# Modern Techniques in Quantum Field Theory

by

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# Dedication

*To my family, who gave me everything expecting nothing in return.*

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# Abstract

In this thesis, I discuss three projects on modern approaches to quantum field theory. Firstly, I present a novel approach to holographic renormalization, using an algorithmic method that utilizes the Hamilton-Jacobi equation and the Hamiltonian formulation of gravity. Secondly, I discuss the validity and applications of soft-subtracted on-shell recursion relations for exceptional effective field theories. In particular, using the soft bootstrap method, I examine the possibility of a supersymmetric non-linear sigma model. I also study Galileon theories, in terms of their compatibility with supersymmetry and possible higher-derivative corrections to the so-called special Galileon. Finally, I calculate the logarithmic contribution to the entanglement entropy of a free scalar or a free fermion in 3 spacetime dimensions.

CHAPTER 1

# Introduction

Quantum field theory (QFT) has been the theoretical foundation of particle physics of the last century. The standard model of theoretical physics is a quantum field theory, whose predictions have been verified by multiple experiments. Most analytic calculations in QFT are performed in the context of perturbation theory. Despite the numerous successes of this method, there are significant limitations that require new lines of attack. Strongly-coupled systems with no small parameters, like low-energy quantum chromodynamics (the theory of strong interactions) cannot be treated perturbatively. Even in weakly-coupled systems, some calculations are too technical to carry out in practice, in part because of redundancy in the traditional Lagrangian description of field theory.

Several new approaches to QFT have been introduced that try to circumvent these and other problems. In this thesis, I discuss three such approaches: gauge-gravity duality, the scattering amplitudes program, and an approach using quantum information theory. Studying different techniques is essential for a complete understanding of quantum field theories.

## 1.1 Gauge-Gravity Duality

*Gauge-gravity duality* has been a major breakthrough of theoretical high-energy physics. It has immediate applications in particle physics and gravitational physics, but it has also found other applications, like the description and study of critical condensed matter systems.

Motivated by string theory, gauge-gravity duality relates physical quantities in quantum field theories to quantities in a theory of gravity (like general relativity or string theory). The underlying principle of the duality, called the *holographic principle*, states that a gravitational theory in the bulk of a spacetime can be described solely in terms of degrees of freedom and interactions of a quantum field theory that lives on the boundary of this spacetime. A well-known example of this is that the leading contribution to the Bekenstein-Hawking entropy (a measure of degrees of freedom) of a black hole is proportional to the area of the black-hole horizon instead of the volume of the enclosed space. The holographic principle is generic and it is expected to be generally true for gravitational theories [1]. Nevertheless, only a handful of pairs of dual gauge-gravity theories are well-understood.

Perhaps the most celebrated example of a gauge-gravity duality is the AdS/CFT correspondence [2, 3, 4, 5]. The original form of the duality conjectured that type IIB string theory on $\mathrm{AdS}_5 \times S^5$ ($\mathrm{AdS}_5$ stands for 5-dimensional anti-de Sitter space and $S^5$ for 5-dimensional sphere) is dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory that lives on the 4-dimensional boundary of AdS. $\mathcal{N} = 4$ supersymmetric Yang-Mills is a conformal field theory (CFT) i.e. it is invariant under an extended group of spacetime symmetries that is larger than the usual Lorentz and translational symmetries of ordinary quantum field theories. The large amount of symmetry and supersymmetry that this system manifests, makes it an ideal playground to understand the properties of gauge-gravity duality.

A very useful feature of the AdS/CFT correspondence is that it is a duality between weak and strong couplings. Under certain conditions, the large-coupling limit of the gauge theory is dual to the limit where the bulk string theory reduces to classical weakly-coupled supergravity, whose dynamics are described by Einstein's equation. Therefore, using the duality we can make predictions about strongly-coupled field theory (normally very cumbersome) by performing relatively simpler calculations in the weakly-coupled gravitational theory. This strong-weak correspondence is a generic feature in gauge-gravity duality, which makes it appealing for describing systems with large couplings like critical systems in condensed matter physics, or potentially low-energy quantum chromodynamics.[1]

[1]There is no exact holographic dual known for quantum chromodynamics.

A common artifact of calculations in quantum field theory is that they are often plagued by infinities. These infinities often arise from integrals over a range of arbitrarily large momenta. In the context of gauge-gravity duality, such infinities of the boundary field theory correspond to infinities of the bulk gravitational theory that connect to the infinite volume of the bulk spacetime.

The systematic procedure for removing infinities from a calculation in QFT and extracting the physically interesting quantities is called *renormalization*. It involves modifying the action of a model by the addition of *counterterms*, which are chosen such that their contribution makes all physical quantities finite. The corresponding procedure in the bulk dual is called *holographic renormalization*. As in the boundary case, it involves modifying the action of the model by the addition of boundary counterterms that cancel the infinities in all calculations of the gravitational model.

The traditional method of calculating the necessary counterterms [6, 7, 8, 9] involves a laborious, yet algorithmic procedure. The first step of this procedure is to write the metric and matter fields of the theory as a power series in the radial coordinate of AdS. This expansion of the fields near the boundary is called the Fefferman-Graham (FG) expansion [10]. The coefficients in the expansion are determined using Einstein's equation and the equations of motion for the matter fields. For each field there remain two undetermined coefficients that correspond to the sources and vacuum expectation values of the operators of the boundary theory. For the next step, one substitutes the Fefferman-Graham expansion of the fields into the action integral of the model. This results in the on-shell action, which is expressed as a power series in the radial coordinate, with coefficients that depend on the free coefficients of the FG expansion and their derivatives. The terms in the expansion that diverge near the boundary are the terms that need to be cancelled in order to cure the infinities of the theory. The last step is to isolate these divergent terms (which recall are written in terms of the free FG parameters) and by reversing the FG expansion write them in terms of boundary values of the fields of the model. The resulting expression is the negative of the necessary counterterm action. Adding this to the original action will guarantee that all infinities are cancelled.

This "brute force" procedure can in principle produce the counterterm action for any holographic model. However, it seems redundant that one has to employ the FG expansion only to reverse it at the end; a different approach to this calculation, one that employs the Hamiltonian formulation of gravity and the Hamilton-Jacobi equation has been proposed by several authors to resolve this redundancy. In fact, since the early days of AdS/CFT, de Boer, Verlinde and Verlinde [11] have related this application of the Hamilton-Jacobi equation to the Callan-Symanzik equation, which describes how fields of the (boundary) quantum field theory behave under renormalization. (See also [12, 13, 14].)

In classical mechanics, the Hamilton-Jacobi equation,

$$\frac{\partial S_{\text{on-shell}}}{\partial t} + H = 0\,, \tag{1.1}$$

where $H$ is the Hamiltonian of the system can be used to determine the on-shell action of a system. Notice that if the Hamiltonian of the system is known, the Hamilton-Jacobi equation is a differential equation that can be directly solved for the on-shell action at a given time $t$. In a holographic system, the time $t$ is replaced by the radial coordinate $r$ of anti-de Sitter space and the Hamilton-Jacobi equation is used to determine the on-shell action for a given value of $r$. Specifically, we are interested in the case where the value of $r$ reaches the boundary value.

The first time the Hamilton-Jacobi equation was used to determine the counterterm action was by Kalkkinen, Martelli, and Mueck in [15, 16]. Their idea was further developed by Papadimitriou and Skenderis in a series of papers that used the Hamilton-Jacobi equation to calculate the counterterm action for specific models [17, 18, 19, 20]. Their method, however, is rather opaque and not as algorithmic as one would desire for a generally applicable method. It moreover has certain limitations, since it cannot be applied to all holographic models because of the authors' choice to organize their calculation using eigenfunctions of the dilatation operator. As we explain in chapter 2, not all fields of a model can be expressed in terms of such functions.

The goal of our work [21] was to present a simple algorithmic method for Hamilton-Jacobi approach for holographic renormalization that is applicable to a generic class of models. Our method significantly simplifies the "brute force" Fefferman-Graham approach to holographic renormalization. We also avoid using the dilatation operator as an organizing principle, to make our method more generically applicable. Instead we organize the different terms in a derivative expansion, as was also suggested in [11, 16, 20]. Moreover, we have provided detailed examples that demonstrate the mechanics of our method and we show that it reproduces well-known results in the literature.

This work is discussed in detail in chapter 2. It is based on the paper titled *"A Practical Approach to the Hamilton-Jacobi Formulation of Holographic Renormalization"* [21], written in collaboration with Henriette Elvang and published in the Journal of High Energy Physics.

## 1.2 The Scattering Amplitudes Program

In particle physics, scattering experiments measure the probability of a final state $f$ given an initial state $i$. The probability is then used to calculate the *scattering cross-section* $\sigma$, which can be directly compared to theoretical predictions. On the theory side, the probability amplitude for transitioning from the initial to the final state,

$$\mathcal{A}_n = \langle f|i\rangle \,, \tag{1.2}$$

is called the *scattering amplitude*. After integration over the phase space of the involved particles,[2] the squared scattering amplitude gives a theoretical prediction for the cross-section

[2] For example, for 2-to-$n$ particle scattering, the integration is taken over all possible values of the momenta of the final-state particles, subject to energy and momentum conservation.

of the interaction,

$$\sigma \sim \int |\mathcal{A}|^2 \,. \tag{1.3}$$

Nowadays, the agreement between theoretical predictions and experimental results is astonishing. The most celebrated example is the anomalous magnetic moment of the electron, whose theoretical prediction and experimental measurements are in agreement with a precision of ten significant digits. This makes clear that, in order to keep testing our theoretical models and possibly uncover new physics, we will need to calculate scattering amplitudes with more and more precision and better understand their properties.

The textbook method of calculating the scattering amplitudes is by the use of Feynman diagrams. In this method, each interaction term in the Lagrangian of the system is represented by a vertex. There are also propagators that connect two vertices and external lines that represent the particles in the initial and final state. Using a well-defined set of rules (the Feynman rules) each diagram is converted to a mathematical expression that contributes to the scattering amplitude. The full amplitude is the sum of contributions from all diagrams that have the correct initial and final states.

Feynman diagrams are a natural way to apply perturbation theory in a model of particle physics with a small coupling constant. Each interaction vertex that enters a diagram corresponds to a power of the coupling constant. Therefore, the leading-order contribution to the amplitude corresponds to Feynman diagrams that have the minimum number of vertices. These are tree diagrams, with no closed loops. Sub-leading contributions correspond to more complicated diagrams with an increasing number of loops, such that the loop expansion of the amplitude corresponds to writing it as a power series in the small coupling.

Despite being a very well-defined and algorithmic method for calculating the scattering amplitude, the Feynman method has drawbacks. In complicated theories, like a quantum theory of gravity there is an infinite number of interaction terms in the Lagrangian, which means an infinite number of interaction vertices to consider. Additionally, for processes that involve a large number of initial and final particles, even for simpler theories, the vast number of diagrams one needs to consider often makes the calculation of the scattering amplitude practically impossible. The canonical example of this is Yang-Mills theory. The amplitude describing the scattering of 4 gluons requires the summation over 4 tree-level Feynman diagrams. However, the one for scattering of 10 gluons requires more than a million diagrams, even at tree-level. Although this situation seems hopeless, in 1985 Parke and Taylor [22], motivated by earlier calculations of the 4, 5 and 6-gluon amplitudes, proposed a simple formula for the squared amplitude of scattering of any number $n$ of gluons in the MHV configuration (Maximum Helicity Violation–2 gluons with positive helicity scatter to $n-2$ gluons of positive helicity). Their formula was soon proven recursively by Berends and Giele [23], who showed that written in terms of the right variables (the so-called helicity basis) the MHV amplitude has a simple one-line expression.

The above discussion makes clear that the Feynman approach to scattering amplitudes is not always ideal, since, often, the calculations are plagued by massive redundancies. Consider the case of vector bosons, for example gluons. In the Lagrangian picture these particles are described by a Lorentz vector field $A_\mu(x)$. However, the physical quantities in the model are invariant under an (infinitesimal) *gauge transformation* of the field $A_\mu \to A_\mu + \partial_\mu \Lambda - ig[A_\mu, \Lambda]$, for any function $\Lambda$. This means that, in reality, each physical state corresponds to an infinite class of field configurations. Often, one may choose to impose an additional condition on the field $A_\mu$, such that this gauge redundancy is resolved. This process is called gauge fixing. In the Feynman calculation, different gauge choices may lead to the vanishing of different subsets of diagrams, while the final result, which is the sum of all the remaining diagrams, does not change. This leads to the conclusion that, as a result of the gauge freedom of the gluon field, the information about the scattering amplitude is redundantly encoded in the Feynman expansion and that there is a more efficient way to calculate gluon scattering amplitudes.

During the last decades, there has been significant progress in finding more efficient ways to calculate scattering amplitudes. This effort is known as the *scattering amplitudes or* *modern S-matrix program.* Its main philosophy is to explore the properties of scattering amplitudes in order to find more efficient techniques to calculate them.

A major breakthrough of the S-matrix program was the development of on-shell recursion relations for tree-level scattering amplitudes. The first example of an on-shell recursive reconstruction of amplitudes was in the form of the Britto-Cachazo-Feng-Witten (BCFW) recursion relations [24, 25]. With these, one can determine recursively the tree-level amplitudes of the Yang-Mills model for any multiplicity of external states without the use of a Lagrangian or Feynman rules.

The basic mechanism of BCFW recursion relies on the contour integral

$$\oint_c \frac{\mathrm{d}z}{2\pi i} \frac{\hat{\mathcal{A}}_n(z)}{z} = 0\,. \tag{1.4}$$

Here, the external momenta are shifted by a quantity that depends on the complex variable $z$, such that different values of $z$ correspond to different kinematic configurations of the external states. The amplitude then becomes a function of the complex parameter $z$; one writes $\mathcal{A}_n \to \hat{\mathcal{A}}_n(z)$. If the momentum shift is chosen appropriately, the integral (1.4) has no contribution from a pole at infinity and by Cauchy's integral theorem it vanishes when the interior of the closed curve $c$ contains all other finite poles. For a theory with local interactions, the amplitude $\hat{\mathcal{A}}_n$ has only simple poles, which occur when the sum of a subset of the external momenta becomes on shell

$$P_I^2 = 0 \quad \text{with} \quad P_I = \sum_{i \in I} p_i\,. \tag{1.5}$$

Unitarity of the S-matrix states that the residues of these simple poles are products of amplitudes with a lower number of external states, for example $\mathcal{A}_3 \times A_{n-1}$, $\mathcal{A}_4 \times \mathcal{A}_{n-2}$ etc. Then, knowing all the poles and residues of the integral in (1.4), one can readily solve for the residue at $z = 0$ (the non-shifted amplitude) in terms of products of lower-multiplicity amplitudes. This can be repeated recursively to construct all the amplitudes of Yang-Mills from only the 3-particle input.

After BCFW, tree-level on-shell recursion relations were soon extended to amplitudes of gravity models [26, 27] and maximally supersymmetric theories [28, 29, 30]. For the maximally suprsymmetric Yang-Mills theory, the notion of recursion was also generalized to all loop orders for amplitude integrands [31]. Moreover, the authors of [32, 33] discuss when and which recursion relations are valid for general classes of renormalizable and non-renormalizable theories.

Going back to the Lagrangian description, recursion relations are expected to work for these theories because their Lagrangian is almost completely fixed by their symmetries. The couplings of the different interaction terms are related to each other in order to preserve the symmetry of the model. For example, the Lagrangian of Yang-Mills

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) = \mathrm{tr}\left(-\partial_\mu A_\nu \partial^\mu A^\nu + \partial_\mu A_\nu \partial^\nu A^\mu + ig\partial_\mu A_\nu [A^\mu, A^\nu] + g^2 A_\mu A_\nu [A^\mu, A^\nu]\right) \tag{1.6}$$

would not be invariant under gauge transformations, were the coefficients of the different terms not related in this fashion. Similarly, for the Lagrangian of a gravitational theory, although there are infinite interaction terms, their coefficients must be related in a way such that the sum is invariant under diffeomorphisms. Since the form and the coefficients of these interactions are fully dictated by the symmetries of the model, the corresponding scattering amplitudes must also be related. One should be able to reconstruct higher-multiplicity amplitudes in the model from lower-multiplicity ones, if the underlying symmetry is taken into account.

This situation is not always true. In fact, our work focuses on effective field theories, which are quantum field theories that are only valid for a specific energy scale. They are often used to describe the low-energy degrees of freedom of a model, without the need to know specifically the high-energy dynamics. An effective field theory can in principle include any number of operators in its Lagrangian, as long as they are compatible with the symmetries of the model. Each operator appears with an independent Wilson coefficient that is fixed by matching to experimental results or the known high-energy completion of the model. In this case, unitarity and locality, the only physical input in a recursion relation like (1.4), are not enough to determine the amplitudes of the model. There needs to be some additional input.

One can gain some mileage for the special case of the so-called *exceptional* effective field theories. These are theories that describe the massless Goldstone modes of spontaneously broken symmetries. What makes these theories exceptional is that their Lagrangian is invariant under shifts of the fields, because of the non-linearly realized symmetry. For example, the Lagrangian of a non-linear sigma model, like the theory describing pions in low-energy quantum chromodynamics, is invariant under a constant shift of the pion fields, $\phi \to \phi + c$. In order to have a shift invariant Lagrangian, the Wilson coefficients of these theories must be related with each other. For the scattering amplitudes of these theories, the shift symmetry translates to low-energy theorems. When the relativistic energy-momentum of an external state goes to zero, the amplitude vanishes. This is known in the literature as the Adler zero [34].

The fact that the Lagrangian of exceptional effective field theories is heavily constrained by the shift symmetry suggests that a recursive reconstruction of their amplitudes is possible. This is achieved by the so-called soft-subtracted recursion relations [35]. In this construction, instead of (1.4), one starts with the integral

$$\oint_c \frac{\mathrm{d}z}{2\pi i} \frac{\hat{\mathcal{A}}_n(z)}{zF(z)} = 0 \,. \tag{1.7}$$

Here, $F(z)$ is a polynomial in $z$ that is chosen to have zeroes exactly on the values of $z$ that correspond to one of the external momenta of the amplitude going to zero; this fully determines the functional form of $F(z)$ up to an insignificant overall constant. Including this additional polynomial in the denominator improves the large $z$ behavior of the integrand and it is possible to avoid the pole at infinity. Moreover, no new poles are introduced, since the amplitude in the numerator also vanishes when $F(z)$ vanishes, because of its low-energy theorems. The rest of the recursion process proceeds as before.

Let us pause for a moment and reiterate why soft-subtracted recursion relations are good for exceptional effective field theories. We stated that a BCFW-like recursion relation, like (1.4), is not suitable for our purpose. We also know that what makes exceptional field theories possibly constructible is the shift symmetry of the Lagrangian, or equivalently the Adler zero of the amplitudes. For this reason, the soft-subtracted recursion relation (1.7) can work because it takes into account information about the low-energy limit of the amplitude.

There are, however, limitations for this. In our work [36], we prove a precise criterion for when soft-subtracted recursion relations are valid. We show that having a valid recursion relation means that there is a unique scattering amplitude that has the correct low-energy behavior and the correct pole structure. On the other hand, soft-subtracted recursion relations fail precisely when there can be independent contributions to the amplitude that can be added with arbitrary coefficients without changing its low-energy behavior. Equivalently, in the Lagrangian picture, when there are interaction terms one can introduce with an arbitrary coupling constant that trivially satisfy the shift symmetry, recursion cannot be applied.

Soft-subtracted recursion relations are not only used for reconstructing the S-matrix of known field theories, but they can also be used to probe the space of exceptional field theories. This method is known as the soft bootstrap and its philosophy is the following. One starts with basic assumptions about the spectrum of a model and the low-energy behavior of its amplitudes and an ansatz for those amplitudes with the lowest possible number of external states. Then, using the recursion relation, one calculates amplitudes with higher number of external states. If the reconstructed amplitude has poles that are non-physical, then the original assumptions were wrong and no theory can exist with the assumed characteristics. On the other hand, if the reconstructed amplitude has no spurious poles, it is strong evidence (yet no proof) for the existence of a theory with the assumed characteristics.

In the work presented in chapter 3, through specific examples, we show how the soft bootstrap method and soft-subtracted recursion relations can be applied to models that include spin-1/2 and spin-1 particles, thus generalizing the work of [37] that focused only on scalar theories. We also provide a general criterion for the validity of soft-subtracted recursion relations, applicable for models that include particles of arbitrary spin. We then proceed to tackle several interesting physical problems, by applying these techniques.

We first examine whether the $\mathbb{CP}^1$ non-linear sigma model is compatible with supersymmetry. Note that, from an on-shell point of view, the constraints of supersymmetry in a model take the form of supersymmetric Ward identities [38, 39]. These are linear relationships between amplitudes with different external states; they are reviewed in section 3.5. A surprising feature we find in our work [36] is the fact that extended supersymmetry enforces us to introduce 3-particle interactions in the model, which are not present in the non-supersymmetric case. The presence of these interactions modifies the low-energy theorems for the scalar fields of the model, such that their amplitudes are no longer generally vanishing when an external momentum goes to zero.

Another problem we study in chapter 3 concerns the special Galileon theory [37, 40]. This is an exceptional scalar effective field theory with an enhanced shift symmetry of the form $\phi \to \phi + c + v_\mu x^\mu + s_{\mu\nu} x^\mu x^\nu$. Specifically, we show that it is possible to have higher-derivative corrections to this theory. We verify our results using the soft bootstrap method and independently using a double-copy construction.

The Bern-Carrasco-Johansson double copy [41, 42] is another remarkable achievement of the S-matrix program. It relies on the so-called color-kinematics duality, a duality between kinematic and group-theory factors in the Yang-Mills amplitude, to calculate amplitudes of gravitons at tree-level, or amplitude integrands at loop-level.

At tree-level, the Bern-Carrasco-Johansson double copy is equivalent to the Kawai-Lewellen-Tye relations [43]. These relations are proved in string theory and state that amplitudes of closed strings can be written as sums of products of amplitudes of open strings. In the low-energy limit of string theory this translates to the fact that gravity amplitudes can be expressed as sums of products of amplitudes of Yang-Mills theory. This fact is known colloquially as

$$(\text{gravity}) = \sum (\text{Yang-Mills})^2 \,. \tag{1.8}$$

It was later shown that other theories satisfy similar relations. In particular, the special Galileon theory amplitudes can be expressed as sums of products of amplitudes of chiral perturbation theory [44], the non-linear sigma model that describes pions. Borrowing the notation of gravity, this can be written as

$$(\text{special Galileon}) = \sum (\text{chiral perturbation theory})^2 \,. \tag{1.9}$$

Assuming that this relation continues to be valid for higher-derivative corrections to these models, in our work [36], we constructed the corrections of the special Galileon by first calculating possible corrections to chiral perturbation theory.

We also compared the output of the double-copy construction to the output of the soft bootstrap method, finding perfect agreement for the orders that are possible to check. Since these results come from two different definitions of the special Galileon, their agreement is a highly non-trivial consistency check. In the soft bootstrap approach, we define the special Galileon as the effective field theory with the special Galilean symmetry, while in the double copy approach, the theory is defined as the double copy of chiral perturbation theory. There is no a priori reason to believe that these two definitions are equivalent. Their matching beyond leading order is by itself remarkable.

All the details of these projects are presented in chapter 3. This work was published in a paper titled *"Soft Bootstrap and Supersymmetry"* [36] in the Journal of High Energy Physics in collaboration with Henriette Elvang, Callum R.T. Jones and Shruti Paranjape.

## 1.3 Entanglement Entropy

In a quantum many-body system, often, the state of the system cannot be written as an outer product of one-body states. For example, consider a system of two spins in the singlet state $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$. This state cannot be written as an outer product of single-spin states. If the particles are indistinguishable it is not even meaningful to talk about the state of one particle. This physical phenomenon is known as *quantum entanglement*. A result of entanglement is that experimental observations on different parts of the system are not independent from each other. For example, in the spin system described above, if one particle is found to have spin up then the other must have spin down and vice versa.

*Entanglement entropy* is a quantitative measure of these correlations between observables on an entangled system. The entropy of entanglement between a sub-system $V$ and its complement is defined as the von Neumann entropy

$$S_V = -\operatorname{tr}\left(\rho_V \log \rho_V\right) , \tag{1.10}$$

where the trace is taken over all degrees of freedom of $V$. Here, $\rho_V = \operatorname{tr}_{\bar{V}} \rho$ is the reduced density matrix of the subsystem $V$, where all degrees of freedom that do not belong in $V$ have been traced out from the full density matrix of the system. Another quantitative measure of entanglement is *Rényi entropies*

$$S_V^{(n)} = \frac{1}{1-n} \log \operatorname{tr} \rho_V^n \, . \tag{1.11}$$

Notice that in the limit $n \to 1$, the Rényi entropy is equal to the entanglement entropy. In fact, it is often easier to first calculate the Rényi entropy $S_V^{(n)}$ and then take the limit $n \to 1$,[3] rather than computing the entanglement entropy directly.

[3]There are subtleties regarding the $n \to 1$ limit; the integer parameter $n$ is discrete and taking a continuous limit is not a priori well-defined.

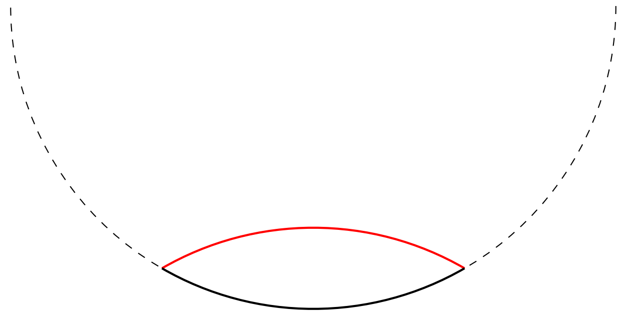



Figure 1.1: Illustration of the Ryu-Takayanagi conjecture. The dashed line corresponds to the boundary of AdS. The entanglement entropy of the black region $V$ in the boundary field theory is proportional to the length of the red extremal curve of the bulk.

Entanglement entropy is a very useful and well-studied physical quantity, with applications in many areas of physics and beyond. For example, it plays a central role in quantum computing and the study of how information is distributed in quantum systems [45]. In the context of high-energy physics and gauge-gravity duality, it provides another example of the holographic principle that was presented in section 1.1 and how boundary field theory quantities are encoded in those of the bulk gravitational theory. The Ryu-Takayanagi conjecture [46, 47] states that the entanglement entropy of a sub-region $V$ of the boundary field theory is proportional to the area of the bulk extremal surface that has the same boundary as $V$ (see Figure 1.1). Entanglement entropy also has applications in black hole physics [48, 49, 50], where the leading contribution to the Bekenstein-Hawking entropy of a black hole is equal to the entanglement entropy of the sub-region of space that is outside the black hole horizon.

In quantum field theories, because of vacuum fluctuations and the constant creation and annihilation of particle-antiparticle pairs, it is useful to talk about entanglement of space itself and calculate the (Rényi) entanglement entropy of a spatial region. Because of short-distance correlations of points that are very close to the region's boundary entanglement entropy is a divergent quantity. In order to regulate the divergence, one is forced to introduce a short-distance cutoff $\epsilon$ in the calculation and express the result as a power series in $\epsilon$ with the following structure

$$S = g_{d-2}\epsilon^{-(d-2)} + \ldots + g_1\epsilon^{-1} + g_0 \log\epsilon + \tilde{g}_0 + \mathcal{O}(\epsilon)\,, \tag{1.12}$$

where $d$ is the number of spacetime dimensions. The coefficients $g_i$ with $i \geq 1$ depend of this expansion depend on the choice of cutoff and therefore cannot contain any physical information. On the other hand, the coefficient of the logarithmic term $g_0$ is independent of the choice of regulator. Consider, for example, rescaling the short-distance cutoff by $\epsilon \to k\epsilon$. Then, in order for the entropy $S$ to stay invariant, the coefficients $g_i$ must also be rescaled by $g_i \to k^i g_i$, for $i \geq 1$. $g_0$ must not change, while $\tilde{g}_0$ is shifted by $\tilde{g}_0 \to \tilde{g}_0 - g_0 \log k$.

For conformal theories in even spacetime dimensions, the logarithmic term is always present and its coefficient is proportional to the *central charge* of the theory. For theories in odd spacetime dimensions, the logarithmic term is absent if the boundary of the entangling region is smooth. In this case, physical information is encoded in the constant term $\tilde{g}_0$.

In 3 spacetime dimensions, in particular, the (Rényi) entanglement entropy has a logarithmic contribution only if the boundary of the entangling region has a sharp cusp (see Figure 1.2); recall that in 3 spacetime dimensions (2-dimensional Euclidean space and time) the boundary of a spatial region is a 1-dimensional closed curve and it has a sharp cusp if the derivative of the curve does not exist at a given point. The coefficient of this logarithmic contribution (usually called the *corner coefficient*) is *universal*, in that it does not depend on the way one chooses to regulate the divergence of the entropy. It does nevertheless depend on the particle spectrum and interactions of the model.

The calculation of the (Rényi) entanglement entropy in a quantum field theory is not remotely straightforward. For models with a holographic dual, one can use the Ryu-Takayanagi formula [46, 47]. For generic theories, however, one has to use the so-called replica trick [50, 51] (see also [52] for a review). This involves taking multiple copies of the spacetime and "sewing" them together to create a complicated manifold with singularities at the boundary

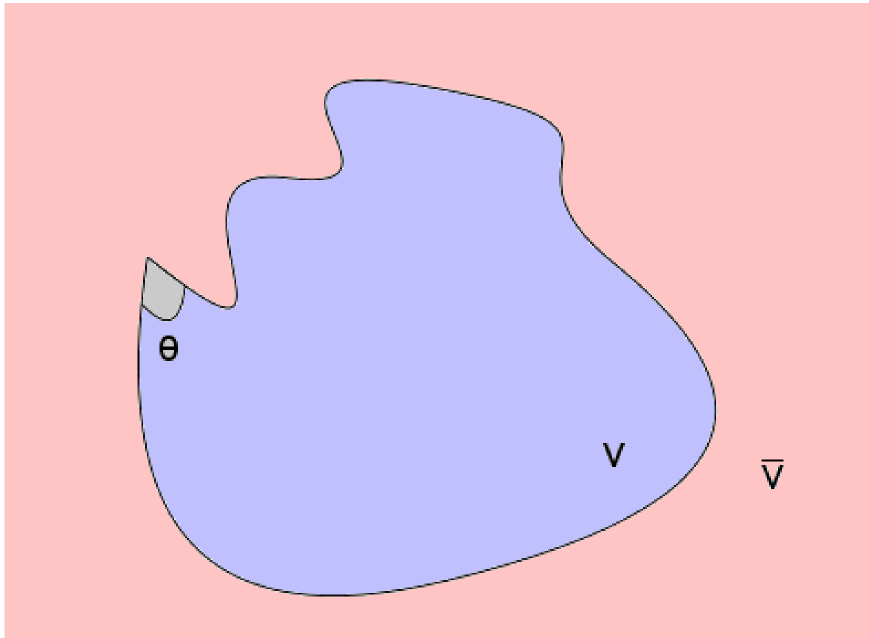



Figure 1.2: Example of a region with a boundary that has a sharp angle. In 3 spacetime dimensions entanglement entropy has a logarithmic contribution only in the presence of such sharp angles.

of the entangling region. Using the replica trick, Casini, Huerta and Leitao [53, 54, 52] were able to express the corner coefficient for the entanglement entropy of a free real scalar or Dirac fermion in 3 spacetime dimensions, in terms of several rather complicated integrals. Numerical evaluation of these integrals [55] by Bueno, Myers and Witczak-Krempa suggested a simple proportionality relation (see equation (4.4)) between the corner coefficient of the entanglement entropy and the *central charge* of these two theories. The authors of [55] conjectured that this relation must hold true for all conformal theories in 3 dimensions. Their conjecture was checked numerically for a number of theories, including models with with a holographic dual and Wilson-Fisher fixed points of the $O(N)$ model [55, 56].

In our work [57], we were able to simplify the integral expressions of Casini et al. [53, 54, 52] and give analytic results for the corner coefficient of (Rényi) entanglement entropies. Our results verified the conjecture of [55] for the case of a free real scalar or fermion. Furthermore, we studied the large-$n$ asymptotic behavior of the corner coefficient. We observed a simple proportionality relation between this value and the large-$n$ asymptotic value of the free energy of a real scalar or fermion on a $n$-covered sphere. This observation was later proved analytically in [58].

The details of our work are presented in chapter 4. They were published in a paper titled *"Exact results for corner contributions to the entanglement entropy and Rényi entropies of free bosons and fermions in 3d"* [57] in Physics Letters B in collaboration with Henriette Elvang.

## 1.4 Other Projects

Beyond the projects I describe in this thesis, during my doctoral studies, I had the opportunity to work on several other aspects of theoretical physics, which I summarize here.

Most recently, using techniques from the scattering amplitudes program like generalized unitarity and the double copy, we studied electromagnetic duality in the context of the Born-Infeld theory. Specifically, we wanted to understand whether electromagnetic duality, a symmetry of the amplitudes of Born-Infeld at tree-level, is preserved by quantum corrections (loop-level amplitudes) or by higher-derivative corrections to the model. This work was published in a preprint titled *"All-Multiplicity One-Loop Amplitudes in Born-Infeld Electrodynamics from Generalized Unitarity"* [59], in collaboration with Henriette Elvang, Callum R.T. Jones and Shruti Paranjape. In this paper, we give exact expressions for an infinite class of 1-loop amplitudes of Born-Infeld, while we further discuss the consequences for electromagnetic duality in a paper in preparation.

Additionally, in a previous paper, also in collaboration with Henriette Elvang, Callum R.T. Jones and Shruti Paranjape, we used the soft bootstrap technique to study the compatibility of Galileon theories with supersymmetry. The title of the paper is *"On the Supersymmetrization of Galileon Theories in Four Dimensions"* [60] and it was published in Physics Letters B. Results of this work are also reviewed in chapter 3 of this thesis.

Finally, during the first years of my studies, I published the work I started during my undergraduate education. In this work, in collaboration with Martha Constantinou of Temple University and Haralambos Panagopoulos and Gregoris Spanoudes of the University of Cyprus, we studied the renormalization of flavor-singlet operators in the context of lattice quantum chromodynamics, a computational method to perform (perturbative and non-perturbative) calculations in QCD. The title of this paper is *"Singlet versus nonsinglet perturbative renormalization of fermion bilinears"* [61] and it was published Physical Review D.

The diverse topics, on which I have worked during the last years, are a reflection of my broad interests in many aspects of quantum field theory. A complete understanding of this rich and complex framework of theoretical physics can only be achieved through probing several directions and using all the available tools. The development and application of new methods that will simplify our existing problems and expand our abilities is and will continue to be my personal endeavour.

CHAPTER 2

# A Practical Approach to the Hamilton-Jacobi Formulation of Holographic Renormalization

## 2.1 Introduction

In many applications of gauge-gravity duality, there is a need to regulate divergences that appear near the boundary of the bulk theory; these are simply associated with UV divergences in the dual quantum field theory. The divergences appear, for example, in calculations of conformal anomalies, correlation functions, and the free energy. The prescription for regulating divergences is to include suitable local counterterms. The resulting process of holographic renormalization is an old subject: it was discussed in the early days of AdS/CFT [3] and implemented in the classic calculations of conformal anomalies [6], the trace of the stress-tensor [7], and since then in countless other examples.

We focus on bulk spacetimes that are asymptotically AdS or Euclidean AdS. This includes duals of conformal theories (CFTs) as well as holographic renormalization group flows with a UV CFT. For a given gravity dual, the local counterterms are universal and one can calculate them once and for all in any given gravitational model. We distinguish between *infinite counterterms* and *finite counterterms*. The former are unambiguous and can be determined using the bulk equations of motion. The finite counterterms, however, can typically only be fixed using further constraints, such as supersymmetry. *In this paper, we are concerned only with the infinite counterterms.*

There is a standard "brute force" procedure for determining the infinite counterterms

[6, 7, 8, 9]. One expands the metric and fields near the AdS boundary using the Fefferman-Graham (FG) expansion [10]. Solving the equations of motion relates various coefficients in the FG expansion, but leaves unfixed the coefficients that correspond to the source and vev rates for each field. Using a suitable cutoff, the on-shell action is evaluated near the AdS boundary by plugging in the FG expansion, subject to the equations of motion. This identifies the divergences, however, they will be expressed in terms of the free coefficients in the FG expansion. This is not sufficient, as local counterterms must be expressed directly in terms of the fields on the cutoff surface. So starting with the most divergent terms, one works systematically backwards to convert each divergence to a local field expression, thus basically reversing the FG expansion. This process identifies the field polynomials that are responsible for the divergences in the on-shell action. The counterterm action is then taken to be exactly minus those field expressions; this ensures that the renormalized action $S_{\text{bulk}} + S_{\text{ct}}$ is finite. (This still leaves the possibility of ambiguities from finite counterterms; we will discuss this briefly in the Discussion section.)

While straightforward for many simple models with just one or two scalar fields, the brute force approach outlined above becomes increasingly tedious for models with multiple fields. Moreover, it is fundamentally unsatisfying that one first abandons the field expressions in favor of Fefferman-Graham only to reverse back to fields after identifying the infinite terms. For this reason, another approach, based on the Hamiltonian formalism for gravity and the Hamilton-Jacobi equation, has been proposed for holographic renormalization.

Early in the studies of holographic renormalization group flows, de Boer, Verlinde, and Verlinde [11] proposed to use the Hamilton-Jacobi equation to derive first-order equations for the supergravity model and they related it to the Callan-Symanzik equation. (See also [12, 13] and the lectures [14].) The specific application of the Hamilton-Jacobi equation to determine infinite counterterms was studied by Kalkkinen, Martelli, and Mueck in [15, 16] and subsequently by Papadimitriou and Skenderis in [17] (see also [18, 19, 20]).

One limitation of the method as formulated in [17] is that the dilatation operator is used to organize the calculation. This requires that the fields are eigenfunctions of the dilatation operator, but that makes it more challenging to handle scalars dual to operators with scaling dimension $\Delta = d/2$, because of their leading log-falloff.[1] This is not an exotic case, but a very common one; for example, in a $d = 4$ field theory, a scalar mass term is a relevant operator of dimension $\Delta = 2$. Another challenge is that, as presented in [17], the Hamilton-Jacobi method looks rather difficult to carry out in practice.

The goal of this work is to straighten out and simplify the Hamilton-Jacobi approach for holographic renormalization. We will show that the application of the Hamilton-Jacobi equation

$$\frac{\partial S_{\text{on-shell}}}{\partial r} + H = 0 \tag{2.1}$$

(with the radial coordinate $r$ playing the role of the usual time-coordinate), can be implemented via an algorithm that significantly simplifies the process of computing the infinite counterterms. To avoid the issue of the dilatation operator and have an approach that applies more generally, we organize the calculation in terms of a derivative expansion (or inverse metric expansion), as also suggested in for example [11, 16, 20].

We will be working with bulk actions of the form

$$S = -\frac{1}{2\kappa^2}\int_M \mathrm{d}^{d+1}x\sqrt{g}\,\Big(\mathcal{R}[g] - g^{\mu\nu}G_{IJ}\partial_\mu\Phi^I\partial_\nu\Phi^J - V(\Phi)\Big)\,, \tag{2.2}$$

where we allow for a general metric $G_{IJ} = G_{IJ}(\Phi)$ on the scalar manifold. We consider domain wall solutions with arbitrary slicing and assume that the asymptotic UV structure of the metric is AdS (or Euclidean AdS). For such a system, we formulate the Hamilton-Jacobi problem for the on-shell action $S_{\text{on-shell}}$; (2.1) is basically a partial differential equation for $S_{\text{on-shell}}$ and once derived, one no longer has to think about the Hamiltonian formulation of general relativity. Instead, one systematically solves the Hamilton-Jacobi differential equation for $S_{\text{on-shell}}$ by writing a suitable Ansatz for its divergent terms and then solving for

[1] One can work around this, see for example [18]. The issue is also addressed in [20].

the coefficients in this Ansatz. The key point here is that scalars dual to relevant operators in the field theory go to zero at the boundary. Therefore there can only be limited powers of each field in the infinite counterterms, and that makes the Ansatz finite.

