JS(Delta.p1[1],p1,p2),
ADRRE.PR(Delta.p1[1],p1,p2),
ADRRE.RR(Delta.p1[1],p1,p2) )

result.p1.1.1 <- round(result.p1.1.1,digits = 2)

result.p1.1.2 <- rbind(
ADRRE.UR(Delta.p1[2],p1,p2),
ADRRE.RE(Delta.p1[2],p1,p2),
ADRRE.PT(Delta.p1[2],p1,p2,0.15),
ADRRE.PT(Delta.p1[2],p1,p2,0.2),
ADRRE.PT(Delta.p1[2],p1,p2,0.25),
ADRRE.JS(Delta.p1[2],p1,p2),
ADRRE.PR(Delta.p1[2],p1,p2),
ADRRE.RR(Delta.p1[2],p1,p2) )

result.p1.1.2 <- round(result.p1.1.2,digits = 2)

result.p1.1 <- cbind(result.p1.1.1,result.p1.1.2)

result.p1.2.1 <- rbind(
ADRRE.UR(Delta.p1[3],p1,p2),
ADRRE.RE(Delta.p1[3],p1,p2),
ADRRE.PT(Delta.p1[3],p1,p2,0.15),
ADRRE.PT(Delta.p1[3],p1,p2,0.2),
ADRRE.PT(Delta.p1[3],p1,p2,0.25),
ADRRE.JS(Delta.p1[3],p1,p2),
ADRRE.PR(Delta.p1[3],p1,p2),
ADRRE.RR(Delta.p1[3],p1,p2) )

result.p1.2.1 <- round(result.p1.2.1,digits = 2)

result.p1.2.2 <- rbind(
ADRRE.UR(Delta.p1[4],p1,p2),
ADRRE.RE(Delta.p1[4],p1,p2),
ADRRE.PT(Delta.p1[4],p1,p2,0.15),
ADRRE.PT(Delta.p1[4],p1,p2,0.2),
ADRRE.PT(Delta.p1[4],p1,p2,0.25),
ADRRE.JS(Delta.p1[4],p1,p2),
ADRRE.PR(Delta.p1[4],p1,p2),
ADRRE.RR(Delta.p1[4],p1,p2) )
```

```
result.p1.2.2 <- round(result.p1.2.2,digits = 2)
result.p1.2 <- cbind(result.p1.2.1,result.p1.2.2)

rownames(result.p1.1)<-rownames(result.p1.2) <-
 c("URE","RE/LASSO","PT(0.15)","PT(0.2)","PT(0.25)","JSE",
 "PRSE","RRE")

result.p1 <- rbind(result.p1.1,result.p1.2)
colnames(result.p1) <- c("p1 = 2", "p1 = 3","p1 = 5",
"p1 = 7","p1 = 2", "p1 = 3","p1 = 5","p1 = 7")
result.p1
```

Tables S6 and S7:

```
Delta.p2 <- c(0,0.5,5)
p1 = 7 # 5 , 7
p2 = c(5,15,25,35,55)

result.p2.1 <- cbind(
ADRRE.UR(Delta.p2[1],p1,p2),
ADRRE.RE(Delta.p2[1],p1,p2),
ADRRE.PT(Delta.p2[1],p1,p2,0.15),
ADRRE.PT(Delta.p2[1],p1,p2,0.2),
ADRRE.PT(Delta.p2[1],p1,p2,0.25),
ADRRE.JS(Delta.p2[1],p1,p2),
ADRRE.PR(Delta.p2[1],p1,p2),
ADRRE.RR(Delta.p2[1],p1,p2)  )

result.p2.1 <- round( result.p2.1,digits = 2)

result.p2.2 <- cbind(
ADRRE.UR(Delta.p2[2],p1,p2),
ADRRE.RE(Delta.p2[2],p1,p2),
ADRRE.PT(Delta.p2[2],p1,p2,0.15),
ADRRE.PT(Delta.p2[2],p1,p2,0.2),
ADRRE.PT(Delta.p2[2],p1,p2,0.25),
ADRRE.JS(Delta.p2[2],p1,p2),
ADRRE.PR(Delta.p2[2],p1,p2),
ADRRE.RR(Delta.p2[2],p1,p2) )

result.p2.2 <- round( result.p2.2,digits = 2)
```

```
result.p2.3 <- cbind(
ADRRE.UR(Delta.p2[3],p1,p2),
ADRRE.RE(Delta.p2[3],p1,p2),
ADRRE.PT(Delta.p2[3],p1,p2,0.15),
ADRRE.PT(Delta.p2[3],p1,p2,0.2),
ADRRE.PT(Delta.p2[3],p1,p2,0.25),
ADRRE.JS(Delta.p2[3],p1,p2),
ADRRE.PR(Delta.p2[3],p1,p2),
ADRRE.RR(Delta.p2[3],p1,p2)  )

result.p2.3 <- round( result.p2.3,digits = 2)
rownames(result.p2.1)<-rownames(result.p2.2)
<- rownames(result.p2.3) <- c("5","15","25","35","55")

result.p2 <- rbind(result.p2.1,result.p2.2,result.p2.3)
colnames(result.p2) <-c("URE","RE/LASSO","PT(0.15)",
"PT(0.2)","PT(0.25)","JSE","PRSE","RRE")
result.p2
```

Application Example: In this section, first the dataset are presented in Table S8. Then, the related codes are given.