Our method departs from previous approaches as follows.[2] We consider $S_{\text{on-shell}}$ as the action on the cut-off boundary; this breaks the general diffeomorphism invariance in the radial direction and therefore we must take seriously the explicit dependence on the radial coordinate in $S_{\text{on-shell}}$. Thus, the $r$ partial-derivative in (2.1) plays a central role in our method. In fact, the coefficients in our Ansatz will be allowed to have explicit $r$-dependence, and the Hamilton-Jacobi equation then yields differential equations for these coefficients that we can solve unambiguously in the near boundary limit.

We illustrate the use of the method in several contexts. To start out, we reproduce the purely gravitational counterterms [8] in $d$-dimensions. To show how the method works for a case with $d$ odd, we reproduce the infinite counterterms of the $d = 3$ ABJM dual model of [63]. We then turn to the example of the $d = 4$ FGPW model [64] whose two scalars have $\Delta = 2$ and $\Delta = 3$.

In the presence of a marginal scalar, more care must be taken. A marginal scalar generically goes to a finite value at the boundary and therefore the associated counterterms do not enjoy the same suppression as the scalars dual to relevant operators. We handle this by allowing the coefficients of our Ansatz for $S_{\text{on-shell}}$ to be functions of the marginal scalar. We have applied this method successfully to calculate the counterterms for a ten-scalar model dual to (a limit of) $\mathcal{N} = 1^*$ theory on $S^4$ [65]; this indeed served as a motivation for us to revisit the subject of holographic renormalization. However, for the purpose of presentation here, we restrict ourselves to simply show how our method reproduce the infinite counterterms for the dilaton-axion system in [20].

This chapter is organized as follows. In Section 2.2, we present the Hamilton-Jacobi equation for the bulk and describe our algorithm for determining the infinite counterterms.

[2] However, see [62] for a similar approach in dS space.

Section 2.3 implements the method for pure gravity in $d$ dimensions. The examples of the ABJM model and FGPW can be found in Sections 2.4 and 2.5; these give very concrete illustrations of how we implement the algorithm. The more advanced case of marginal scalars is treated in Section 2.6. The three appendices relevant to this chapter contain various technical details. Appendix A is a short list of useful identities for the metric variations of gravitational curvatures. Appendix B gives details of the calculation of the gravitational six-derivative terms needed for counterterms in $d = 6$. Finally, Appendix C offers explicit calculation of the one-point functions in FGPW to illustrate that the one-point functions determined from the renormalized action with our infinite counterterms are indeed all finite.

## 2.2 Hamiltonian Approach to Holographic Renormalization

We start with a brief description of the essential parts of the Hamiltonian formulation needed for holographic renormalization. We then formulate the problem of determining the on-shell action in terms of the Hamilton-Jacobi equation and we present our algorithm for calculating the divergent part of the on-shell action.

### 2.2.1 Hamiltonian Formalism of Gravity

We consider a general form of the bulk gravitational action:

$$S = -\frac{1}{2\kappa^2}\int_M \mathrm{d}^{d+1}x\,\sqrt{g}\,\big(\mathcal{R}[g] - g^{\mu\nu}G_{IJ}\partial_\mu\Phi^I\partial_\nu\Phi^J - V(\Phi)\big) - \frac{1}{\kappa^2}\int_{\partial M}\mathrm{d}^d x\,\sqrt{\gamma}K\,. \qquad (2.3)$$

The last term in (2.3) is the Gibbons-Hawking boundary term which ensures that the variational problem is well-defined. In this term, $\gamma_{ij}$ is the induced metric on the boundary and $K$ is its extrinsic curvature.

We choose a gauge for the bulk metric $g_{\mu\nu}$ such that the line element takes the form

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \gamma_{ij}(r,x)\mathrm{d}x^i\mathrm{d}x^j\,, \tag{2.4}$$

where latin indices $i, j, \ldots$ are in the range $i, j = 1, 2, \ldots, d$ and will denote boundary coordinates. This allows us to decompose the Ricci scalar in the action to get

$$S = -\frac{1}{2\kappa^2}\int_M \mathrm{d}^d x\,\mathrm{d}r\sqrt{\gamma}\Big(R[\gamma] + K^2 - K_{ij}K^{ij} - G_{IJ}\dot{\Phi}^I\dot{\Phi}^J - \gamma^{ij}G_{IJ}\partial_i\Phi^I\partial_j\Phi^J - V(\Phi)\Big)\,, \tag{2.5}$$

where the extrinsic curvatures are

$$K^i{}_j = \frac{1}{2}\gamma^{ik}\dot{\gamma}_{kj} \qquad \text{and} \qquad K = \frac{1}{2}\gamma^{ij}\dot{\gamma}_{ij}\,. \tag{2.6}$$

The dots denote derivatives with respect to $r$. The boundary Gibbons-Hawking term does not appear in the expression (2.5), since it has been canceled by boundary terms that occur from partial integration of second derivative terms in the expansion of $\mathcal{R}[g]$.

In the Hamiltonian formulation of holographic renormalization, the radial coordinate $r$ plays the role of the time coordinate. Therefore, the conjugate momenta to the fields are given by

$$\pi^{ij} = \frac{\delta S}{\delta\dot{\gamma}_{ij}} = \frac{1}{2\kappa^2}\sqrt{\gamma}\left(K^{ij} - K\gamma^{ij}\right) \qquad \text{and} \qquad \pi_I = \frac{\delta S}{\delta\dot{\Phi}^I} = \frac{1}{\kappa^2}\sqrt{\gamma}G_{IJ}\dot{\Phi}^J\,, \tag{2.7}$$

and the Hamiltonian is

$$\begin{aligned} H &= \int_{\partial M}\mathrm{d}^d x\left(\pi^{ij}\dot{\gamma}_{ij} + \pi_I\dot{\Phi}^I - \mathcal{L}\right) \\ &= \frac{1}{2\kappa^2}\int_{\partial M}\mathrm{d}^d x\sqrt{\gamma}\left(R[\gamma] - K^2 + K_{ij}K^{ij} + G^{IJ}p_Ip_J - \gamma^{ij}G_{IJ}\partial_i\Phi^I\partial_j\Phi^J - V(\Phi)\right)\,, \end{aligned} \tag{2.8}$$

where, for simplicity, we have introduced $p_I \equiv \frac{\kappa^2}{\sqrt{\gamma}}\pi_I$.

### 2.2.2 Hamilton-Jacobi Formulation

The Hamilton-Jacobi formulation is well-known in classical mechanics [66]. With the radial coordinate $r$ playing the role of time, the Hamilton-Jacobi equation takes the form

$$H + \frac{\partial S_{\text{on-shell}}}{\partial r} = 0\,. \tag{2.9}$$

Just as in classical mechanics, it is key to emphasize that in the Hamilton-Jacobian formalism, the Hamiltonian is a functional of canonical momenta *defined* by

$$\pi^{ij} = \frac{\delta S_{\text{on-shell}}}{\delta \gamma_{ij}} \quad \text{and} \quad p_I = \frac{\kappa^2}{\sqrt{\gamma}}\pi_I = \frac{\kappa^2}{\sqrt{\gamma}}\frac{\delta S_{\text{on-shell}}}{\delta \Phi^I}\,, \tag{2.10}$$

as opposed to the canonical definitions (2.7). When the momenta are defined via equation (2.7) with the extrinsic curvature given by (2.6), the Hamiltonian constraint of Einstein's equation is simply $H = 0$. If this were used with the Hamilton-Jacobi equation (2.9), it would imply that the action has no explicit $r$-dependence; this is of course true for the diffeomorphism-invariant gravitational bulk action whose metric equations-of-motion imply the Hamiltonian constraint. However, it is not true for the *on-shell action*, which is an action on the cut-off boundary. It has explicit $r$-dependence, as we shall see, and to determine it via the Hamilton-Jacobi equation we must use the definitions (2.10). With (2.10), the Hamilton-Jacobi equation (2.9) should be thought of as a first-order partial differential equation for $S_{\text{on-shell}}$ with respect to the fields, the metric, and $r$.

A practical approach is to use an Ansatz for the on-shell action: below we will be more explicit about how we choose an appropriate Ansatz, but for now we will develop the general formalism further. Let us write the Ansatz as

$$S_{\text{on-shell}} = \frac{1}{\kappa^2}\int_{\partial M_\epsilon} \mathrm{d}^d x \sqrt{\gamma}\, U(\gamma, \Phi, r)\,. \tag{2.11}$$

The function $U$ is a function of the induced (inverse) metric $\gamma^{ij}$ on the boundary and the scalar fields $\Phi^I$, and it has also explicit dependence on $r$. The cutoff surface $\partial M_\epsilon$ becomes the boundary of the spacetime when $\epsilon \to 0$.

Using the above Ansatz, the Hamilton-Jacobi equation takes the form

$$R[\gamma] + K_{ij}K^{ij} - K^2 + G^{IJ}p_I p_J - \gamma^{ij}G_{IJ}\partial_i\Phi^I\partial_j\Phi^J - V(\Phi) + 2\frac{\partial U}{\partial r} \;=\; 0\,. \tag{2.12}$$

We emphasize that this equation is to be understood as an integral equation, i.e. it holds up to total derivatives and we can manipulate it using partial integration in the boundary coordinates.

As discussed above, the conjugate momenta in (2.12) will be given by derivatives of $U$. For the scalar field conjugates, this straightforwardly gives

$$p_I = \frac{\kappa^2}{\sqrt{\gamma}}\frac{\delta S_{\text{on-shell}}}{\delta\Phi^I} \quad \Rightarrow \quad \boxed{p_I = \frac{\delta U}{\delta\Phi^I}\,.} \tag{2.13}$$

The conjugate momentum of the metric enters (2.12) via the extrinsic curvatures, since $K^{ij} = \frac{2\kappa^2}{\sqrt{\gamma}}\big(\pi^{ij} - \frac{1}{d-1}\gamma^{ij}\pi^{kl}\gamma_{kl}\big)$, as follows from (2.7). Now in the context of the Hamilton-Jacobi formalism, the extrinsic curvatures $K^{ij}$ in (2.12) must then be expressed in terms of $\pi^{ij}$ as given by (2.10). This gives

$$K^i{}_j = -2\gamma^{ik}\frac{\delta U}{\delta\gamma^{kj}} - \frac{1}{d-1}\left(U - 2\gamma^{mn}\frac{\delta U}{\delta\gamma^{mn}}\right)\delta^i{}_j\,, \tag{2.14}$$

where we have used $\gamma_{ij}\gamma^{jk} = \delta_i{}^k \quad\Longrightarrow\quad (\delta\gamma_{ij})\gamma^{jk} = -\gamma_{ij}(\delta\gamma^{jk})$ to express $K^i{}_j$ in terms of derivatives with respect to the *inverse* metric rather than the metric; this will be useful later.

It is convenient to define

$$\boxed{Y_{ij} = \frac{\delta U}{\delta\gamma^{ij}} \quad \text{and} \quad Y = \gamma^{ij}Y_{ij}\,.} \tag{2.15}$$

One then finds from (2.14) that the dependence on extrinsic curvatures in the Hamilton-Jacobi equation (2.12) is given in terms of $U$ as

$$\boxed{\mathcal{K} \;\equiv\; K_{ij}K^{ij} - K^2 \;=\; 4Y_{ij}Y^{ij} - \frac{1}{d-1}(U-2Y)^2 - U^2\,.} \tag{2.16}$$

To summarize, our strategy for computing the on-shell action $S_{\text{on-shell}}$ is to use the Ansatz (2.11) and solve the Hamilton-Jacobi equation

$$\boxed{R[\gamma] + \mathcal{K} + G^{IJ}p_I p_J - \gamma^{ij}G_{IJ}\partial_i\Phi^I\partial_j\Phi^J - V(\Phi) + 2\frac{\partial U}{\partial r} = 0\,.} \tag{2.17}$$

with conjugate momenta given by (2.13) and $\mathcal{K}$ defined in (2.16). We remind the reader that equation (2.17) has to hold only as an integral equation, so we are free to manipulate it using partial integration. While this was derived using the Hamiltonian formalism of gravity, we no longer need to think of the problem that way. Rather, we now have differential equation (2.17) for the on-shell action $S_{\text{on-shell}}$. Next, we explain how to solve it systematically.

### 2.2.3 Algorithm to Determine the Divergent Part of the On-shell Action

Let us next address how we propose to use the Hamilton-Jacobi formulation to determine the divergent part of the on-shell action and thereby the counterterms needed for a finite result. We outline here the general approach, however the method is much better illustrated by concrete examples; these follow in the next sections.

We assume that asymptotically the bulk metric approaches AdS space: in terms of the choice of coordinates (2.4), $\mathrm{d}s^2 = \mathrm{d}r^2 + \gamma_{ij}(r,x)\mathrm{d}x^i\mathrm{d}x^j$, this means that

$$\gamma_{ij} \to e^{2r/L}\gamma_{(0)ij} + \ldots \qquad \text{as} \qquad r \to \infty\,, \tag{2.18}$$

where $L$ is the AdS radius. The boundary metric $\gamma_{(0)ij}$ can be Lorentzian or Euclidean, it can be flat or curved. For example, recent applications of holography considered the dual field theory on $d$-dimensional compact Euclidean spaces, such as spheres. In the following, $\gamma_{(0)ij}$ will be general.

The asymptotic behavior (2.18), gives $\sqrt{\gamma} \sim e^{dr}\sqrt{\gamma_{(0)}}$. We are focusing only on the divergent parts of the on-shell action, so we need terms in $U$ only up to orders $e^{-dr}$ (possibly including also terms polynomial in $r$). Since the inverse metric $\gamma^{ij}$ scales as $e^{-2r}$, we can ignore any terms with more than $\lfloor \frac{d}{2} \rfloor$ inverse metrics. Any (boundary) derivatives that appear in terms in $U$ must necessarily be contracted pairwise by inverse metrics $\gamma^{ij}$, so we do not consider terms with more than $d$-derivatives. All in all, this makes it natural to organize the Ansatz for $U$ in a derivative expansion:

$$U = U_{(0)} + U_{(2)} + \ldots + U_{\left(2\lfloor \frac{d}{2} \rfloor\right)}\,, \tag{2.19}$$

where the subscript represents the number of derivatives in each term. Curvature terms such as the boundary Ricci scalar, Ricci tensor, and Riemann tensor are each order 2 (i.e. they have two derivatives). Previous work, for example [11] and [20], have also organized the on-shell action as a derivative expansion.

For the 0th order in the derivative expansion, we have $Y_{(0)ij} = \frac{\delta U_{(0)}}{\delta \gamma^{ij}} = 0$, so (2.16) simply gives

$$\mathcal{K}_{(0)} \;=\; -\frac{d}{d-1}U_{(0)}^2\,. \tag{2.20}$$

Thus at 0th order, the Hamilton-Jacobian equation (2.12) becomes

$$V(\Phi) = G^{IJ}\frac{\delta U_{(0)}}{\delta \Phi^I}\frac{\delta U_{(0)}}{\delta \Phi^J} - \frac{d}{d-1}U_{(0)}^2 + 2\frac{\partial U_{(0)}}{\partial r}\,. \tag{2.21}$$

Without the last $r$-derivative term, we see that $U_{(0)}$ is essentially like a (fake) superpotential for the scalar potential $V$; this was also noted in [11] (see also [67, 20]). In general, it is not easy to solve for a superpotential for a given $V$; however, we will not need to since our focus is on the generic asymptotically divergent terms only. As noted in the discussion below (2.10) the presence of the explicit $r$-derivative term in the Hamilton-Jacobi equation, and hence in (2.21), is crucial — this point does not seem to have been appreciated in previous discussions of the method.

Let us for later convenience also record the results for $\mathcal{K}$ at two- and four-derivative order:

$$\begin{aligned}
\mathcal{K}_{(2)} &= -\frac{2}{d-1} U_{(0)} \left[ U_{(2)} - 2Y_{(2)} \right] - 2U_{(0)} U_{(2)} \,, \\
\mathcal{K}_{(4)} &= 4Y_{(2)ij} Y_{(2)}^{ij} - \frac{1}{d-1} \left[ U_{(2)} - 2Y_{(2)} \right]^2 - \frac{2}{d-1} U_{(0)} \left[ U_{(4)} - 2Y_{(4)} \right] - U_{(2)}^2 - 2U_{(0)} U_{(4)} \,,
\end{aligned} \tag{2.22}$$

where $Y_{(k)ij} = \frac{\delta U_{(k)}}{\delta \gamma^{ij}}$.

*We propose the following algorithm to determine the infinite terms in the on-shell action:*

**Step 1: Ansatz for $U_{(2n)}$.** For each $U_{(2n)}$, we write a systematic Ansatz that includes all potentially divergent terms of this order with undetermined coefficients,[3] for example

$$U_{(0)} = A_0 + A_1 \phi + A_3 \phi^2 + \dots \quad \text{and} \quad U_{(2)} = B_0 R + B_1 R \phi + B_2 \phi \Box \phi + \dots \tag{2.23}$$

where the coefficients $A_i$ and $B_i$ can have explicit dependence on $r$. The Hamilton-Jacobi equations will therefore give us differential equations of these coefficients which we solve asymptotically, keeping only terms that give divergent contributions to the on-shell action.

Recall that the asymptotic behavior of a scalar with bulk mass $m_I^2$ is

$$\Phi^I \to \Phi^I_{(0)} e^{-(d-\Delta_I)r/L},$$

where $m_I^2 L^2 = \Delta_I(\Delta_I - d)$. The two solutions for $\Delta_I$ correspond to the source and vev-rate falloffs. When a scalar approaches zero at the boundary, as is the case in many applications,

[3] Terms are considered equivalent if related by partial integration.

we can immediately read off how many powers of the scalar can possibly appear in $U_{(2n)}$; the number of possible terms is finite and limited by the fact that we are only interested in the divergent terms.[4] For example, if $\phi$ is a scalar with dimension $\Delta_\phi = 3$ in $d = 4$, then $\phi \sim e^{-r}$, and we have to include powers up to $\phi^4$ in $U_{(0)}$ and $\phi\Box\phi$ can appear in $U_{(2)}$. (Note: such terms with $e^{-dr}$ falloff will be finite unless the $r$-dependence in the coefficient makes it divergent.) On the other hand, if $\phi$ in (2.23) is a $\Delta_\phi = 2$ scalar in $d = 4$, there can at most be quadratic powers of $\phi$ in $U_{(0)}$ and the term $\phi\Box\phi$ is not divergent, so it is not included in the Ansatz for $U_{(2)}$.

One can impose symmetries of the theory in order to further simplify the Ansatz for $U_{(2n)}$. If, for example, the bulk action has a symmetry $\phi \to -\phi$, we can drop any terms odd under this symmetry in the Ansatz.

**Step 2: Conjugate momenta.** Next, using the leading asymptotic behaviors of the fields, we determine the leading asymptotics of the conjugate momenta. Using this together with $p_I = \frac{\delta U}{\delta \Phi^I}$ fixes some of the coefficients in $U_{(0)}$ quite easily.

**Step 3: Solving the Hamilton-Jacobi equation.** We plug the Ansatz for $U_{(2n)}$ into the Hamilton-Jacobi equation and we solve it order by order by demanding that the coefficients of the different field monomials vanish independently. When necessary, use partial integration to eliminate potentially non-independent terms that appeared by varying $U$. We start with $U_{(0)}$, then use those results to determine $U_{(2)}$, then $U_{(4)}$ etc.

**Step 4: Counterterm action.** Once the divergent terms in $S_{\text{on-shell}}$ have been determined, the counterterm action is simply

$$S_{\text{ct}} = -S_{\text{on-shell}}\big|_{\text{div}} \,. \tag{2.24}$$

[4] We will also discuss cases with a marginal scalar $m_I^2 = 0$, for which there is no suppression near the boundary and generically the scalar goes to a non-zero constant. For such cases, we allow the coefficients $A_i$ in our Ansatz to be functions of the marginal scalar. An example is presented in Section 2.6.

This is added to the bulk action to get the regularized action $S_{\text{reg}} = S_{\text{bulk}} + S_{\text{GH}} + S_{\text{ct}}$ from which correlation functions can be computed and by construction are guaranteed to be finite. In many cases, counterterm actions are presented in term of the Fefferman-Graham radial coordinate $\rho$ related to $r$ via $\rho = e^{-2r/L}$, so that the line element is

$$\mathrm{d}s^2 = L^2 \frac{\mathrm{d}\rho^2}{4\rho^2} + \gamma_{ij}\,\mathrm{d}x^i \mathrm{d}x^j\,. \tag{2.25}$$

We determine the divergent terms in the on-shell action using the $r$-coordinate, but convert to $\rho$-coordinates for the final presentation of our counterterm actions. In terms of the $\rho$-coordinate, the cutoff surface $\partial M_\epsilon$, introduced in (2.11), is then located at $\rho = \epsilon$.

In the following sections, we demonstrate the procedure explicitly in a set of representative explicit examples. We start with pure gravity in $d$-dimensions with $d = 2, 3, 4, 5, 6$, then move on to a $d = 3$ ABJM dual model and the $d = 4$ two-scalar model known as FGPW. Finally, we illustrate how our method works with marginal scalars (dilaton + axion in $d = 4$).

## 2.3 Pure Gravity

The simplest model one can consider is pure AdS gravity with no matter content in $D = d+1$ dimensions. Counterterms obtained by renormalizing this model will be present in every other model and it is therefore useful to deal with them once and for all. The action we consider is given by (2.3) with no scalar fields and constant scalar potential

$$V = -\frac{d(d-1)}{L^2}\,. \tag{2.26}$$

The Hamilton-Jacobi equation (2.17) simplifies to

$$R[\gamma] + \mathcal{K} + \frac{d(d-1)}{L^2} + 2\frac{\partial U}{\partial r} = 0\,, \tag{2.27}$$

with $\mathcal{K}$ given by (2.16). Let us now apply the algorithmic procedure described in the previous section in order to determine the necessary counterterms for this class of theories.

**Step 1:** Since there are no scalars, the general Ansatz for each order of the expansion of $U$ is

$$U_{(0)} = A(r)\,, \qquad U_{(2)} = B(r)R\,, \qquad U_{(4)} = C_1(r)R_{ij}R^{ij} + C_2(r)R^2\,, \tag{2.28}$$

where the four-derivative terms are only needed for $d \geq 4$.[5] We are not including terms like $\Box R$ since it is a total derivative and it will not contribute in the on-shell action. For $d \geq 6$, we need

$$U_{(6)} = D_1 R^3 + D_2 R R_{ij}R^{ij} + D_3 R_i^{\ j} R_j^{\ k} R_k^{\ i} + D_4 R^{ij}R^{kl}R_{ikjl} + D_5 R\Box R + D_6 R_{ij}\Box R^{ij}\,. \tag{2.29}$$

This is not a complete list of independent six-derivative terms, but it turns out to be a sufficient list.

It is important that all the coefficients in the above expressions for $U$ depend on the radial coordinate $r$, as this will capture the explicit $r$-dependence of the on-shell action.

**Step 2:** This step is irrelevant for the pure gravity case since there are no matter fields.

**Step 3:** We now solve Hamilton-Jacobi equation (2.27) order by order to determine the unknown coefficients $A$, $B$, $C_{1,2}$ and $D_i$.

At zero-derivatives, (2.27) with $\mathcal{K}_{(0)}$ given by (2.20) gives

$$2\dot{A} - \frac{d}{d-1}A^2 + \frac{d(d-1)}{L^2} = 0\,, \tag{2.30}$$

where the dot denotes differentiation with respect to $r$. For large $r$, the solution to the

[5] In $U_{(4)}$, one could also have included a term with the square of the Riemann tensor. However, it is not hard to see that its coefficient will be set to zero in the HJ equation.

differential equation is

$$A(r) = -\frac{d-1}{L} + \mathcal{O}\big(e^{-dr/L}\big)\,. \tag{2.31}$$

The subleading terms in the large-$r$ expansion of $A$ give only finite contribution to the on-shell action and we can drop it to simply have

$$U_{(0)} = -\frac{d-1}{L}\,. \tag{2.32}$$

This captures the leading divergence associated with the cosmological constant.

At two-derivative order, the HJ equation (2.27) with (2.22) gives

$$R - \frac{2}{d-1}U_{(0)}\left(U_{(2)} - 2Y_{(2)}\right) - 2U_{(0)}U_{(2)} + 2\frac{\partial U_{(2)}}{\partial r} = 0\,. \tag{2.33}$$

The inverse-metric variation of $U_{(2)}$ simply gives $Y_{(2)ij} = \frac{\delta U_{(2)}}{\delta \gamma^{ij}} = BR_{ij}$, so $Y_{(2)} = BR$. With the solution for $U_{(0)}$ in (2.32), we obtain the following differential equation for $B$:

$$2\dot{B} + 2\frac{d-2}{L}B + 1 = 0\,. \tag{2.34}$$

The differential equation for $B$ has solution

$$B(r) = \begin{cases} -\frac{r}{2} + \mathcal{O}(1) & \text{for } d = 2 \\ -\frac{L}{2(d-2)} + \mathcal{O}\left(e^{-(d-2)r/L}\right) & \text{for } d > 2 \end{cases} \tag{2.35}$$

In both cases, the subleading terms are not important since they give finite contributions to the on-shell action. The result is therefore

$$U_{(2)} = \begin{cases} -\frac{r}{2}R & \text{for } d = 2 \\ -\frac{L}{2(d-2)}R & \text{for } d > 2 \end{cases} \tag{2.36}$$

The linear $r$ behavior in the $d = 2$ case is our first illustration of the explicit $r$-dependence in the on-shell action and the importance of keeping the $\frac{\partial S_{\text{on-shell}}}{\partial r}$-term in the Hamilton-Jacobi equation.

For the four-derivative terms, we calculate the inverse-metric variation of $U_4$ using the formulae in Appendix A. In particular, we find $Y_{(4)} = 2C_1 R_{ij}R^{ij} + 2C_2 R^2$ (up to total derivatives that can be dropped). Using this together with the results for $Y_{(2)}$ above, we can calculate $\mathcal{K}_{(4)}$ given in (2.22). At 4th order, the HJ equation (2.27) is simply $\mathcal{K}_{(4)} + 2\frac{\partial U_{(4)}}{\partial r} = 0$ and collecting terms gives

$$\left[2\dot{C}_1 + \frac{2(d-4)}{L}C_1 + \left(\frac{L}{d-2}\right)^2\right] R_{ij}R^{ij} \\ + \left[2\dot{C}_2 + \frac{2(d-4)}{L}C_2 - \frac{dL^2}{4(d-1)(d-2)^2}\right] R^2 = 0\,.$$

Demanding the coefficients of the $R_{ij}R^{ij}$ and $R^2$ terms to vanish independently results in two differential equation for the coefficients $C_1$ and $C_2$, which have solutions

$$C_1 = \begin{cases} -\frac{L^2 r}{8} + \mathcal{O}(1) & \text{for } d=4 \\ -\frac{L^3}{2(d-2)^2(d-4)} + \mathcal{O}\left(e^{-(d-4)r/L}\right) & \text{for } d>4 \end{cases} \tag{2.37}$$

$$C_2 = \begin{cases} \frac{L^2 r}{24} + \mathcal{O}(1) & \text{for } d=4 \\ \frac{dL^3}{8(d-1)(d-2)^2(d-4)} + \mathcal{O}\left(e^{-(d-4)r/L}\right) & \text{for } d>4 \end{cases} \tag{2.38}$$

Again, the subleading terms can be dropped because they give only finite contributions to the on-shell action. Thus, the result for $U_{(4)}$ is

$$U_{(4)} = \begin{cases} -\frac{L^2 r}{8}\left(R_{ij}R^{ij} - \frac{1}{3}R^2\right) & \text{for } d=4 \\ -\frac{L^3}{2(d-2)^2(d-4)}\left(R_{ij}R^{ij} - \frac{d}{4(d-1)}R^2\right) & \text{for } d>4 \end{cases} \tag{2.39}$$

**Step 4:** We now have all information needed to write the counterterm action.

$$S_{\rm ct} = -\frac{1}{\kappa^2}\int_{\partial M_\epsilon} \mathrm{d}^d x\sqrt{\gamma}\, U = -\frac{1}{\kappa^2}\int_{\partial M_\epsilon} \mathrm{d}^d x\sqrt{\gamma}\left[U_{(0)} + U_{(2)} + \ldots + U_{\left(2\left\lfloor\frac{d}{2}\right\rfloor\right)}\right]. \qquad (2.40)$$

Summarizing the above results, the purely gravitational counterterms are

$$\boxed{\begin{aligned}
d=2:&\qquad S_{\rm ct} = \frac{1}{\kappa^2}\int_{\partial M_\epsilon}\mathrm{d}^d x\sqrt{\gamma}\left[\frac{1}{L} - \log\rho\,\frac{L}{4}R\right],\\
d=3:&\qquad S_{\rm ct} = \frac{1}{\kappa^2}\int_{\partial M_\epsilon}\mathrm{d}^d x\sqrt{\gamma}\left[\frac{2}{L} + \frac{L}{2}R\right],\\
d=4:&\qquad S_{\rm ct} = \frac{1}{\kappa^2}\int_{\partial M_\epsilon}\mathrm{d}^d x\sqrt{\gamma}\left[\frac{3}{L} + \frac{L}{4}R - \log\rho\,\frac{L^3}{16}\left(R_{ij}R^{ij} - \frac{1}{3}R^2\right)\right],\\
d=5:&\qquad S_{\rm ct} = \frac{1}{\kappa^2}\int_{\partial M_\epsilon}\mathrm{d}^d x\sqrt{\gamma}\left[\frac{4}{L} + \frac{L}{6}R + \frac{L^3}{18}\left(R_{ij}R^{ij} - \frac{5}{16}R^2\right)\right],\\
d=6:&\qquad S_{\rm ct} = \frac{1}{\kappa^2}\int_{\partial M_\epsilon}\mathrm{d}^d x\sqrt{\gamma}\left[\frac{5}{L} + \frac{L}{8}R + \frac{L^3}{64}\left(R_{ij}R^{ij} - \frac{3}{10}R^2\right)\right.\\
&\qquad\qquad - \log\rho\,\frac{L^5}{256}\left(R_{ij}\Box R^{ij} - \frac{1}{20}R\Box R\right.\\
&\qquad\qquad\left.\left. +2R^{ij}R^{kl}R_{ikjl} + \frac{1}{5}RR_{ij}R^{ij} - \frac{3}{100}R^3\right)\right],
\end{aligned}} \qquad (2.41)$$

where we have used $\rho = e^{-2r/L}$. The results for the six-derivative terms displayed for $d = 6$ are derived in Appendix B.

These purely gravitational counterterms reproduce results well-known in the literature, see for example [8], but it is relevant to present them here in the context of our approach to holographic renormalization. In particular, they will appear in the following examples.

## 2.4 Renormalization for the ABJM Model

ABJM theory [68] is the $\mathcal{N} = 6$ superconformal Chern-Simons theory in $d = 3$ dimensions with gauge group $U(N)\times U(N)$ and Chern-Simons levels $k$ and $-k$. Its holographic dual is M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$. In the limit of large t'Hooft coupling ($\lambda = N/k$), M-theory reduces to eleven dimensional supergravity on $AdS_4 \times S^7/\mathbb{Z}_k$. The recent paper [63] by Freedman and Pufu explores the gauge-gravity dual description of $F$-maximization for ABJM theory on a 3-sphere using a 4-dimensional holographic dual. We will use the model of [63] as a very simple example to illustrate our approach to holographic renormalization.

The ABJM holographic model [63] is described by the Euclidean bulk action

$$S_{\rm bulk} = -\frac{1}{2\kappa^2}\int_M \mathrm{d}^3x\,\mathrm{d}r\sqrt{g}\big(\mathcal{R}[g] - \mathcal{L}_m\big)\,, \tag{2.42}$$

where $\kappa^2 = 8\pi G_4$ and the matter Lagrangian is

$$\mathcal{L}_m = 2\sum_{a=1}^{3}\frac{\partial_\mu z^a \partial^\mu \bar{z}^a}{(1-z^a\bar{z}^a)^2} + V(z,\bar{z})\,, \qquad V(z,\bar{z}) = \frac{1}{L^2}\bigg(6 - \sum_{a=1}^{3}\frac{4}{1-z^a\bar{z}^a}\bigg)\,. \tag{2.43}$$

In the Euclidean theory, the scalars $z^a$ and $\bar{z}^a$ are independent complex fields, not related by complex conjugation. However, since only products of $z^a$ and $\bar{z}^a$ appear in this Lagrangian, it is useful to define $z^a \to \frac{1}{\sqrt{2}}\left(\chi^a + i\psi^a\right), \bar{z}^a \to \frac{1}{\sqrt{2}}\left(\chi^a - i\psi^a\right)$, where $\chi^a$ and $\psi^a$ are fields that can take complex values.

Under this, the matter Lagrangian becomes

$$\mathcal{L}_m = \sum_{a=1}^{3}\frac{\partial_\mu\chi^a\partial^\mu\chi^a + \partial_\mu\psi^a\partial^\mu\psi^a}{\big[1-\frac{1}{2}(\chi^a)^2 - \frac{1}{2}(\psi^a)^2\big]^2} + V\,, \qquad V = \frac{1}{L^2}\bigg(6 - \sum_{a=1}^{3}\frac{4}{1-\frac{1}{2}(\chi^a)^2-\frac{1}{2}(\psi^a)^2}\bigg)\,. \tag{2.44}$$

Expanding the potential for small fields, we find

$$V = \frac{1}{L^2}\Big(-6 - 2(\chi^a\chi^a + \psi^a\psi^a) - (\chi^a\chi^a + \psi^a\psi^a)^2 + \dots\Big)\,, \tag{2.45}$$

so the six fields $\chi^a$ and $\psi^a$ all have mass $-2/L^2$. By our general discussion, this means that their asymptotic falloff is generically $e^{-r/L}$.

For simplicity, let us start out with a model with just one pair of the fields $\chi$ and $\psi$; since the ABJM dual has the three pairs appear the same way and they do not mix, it is easy to generalize the result back to that case. Thus setting the fields with $a = 2, 3$ to zero, we will consider the model described by the potential

$$V = \frac{1}{L^2}\left( -2 - \frac{4}{1 - \frac{1}{2}\chi^2 - \frac{1}{2}\psi^2} \right). \tag{2.46}$$

In the notation (2.3), we have scalars $\Phi^I = (\chi, \psi)$ and the metric on the scalar target space is $G_{IJ} = \left(1 - \frac{1}{2}\chi^2 - \frac{1}{2}\psi^2\right)^{-2} \delta_{IJ}$ with $I, J = 1, 2$. The Hamilton-Jacobi equation (2.17) for this model is then

$$\begin{aligned} R + \mathcal{K} - \left(1 - \frac{1}{2}\chi^2 - \frac{1}{2}\psi^2\right)^{-2} \gamma^{ij}(\partial_i\chi\partial_j\chi + \partial_i\psi\partial_j\psi) \\ + \left(1 - \frac{1}{2}\chi^2 - \frac{1}{2}\psi^2\right)^2 (p_\chi^2 + p_\psi^2) - \frac{1}{L^2}\left(-2 - \frac{4}{1 - \frac{1}{2}\chi^2 - \frac{1}{2}\psi^2}\right) + 2\frac{\partial U}{\partial r} = 0\,, \end{aligned} \tag{2.47}$$

where $\mathcal{K}$ is given by equation (2.16) and the conjugate momenta $p_\chi$ and $p_\psi$ are the $\chi$ and $\psi$ derivatives of the on-shell action (2.13). We now proceed to determine the infinite counterterms for this model.

**Step 1:** Since we are working in $d = 3$ dimensions we need to include in our Ansatz only terms with up to two derivatives:

$$U = U_{(0)} + U_{(2)}\,. \tag{2.48}$$

Terms with four or more derivatives give finite contributions to the on-shell action.

Keeping only potentially divergent contributions means that for $U_{(0)}$ we only need to consider terms up to cubic order in the scalar fields. However, we get strong constraints on the Ansatz from the symmetries of the model: it is invariant under the transformations $\chi \to -\chi$, $\psi \to -\psi$, and $\chi \leftrightarrow \psi$. With these symmetries imposed, the most general Ansatz at zero-derivative order is

$$U_{(0)} = -\frac{2}{L} + A(r)(\chi^2 + \psi^2)\,. \tag{2.49}$$

The constant term is fixed from the purely gravitational calculation of Section 2.3. At two-derivative order, the only potentially divergent term that preserves the symmetries of the theory is purely gravitational and it was calculated in Section 2.3:

$$U_{(2)} = -\frac{L}{2}R\,. \tag{2.50}$$

We can skip **Step 2** because the model is so simple.

**Step 3:** We are now able to solve Equation (2.47). Keeping only zero-derivative terms and using that $\mathcal{K}_{(0)} = -\frac{3}{2}U_{(0)}^2$ from (2.16) we find that

$$-\frac{3}{2}U_{(0)}^2 + \left(1 - \frac{1}{2}\chi^2 - \frac{1}{2}\psi^2\right)^2 (p_{\chi(0)}^2 + p_{\psi(0)}^2) - V(\chi,\psi) + 2\frac{\partial U_{(0)}}{\partial r} = 0\,, \tag{2.51}$$

where,

$$p_{\chi(0)} = \frac{\delta U_{(0)}}{\delta \chi} = 2A\chi\,, \qquad p_{\psi(0)} = \frac{\delta U_{(0)}}{\delta \psi} = 2A\psi\,. \tag{2.52}$$

Putting everything together and collecting terms that are proportional to $(\chi^2 + \psi^2)$ gives the following differential equation for $A(r)$:

$$\dot{A} + 2A^2 + \frac{3}{L}A + \frac{1}{L^2} = 0\,. \tag{2.53}$$

This has solution

$$A = -\frac{1}{2L} + \mathcal{O}\left(e^{-r/L}\right)\,. \tag{2.54}$$

Since $A$ was the only unknown coefficient in the Ansatz for $U$, this concludes the calculation of the infinite contributions in the on-shell action. Specifically, we have found that

$$U_{(0)} = -\frac{1}{L}\left(2 + \frac{1}{2}\chi^2 + \frac{1}{2}\psi^2\right) = -\frac{1}{L}\left(2 + z\bar{z}\right)\,. \tag{2.55}$$

**Step 4:** The counterterm action for the ABJM model is obtained by generalizing our result to the three flavors of $z^a$ and $\bar{z}^a$ fields:

$$\boxed{S_{ct} = \frac{1}{\kappa^2} \int_{\partial M_\epsilon} \mathrm{d}^3x \sqrt{\gamma} \left[ \frac{1}{L} \left( 2 + \sum_{a=1}^{3} z^a \bar{z}^a \right) + \frac{L}{2} R \right] .} \tag{2.56}$$

This result is in perfect agreement with the counterterm action given in equations (6.4)-(6.5) in [63]. For the applications in [63] one further needs to use supersymmetry to determine the finite counterterms; we do not discuss this here.