```
library(xlsx); library(optimx)
mydata = read.xlsx('C:\\Pollution.xlsx',1)
XX=mydata[,-16]
a=princomp(XX,cor=T)

pdf("PCA.pdf", width = 16, height = 8, onefile = TRUE)
plot(a,main="Explained variance by the principal components")
dev.off()

X=a$scores
Y=mydata$MORT
summary(lm(Y~X))
poc=lm(Y~X)$coef[-1]

x1=X[,1]; x2=X[,2]; x3=X[,3]; x4=X[,4]; x5=X[,5]; x6=X[,6];
x7=X[,7]; x8=X[,8]; x9=X[,9]; x10=X[,10]; x11=X[,11];
x12=X[,12]; x13=X[,13]; x14=X[,14]; x15=X[,15]


L=function(b){
R=rank(Y-x1*b[1]-x2*b[2]-x3*b[3]-x4*b[4]-x5*b[5]-x6*b[6]
```

Table S8: The Dataset (McDonald and Schwing, 1973) used in Section 5

PREC	JANT	JULT	OVR65	POPN	EDUC	HOUS	DENS	NONW	WWDRK	POOR	HC	NOX	SO2	HUMID	MORT
36	27	71	8.1	3.34	11.4	81.5	3243	8.80	42.6	11.7	21	15	59	59	921.87
35	23	72	11.1	3.14	11	78.8	4281	3.5	50.7	14.4	8	10	39	57	997.875
44	29	74	10.4	3.21	9.80	81.60	4260	0.8	39.4	12.4	6	6	33	54	962.35
47	45	79	6.5	3.41	11.1	77.5	3125	27.1	50.2	20.6	18	8	24	56	982.29
43	35	77	7.6	3.44	9.6	84.6	6441	24.4	43.7	14.3	43	38	206	55	1071.289
53	45	80	7.7	3.45	10.20	66.8	3325	38.5	43.1	25.5	30	32	72	54	1030.38
43	30	74	10.9	3.23	12.1	83.9	4679	3.5	49.2	11.3	21	32	62	56	934.7
45	30	73	9.30	3.29	10.6	86	2140	5.3	40.4	10.5	6	4	4	56	899.529
36	24	70	9	3.31	10.5	83.2	6582	8.1	42.5	12.6	18	12	37	61	1001.902
36	27	72	9.5	3.36	10.7	79.3	4213	6.7	41	13.2	12	7	20	59	912.35
52	42	79	7.7	3.39	9.6	69.2	2302	22.2	41.3	24.2	18	8	27	56	1017.61
33	26	76	8.6	3.2	10.9	83.4	6122	16.3	44.9	10.7	88	63	278	58	1024.885
40	34	77	9.20	3.21	10.20	77	4101	13	45.7	15.1	26	26	146	57	970.47
35	28	71	8.80	3.29	11.1	86.3	3042	14.7	44.6	11.4	31	21	64	60	985.95
37	31	75	8	3.26	11.9	78.40	4259	13.1	49.6	13.9	23	9	15	58	958.84

Table S8 (Continued). The Dataset (McDonald and Schwing, 1973) used in Section 5

PREC	JANT	JULT	OVR65	POPN	EDUC	HOUS	DENS	NONW	WWDRK	POOR	HC	NOX	SO2	HUMID	MORT
35	46	85	7.1	3.22	11.8	79.90	1441	14.8	51.2	16.10	1	1	1	54	860.101
36	30	75	7.5	3.35	11.4	81.90	4029	12.4	44	12	6	4	16	58	936.23
15	30	73	8.20	3.15	12.2	84.2	4824	4.7	53.1	12.7	17	8	28	38	871.76
31	27	74	7.2	3.44	10.8	87	4834	15.8	43.5	13.6	52	35	124	59	959.221
30	24	72	6.5	3.53	10.8	79.5	3694	13.1	33.80	12.4	11	4	11	61	941.18
31	45	85	7.3	3.22	11.4	80.7	1844	11.5	48.1	18.5	1	1	1	53	891.71
31	24	72	9	3.37	10.9	82.8	3226	5.10	45.2	12.3	5	3	10	61	871.34
42	40	77	6.1	3.45	10.4	71.8	2269	22.7	41.4	19.5	8	3	5	53	971.12
43	27	72	9	3.25	11.5	87.1	2909	7.2	51.6	9.5	7	3	10	56	887.47
46	55	84	5.6	3.35	11.4	79.7	2647	21	46.9	17.90	6	5	1	59	952.529
39	29	75	8.70	3.23	11.4	78.60	4412	15.6	46.6	13.2	13	7	33	60	968.66