## 2.5 Renormalization for the FGPW Model

The FGPW model [64] is the holographic dual of the single-mass limit of $\mathcal{N} = 1^*$ gauge theory in flat space. This non-conformal field theory is obtained from $\mathcal{N} = 4$ SYM theory by softly breaking the supersymmetry to $\mathcal{N} = 1$ as follows. In $\mathcal{N} = 1$ language, $\mathcal{N} = 4$ SYM consists of a vector multiplet and three chiral multiplets. The field theory dual to FGPW is obtained by giving a mass to one of the chiral multiplets. In the UV, the conformal theory of $\mathcal{N} = 4$ SYM is recovered, while in the infrared, the theory flows to a Leigh-Strassler fixed point. The holographic dual FGPW model captures the RG flow of this theory via a flat-space sliced domain wall solution which approaches asymptotic $AdS_5$ in the UV and another $AdS_5$ in the IR. The ratio of the AdS radii in the UV and IR translates to the ratio of UV and IR central charges $a$ in the field theory. More generally, the authors of [69, 64] derived the first version of a holographic version of the $c$-theorem.

The holographic FGPW model is described by a $D = 4 + 1$-dimensional bulk action

$$S = -\frac{1}{2\kappa^2} \int_M \mathrm{d}^4x \, \mathrm{d}r \sqrt{g} \left( \mathcal{R}[g] - \mathcal{L}_m \right) , \tag{2.57}$$

with matter Lagrangian given by[6]

$$\mathcal{L}_m = \partial_\mu\phi\partial^\mu\phi + \partial_\mu\psi\partial^\mu\psi + V(\phi,\psi) = \dot{\phi}^2 + \dot{\psi}^2 + \gamma^{ij}\partial_i\phi\partial_j\phi + \gamma^{ij}\partial_i\psi\partial_j\psi + V(\phi,\psi)\,. \quad (2.60)$$

The scalars $\psi$ and $\phi$ are dimension $\Delta_\psi = 3$ and $\Delta_\phi = 2$ fields dual to the fermion and scalar mass deformations of $\mathcal{N} = 4$ SYM. They approach zero near the UV boundary as

$$\psi \sim \psi_0\, e^{-r/L} \qquad \text{and} \qquad \phi \sim (\phi_0 r + \tilde{\phi}_0)\, e^{-2r/L}\,, \quad (2.61)$$

as $r \to \infty$. For the purpose of holographic renormalization, we only need to keep the terms in the potential that can give divergent terms in this limit, so we expand the potential in small fields to find

$$V(\phi,\psi) = \frac{1}{L^2}\left(-12 - 4\phi^2 - 3\psi^2 + c\psi^4 + \ldots\right)\,. \quad (2.62)$$

The masses of the scalars, $m_\psi^2 = -3/L^2$ and $m_\phi^2 = -4/L^2$, are directly related to the scaling dimensions $\Delta_\psi = 3$ and $\Delta_\phi = 2$ via $m_I^2 L^2 = \Delta_I(\Delta_I - 4)$.

The actual FGPW model has $c = 1$ in (2.62), but here we keep the coefficients general. This will serve to illustrate how the counterterms carry information that is specifically dependent on coefficients in the scalar potential; i.e. one should in general expect model-dependent terms in the counterterm action.

[6] In the paper [64], the scalar potential $V$ is given in terms of a superpotential $W$ as

$$V_{\text{FGPW}} = \frac{1}{L^2}\left(\frac{1}{2}\left|\frac{\partial W}{\partial\phi_1}\right|^2 + \frac{1}{2}\left|\frac{\partial W}{\partial\phi_3}\right|^2 - \frac{4}{3}W^2\right)\,, \quad (2.58)$$

with

$$W = \frac{1}{4\rho^2}\Big[\cosh(2\phi_1)(\rho^6 - 2) - (3\rho^6 + 2)\Big] \qquad \text{and} \qquad \rho = e^{\phi_3/\sqrt{6}}\,. \quad (2.59)$$

Here, we have conformed to our normalization conventions by rescaling the scalars $\phi_1 = \psi/\sqrt{2}$ and $\phi_3 = \phi/\sqrt{2}$, and taken the potential to be $V = 4V_{\text{FGPW}}$.

The HJ equation (2.17) for the FGPW model takes the form

$$R[\gamma] + \mathcal{K} + p_\phi^2 + p_\psi^2 - \gamma^{ij}\partial_i\phi\partial_j\phi - \gamma^{ij}\partial_i\psi\partial_j\psi - V(\phi,\psi) + 2\frac{\partial U}{\partial r} = 0\,. \tag{2.63}$$

with $\mathcal{K}$ defined in (2.16) and momenta

$$p_\phi = \frac{\delta U}{\delta \phi} \qquad p_\psi = \frac{\delta U}{\delta \psi}\,. \tag{2.64}$$

Since we are working in $d = 4$ dimensions we need to keep terms with up to four derivatives, so we write

$$U = U_{(0)} + U_{(2)} + U_{(4)}\,. \tag{2.65}$$

We now proceed with solving for the divergent terms of the on-shell action following the algorithmic procedure described in Section 2.2.3:

**Step 1:** We begin by writing the most general Ansatz for each $U_{(i)}$. We only keep terms that can give divergent contributions. With the scalar falloffs (2.61) and each inverse metric giving $e^{-2r}$, the most general Ansatz at 0th order is

$$U_{(0)} = -\frac{3}{L} + A_1\psi + A_2\phi + A_3\psi^2 + A_4\phi\psi + A_5\psi^3 + A_6\phi^2 + A_7\phi\psi^2 + A_8\psi^4\,, \tag{2.66}$$

where the constant term is fixed by the purely gravitational analysis in Section 2.3. Each of the coefficients $A_i$ is considered a function of $r$.

At order 2 we use the Ansatz

$$U_{(2)} = -\frac{L}{4}R + B_1R\psi + B_2R\phi + B_3R\psi^2 + B_4\psi\Box\psi\,. \tag{2.67}$$

We did not include $(\partial\psi)^2$, since it is equivalent to $\psi\Box\psi$ after partial integration.

At order 4, the only option are the purely gravitational terms we have already solved, so we have

$$U_{(4)} = -\frac{L^2 r}{8}\left(R_{ij}R^{ij} - \frac{1}{3}R^2\right) . \tag{2.68}$$

Since the full FGPW model (2.58)-(2.59) is symmetric under $\psi \to -\psi$, we can immediately set the following coefficients in the Ansatz to zero:

$$A_1 = A_4 = A_5 = B_1 = 0 . \tag{2.69}$$

**Step 2:** At the leading order, the conjugate momenta obtained from (2.7) must agree with those in (2.64). From (2.7), we have

$$p_\phi = \dot{\phi} \qquad p_\psi = \dot{\psi} , \tag{2.70}$$

and via (2.61) this gives

$$p_\phi = -\frac{2}{L}\Big(1 - \frac{L}{2r}\Big)\phi + \mathcal{O}\big(e^{-2r/L}/r\big) , \qquad p_\psi = -\frac{1}{L}\psi + \mathcal{O}\big(e^{-3r/L}\big) . \tag{2.71}$$

On the other hand (2.64) gives

$$p_{\phi(0)} = \frac{\delta U_{(0)}}{\delta \phi} = A_2 + 2A_6\phi + A_7\psi^2 , \qquad p_{\psi(0)} = \frac{\delta U_{(0)}}{\delta \psi} = 2A_3\psi + 2A_7\phi\psi + 4A_8\psi^3 . \tag{2.72}$$

Comparing (2.71) to terms in (2.72) at similar orders, we can directly infer that some of the coefficients $A_i$ must vanish:

$$A_2 = A_7 = 0 . \tag{2.73}$$

Furthermore, we learn that $A_3 = -\frac{1}{2L}$ and $A_6 = -\frac{1}{L}\big(1 - \frac{L}{2r}\big)$. However, let us leave $A_3$ and $A_6$ unfixed for now for the purpose of illustrating how they are fixed using the HJ equation.

**Step 3:** We proceed to solve the HJ equation (2.63). We start from the terms at 0th order. Keeping only terms without spatial derivatives and using $\mathcal{K}_{(0)} = -\frac{4}{3}U_{(0)}^2$ from (2.20) we find that

$$- \frac{4}{3} U_{(0)}^2 + p_{\phi(0)}^2 + p_{\psi(0)}^2 - V(\phi, \psi) + 2 \frac{\partial U_{(0)}}{\partial r} = 0 \,. \tag{2.74}$$

To solve this, we set the coefficient of each combination of fields to zero. For example, collecting the terms proportional to $\psi^2$ gives

$$\dot{A}_3 + \frac{4}{L} A_3 + 2A_3^2 + \frac{3}{2L^2} = 0 \qquad \Longrightarrow \qquad A_3 = -\frac{1}{2L} + \mathcal{O}\big(e^{-2r/L}\big) \,. \tag{2.75}$$

This is the solution for $A_3$ we anticipated from comparing (2.71) and (2.72).

Similarly, one finds

$$\begin{aligned} &\phi^2\text{-terms:} \quad && \dot{A}_6 + \frac{4}{L} A_6 + 2A_6^2 + \frac{2}{L^2} = 0 \quad && \Longrightarrow \quad A_6 = -\frac{1}{L} + \frac{1}{2r} + \mathcal{O}\Big(\frac{L^2}{r^2}\Big) \,, \\ &\psi^4\text{-terms:} && \dot{A}_8 - \frac{1}{6L^2} (1 + 3c) = 0 && \Longrightarrow \quad A_8 = \frac{1}{6L^2} (1 + 3c) \, r + \mathcal{O}\big(1\big) \,. \end{aligned} \tag{2.76}$$

Terms proportional to $\phi\psi^2$ vanish directly; had we had a term $b\phi\psi^2$ in the expansion of the scalar potential, the HJ equation would have shown that $b \neq 0$ is not consistent with the EOM.

Having calculated all the unknown coefficients in the $U_{(0)}$ Ansatz, let us write down the final result (with $r = -\frac{L}{2} \log \rho$):

$$U_{(0)} = -\frac{1}{L} \left[ 3 + \left( 1 + \frac{1}{\log \rho} \right) \phi^2 + \frac{1}{2} \psi^2 + \frac{1}{12} \big( 1 + 3c \big) \psi^4 \log \rho \right] \,. \tag{2.77}$$

We can identify each of the contributions. The first one is related to the cosmological constant and it is fixed for all models in $D = 4 + 1$ dimensions, as we saw in the pure gravity case in Section 2.3. The terms that are quadratic in the fields are uniquely fixed by the mass terms in the scalar potential and are as such universal for all models. Finally, the $\psi^4$-terms are clearly model-dependent, as can be seen from the explicit dependence on $c$.

With the 0th order result in hand, we are now able to continue solving HJ equation for the two-derivative terms. Keeping only such terms from equation (2.63) gives

$$R - \frac{8}{3}U_{(0)}\left(U_{(2)} - \frac{1}{2}Y_{(2)}\right) + 2p_{\phi(0)}p_{\phi(2)} + 2p_{\psi(0)}p_{\psi(2)} - \gamma^{ij}\partial_i\phi\partial_j\phi - \gamma^{ij}\partial_i\psi\partial_j\psi + 2\frac{\partial U_{(2)}}{\partial r} = 0\,, \tag{2.78}$$

where we used $\mathcal{K}_{(2)}$ from (2.22). $U_{(0)}$, $p_{\phi(0)}$ and $p_{\psi(0)}$ are known from (2.72) and (2.77), while we calculate $p_{\phi(2)}$ and $p_{\psi(2)}$, and $Y_{(2)}$ from the Ansatz (2.67) for $U_{(2)}$:

$$\begin{aligned} p_{\phi(2)} &= \frac{\delta U_{(2)}}{\delta\phi} = B_2 R\,, \\ p_{\psi(2)} &= \frac{\delta U_{(2)}}{\delta\psi} = 2B_3 R\psi + 2B_4\Box\psi\,, \\ Y_{(2)ij} &= \frac{\delta U_{(2)}}{\delta\gamma^{ij}} = -\frac{L}{4}R_{ij} + B_2 R_{ij}\phi + B_3 R_{ij}\psi^2 + B_4\psi\nabla_i\nabla_j\psi\,, \end{aligned} \tag{2.79}$$

where we are dropping total derivatives. The result for $Y_{(2)ij}$ implies $Y_{(2)} = U_{(2)}$. In the HJ equation (2.78), we organize the terms according to the field monomials and set the coefficients of divergent terms to zero. The terms simply proportional to $R$ directly vanish because we have already solved the purely gravitational part of the problem. The remaining terms allow us to solve for the coefficients $B_{2,3,4}$ :

$$\begin{aligned} &R\phi\text{-terms:} && \dot{B}_2 + \frac{1}{r}B_2 = 0 && \Longrightarrow && B_2 = \mathcal{O}\Big(\frac{1}{r}\Big)\,, \\ &R\psi^2\text{-terms:} && \dot{B}_3 - \frac{1}{12} = 0 && \Longrightarrow && B_3 = \frac{1}{12}r + \mathcal{O}\big(1\big)\,, \\ &\psi\Box\psi\text{-terms:} && \dot{B}_4 + \frac{1}{2} = 0 && \Longrightarrow && B_4 = -\frac{1}{2}r + \mathcal{O}\big(1\big)\,. \end{aligned} \tag{2.80}$$

As in the zero weight case the subleading terms related to integration constants are not important because they lead to finite contributions to the action. The final expression for $U_{(2)}$ is then

$$U_{(2)} = -L\left[\frac{1}{4}R - \frac{1}{4}\psi\left(\Box - \frac{1}{6}R\right)\psi\log\rho\right]\,. \tag{2.81}$$

The first term is purely gravitational. The second term is independent of details of the higher order terms in the potential and thus fixed for all models that contain a scalar with

$m^2L^2 = -3$. Finally, notice that the combination of the Laplace operator $\Box$ and the Ricci scalar $R$ that appears in the last term is proportional, up to an overall constant to the conformal Laplacian.

**Step 4:** We have now fully determined the counterterm action necessary to cancel the divergences of the on-shell action. In particular we will have $S_{\rm ct} = -\frac{1}{\kappa^2}\int d^4x\sqrt{\gamma}\,U$ and therefore,

$$\boxed{\begin{aligned} S_{\rm ct} = \frac{1}{\kappa^2}\int_{\partial M_\epsilon} \mathrm{d}^4x\sqrt{\gamma}\Bigg\{&\frac{1}{L}\left[3+\left(1+\frac{1}{\log\rho}\right)\phi^2+\frac{1}{2}\psi^2+\frac{1}{12}\left(1+3c\right)\psi^4\log\rho\right] \\ &+L\left[\frac{1}{4}R-\frac{1}{4}\psi\left(\Box-\frac{1}{6}R\right)\psi\log\rho\right]-\frac{1}{16}L^3\left(R_{ij}R^{ij}-\frac{1}{3}R^2\right)\log\rho\Bigg\}\,. \end{aligned}} \tag{2.82}$$

This is our final result for the FGPW model.

As a test, we have calculated the one-point functions of the QFT operators that are dual to the fields of the FGPW model. The one-point function of the operator dual to field $\phi^I$ will be given by[7]

$$\langle O_{\phi^I}\rangle = -\lim_{\rho\to 0}\frac{\rho^{-\Delta_I/2}}{\sqrt{\gamma}}\frac{\delta S_{\rm ren}}{\delta\phi^I}\,, \tag{2.83}$$

where the regularized action (ignoring possible finite counterterms) is

$$S_{\rm reg} = S_{\rm bulk} + S_{\rm GH} + S_{\rm ct}\,. \tag{2.84}$$

In order to check that the expressions obtained are indeed finite, one must impose the equations of motion on the coefficients in the Fefferman-Graham expansion of the fields. We find that with our infinite counterterms, all three one-point functions in FGPW are indeed finite. Details are presented in Appendix C.

[7] In the special case where $\Delta_I = d/2$ the one-point function has an extra factor of $\log\rho$.

## 2.6 Renormalization of a Dilaton-Axion Model

In this section we present the procedure of renormalization of a dilaton-axion model. The purpose of this example is to illustrate how the procedure for holographic renormalization applies to theories that include marginal scalars. Specifically, we examine the renormalization of the dilaton-axion model previously studied in [20]: the 5d bulk action is

$$S_{\text{bulk}} = -\frac{1}{2\kappa^2}\int_M \mathrm{d}^4x\,\mathrm{d}r\sqrt{g}\big(\mathcal{R}[g]-\mathcal{L}_m\big)\,, \tag{2.85}$$

with

$$\mathcal{L}_m = \partial_\mu\varphi\partial^\mu\varphi + Z(\varphi)\partial_\mu\chi\partial^\mu\chi - \frac{12}{L^2}\,. \tag{2.86}$$

The fields $\varphi$ and $\chi$ are massless and therefore correspond to marginal QFT operators with scaling dimension $\Delta = 4$. $Z$ denotes an arbitrary function of the dilaton field $\varphi$. Near the asymptotic boundary, these scalars generically do not vanish but instead approach a finite value. In particular, their asymptotic behavior is given by

$$\varphi(x,r) = \varphi_{(0)}(x) + \mathcal{O}\Big(e^{-2r/L}\Big)\,, \qquad \chi(x,r) = \chi_{(0)}(x) + \mathcal{O}\Big(e^{-2r/L}\Big)\,. \tag{2.87}$$

As a consequence, we cannot regard the effective action as a power-expansion in these fields, as higher powers are not suppressed. Instead, we will take the Ansatz to involve general functions of $\varphi$ and $\chi$.

By defining the field $\Phi$ to be $\Phi = (\varphi,\chi)$ and the Kähler metric to be $G = \left(\begin{smallmatrix}1 & 0\\ 0 & Z(\varphi)\end{smallmatrix}\right)$, we conclude that the HJ Equation (2.17) now becomes

$$R[\gamma] + \mathcal{K} + p_\varphi^2 + \frac{1}{Z(\varphi)}p_\chi^2 - \gamma^{ij}\partial_i\varphi\partial_j\varphi - Z(\varphi)\gamma^{ij}\partial_i\chi\partial_j\chi + \frac{12}{L^2} + 2\frac{\partial U}{\partial r} = 0\,. \tag{2.88}$$

The momenta are defined, in the usual way (2.13), as derivatives of $U$.

Let us now examine step-by-step the procedure introduced in the previous sections and spot any important differences.

**Step 1:** With $d = 4$, we need to keep terms with up to four derivatives:

$$U = U_{(0)} + U_{(2)} + U_{(4)} \,. \tag{2.89}$$

Taking into account that any possible function of the fields could give divergent contributions in the on-shell action we write the following Ansatz for the zero, two and four derivative parts of $U$ respectively:

$$U_{(0)} = A(\varphi, \chi, r) \,, \tag{2.90}$$

$$U_{(2)} = B_0 R + B_1 (\nabla\varphi)\cdot(\nabla\chi) + B_2 (\nabla\varphi)^2 + B_3 (\nabla\chi)^2 \,, \tag{2.91}$$

$$\begin{aligned} U_{(4)} = {} & C_1 R^2 + C_2 R_{ij} R^{ij} + C_3 R\Box\varphi + C_4 R\Box\chi + C_5 R(\nabla\varphi)^2 + C_6 R(\nabla\chi)^2 \\ & + C_7 R(\nabla\varphi)\cdot(\nabla\chi) + C_8 R^{ij}\nabla_i\varphi\nabla_j\varphi + C_9 R^{ij}\nabla_i\chi\nabla_j\chi + C_{10} R^{ij}\nabla_i\varphi\nabla_j\chi \\ & + C_{11}(\Box\varphi)^2 + C_{12}(\Box\chi)^2 + C_{13}\Box\varphi\Box\chi + C_{14}\nabla_i\nabla_j\varphi\nabla^i\nabla^j\varphi + C_{15}\nabla_i\nabla_j\chi\nabla^i\nabla^j\chi \\ & + C_{16}\nabla_i\nabla_j\varphi\nabla^i\nabla^j\chi + C_{17}\Box\varphi(\nabla\varphi)^2 + C_{18}\Box\chi(\nabla\chi)^2 + C_{19}\Box\varphi(\nabla\chi)^2 \\ & + C_{20}\Box\varphi(\nabla\varphi)\cdot(\nabla\chi) + C_{21}\Box\chi(\nabla\varphi)^2 + C_{22}\Box\chi(\nabla\varphi)\cdot(\nabla\chi) + C_{23}\left((\nabla\varphi)^2\right)^2 \\ & + C_{24}\left((\nabla\chi)^2\right)^2 + C_{25}(\nabla\varphi)^2(\nabla\chi)^2 + C_{26}((\nabla\varphi)\cdot(\nabla\chi))^2 + C_{27}(\nabla\varphi)^2(\nabla\varphi)\cdot(\nabla\chi) \\ & + C_{28}(\nabla\chi)^2(\nabla\varphi)\cdot(\nabla\chi) \,. \end{aligned} \tag{2.92}$$

The coefficients $A$, $B_i$ and $C_i$ are all considered functions of the radial coordinate $r$ as well as the fields $\varphi$ and $\chi$. We have omitted terms that up to total derivatives can be decomposed to the ones already included. For example, since $B\Box\varphi = \nabla_i(B\nabla^i\varphi) - \partial_\varphi B(\nabla\varphi)^2 - \partial_\chi B(\nabla\varphi)\cdot(\nabla\chi)$, such a term can be absorbed in $B_1$ and $B_2$, so it is redundant to include it in the Ansatz.

**Step 2:** We use equation (2.7) and the asymptotic behavior of the fields (2.87) to determine the leading behavior of $p_\varphi$ and $p_\chi$ to be

$$p_\varphi = \dot{\varphi} = \mathcal{O}\Big(e^{-2r/L}\Big)\,, \qquad p_\chi = Z(\varphi)\dot{\chi} = \mathcal{O}\Big(e^{-2r/L}\Big)\,. \tag{2.93}$$

On the other hand, our Ansatz for $U_{(0)}$ gives

$$p_{\varphi(0)} = \frac{\delta U_{(0)}}{\delta\varphi} = \partial_\varphi A\,, \qquad p_{\chi(0)} = \frac{\delta U_{(0)}}{\delta\chi} = \partial_\chi A\,. \tag{2.94}$$

By comparing the two sets of expressions for the momenta, we understand that the coefficient $A$ can neither depend on $\varphi$ nor $\chi$, and thus $p_{\varphi(0)}$ and $p_{\chi(0)}$ vanish. This leaves $U_{(0)}$ to be purely gravitational and thus we can use directly our result from Section 2.3:

$$U_{(0)} = -\frac{3}{L}\,. \tag{2.95}$$

**Step 3:** We now proceed to solve HJ equation and determine the unknown coefficients of our Ansatz. Since the zero-derivatives contribution has already been fixed, we start our analysis with the two-derivative terms. At this order, the HJ equation simplifies to

$$R - \frac{8}{3}U_{(0)}\left(U_{(2)} - \frac{1}{2}Y_{(2)}\right) - (\nabla\varphi)^2 - Z(\varphi)(\nabla\chi)^2 + 2\frac{\partial U_{(2)}}{\partial r} = 0 \tag{2.96}$$

using $p_{\varphi(0)} = p_{\chi(0)} = 0$. Here, $Y_{(2)} = \gamma^{ij}Y_{(2)ij}$ is the trace of the tensor

$$\begin{aligned} Y_{(2)ij} = \frac{\delta U_{(2)}}{\delta\gamma^{ij}} &= B_0 R_{ij} - \nabla_i\nabla_j B_0 + \Box B_0 \gamma_{ij} + \frac{1}{2}B_1\nabla_i\varphi\nabla_j\chi \\ &+ \frac{1}{2}B_1\nabla_i\chi\nabla_j\varphi + B_2\nabla_i\varphi\nabla_j\varphi + B_3\nabla_i\chi\nabla_j\chi\,. \end{aligned} \tag{2.97}$$

After plugging everything into the HJ equation, one uses partial integration to eliminate terms that were not in our original Ansatz and therefore were not independent. Demanding that the coefficient of each independent term in the resulting HJ equation is zero, one finds that the two-derivative contribution to the on-shell action is

$$U_{(2)} = -\frac{L}{4}\big[R - (\nabla\varphi)^2 - Z(\varphi)(\nabla\chi)^2\big]\,. \tag{2.98}$$

For terms with four spatial derivatives equation (2.88) simplifies to

$$-\frac{8}{3}U_{(0)}\left(U_{(4)} - \frac{1}{2}Y_{(4)}\right) + 4Y_{(2)ij}Y^{ij}_{(2)} - \frac{4}{3}\left(U_{(2)} - \frac{1}{2}Y_{(2)}\right)^2 - Y^2_{(2)} \\ + p^2_{\varphi(2)} + \frac{1}{Z(\varphi)}p^2_{\chi(2)} + 2\frac{\partial U_{(2)}}{\partial r} = 0\,. \tag{2.99}$$

The canonical momenta that appear in this equation are

$$\begin{aligned} p_{\varphi(2)} &= \frac{\delta U_{(2)}}{\delta\varphi} = -\frac{L}{2}\Box\varphi + \frac{L}{4}Z'(\varphi)(\nabla\chi)^2 \\ p_{\chi(2)} &= \frac{\delta U_{(2)}}{\delta\chi} = -\frac{L}{2}\Box\chi - \frac{L}{2}Z'(\varphi)(\nabla\varphi)\cdot(\nabla\chi)\,. \end{aligned} \tag{2.100}$$

It is useful to notice that

$$Y_{(4)} = \gamma^{ij}\frac{\delta U_{(4)}}{\delta\gamma^{ij}} = 2U_{(4)} + \text{total derivatives}\,, \tag{2.101}$$

and the complicated tensor $Y_{(4)ij}$ is not needed for the calculation. The total derivatives of $Y_{(4)}$ will not contribute to HJ equation since they are multiplied by $U_{(0)}$, which is a constant, and total derivatives can be dropped by the equation.

Demanding that the different kinds of terms that appear in the four-derivative equation vanish independently yields the following solution for $U_{(4)}$:

$$U_{(4)} = \frac{L^3}{16}\Bigg[R_{ij}R^{ij} - \frac{1}{3}R^2 - 2\left(R^{ij} - \frac{1}{3}R\gamma^{ij}\right)(\nabla_i\varphi\nabla_j\varphi + Z(\varphi)\nabla_i\chi\nabla_j\chi) \\ + \left(\Box\varphi - \frac{1}{2}Z'(\varphi)(\nabla\chi)^2\right)^2 + Z(\varphi)\left(\Box\chi + \frac{Z'(\varphi)}{Z(\varphi)}(\nabla\varphi)\cdot(\nabla\chi)\right)^2 \\ + \frac{2}{3}\left((\nabla\varphi)^2 + Z(\varphi)(\nabla\chi)^2\right)^2 + 2Z(\varphi)\left(((\nabla\varphi)\cdot(\nabla\chi))^2 - (\nabla\varphi)^2(\nabla\chi)^2\right)\Bigg]\log\rho\,. \tag{2.102}$$

**Step 4:** This concludes the calculation of the counterterms that cancel the infinities of the on-shell action for the dilaton-axion model. For completeness, let us write down the general result.

$$
\boxed{
\begin{aligned}
S_{ct} = \frac{1}{\kappa^2} \int_{\partial M_\epsilon} \mathrm{d}^4 x \sqrt{\gamma} \Bigg\{ & \frac{3}{L} + \frac{L}{4} \left[ R - (\nabla\varphi)^2 - Z(\varphi)(\nabla\chi)^2 \right] \\
& - \frac{L^3}{16} \Bigg[ R_{ij}R^{ij} - \frac{1}{3}R^2 - 2\left( R^{ij} - \frac{1}{3}R\gamma^{ij} \right) \left( \nabla_i\varphi\nabla_j\varphi + Z(\varphi)\nabla_i\chi\nabla_j\chi \right) \\
& + \left( \Box\varphi - \frac{1}{2}Z'(\varphi)(\nabla\chi)^2 \right)^2 + Z(\varphi)\left( \Box\chi + \frac{Z'(\varphi)}{Z(\varphi)}(\nabla\varphi)\cdot(\nabla\chi) \right)^2 \\
& + \frac{2}{3}\left( (\nabla\varphi)^2 + Z(\varphi)(\nabla\chi)^2 \right)^2 + 2Z(\varphi)\left( ((\nabla\varphi)\cdot(\nabla\chi))^2 - (\nabla\varphi)^2(\nabla\chi)^2 \right) \Bigg] \log\rho \Bigg\} \, .
\end{aligned}
}
\tag{2.103}
$$

This result for the counterterm action agrees with the one found by a more complicated route in [20].

## 2.7 Discussion

We have presented a simple implementation of the Hamiltonian approach to holographic renormalization. The idea of using the Hamilton-Jacobi equation is not new, but we hope that our presentation and algorithm makes the method more accessible and useful for others to use. For our own purposes, it has shown great value in the application to the holographic renormalization of a 10 scalar model dual to $\mathcal{N} = 1^*$ gauge theory on $S^4$, an analysis that will be presented elsewhere [65].

Determining the infinite counterterms is typically only one part of holographic renormalization. One often needs the finite counterterms too, but just as in standard quantum field theory, this typically amounts to being a scheme-dependent question. However, in the presence of supersymmetry, one can fix the finite counterterms to be compatible with the supersymmetries in the problem. In the case of flat-sliced domain walls, this can be done using the Bogomolnyi-trick of writing the bulk action in terms of sums of squares that each vanish on the BPS equations. This rewriting requires a partial integration that leaves a boundary term that exactly becomes the counterterm action and encodes both infinite and finite counterterms. In the case of non-flat slicing, one can then argue that the universality of the counterterms allows one to pick the finite counterterms of the flat-space Bogomolnyi boundary term and use them in conjunction with the more general infinite counterterms discussed in this paper. This has worked successfully in several cases, for example [63] and [70]. The prescriptions does, however, have a bit of an ad hoc feel to it and it would be interesting to understand better the relationship between the BPS equations for curved domain walls and how/if they can be used to determine directly the infinite and finite counterterms.

## CHAPTER 3

# Soft Bootstrap and Supersymmetry

## 3.1 Motivation and Results

Effective field theories (EFTs) encode the low-energy dynamics of the light degrees of freedom in a physical system. The general principle of EFTs is to include all possible local interaction terms permissible by symmetries up to a certain order in the derivative expansion. Irrelevant operators are suppressed by powers of the UV cutoff and have dimensionless Wilson coefficients that parameterize the (possibly unknown) UV physics. Of particular interest, both for formal and phenomenological applications, are the EFTs describing the low-energy interactions of Goldstone modes of spontaneously broken symmetries. Traditionally, such effective actions are constructed explicitly from the underlying symmetry breaking pattern using the method of *nonlinear realization* [71, 72, 73].

However, constructing effective actions one by one is not an efficient approach to the problem of classifying such models and studying the properties of the associated scattering amplitudes. Similar to gauge and gravity theories, the Lagrangian description of EFTs has an enormous redundancy in the form of nonlinear field redefinitions which are completely invisible in the S-matrix[74, 75]. The modern *on-shell* approach completely avoids both the redundant description and the associated process of calculating observables from explicitly given Lagrangians. Instead one uses the required physical and mathematical properties of the on-shell scattering amplitudes to constrain the underlying models and directly calculate the physical scattering amplitudes.

The effective actions for Goldstone modes typically have the unusual property that while there may be an *infinite* number of gauge invariant local operators at a fixed order in the derivative expansion, the associated infinite set of Wilson coefficients is determined in terms of a *finite* number of independent parameters. How can this be understood in purely on-shell terms? The traditional explanation is that the spontaneously broken symmetries are nonlinearly realized on the fundamental fields and therefore mix operators in the effective action of different valence. From a more physical perspective, the spontaneously broken symmetries manifest themselves on the physical observables via low-energy or soft theorems. The non-independence of the Wilson coefficients is required to produce a cancellation between Feynman diagrams that ensures the low-energy theorem to hold. This is a redundant statement: while the number of independent parameters required to specify the effective action at a given order is reparametrization invariant, the actual Wilson coefficients are not. As we will see, from a purely on-shell perspective the collapse from an infinite number of free parameters to a finite number is a symptom of the underlying *recursive constructiblility* of the S-matrix, which itself can be understood as a consequence of the low-energy theorems.

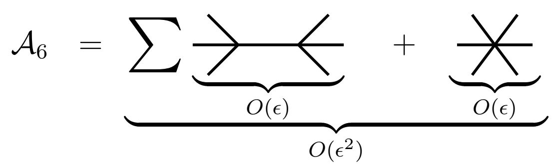
It is instructive to consider an explicit example that illustrates these ideas. Consider a flat 3-brane in 5d Minkowski space. There is a Goldstone mode $\phi$ associated with the spontaneous breaking of translational symmetry in the direction transverse to the brane, and it is well-known that the leading low-energy dynamics is governed by the Dirac-Born-Infeld (DBI) action. In static gauge, it takes the form

$$S_{\text{DBI}} = \Lambda^4 \int d^4x \left( \sqrt{\det \left( \eta_{\mu\nu} + \tfrac{1}{\Lambda^4} \partial_\mu \phi \partial_\nu \phi \right)} - 1 \right), \tag{3.1}$$

where $\Lambda^4$ is the brane tension. The action trivially has a constant shift symmetry $\phi \to \phi + c$ which implies that the DBI amplitudes have vanishing single-soft limits. In particular, when one of its momentum lines is taken soft,

$$p^\mu_{\text{soft}} \to \epsilon\, p^\mu_{\text{soft}} \quad \text{with} \quad \epsilon \to 0\,, \tag{3.2}$$

the Feynman vertex it sits on goes to zero as $O(\epsilon)$. There are no cubic interactions, so propagators remain finite. Hence, every tree-level Feynman diagram goes to zero as $O(\epsilon)$. What may be surprising is that a cancellation occurs between Feynman diagrams such that the soft behavior of any tree-level DBI $n$-point amplitude is enhanced to $O(\epsilon^2)$. For example for the 6-point amplitude, the $O(\epsilon)$-contributions of the pole diagrams cancel against those of the 6-point contact term, leaving an overall $O(\epsilon^2)$ soft behavior:




(3.3)

The cancellation of the $O(\epsilon)$-contributions requires the coefficients of the 4- and 6-particle interactions $(\partial\phi)^4$ and $(\partial\phi)^6$ to be uniquely related. Interestingly we can invert the logic of this argument. Begin with the most general effective action constructed from the operators present in the DBI action, but now with *a priori* independent Wilson coefficients $c_i$, schematically

$$S_{\text{eff}} \sim \int \mathrm{d}^4x \, \left[(\partial\phi)^2 + \frac{c_1}{\Lambda^4}\partial^4\phi^4 + \frac{c_2}{\Lambda^8}\partial^6\phi^6 + \ldots\right]. \tag{3.4}$$

Imposing that the amplitudes of this model satsify $O(\epsilon^2)$ low-energy theorems generates an infinite set of relations among the $c_i$. Up to non-physical ambiguities related to field redefinitions, the unique solution to these constraints is the DBI action. In that sense, DBI is the unique leading-order 4d real single-scalar theory with $O(\epsilon^2)$ low-energy theorems [76].

The cancellation of the $O(\epsilon)$-terms in the DBI amplitudes is a manifestation of a less obvious symmetry of the action. The broken Lorentz transformations transverse to the brane induce an enhanced shift symmetry on the brane action of the form $\phi \to \phi + c_\mu x^\mu + \ldots$, where the "$+\ldots$" stand for field-dependent terms. A theory with interaction terms built from scalar fields with at least two derivatives on every field would trivially have the enhanced shift symmetry that leads to the $O(\epsilon^2)$ soft behavior, but this is not the case for DBI. Therefore

DBI is in a class of EFTs that have been described in previous work as *exceptional*[76]. This example illustrates the Lagrangian-based description of what is meant by an exceptional EFT: a local field theory of massless particles with shift symmetries that lead to an enhanced soft behavior of the scattering amplitudes beyond what is obvious from simple counting of derivatives on the fields.[1]

The on-shell significance of the exceptional EFTs was first described in [35, 37]. It was shown, for the case of scalar effective field theories, that the class of exceptional EFTs as defined above coincides precisely with the class of EFTs for which there exists a valid method of on-shell recursion. On-shell recursion for scattering amplitudes in the form of BCFW [24, 25] or those based on various types of multi-line shifts [77, 30, 32, 33] have been around for several years now, but they are often not valid in EFTs. Technically, this is because higher-derivative interactions tend to give "bad" large-$z$ behavior of the amplitudes under the complex momentum shifts and as a result there are non-factorizable contributions from a pole at $z = \infty$. A more physical reason is that in order for a recursive approach to have a chance, it has to be given information about how higher-point terms are possibly connected to the lower-point interactions. Standard recursion relations basically only 'know' gauge-invariance, so in the DBI example they have no opportunity to know about any relation between the couplings of $(\partial\phi)^4$ and $(\partial\phi)^6$. So, naturally, a recursive approach to calculate amplitudes in exceptional EFTs needs to know about the low-energy theorems, since — as illustrated for DBI — this is what ties the higher-point interactions to the lower-point ones. This is exactly the additional input introduced to define the *soft subtracted recursion relations* presented in [35]; they provide a tool to calculate the leading (and possibly next-to-leading) order contribution to the S-matrix of an exceptional EFT without explicit reference to the action.

[1] This definition is a little imprecise. In standard usage, an EFT is defined by some physical data including the spectrum of particles and associated symmetries and corresponds to an effective action with operators at all orders in the derivative expansion. The defining property of an exceptional EFT however is typically only valid at leading or next-to-leading order. The equivalent on-shell statement is that the scattering amplitudes of the EFT are only recursively constructible at the same order in the expansion.

The existence of valid recursion relations gives us our sought-after on-shell characterization of the relation among the Wilson coefficients of Goldstone EFTs. The infinite set of *a priori* independent local operators at leading order in the derivative expansion determine the leading-order part of the S-matrix. For a generic EFT, the presence of independent operators of valence $n$ corresponds to the appearance of independent coefficients on contact contributions for amplitudes with $n$ external particles. If the scattering amplitudes are recursively constructible at a given order, then no such *independent* coefficients can appear since the entire amplitude must be determined by factorization into amplitudes with fewer external particles. Furthermore, the recursion must take as its input a finite set of seed amplitudes that depend on only a finite number of parameters.

Beyond being an efficient method for calculating explicit scattering amplitudes in known models, the subtracted recursion relations can be implemented as a numerical algorithm to explore and classify the landscape of possible EFTs. We term this program the *soft bootstrap* due to the structural similarity of the method with the conformal bootstrap [78, 79]. The method is described in detail in Section 3.3.5, here we give a simplified description. We consider EFTs as defined by a set of on-shell *soft data*: a spectrum of massless states, linearly realized symmetries and low-energy theorems. We use general ansätze for scattering amplitudes of low valence and low mass dimension, consistent with the assumed spectrum and linear symmetries, as input for subtracted recursion. If the ansätze satisfy a certain criterion guaranteeing the validity of the subtracted recursion relations and if the assumed soft data corresponds to a valid EFT, then the output of the recursion should correspond to a physical scattering amplitude. Here *valid EFT* means the existence of the assumed EFT as a local, unitary, Poincaré invariant quantum field theory.

For tree-level scattering amplitudes this includes the requirement that the only singularities of the amplitude correspond to factorization on a momentum channel. Conversely if no such valid EFT exists, or equivalently if the assumed soft data is inconsistent, then the output of the recursion generically will not correspond to a physical scattering amplitude and this may be detected through the presence of non-physical or spurious singularities. In practice, the ansätze are parametrized by a finite number of coefficients, and the removal of spurious singularities often places constraints on these coefficients.

The soft bootstrap program was initiated in [37], where it was used to explore the landscape of real scalar EFTs with vanishing low-energy theorems. The results are reviewed and extended in Section 3.4. This work should be understood as a continuation and generalization of this program, incorporating richer soft data including spinning particles and linearly realized supersymmetry. In Section 3.1.1 we provide a brief overview of exceptional EFTs studied in our work before summarizing our main results in Section 3.1.2 that also provides an outline of this chapter.

### 3.1.1 Overview of EFTs

In our work, we extend the application of the soft bootstrap from real scalars to any massless helicity-$h$ particle and we derive a precise criterion for the validity of the soft subtracted recursion relations. By the new validity criterion, the on-shell characterization of an exceptional EFT will precisely be that its amplitudes are constructible using soft recursion.

Our work requires a precise definition of the degree of softness of the amplitude. This is given in Section 3.3.1. For now, let us simply introduce the *soft weight* $\sigma$ as

$$\mathcal{A}_n(\epsilon p_1, p_2, \ldots) = \epsilon^{\sigma}\, \mathcal{S}_n^{(0)} + O(\epsilon^{\sigma+1}) \quad \text{as} \;\; \epsilon \to 0\,, \tag{3.5}$$

where $\mathcal{S}_n^{(0)} \neq 0$. Table 3.1.1 summarizes the soft weights for various known cases of spontaneous symmetry breaking. The earlier example of DBI corresponds to the case of spontaneously broken higher-dimensional Poincaré symmetry; only the breaking of the translational symmetry actually gives rise to a Goldstone mode [80] and it will have $\sigma = 2$.

Here follows a brief overview of exceptional EFTs that appear in this paper. We include the connection between their soft behavior and Lagrangian shift symmetries:

| Soft degree $\sigma$ | Spin $s$ | Type of symmetry breaking |
|---|---|---|
| 1 | 0 | Internal symmetry (symmetric coset) |
| 0 | 0 | Internal symmetry (non-symmetric coset) |
| 1 | 1/2 | Supersymmetry |
| 0 | 0 | Conformal symmetry |
| 0 | 1/2 | Superconformal symmetry |
| 2 | 0 | Higher-dimensional Poincaré symmetry |
| 0 | 0 | Higher-dimensional AdS symmetry |
| 3 | 0 | Special Galileon symmetry |

Table 3.1: The table lists soft weights $\sigma$ associated with the soft theorems $\mathcal{A}_n \to O(\epsilon^\sigma)$ as $\epsilon \to 0$ for several known cases. The soft limit is taken holomorphically in 4d spinor helicity, see Section 3.3.1 for a precise definition. Conformal and superconformal breaking is discussed in Section 3.5.3.

- **DBI** can be extended to a complex scalar Dirac-Born-Infeld theory and coupled supersymmetrically to a fermion sector described by the **Akulov-Volkov** action of Goldstinos from spontaneous breaking of supersymmetry. In extended supersymmetric DBI, the vector sector is **Born-Infeld (BI)** theory. The soft weights are $\sigma_Z = 2$ for the complex scalars $Z$ of DBI, $\sigma_\psi = 1$ for the fermions of Akulov-Volkov, and $\sigma_\gamma = 0$ for the BI photon. The soft behaviors can be associated with shift symmetries $Z \to Z + c + v_\mu x^\mu$ and $\psi \to \psi + \xi$, where $\xi$ is a constant Grassmann-number.[2] $\mathcal{N} = 1$ **supersymmetric Born-Infeld** couples the BI vector to the Goldstino of Akulov-Volkov.

- **Nonlinear sigma models (NLSM)** describe the Goldstone modes of sponteneously broken internal symmetries and have scalars with constant shift symmetries that give $\sigma = 1$ soft weights in the low-energy theorems. A common example of an NLSM is **chiral perturbation theory** in which the scalars live in a coset space $U(N) \times U(N)/U(N)$.

  The complex scalar $\mathbb{CP}^1$ NLSM can be supersymmetrized with a fermion sector that is **Nambu-Jona-Lasinio (NJL)** model. The complex scalars have shift symmetry $Z \to Z + c$ and $\sigma_Z = 1$ while the fermions have no shift symmetry and $\sigma_\psi = 0$. We

[2] We leave out field-dependent terms for simplicity when stating the shift symmetries.

study both the $\mathcal{N}=1$ and 2 supersymmetric $\mathbb{CP}^1$ NLSM.[3]

- A NLSM can have a non-trivial subleading operator that respects the shift symmetry and hence also the low-energy theorems with $\sigma = 1$. This operator is known as the **Wess-Zumino-Witten (WZW)** term and has a leading 5-point interaction.

- **Galileon** scalar EFTs arise in various contexts and have the extended shift symmetry $\phi \to \phi + c + v_\mu x^\mu$ that gives low-energy theorems with $\sigma = 2$. As such they can be thought of as subleading operators of the DBI action, and are called **DBI-Galileons**. They can also be decoupled from DBI (at the cost of having no UV completion).

  In 4d there are two independent Galileon operators: the quartic and quintic Galileon. (By a field redefinition, the cubic Galileon is not independent from the quartic and quintic.) When decoupled from DBI, the quartic Galileon has an even further enhanced shift symmetry $\phi \to \phi + c + v_\mu x^\mu + s_{\mu\nu} x^\mu x^\nu$ that gives low-energy theorems with soft weight $\sigma = 3$ and is then called the **Special Galileon** [37, 40].

- The quartic Galileon has a complex scalar version with $\sigma_Z = 2$ (but it cannot have $\sigma_Z = 3$). It has an $\mathcal{N}=1$ supersymmetrization [81, 60] in which the fermion sector trivially realizes a constant shift symmetry that gives $\sigma_\psi = 1$.

- There is evidence [60] that the quintic Galileon may have an $\mathcal{N}=1$ supersymmetrization. This involves a complex scalar whose real part is a Galileon with $\sigma = 2$ and imaginary part is an R-axion with $\sigma = 1$.

We now summarize the main results of this chapter.

### 3.1.2 Outline of Results

In Section 3.2 a brief review is given of the Wilsonian effective action. The notion of the *reduced dimension* of an operator is defined and the relevance to power-counting in the

[3]In Section 3.6.2 we show that the $\mathcal{N}=2$ $\mathbb{CP}^1$ NLSM requires the presence of 3-point interactions and the soft weight of the scalar is reduced to $\sigma_Z = 0$.

derivative expansion is explained.

In Section 3.3 we present a review and elaboration on the method of soft subtracted recursion. The asymptotic (large-$z$) behavior of a scattering amplitude under the momentum deformation is determined using a novel method exploiting the properties of tree amplitudes of massless particles under complex scale transformations. This result is then used to formulate a precise *constructibility criterion* (3.20) for the applicability of the method. The failure of an EFT (at some order in the derivative expansion) to satisfy the criterion is shown to be equivalent to the existence of independent local operators which are "trivially" invariant under an extended shift symmetry. The systematics of the soft bootstrap algorithm for constraining EFTs is described.

In Section 3.4 several numerical applications of the soft bootstrap are presented. The landscape of constructible EFTs with simple spectra consisting of a single massless complex scalar, Weyl fermion, or vector boson is exhaustively explored. In particular, our analysis shows that there can be no vector Goldstone bosons with vanishing soft theorems. A similar result follows from an algebraic analysis that appeared around the same time as this paper [82].

In Section 3.5 we describe the interplay between soft behavior and supersymmetry. From the supersymmetry Ward identities we show that the soft weights of the states in an $\mathcal{N} = 1$ multiplet can differ by at most one. Implications for superconformal symmetry breaking and constraints on low-energy theorems in extended supergravity are presented as examples.

In Section 3.6, we apply recursion to construct the scattering amplitudes of the $\mathcal{N} = 1, 2$ $\mathbb{CP}^1$ nonlinear sigma models at leading (two-derivative) order. For the $\mathcal{N} = 1$ case, it is shown that recursive constructibility together with the conservation of $U(1)$ charges by the seed amplitudes implies that (at two-derivative order) all tree amplitudes of this model conserve an additional accidental $U(1)$ charge. For the $\mathcal{N} = 2$ model, recursive constructibility is non-trivial due to the presence of 3-point interactions and non-vanishing scalar soft limits, but can be achieved using the supersymmetry Ward identities (see Appendix G). Using this, we show that all tree amplitudes satisfy the Ward identities of $SU(2)_R$ and conserve an additional $U(1)_R$ under which the vector bosons are charged. (A detailed inductive proof of the $SU(2)_R$ Ward identities is given in Appendix F.) The connection between the existence of such chiral charges for vector bosons and known results about special Kähler geometry are described, in particular we highlight the emergence of electric-magnetic duality. Finally, an explicit form of the singular low-energy theorem for the vector bosons of the $\mathcal{N} = 2$ model is presented.

Section 3.7 contains brief comments on supersymmetrizations of DBI and Born-Infeld.

In Section 3.8 various applications of the soft bootstrap algorithm to Galileon-like models are presented. Previous results on the $\mathcal{N} = 1$ supersymmetrization of the quartic and quintic Galileon are elaborated upon, in particular the various possible soft weight assignments to the states in the multiplet are described in detail.

The existence of an extension of the special Galileon with non-trivial couplings to a massless vector is considered and evidence is given in favor of the existence of such a model. The soft bootstrap algorithm is applied to the problem of classifying higher-derivative corrections to the special Galileon effective action that preserve the low-energy theorem via the associated on-shell matrix elements. Compatible amplitudes are classified up to couplings of dimension $-12$ for quartic interactions and $-17$ for quintic interactions. These results are compared with the output of the double-copy in the form of the field theory KLT relations as applied to chiral perturbation theory. These two constructions are found to agree for quartic interactions but not for quintic.

In Appendix E many explicit forms of calculated amplitudes for various models considered in this paper are presented.

## 3.2 Structure of the Effective Action

The low-energy dynamics of a physical system can be described by a Wilsonian effective action containing a set of local quantum fields for each of the on-shell asymptotic states with *all possible* local interactions allowed by the assumed symmetries:

$$S_{\text{effective}} = S_0 + \sum_{\mathcal{O}} \frac{c_{\mathcal{O}}}{\Lambda^{\Delta[\mathcal{O}]-4}} \int \mathrm{d}^4x \, \mathcal{O}(x) \,. \tag{3.6}$$

Here $S_0$ denotes the free theory, i.e. the kinetic terms, $\Lambda$ is a characteristic scale of the problem, and $c_{\mathcal{O}}$ are dimensionless constants. The sum is over all local Lorentz invariant operators $\mathcal{O}(x)$ of the schematic form

$$\mathcal{O}(x) \sim \partial^A \phi(x)^B \psi(x)^C F(x)^D \,, \tag{3.7}$$

where $A, \ldots, D$ are integer exponents. In this paper we focus on EFTs in which the operators $\mathcal{O}$ are manifestly gauge invariant.[4]

We assign the following quantities to a local operator

- *Dimension:* $\Delta[\mathcal{O}]$ defined as the engineering dimension with bosonic fields of dimension 1 and fermionic fields of dimension 3/2.

- *Valence:* $N[\mathcal{O}]$ defined as the sum of the total number of field operators appearing. Equivalently, this is the valence of the Feynman vertex derived from such an interaction.

The schematic operator in (3.7) has $\Delta[\mathcal{O}] = A + B + \frac{3}{2}C + 2D$ and $N[\mathcal{O}] = B + C + D$.

In standard EFT lore, operators of lowest dimension dominate in the IR. In many cases this means the marginal and relevant interactions dominate and the irrelevant interactions

[4] This need not be the case in more general scenarios (though of course we insist on overall gauge invariance). For example in Yang-Mills theory, the gauge invariant operator $\operatorname{tr} F^2$ has a quadratic term which we group into the free part $S_0$ of the action while the interaction terms would be accounted for in the sum of all operators $\mathcal{O}$ in (3.6). Similarly, for massless spin-2 fields when $\sqrt{-g}R$ is expanded around flat space.

are sub-dominant and suppressed by powers of the UV scale $\Lambda$. In other cases, such as effective field theories describing the dynamics of Goldstone modes, there are only irrelevant interactions and it may be less clear which operators dominate. It is therefore useful to introduce the *reduced* dimension

$$\tilde{\Delta}[\mathcal{O}] = \frac{\Delta[\mathcal{O}] - 4}{N[\mathcal{O}] - 2} \tag{3.8}$$

for the operator basis (3.6). Operators that minimize $\tilde{\Delta}$ dominate in the IR.

The authors of [76, 35, 37] consider only scalar EFTs and therefore operators of the form $\mathcal{O} \sim \partial^m \phi^n$. They define a quantity

$$\rho \equiv \frac{m-2}{n-2} = \tilde{\Delta}[\mathcal{O}] - 1\,, \tag{3.9}$$

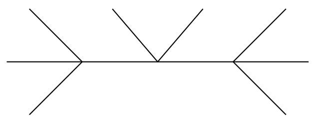
to determine when two operators of this form produce tree-level diagrams with couplings of the same mass dimension. Morally $\rho$ is the same as the reduced dimension $\tilde{\Delta}[\mathcal{O}]$. The latter is the natural generalization of $\rho$ to operators containing particles of all spins.

The quantity $\tilde{\Delta}$ is useful for clarifying the notion of what it means for an interaction to be leading order in an EFT with only irrelevant interactions. In the deep IR, the relative size of the dimensionless Wilson coefficients in the effective action is unimportant since lower dimension operators will *always* dominate over higher dimension operators. It is therefore only necessary to isolate the contributions that are leading in a power series expansion of the amplitudes in the inverse UV cutoff scale $\Lambda^{-1}$. The dominant interactions in the deep IR are generated by operators that *minimize* this quantity. As an illustrative example, consider an effective action for scalars with interaction terms of the form

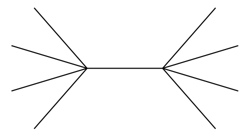
$$S_{\text{effective}} \supset \int \mathrm{d}^4 x \left[\frac{c_4}{\Lambda^4}\partial^4\phi^4 + \frac{c_5}{\Lambda^5}\partial^4\phi^5\right]. \tag{3.10}$$

The reduced dimensions $\tilde{\Delta}$ are 2 and 5/3 for the quartic and quintic interactions respectively.

The quintic interaction should therefore dominate over the quartic in the deep IR. To see this explicitly we have to compare amplitudes with the *same* number of external states, so we compare the contributions from tree-level Feynman diagrams to the 8-point amplitude:



$\sim \frac{1}{\Lambda^{12}}$



$\sim \frac{1}{\Lambda^{10}}$

This confirms that the diagrams arising from the quintic interaction dominate the 8-point amplitude.

It is useful to introduce the notion of *fundamental interactions* (or *fundamental operators*) in an EFT. These are the lowest dimension operator(s) whose on-shell matrix elements can be recursed to define all matrix elements of the theory at leading order in the low-energy expansion.

Consider the DBI action. The leading interaction comes from an operator of the form $\frac{1}{\Lambda^4}\partial^4\phi^4$ and as discussed in the introduction, with the associated 4-point amplitude as input, all other $n$-point amplitudes in DBI can be constructed with soft subtracted recursion relations. If the action had contained an interaction term of the form $\frac{c_5}{\Lambda^5}\partial^5\phi^4$, then $\frac{1}{\Lambda^4}\partial^4\phi^4$ would not be sufficient to determine dominating contributions at $n$-point order, i.e. both interactions would need to be considered fundamental for soft recursion.

The operators immediately subleading to DBI in the brane-effective action are encoded in the DBI-Galileon. In 4d, there are two such independent couplings,[5] namely for a quartic interaction of the schematic form $\frac{b_4}{\Lambda^6}\partial^6\phi^4$ and a quintic interaction of the form $\frac{b_5}{\Lambda^9}\partial^8\phi^5$; these both have $\tilde{\Delta} = 3$ whereas DBI has $\tilde{\Delta} = 2$. Thus the DBI-Galileon has a total of three fundamental operators: the 4-point DBI interaction and the 4- and 5-point Galileon interactions.

[5]The cubic Galileon interaction is equivalent to a particular linear combination of the quartic and quintic Galileon after a field redefinition.

## 3.3 Subtracted Recursion Relations

We review on-shell subtracted recursion relations for scattering amplitudes of Goldstone modes [83, 76, 35, 37, 84] and derive a new precise criterion for their validity.

### 3.3.1 Holomorphic Soft Limits and Low-Energy Theorems

We rely on the 4d spinor helicity formalism (for reviews, see [85, 86, 87, 88]) in which a massless on-shell momentum is written $p = -|p\rangle[p|$. This presents an ambiguity in how to take the soft limit (3.2): it could for example be taken democratically as $\{|p\rangle, |p]\} \to \{\epsilon^{1/2}|p\rangle, \epsilon^{1/2}|p]\}$, holomorphically $\{|p\rangle, |p]\} \to \{\epsilon|p\rangle, |p]\}$, or anti-holomorphically $\{|p\rangle, |p]\} \to \{|p\rangle, \epsilon|p]\}$. These are all equivalent choices, because the momentum $p$ is invariant under *little group scaling* $\{|p\rangle, |p]\} \to \{t|p\rangle, t^{-1}|p]\}$. Amplitudes scale homogeneously under the little group,

$$\mathcal{A}_n\big(\{|1\rangle, |1]\} \ldots \{t|i\rangle, t^{-1}|i]\}_+ \ldots\big) = t^{-2h_i}\mathcal{A}_n\big(\{|1\rangle, |1]\} \ldots \{|i\rangle, |i]\}_+ \ldots\big)\,,. \tag{3.11}$$

so the choice of soft limit is simply reflected in a helicity-dependent overall scaling factor. We choose to minimize the power of $\epsilon$ in the soft limit by letting the choice depend on the sign of the helicity of the particle: specfically, we take $p_{\text{soft}} \to \epsilon\, p_{\text{soft}} = -\epsilon|s\rangle[s|$ holomorphically for any state with non-negative helicity:[6]

$$|s\rangle \to \epsilon|s\rangle \qquad \text{for } h_s \geq 0\,. \tag{3.12}$$

For a negative-helicity particle, we use the anti-holomorphic prescription $|s] \to \epsilon|s]$. For scalars, it makes no difference which choice is made.

[6] Taking the soft limit as simply as in (3.12) is not compatible with overall momentum conservation. To stay on the algebraic locus of momentum conservation in momentum space, we take the limit with appropriate shifts in a subset of the $n-1$ other momentum variables. The precise prescription can be found in equation (6) of [89]. The details will not affect the main line of the discussion in this paper, but we note that all calculations are done manifestly on-shell, including the soft limits.

We characterize the soft behavior of amplitudes of massless particles in terms of a *holomorphic soft weight* $\sigma$ (or, for brevity, just *soft weight*). It is defined in terms of the holomorphic soft limit (3.12) as

$$\mathcal{A}_n\big(\{|1\,,|1]\}\ldots\{\epsilon|s\rangle,|s]\}_+\ldots\big) = \epsilon^\sigma\,\mathcal{S}_n^{(0)} + O(\epsilon^{\sigma+1}) \qquad \text{as } \epsilon\to 0\,, \tag{3.13}$$

where $\mathcal{S}_n^{(0)} \neq 0$. This way of taking the soft limit is closely correlated with the shifts introduced for the soft subtracted recursion relations in the following.

### 3.3.2 Review of Soft Subtracted Recursion Relations

We consider complex momentum deformations of the form

$$p_i \to \hat{p}_i = (1-a_iz)p_i \qquad \text{with } \sum_{i=1}^{n} a_ip_i = 0\,. \tag{3.14}$$

The label $i = 1,2,\ldots,n$ runs over the $n$ massless particles in the scattering amplitude. The shifted momenta $\hat{p}_i$ are on-shell by virtue of $p_i^2 = 0$ and satisfy momentum conservation when the shift coefficients $a_i$ satisfy the condition in (3.14). (We discuss the solutions to this condition in Section 3.3.5.) When evaluated on the shifted momenta $\hat{p}_i$, an $n$-point amplitude becomes a function of $z$ and we write it as $\hat{\mathcal{A}}_n(z)$.

The *subtracted recursion relations* for an $n$-point tree-level amplitude $\mathcal{A}_n$ are derived from the Cauchy integral

$$\oint \frac{\mathrm{d}z}{z}\frac{\hat{\mathcal{A}}_n(z)}{F(z)} = 0\,, \tag{3.15}$$

where the contour surrounds all the poles at finite $z$ and the function $F$ is defined as

$$F(z) = \prod_{i=1}^{n}(1-a_iz)^{\sigma_i}\,. \tag{3.16}$$

The vanishing of the integral in (3.15) requires absence of a simple pole at $z=\infty$. We derive a sufficient criterion for this behavior in Section 3.3.3.

The shift (3.14) is implemented on the spinor helicity variables according to the sign of the helicity $h_i$ of particle $i$ as

$$\begin{aligned} h_i \geq 0: \qquad & |i\rangle \to (1-a_i z)|i\rangle\,, \quad |i] \to |i]\,, \\ h_i < 0: \qquad & |i\rangle \to |i\rangle\,, \qquad\qquad\quad |i] \to (1-a_i z)|i]\,. \end{aligned} \tag{3.17}$$

The limit $z \to 1/a_i$ is then precisely the soft limit $\hat{p}_i \to 0$ of the $i$th particle in the deformed amplitude. Hence, *if the amplitude satisfies low-energy theorems of the form* (3.13) *with weights $\sigma_i$ for each particle $i$*, the integral (3.15) will not pick up any non-zero residues from poles arising from the function $F$ when it is chosen as in (3.16). Therefore the only simple poles in (3.15) arise from $z = 0$ and factorization channels in the deformed tree amplitude. They occur where internal momenta go on-shell, $\hat{P}_I^2 = 0$. The residue theorem then states that the residue at $z = 0$ equals minus the sum of all such residues, and factorization on these poles gives

$$\mathcal{A}_n = \hat{\mathcal{A}}_n(z=0) = \sum_I \sum_{|\psi^{(I)}\rangle} \sum_{\pm} \frac{\hat{\mathcal{A}}_L^{(I)}(z_I^\pm)\hat{\mathcal{A}}_R^{(I)}(z_I^\pm)}{F(z_I^\pm)P_I^2(1-z_I^\pm/z_I^\mp)}\,. \tag{3.18}$$

The sums are over all factorization channels $I$, the two solutions $z_I^\pm$ to $\hat{P}_I^2 = 0$, and all possible particle types $|\psi^{(I)}\rangle$ that can be exchanged in channel $I$. These recursion relations are called *soft subtracted recursion relations.* When $F = 1$, the recursion is called *unsubtracted.*

The expression for the solutions $z_I^\pm$ to the quadratic equation $\hat{P}_I^2 = 0$ involves square roots, but those must cancel since the tree amplitude is a rational function of the kinematic variables. On channels where the amplitude factorizes into two *local* lower-point amplitudes (meaning that they have no poles), the cancellations of the square roots can be made manifest. This is done by a second application of Cauchy's theorem, which for each channel $I$ converts the sum of residues at $z = z_I^\pm$ to the sum of the residues at $z = 0$ and $z = 1/a_i$ for all $i$. Details are provided in Appendix D, here we simply state the result: *if $\mathcal{A}_L^{(I)}$ and $\mathcal{A}_R^{(I)}$*

*are local for all factorization channels*, the soft recursion relations take the form

$$\mathcal{A}_n = \sum_I \sum_{|\psi^{(I)}\rangle} \left( \frac{\hat{\mathcal{A}}_L^{(I)}(0)\hat{\mathcal{A}}_R^{(I)}(0)}{P_I^2} + \sum_{i=1}^{n} \underset{z=\frac{1}{a_i}}{\text{Res}} \frac{\hat{\mathcal{A}}_L^{(I)}(z)\hat{\mathcal{A}}_R^{(I)}(z)}{z\,F(z)\,\hat{P}_I^2} \right). \tag{3.19}$$

This form of the recursion relations is manifestly rational in the kinematic variables, and we will be using (3.19) for the applications in this paper. Note that only the first term in (3.19) has pole terms. Therefore the sum of the $1/a_i$ residues over all channels must be a local polynomial in the momenta.

### 3.3.3 Validity Criterion

The purpose of including $F(z)$ in (3.15) is to improve the large-$z$ behavior of the integrand so that one can avoid a pole at $z = \infty$. This is necessary in EFTs, where the large-$z$ behavior of the amplitude typically does not allow for unsubtracted recursion relations with $F(z) = 1$ to be valid without a boundary term from $z = \infty$. A sufficient condition for absence of a simple pole at infinity is that the deformed amplitude vanishes as $z \to \infty$. Below we show that for a theory with a single fundamental interaction (see Section 3.2) of valence $v$ and coupling of mass-dimension $[g_v]$ the criterion for validity of the subtracted recursion relations is

$$\boxed{4 - n - \frac{n-2}{v-2}[g_v] - \sum_{i=1}^{n} s_i - \sum_{i=1}^{n} \sigma_i < 0\,.} \tag{3.20}$$

Here $s_i$ is the spin (not helicity) of particle $i$ and $\sigma_i$ is its soft behavior (3.13). Alternatively, one can write the constructibility criterion in terms of the reduced dimension $\tilde{\Delta}$, introduced in (3.8), as

$$\boxed{4 - n + (n-2)\tilde{\Delta} - \sum_{i=1}^{n} s_i - \sum_{i=1}^{n} \sigma_i < 0\,.} \tag{3.21}$$

The criterion generalizes to theories with more than one fundamental coupling by replacing $\frac{n-2}{v-2}[g_v]$ in (3.20) by the sum over all couplings contributing to the diagrammatic expansion of the amplitude in question; the precise criterion is given in (3.29).

**Proof of the criterion** (3.20). To avoid a pole at infinity in the Cauchy integral (3.15), it is sufficient to require $\hat{\mathcal{A}}_n(z)/F(z) \to 0$ as $z \to \infty$. To start with, we determine the large-$z$ behavior of the deformed amplitude $\hat{\mathcal{A}}_n(z)$. Generically, in a theory of massless particles with couplings $g_k$, a tree-level amplitude takes the form

$$\mathcal{A}_n = \sum_j \left( \prod_k g_k^{n_{jk}} \right) M_j \,, \tag{3.22}$$

where $\prod_k g_k^{n_{jk}}$ is a product of coupling constants and $M_j$ is a function of spinor brackets only. Since there can be no other dimensionful quantities entering $M_j$, the mass dimension $[M_j]$ can be determined via a homogenous scaling of all spinors:

$$|i\rangle \to \lambda^{1/2}|i\rangle \quad \text{and} \quad |i] \to \lambda^{1/2}|i] \qquad \Longrightarrow \qquad M_j \to \lambda^{[M_j]} M_j \,. \tag{3.23}$$

The mass dimension is also fixed by simple dimensional analysis to be

$$[M_j] = 4 - n - \sum_k n_{jk}[g_k] \,, \tag{3.24}$$

since an $n$-point scattering amplitude in 4d has to have mass-dimension $4 - n$.

It is useful to consider a modified scale transformation defined as

$$\begin{aligned} h_i \geq 0: \qquad & |i\rangle \to \lambda|i\rangle \,, \quad |i] \to |i] \,, \\ h_i < 0: \qquad & |i\rangle \to |i\rangle \,, \quad |i] \to \lambda|i] \,. \end{aligned} \tag{3.25}$$

The effect of this scaling can be obtained from the uniform scaling (3.23) via a little group transformation (3.11) on all momenta with $t = \lambda^{1/2}$. Therefore under (3.25), $M_j$ scales as $M_j \to \lambda^{[M_j] - \sum_i s_i} M_j$, where $s_i$ is the spin (*not* helicity) of particle $i$.

For the case of a theory with a single fundamental interaction of valence $v$ with coupling

$g_v$, the number of couplings appearing in an $n$-point amplitude is $\frac{n-2}{v-2}$, and therefore we have

$$\mathcal{A}_n \to \lambda^D \mathcal{A}_n \,, \qquad D = 4 - n - \frac{n-2}{v-2}[g_v] - \sum_i s_i \tag{3.26}$$

under the modified scale transformation (3.25).

Under the momentum shift (3.17), the deformed tree amplitude $\hat{\mathcal{A}}_n(z)$ can be written

$$\begin{aligned}
\hat{\mathcal{A}}_n(z) &= \hat{\mathcal{A}}_n\big(\ldots\{(1-a_i z)|i\,,|i]\}_+ \ldots \{|j\,,(1-a_j z)|j]\}_-\big) \\
&= \hat{\mathcal{A}}_n\big(\ldots\{z(1/z-a_i)|i\,,|i]\}_+ \ldots \{|j\,,z(1/z-a_j)|j]\}_-\big) \\
&= z^D\,\hat{\mathcal{A}}_n\big(\ldots\{(1/z-a_i)|i\,,|i]\}_+ \ldots \{|j\,,(1/z-a_j)|j]\}_-\big)\,,
\end{aligned} \tag{3.27}$$

where the subscripts $\pm$ refer to the sign of the helicity of each particle. In the last line we used the behavior (3.26) under the modified scaling (3.25).

At large $z$, the amplitude in the last line of (3.27) is the original unshifted amplitude evaluated at a momentum configuration with $q_i = -a_i p_i$. These momenta are all on-shell and satisfy, via (3.14), momentum conservation. The only way the tree amplitude could have a singularity at this momentum configuration would be if an internal line went on-shell. This can always be avoided for generic momenta.[7] Thus we conclude from (3.27) that for large $z$, the deformed amplitude behaves as

$$\hat{\mathcal{A}}_n(z) \to z^N \quad \text{with} \quad N \le D\,, \tag{3.28}$$

where $D$ is given in (3.26). The inequality allows for the possibility that $\mathcal{A}_n$ could have a zero at $q_i = -a_i p_i$.

Our mission was to find a criterion for $\hat{\mathcal{A}}_n(z)/F(z) \to 0$ as $z \to \infty$. By the definition (3.16), we have $F(z) \to z^{\sum_i \sigma_i}$ for large $z$. From our analysis of the large-$z$ behavior of $\hat{\mathcal{A}}_n(z)$,

[7]The condition (3.14) has a trivial solution with all $a_i$ equal. Therefore any solution to (3.14) can be shifted uniformly $a_i \to a_i + a$ for any real number $a$. Hence, we can always avoid the discrete set of momentum configurations for which an internal line in $\mathcal{A}_n$ goes on-shell.

we can therefore conclude that, at worst, $\hat{\mathcal{A}}_n(z)/F(z) \to z^{D-\sum_i \sigma_i}$. The sufficient criterion for absence of a pole at infinity, and hence for validity of the subtracted recursion relation, is then $D - \sum_i \sigma_i < 0$. This is precisely the condition (3.20). This concludes the proof.

It is straightforward to generalize the constructibility criterion to EFTs with more than one fundamental interaction,

$$4 - n - \min_j \Big( \sum_k n_{jk}[g_k] \Big) - \sum_{i=1}^{n} s_i - \sum_{i=1}^{n} \sigma_i < 0 \,. \tag{3.29}$$

Recall that in effective field theories, the couplings have negative mass-dimension. This means that the constructibility criterion tends to be dominated by the fundamental interactions associated with operators of the highest mass-dimension that can contribute to the $n$-point amplitude.

**Example 1.** Let us once again return to the example of DBI. The action has a fundamental quartic vertex $g_4(\partial\phi)^4$ with a coupling of mass-dimension $[g_4] = -4$. The constructibility criterion (3.20) for the $n$-scalar amplitude is $n(1 - \sigma_S) < 0$, where $\sigma_S$ is the soft behavior of the scalar $\phi$. Since $\sigma_S = 2$ in DBI, all DBI tree amplitudes are constructible via the subtracted soft recursion relations, as claimed in the introduction.

The failure of the constructibility criterion for $\sigma_S = 1$ is simply the statement that an EFT whose interactions are built from powers of $(\partial\phi)^2$ trivially has a constant shift symmetry and hence $\sigma_S = 1$, so there are no constraints from shift symmetry on the coefficients of $(\partial\phi)^{2k}$ in terms of that of $(\partial\phi)^4$ and then one has no chance of recursing $\mathcal{A}_4$ to get all-point amplitudes.

**Example 2.** Consider a theory of massless fermions with quartic coupling of mass dimension $[g_4] = -2$. The criterion (3.20) says that the $n$-fermion amplitudes are constructible when $4 < n(1 + 2\sigma_\psi)$. Thus all $n > 4$ point tree-amplitudes are constructible by (3.18) for any soft weight $\sigma_\psi \geq 0$. No such theory exists for $\sigma_\psi > 0$ (as we prove in Section 3.4.2), but for $\sigma_\psi = 0$ this is exactly the Nambu–Jona-Lasinio (NJL) model, which consists of the simple 4-fermion interaction $\psi^2\bar{\psi}^2$ [90].

### 3.3.4 Non-Constructibility = Triviality

We have derived a constructibility criterion, but what does it mean? The answer is quite simple: if an $n$-point amplitude can be constructed recursively from lower-point on-shell amplitudes, there cannot exist a local gauge-invariant $n$-field operator that contributes to the amplitude without modifying its soft behavior. We define a *trivial operator* to be one with at least 4 fields whose matrix elements manifestly have a given soft weight $\sigma$. Let us now assess what it takes to make an operator of scalar, fermion, and vector fields trivial.

**Triviality.** *Scalars:* Operators with at least $m$ derivatives on each scalar field will trivially have single-soft scalar limits with $\sigma_S = m$.

*Fermions:* We have chosen the soft limit (3.12) according to the helicity such that the fermion wave-functions do not generate any soft factors of $\epsilon$. Thus a trivial soft behavior must come from derivatives on each fermion field in the Lagrangian. We conclude that the trivial soft behavior $\sigma_F$ = smallest number of derivatives on each fermion field.

*Photons:* Gauge invariance tells us that we should construct the interaction terms using the field strength $F_{\mu\nu}$.[8] When associated with an external photon, the Feynman rule for $F_{\mu\nu}$ gives $p_\mu\epsilon_\nu - p_\nu\epsilon_\mu$. Naively, it may seem to be linear in the soft momentum, but under the holomorphic soft shift (3.17) it is actually $O(\epsilon^0)$. Recall that in spinor helicity formalism, a positive helicity vector polarization takes the form $\epsilon_+^\mu \bar{\sigma}_\mu^{\dot{a}b} = \epsilon_+^{\dot{a}b} = |q\rangle^{\dot{a}}[p|^b/\langle pq\rangle$, where $q$ is a reference spinor. Hence, for a positive helicity photon we have

$$(F_+)_a{}^b \equiv (\sigma^{\mu\nu})_a{}^b F_{\mu\nu} \longrightarrow (\sigma^{\mu\nu})_a{}^b(p_\mu\epsilon_{+\nu} - p_\nu\epsilon_{+\mu}) \sim |p]_a\langle p|_{\dot{c}}\frac{|q\rangle^{\dot{c}}[p|^b}{\langle pq\rangle} = |p]_a[p|^b\,. \tag{3.30}$$

[8] Or covariant derivatives $D_\mu = \partial_\mu + igA_\mu$. In this paper, we focus on scalars and fermions that do not transform under any gauge-$U(1)$, therefore photons must couple via $F_{\mu\nu}$.

This is explicitly independent of the reference spinor $q$ because $F_{\mu\nu}$ is gauge invariant. For a positive helicity particle, we take the soft limit holomorphically as $|p\rangle \to \epsilon|p\rangle$ (while $|p] \to |p]$), so we explicitly see that $F_{\mu\nu} \longrightarrow |p][p|$ is $O(\epsilon^0)$ when $p$ is taken soft. Likewise, for a negative helicity photon, $(F_-)^{\dot{a}}{}_{\dot{b}} \longrightarrow |p\rangle\langle p|$. We conclude that an operator with photons has trivial soft behavior that is determined by the smallest number of derivatives on each field strength $F_{\mu\nu}$. In an EFT where photon interactions are built only from the field strengths, the matrix elements are $O(1)$ when a photon is taken soft. This, for example, is exactly the case for Born-Infeld theory in which the photons have $\sigma = 0$.

**Constructibility.** Suppose we study an $n$-particle amplitude with $n_s$ scalars, $n_f$ fermions, and $n_\gamma$ photons in an EFT whose fundamental $v$-particle interactions all have couplings of the same mass-dimension $[g_v]$. The criterion (3.20) for constructibility via subtracted soft recursion relations can be written as

$$4 - n - n_v[g_v] - \frac{1}{2}n_f - n_\gamma - n_s\sigma_s - n_f\sigma_f - n_\gamma\sigma_\gamma < 0\,, \tag{3.31}$$

where $n_v = (n-2)/(v-2)$ is the number of vertices needed at $n$-point.

**Non-constructibility = Triviality.** Let us assess if there can be a local contact term for an $n$-particle amplitude with $n_s$ scalars, $n_f$ fermions, and $n_\gamma$ photons and soft behaviors $\sigma_s$, $\sigma_f$, and $\sigma_\gamma$, respectively. As discussed above, a contact term that has such trivial soft behavior takes the form

$$g_n\, \underbrace{(\partial^{\sigma_s}\phi)\cdots(\partial^{\sigma_s}\phi)}_{n_s}\, \underbrace{(\partial^{\sigma_f}\psi)\cdots(\partial^{\sigma_f}\psi)}_{n_f}\, \underbrace{(\partial^{\sigma_\gamma}F)\cdots(\partial^{\sigma_\gamma}F)}_{n_\gamma} \tag{3.32}$$

(for brevity we have not distinguished $\psi$ and $\bar{\psi}$). In 4d, the mass-dimension of the coupling $g_n$ is easily computed as

$$[g_n] = 4 - \big(n_s + n_s\sigma_s\big) - \big(\tfrac{3}{2}n_f + n_f\sigma_f\big) - \big(2n_\gamma + n_\gamma\sigma_\gamma\big)\,. \tag{3.33}$$

Using $n = n_s + n_f + n_\gamma$, we can rewrite this as

$$4 - n - [g_n] - \frac{1}{2}n_f - n_\gamma - n_s\sigma_s - n_f\sigma_f - n_\gamma\sigma_\gamma = 0\,. \tag{3.34}$$

Compare this with (3.31); we note that the constructibility criterion is simply that $n_v[g_v] > [g_n]$, or maybe more intuitively, that $g_n$ has more negative mass-dimension than $n_v$ $g_v$-vertices. So, when constructibility holds, the $n$-particle amplitude constructed from the $n_v$ $v$-valent vertices *cannot* be influenced by a contact term that trivially has the soft behavior: such a contact term would be too high order in the EFT due to all the derivatives needed to trivialize the soft behavior. That of course makes sense; were there such an independent local contact term, it could be added to the result of recursion with any coefficient without changing any of the properties of the amplitude. Hence recursion cannot possibly work in that case. (This is analogous to the example in [85, 86] for constructibility in scalar-QED via BCFW; the difference here is that the subtracted soft recursion relations "know" about the soft behavior in addition to gauge-invariance.)

The argument is easily extended to the case where the theory has fundamental vertices of different valences and mass-dimensions. We conclude that the constructibility criterion (3.20) is equivalent to the non-existence of local $n$-particle operators with couplings of the same mass-dimension and trivial soft behavior: Non-constructibility = Triviality.

### 3.3.5 Implementation of the Subtracted Recursion Relations

Here we present details relevant for the practical implementation of the soft subtracted recursion relations.

**Solving the shift constraints.** Conservation of the momentum for the shifted momenta $\hat{p}_i$ (3.14) requires the shift variables $a_i$ to satisfy

$$\sum_i a_i p_i^\mu = 0. \tag{3.35}$$

In 4d, the LHS can be viewed as a $4 \times n$ matrix $p_i^\mu$ of rank 4 (if $n \geq 5$) multiplying a $n$-component vector $a_i$. Hence the valid choices of parameters $a_i$ form a vector space given by the kernel of the matrix $p_i^\mu$. For $n \geq 5$ any subset of four momenta are generically linearly independent, so the $p_i^\mu$-matrix has full rank. By the rank-nullity theorem, the dimension of the kernel is therefore $n-4$. However, there is always a trivial solution which consists of all $a_i$'s equal, hence non-trivial solutions to (3.35) exist only when $n \geq 6$.