35	31	81	9.20	3.1	12	78.3	3262	12.6	48.6	13.9	7	4	4	55	919.73
43	32	74	10.1	3.38	9.5	79.2	3214	2.9	43.7	12	11	7	32	54	844.053
11	53	68	9.20	2.99	12.1	90.6	4700	7.8	48.9	12.3	648	319	130	47	861.83
30	35	71	8.30	3.37	9.9	77.40	4474	13.1	42.6	17.7	38	37	193	57	989.26

Table S8 (Continued). The Dataset (McDonald and Schwing, 1973) used in Section 5

PREC	JANT	JULT	OVR65	POPN	EDUC	HOUS	DENS	NONW	WWDRK	POOR	HC	NOX	SO2	HUMID	MORT
50	42	82	7.3	3.49	10.4	72.5	3497	36.70	43.3	26.4	15	18	34	59	1006.49
60	67	82	10	2.98	11.5	88.6	4657	13.5	47.3	22.4	3	1	1	60	861.44
30	20	69	8.80	3.26	11.1	85.4	2934	5.8	44	9.4	33	23	125	64	929.15
25	12	73	9.20	3.28	12.1	83.1	2095	2	51.9	9.80	20	11	26	58	857.62
45	40	80	8.30	3.32	10.1	70.3	2682	21	46.1	24.1	17	14	78	56	961.01
46	30	72	10.20	3.16	11.3	83.2	3327	8.80	45.3	12.2	4	3	8	58	923.23
54	54	81	7.4	3.36	9.70	72.8	3172	31.4	45.5	24.2	20	17	1	62	1113.15
42	33	77	9.70	3.03	10.7	83.5	7462	11.3	48.7	12.4	41	26	108	58	994.65
42	32	76	9.1	3.32	10.5	87.5	6092	17.5	45.3	13.2	29	32	161	54	1015.023
36	29	72	9.5	3.32	10.6	77.60	3437	8.1	45.5	13.8	45	59	263	56	991.29
37	38	67	11.3	2.99	12	81.5	3387	3.6	50.3	13.5	56	21	44	73	893.99
42	29	72	10.7	3.19	10.1	79.5	3508	2.20	38.80	15.7	6	4	18	56	938.5
41	33	77	11.2	3.08	9.6	79.90	4843	2.7	38.6	14.1	11	11	89	54	946.18
44	39	78	8.20	3.32	11	79.90	3768	28.6	49.5	17.5	12	9	48	53	1025.502
32	25	72	10.9	3.21	11.1	82.5	4355	5	46.4	10.8	7	4	18	60	874.28

Table S8 (Continued). The Dataset (McDonald and Schwing, 1973) used in Section 5

PREC	JANT	JULT	OVR65	POPN	EDUC	HOUS	DENS	NONW	WWDRK	POOR	HC	NOX	SO2	HUMID	MORT
34	32	79	9.30	3.23	9.70	76.8	5160	17.2	45.1	15.3	31	15	68	57	953.56
10	55	70	7.3	3.11	12.1	88.9	3033	5.9	51	14	144	66	20	61	839.71
18	48	63	9.20	2.92	12.2	87.7	4253	13.7	51.2	12	311	171	86	71	911.70
13	49	68	7	3.36	12.2	90.7	2702	3	51.9	9.70	105	32	3	71	790.73
35	40	64	9.6	3.02	12.2	82.5	3626	5.7	54.3	10.1	20	7	20	72	899.26
45	28	74	10.6	3.21	11.1	82.6	1883	3.4	41.9	12.3	5	4	20	56	904.15
38	24	72	9.80	3.34	11.4	78	4923	3.8	50.5	11.1	8	5	25	61	950.67
31	26	73	9.30	3.22	10.7	81.3	3249	9.5	43.9	13.6	11	7	25	59	972.46
40	23	71	11.3	3.28	10.3	73.8	1671	2.5	47.4	13.5	5	2	11	60	912.202
41	37	78	6.2	3.25	12.3	89.5	5308	25.9	59.7	10.3	65	28	102	52	967.803
28	32	81	7	3.27	12.1	81	3665	7.5	51.6	13.2	4	2	1	54	823.76
45	33	76	7.7	3.39	11.3	82.2	3152	12.1	47.3	10.9	14	11	42	56	1003.502
45	24	70	11.08	3.25	11.1	79.8	3678	1	44.8	14	7	3	8	56	895.7
42	33	76	9.70	3.22	9	76.2	9699	4.8	42.2	14.5	8	8	49	54	911.82
38	28	72	8.9	3.48	10.7	79.8	3451	11.7	37.5	13	14	13	39	58	954.44