Practically, the linear system of equations is solved by dotting in $p_j$, i.e. we have

$$\sum_i s_{ji}\, a_i = 0 \qquad \text{for } j = 1, 2, \ldots, n\,. \tag{3.36}$$

The symmetric $n \times n$-matrix with entries $s_{ji}$ has rank 4, so the linear system (3.36) can be solved for say $a_1$, $a_2$, $a_3$, and $a_4$ in terms of the $n-4$ other $a_i$'s.

**Soft bootstrap.** Subtracted recursion relations can be used to calculate tree amplitudes in EFTs of Goldstone modes in theories we already know well, such as DBI, Akulov-Volkov etc. However, the soft subtracted recursion relations can also be used as a tool to *classify* and assess the existence of exceptional EFTs with a given spectrum of massless particles and low-energy theorems with given weights $\sigma$.

The approach to the classification of special EFTs is as follows:

(1) Model input: the spectrum of massless particles and the coupling dimensions of the fundamental interactions in the model.

(2) Symmetry assumptions: the $n$-particle amplitudes have soft behavior with weight $\sigma_i$ for the $i$th particle.

If the constructibility criterion (3.20) is *not* satisfied, the assumptions (1) and (2) are trivially satisfied and we cannot constrain the couplings in the EFTs; it is not exceptional.

If the constructibility criterion (3.20) *is* satisfied for input (1) and (2), one can use the soft subtracted recursion relations to test whether a theory can exist with the above assumptions. One proceeds as follows.

The fundamental vertices give rise to local amplitudes which must be polynomials[9] in the spinor helicity brackets, and it is simple to construct the most general such ansatz for the local input amplitudes. One can further restrict this ansatz by imposing on it the soft behaviors associated with the assumed symmetries. The result of recursing this input from the fundamental vertices is supposed to be a physical amplitude and therefore it must necessarily be independent of the $n-4$ parameters $a_i$ that are unfixed by (3.35). If that is *not* the case for any ansatz of the fundamental input amplitudes (vertices), we learn that there cannot exist a theory with the properties (1) and (2) above. On the other hand, an $a_i$-independent result is evidence (but not proof) of the existence of such a theory. It may well be that $a_i$-independence requires some of the free parameters in the input amplitudes to be fixed in certain ways and this can teach us important lessons about the underlying theory. The test of $a_i$-independence can be done efficiently numerically, and this way one can scan through theory-space to test which symmetries are compatible with a given model input.

Additionally, one can impose further constraints from unbroken global symmetries, for example, one can restrict the input from the fundamental amplitudes by imposing the supersymmetry Ward identities. We shall see examples of this in later sections.

**4d and 3d consistency checks.** There is a subtlety that must be addressed for $n = 6$. In this case, the solution space is 2-dimensional, but one solution is the trivial one with all $a_i$ equal. Furthermore, one can rescale all $a_i$. This means that if the recursed result for the

[9]This is true at 4-point and higher; for 3-point, massless particle amplitudes are uniquely fixed by the little group scaling.

amplitude depends on the $a_i$ only through ratios of the form

$$\frac{(a_i - a_j)}{(a_k - a_l)}\,, \tag{3.37}$$

it will appear to be $a_i$-independent numerically, but the result will nonetheless have spurious poles. To detect this problem numerically, we dimensionally reduce the recursed result to 3d.[10] Then the space of solutions to (3.35) is $(n-3)$-dimensional, so there are non-trivial solutions and a numerical 3d test will reveal dependence on ratios such as (3.37) for $n = 6$.

We refer to the consistency checks of $a_i$-independence as *4d and 3d consistency checks*, respectively, or simply as *n-point tests* when applied to construction of $n$-point amplitudes. In this paper, we use 6-, 7- and 8-point tests. In Section 3.4, we present an overview of the resulting space of exceptional pure real and complex scalar, fermion, and vector EFTs.

**Special requirements for non-trivial 5-point interactions.** Consider 5-particle interactions which are non-trivial with respect to a given soft behavior. This could for example be the Wess-Zumino-Witten (WZW) term, which with 4 derivatives on 5 scalars has a non-trivial $\sigma = 1$ soft behavior. Or the 5-point Galileon, which with 8 derivatives on 5 scalars has a non-trivial $\sigma = 2$. Constructibility tells us that one must be able to calculate such 5-point amplitudes from soft recursion relations via factorization, i.e.

$$\mathcal{A}_5 = \sum_I \frac{\hat{\mathcal{A}}_3 \hat{\mathcal{A}}_4}{P_I^2}\,. \tag{3.38}$$

However, there are no 3-point amplitudes available that could possibly make this work. The reason is that the only 3-scalar interaction with a non-zero on-shell amplitude is $\phi^3$, which gives rise to amplitudes with $\sigma = -1$ [89]. So we appear to have a contradiction: the constructibility criterion tells us that these 5-particle amplitudes are recursively constructible, but it is obviously impossible to construct them from lower-point input.

[10] The dimensional reduction from 4d to 3d is carried out by simply replacing all square spinors by angle spinors.

What goes wrong is that at 5-points, there are no non-trivial choices of the $a_i$ parameters that give valid recursion relations in 4d. So we have to go to 3d kinematics to resolve this issue. The above contradiction persists in 3d, so the only resolution is that these non-trivial constructible 5-point amplitudes must vanish in 3d kinematics.

Indeed they do: for WZW term and the quintic Galileon, the 5-point matrix elements are

$$A_5^{\rm WZW} = g_5\, \epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma\,, \qquad A_5^{\rm Gal} = g_5'\, (\epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma)^2\,. \tag{3.39}$$

The Levi-Civita contraction makes it manifest that these amplitudes vanish in 3d.

We conclude that *any* non-trivial (in the sense of soft behavior) 5-particle interaction must vanish in 3d. Thus, it is no coincidence that the WZW and quintic Galileon 5-point amplitudes are proportional to Levi-Civita contractions.

## 3.4 Soft Bootstrap

We now turn to examples of how the soft recursion relations can be used to examine the existence of exceptional EFTs. The landscape of real scalar theories was previously studied in [76, 33, 35, 37]. We outline it briefly below for completeness, but otherwise focus on new results, in particular for complex scalars, fermions, and vectors. This section considers only theories with one kind of massless particle. One can of course also couple scalars, fermions, and vectors in EFTs, and this is discussed in Sections 3.6, 3.7, and 3.8.

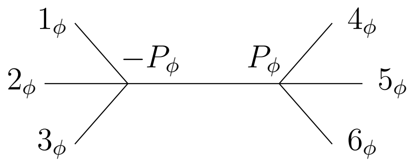
### 3.4.1 Pure Scalar EFTs

Consider an EFT with a single real scalar field $\phi$. There can only be non-vanishing 3-point amplitudes in $\phi^3$-theory and this gives amplitudes with soft weight $\sigma = -1$. Focusing on EFTs with soft weights $\sigma \geq 0$, the lowest-point amplitude is 4-point.

The on-shell factorization diagrams that contribute in the recursion relations (3.19) for $\mathcal{A}_6(1_\phi\, 2_\phi\, 3_\phi\, 4_\phi\, 5_\phi\, 6_\phi)$ are composed of a product of two 4-point amplitudes, for example the

123-channel diagram is

$$\mathcal{A}_6^{(123)} =$$

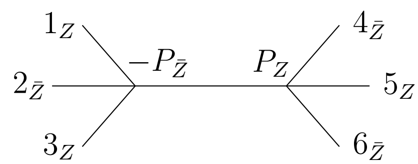



$$= \frac{\hat{\mathcal{A}}_L(0)\hat{\mathcal{A}}_R(0)}{P_{123}^2} + \sum_{i=1}^{6} \text{Res}_{z=\frac{1}{a_i}} \frac{\hat{\mathcal{A}}_L(z)\hat{\mathcal{A}}_R(z)}{z\,F(z)\,\hat{P}_{123}^2}\,,$$

where $\hat{\mathcal{A}}_L = \hat{\mathcal{A}}_4(1_\phi\,2_\phi\,3_\phi\,-P_\phi)$ and $\hat{\mathcal{A}}_R = \hat{\mathcal{A}}_4(P_\phi\,4_\phi\,5_\phi\,6_\phi)$.[11] One sums over the 10 independent permutations corresponding to the 10 distinct factorization channels.[12]

For complex scalars, we assume that the input 4-point amplitudes are of the form $\mathcal{A}_4(1_Z\,2_{\bar{Z}}\,3_Z\,4_{\bar{Z}})$;[13] one can also consider more general input but it would not be compatible with supersymmetry, so in the present paper we do not discuss such options. At 6-point, there is only one type of amplitude that can arise from such 4-point input via recursion, and that is $\mathcal{A}_6(1_Z\,2_{\bar{Z}}\,3_Z\,4_{\bar{Z}}\,5_Z\,6_{\bar{Z}})$. The 123-channel diagram is

$$\mathcal{A}_6^{(123)} = \tag{3.40}$$




To get the full amplitude, one must sum over all factorization channels:

$$\mathcal{A}_6(1_Z\,2_{\bar{Z}}\,3_Z\,4_{\bar{Z}}\,5_Z\,6_{\bar{Z}}) = \Big(\mathcal{A}_6^{(123)} + (2 \leftrightarrow 4) + (2 \leftrightarrow 6)\Big) + (1 \leftrightarrow 5) + (3 \leftrightarrow 5)\,. \tag{3.41}$$

In the following we consider real and complex scalar theories with 4- and 5-point fundamental vertices.

[11]The momenta in the hatted amplitudes are shifted; for simplicity, we do not write the hats on the momentum variables explicitly. Note that in particular $P_\phi$ should really be understood as $\hat{P}_\phi$ with $\hat{P}_\phi^2 = 0$.

[12]We do not consider color-ordering in this section. With color-ordering, one only includes the factorization diagrams from cyclic permutations of the external lines.

[13]There is no color-ordering implied in any of the amplitudes here. We simply alternate $Z$ and $\bar{Z}$ states as odd/even numbered momentum lines. In later sections, other helicity states are grouped similarly, in particular for supersymmetric cases, states that belong to the positive helicity sector sit on odd-numbered lines and negative helicity sector states on even-numbered lines. This is convenient for the practical implementation but should not be misunderstood as an indication of color-ordering.

| | | | Assumed value for $\sigma$ | | | | |
|---|---|---|---|---|---|---|---|
| $m$ | $-[g]$ | $\mathcal{A}_4^{\text{ansatz}}(1_\phi 2_\phi 3_\phi 4_\phi)$ | 0 | 1 | 2 | 3 | 4 |
| 0 | 0 | $g$ | $\phi^4$-theoroy | F | F | F | F |
| 1 | 2 | 0 | – | F | F | F | F |
| 2 | 4 | $g\,(s^2+t^2+u^2)$ | – | – | DBI | F | F |
| 3 | 6 | $g\ stu$ | – | – | $\text{Gal}_4$ | $\text{Spec.Gal}_4$ | F |
| 4 | 8 | $g\,(s^4+t^4+u^4)$ | – | – | – | F | F |

Table 3.2: Ansatz for fundamental interactions corresponding to a schematic operator of the form $g\ \partial^{2m}\phi^4$ and the results of the recrsion test for different values of the soft degree $\sigma$.

**Fundamental 4-point Interactions**

Consider a theory of a single real scalar with fundamental 4-point interactions. We parameterize $\mathcal{A}_4^{\text{ansatz}}$ as the most general polynomial in the Mandelstam variables $s, t, u$ (with $s+t+u=0$) and full Bose symmetry. We subject the recursed result for $\mathcal{A}_6$ to the test of $a_i$-independence, as described in Section 3.3.5. The results of the test for different values of the soft degree $\sigma$ are shown in Table 3.2.

In the table, we list the coupling dimension $[g]$ of the fundamental quartic couplings along with the most general ansatz for the corresponding 4-point amplitude. The dash, –, indicates that the constructibility criterion (3.20) fails; this means "triviality" in the sense described in Section 3.3.4). "F" indicates that the soft recursion fails to give an $a_i$-independent result, and hence no such theory can exist with the given assumptions. When a case passes the 6-point test, we are able to uniquely identify which theory it is. In the above table, the non-trivial theories that pass the 6-point test are: $\phi^4$-theory, DBI, and the quartic Galileon. The latter automatically has $\sigma = 3$ (which is called the Special Galileon) and passes 6-point test for both $\sigma = 2$ and $\sigma = 3$.

The analysis for complex scalars proceeds similarly and the results are summarized in Table 3.3. The non-trivial theories are $|Z|^4$-theory, the $\mathbb{CP}^1$ NLSM (which is studied in further detail in Section 3.6), and the complex scalar versions of DBI and the quartic Galileon. Note that there does not exist a complex scalar version of the Special Galileon with $\sigma = 3$. The results for the 6-point amplitudes of each of the theories with $\sigma > 0$ can be found in

| $m$ | $-[g]$ | $\mathcal{A}_4^{\text{ansatz}}(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}})$ | Assumed value for $\sigma$: 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| 0 | 0 | $g$ | $\lvert Z\rvert^4$-theoroy | F | F | F |
| 1 | 2 | $gt$ | – | $\mathbb{CP}^1$ NLSM | F | F |
| 2 | 4 | $gt^2 + g'su$ | – | – | $g' = 0$ complex DBI | F |
| 3 | 6 | $gt^3 + g'stu$ | – | – | $g = 0$ complex Gal$_4$ | F |
| 4 | 8 | $gt^4 + g'st^2u + g''s^2u^2$ | – | – | – | F |

Table 3.3: Ansatz for fundamental interactions corresponding to a schematic operator of the form $g\,\partial^{2m} Z^2 \bar{Z}^2$ and the results of the recrsion test for different values of the soft degree $\sigma$.

Appendix E.

**Fundamental 5-point Interactions**

At 5-point, the input amplitudes are constructed as polynomials of Mandelstam variables $s_{ij}$ and Levi-Civita contractions of momenta. They must obey (1) momentum conservation, (2) Bose symmetry, and (3) assumed soft behavior $\sigma$. In many cases, these constraints on the 5-point input amplitudes are sufficient to rule out such theories (assuming no other interactions) without even applying soft recursion.

As discussed at the end of Section 3.3.5, non-trivial 5-point amplitudes must vanish in 3d kinematics, so they are naturally written using the Levi-Civita tensor, as in the two cases of WZW and the quintic Galileon (3.39).

We can summarize the results in the following:

- **1 real scalar.** There are only two non-trivial theories based on a fundamental 5-point interaction, namely $\phi^5$-theory, which has $[g_5] = -1$ and $\sigma = 0$, and the quintic Galileon, which has $[g_5] = -9$ and $\sigma = 2$.

- **1 complex scalar.** We assume input amplitudes of the form $\mathcal{A}_5(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}} 5_Z)$. Two cases pass the 8-point test: The quintic $g_5(Z^3\bar{Z}^2 + Z^2\bar{Z}^3)$-theory with $[g_5] = -1$ has $\sigma_Z = 0$. The complex-scalar version of the quintic Galileon with $[g_5] = -9$ and $\sigma_Z = 2$.

The 5-point amplitude is

$$\mathcal{A}_5(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}} 5_Z) = g_5(\epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma)^2 \,, \tag{3.42}$$

same as for the real-scalar quintic Galileon. The fact that it passes the 8-point test is somewhat trivial: because of the two explicit factors of momentum for 4 out of 5 particles, the residues at $1/a_i$ vanish identically for each factorization channel. The same is true for the real Galileon, so the 8-point test is not really effective as an indicator of whether such a theory may exist.

Suppose the putative complex-scalar quintic Galileon is coupled to the complex scalar DBI. Then we can conduct a 7-point test based on factorization into a quantic Galileon and a quartic DBI subamplitude. The test of $a_i$-independence requires the coupling constant $g_5$ to vanish. This means that the DBI-Galileon with a complex scalar cannot have a 5-point interaction.

At $[g_5] = -9$, there is a 6-parameter family of 5-point amplitudes with $\sigma_Z = 1$. The EFT with such amplitudes is generally non-constructible. However, a 1-parameter subfamily is compatible with the constraints of supersymmetry. As discussed in [60] and further in Section 3.8.1 this may be a candidate for a supersymmetric quintic Galileon with a limited sector of constructible amplitudes.

### 3.4.2 Pure Fermion EFTs

Let us now consider EFTs with only fermions and fundamental interactions of the form $\partial^{2m}\psi^2\bar{\psi}^2$. This is not the only choice, but it is the option compatible with supersymmetry. Moreover, we have found that couplings of "helicity violating" 4-point interactions in the fermion sector must vanish by the 6-point test in all pure-fermion cases we tested. The calculations proceed much the same way as for scalars, except that one must be more careful with signs when inserting fermionic states on the internal line. The diagrams needed for the

| $m$ | $-[g]$ | $\mathcal{A}_4^{\text{ansatz}}(1_\psi 2_{\bar\psi} 3_\psi 4_{\bar\psi}) = \langle 24\rangle[13]\times$ | Assumed value for $\sigma$ 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| 0 | 2 | $g$ | NJL | F | F | F |
| 1 | 4 | $gt$ | – | AV | F | F |
| 2 | 6 | $gt^2 + g'su$ | – | – | F | F |
| 3 | 8 | $gt^3 + g'stu$ | – | – | $g = 0$ new | F |

Table 3.4: Ansatz for fundamental interactions corresponding to a schematic operator of the form $g\,\partial^{2m}\psi^2\bar\psi^2$ and the results of the recrsion test for different values of the soft degree $\sigma$.

recursive calculation of the 6-fermion amplitude $\mathcal{A}_6(1^+_\psi\, 2^-_{\bar\psi}\, 3^+_\psi\, 4^-_{\bar\psi}\, 5^+_\psi\, 6^-_{\bar\psi})$ are just like those in the scalar case (3.40), but now the permutations have to be taken with a sign:

$$\mathcal{A}_6(1^+_\psi\, 2^-_{\bar\psi}\, 3^+_\psi\, 4^-_{\bar\psi}\, 5^+_\psi\, 6^-_{\bar\psi}) = \Big(\mathcal{A}_6^{(123)} - (1\leftrightarrow 5) - (3\leftrightarrow 5)\Big) - (2\leftrightarrow 4) - (2\leftrightarrow 6). \tag{3.43}$$

The input 4-point amplitudes $\mathcal{A}_4(1^+_\psi\, 2^-_{\bar\psi}\, 3^+_\psi\, 4^-_{\bar\psi})$ are fixed by little group scaling to be $\langle 24\rangle[13]$ times a Mandelstam polynomial of degree $m$ that must be symmetric under $s \leftrightarrow u$ to ensure Fermi antisymmetry for identical fermions. The most general input amplitudes for low values of $m$ are summarized in Table 3.4 that also shows the result of the recursive 6-point test.

We comment briefly on these results:

- The **NJL model** has the fundamental 4-fermion interaction $\bar\psi^2\psi^2$ and the result of recursing it to 6-point is given in Appendix E.1. The relevance of this model will for our purposes be as part of the supersymmetrization of the NLSM (see Section 3.6).

- **Akulov-Volkov theory** of Goldstinos is the only non-trivial EFT with coupling of mass-dimension $-4$. The Goldstinos in this theory have low-energy theorems with $\sigma = 1$. The 6-fermion amplitude is given in (E.14) in Appendix E.2.

- There are no constructible purely fermionic EFTs with fundamental quartic coupling $[g_4] = -6$. Nonetheless, as was shown in [60], the quartic Galileon has a supersymmetrization with a 4-fermion fundamental interaction, however, the fermion has $\sigma = 1$, so the all-fermion amplitudes in that theory are not constructible by soft recursion: one needs additional input from supersymmetry. We refer the reader to [60] and present some further details in Section 3.8.1.

- For $[g] = -8$ and $\sigma = 2$, the 6-point numerical test is passed in 4d kinematics without constraints on $g$ and $g'$; that is because the recursed result depends only on ratios (3.37). When the 3d consistency check is employed, we learn that we must set $g = 0$ to ensure $a_i$-independence. (This is not a strong test since the particular form of the interaction, $stu$, ensures that all $1/a_i$-poles cancel in each factorization individual diagram.) Hence, the theory that passes the 6-point test with $\sigma = 2$ has $\mathcal{A}_4(1_\psi, 2_{\bar{\psi}}, 3_\psi, 4_{\bar{\psi}}) = g'\langle 24\rangle[13]stu$. The subtracted recursion relations fail at $n > 6$, which means that at 8-point and higher, this model is not uniquely determined by its symmetries. The Lagrangian construction of this theory has been studied as a fermionic generalization of the scalar Galileon [91].

### 3.4.3 Pure Vector EFTs

Pure abelian vector EFTs consist of interaction terms built from $F_{\mu\nu}$-contractions, possibly dressed with extra derivatives. In 4d, the Cayley-Hamilton relations imply that theories built from just field strengths $F_{\mu\nu}$ can be constructed from two types of index-contractions, namely (see for example [92])

$$f = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad \text{and} \quad g = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\,, \tag{3.44}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. If one assumes parity, the Lagrangian can only contain even powers of $g$. One can then write an ansatz for the Lagrangian as

$$\mathcal{L} = f + \frac{b_1}{\Lambda^4}f^2 + \frac{b_2}{\Lambda^4}g^2 + \frac{b_3}{\Lambda^8}f^3 + \frac{b_4}{\Lambda^8}fg^2 + \dots \tag{3.45}$$

As established in Section 3.3.4, a model with photon interactions built of $F_{\mu\nu}$-contractions only have soft behavior $\sigma = 0$. The simplest 4-photon interactions may naively look like the vector equivalent of the constructible $\phi^4$ scalar EFT. However, that is not the case. For the scalar, the 6-particle operator $\frac{1}{\Lambda^2}\phi^6$ is subleading to the pole contributions with two $\phi^4$-vertices. However, for photons the pole terms with two $\frac{1}{\Lambda^4}F^4$ vertices are exactly the same order as $\frac{1}{\Lambda^8}F^6$. Therefore amplitudes in a theory with $F^n$ interactions and $\sigma = 0$ are non-constructible, in other words it is trivial to have $\sigma = 0$ for any choice of coefficients $b_i$. One may ask if it is possible to choose the parameters $b_i$ in (3.45) such that the amplitudes have enhanced soft behavior $\sigma > 0$. The 6-point soft recursive test shows that this is impossible, i.e. no models exist with Lagrangians of the form (3.45) and $\sigma > 0$.

Nonetheless, the class of theories with pure $F^n$ interactions do include one particularly interesting case, namely Born-Infeld (BI) theory. The BI Lagrangian can be written in 4d as

$$\mathcal{L}_{\mathrm{BI}} = \Lambda^4\left(1 - \sqrt{-\det\left(\eta_{\mu\nu} + F_{\mu\nu}/\Lambda^2\right)}\right). \tag{3.46}$$

Upon expansion, the Lagrangian will take the form (3.45) with some particular coefficients $b_i$. As noted, these particular coefficients do not change the single-soft behavior of amplitudes, the BI photon also has $\sigma = 0$. Nonetheless, BI theory does have the distinguishing feature of being the vector part of a supersymmetric EFT. In particular, $\mathcal{N} = 1$ supersymmetric Born-Infeld theory couples the BI vector to a Goldstino mode whose self-interactions are described by the Akulov-Volkov action. One can also view Born-Infeld as the vector part of the $\mathcal{N} = 2$ or $\mathcal{N} = 4$ supersymmetrization of DBI. It was argued recently [92] that supersymmetry ensures BI amplitudes to vanish in certain multi-soft limits. Based on that, the BI amplitudes can be calculated unambiguously using on-shell techniques [92]. Alternatively, one can show that the $\mathcal{N} = 1$ supersymmetry Ward identities uniquely fix the BI amplitudes in terms of amplitudes with Goldstinos.

Next, one can consider EFTs in which the field strengths are dressed with derivatives, for example

$$\mathcal{L} = -\frac{1}{4}F^2 + \frac{c_1}{\Lambda^6}\partial^2 F^4 + \frac{c_1}{\Lambda^{12}}\partial^4 F^6 + \dots \tag{3.47}$$

Theories with fundamental 4-point interactions are non-constructible for $\sigma = 0$ and fail the soft recursion $a_i$-independence 6-point test for $\sigma > 0$. One implication of this is that there can be no vector Goldstone bosons with vanishing low-energy theorems. This conclusion was also reached in [82], but from a very different algebraically-based analysis. A second implication is that the pure vector sector of an $\mathcal{N} \geq 2$ Galileon model is non-constructible with the basic soft recursion, and other properties (such as supersymmetry) have to be specified in order to determine those amplitudes recursively.

There are other interesting vector EFTs: we study in detail the $\mathcal{N} = 2$ supersymmetric NLSM in Section 3.6. Furthermore, massive gravity [93, 94, 95] motivates the existence of a vector-scalar theory coupling Galileons to a vector field; we explore this in Section 3.8.2.

## 3.5 Soft Limits and Supersymmetry

For models with unbroken supersymmetry, the on-shell amplitudes satisfy a set of linear relations known as the *supersymmetry Ward identities* [38, 39]. (For recent reviews and results, see [96, 85, 86].) In this section, we use $\mathcal{N} = 1$ supersymmetry to derive general consequences for the soft behavior for massless particles in the same supermultiplet. It is not assumed that these particles are Goldstone or quasi-Goldstone modes; the results apply to all $\mathcal{N} = 1$ supermultiplets of massless particles. The consequences for extended supersymmetry are directly inferred from the $\mathcal{N} = 1$ constraints.

### 3.5.1 $\mathcal{N} = 1$ Supersymmetry Ward Identities

We consider $\mathcal{N} = 1$ *chiral* and *vector* supermultiplets. We use the following shorthand for the action of the supercharges on individual particles with momentum label $i$: for chiral multiplets

| state $i$ | $\mathcal{Q}\cdot i$ | $\mathcal{A}_n$ prefactor | $\mathcal{Q}^\dagger\cdot i$ | $\mathcal{A}_n$ prefactor |
|---|---|---|---|---|
| $\psi^+$ | $Z$ | $\|i]$ | $0$ | $0$ |
| $Z$ | $0$ | $0$ | $\psi^+$ | $-\|i\rangle$ |
| $\overline{Z}$ | $\psi^-$ | $\|i]$ | $0$ | $0$ |
| $\psi^-$ | $0$ | $0$ | $\overline{Z}$ | $-\|i\rangle$ |

(3.48)

where $Z$ is a complex scalar and $\psi$ is a Weyl fermion. The superscripts $\pm$ refer to the helicity of the particle. $\mathcal{Q}^\dagger$ raises helicity by 1/2 while $\mathcal{Q}$ lowers it by 1/2. The prefactor is what goes outside the amplitude when the supercharge acts on it, e.g.

$$\mathcal{Q}\cdot\mathcal{A}_n\big(1_Z 2^+_\psi 3^+_\psi 4_{\overline{Z}}\dots\big) = 0 + |2]\mathcal{A}_n\big(1_Z 2_Z 3^+_\psi 4_{\overline{Z}}\dots\big) - |3]\mathcal{A}_n\big(1_Z 2^+_\psi 3_Z 4_{\overline{Z}}\dots\big) + |4]\mathcal{A}_n\big(1_Z 2^+_\psi 3^+_\psi 4^-_\psi\dots\big) + \dots \quad (3.49)$$

Due to the Grassmann nature of the supercharges, there is a minus sign for each fermion that the supercharge has to move past to get to the $i$th state.

Similarly for a vector multiplet:

| state $i$ | $\mathcal{Q}\cdot i$ | $\mathcal{A}_n$ prefactor | $\mathcal{Q}^\dagger\cdot i$ | $\mathcal{A}_n$ prefactor |
|---|---|---|---|---|
| $\gamma^+$ | $\psi^+$ | $\|i]$ | $0$ | $0$ |
| $\psi^+$ | $0$ | $0$ | $\gamma^+$ | $-\|i\rangle$ |
| $\psi^-$ | $\gamma^-$ | $-\|i]$ | $0$ | $0$ |
| $\gamma^-$ | $0$ | $0$ | $\psi^-$ | $\|i\rangle$ |

(3.50)

where $\psi$ is a Weyl fermion and $\gamma$ is a vector boson.

In this notation, the supersymmetry Ward identities are equivalent to the statement that the following action of the supercharges annihilates the amplitude [85, 86, 96]

$$\begin{aligned} 0 = \mathcal{Q} \cdot \mathcal{A}_n\left(1, \ldots, n\right) &= \sum_{i=1}^{n} (-1)^{L_i + P_i} |i] \mathcal{A}_n\left(1, \ldots, \mathcal{Q} \cdot i, \ldots, n\right), \\ 0 = \mathcal{Q}^\dagger \cdot \mathcal{A}_n\left(1, \ldots, n\right) &= \sum_{i=1}^{n} (-1)^{L_i + P_i} |i\rangle \mathcal{A}_n\left(1, \ldots, \mathcal{Q}^\dagger \cdot i, \ldots, n\right), \end{aligned} \tag{3.51}$$

where $L_i$ is equal to the number of fermions to the *left* of $\mathcal{Q}^{(\dagger)} \cdot i$ and the factors $P_i = 0$ or 1 correspond to the additional minus signs associated with the spinor prefactors as described in equations (3.48) and (3.50). Note that the action of the supercharges always changes the number of fermions by $\pm 1$, but that amplitudes are non-vanishing only if the number of fermions is even. So to get an interesting relation among amplitudes on the right-hand-side, the amplitude on the left-hand-side must vanish identically.

### 3.5.2 Soft Limits and Supermultiplets

We consider the chiral multiplet and vector multiplet separately and then extend the results to enhanced supersymmetry.

**Chiral multiplet.** Define the *soft factors* $\mathcal{S}_n^{(i)}$ as the momentum dependent coefficients in the holomorphic soft expansion taken here for simplicity on the first particle

$$\begin{aligned} \mathcal{A}_n\left(\{\epsilon|1\rangle, |1]\}_Z, \ldots\right) &\rightarrow \mathcal{S}_n^{(0)}(1_Z, \ldots)\, \epsilon^{\sigma_Z} + \mathcal{S}_n^{(1)}(1_Z, \ldots)\, \epsilon^{\sigma_Z + 1} + \mathcal{O}\left(\epsilon^{\sigma_Z + 2}\right), \\ \mathcal{A}_n\left(\{\epsilon|1\rangle, |1]\}_\psi^+, \ldots\right) &\rightarrow \mathcal{S}_n^{(0)}(1_\psi^+, \ldots)\, \epsilon^{\sigma_\psi} + \mathcal{S}_n^{(1)}(1_\psi^+, \ldots)\, \epsilon^{\sigma_\psi + 1} + \mathcal{O}\left(\epsilon^{\sigma_\psi + 2}\right). \end{aligned} \tag{3.52}$$

The soft weights are $\sigma_Z$ and $\sigma_\psi$ for the scalar and fermion, respectively. To see how supersymmetry forces relations among the soft weights and soft factors we use (3.51) to write

$$\begin{aligned} \mathcal{A}_n\left(1_Z, \ldots, n\right) &= \sum_{i=2}^{n} (-1)^{L_i + P_i + 1} \frac{[Xi]}{[X1]}\, \mathcal{A}_n\left(1_\psi^+, \ldots, \mathcal{Q} \cdot i, \ldots, n\right), \\ \mathcal{A}_n\left(1_\psi^+, \ldots, n\right) &= \sum_{i=2}^{n} (-1)^{L_i + P_i + 1} \frac{\langle Xi\rangle}{\langle X1\rangle}\, \mathcal{A}_n\left(1_Z, \ldots, \mathcal{Q}^\dagger \cdot i, \ldots, n\right), \end{aligned} \tag{3.53}$$

where the arbitrary $X$-spinor cannot be proportional to $|1\rangle$ or $|1]$.

Taking the holomorphic soft expansion on the right-hand-side of these expressions, in the second line only, an extra power of $\epsilon$ appears in the denominator and we find

$$\begin{aligned}
\mathcal{S}_n^{(0)}(1_Z,\ldots)\,\epsilon^{\sigma_Z}+\mathcal{O}\left(\epsilon^{\sigma_Z+1}\right) &= \sum_{i=2}^{n}(-1)^{L_i+P_i+1}\frac{[Xi]}{[X1]}\mathcal{S}_n^{(0)}(1_\psi^+,\ldots,\mathcal{Q}\cdot i,\ldots)\,\epsilon^{\sigma_\psi}+\mathcal{O}\left(\epsilon^{\sigma_\psi+1}\right),\\
\mathcal{S}_n^{(0)}(1_\psi^+,\ldots)\,\epsilon^{\sigma_\psi}+\mathcal{O}\left(\epsilon^{\sigma_\psi+1}\right) &= \sum_{i=2}^{n}(-1)^{L_i+P_i+1}\frac{\langle Xi\rangle}{\langle X1\rangle}\mathcal{S}_n^{(0)}(1_Z,\ldots,\mathcal{Q}^\dagger\cdot i,\ldots)\,\epsilon^{\sigma_Z-1}+\mathcal{O}\left(\epsilon^{\sigma_Z}\right).
\end{aligned}$$

The leading power of $\epsilon$ on the right-hand-side must match the leading power on the left. It is possible that cancellations among the terms on the right-hand-side may effectively increase the leading power but never decrease it. This then gives the following inequalities

$$\sigma_Z \geq \sigma_\psi \qquad \text{and} \qquad \sigma_\psi \geq \sigma_Z - 1\,, \tag{3.54}$$

for which there are only two solutions

$$\boxed{\sigma_Z = \sigma_\psi + 1 \quad \text{or} \quad \sigma_Z = \sigma_\psi\,.} \tag{3.55}$$

These two options have different consequences for the soft factors. For $\sigma_Z = \sigma_\psi + 1$, we have

$$\begin{aligned}
0 &= \sum_{i=2}^{n}(-1)^{L_i+P_i}[Xi]\,\mathcal{S}_n^{(0)}\left(1_\psi^+,\ldots,\mathcal{Q}\cdot i,\ldots\right),\\
\mathcal{S}_n^{(0)}\left(1_\psi^+,\ldots\right) &= \sum_{i=2}^{n}(-1)^{L_i+P_i+1}\frac{\langle Xi\rangle}{\langle X1\rangle}\,\mathcal{S}_n^{(0)}\left(1_Z,\ldots,\mathcal{Q}^\dagger\cdot i,\ldots\right),
\end{aligned}\tag{3.56}$$

while for $\sigma_\phi = \sigma_\psi$, we have

$$\begin{aligned}
0 &= \sum_{i=2}^{n}(-1)^{L_i+P_i}\langle Xi\rangle\mathcal{S}_n^{(0)}\left(1_Z,\ldots,\mathcal{Q}^\dagger\cdot i,\ldots\right),\\
\mathcal{S}_n^{(0)}\left(1_Z,\ldots\right) &= \sum_{i=2}^{n}(-1)^{L_i+P_i+1}\frac{[Xi]}{[X1]}\mathcal{S}_n^{(0)}\left(1_\psi^+,\ldots,\mathcal{Q}\cdot i,\ldots\right).
\end{aligned}\tag{3.57}$$

In addition there will be an infinite number of similar relations which come from matching higher powers in $\epsilon$.

**Vector multiplet.** We define the soft factors as

$$\mathcal{A}_n\left(\{\epsilon|1\rangle, |1]\}_\gamma^+, \ldots\right) \;\to\; \mathcal{S}_n^{(0)}(1_\gamma^+, \ldots)\,\epsilon^{\sigma_\gamma} + \mathcal{S}_n^{(1)}(1_\gamma^+, \ldots)\,\epsilon^{\sigma_\gamma+1} + \mathcal{O}\left(\epsilon^{\sigma_\gamma+2}\right). \tag{3.58}$$

The analysis of the supersymmetry Ward identities proceeds similarly to that of the chiral multiplet and results in only two options for the soft weights:

$$\boxed{\sigma_\psi = \sigma_\gamma + 1, \qquad \text{or} \qquad \sigma_\psi = \sigma_\gamma\,.} \tag{3.59}$$

The consequences for the soft factors are for $\sigma_\psi = \sigma_\gamma + 1$

$$\begin{aligned} 0 &= \sum_{i=2}^{n}(-1)^{L_i+P_i}[Xi]\mathcal{S}_n^{(0)}\left(1_\gamma^+, \ldots, \mathcal{Q}\cdot i, \ldots\right), \\ \mathcal{S}_n^{(0)}\left(1_\gamma^+, \ldots\right) &= \sum_{i=2}^{n}(-1)^{L_i+P_i+1}\frac{\langle Xi\rangle}{\langle X1\rangle}\mathcal{S}_n^{(0)}\left(1_\psi^+, \ldots, \mathcal{Q}^\dagger\cdot i, \ldots\right), \end{aligned} \tag{3.60}$$

and for $\sigma_\gamma = \sigma_\psi$

$$\begin{aligned} 0 &= \sum_{i=2}^{n}(-1)^{L_i+P_i}\langle Xi\rangle\mathcal{S}_n^{(0)}\left(1_\psi^+, \ldots, \mathcal{Q}^\dagger\cdot i, \ldots\right), \\ \mathcal{S}_n^{(0)}\left(1_\psi^+, \ldots\right) &= \sum_{i=2}^{n}(-1)^{L_i+P_i+1}\frac{[Xi]}{[X1]}\mathcal{S}_n^{(0)}\left(1_\gamma^+, \ldots, \mathcal{Q}\cdot i, \ldots\right). \end{aligned} \tag{3.61}$$

Note that we have made no assumptions about the sign of $\sigma$, so the relations derived here are totally general. Also, the supersymmetry Ward identities hold at all orders in perturbation theory, so the relations among the soft behaviors remain true at loop-level.

**Extended supersymmetry.** Relations between the soft weights of particles in the same massless supermultiplets in extended supersymmetry follow directly from the $\mathcal{N} = 1$ results above, since the supersymmetry Ward identities take the same form for each pair of $(s, s+\frac{1}{2})$-multiplets. In particular, the soft weights of the boson ($\sigma_B$) and fermion ($\sigma_F$) in a $(s, s+\frac{1}{2})$-multiplet are related as

$$\begin{cases} \sigma_B = \sigma_F + 1 \quad \text{or} \quad \sigma_B = \sigma_F \quad \text{for } s \text{ integer}\,, \\ \sigma_B = \sigma_F - 1 \quad \text{or} \quad \sigma_B = \sigma_F \quad \text{for } s \text{ half-integer}\,. \end{cases} \tag{3.62}$$

These relations will be useful in later applications in this paper. For now, we make a small aside and demonstrate the application of (3.62) to the case of spontaneously broken superconformal symmetry and for unbroken extended supergravity.

### 3.5.3 Application to Superconformal Symmetry Breaking

The breaking of conformal symmetry gives rise to a single Goldstone mode [80], often called the dilaton. It has been established in the literature [97, 98, 99] that this dilaton obeys low-energy theorems with $\sigma = 0$. In a superconformal theory, breaking of conformal invariance must be accompanied by breaking of the superconformal symmetries. This follows from the algebra: $\{\mathcal{S}, \mathcal{S}^\dagger\} = \mathcal{K}$, $[\mathcal{Q}, \mathcal{K}] = \mathcal{S}^\dagger$ and $[\mathcal{Q}^\dagger, \mathcal{K}] = \mathcal{S}$, where $\mathcal{K}$ are the generators of conformal boosts, $\mathcal{S}$ and $\mathcal{S}^\dagger$ are the superconformal fermionic generators, and $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ are the regular supercharges with $\{\mathcal{Q}, \mathcal{Q}^\dagger\} = \mathcal{P}$.