```
-x7*b[7]-x8*b[8]-x9*b[9]-x10*b[10]-x11*b[11]-x12*b[12]
-x13*b[13]-x14*b[14]-x15*b[15])
L=0
L[1]=sum((x1-mean(x1))*R); L[2]=sum((x2-mean(x2))*R)
L[3]=sum((x3-mean(x3))*R); L[4]=sum((x4-mean(x4))*R)
L[5]=sum((x5-mean(x5))*R); L[6]=sum((x6-mean(x6))*R)
L[7]=sum((x7-mean(x7))*R); L[8]=sum((x8-mean(x8))*R)
L[9]=sum((x9-mean(x9))*R); L[10]=sum((x10-mean(x10))*R)
L[11]=sum((x11-mean(x11))*R); L[12]=sum((x12-mean(x12))*R)
L[13]=sum((x13-mean(x13))*R); L[14]=sum((x14-mean(x14))*R)
L[15]=sum((x15-mean(x15))*R)
K=L[1]^2+L[2]^2+L[3]^2+L[4]^2+L[5]^2+L[6]^2+L[7]^2+L[8]^2
+L[9]^2+L[10]^2+L[11]^2+L[12]^2+L[13]^2+L[14]^2+L[15]^2
return(K) }

O=optimx(poc,L); O
betaR=c(O$XComp.1[1],O$XComp.2[1],O$XComp.3[1],O$XComp.4[1],
O$XComp.5[1],O$XComp.6[1],O$XComp.7[1],O$XComp.8[1],
O$XComp.9[1],O$XComp.10[1],O$XComp.11[1],O$XComp.12[1],
O$XComp.13[1],O$XComp.14[1],O$XComp.15[1])

O=optimx(betaR,L); O
betaR=c(O$p1[1],O$p2[1],O$p3[1],O$p4[1],O$p5[1],O$p6[1],
O$p7[1],O$p8[1],O$p9[1],O$p10[1],O$p11[1],O$p12[1],O$p13[1],
O$p14[1],O$p15[1])

O=optimx(betaR,L); O

Jaeckel=function(b){
R=rank(Y-x1*b[1]-x2*b[2]-x3*b[3]-x4*b[4]-x5*b[5]-x6*b[6]
-x7*b[7]-x8*b[8]-x9*b[9]-x10*b[10]-x11*b[11]-x12*b[12]
-x13*b[13]-x14*b[14]-x15*b[15])
Jaeckel=sum((Y-x1*b[1]-x2*b[2]-x3*b[3]-x4*b[4]-x5*b[5]
-x6*b[6]-x7*b[7]-x8*b[8]-x9*b[9]-x10*b[10]-x11*b[11]
-x12*b[12]-x13*b[13]-x14*b[14]-x15*b[15])*R)
return(Jaeckel)}

O=optimx(poc,Jaeckel); O
betaR=c(O$XComp.1[1],O$XComp.2[1],O$XComp.3[1],O$XComp.4[1],
O$XComp.5[1],O$XComp.6[1],O$XComp.7[1],O$XComp.8[1],
O$XComp.9[1],O$XComp.10[1],O$XComp.11[1],O$XComp.12[1],
O$XComp.13[1],O$XComp.14[1],O$XComp.15[1])
```

```
O=optimx(betaR,Jaeckel); O
betaR=c(O$p1[1],O$p2[1],O$p3[1],O$p4[1],O$p5[1],O$p6[1],
O$p7[1],O$p8[1],O$p9[1],O$p10[1],O$p11[1],O$p12[1],O$p13[1],
O$p14[1],O$p15[1])
O=optimx(betaR,Jaeckel); O

betaR=c(O$p1[1],O$p2[1],O$p3[1],O$p4[1],O$p5[1],O$p6[1],
O$p7[1],O$p8[1],O$p9[1],O$p10[1],O$p11[1],O$p12[1],O$p13[1],
O$p14[1],O$p15[1])
O=optimx(betaR,Jaeckel); O

poc2=betaR

#bootstrap
beta1=0; beta2=0; beta3=0; beta4=0; beta5=0; beta6=0;
beta7=0; beta8=0; beta9=0; beta10=0; beta11=0; beta12=0;
beta13=0; beta14=0; beta15=0

beta1P1=0; beta2P1=0; beta3P1=0; beta4P1=0; beta5P1=0;
beta6P1=0; beta7P1=0; beta8P1=0; beta9P1=0; beta10P1=0
beta11P1=0; beta12P1=0; beta13P1=0; beta14P1=0; beta15P1=0

beta1P2=0; beta2P2=0; beta3P2=0; beta4P2=0; beta5P2=0;
beta6P2=0; beta7P2=0; beta8P2=0; beta9P2=0; beta10P2=0;
beta11P2=0; beta12P2=0; beta13P2=0; beta14P2=0; beta15P2=0

beta1P3=0; beta2P3=0; beta3P3=0; beta4P3=0; beta5P3=0;
beta6P3=0; beta7P3=0; beta8P3=0; beta9P3=0; beta10P3=0;
beta11P3=0; beta12P3=0; beta13P3=0; beta14P3=0; beta15P3=0

beta1RE=0; beta2RE=0; beta3RE=0; beta4RE=0; beta5RE=0;
beta6RE=0; beta7RE=0; beta8RE=0; beta9RE=0; beta10RE=0;
beta11RE=0; beta12RE=0; beta13RE=0; beta14RE=0; beta15RE=0

beta1JS=0; beta2JS=0; beta3JS=0; beta4JS=0; beta5JS=0;
beta6JS=0; beta7JS=0; beta8JS=0; beta9JS=0; beta10JS=0;
beta11JS=0; beta12JS=0; beta13JS=0; beta14JS=0; beta15JS=0;

beta1JS2=0; beta2JS2=0; beta3JS2=0; beta4JS2=0;
beta5JS2=0; beta6JS2=0; beta7JS2=0; beta8JS2=0;
beta9JS2=0; beta10JS2=0; beta11JS2=0; beta12JS2=0
beta13JS2=0; beta14JS2=0; beta15JS2=0;
```