Assuming $\mathcal{Q}$-supersymmetry to be unbroken, the dilaton will be joined by a Goldstone mode from the broken R-symmetry to form a complex scalar $Z$ with $\sigma_Z = 0$.[14] It follows from our general analysis that the fermionic partner of $Z$ will have $\sigma = 0$ or $\sigma = -1$. For the latter, Yukawa-interactions are necessary [89] and supersymmetry then requires cubic scalar interactions $Z|Z|^2$ + h.c. which would imply $\sigma = -1$ for the dilaton. Since $\sigma_Z = 0$, $\sigma = -1$ is not possible for the dilaton and we conclude that the Goldstino mode associated with the breaking of the superconformal fermionic symmetries generated by $\mathcal{S}$ and $\mathcal{S}^\dagger$ must

[14] An example of the bosonic part of an $\mathcal{N} = 1$ effective action of the dilaton and a $U(1)_R$ Goldstone boson can be found in [100].

| helicity state | $\sigma$ |
|---|---|
| +2 graviton | $-3$ |
| +3/2 gravitino | $-2$ |
| +1 graviphoton | $-1$ |
| +1/2 fermion | 0 |
| 0 scalar | 0 or +1 |
| -1/2 fermion | +1 |
| -1 graviphoton | +1 |
| -3/2 gravitino | +1 |
| -2 graviton | +1 |

Table 3.5: Holomorphic soft weights $\sigma$ for the $\mathcal{N} = 8$ supermultiplet.

have low-energy theorems with soft weight $\sigma = 0$.

An example is $\mathcal{N} = 4$ SYM on the Coulomb branch with the simplest breaking pattern.[15] The $R$-symmetry is broken from $SO(6)$ to $SO(5)$ and the five broken generators give rise to five Goldstone modes which join the dilaton of the conformal breaking to be the 6 real scalars of an $\mathcal{N} = 4$ massless multiplet. The supermultiplet also contains the 4 Goldstinos associated with the four broken superconformal generators. The supermultiplet is capped off by a $U(1)$ vector whose soft weight, by the above analysis, must be either $\sigma = 0$ or $-1$. The states that are charged under this $U(1)$ are the massive $W$-multiplets and in their presence, one can have $\sigma = -1$, otherwise $\sigma = 0$ for the vector.

### 3.5.4 Application to Supergravity

It is well-known that gravitons have a universal soft behavior [102]: when the soft limit (3.12) is applied to a single graviton, the amplitude diverges as $1/\epsilon^3$, i.e. the soft weight is $\sigma_2 = -3$. (In this section, we use a subscript on the soft weight to indicate the spin of the particle.) Applying (3.62) shows that the gravitino can have $\sigma_{3/2} = -2$ or $-3$. However, unitarity and locality constraints show [89] that amplitudes cannot be more singular than $1/\epsilon^2$ for a single soft gravitino, so it must be that $\sigma_{3/2} = -2$. This must be true in any supergravity theory.

Consider now a graviphoton in $\mathcal{N} \geq 2$ supergravity. Its supersymmetry Ward identi-

[15] See [101, 99] for explicit amplitudes on the Coulomb branch of $\mathcal{N} = 4$ SYM.

ties with the gravitino imply $\sigma_1 = -2$ or $\sigma_1 = -1$. The $\sigma_1 = -2$ behavior requires the graviphoton, and by supersymmetry also the gravitino, to interact with a pair of electrically charged particles via a dimensionless coupling; however, for the gravitino such a coupling is inconsistent with unitarity and locality [89]. So there is only one option, namely $\sigma_1 = -1$.

In pure $\mathcal{N} \geq 3$ supergravity, we also have spin-$\frac{1}{2}$ fermions in the graviton supermultiplet. By (3.62) and the previous results, they can have either $\sigma_{1/2} = -1$ or 0. The analysis in [89] shows that $\sigma_{1/2} = -1$ requires a dimensionless coupling of the spin-$\frac{1}{2}$ particle with two other particles, for example via a Yukawa coupling. Since there are no dimensionless couplings in pure supergravity, it follows from [89] that the amplitude has to be $O(\epsilon^0)$ or softer. This leaves only one option, namely that $\sigma_{1/2} = 0$ in pure supergravity.

In pure $\mathcal{N} \geq 4$ supergravity, the scalars in the supermultiplet can have $\sigma_0 = 0$ or $\sigma_0 = 1$. If we focus on the MHV sector, the supersymmetry Ward identities give

$$\mathcal{A}_n(1_Z\, 2_{\bar{Z}}\, 3_h^-\, 4_h^+ \dots n_h^+) = \frac{\langle 13\rangle^4}{\langle 23\rangle^4}\, \mathcal{A}_n(1_h^+\, 2_h^-\, 3_h^-\, 4_h^+ \dots n_h^+)\,, \tag{3.63}$$

where $Z$ and $\bar{Z}$ denote any pair of conjugate scalars and $h$ are gravitons. Taking line 1 soft holomorphically, $|1\rangle \to \epsilon|1\rangle$, the graviton amplitude on the RHS diverges as $1/\epsilon^3$ but the prefactor vanishes as $\epsilon^4$. It follows that the MHV amplitude vanishes as $O(\epsilon)$ in the single soft-scalar limit. In other words, for MHV amplitudes $\sigma_0 = 1$. It is tempting to conclude that one must have $\sigma_0 = 1$ for all amplitudes, but that is too glib, as we now explain.

It is known that the scalar cosets of $\mathcal{N} \geq 4$ pure supergravity theories in 4d are symmetric, and therefore lead to $\sigma_0 = 1$ vanishing low-energy theorems. But at the level of the on-shell amplitudes, this conclusion does not follow from the supersymmetry Ward identities alone: as we have seen, they give $\sigma_0 = 1$ or $\sigma_0 = 0$. That analysis has to remain true at all loop-orders. In $\mathcal{N} = 4$ supergravity, for example, the anomaly of the $U(1)$ R-symmetry can be expected to affect the soft behavior at some order. Our arguments show that it cannot happen in the MHV sector, but does not rule it out beyond MHV; this is what the $\sigma_0 = 0$ accounts for. Furthermore, one can add higher-derivative operators to the supergravity action such that supersymmetry is preserved but the low-energy theorems are not. Indeed, string theory does this in the $\alpha'$-expansion by adding to the $\mathcal{N}=8$ tree-level action a supersymmetrizable operator $\alpha'^3 e^{-6\phi} R^4$. This operator does not affect the soft behavior of MHV amplitudes, but it is known that it does result in non-vanishing single soft scalar limits for 6-particle NMHV amplitudes at order $\alpha'^3$ [103, 104].

The results for $\mathcal{N}=8$ supersymmetry are summarized in Table 3.5. Note that the soft weights in this table follow from taking the soft limit holomorphically, $|i\rangle \to \epsilon|i\rangle$ for all states, independently of the sign of their helicity. At each step in the spectrum, the soft weight either changes by 1 or not at all. Note that one could also have used the anti-holomorphic definition $|i] \to \epsilon|i]$ of taking the soft limit; in that case the soft weights would just have reversed, to start with $\sigma = -3$ for the negative helicity graviton, but no new constraints would have been obtained on the scalar soft weights. In $\mathcal{N}=8$ supergravity, the 70 scalars are Goldstone bosons of the coset $E_{7(7)}/SU(8)$ and hence $\sigma = 1$. Including higher-derivative corrections may change this behavior to $\sigma = 0$ depending on whether the added terms are compatible with the coset structure.

### 3.5.5 MHV Classification and Examples of Supersymmetry Ward Identities

For later convenience, we state here the explicit form of the supersymmetry Ward identities (3.51) for a few particularly useful cases. We focus on the chiral multiplet, but similar results apply to the vector multiplet.

First we make the simple observation that amplitudes with all $Z$'s or only one $\bar{Z}$ and rest $Z$'s vanish:

$$\mathcal{A}_n\big(1_Z\, 2_Z\, 3_Z\, 4_Z \ldots n_Z\big) = 0 \quad \text{and} \quad \mathcal{A}_n\big(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_Z \ldots\big) = 0\,. \tag{3.64}$$

This follows from the supersymmetry Ward identities such as

$$0 = \mathcal{Q} \cdot \mathcal{A}_n\big(1^+_\psi\ 2_Z\ 3_Z\ 4_Z \ldots n_Z\big) = |1]\,\mathcal{A}_n\big(1_Z\ 2_Z\ 3_Z\ 4_Z \ldots n_Z\big)\,,$$
$$0 = \mathcal{Q} \cdot \mathcal{A}_n\big(1^+_\psi\ 2_{\bar{Z}}\ 3_Z\ 4_Z \ldots n_Z\big) = |1]\,\mathcal{A}_n\big(1_Z\ 2_{\bar{Z}}\ 3_Z\ 4_Z \ldots n_Z\big) - |2]\,\mathcal{A}_n\big(1^+_\psi\ 2^-_\psi\ 3_Z\ 4_Z \ldots n_Z\big)\,.$$

Dotting in $[2|$ gives (3.64). Similarly $\mathcal{A}_n\big(1^+_\psi\ 2^-_\psi\ 3_Z\ 4_Z \ldots n_Z\big) = 0$ and so on. In the context of gluon scattering, the equivalent statements are that amplitudes with helicity structure $+ + + \ldots +$ or $- + + \ldots +$ vanish. These helicity configurations are often called "helicity violating".

The simplest non-vanishing amplitudes are often denoted MHV (Maximally Helicity Violating) in the context of gluon scattering and we adapt the same nomenclature here. MHV amplitudes obey the simplest supersymmetry Ward identities in that they are just linear proportionality relations. For example, it follows from

$$\begin{aligned} 0 &= \mathcal{Q} \cdot \mathcal{A}_n\big(1^+_\psi\ 2_{\bar{Z}}\ 3_Z\ 4_{\bar{Z}}\ 5_Z \ldots n_Z\big) \\ &= |1]\,\mathcal{A}_n\big(1_Z\ 2_{\bar{Z}}\ 3_Z\ 4_{\bar{Z}} \ldots\big) - |2]\,\mathcal{A}_n\big(1^+_\psi\ 2^-_\psi\ 3_Z\ 4_{\bar{Z}} \ldots\big) - |4]\,\mathcal{A}_n\big(1^+_\psi\ 2_{\bar{Z}}\ 3_Z\ 4^-_\psi \ldots\big) \end{aligned} \tag{3.65}$$

upon dotting in $[4|$ that

$$\mathcal{A}_n(1^+_\psi\ 2^-_\psi\ 3_Z\ 4_{\bar{Z}}\ 5_Z \ldots n_Z) = \frac{[14]}{[24]}\,\mathcal{A}_n(1_Z\ 2_{\bar{Z}}\ 3_Z\ 4_{\bar{Z}}\ 5_Z \ldots n_Z). \tag{3.66}$$

Similarly, one finds that the MHV amplitude with four fermions is proportional to the one with two fermions. To summarize, MHV amplitudes satisfy

$$\begin{aligned} \mathcal{A}_n(1^+_\psi\ 2^-_\psi\ 3^+_\psi\ 4^-_\psi\ 5_Z \ldots n_Z) &= \frac{[13]}{[14]}\,\mathcal{A}_n(1^+_\psi\ 2^-_\psi\ 3_Z\ 4_{\bar{Z}}\ 5_Z \ldots n_Z) \\ &= \frac{[13]}{[24]}\,\mathcal{A}_n(1_Z\ 2_{\bar{Z}}\ 3_Z\ 4_{\bar{Z}}\ 5_Z \ldots n_Z)\,. \end{aligned} \tag{3.67}$$

The second-simplest class of supersymmetric Ward identities relate amplitudes in the NMHV class. In this paper, the 6-particle amplitudes play a central role, so we write down the 6-

point NMHV supersymmetry Ward identities explicitly:

$$\begin{aligned}&|1]\mathcal{A}_6(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}\, 5_Z\, 6_{\bar{Z}}) - |2]\mathcal{A}_6(1^+_\psi\, 2^-_\psi\, 3_Z\, 4_{\bar{Z}}\, 5_Z\, 6_{\bar{Z}}) \\ &\qquad - |4]\mathcal{A}_6(1^+_\psi\, 2_{\bar{Z}}\, 3_Z\, 4^-_\psi\, 5_Z\, 6_{\bar{Z}}) - |6]\mathcal{A}_6(1^+_\psi\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}\, 5_Z\, 6^-_\psi) = 0\,,\end{aligned} \tag{3.68}$$

$$\begin{aligned}&|1]\mathcal{A}_6(1_Z\, 2^-_\psi\, 3^+_\psi\, 4_{\bar{Z}}\, 5_Z\, 6_{\bar{Z}}) + |3]\mathcal{A}_6(1^+_\psi\, 2^-_\psi\, 3_Z\, 4_{\bar{Z}}\, 5_Z\, 6_{\bar{Z}}) \\ &\qquad - |4]\mathcal{A}_6(1^+_\psi\, 2^-_\psi\, 3^+_\psi\, 4^-_\psi\, 5_Z\, 6_{\bar{Z}}) - |6]\mathcal{A}_6(1^+_\psi\, 2^-_\psi\, 3^+_\psi\, 4_{\bar{Z}}\, 5_Z\, 6^-_\psi) = 0\,,\end{aligned} \tag{3.69}$$

$$\begin{aligned}&|1]\mathcal{A}_6(1_Z\, 2^-_\psi\, 3^+_\psi\, 4^-_\psi\, 5^+_\psi\, 6_{\bar{Z}}) + |3]\mathcal{A}_6(1^+_\psi\, 2^-_\psi\, 3_Z\, 4^-_\psi\, 5^+_\psi\, 6_{\bar{Z}}) \\ &\qquad + |5]\mathcal{A}_6(1^+_\psi\, 2^-_\psi\, 3^+_\psi\, 4^-_\psi\, 5_Z\, 6_{\bar{Z}}) - |6]\mathcal{A}_6(1^+_\psi\, 2^-_\psi\, 3^+_\psi\, 4^-_\psi\, 5^+_\psi\, 6^-_\psi) = 0\,.\end{aligned} \tag{3.70}$$

We now turn to applications of these results.

## 3.6 Supersymmetric Non-linear Sigma Model

Perhaps the simplest and most familiar class of models that exhibit both linearly realized supersymmetry and interesting low-energy theorems are the *supersymmetric non-linear sigma models*. Of particular interest are the *coset* sigma models for which the target manifold is a homogeneous space $G/H$. At lowest order, the coset sigma model captures the universal low-energy behavior of the scalar Goldstone modes of a spontaneous symmetry breaking pattern $G \to H$, where $G$ and $H$ are the *isometry* and *isotropy* groups of the target manifold respectively. If the target manifold is additionally a *symmetric* space and there are no 3-point interactions, then the off-shell Ward-Takahashi identities for the spontaneously broken currents imply $\sigma = 1$ vanishing low-energy theorems for the Goldstone scalars. An interesting recent perspective on coset sigma models can be found in [105].

At leading order it is fairly straightforward to calculate the on-shell scattering amplitudes for such a model from the (two-derivative) non-linear sigma model effective action. Using the methods of on-shell recursion, the use of an effective action is unnecessary. Instead, we may assume low-energy theorems and on-shell Ward identities of the isotropy group $H$ as the on-shell data that defines the model. Using the procedure of the soft bootstrap described in Section 3.3.5, we may apply subtracted recursion to construct the contributions to the S-matrix at leading order.

A particularly simple and well-studied example of such a construction has previously been given for the $\frac{U(N)\times U(N)}{U(N)}$ coset sigma model [35, 83]. There are several nice features of this model which make it an appealing toy-model to study on-shell. As will be discussed in Section 3.8.4, at leading order ($\tilde{\Delta} = 1$ or equivalently two-derivative) the isotropy $U(N)$ symmetry allows for the construction of *flavor*-ordered partial amplitudes with only $(n-3)!$ independent amplitudes for the scattering of $n$ Goldstone scalars.

The situation is somewhat less straightforward for models describing the low-energy dynamics of the Goldstone modes of internal symmetry breaking with some amount of linearly realized supersymmetry.[16] There are several interesting consequences of this combination of symmetries. The states must form mass degenerate multiplets of the supersymmetry algebra, which in this case means that the Goldstone scalars must always transform together with additional massless spinning states. As discussed in Section 3.5.2, the low-energy theorems of each of the particles in these *Goldstone* multiplets are not independent.

It is well-known in the literature of supersymmetric field theories that to construct a supersymmetric action, the massless scalar modes must parametrize a target space manifold with *Kähler* structure for $\mathcal{N} = 1$ supersymmetry [106]. For $\mathcal{N} = 2$ supersymmetry the target space manifold must have the structure

$$\mathcal{M}_{\mathcal{N}=2} = \mathcal{M}_{\text{V}} \times \mathcal{M}_{\text{H}}, \tag{3.71}$$

where the scalars of the *vector* multiplets parametrize the *special-Kähler* manifold $\mathcal{M}_{\text{V}}$ while the scalars belonging to *hyper* multiplets parametrize the *hyper-Kähler* manifold $\mathcal{M}_{\text{H}}$ [107]. As a consequence, despite the obvious virtues of a flavor ordered representation, this makes

[16] In this more general context *internal* symmetry includes R-symmetry. For our purposes the relevant property is that the conserved charges are Lorentz scalars and so correspond to a spectrum of spin-0 Goldstone modes.

studying the supersymmetrization of the $\frac{U(N)\times U(N)}{U(N)}$ coset sigma model using subtracted recursion more difficult, since even in the $\mathcal{N}=1$ case the target manifold is not Kähler. This does not mean that the internal symmetry breaking pattern $U(N)\times U(N)\to U(N)$ is impossible in an $\mathcal{N}=1$ supersymmetric model. Rather it means that the target space contains $\frac{U(N)\times U(N)}{U(N)}$ as a non-Kähler submanifold and includes additional directions in field space or equivalently includes additional massless *quasi-Goldstone* scalars [108]. In general there is no unique way to extend the symmetry breaking coset to a Kähler manifold, because in any given example the spectrum of quasi-Goldstone modes depends on the details of the UV physics. Correspondingly, the quasi-Goldstone scalars do not satisfy the kind of universal low-energy theorems necessary for us to construct the scattering amplitudes recursively.

Instead, in this section we will study the interplay of low-energy theorems and supersymmetry by considering the simplest symmetric coset that is both Kähler and special-Kähler

$$\frac{SU(2)}{U(1)}\cong\mathbb{CP}^1\,, \tag{3.72}$$

and therefore should admit both an $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetrization. Our assumption here is that the target manifold *is* the coset manifold and therefore the massless spectrum should contain only two real scalar degrees of freedom, both Goldstone modes. They form a single complex scalar field $Z,\overline{Z}$ which carries a conserved charge associated with the isotropy $U(1)$. These properties uniquely determine the Goldstone multiplets as an $\mathcal{N}=1$ chiral and $\mathcal{N}=2$ vector multiplet respectively.

The main results of this section are (1) the demonstration that both the $\mathcal{N}=1$ and $\mathcal{N}=2$ $\mathbb{CP}^1$ non-linear sigma models are constructible on-shell using recursion without the need to explicitly construct an effective action. And (2) this construction gives a new *on-shell* perspective on the relationship between the linearly realized target space isotropies of $\mathcal{M}_{\mathrm{V}}$ and electric-magnetic duality transformations of the associated vector bosons.

### 3.6.1 $\mathcal{N} = 1$ $\mathbb{CP}^1$ NLSM

The $\mathcal{N} = 1$ $\mathbb{CP}^1$ non-linear sigma model is defined by the following on-shell data:

- A spectrum consisting of a massless $\mathcal{N} = 1$ chiral multiplet $(Z, \bar{Z}, \psi^+, \psi^-)$.
- Scattering amplitudes satisfy $\mathcal{N} = 1$ supersymmetry Ward identities.
- Scattering amplitudes satisfy isotropy $U(1)$ Ward identities under which $Z, \bar{Z}$ are charged.
- $\sigma_Z = \sigma_{\bar{Z}} = 1$ soft weight for the scalars.

Using the approach of the soft bootstrap, we begin by constructing the most general on-shell amplitudes at lowest valence that are consistent with the above data and minimize $\tilde{\Delta}$. There are no possible 3-point amplitudes consistent with the assumptions and so we must begin at 4-point. A $|Z|^4$ interaction, corresponding to $\tilde{\Delta} = 0$, is consistent with $U(1)$ conservation but violates the assumed low-energy theorem. The next-to-lowest reduced dimension interactions correspond to $\tilde{\Delta} = 1$ and have a unique 4-point amplitude consistent with the assumptions

$$\mathcal{A}_4(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}) = \frac{1}{\Lambda^2} s_{13}. \tag{3.73}$$

Note that at 4-point, the conservation of the $U(1)$-charge for the complex scalar is automatically enforced as a consequence of the supersymmetry Ward identitites. We will see that this implies the conservation of the $U(1)$ charge for amplitudes with arbitrary number of external particles corresponding to $\tilde{\Delta} = 1$. Note that this is not automatic for higher order ($\tilde{\Delta} > 1$) corrections and must be imposed as a separate constraint. Using (3.67) the remaining 4-point amplitudes are completely determined by supersymmetry; it is convenient to summarize the component amplitudes in a single *superamplitude* [109]

$$\mathcal{A}_4(1_{\Phi^+} 2_{\Phi^-} 3_{\Phi^+} 4_{\Phi^-}) = \frac{1}{\Lambda^2}[13]\delta^{(2)}(\tilde{Q}) = \frac{1}{2\Lambda^2}[13] \sum_{i,j=1}^{4} \langle ij \rangle \eta_i \eta_j\, . \tag{3.74}$$

Here we have introduced two chiral superfields $\Phi^+$ and $\Phi^-$ that contain the positive and negative helicity fields of the $\mathcal{N}=1$ chiral multiplet as

$$\Phi^+ = \psi^+ + \eta Z \ , \qquad \Phi^- = \bar{Z} - \eta\,\psi^- . \tag{3.75}$$

$\eta$ is the Grassmann coordinate of $\mathcal{N}=1$ on-shell superspace and $\eta_i$ denotes the $\eta$-coordinate of the $i^{\text{th}}$ superfield. We can obtain all the component amplitudes by projecting out components of the superfield. For example, the all-fermion amplitude can be derived as follows

$$\mathcal{A}_4(1^+_\psi 2^-_\psi 3^+_\psi 4^-_\psi) = \frac{\partial}{\partial\eta_2}\frac{\partial}{\partial\eta_4}\mathcal{A}_4(1_{\Phi^+}2_{\Phi^-}3_{\Phi^+}4_{\Phi^-}) = -\frac{1}{\Lambda^2}[13]\langle 24\rangle . \tag{3.76}$$

It is useful to note that the expression (3.74) is manifestly local. It follows that all component amplitudes are free of factorization singularities, indicating the absence of 3-point interactions in this theory. Note also that the pure fermion sector is exactly the NJL model detected by the soft bootstrap in Section 3.4.2.

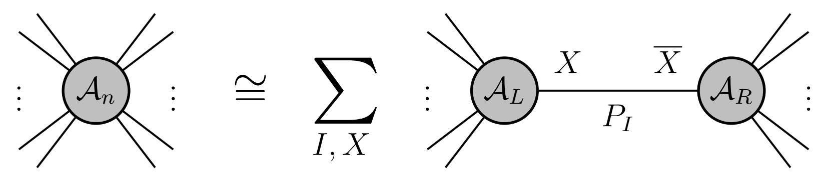
Next, we use these 4-point amplitudes to recursively construct $n$-point amplitudes. Following the discussion in Section 3.5, we note that the soft weight of the fermion must be either $\sigma_\psi = 0$ or $\sigma_\psi = 1$. Making the conservative choice $\sigma_\psi = 0$, we evaluate the constructibility criterion on the above on-shell data,

$$4 < 2n_s + n_f, \tag{3.77}$$

where $n_f$ is the number of external fermion states of the $n$-point amplitude and $n_s = n - n_f$ is the number of external scalar states. For $n > 4$, this condition is satisfied for all $n$-point amplitudes. We find that recursively constructing the 6-point amplitudes yields an $a_i$-independent expression. All the 6-point amplitudes can be found in Appendix E.1. Since our input 4-point amplitudes are MHV, the only non-zero constructible amplitudes at 6-point are NMHV and can be verified to satisfy the NMHV 6-point Ward identities (3.68), (3.69), (3.70).

If however we make the stronger assumption $\sigma_\psi = 1$, the recursively constructed 6-point amplitude is $a_i$-dependent and therefore fails the consistency checks. As a result we conclude that the true soft weight of the fermion of our theory is $\sigma_\psi = 0$ and this is sufficient to construct the S-matrix at leading order from the 4-point seed amplitudes (3.74).

The recursive constructibility of the S-matrix has non-trivial consequences for the possible conserved additive quantum numbers. In a recursive model the only non-zero amplitudes are those which can be constructed by gluing together lower-point on-shell amplitudes




where the states $X, \bar{X}$ on either side of the factorization channel $I$ have CP conjugate quantum numbers. As discussed further in Appendix F, if an additive quantum number is conserved by all seed amplitudes then it must be conserved by all recursively constructible amplitudes.

For example, in the present context the seed amplitudes conserve two *independent* $U(1)$ charges:

| | $U(1)_A$ | $U(1)_B$ |
|---|---|---|
| $Z$ | $q_A$ | $0$ |
| $\bar{Z}$ | $-q_A$ | $0$ |
| $\psi^+$ | $0$ | $q_B$ |
| $\psi^-$ | $0$ | $-q_B$ |
| $\eta$ | $-q_A$ | $q_B$ |
| $\Phi^+$ | $0$ | $q_B$ |
| $\Phi^-$ | $-q_A$ | $0$ |

We know to expect the existence of an isotropy $U(1)$ under which the scalars are charged, but from our on-shell construction it is unclear whether this should be $U(1)_A$ or a combination of $U(1)_A$ and $U(1)_B$. We have presented the charges as two independent R-symmetries but more correctly we should consider them as a single *global* $U(1)$ and a $U(1)_R$. The presence of a second conserved quantum number is not part of the definition of the $\mathbb{CP}^1$ non-linear sigma model but is instead an *emergent* or *accidental* symmetry at lowest order in the EFT. In general one would expect $U(1)_A \times U(1)_B$ to be explicitly broken to the isotropy $U(1)$ by higher dimension operators.

### 3.6.2 $\mathcal{N} = 2$ $\mathbb{CP}^1$ NLSM

The $\mathcal{N} = 2$ $\mathbb{CP}^1$ NLSM is defined by the following on-shell data:

- A spectrum consisting of a massless $\mathcal{N} = 2$ vector multiplet $(Z, \bar{Z}, \psi^{a+}, \psi_a^-, \gamma^+, \gamma^-)$, where $a = 1, 2$.

- Scattering amplitudes satisfy $\mathcal{N} = 2$ supersymmetry Ward identities.

- Scattering amplitudes satisfy isotropy $U(1)$ Ward identities under which $Z, \bar{Z}$ are charged.

Note that, importantly, we do *not* impose the the soft weight of the scalars $\sigma_Z = \sigma_{\bar{Z}} = 1$. As we will explain further below, no model with the above properties *and* vanishing scalar soft limits exists.

To proceed, interactions with reduced dimension $\tilde{\Delta} = 0$ (such as Yukawa interactions) are incompatible with $\mathcal{N} = 2$ supersymmetry for a single vector multiplet. Thus, the minimal value is $\tilde{\Delta} = 1$; that is of course also the value for the $\mathcal{N} = 1$ model. It is curious to note that $\mathcal{N} = 2$ supersymmetry is sufficient to uniquely construct the S-matrix at this order in $\tilde{\Delta}$. As we show in the following, *without* assuming vanishing scalar soft limits, the restriction of the external states to a single chiral multiplet $(Z, \bar{Z}, \psi^{1+}, \psi_1^-)$ reproduces the $\mathcal{N} = 1$ $\mathbb{CP}^1$ sigma model.

As in the previous section, for $\tilde{\Delta} = 1$ the 4-point scalar amplitude takes the form (3.73). All 4-point component amplitudes are uniquely fixed by the 4-scalar amplitudes by the $\mathcal{N} = 2$

supersymmetry Ward identities and they can be encoded compactly into superamplitudes using two chiral superfields [109]

$$\begin{aligned}\Phi^+ &= \gamma^+ + \eta_1\psi^{1+} + \eta_2\psi^{2+} - \eta_1\eta_2 Z,\\ \Phi^- &= \bar{Z} + \eta_1\psi_2^- - \eta_2\psi_1^- - \eta_1\eta_2\gamma^- \,.\end{aligned} \tag{3.78}$$

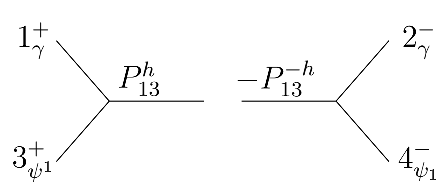
Here $\eta_1$ and $\eta_2$ are the Grassmann coordinates of $\mathcal{N}=2$ on-shell superspace. The $R$-indices on $\psi^a$ are raised and lowered using $\epsilon_{ab}$, so $\psi_2^- = \epsilon_{21}\psi^{1-} = \psi^{1-}$ and $\psi_1^- = \epsilon_{12}\psi^{2-} = -\psi^{2-}$. In terms of the superfields, the 4-point superamplitude can be expressed as

$$\mathcal{A}_4(1_{\Phi^+}2_{\Phi^-}3_{\Phi^+}4_{\Phi^-}) = \frac{1}{\Lambda^2}\frac{[13]}{\langle 13\rangle}\delta^{(4)}(\tilde{Q}) = \frac{1}{4\Lambda^2}\frac{[13]}{\langle 13\rangle}\prod_{a=1}^{2}\sum_{i,j=1}^{4}\langle ij\rangle\eta_{ia}\eta_{ja}. \tag{3.79}$$

We use $\eta_{ia}$ to denote the $a^{\text{th}}$ Grassmann coordinate of the $i^{\text{th}}$ external superfield. In contrast to (3.74), the superamplitude (3.79) generates component amplitudes that are not local due to the factorization singularity at $P_{13}^2 \to 0$. For example, consider the following component amplitude

$$\mathcal{A}_4(1_\gamma^+ 2_\gamma^- 3_{\psi^1}^+ 4_{\psi_1}^-) = -\frac{\partial}{\partial\eta_{21}}\frac{\partial}{\partial\eta_{22}}\frac{\partial}{\partial\eta_{31}}\frac{\partial}{\partial\eta_{42}}\mathcal{A}_4(1_{\Phi^+}2_{\Phi^-}3_{\Phi^+}4_{\Phi^-}) = -\frac{1}{\Lambda^2}\frac{[13][14]\langle 24\rangle}{[24]}. \tag{3.80}$$

Locality and unitarity imply that this 4-point amplitude must factorize into 3-point amplitudes on the singularity at $P_{13}^2 \to 0$. Denoting the helicity of the exchanged particle $h$, the amplitude factorizes as



The contribution to the residue on the singularity takes the form

$$\begin{aligned}
P_{13}^2 \mathcal{A}_4(1_\gamma^+ 2_\gamma^- 3_{\psi^1}^+ 4_{\psi_1}^-)\Big|_{P_{13}^2=0} &= \mathcal{A}_3\big(1_\gamma^+ 3_{\psi^1}^+ (P_{13})_h\big)\mathcal{A}_3\big((-P_{13})_{-h} 2_\gamma^- 4_{\psi_1}^-\big) \\
&= \left(\frac{g_1}{\Lambda}[13]^{3/2-h}[1P_{13}]^{1/2+h}[3P_{13}]^{-1/2+h}\right)\left(\frac{g_2}{\Lambda}\langle 24\rangle^{3/2-h}\langle 2P_{13}\rangle^{1/2+h}\langle 4P_{13}\rangle^{-1/2+h}\right) \\
&= \frac{g_1 g_2}{\Lambda^2}(-1)^{2h}[13]^{3/2-h}\langle 24\rangle^{3/2+h}[23]^{-1/2+h}[14]^{1/2+h}\,,
\end{aligned} \tag{3.81}$$

with the 3-point amplitudes completely determined by Poincaré invariance and little group scaling. Comparing with the explicit form of the residue calculated from (3.80)

$$P_{13}^2 \mathcal{A}_4(1_\gamma^+ 2_\gamma^- 3_{\psi^1}^+ 4_{\psi_1}^-)\Big|_{P_{13}^2=0} = \frac{1}{\Lambda^2}[13][14]\langle 24\rangle^2, \tag{3.82}$$

we find that $h = 1/2$ and $g_1 g_2 = -1$. The exchanged particle of helicity $h = 1/2$ can be either $\psi^{1+}$ or $\psi^{2+}$. The locality of the $\mathcal{A}_4(1_{\psi^1}^+ 2_{\psi_1}^- 3_{\psi^1}^+ 4_{\psi_1}^-)$ and $\mathcal{A}_4(1_{\psi^2}^+ 2_{\psi_2}^- 3_{\psi^2}^+ 4_{\psi_2}^-)$ tells us that they do not factorize on the $(P_{13})^2 \to 0$ pole. We conclude that $\mathcal{A}_3(1_\gamma^+ 2_{\psi_1}^+ 3_{\psi_1}^+) = \mathcal{A}_3(1_\gamma^+ 2_{\psi_2}^+ 3_{\psi_2}^+) = 0$, while

$$\mathcal{A}_3(1_\gamma^+ 2_{\psi_1}^+ 3_{\psi_2}^+) = \frac{g_1}{\Lambda}[12][13]\,, \qquad \mathcal{A}_3(1_\gamma^- 2_{\psi_1}^- 3_{\psi_2}^-) = \frac{g_2}{\Lambda}\langle 12\rangle\langle 13\rangle\,. \tag{3.83}$$

We carry out a similar exercise with $\mathcal{A}_4(1_\gamma^+ 2_\gamma^- 3_\gamma^+ 4_\gamma^-)$ for a particle of helicity $h$ in the $P_{13}^2 \to 0$ factorization channel. Comparing with the 4-point amplitude (3.79) fixes $h = 0$. This could correspond to either $Z$ or $\bar{Z}$ exchange. The absence of a $P_{14}^2 \to 0$ pole in $\mathcal{A}_4(1_\gamma^+ 2_\gamma^- 3_Z 4_{\bar{Z}})$ shows that $\mathcal{A}_3(1_\gamma^+ 2_\gamma^+ 3_{\bar{Z}}) = 0$ and

$$\mathcal{A}_3(1_\gamma^+ 2_\gamma^+ 3_Z) = \frac{g_3}{\Lambda}[12]^2\,, \qquad \mathcal{A}_3(1_\gamma^- 2_\gamma^- 3_{\bar{Z}}) = \frac{g_4}{\Lambda}\langle 12\rangle^2\,, \tag{3.84}$$

where $g_3 g_4 = 1$. Demanding that all non-local 4-point amplitudes factorize correctly fixes

$-g_1 = g_2 = g_3 = g_4 = -1$. The 3-point superamplitudes are

$$\begin{aligned}
\mathcal{A}_3(1_{\Phi^-}2_{\Phi^-}3_{\Phi^-}) &= \delta^{(4)}(\tilde{Q}) = \frac{1}{4\Lambda}\prod_{a=1}^{2}\sum_{i,j=1}^{3}\langle ij\rangle\eta_{ia}\eta_{ja}\,, \\
\mathcal{A}_3(1_{\Phi^+}2_{\Phi^+}3_{\Phi^+}) &= \frac{1}{\Lambda}\delta^{(2)}(\eta_1[23]+\eta_2[31]+\eta_3[12]) = \frac{1}{\Lambda}\prod_{a=1}^{2}(\eta_{1a}[23]+\eta_{2a}[31]+\eta_{3a}[12])\,,
\end{aligned} \tag{3.85}$$

where $\prod_{a=1}^{2} f_a$ is defined as $f_1 f_2$. It is interesting to observe that even though the $\mathcal{N}=0,1$ and 2 $\mathbb{CP}^1$ NLSM have the pure scalar 4-point amplitude in common, in the latter case the extended supersymmetry together with locality require the presence 3-point interactions.

We are now in a position to address the constructibility of general $n$-point amplitudes. Since we are not assuming vanishing soft limits as part of our on-shell data, we are not able to make use of subtracted recursion. This is only problematic for a subset of the amplitudes in this model, at least at leading order. The unsubtracted constructibility criterion for this model reads

$$4 < n_f + 2n_v, \tag{3.86}$$

where $n_f$ and $n_v$ are the number of fermions and vector bosons respectively. It turns out that the amplitudes that do not satisfy this criterion can be determined from the $\mathcal{N}=2$ supersymmetry Ward identities in terms of those that do; explicit formulae are given in Appendix G. Remarkably, without making any strong assumptions about the structure of low-energy theorems for the scalars, which usually characterize the sigma model coset structure, the $\mathcal{N}=2$ supersymmetry is sufficient at leading order to both construct the entire S-matrix and reproduce the amplitudes of the $\mathcal{N}=1$ and $\mathcal{N}=0$ models as special cases.

This same statement can be made in the perhaps more familiar language of local field theory. At this order in the EFT expansion, the S-matrix elements should be calculable from some effective action, the bosonic sector of which should be described by a two-derivative

Lagrangian of the general form

$$\mathcal{L}_{\text{eff}} = P\left(|Z|^2\right)|\partial_\mu Z|^2 + Q\left(|Z|^2\right) Z\, F_+^2 + \text{h.c.} \tag{3.87}$$

where $P(|Z|^2)$ and $Q(|Z|^2)$ are some functions analytic around $Z \sim 0$. Insisting that the S-matrix elements satisfy the on-shell $\mathcal{N}=2$ supersymmetry Ward identities is equivalent to requiring the existence of off-shell $\mathcal{N}=2$ supersymmetry transformations under which the effective action is invariant. The on-shell uniqueness result is equivalent to the statement that the off-shell $\mathcal{N}=2$ supersymmetry uniquely (up to field redefinitions) determines the form of the two-derivative effective action. In particular, the function $P(|Z|^2)$ is uniquely determined to be

$$P\left(|Z|^2\right) = \left(\frac{1}{1+|Z|^2}\right)^2, \tag{3.88}$$

corresponding to the Fubini-Study metric on $\mathbb{CP}^1$.

Since the entire S-matrix is determined, we can explicitly demonstrate how the presence of the vector bosons modifies the structure of the low-energy theorems from the naive vanishing soft limits suggested by the coset structure. Consider the following relation among 5-point amplitudes given by the $\mathcal{N}=2$ supersymmetry Ward identities

$$\mathcal{A}_5\left(1_\gamma^+, 2_\gamma^+, 3_Z, 4_Z, 5_{\bar{Z}}\right) = \frac{\langle 34\rangle^2}{\langle 45\rangle^2}\mathcal{A}_5\left(1_\gamma^+, 2_\gamma^+, 3_\gamma^+, 4_Z, 5_\gamma^-\right). \tag{3.89}$$

The amplitude on the right-hand-side satisfies (3.86) and therefore is constructible using unsubtracted recursion. This gives the non-constructible amplitude on the left-hand-side as

$$\mathcal{A}_5\left(1_\gamma^+, 2_\gamma^+, 3_Z, 4_Z, 5_{\bar{Z}}\right) = \frac{1}{\Lambda^3}\langle 34\rangle^2\left(\frac{[12][34]}{\langle 12\rangle\langle 34\rangle} + \frac{[23][14]}{\langle 23\rangle\langle 14\rangle} + \frac{[31][24]}{\langle 31\rangle\langle 24\rangle}\right). \tag{3.90}$$

The soft limits on particles 1, 2, 3 and 4 vanish, as expected. The soft limit on particle 5, however, is $\mathcal{O}(1)$, contrary to the expected soft behavior for a Goldstone mode of a symmetric coset. Explicitly

$$\mathcal{A}_5\left(1_\gamma^+, 2_\gamma^+, 3_Z, 4_Z, 5_{\bar{Z}}\right) \xrightarrow{|5]\to\epsilon|5]} \frac{1}{\Lambda^3}[12]^2 + \mathcal{O}(\epsilon). \tag{3.91}$$

It is interesting that the coupling to the photons, required by $\mathcal{N}=2$ supersymmetry, results in non-vanishing soft scalar limits for a theory with a symmetric coset. In principle, this amplitude could have had a contact contribution of the form $\propto [12]^2$, but our calculation shows that such a term would be incompatible with $\mathcal{N}=2$ supersymmetry.

The maximal $R$-symmetry group that this model can realize is $U(2)_R = U(1)_R \times SU(2)_R$. We will now verify that the $SU(2)_R$ symmetry Ward identities hold for the seed amplitudes, the $U(1)_R$ we will address separately. To do this we choose a basis for the generators of $SU(2)_R$. The scalars and vectors both transform as $SU(2)$ singlets. The positive helicity fermion species $\psi^{1,2+}$ will transform in the fundamental representation under

$$\mathcal{T}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{T}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{T}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.92}$$

The negative helicity fermions transform in the anti-fundamental with $\bar{\mathcal{T}}_i = -\mathcal{T}_i^\dagger$. This tells us that the $\mathcal{T}_0$-Ward identity is satisfied as long as the fermion species appear in pairs of (a) different helicity, same species or (b) same helicity, different species. This is true of all the non-zero amplitudes in this model. The action of $\mathcal{T}_+$ and $\mathcal{T}_-$ are

| state $i$ | $\mathcal{T}_+ \cdot i$ | $\mathcal{A}_n$ prefactor | $\mathcal{T}_- \cdot i$ | $\mathcal{A}_n$ prefactor | $\mathcal{T}_0 \cdot i$ | $\mathcal{A}_n$ prefactor |
|---|---|---|---|---|---|---|
| $\psi^{1+}$ | 0 | 0 | $\psi^{2+}$ | 1 | $\psi^{1+}$ | 1 |
| $\psi^{2+}$ | $\psi^{1+}$ | 1 | 0 | 0 | $\psi^{2+}$ | $-1$ |
| $\psi_1^-$ | $\psi_2^-$ | $-1$ | 0 | 0 | $\psi_1^-$ | $-1$ |
| $\psi_2^-$ | 0 | 0 | $\psi_1^-$ | $-1$ | $\psi_2^-$ | 1 |

(3.93)

We find that all 3-point and 4-point amplitudes in this model satisfy the $SU(2)_R$ Ward identities, for example

$$\begin{aligned}\mathcal{T}_- \cdot \mathcal{A}_4(1^+_{\psi^1}2^-_{\psi_2}3^+_{\psi^1}4^-_{\psi_1}) &= \mathcal{A}_4(1^+_{\psi^2}2^-_{\psi_2}3^+_{\psi^1}4^-_{\psi_1}) - \mathcal{A}_4(1^+_{\psi^1}2^-_{\psi_1}3^+_{\psi^1}4^-_{\psi_1}) + \mathcal{A}_4(1^+_{\psi^1}2^-_{\psi_2}3^+_{\psi^2}4^-_{\psi_1}) \\ &= -\frac{[13]}{[24]}(s+t+u) = 0\,.\end{aligned} \tag{3.94}$$

As discussed above, we conclude that at leading order the $SU(2)_R$ Ward identities are satisfied by all amplitudes in the $\mathcal{N}=2$ model.

Following the same approach as described for the $\mathcal{N}=1$ model, conservation laws satisfied by the seed amplitudes imply that the same quantities are conserved by all leading-order amplitudes if they are recursively constructible (see Appendix F). This result extends to non-Abelian symmetries, which in the on-shell language correspond to Ward identities for non-diagonal generators; this is shown for $SU(2)$ in Appendix F. The amplitudes that are not constructible using recursion are fixed by supersymmetry in terms of those that are. Therefore, they will also respect the conservation laws and non-Abelian symmetries of the seed amplitudes.

This model also conserves a separate $U(1)_R$ charge. We know to expect the conservation of the charge associated with the $U(1)$ isotropy group. In the $\mathcal{N}=1$ case we found that the scattering amplitudes conserve an R-charge $U(1)_A$ assigned only to the complex scalar but it was consistent with the existence of $U(1)_B$ that the isotropy $U(1)$ might also assign a charge to the fermion or even to assign equal charges in the form of a global symmetry. In the present context we also have two independent $U(1)$ symmetries. The first is the $U(1) \subset SU(2)_R$ which assigns opposite charges to the fermions $\psi^{1+}$ and $\psi^{2+}$. The second assigns charges to each of the states which, up to overall normalization can be deduced from the 3- and 4-point seed amplitudes and are summarized in the following table:

| | $U(1)_R$ | $SU(2)_R$ |
|---|---|---|
| $Z$ | $-4$ | $\mathbf{1}$ |
| $\bar{Z}$ | $4$ | $\mathbf{1}$ |
| $\psi^{a+}$ | $-1$ | $\mathbf{2}$ |
| $\psi_a^-$ | $1$ | $\mathbf{2}$ |
| $\gamma^+$ | $2$ | $\mathbf{1}$ |
| $\gamma^-$ | $-2$ | $\mathbf{1}$ |
| $\eta_a$ | $3$ | $\mathbf{2}$ |
| $\Phi^+$ | $2$ | $\mathbf{1}$ |
| $\Phi^-$ | $4$ | $\mathbf{1}$ |

These are the *only* linear symmetries compatible with the seed amplitudes. The isotropy $U(1)$ must therefore be identified with some linear combination of $U(1)_R$ and $U(1) \subset SU(2)_R$. This is perhaps surprising, it tells us that the massless vector boson must also be charged under the isotropy $U(1)$. Just as for the fermions, the vector charges are *chiral* meaning that the positive and negative helicity states have opposite charges. Such charges for vectors are associated with electric-magnetic duality symmetries.

Such an extra $U(1)_R$ symmetry is possible because the maximal outer-automorphism group of the $\mathcal{N} = 2$ supersymmetry algebra is $U(2)_R$. The assignment of the associated charges is, up to normalization, fixed by the charge of the highest helicity state in the multiplet. It is interesting to observe that in the present context, knowledge of the non-vanishing 4-point amplitudes is *insufficient* to determine the $U(1)_R$ charge assignments. It is only from considering the 3-point amplitudes that we find the assignment of a non-zero chiral charge for the vector bosons unavoidable. Consider for example the amplitudes (3.84). Since the scalar is required to be charged under the isotropy $U(1)$, which in this case must be the $U(1)_R$ since there are no other symmetries under which the scalar is charged, we see that the vector must also be charged and satisfy $2q[\gamma^+] = -q[Z]$. The existence of fundamental 3-point interactions in this model was deduced by demanding that the singularities of the

4-point amplitudes be identified with physical factorization channels. From an on-shell point of view, it is therefore an *unavoidable* consequence of locality, unitarity and supersymmetry that the $\mathcal{M}_V$ isotropy group of an $\mathcal{N}=2$ non-linear sigma model acts on the vector bosons as an electric-magnetic duality transformation.

The necessary existence of the fundamental 3-point amplitudes (3.83) and (3.84) has a further interesting consequence for the low-energy behavior of the vector boson. In [89] it was shown that *singular* low-energy theorems arise from the presence of certain 3-point amplitudes. In the notation used in [89] the 3-point amplitudes (3.83) and (3.84) are classified as $\mathtt{a}=1$ in the soft limit of a positive helicity vector boson. Therefore a vector boson present in amplitudes which contain at least one of the following other particles: $Z$, $\psi^{a+}$ or $\gamma^+$ has soft weight $\sigma_\gamma=-1$. Using the general formalism developed in [89], we can write down the low-energy theorem of the vector bosons in this subclass of amplitudes

$$\mathcal{A}_{n+1}\left(s_\gamma^+,1,2,...,n\right) \xrightarrow{p_s\to\epsilon p_s \text{ as } \epsilon\to 0} \sum_{k=1}^{n}\frac{[sk]}{\epsilon\langle sk\rangle}\mathcal{A}_n\left(1,2,...,\mathcal{F}_+\cdot k,...,n\right)+\mathcal{O}\left(\epsilon^0\right). \tag{3.95}$$

Here we are using a notation similar to [110] with the introduction of an operator $\mathcal{F}_+$ which acts on the one-particle states as

| state $i$ | $\mathcal{F}_+\cdot i$ | $\mathcal{A}_n$ prefactor |
|---|---|---|
| $Z$ | $\gamma^-$ | 1 |
| $\psi^{1+}$ | $\psi_2^-$ | $-1$ |
| $\psi^{2+}$ | $\psi_1^-$ | $-1$ |
| $\gamma^+$ | $\overline{Z}$ | $-1$ |

(3.96)

and annihilates the states of the negative helicity multiplet. A similar operator $\mathcal{F}_-$ can be defined for the soft limit of a negative helicity vector. Using equation (3.60) in conjunction with the soft behavior (3.95) of the $n+1$-point amplitude results in the following identity for the residual $n$-point amplitudes

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{L_i+P_i}\frac{[Xi][Yj]}{\langle Yj\rangle}\mathcal{A}_n\left(1,2,...,\mathcal{Q}_1\cdot i,...,\mathcal{F}_+\cdot j,...,n\right)=0\,, \tag{3.97}$$

where here $P_i = 0$ or 1 corresponds to the additional signs associated with the prefactors of both the supersymmetry Ward identities and the operator $\mathcal{F}_+$ given in Table 3.96. Note that the action of $\mathcal{Q}_1$ and $\mathcal{F}_+$ commute on all physical states, so there is no ambiguity when $i = j$ in the sums. Moreover, rearranging the order of the sums, it becomes clear that for each fixed $j$, the sum over $i$ expresses a supersymmetry Ward identity for the $n$-point amplitudes. As such, the identity (3.97) does not impose further constraints beyond supersymmetry.

## 3.7 Super Dirac-Born-Infeld and Super Born-Infeld

In the soft bootstrap analysis of Section 3.4, we encountered three theories with a fundamental quartic interaction whose couplings are of mass-dimension $-4$: DBI, Akulov-Volkov, and Born-Infeld. These EFTs can all be related by supersymmetry. We will discuss them in further detail in future work, so for now we simply note the following:

- The $\mathcal{N} = 1$ supersymmetric Dirac-Born-Infeld model has as its pure scalar sector the complex scalar DBI theory with $\sigma_Z = 2$ and as its pure fermion sector Akulov-Volkov theory with $\sigma_\psi = 1$. All amplitudes are constructible with soft subtracted recursion. We present the expressions for the 4- and 6-point amplitudes in Appendix E.2.

- The $\mathcal{N} = 1$ supersymmetric Born-Infeld model combines Akulov-Volkov theory with Born-Infeld theory with $\sigma_\gamma = 0$. All amplitudes are constructible with the soft subtracted recursion relations of Section 3.3, except the pure vector ones, but they are uniquely fixed by the supersymmetry Ward identities. The 4- and 6-point amplitudes are given in Appendix E.3.

- Extended supersymmetry binds BI, Akulov-Volkov, and DBI into one supersymmetric exceptional EFT. For the case with $\mathcal{N} = 4$ supersymmetry, the amplitudes can be constructed using the CHY approach [111].

## 3.8 Galileons

Galileons are scalar effective field theories that arise in a multitude of contexts and as a result can be defined in different ways. In 4d, Galileons are

1. Higher-derivative scalar field theories with second-order equations of motion and absence of Ostrogradski ghosts. These theories have three free parameters: the cubic, quartic and quintic interaction coupling constants. A field redefinition removes the cubic interaction in favor of a linear combination of the quartic and quintic. The scattering amplitudes are of course invariant under the field redefinition, so for the purpose of studying perturbative scattering amplitudes, we consider only the quartic and quintic Galileons.

2. The non-linear realization of the algebra $\mathfrak{Gal}(4,1)$ which is an İnönü-Wigner contraction of the ISO$(4,1)$ symmetry algebra [112]. Truncated to leading order in the reduced dimension $\tilde{\Delta}$, this gives an effective field theory of a real massless scalar $\phi$ with $\sigma = 2$ vanishing soft limits and coupling dimensions $[g_4] = -6$ and $[g_5] = -9$ for the quartic and quintic interactions respectively.

3. Subleading contributions to the low-energy effective action on a 3-brane embedded in a $5d$ Minkowski space. The leading contribution to this EFT is the DBI action and including the Galileon terms, the model is often called the DBI-Galileon. In the limit of infinite brane tension, the Galileons decouple from DBI. The non-$\mathbb{Z}_2$-symmetric cubic and quintic interactions arise from considering the effective action on an end-of-the-world brane.

4. Scalar effective field theories that arise from the massless decoupling limit of Fierz-Pauli-type massive gravity [94, 93] and from the decoupling limit of Proca theories.

It is not obvious if these definitions are equivalent. The equivalence between Definitions 2 and 3 is straightforward since ISO$(4,1)$ is the Poincaré symmetry of the 5d embedding space. In the brane picture of Definition 3, the DBI-Galileon scalar is a Goldstone boson that arises from the spontaneous breaking of translational symmetry transverse to the brane, with the contraction of the 5d Poincaré algebra equivalent to the non-relativistic limit of the fluctuations of the brane into the extra dimension [113].

In an approach based on scattering amplitudes, it is natural to use the second definition of Galileon theories, based on their soft weight $\sigma = 2$ and fundamental coupling dimension. This is what we do in the following, however, we do comment on the connections to the other definitions. In Section 3.8.1, we briefly review our recent results about the supersymmetrization of (DBI-)Galileon theories in 4d and cover some details that were left out in [60]. Motivated by Definition 4, we investigate the possibility of a scalar-vector Galileon theory in Section 3.8.2. In Sections 3.8.3 and 3.8.4, we focus our attention on the Special Galileon. In Section 3.8.3 we address the question of subleading operators respecting the enhanced $\sigma = 3$ soft behavior. In Section 3.8.4, we approach the same question from a double-copy construction.

### 3.8.1 Galileons and Supersymmetry

This section reviews and expands on the results of [60] for $\mathcal{N} = 1$ supersymmetrization of Galileon models. Two approaches to forming a complex scalar $Z = \phi + i\chi$ are considered:

(a) Both $\phi$ and $\chi$ are Galileons so that the complex scalar $Z$ has soft weight $\sigma_Z = 2$, or

(b) $\phi$ is a Galileon but $\chi$ only has constant shift symmetry; then $\sigma_\phi = 2$ and $\sigma_\chi = 1$, and hence $\sigma_Z = 1$. A natural interpretation of $\chi$ is as an R-axion.

Both options were considered in [60].

**Option (a).** Consider first the *quartic* Galileon. As discussed in Section 3.5.5, to be compatible with supersymmetry, the 4-point complex scalar amplitudes must have two $Z$'s and two $\bar{Z}$'s; such an amplitude is in the MHV class. It is also clear from the table of "soft bootstrap" results in Table 3.3 that there is a unique complex scalar quartic Galileon theory[17] with $\sigma_Z = 2$ based on the 4-point interaction with $\mathcal{A}_4(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}) = g_4 stu$. The other 4-point amplitudes in a supersymmetric theory are fixed by $\mathcal{A}_4(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}})$ using the supersymmetry Ward identity (3.67).

By (3.55), the soft behavior of the fermion must be either $\sigma_\psi = 1$ or 2. The all-fermion amplitudes are constructible when $\sigma_\psi = 2$, and our soft bootstrap results for fermion theories (see Table 3.4) show that no such theory exists. Therefore, the fermions in a supersymmetric Galileon theory with $\sigma_Z = 2$ must have $\sigma_\psi = 1$.

In a supersymmetric quartic Galileon theory with $\sigma_Z = 2$ and $\sigma_\psi = 1$, the constructibility criterion (3.20) for $n$-point amplitudes with $n_s$ scalars and $n_f$ fermions is $n_f < 4$. Thus at 6-point, we can only use soft subtracted recursion to compute the amplitudes with at most two fermions. However, as discussed in [60], two of the six supersymmetry Ward identities (3.68)-(3.70) uniquely determine the 4- and 6-fermion amplitudes. The remaining four identities in (3.68)-(3.70) are used as consistency checks. The expressions for the 6-point amplitudes of the supersymmetric quartic Galileon can be found in Appendix E.4. We have checked that the recursively constructed 4- and 6-point amplitudes match those that we calculate from the Lagrangian superspace construction of the quartic Galileon in [81].

The supersymmetry Ward identities at 8-point and higher do not uniquely determine the non-constructible amplitudes of the supersymmetric quartic Galileon. We therefore suspect that the quartic Galileon fails to be unique at 8-point and higher [60].

The *quintic* Galileon does not admit a supersymmetrization with $\sigma_Z = 2$ for the complex scalar. As discussed at the end of Section 3.4.1, there are no obvious obstructions from the soft-recursion tests to a complex scalar decoupled quintic Galileon with $\mathcal{A}_5(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}\, 5_Z) =$

[17] That analysis also shows that it is impossible for this kind of model to have special Galileon symmetry with $\sigma_Z = 3$.

$(\epsilon_{\mu\nu\rho\sigma}p_1^\mu p_2^\nu p_3^\rho p_4^\sigma)^2$. However, it is not compatible with the 5-point supersymmetry Ward identities. It follows that the cubic Galileon also cannot be supersymmetrized with $\sigma_Z = 2$.

**Option (b).** Consider a *quartic* complex scalar theory where the real part of the complex scalar $Z$ is the Galileon $\phi$ and the imaginary part is an R-axion $\chi$. The constructibility criterion with $\sigma_\phi = 2$ and $\sigma_\chi = \sigma_\psi = 1$ is $2n_\chi + n_f < 4$, so there are only two mixed amplitudes to check; they do not restrict the 2-parameter family of input amplitudes [60]. We have checked that the constructible 6-point amplitudes are compatible with DBI.

For a *quintic* Galileon with $\sigma_Z = 1$, we found [60] a unique solution to the supersymmetry Ward identities

$$\mathcal{A}_5(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}\, 5_Z) = -\frac{[24]}{[25]}\mathcal{A}_5(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{\psi}}\, 5_\psi) = \frac{[24]}{[35]}\mathcal{A}_5(1_Z\, 2_{\bar{\psi}}\, 3_\psi\, 4_{\bar{\psi}}\, 5_\psi)\,, \tag{3.98}$$

namely

$$\begin{aligned}\mathcal{A}_5(1_Z\, 2_{\bar{Z}}\, 3_Z\, 4_{\bar{Z}}\, 5_Z) = s_{24}\big[&6s_{24}s_{25}s_{45} + \big(4s_{12}s_{23}s_{45} + 2s_{12}s_{24}s_{34} + 2s_{25}^2 s_{45} + s_{24}s_{25}^2 + (2 \leftrightarrow 4)\big) \\ &+ (1 \leftrightarrow 5) + (3 \leftrightarrow 5)\big] - 4s_{24}^2\,.\end{aligned} \tag{3.99}$$

The amplitudes $\mathcal{A}_5(1_{\bar{Z}}\, 2_Z\, 3_{\bar{Z}}\, 4_Z\, 5_{\bar{Z}})$, $\mathcal{A}_5(1_{\bar{Z}}\, 2_Z\, 3_{\bar{Z}}\, 4_\psi 5_{\bar{\psi}})$, and $\mathcal{A}_5(1_{\bar{Z}}\, 2_\psi\, 3_{\bar{\psi}}\, 4_\psi\, 5_{\bar{\psi}})$ follow from conjugation of the above.[18] It is interesting that the fermions in these 5-point amplitudes automatically have $\sigma_\psi = 1$.

To test consistency of a supersymmetric quintic Galileon with $\sigma_\phi = 2$, $\sigma_\chi = 1$, and $\sigma_\psi = 1$, we consider the 7-point and 8-point amplitudes in the decoupled Galileon theory. In both cases, the constructibility criterion is $2n_\chi + n_f < 4$. The (few) non-trivial constructible amplitudes pass the soft subtraction recursive tests of $a_i$-independence. We have also tested compatibility with the supersymmetric DBI interactions: at 7-point the constructibility

[18] These 5-point amplitudes are not required to vanish in $3d$ kinematics (and they do not) because they do not satisfy the constructibility criterion.

criterion is $2n_\chi + n_f < 8$ and again the constructible 7-point amplitudes pass the test. This indicates that there may indeed be a supersymmetric brane-theory with both quartic and quintic terms subleading to DBI. The scalar $\phi$ is the Goldstone mode of the broken transverse translational symmetry whereas the scalar $\chi$ is an R-axion. The fermion $\psi$ is a genuine Goldstino of partial broken supersymmetry. We discuss such scenarios further in forthcoming work.

### 3.8.2 Vector-Scalar Special Galileon

It is known that scalar Galileon theories arise in certain limits of massive gravity [93, 94] (for a review, see [95]). An on-shell massive graviton in 4d has 5 polarization states and the decoupling limit gives one real massless scalar (the Galileon) and a massless photon in addition to the massless graviton. So we expect there to be an EFT of a real Galileon scalar coupled to vector.[19] The vector couples quadratically to the scalar and was consistently truncated off in [94]. Some subsequent studies have discussed the photon-scalar coupling of Galileons, see for example [114]. Here, we use soft recursion to give some definitive results about the possible scattering amplitudes in such a theory.

If the scalar has $\sigma_\phi = 2$, only the scalar amplitudes are constructible, and we are not able to say anything about the vector sector and its couplings to the scalar. If however the couplings are tuned in such a way that the cubic and quintic Galileon interactions are set to zero then in the scalar sector the soft weight of the scalar is enhanced to $\sigma_\phi = 3$, the *special Galileon* scenario. At present it is unknown whether this enhancement of symmetry can be understood in some natural way from the decoupling limit of some model of massive gravity. Moreover, it is not a priori clear if the $\sigma_\phi = 3$ enhancement can survive coupling to other particles.

We use the power of the soft bootstrap to construct the most general amplitudes consistent with the special Galileon low-energy theorem. We use the 6-point test to exclude EFTs

[19] The decoupling of these interactions from the graviton is not clear [95].

with a special Galileon coupled non-trivially to a photon with $\sigma_\gamma > 0$. For the model with $\sigma_\phi = 3$ and $\sigma_\gamma = 0$, we find that the soft recursion 6-point test reduces the most general 4 real-parameter ansatz for the scalar and scalar-vector interactions to a 3 real-parameter family:

$$\begin{aligned}
\mathcal{A}_4(1_\phi\, 2_\phi\, 3_\phi\, 4_\phi) &= g_1 stu\,, \\
\mathcal{A}_4(1_\phi\, 2_\phi\, 2^+_\gamma\, 4^+_\gamma) &= 0\,, \\
\mathcal{A}_4(1^-_\gamma\, 2_\phi\, 3_\phi\, 4^+_\gamma) &= g_1 \langle 12\rangle[24]\langle 13\rangle[34]u\,, \\
\mathcal{A}_4(1_\phi\, 2_\phi\, 3^-_\gamma\, 4^-_\gamma) &= 0\,.
\end{aligned} \tag{3.100}$$

The couplings of the pure vector sector are unconstrained; the most general ansatz is

$$\begin{aligned}
\mathcal{A}_4(1^+_\gamma\, 2^+_\gamma\, 3^+_\gamma\, 4^+_\gamma) &= g_3\Big([12]^2[34]^2 s + [13]^2[24]^2 t + [14]^2[23]^2 u\Big)\,, \\
\mathcal{A}_4(1^-_\gamma\, 2^-_\gamma\, 3^+_\gamma\, 4^+_\gamma) &= g_4 \langle 12\rangle^2[34]^2 s\,, \\
\mathcal{A}_4(1^-_\gamma\, 2^-_\gamma\, 3^-_\gamma\, 4^-_\gamma) &= g_3^*\Big(\langle 12\rangle^2\langle 34\rangle^2 s + \langle 13\rangle^2\langle 24\rangle^2 t + \langle 14\rangle^2\langle 23\rangle^2 u\Big)\,.
\end{aligned} \tag{3.101}$$

The most interesting feature of the above result is the relation between the coefficients of the amplitudes $\mathcal{A}_4(1_\phi\, 2_\phi\, 3_\phi\, 4_\phi)$ and $\mathcal{A}_4(1^-_\gamma\, 2_\phi\, 3_\phi\, 4^+_\gamma)$. The former is the familiar quartic Galileon, while the latter would arise from an operator of the form

$$\mathcal{O} \sim g_1 (\partial_\mu F_+^{\alpha\beta})(\partial^\mu F_-^{\dot\alpha\dot\beta})(\sigma^\nu_{\alpha\dot\alpha}\partial_\nu\phi)(\sigma^\rho_{\beta\dot\beta}\partial_\rho\phi), \tag{3.102}$$

where $F_\pm$ are as defined in and below (3.30)

The relation between the couplings strongly indicates the existence of a non-linear symmetry which mixes the scalar and vector modes. Describing the action of this symmetry and its consequences is left for future work.

### 3.8.3 Higher Derivative Corrections to the Special Galileon

The real quartic Galileon has low-energy theorems with $\sigma = 3$ soft weight. Being agnostic about the origin of the special Galileon, from an EFT perspective, one should write a Lagrangian with all possible operators that respect the symmetries of the theory in a derivative expansion. The authors of [115] found that among a specific subclass of Lagrangian operators, namely those with the schematic form $\partial^4\phi^4$, $\partial^6\phi^4$ and $\partial^8\phi^5$, the special Galileon is the unique choice that can give enhanced soft limits with $\sigma = 3$ soft weight. In this section, we investigate much more exhaustively the possible higher-derivative quartic and quintic operators compatible with $\sigma = 3$ soft behavior. This is done using soft-subtracted recursion relations to calculate the 6- and 7-point scattering amplitudes of the model.

Let us start our discussion with the 6-point case. The constructibility criterion (3.31) implies that recursion relations are valid if the coupling constant $g_6$ of the 6-point amplitude satisfies

$$[g_6] > -20\,. \tag{3.103}$$

Given that this coupling is the product of two quartic couplings and that the leading order quartic coupling has mass dimension $-6$ recursion relations can probe contributions to the 4-point amplitude with mass dimension in the range

$$-14 < [g_4] \leq -6\,. \tag{3.104}$$

Taking into account Bose symmetry, the most general ansatz one can write down for the

4-point matrix element of local operators is

$$\begin{aligned}\mathcal{A}_4(1_\phi 2_\phi 3_\phi 4_\phi) =& \frac{c_0}{\Lambda^6} stu \\ &+ \frac{c_1}{\Lambda^8}\left(s^4+t^4+u^4\right) \\ &+ \frac{c_2}{\Lambda^{10}}\left(s^5+t^5+u^5\right) \\ &+ \frac{1}{\Lambda^{12}}\left(c_3\left(s^6+t^6+u^6\right)+c_3' s^2t^2u^2\right)+\mathcal{O}(\Lambda^{-14})\,.\end{aligned} \tag{3.105}$$

The leading term with coupling $c_0/\Lambda^6$ is the usual quartic Galileon. The terms suppressed by higher powers of the the UV cutoff $\Lambda$ encode all possible higher-derivative quartic operators of the scalar field up to order $\Lambda^{-14}$.

We apply the 6-point test with $\sigma = 3$ and find that consistency requires $c_1 = c_3 = 0$ in the ansatz (3.105). The 4-point amplitude then becomes

$$\mathcal{A}_4(1_\phi\, 2_\phi\, 3_\phi\, 4_\phi) = \frac{c_0}{\Lambda^6} stu + \frac{c_2}{\Lambda^{10}}\left(s^5+t^5+u^5\right) + \frac{c_3'}{\Lambda^{12}} s^2t^2u^2 + \mathcal{O}(\Lambda^{-14})\,. \tag{3.106}$$

From this, we understand that there cannot exist an 8-derivative Lagrangian operator that preserves the special Galileon symmetry. Additionally, at 6-, 10- and 12-derivative order there exist unique quartic operators compatible with $\sigma = 3$. In Section 3.8.4, we show explicitly that the result (3.106) can also be obtained from an application of the BCJ double-copy.

Next we examine the possible existence of quintic operators compatible with $\sigma = 3$. We combine input from the quartic Galileon with the most general possible ansatz for the 5-point matrix elements and use the 7-point test to assess compatibility with $\sigma = 3$. The soft subtracted recursion relations at 7 points are valid if

$$[g_7] > -24\,. \tag{3.107}$$

Since the 7-point coupling constant is the product of a quartic (with mass dimension $-6$ or lower) and a quintic coupling, the latter must then satisfy

$$[g_5] > -18\,. \tag{3.108}$$

With Bose symmetry and the requirement that the ansatz for the 5-point amplitude must have soft weight $\sigma = 3$, we are left with

$$\begin{aligned}\mathcal{A}_5(1_\phi\, 2_\phi\, 3_\phi\, 4_\phi\, 5_\phi) &= \frac{d_1}{\Lambda^{15}}\epsilon(1234)\sum_P (-1)^{|P|} s_{P_1P_2}s_{P_2P_3}s_{P_3P_4}s_{P_4P_5}s_{P_5P_1} \\ &+ \frac{1}{\Lambda^{17}}\Bigg[d_2\ \epsilon(1234)^4 + d_3\epsilon(1234)\sum_P (-1)^{|P|} s_{P_1P_2}s^2_{P_2P_3}\left(s^2_{P_2P_3}s_{P_3P_4} - s^2_{P_1P_2}s_{P_2P_4}\right) \\ &+ d_4\left(\frac{4}{5}\sum_{i<j}s^3_{ij}\sum_{i<j}s^5_{ij} + \sum_{i<j}\sum_{k\neq i,j}\left(20s^2_{ij}s^3_{ik}s^3_{jk} + 9s^4_{ij}s^2_{ik}s^2_{jk} - 2s^6_{ij}s_{ik}s_{jk}\right)\right)\Bigg] + \mathcal{O}(\Lambda^{-19})\,.\end{aligned} \tag{3.109}$$

In the above, $\epsilon(1234) = \epsilon_{\mu\nu\rho\sigma}p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$, the sum $\sum_{i<j}$ means $\sum_{i=1}^4\sum_{j=i+1}^5$, while the sum $\sum_P$ is over all permutations of $\{1,2,3,4,5\}$, $(-1)^{|P|}$ is the signature of the permutation and $P_i$ is its $i$th element. There are no contributions to the amplitude that have less than 14 derivatives. The $1/\Lambda^{14}$-term satisfies the constructibility criterion and vanishes in 3d kinematics, in agreement with the discussion of Section 3.3.5. Two of the $1/\Lambda^{17}$-terms also vanish in 3d kinematics, but this was not a priori expected since they are too high order to satisfy constructibility.

The 7-point test implies no constraints on the coefficients $d_1$, $d_2$, $d_3$ and $d_4$. This is evidence in favor of the existence of four 5-point operators that preserve the special Galileon symmetry. Next, in Section 3.8.4, we investigate whether this result can be obtained from a double-copy prescription, similar to the 4-point case.

### 3.8.4 Comparison with the Field Theory KLT Relations

The significance of the special Galileon extends well beyond the contraction limit of the 3-brane effective field theory and the decoupling limit of massive gravity. The enhancement of the soft behavior to $\sigma = 3$ (which degenerates to $\sigma = 2$ when the DBI interactions are re-introduced) or correspondingly the extension of the non-linearly realized symmetry algebra suggests that this model has a fundamental significance of its own that is at present only partially understood. Perhaps one of the deepest and least understood aspects of the special Galileon is its role in the (field theory) *KLT algebra* as the product of two copies of the $\frac{U(N)\times U(N)}{U(N)}$ non-linear sigma model. For $N = 2, 3$ this coset sigma model has been intensively studied as a phenomenological model of the lightest mesons under the name *Chiral Perturbation Theory* ($\chi$PT). Henceforth we will use this name to avoid confusion with the $\mathbb{CP}^1$ non-linear sigma model discussed in Section 3.6.

The *double-copy* relation between $\chi$PT and the special Galileon was first understood in the CHY auxilliary world-sheet formalism[44]. Specifically, it was shown in the CHY formalism that the *leading order* contribution to scattering in the special Galileon model can be obtained from the KLT product

$$\mathcal{A}_n^{\text{sGal}} = \sum_{\alpha,\beta} \mathcal{A}_n^{\chi\text{PT}}[\alpha] S_{\text{KLT}}[\alpha|\beta] \mathcal{A}_n^{\chi\text{PT}}[\beta] \,, \tag{3.110}$$

where $\alpha, \beta$ index the $(n-3)!$ independent color(flavor) orderings.[20] The KLT summation kernel $S_{\text{KLT}}[\alpha|\beta]$ is universal in the sense that the explicit form of the relations (3.110) are identical to the perhaps more familiar field theory KLT relations giving a double-copy construction of Einstein-dilaton-$B_{\mu\nu}$ gravity from two copies of Yang-Mills theory. Concretely,

[20]We use square brackets for the arguments of a color-ordered amplitude.

the first few relations have the form

$$\begin{aligned}
\mathcal{A}_4^{\rm sGal}(1,2,3,4) &= -s_{12}\mathcal{A}_4^{\chi\rm PT}[1,2,3,4]\,\mathcal{A}_4^{\chi\rm PT}[1,2,4,3]\ ,\\
\mathcal{A}_5^{\rm sGal}(1,2,3,4,5) &= s_{23}s_{45}\mathcal{A}_5^{\chi\rm PT}[1,2,3,4,5]\,\mathcal{A}_5^{\chi\rm PT}[1,3,2,5,4] + (3\leftrightarrow 4)\,,\\
\mathcal{A}_6^{\rm sGal}(1,2,3,4,5,6) &= -s_{12}s_{45}\mathcal{A}_6^{\chi\rm PT}[1,2,3,4,5,6]\Big(s_{35}\mathcal{A}_6^{\chi\rm PT}[1,5,3,4,6,2]\\
&\qquad +(s_{34}+s_{35})\mathcal{A}_6^{\chi\rm PT}[1,5,4,3,6,2]\Big) + \mathcal{P}(2,3,4)\,,
\end{aligned} \tag{3.111}$$

where $\mathcal{P}(2,3,4)$ denotes the sum of all permutations of legs 2, 3 and 4.

For the formulae (3.110) and (3.111) to even be well-defined, the color-ordered amplitudes on the right-hand-side must satisfy a number of non-trivial relations to reduce the number of independent partial amplitudes to $(n-3)!$ for the scattering of $n$ particles. The existence of a color-ordered representation is itself non-trivial and not guaranteed to be satisfied in all models with color structure[116]. In all known cases where the double-copy relations (3.110) give a sensible, physical output, the reduction to a reduced basis of size $(n-3)!$ is accomplished by two sets of identities among the partial amplitudes, namely the *Kleiss-Kuijf* and *fundamental Bern-Carrasco-Johansson* relations. That these identites obtain for amplitudes calculated in the leading two-derivative action of $\chi$PT was first established in [117] using semi-on-shell recursion techniques developed in [118].

Our goal in this section is to connect two (possibly discrepant) definitions of the special Galileon model:

1. The special Galileon is the most general effective field theory of a real massless scalar with $\sigma = 3$ vanishing soft limits.

2. The special Galileon is the double-copy of two copies of $\chi$PT.

What we have described above is the known fact that these definitions agree at the lowest non-trivial order. In the previous section we used soft subtracted recursion to construct the most general 4- and 5-point amplitudes consistent with the first definition up to order $\Lambda^{-12}$

and $\Lambda^{-17}$ respectively. To determine if these results agree with the second definition we must first construct the most general 4- and 5-point amplitudes in $\chi$PT compatible with the requirements of the double-copy. Here we are following the approach of [116] and making the most conservative possible assumptions. Specifically we assume that both the explicit form of the double-copy (3.111) *and* the relations the amplitudes must satisfy to reduce the basis of partial amplitudes to size $(n-3)!$ are *identical* to what is required at leading order.

Let us begin with the 4-point amplitudes. The relations we impose are cyclicity (C)

$$\mathcal{A}_4^{\chi\mathrm{PT}}[1,2,3,4] = \mathcal{A}_4^{\chi\mathrm{PT}}[2,3,4,1]\,, \tag{3.112}$$

Kleiss-Kuijf (KK) or $U(1)$-decoupling

$$\mathcal{A}_4^{\chi\mathrm{PT}}[1,2,3,4] + \mathcal{A}_4^{\chi\mathrm{PT}}[2,1,3,4] + \mathcal{A}_4^{\chi\mathrm{PT}}[2,3,1,4] = 0\,, \tag{3.113}$$

and the fundamental BCJ relation

$$(-s-t)\mathcal{A}_4^{\chi\mathrm{PT}}[1,2,3,4] - t\mathcal{A}_4^{\chi\mathrm{PT}}[1,2,4,3] = 0\,. \tag{3.114}$$

Since there are no additional quantum number labels in the partial amplitudes, at each order the 4-point amplitude is determined by a single polynomial function of the available Lorentz singlets

$$\mathcal{A}_4^{\chi\mathrm{PT}}[1,2,3,4] = F^{(0)}(s,t) + \frac{1}{\Lambda^2}F^{(2)}(s,t) + \frac{1}{\Lambda^4}F^{(4)}(s,t) + \ldots \tag{3.115}$$

The superscript $k$ counts both the mass dimension of the function and the number of derivatives in the underlying effective operator. In this language, the double-copy-compatibility conditions take the form

$$\begin{aligned}
&\text{C:} && F^{(k)}(s,t) = F^{(k)}(-s-t,t)\,, \\
&\text{KK:} && F^{(k)}(s,t) + F^{(k)}(s,-s-t) + F^{(k)}(-s-t,s) = 0\,, \\
&\text{BCJ:} && (-s-t)F^{(k)}(s,t) - tF^{(k)}(s,-s-t) = 0\,.
\end{aligned} \tag{3.116}$$

We make a general parametrization of the polynomial functions as

$$\begin{aligned}
F^{(0)}(s,t) &= c_1^{(0)}, \\
F^{(2)}(s,t) &= c_1^{(2)}s + c_2^{(2)}t, \\
F^{(4)}(s,t) &= c_1^{(4)}s^2 + c_2^{(4)}st + c_3^{(4)}t^2, \\
F^{(6)}(s,t) &= c_1^{(6)}s^3 + c_2^{(6)}s^2t + c_3^{(6)}st^2 + c_4^{(6)}t^3, \\
F^{(8)}(s,t) &= c_1^{(8)}s^4 + c_2^{(8)}s^3t + c_3^{(8)}s^2t^2 + c_4^{(8)}st^3 + c_5^{(8)}t^4,
\end{aligned} \tag{3.117}$$

and so on. Imposing the conditions (3.116) gives a system of linear relations among the coefficients $c_i^{(k)}$. These are straightforward to solve and give

$$\mathcal{A}_4^{\chi\text{PT}}[1,2,3,4] = \frac{g_2}{\Lambda^2}t + \frac{g_6}{\Lambda^6}t(s^2+t^2+u^2) + \frac{g_8}{\Lambda^8}t(stu) + \ldots \tag{3.118}$$

A few comments about this result. As expected, the leading 2-derivative contribution is compatible with the conditions (3.116). Surprisingly, there are no compatible contributions from 4-derivative operators, but there are unique contributions at 6- and 8-derivative order. Moreover, the structure of the result here agrees with the 4-point amplitude of *Abelian Z-theory* [119]. The Z-theory model is a top-down construction which gives open string scattering amplitudes as the field theory double-copy of Yang-Mills and a higher-derivative extension of $\chi$PT. The Z-amplitudes are by construction guaranteed to satisfy the double-copy-compatibility conditions but with Wilson coefficients $g_i$ having precise values calculated from the known string amplitudes. The method of this section can be understood as the bottom-up converse of the Z-theory construction, and at 4-point we find agreement.

To summarize, we have shown that up to 8-derivative order there is a 3-parameter family of operators that generate 4-point matrix elements compatible with the conditions required for the double-copy to be well-defined. We could continue this to higher order, but our ability to compare with the methods of Section 3.8.3 are bounded above at this order by the constructibility criterion.