```
n=60
b=betaR
e=Y-x1*b[1]-x2*b[2]-x3*b[3]-x4*b[4]-x5*b[5]-x6*b[6]
-x7*b[7]-x8*b[8]-x9*b[9]-x10*b[10]-x11*b[11]-x12*b[12]
-x13*b[13]-x14*b[14]-x15*b[15]

opak=1000
for (i in 1:opak){
E=sample(e,n,replace=T)
Y2=-(-x1*b[1]-x2*b[2]-x3*b[3]-x4*b[4]-x5*b[5]-x6*b[6]
-x7*b[7]-x8*b[8]-x9*b[9]-x10*b[10]-x11*b[11]-x12*b[12]
-x13*b[13]-x14*b[14]-x15*b[15])+E

Jaeckel=function(b){
R=rank(Y2-x1*b[1]-x2*b[2]-x3*b[3]-x4*b[4]-x5*b[5]-x6*b[6]
-x7*b[7]-x8*b[8]-x9*b[9]-x10*b[10]-x11*b[11]-x12*b[12]
-x13*b[13]-x14*b[14]-x15*b[15])
Jaeckel=sum((Y2-x1*b[1]-x2*b[2]-x3*b[3]-x4*b[4]-x5*b[5]
-x6*b[6]-x7*b[7]-x8*b[8]-x9*b[9]-x10*b[10]-x11*b[11]
-x12*b[12]-x13*b[13]-x14*b[14]-x15*b[15])*R)
return(Jaeckel)}

O=optimx(betaR,Jaeckel)
betaR=c(O$p1[1],O$p2[1],O$p3[1],O$p4[1],O$p5[1],O$p6[1],
O$p7[1],O$p8[1],O$p9[1],O$p10[1],O$p11[1],O$p12[1],
O$p13[1],O$p14[1],O$p15[1])
O=optimx(betaR,Jaeckel)
betaR=c(O$p1[1],O$p2[1],O$p3[1],O$p4[1],O$p5[1],O$p6[1],
O$p7[1],O$p8[1],O$p9[1],O$p10[1],O$p11[1],O$p12[1],
O$p13[1],O$p14[1],O$p15[1])
O=optimx(betaR,Jaeckel)
betaR=c(O$p1[1],O$p2[1],O$p3[1],O$p4[1],O$p5[1],O$p6[1],
O$p7[1],O$p8[1],O$p9[1],O$p10[1],O$p11[1],O$p12[1],
O$p13[1],O$p14[1],O$p15[1])
O=optimx(betaR,Jaeckel)
betaR=c(O$p1[1],O$p2[1],O$p3[1],O$p4[1],O$p5[1],O$p6[1],
O$p7[1],O$p8[1],O$p9[1],O$p10[1],O$p11[1],O$p12[1],
O$p13[1],O$p14[1],O$p15[1])
O=optimx(betaR,Jaeckel)
betaR=c(O$p1[1],O$p2[1],O$p3[1],O$p4[1],O$p5[1],O$p6[1],
O$p7[1],O$p8[1],O$p9[1],O$p10[1],O$p11[1],O$p12[1],
O$p13[1],O$p14[1],O$p15[1])
```

```
beta1[i]=betaR[1]; beta2[i]=betaR[2]; beta3[i]=betaR[3];
beta4[i]=betaR[4]; beta5[i]=betaR[5]; beta6[i]=betaR[6]
beta7[i]=betaR[7]; beta8[i]=betaR[8]; beta9[i]=betaR[9];
beta10[i]=betaR[10]; beta11[i]=betaR[11];
beta12[i]=betaR[12]; beta13[i]=betaR[13];
beta14[i]=betaR[14]; beta15[i]=betaR[15]

beta1RE[i]=betaR[1]; beta2RE[i]=0; beta3RE[i]=betaR[3];
beta4RE[i]=0;  beta5RE[i]=0; beta6RE[i]=betaR[6]
beta7RE[i]=betaR[7]; beta8RE[i]=0; beta9RE[i]=betaR[9]
beta10RE[i]=0; beta11RE[i]=0; beta12RE[i]=betaR[12]
beta13RE[i]=0; beta14RE[i]=0; beta15RE[i]=0

R=rank(Y)
An=var(R)
Ln=(sum((x2-mean(x2))*R))^2+(sum((x4-mean(x4))*R))^2
+(sum((x5-mean(x5))*R))^2+(sum((x8-mean(x8))*R))^2
+(sum((x10-mean(x10))*R))^2+(sum((x11-mean(x11))*R))^2
+(sum((x13-mean(x13))*R))^2+(sum((x14-mean(x14))*R))^2
+(sum((x15-mean(x15))*R))^2
Ln2=Ln/n/An

beta1P1[i]=betaR[1]
beta2P1[i]=betaR[2]*(Ln2>qchisq(0.85,p2))
beta3P1[i]=betaR[3]
beta4P1[i]=betaR[4]*(Ln2>qchisq(0.85,p2))
beta5P1[i]=betaR[5]*(Ln2>qchisq(0.85,p2))
beta6P1[i]=betaR[6]
beta7P1[i]=betaR[7]
beta8P1[i]=betaR[8]*(Ln2>qchisq(0.85,p2))
beta9P1[i]=betaR[9]
beta10P1[i]=betaR[10]*(Ln2>qchisq(0.85,p2))
beta11P1[i]=betaR[11]*(Ln2>qchisq(0.85,p2))
beta12P1[i]=betaR[12]
beta13P1[i]=betaR[13]*(Ln2>qchisq(0.85,p2))
beta14P1[i]=betaR[14]*(Ln2>qchisq(0.85,p2))
beta15P1[i]=betaR[15]*(Ln2>qchisq(0.85,p2))

beta1P2[i]=betaR[1]
beta2P2[i]=betaR[2]*(Ln2>qchisq(0.80,p2))
beta3P2[i]=betaR[3]
beta4P2[i]=betaR[4]*(Ln2>qchisq(0.80,p2))
```

```
beta5P2[i]=betaR[5]*(Ln2>qchisq(0.80,p2))
beta6P2[i]=betaR[6]
beta7P2[i]=betaR[7]
beta8P2[i]=betaR[8]*(Ln2>qchisq(0.80,p2))
beta9P2[i]=betaR[9]
beta10P2[i]=betaR[10]*(Ln2>qchisq(0.80,p2))
beta11P2[i]=betaR[11]*(Ln2>qchisq(0.80,p2))
beta12P2[i]=betaR[12]
beta13P2[i]=betaR[13]*(Ln2>qchisq(0.80,p2))
beta14P2[i]=betaR[14]*(Ln2>qchisq(0.80,p2))
beta15P2[i]=betaR[15]*(Ln2>qchisq(0.80,p2))

beta1P3[i]=betaR[1]
beta2P3[i]=betaR[2]*(Ln2>qchisq(0.75,p2))
beta3P3[i]=betaR[3]
beta4P3[i]=betaR[4]*(Ln2>qchisq(0.75,p2))
beta5P3[i]=betaR[5]*(Ln2>qchisq(0.75,p2))
beta6P3[i]=betaR[6]
beta7P3[i]=betaR[7]
beta8P3[i]=betaR[8]*(Ln2>qchisq(0.75,p2))
beta9P3[i]=betaR[9]
beta10P3[i]=betaR[10]*(Ln2>qchisq(0.75,p2))
beta11P3[i]=betaR[11]*(Ln2>qchisq(0.75,p2))
beta12P3[i]=betaR[12]
beta13P3[i]=betaR[13]*(Ln2>qchisq(0.75,p2))
beta14P3[i]=betaR[14]*(Ln2>qchisq(0.75,p2))
beta15P3[i]=betaR[15]*(Ln2>qchisq(0.75,p2))

beta1JS[i]=betaR[1]
beta2JS[i]=betaR[2]*(1-1/Ln2)
beta3JS[i]=betaR[3]
beta4JS[i]=betaR[4]*(1-1/Ln2)
beta5JS[i]=betaR[5]*(1-1/Ln2)
beta6JS[i]=betaR[6]
beta7JS[i]=betaR[7]
beta8JS[i]=betaR[8]*(1-1/Ln2)
beta9JS[i]=betaR[9]
beta10JS[i]=betaR[10]*(1-1/Ln2)
beta11JS[i]=betaR[11]*(1-1/Ln2)
beta12JS[i]=betaR[12]
beta13JS[i]=betaR[13]*(1-1/Ln2)
beta14JS[i]=betaR[14]*(1-1/Ln2)
```

```
beta15JS[i]=betaR[15]*(1-1/Ln2)

beta1JS2[i]=betaR[1]
beta2JS2[i]=betaR[2]*(1-1/Ln2)*(Ln2>1)
beta3JS2[i]=betaR[3]
beta4JS2[i]=betaR[4]*(1-1/Ln2)*(Ln2>1)
beta5JS2[i]=betaR[5]*(1-1/Ln2)*(Ln2>1)
beta6JS2[i]=betaR[6]
beta7JS2[i]=betaR[7]
beta8JS2[i]=betaR[8]*(1-1/Ln2)*(Ln2>1)
beta9JS2[i]=betaR[9]
beta10JS2[i]=betaR[10]*(1-1/Ln2)*(Ln2>1)
beta11JS2[i]=betaR[11]*(1-1/Ln2)*(Ln2>1)
beta12JS2[i]=betaR[12]
beta13JS2[i]=betaR[13]*(1-1/Ln2)*(Ln2>1)
beta14JS2[i]=betaR[14]*(1-1/Ln2)*(Ln2>1)
beta15JS2[i]=betaR[15]*(1-1/Ln2)*(Ln2>1)}

bb=b
sd(beta1)+sd(beta2)+sd(beta3)+sd(beta4)+sd(beta5)+sd(beta6)
+sd(beta7)+sd(beta8)+sd(beta9)+sd(beta10)+sd(beta11)+sd(beta12)
+sd(beta13)+sd(beta14)+sd(beta15)
sqrt(sum((beta1-b[1])^2)/opak )+
sqrt(sum((beta2-b[2])^2)/opak )+
sqrt(sum((beta3-b[3])^2)/opak )+
sqrt(sum((beta4-b[4])^2)/opak )+
sqrt(sum((beta5-b[5])^2)/opak )+
sqrt(sum((beta6-b[6])^2)/opak )+
sqrt(sum((beta7-b[7])^2)/opak )+
sqrt(sum((beta8-b[8])^2)/opak )+
sqrt(sum((beta9-b[9])^2)/opak )+
sqrt(sum((beta10-b[10])^2)/opak )+
sqrt(sum((beta11-b[11])^2)/opak )+
sqrt(sum((beta12-b[12])^2)/opak )+
sqrt(sum((beta13-b[13])^2)/opak )+
sqrt(sum((beta14-b[14])^2)/opak )+
sqrt(sum((beta15-b[15])^2)/opak )

b=c(b[1],0,b[3],0,0,b[6],b[7],0,b[9],0,0,b[12],0,0,0)

sqrt(sum((beta1RE-b[1])^2)/opak )+
sqrt(sum((beta2RE-b[2])^2)/opak )+
```

```
sqrt(sum((beta3RE-b[3])^2)/opak )+
sqrt(sum((beta4RE-b[4])^2)/opak )+
sqrt(sum((beta5RE-b[5])^2)/opak )+
sqrt(sum((beta6RE-b[6])^2)/opak )+
sqrt(sum((beta7RE-b[7])^2)/opak )+
sqrt(sum((beta8RE-b[8])^2)/opak )+
sqrt(sum((beta9RE-b[9])^2)/opak )+
sqrt(sum((beta10RE-b[10])^2)/opak )+
sqrt(sum((beta11RE-b[11])^2)/opak )+
sqrt(sum((beta12RE-b[12])^2)/opak )+
sqrt(sum((beta13RE-b[13])^2)/opak )+
sqrt(sum((beta14RE-b[14])^2)/opak )+
sqrt(sum((beta15RE-b[15])^2)/opak )

b=bb
b[nind]=b[nind]*(1-1/Ln2)

sqrt(sum((beta1JS-b[1])^2)/opak )+
sqrt(sum((beta2JS-b[2])^2)/opak )+
sqrt(sum((beta3JS-b[3])^2)/opak )+
sqrt(sum((beta4JS-b[4])^2)/opak )+
sqrt(sum((beta5JS-b[5])^2)/opak )+
sqrt(sum((beta6JS-b[6])^2)/opak )+
sqrt(sum((beta7JS-b[7])^2)/opak )+
sqrt(sum((beta8JS-b[8])^2)/opak )+
sqrt(sum((beta9JS-b[9])^2)/opak )+
sqrt(sum((beta10JS-b[10])^2)/opak )+
sqrt(sum((beta11JS-b[11])^2)/opak )+
sqrt(sum((beta12JS-b[12])^2)/opak )+
sqrt(sum((beta13JS-b[13])^2)/opak )+
sqrt(sum((beta14JS-b[14])^2)/opak )+
sqrt(sum((beta15JS-b[15])^2)/opak )

b=bb
b[nind]=b[nind]*(1-1/Ln2)*(Ln2>1)

sqrt(sum((beta1JS2-b[1])^2)/opak )+
sqrt(sum((beta2JS2-b[2])^2)/opak )+
sqrt(sum((beta3JS2-b[3])^2)/opak )+
sqrt(sum((beta4JS2-b[4])^2)/opak )+
sqrt(sum((beta5JS2-b[5])^2)/opak )+
sqrt(sum((beta6JS2-b[6])^2)/opak )+
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sqrt(sum((beta7JS2-b[7])^2)/opak )+
sqrt(sum((beta8JS2-b[8])^2)/opak )+
sqrt(sum((beta9JS2-b[9])^2)/opak )+
sqrt(sum((beta10JS2-b[10])^2)/opak )+
sqrt(sum((beta11JS2-b[11])^2)/opak )+
sqrt(sum((beta12JS2-b[12])^2)/opak )+
sqrt(sum((beta13JS2-b[13])^2)/opak )+
sqrt(sum((beta14JS2-b[14])^2)/opak )+
sqrt(sum((beta15JS2-b[15])^2)/opak )

b=bb
b[nind]=b[nind]*(Ln2>qchisq(0.85,p2))

sqrt(sum((beta1P1-b[1])^2)/opak )+
sqrt(sum((beta2P1-b[2])^2)/opak )+
sqrt(sum((beta3P1-b[3])^2)/opak )+
sqrt(sum((beta4P1-b[4])^2)/opak )+
sqrt(sum((beta5P1-b[5])^2)/opak )+
sqrt(sum((beta6P1-b[6])^2)/opak )+
sqrt(sum((beta7P1-b[7])^2)/opak )+
sqrt(sum((beta8P1-b[8])^2)/opak )+
sqrt(sum((beta9P1-b[9])^2)/opak )+
sqrt(sum((beta10P1-b[10])^2)/opak )+
sqrt(sum((beta11P1-b[11])^2)/opak )+
sqrt(sum((beta12P1-b[12])^2)/opak )+
sqrt(sum((beta13P1-b[13])^2)/opak )+
sqrt(sum((beta14P1-b[14])^2)/opak )+
sqrt(sum((beta15P1-b[15])^2)/opak )

b=bb
b[nind]=b[nind]*(Ln2>qchisq(0.80,p2))

sqrt(sum((beta1P2-b[1])^2)/opak )+
sqrt(sum((beta2P2-b[2])^2)/opak )+
sqrt(sum((beta3P2-b[3])^2)/opak )+
sqrt(sum((beta4P2-b[4])^2)/opak )+
sqrt(sum((beta5P2-b[5])^2)/opak )+
sqrt(sum((beta6P2-b[6])^2)/opak )+
sqrt(sum((beta7P2-b[7])^2)/opak )+
sqrt(sum((beta8P2-b[8])^2)/opak )+
sqrt(sum((beta9P2-b[9])^2)/opak )+
sqrt(sum((beta10P2-b[10])^2)/opak )+
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sqrt(sum((beta11P2-b[11])^2)/opak )+
sqrt(sum((beta12P2-b[12])^2)/opak )+
sqrt(sum((beta13P2-b[13])^2)/opak )+
sqrt(sum((beta14P2-b[14])^2)/opak )+
sqrt(sum((beta15P2-b[15])^2)/opak )

b=bb
b[nind]=b[nind]*(Ln2>qchisq(0.75,p2))

sqrt(sum((beta1P3-b[1])^2)/opak )+
sqrt(sum((beta2P3-b[2])^2)/opak )+
sqrt(sum((beta3P3-b[3])^2)/opak )+
sqrt(sum((beta4P3-b[4])^2)/opak )+
sqrt(sum((beta5P3-b[5])^2)/opak )+
sqrt(sum((beta6P3-b[6])^2)/opak )+
sqrt(sum((beta7P3-b[7])^2)/opak )+
sqrt(sum((beta8P3-b[8])^2)/opak )+
sqrt(sum((beta9P3-b[9])^2)/opak )+
sqrt(sum((beta10P3-b[10])^2)/opak )+
sqrt(sum((beta11P3-b[11])^2)/opak )+
sqrt(sum((beta12P3-b[12])^2)/opak )+
sqrt(sum((beta13P3-b[13])^2)/opak )+
sqrt(sum((beta14P3-b[14])^2)/opak )+
sqrt(sum((beta15P3-b[15])^2)/opak )

summary(lm(Y~X))

p=15
p1=6
p2=9
#R=rank(Y-x1*b[1]-x2*b[2]-x3*b[3]-x4*b[4]-x5*b[5]-x6*b[6]
-x7*b[7]-x8*b[8]-x9*b[9]-x10*b[10])

#1,3,6,7,9,12
ind=c(1,3,6,7,9,12)
nind=c(2,4,5,8,10,11,13,14,15)

n=60
R=rank(Y)
An=var(R)
Ln=(sum((x2-mean(x2))*R))^2+(sum((x4-mean(x4))*R))^2
+(sum((x5-mean(x5))*R))^2+(sum((x8-mean(x8))*R))^2
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+(sum((x10-mean(x10))*R))^2+(sum((x11-mean(x11))*R))^2
+(sum((x13-mean(x13))*R))^2+(sum((x14-mean(x14))*R))^2
+(sum((x15-mean(x15))*R))^2
Ln2=Ln/n/An
Ln2
qchisq(0.90,p2)

betaRE=c(betaR[1],0,betaR[3],0,0,betaR[6],betaR[7],0,
betaR[9],0,0,,betaR[12],0,0,0)

betaJS=betaR; betaJS[nind]=betaJS[nind]*(1-1/Ln2)

betaPRSE=betaR
betaPRSE[nind]=betaPRSE[nind]*(1-1/Ln2)*(Ln2>1)

betaP1=betaR
betaP1[nind]=betaP1[nind]*(Ln2>qchisq(0.85,p2))

betaP2=betaR
betaP2[nind]=betaP2[nind]*(Ln2>qchisq(0.80,p2))

betaP3=betaR
betaP3[nind]=betaP3[nind]*(Ln2>qchisq(0.75,p2))

betaP1=c(betaR[1:p1],betaR[6:15]*(Ln2>qchisq(0.85,p2)))
betaP2=c(betaR[1:p1],betaR[6:15]*(Ln2>qchisq(0.8,p2)))
betaP3=c(betaR[1:p1],betaR[6:15]*(Ln2>qchisq(0.75,p2)))

#LASSO
LA=function(b){
R=rank(Y-x1*b[1]-x2*b[2]-x3*b[3]-x4*b[4]-x5*b[5]-x6*b[6]
-x7*b[7]-x8*b[8]-x9*b[9]-x10*b[10]-x11*b[11]-x12*b[12]
-x13*b[13]-x14*b[14]-x15*b[15])
LA=sum((Y-x1*b[1]-x2*b[2]-x3*b[3]-x4*b[4]-x5*b[5]-x6*b[6]
-x7*b[7]-x8*b[8]-x9*b[9]-x10*b[10]-x11*b[11]-x12*b[12]
-x13*b[13]-x14*b[14]-x15*b[15])*R)+sum(b^2)*1000
return(LA) }

O=optimx(poc,LA); O
betaL=c(O$XComp.1[1],O$XComp.2[1],O$XComp.3[1],
O$XComp.4[1],O$XComp.5[1],O$XComp.6[1],O$XComp.7[1],
O$XComp.8[1],O$XComp.9[1],O$XComp.10[1],O$XComp.11[1],
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O$XComp.12[1],O$XComp.13[1],O$XComp.14[1],O$XComp.15[1])
O=optimx(betaL,LA); O
betaL=c(O$p1[1],O$p2[1],O$p3[1],O$p4[1],O$p5[1],O$p6[1],
O$p7[1],O$p8[1],O$p9[1],O$p10[1],O$p11[1],O$p12[1],O$p13[1],
O$p14[1],O$p15[1])
O=optimx(betaL,LA)
```