To construct the associated amplitudes in the special Galileon model (according to the second definition described above) we use the first relation in (3.111). The result is

$$\mathcal{A}_4^{\text{sGal}}(1,2,3,4) = \frac{c_1}{\Lambda^6} stu + \frac{c_2}{\Lambda^{10}}\left(s^5+t^5+u^5\right) + \frac{c_3}{\Lambda^{12}} s^2t^2u^2 + \dots\,, \tag{3.119}$$

in precise agreement with the special Galileon amplitude (3.106).

As an additional check to the results obtained above, we calculate the 6-point amplitudes of both $\chi$PT and the special Galileon. Up to order $\mathcal{O}(\Lambda^{-6})$ the $\chi$PT amplitude can be calculated using soft subtracted recursion with (3.118) as input. Note that only three factorization channels contribute to this calculation because the rest do not preserve color ordering. The resulting amplitude,

$$\mathcal{A}_6^{\chi\text{PT}}[1,2,3,4,5,6] = \frac{g_2^2}{\Lambda^4}\left[\frac{s_{13}s_{46}}{p_{123}^2} + \frac{s_{24}s_{15}}{p_{234}^2} + \frac{s_{35}s_{26}}{p_{345}^2} - s_{246}\right] + \mathcal{O}(\Lambda^{-8})\,, \tag{3.120}$$

satisfies all C, KK and BCJ constraints. Contributions subleading to the ones listed above do not satisfy the constructibility criterion (3.31) and cannot be calculated using soft subtracted recursion. However, we were able to uniquely determine them up to order $\mathcal{O}(\Lambda^{-10})$, by demanding that they have the correct pole structure, consistent with unitarity and locality, have $\sigma = 1$ soft weight and satisfy C, KK and BCJ conditions. The result of this calculation is listed in (E.30).

We are now in position to calculate the 6-point special Galileon amplitude with two different methods. We can either use the 6-point KLT relation in (3.111) or use soft subtracted recursion with (3.119) as input. The results of these calculations match perfectly up to order $\mathcal{O}(\Lambda^{-18})$, which is the furthest the recursive calculation can go.

Shifting our focus to 5-point amplitudes, we find that it is *not* possible to reproduce (3.109) as a double-copy of two (identical or non-identical) color-ordered scalar amplitudes, despite the perfect agreement at 4- and 6-points. Starting from a general ansatz for the scalar color-ordered amplitude, we find that the leading contribution that satisfies all C, KK and BCJ constraints is $\mathcal{O}(\Lambda^{-15})$ corresponding to a valence 5 scalar-field operator with 14 derivatives. The existence of such an operator at all is interesting since there are apparently *no odd point amplitudes in Z-theory* [119]! At this order we find that the kinematic structure of Z-theory does not coincide with the most general possible double-copy-compatible higher-derivative extension of $\chi$PT. Or perhaps said differently, just like string theory fixes the Wilson coefficients in the 4-point result (3.118) to take particular (non-zero) values, it appears to fix the Wilson coefficients of the odd-point amplitudes to be zero.

When we use the second relation of (3.111) with this result, we obtain a 5-point scalar amplitude of order $\mathcal{O}(\Lambda^{-33})$, which is significantly subleading to the amplitude (3.109) we calculated in the previous section for the special Galileon.

## 3.9 Outlook

There are several interesting questions that remain unanswered in this work. In Section 3.4 we applied the soft bootstrap to classes of models with simple spectra consisting of a single particle of a particular spin. Furthermore, we gave a limited examination of classes of models with linearly realized supersymmetry with spectra consisting of a single multiplet. There is a potentially vast landscape of constructible models with more complicated spectra and possible futher interesting linearly realized symmetries.

We have already seen examples of this; in Section 3.6 further symmetry (in this case electromagnetic duality symmetry) emerges as an unavoidable consequence of the combination of low-energy theorems and linear $\mathcal{N} = 2$ supersymmetry. Similarly we should expect the soft bootstrap to reveal models with complicated *non-linear* symmetries. In Section 3.8.2 we have given evidence in favor of the existence of such a symmetry underlying a vector-scalar extension of the special Galileon.

Our results also suggest two additional applications for the soft bootstrap. The first is to the classification of higher-derivative operators. The method applied in Sections 3.8.3 and 3.8.4 to the special Galileon and $\chi$PT is generalizable to a large class of EFTs with manifest advantages over traditional methods. The second is as a useful cross-check on results concerning exceptional EFTs obtained via the double copy. In Section 3.8.4 we found the puzzling result that there exist valence 5 operators invariant under the special Galileon symmetry which apparently cannot be constructed as the double copy of subleading $\chi$PT operators.

It would be reasonable to expect further, similarly rich and unexpected, phenomena to be present throughout the landscape of constructible EFTs.

CHAPTER 4

# Exact Results for Corner Contributions to the Entanglement Entropy and Rényi Entropies of Free Bosons and Fermions in 3d

## 4.1 Motivation and Results

For a $3d$ conformal field theory (CFT) in the ground state, the entanglement entropy $S$ for a region whose boundary has a sharp corner with angle $\theta$ can be written as

$$S = B\,\frac{L}{\epsilon} - a(\theta)\log\left(\frac{L}{\epsilon}\right) + O(1)\,. \tag{4.1}$$

Here $L$ is a length scale associated with the size of the entangling region, $\epsilon$ is a short distance cutoff, and $B$ is a non-universal constant. The *corner contribution* to the entanglement entropy is the scheme-independent positive function $a(\theta)$ of the opening angle $\theta$ [120, 53, 121]. Since the entanglement entropy of the region equals that of the complement region, the corner contribution satisfies $a(2\pi-\theta) = a(\theta)$. If the curve bounding the entangling region is smooth, the logarithmic term is absent, hence $a(\theta)$ must vanish in the limit $\theta \to \pi$ and it does so quadratically as

$$a(\theta) = \sigma\,(\theta-\pi)^2 + \ldots \qquad \text{for} \qquad \theta \to \pi\,. \tag{4.2}$$

The value of the *corner coefficient* $\sigma$ depends on the theory.

For the theory of a free real scalar or a Dirac fermion, Casini, Huerta, and Leitao [54, 52, 53] derived expressions that give $a(\theta)$ implicitly in terms of some rather involved integrals. In the limit $\theta \to \pi$ one can extract double-integral expressions for the corner coefficient $\sigma$ in (4.2). These integrals have been evaluated numerically [52, 55] and the results indicate that the exact values are [55]

$$\boxed{\sigma^{(B)} = \frac{1}{256} \quad \text{and} \quad \sigma^{(F)} = \frac{1}{128}} \tag{4.3}$$

for the free boson and free fermion, respectively.

Bueno, Myers, and Witczak-Krempa [55] conjectured that the ratio of the coefficient $\sigma$ in (4.2) to the central charge $C_T$ is universal in 3d CFTs and that it takes the value

$$\text{conjecture [55]:} \qquad \frac{\sigma}{C_T} = \frac{\pi^2}{24}\,. \tag{4.4}$$

The conjecture (4.4) has passed non-trivial holographic tests for gravity models with a family of higher derivative corrections [55, 56]. The central charge $C_T$ is defined as the coefficient of the vacuum 2-point function of the stress tensor (see eq. (3) in [55]). For free bosons and fermions, Osborn and Petkou [122] found that $C_T^{(B)} = 3/(32\pi^2)$ and $C_T^{(F)} = 3/(16\pi^2)$ in 3d. So with the values (4.3), the ratio $\sigma/C_T$ is indeed $\pi^2/24$ for both free bosons and fermions.

In our work, we evaluate analytically the integral expressions [55] of Casini, Huerta, and Leitao [54, 52, 53] for $\sigma^{(B)}$ and $\sigma^{(F)}$ and prove that their exact values are indeed those in (4.3). This verifies the universality conjecture (4.4) for the case of free bosons and fermions. One way of viewing the conjecture is simply as the statement that the corner coefficient $\sigma$ in (4.2) does not contain independent information about the CFT, but is fixed in terms of the central charge $C_T$.

Turning to the Rényi entropies $S_n$, one can define a similar corner contribution $a_n(\theta)$ which in the smooth limit $\theta \to \pi$ goes to zero as $a_n(\theta) = \sigma_n\,(\theta - \pi)^2 + \dots$ for $n = 2, 3, 4, \dots$. (The $n \to 1$ limit of the Rényi entropy is the entanglement entropy.) It is not known if $\sigma_n/C_T$ has any universal properties.

We calculate $\sigma_n$ analytically for the free boson and free fermion using integral expressions

| $n$ | $\sigma_n^{(\mathrm{B})}$ | Numerical approximation |
|---|---|---|
| 2 | $\frac{1}{48\pi^2}$ | 0.00211086 |
| 3 | $\frac{1}{108\sqrt{3}\pi}$ | 0.00170163 |
| 4 | $\frac{8+3\pi}{1152\pi^2}$ | 0.00153255 |
| 5 | $\frac{\sqrt{25-2\sqrt{5}}}{1000\pi}$ | 0.00144219 |
| 6 | $\frac{81+34\sqrt{3}\pi}{19440\pi^2}$ | 0.00138643 |
| 7 | $\frac{2\cot\frac{\pi}{14}+5\cot\frac{3\pi}{14}+5\tan\frac{\pi}{7}}{4116\pi}$ | 0.00134874 |
| 8 | $\frac{32+9\pi\left(1+\sqrt{2}\right)}{10752\pi^2}$ | 0.00132161 |
| 9 | $\frac{27\sqrt{3}+10\cot\frac{\pi}{18}+28\tan\frac{\pi}{9}+35\tan\frac{2\pi}{9}}{34992\pi}$ | 0.00130116 |
| 10 | $\frac{125+6\pi\sqrt{565+142\sqrt{5}}}{54000\pi^2}$ | 0.00128522 |

Table 4.1: Exact results and approximate numerical values for the corner coefficient of the first 9 Rényi entropies of a free scalar field.

for $\sigma_n$ derived in [54, 52, 53].[1] For the free scalar we find

$$\boxed{\sigma_n^{(B)} = \sum_{k=1}^{n-1} \frac{k(n-k)(n-2k)\tan\left(\frac{k\pi}{n}\right)}{24\pi\, n^3(n-1)}\,.} \tag{4.5}$$

Note that when $n$ is even, the contribution from $k = n/2$ must be evaluated carefully using $\lim_{k\to n/2}(n-2k)\tan\left(\frac{k\pi}{n}\right) = 2n/\pi$. The result for the free fermion is

$$\boxed{\sigma_n^{(F)} = \sum_{k=-(n-1)/2}^{(n-1)/2} \frac{k(n^2-4k^2)\tan\left(\frac{k\pi}{n}\right)}{24\pi\, n^3(n-1)}\,,} \tag{4.6}$$

where the sum is to be taken in integer steps from $-\frac{n-1}{2}$ to $\frac{n-1}{2}$.

For low values of $n$, the finite sums of the trigonometric functions in (4.5) and (4.6) simplify quite nicely. The results and approximate numerical values for the first nine values of $\sigma_n$ are presented in Tables 4.1 and 4.2, for the free scalar field and fermion respectively. In the case of the scalar, the exact $n = 2, 3$ results were guessed by the authors of [55] based

[1] We are grateful to Horacio Casini for sharing with us the integral expression for $\sigma_n^{(F)}$.

| $n$ | $\sigma_n^{(\mathrm{F})}$ | Numerical approximation |
|---|---|---|
| 2 | $\frac{1}{64\pi}$ | 0.00497359 |
| 3 | $\frac{5}{216\sqrt{3}\pi}$ | 0.00425408 |
| 4 | $\frac{1+6\sqrt{2}}{768\pi}$ | 0.00393133 |
| 5 | $\frac{\sqrt{425+58\sqrt{5}}}{2000\pi}$ | 0.00374840 |
| 6 | $\frac{261+20\sqrt{3}}{25920\pi}$ | 0.003630613 |
| 7 | $\frac{13\cot\frac{\pi}{14}+22\cot\frac{3\pi}{14}+15\tan\frac{\pi}{7}}{8232\pi}$ | 0.00354841 |
| 8 | $\frac{1+6\sqrt{2}+4\sqrt{274+17\sqrt{2}}}{7168\pi}$ | 0.00348777 |
| 9 | $\frac{135\sqrt{3}+68\cot\frac{\pi}{18}+77\tan\frac{\pi}{9}+130\tan\frac{2\pi}{9}}{69984\pi}$ | 0.00344118 |
| 10 | $\frac{5+300\sqrt{5}+4\sqrt{425+58\sqrt{5}}}{72000\pi}$ | 0.00340427 |

Table 4.2: Exact results and approximate numerical values for the corner coefficient of the first 9 Rényi entropies of a free fermion.

on their high precision numerical evaluation of the integrals.

Since the ratios of the central charges of free fermions and bosons differ only by a factor of 2, universality of the ratio $\sigma_n/C_T$ would require that $\sigma_n^{(B)}/\sigma_n^{(F)}$ obeys some simple, possibly $n$-dependent, relation. Based on our results above, there is no hint of such a simple relationship. Of course to fully exclude this, one would need values of $\sigma_n$ for other 3d CFTs.

As a function of $n$, the Rényi corner coefficient $\sigma_n$ decreases monotonically, as shown on the left in figure 4.1. When $n$ is large, $\sigma_n$ asymptotes to a constant value, which we calculate analytically:

$$\boxed{\sigma_\infty^{(B)} = \frac{3\zeta(3)}{32\pi^4} \qquad \text{and} \qquad \sigma_\infty^{(F)} = \frac{\zeta(3)}{4\pi^4}\,.} \tag{4.7}$$

The appearance of the Riemann zeta-function is intriguing since $\zeta(3) \approx 1.20206$ also shows up in the free energies and Rényi entropies for free scalars/fermions on a 3-sphere, as shown by Klebanov, Pufu, Sachdev, and Safdi [123]. Specifically, the free energy of a free real scalar

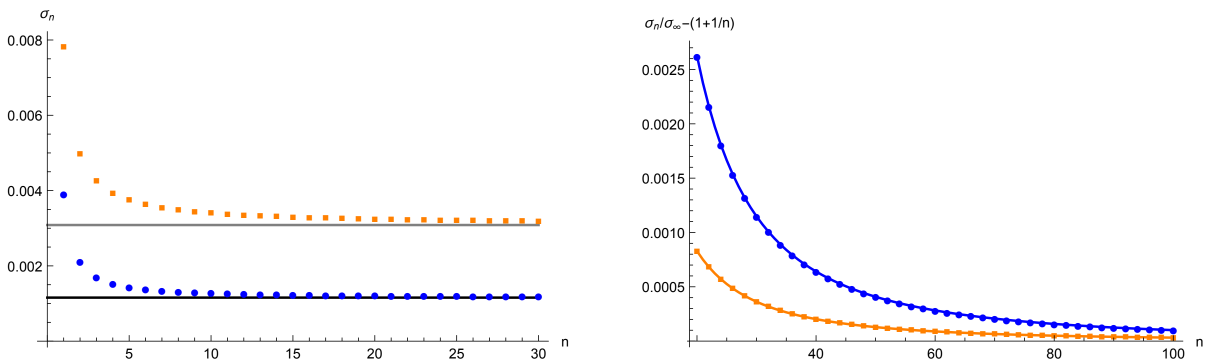



Figure 4.1: *Left:* Plot showing that $\sigma_n$ decreases monotonically from the entanglement entropy value included for $n = 1$ to the asymptotic value $\sigma_\infty$ for free scalars (blue circles) and free Dirac fermions (maize squares). The asymptotic values $\sigma_\infty^{(B)} = \frac{3\zeta(3)}{32\pi^4} \approx 0.0011569$ (black) and $\sigma_\infty^{(F)} = \frac{\zeta(3)}{4\pi^4} \approx 0.00308507$ (gray) are indicated as horizontal lines. *Right:* The plot illustrates our numerical fit $\sigma_n = \sigma_\infty\big(1+\frac{b_1}{n}+\frac{b_2}{n^2}+\frac{b_3}{n^3}+\dots\big)$, for which we find $b_1 = b_2 = b_3 = 1$ for the free scalar, and $b_1 = 1$ and $b_2 = b_3 = 1 - \frac{\pi^2}{12\zeta(3)} \approx 0.31578$ for the free fermion; the solid curves are $\frac{b_2}{n^2} + \frac{b_3}{n^3}$ for those respective values of $b_2$ and $b_3$.

or free fermion on an $n$-covered 3-sphere behaves as[2] $\mathcal{F}_n \to n\mathcal{F}_\infty$ for $n \to \infty$ with

$$\mathcal{F}_\infty^{(B)} = \frac{3\zeta(3)}{8\pi^2} \quad \text{and} \quad \mathcal{F}_\infty^{(F)} = \frac{\zeta(3)}{\pi^2}\,. \tag{4.8}$$

Thus, for both free scalars and fermions we have

$$\sigma_\infty^{(B/F)} = \frac{1}{4\pi^2}\mathcal{F}_\infty^{(B/F)}\,. \tag{4.9}$$

For finite $n$, there is no apparent relation between $\mathcal{F}_n$ and $\sigma_n$, however there are some similarities in the subleading large-$n$ behaviors, as we discuss in section 4.4. The plot on the right in figure 4.1 shows the large-$n$ behaviors of the Rényi corner coefficients $\sigma_n$. A priori it is not clear if there is any relation at large $n$ between $\sigma_n$ and $\mathcal{F}_n$, but it would be curious to test (4.9) in other examples.

The remainder of this chapter details the derivations of the results summarized above. In section 4.2, we derive the results (4.3) for the entanglement entropy corner coefficient $\sigma$.

[2]The authors of [123] work with a complex scalar, so the free energy there is twice that of a real scalar.

We then evaluate the Rényi entropy corner coefficients $\sigma_n$ in section 4.3. In section 4.4, we discuss the asymptotic behavior at large $n$.

## 4.2 Evaluation of the EE Integrals

In this section we describe the procedure for analytically evaluating the integrals for the coefficients $\sigma^{(B)}$ and $\sigma^{(F)}$ of the entanglement entropy. Our starting point is the integrals [54, 52, 53] presented in equations (B1)-(B3) of [55]. After a change of integration variable from $m$ to $\mu = \sqrt{4m^2 - 1}$, the integrals take the form

$$\sigma^{(B)} = -\frac{1}{2}\int_0^\infty \mathrm{d}\mu \int_0^\infty \mathrm{d}b\, \mu^2 H a\,(1-a)\,\frac{\pi}{\cosh^2(\pi b)}\,, \tag{4.10}$$

$$\sigma^{(F)} = -\int_0^\infty \mathrm{d}\mu \int_0^\infty \mathrm{d}b \left[\mu^2 H a\,(1-a) - \frac{\mu F}{4\pi}\right] \frac{\pi}{\sinh^2(\pi b)}\,, \tag{4.11}$$

where $a = 1/2 + ib$ for the scalar and $a = ib$ for the fermion. The functions $H$ and $F$ are defined as

$$H = -\frac{T}{2}\left(\frac{c}{h}X_1 + \frac{1}{c}X_2\right) + \frac{1}{16\pi a\,(a-1)}\,, \qquad F = -\frac{F_1}{F_2}\,, \tag{4.12}$$

with[3]

$$F_1 = 4\pi c h H a\,(1-a)\left[(2a-1)^2 + \mu^2\right] - \frac{1}{4}ch^2\left(\mu^2+1\right)\,, \tag{4.13}$$

$$F_2 = \frac{ch\left[(2a-1)^2+\mu^2\right]^2}{2\,(2a-1)\,\mu}\,. \tag{4.14}$$

[3]We simplified the expression for $F_1$ in [55] by writing it in terms of $H$.

The functions $h$, $c$, $X_1$, $X_2$, and $T$ are defined as follows:

$$\begin{aligned}
h &= \frac{2\left(\mu^2+(2a-1)^2\right)\sin^2(\pi a)}{\left(\mu^2+1\right)\left(\cos(2\pi a)+\cosh(\pi\mu)\right)}\,,\\
c &= \frac{2^{2a}\pi a\,(1-a)\sec\left(\pi a+\frac{i\pi\mu}{2}\right)\Gamma\left(\frac{3}{2}-a+\frac{i\mu}{2}\right)}{\sqrt{\mu^2+1}\,(\Gamma(2-a))^2\,\Gamma\left(-\frac{1}{2}+a+\frac{i\mu}{2}\right)}\,,\\
X_1 &= -\frac{\Gamma(-a)\left[\pi\sinh\left(\frac{\pi\mu}{2}\right)+i\cosh\left(\frac{\pi\mu}{2}\right)\left(\psi\left(\frac{1}{2}+a+\frac{i\mu}{2}\right)-\psi\left(\frac{1}{2}+a-\frac{i\mu}{2}\right)\right)\right]}{2^{2a+1}\mu\Gamma(a+1)\,\Gamma\left(\frac{1}{2}-a+\frac{i\mu}{2}\right)\Gamma\left(\frac{1}{2}-a-\frac{i\mu}{2}\right)\left(\cos(2\pi a)+\cosh(\pi\mu)\right)}\,,\\
X_2 &= \left(X_1 \text{ with } a \text{ replaced by } (1-a)\right),\\
T &= \frac{1}{2}\sqrt{h\left[(h+1)\left(\mu^2+1\right)-4a\,(1-a)\right]}\,.
\end{aligned} \tag{4.15}$$

Here $\psi$ denotes the digamma function, $\psi(z)=\frac{d}{dz}\log\Gamma(z)$.

Our first line of attack involves calculating the quantities $cX_1/h$ and $X_2/c$ that appear in $H$ in (4.12). Beyond the immediate cancellations that occur in these ratios, one can perform further simplifications using identities involving gamma functions. Namely, one can use the recurrence relation

$$\Gamma(1+z)=z\Gamma(z) \tag{4.16}$$

and the reflection relation

$$\Gamma(1-z)\,\Gamma(z)=\frac{\pi}{\sin(\pi z)}. \tag{4.17}$$

Surprisingly, all the gamma functions cancel after a series of such substitutions, giving

$$\begin{aligned}
\frac{c}{h}X_1 &= \frac{\sqrt{\mu^2+1}\csc(\pi a)}{16\pi\mu\,(a-1)\,a}\left[\pi\sinh\left(\frac{\pi\mu}{2}\right)+i\cosh\left(\frac{\pi\mu}{2}\right)\left(\psi\left(\frac{1}{2}+a+\frac{i\mu}{2}\right)-\psi\left(\frac{1}{2}+a-\frac{i\mu}{2}\right)\right)\right],\\
\frac{1}{c}X_2 &= \frac{\sqrt{\mu^2+1}\csc(\pi a)}{16\pi\mu\,(a-1)\,a}\left[\pi\sinh\left(\frac{\pi\mu}{2}\right)+i\cosh\left(\frac{\pi\mu}{2}\right)\left(\psi\left(\frac{3}{2}-a+\frac{i\mu}{2}\right)-\psi\left(\frac{3}{2}-a-\frac{i\mu}{2}\right)\right)\right].
\end{aligned} \tag{4.18}$$

It is suggestive that the pre-factors and the form of these two results are the same. We then proceed by adding them together as in (4.12). The linear combination of digamma functions that appears in the result can be simplified using properties easily derived from (4.16) and (4.17). In the form that is useful for our purpose, these identities are

$$\psi\left(\tfrac{3}{2} - a \pm \tfrac{i\mu}{2}\right) = \psi\left(\tfrac{1}{2} - a \pm \tfrac{i\mu}{2}\right) + \frac{1}{\frac{1}{2} - a \pm \frac{i\mu}{2}}$$

and

$$\psi\left(\tfrac{1}{2} + a \pm \tfrac{i\mu}{2}\right) - \psi\left(\tfrac{1}{2} - a \mp \tfrac{i\mu}{2}\right) = \pi \tan\left(\pi a \pm i\tfrac{\pi\mu}{2}\right) .$$

Then the combination of $X_1$ and $X_2$ that appears in $H$ simplifies to

$$\frac{c}{h}X_1 + \frac{1}{c}X_2 = \frac{\sqrt{\mu^2+1}}{4\pi a(1-a)}\left(\frac{\pi \sin(\pi a)\sinh\left(\frac{\pi\mu}{2}\right)}{\mu\left[\cos(2\pi a) + \cosh(\pi\mu)\right]} - \frac{\csc(\pi a)\cosh\left(\frac{\pi\mu}{2}\right)}{(1-2a)^2+\mu^2}\right) . \tag{4.19}$$

The last ingredient we need to construct $H$ in (4.12) is $T$. Using (4.15), it is

$$T = \sqrt{\frac{\left((1-2a)^2+\mu^2\right)^2 \sin^2(\pi a)\cosh^2\left(\frac{\pi\mu}{2}\right)}{(\mu^2+1)\left(\cos(2\pi a)+\cosh(\pi\mu)\right)^2}} . \tag{4.20}$$

Further simplifications of $H$ depend on the nature of variable $a$, as we will see when we specialize to the cases of the free scalar and the free fermion.

**Free scalar.** To proceed with the evaluation of the integral $\sigma^{(B)}$, we set $a = 1/2 + ib$ as prescribed for the free scalar. It is furthermore convenient to change integration variable $b \to b/2$. Using that both $\mu$ and $b$ are positive, the integrand of $\sigma^{(B)}$ simplifies dramatically and becomes

$$\sigma^{(B)} = \int_0^\infty \mathrm{d}\mu \int_0^\infty \mathrm{d}b \; \frac{\mu\left[\pi\left(\mu^2-b^2\right)\sinh(\pi\mu) + 2\mu\cosh(\pi b) - 2\mu\cosh(\pi\mu)\right]}{64[\cosh(\pi b)-\cosh(\pi\mu)]^2} . \tag{4.21}$$

Next, we integrate by parts. Writing

$$\sigma^{(B)} = \frac{1}{64}\int_0^\infty \mathrm{d}\mu \int_0^\infty \mathrm{d}b \; \left[\frac{\partial}{\partial\mu}\left(\frac{\mu\left(\mu^2-b^2\right)}{\cosh\left(\pi b\right)-\cosh\left(\pi\mu\right)}\right) + \frac{b^2-\mu^2}{\cosh\left(\pi b\right)-\cosh\left(\pi\mu\right)}\right] , \tag{4.22}$$

we see that the boundary term vanishes and we get

$$\sigma^{(B)} = \frac{1}{256} \int_{-\infty}^{+\infty} \mathrm{d}\mu \int_{-\infty}^{+\infty} \mathrm{d}b \; \frac{b^2 - \mu^2}{\cosh(\pi b) - \cosh(\pi\mu)} \,. \tag{4.23}$$

We have extended the limits of integration to facilitate the change of integration variables

$$\mu = x - y \quad \text{and} \quad b = x + y \,. \tag{4.24}$$

This separates the two integrations and reduces the expression to

$$\sigma^{(B)} = \left( \frac{1}{8} \int_{-\infty}^{+\infty} \mathrm{d}x \; \frac{x}{\sinh(\pi x)} \right)^2 = \frac{1}{256} \,. \tag{4.25}$$

This completes the derivation of the result (4.3) for the free scalar.

**Free fermion.** With $F_1$ given in terms of $H$ as in (4.13), we have already done most of the leg-work needed to compute $\sigma^{(F)}$. For the free fermion, we have to take $a = ib$ and it is again convenient to change integration variable $b \to b/2$. After putting everything together, we have

$$\sigma^{(F)} = -\frac{1}{32} \int_0^{\infty} \mathrm{d}\mu \int_0^{\infty} \mathrm{d}b \; \frac{\mu\big[\pi\,(\mu^2 - b^2 - 1)\sinh(\pi\mu) - 2\mu\cosh(\pi b) - 2\mu\cosh(\pi\mu)\big]}{[\cosh(\pi b) + \cosh(\pi\mu)]^2} \,. \tag{4.26}$$

We can express the integrand as a total derivative plus remaining terms as

$$\sigma^{(F)} = \frac{1}{32} \int_0^{\infty} \mathrm{d}\mu \int_0^{\infty} \mathrm{d}b \; \left[ \frac{\partial}{\partial\mu} \left( \frac{\mu\,(\mu^2 - b^2 - 1)}{\cosh(\pi b) + \cosh(\pi\mu)} \right) + \frac{1 - \mu^2 + b^2}{\cosh(\pi b) + \cosh(\pi\mu)} \right] . \tag{4.27}$$

As before, the boundary term vanishes and we are left with the expression (after extending the limits of integration)

$$\sigma^{(F)} = \frac{1}{128} \int_{-\infty}^{+\infty} \mathrm{d}\mu\,\mathrm{d}b \; \frac{1 - \mu^2 + b^2}{\cosh(\pi b) + \cosh(\pi\mu)} = \frac{1}{128} \int_{-\infty}^{+\infty} \mathrm{d}x\,\mathrm{d}y \; \frac{1 + 4xy}{\cosh(\pi x)\cosh(\pi y)} \,. \tag{4.28}$$

In the last step, we changed integration variables using (4.24). Since $x/\cosh(\pi x)$ is odd, that part of the integral vanishes and the result is therefore simply

$$\sigma^{(F)} = \frac{1}{128}\left(\int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{1}{\cosh(\pi x)}\right)^2 = \frac{1}{128} \,. \tag{4.29}$$

Thus we have derived the result (4.3) for the free fermion.

## 4.3 Rényi Entropies

We now proceed to calculate the corner coefficients $\sigma_n$ for the Rényi entropies.

**Free scalar.** For the scalar field, the Rényi corner coefficient is given by the integral (B7) in [55]. We change the integration variable $m$ to $\mu = \sqrt{4m^2 - 1}$ to write it as

$$\sigma_n^{(B)} = -\sum_{k=1}^{n-1} \frac{k\,(n-k)}{2n^2\,(n-1)} \int_0^\infty \mathrm{d}\mu \; \mu^2 H_{k/n} \,, \tag{4.30}$$

where $H_{k/n}$ is $H$ in (4.12) with $a$ replaced by $k/n$. With the simplified expression for $H$ from section 4.2, we get

$$\begin{aligned}\sigma_n^{(B)} = &\sum_{k=1}^{n-1} \frac{\sin^2\left(\frac{\pi k}{n}\right)}{32\pi n^2 (n-1)} \\ &\times \int_0^\infty \mathrm{d}\mu \; \frac{\mu\left[(n-2k)^2 + \mu^2 n^2\right]\pi \sinh(\pi\mu) - 2\mu^2 n^2 \left[\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)\right]}{\left[\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)\right]^2} \,.\end{aligned} \tag{4.31}$$

As before, we write the integrand as a total derivative plus remaining terms:

$$\begin{aligned}\sigma_n^{(B)} = -&\sum_{k=1}^{n-1} \frac{\sin^2\left(\frac{\pi k}{n}\right)}{32\pi n^2 (n-1)} \\ &\times \int_0^\infty \mathrm{d}\mu \left[\frac{\partial}{\partial\mu}\left(\frac{\mu\left[(n-2k)^2 + \mu^2 n^2\right]}{\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)}\right) - \frac{(n-2k)^2 + \mu^2 n^2}{\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)}\right] .\end{aligned} \tag{4.32}$$

The boundary term vanishes and the expression simplifies to

$$\sigma_n^{(B)} = \sum_{k=1}^{n-1} \frac{\sin^2\left(\frac{\pi k}{n}\right)}{32\pi n^2(n-1)} \int_0^\infty \mathrm{d}\mu \; \frac{(n-2k)^2 + \mu^2 n^2}{\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)} \,. \tag{4.33}$$

The contribution of $k = n/2$ is easy to calculate and is equal to

$$\frac{1}{64\pi\,(n-1)} \int_0^\infty \mathrm{d}\mu \; \frac{\mu^2}{\sinh^2\left(\frac{\pi\mu}{2}\right)} = \frac{1}{48\pi^2\,(n-1)} \,. \tag{4.34}$$

For $k \neq n/2$, there are contributions from two integrals:

$$I^{(1)}_{n;k} = \int_0^\infty \frac{\mathrm{d}\mu}{\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)} = \frac{2\tan^{-1}\left(\tan\left(\frac{\pi k}{n}\right)\right)}{\pi \sin\left(\frac{2k\pi}{n}\right)} = \frac{2}{\sin\left(\frac{2k\pi}{n}\right)} \times \begin{cases} \frac{k}{n}\,, & k < n/2 \\ \frac{k}{n} - 1\,, & k > n/2 \end{cases} \tag{4.35}$$

and

$$\begin{aligned} I^{(2)}_{n;k} &= \int_0^\infty \frac{\mu^2\,\mathrm{d}\mu}{\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)} = \frac{2i\left[\,\mathrm{Li}_3\left(-e^{\frac{2ik\pi}{n}}\right) - \mathrm{Li}_3\left(-e^{-\frac{2ik\pi}{n}}\right)\right]}{\pi^3 \sin\left(\frac{2k\pi}{n}\right)} \\ &= -\frac{i\,\log\left(e^{\frac{2ik\pi}{n}}\right)\left[\pi^2 + \log^2\left(e^{\frac{2ik\pi}{n}}\right)\right]}{3\pi^3\sin\left(\frac{2k\pi}{n}\right)} = \frac{2}{\sin\left(\frac{2k\pi}{n}\right)} \times \begin{cases} \frac{k(n^2-4k^2)}{3n^3}\,, & k < n/2 \\ -\frac{(n-k)(n-2k)(3n-2k)}{3n^3}\,, & k > n/2\,. \end{cases} \end{aligned} \tag{4.36}$$

Above, we manipulated the tri-logarithm $\mathrm{Li}_3$ using the polylog identity

$$\mathrm{Li}_3(z) - \mathrm{Li}_3\left(z^{-1}\right) = -\frac{1}{6}\log^3(-z) - \frac{\pi^2}{6}\log(-z)\;, \tag{4.37}$$

which holds for $z \notin\, ]0,1[$.

Combining the results (4.35) and (4.36), we find that the result is the same for $1 < k <$

$n/2$ and $n/2 < k < n$, namely

$$\int_0^\infty \mathrm{d}\mu\ \frac{(n-2k)^2+\mu^2 n^2}{\cos\left(\frac{2\pi k}{n}\right)+\cosh(\pi\mu)} = (n-2k)^2 I^{(1)}_{n;k} + n^2 I^{(2)}_{n;k} = \frac{8k(n-k)(n-2k)}{3n\ \sin\left(\frac{2k\pi}{n}\right)}\,. \tag{4.38}$$

Thus, having evaluated the integral in (4.33), we can write $\sigma_n^{(B)}$ as the finite sum

$$\sigma_n^{(B)} = \frac{1}{24\pi\ n^3\,(n-1)} \sum_{k=1}^{n-1} k\,(n-k)\,(n-2k)\tan\left(\tfrac{\pi k}{n}\right)\ . \tag{4.39}$$

Note that taking the limit $k \to n/2$ as described below (4.5), the summand evaluates precisely to the special case (4.34). The expression (4.39) is the result for the Rényi corner coefficient presented in (4.5), so this completes our evaluation for the free scalar.

**Free fermion.** For the fermion field, the Rényi corner coefficient is given by the integral

$$\sigma_n^{(F)} = -\frac{2}{n-1} \sum_{k>0}^{\frac{1}{2}(n-1)} \int_0^\infty \mathrm{d}\mu\ \left[a\,(1-a)\,\mu^2 H - \frac{\mu F}{4\pi}\right]_{a=k/n}\,, \tag{4.40}$$

where the sum is over $k$ from $1/2$ ($n$ even) or 1 ($n$ odd) in integer steps to $\frac{1}{2}(n-1)$. Substituting the expressions for $H$ and $F$ obtained earlier gives

$$\begin{aligned}\sigma_n^{(F)} = &\sum_{k>0}^{\frac{1}{2}(n-1)} \frac{\sin^2\left(\frac{\pi k}{n}\right)}{8\pi(n-1)} \\ &\times \int_0^\infty \mathrm{d}\mu\ \frac{2\mu^2\left[\cos\left(\frac{2\pi k}{n}\right)+\cosh(\pi\mu)\right] - \mu\left(\frac{4k^2}{n^2}+\mu^2-1\right)\pi\sinh(\pi\mu)}{\left[\cos\left(\frac{2\pi k}{n}\right)+\cosh(\pi\mu)\right]^2}\,.\end{aligned} \tag{4.41}$$

We then use integration by parts to simplify the integral

$$\sigma_n^{(F)} = \sum_{k>0}^{\frac{1}{2}(n-1)} \frac{\sin^2\left(\frac{\pi k}{n}\right)}{8\pi(n-1)} \int_0^\infty \mathrm{d}\mu\ \left[\frac{\partial}{\partial\mu}\left(\frac{\mu\left(\frac{4k^2}{n^2}+\mu^2-1\right)}{\cos\left(\frac{2\pi k}{n}\right)+\cosh(\pi\mu)}\right) - \frac{\frac{4k^2}{n^2}+\mu^2-1}{\cos\left(\frac{2\pi k}{n}\right)+\cosh(\pi\mu)}\right]. \tag{4.42}$$

The boundary term integrates to zero and the expression simplifies to

$$\sigma_n^{(F)} = \sum_{k>0}^{\frac{1}{2}(n-1)} \frac{\sin^2\left(\frac{\pi k}{n}\right)}{8\pi(n-1)} \int_0^\infty \mathrm{d}\mu \, \frac{1-\mu^2-\frac{4k^2}{n^2}}{\cos\left(\frac{2\pi k}{n}\right)+\cosh(\pi\mu)} \,. \tag{4.43}$$

The result of the integral again involves a difference of two tri-logarithms and it can be simplified using equation (4.37). The result is even in $k \to -k$, so we can write the final answer as

$$\sigma_n^{(F)} = \frac{1}{24\pi \; n^3\,(n-1)} \sum_{k=-\frac{1}{2}(n-1)}^{\frac{1}{2}(n-1)} k\left(n^2-4k^2\right)\tan\left(\frac{\pi k}{n}\right). \tag{4.44}$$

This is the formula we presented in (4.6). Values for low $n$ were tabulated in section 4.1 for both $\sigma_n^{(B)}$ and $\sigma_n^{(F)}$.

## 4.4 Asymptotic Behavior of the Rényi Entropies

Let us now study the large $n$ behavior of the Rényi entropy corner coefficients $\sigma_n$. In particular, we evaluate analytically the value for the coefficients $\sigma_n$ in the limit where $n \to \infty$. This is done by introducing a new variable $x = k/n$ and multiplying by $n\Delta x = 1$. Then in the $n \to \infty$ limit, the sum becomes an integral and we have

$$\sigma_\infty^{(B)} = \frac{1}{24\pi}\int_0^1 \mathrm{d}x\; x\,(x-1)\,(2x-1)\tan\left(\pi x\right) = \frac{3\zeta(3)}{32\pi^4}\,, \tag{4.45}$$

$$\sigma_\infty^{(F)} = \frac{1}{24\pi}\int_{-1/2}^{1/2} \mathrm{d}x\; x\left(1-4x^2\right)\tan\left(\pi x\right) = \frac{\zeta(3)}{4\pi^4}\,. \tag{4.46}$$

These values turn out to be proportional to the asymptotic values of the $\mathcal{F}_n \to n\mathcal{F}_\infty$ calculated on the $n$-covered 3-sphere [123]; as noted in Equation (4.9) we have $\sigma_\infty^{(B/F)} = \frac{1}{4\pi^2}\mathcal{F}_\infty^{(B/F)}$.

On the right of figure 4.1, we illustrated the asymptotic behavior of the corner coefficient which we find to be

$$\sigma_n = \sigma_\infty\left(1+\frac{b_1}{n}+\frac{b_2}{n^2}+\frac{b_3}{n^3}+\dots\right). \tag{4.47}$$

Numerical fits show that $b_1$, $b_2$, and $b_3$ are 1 for the free boson while $b_1$ is 1 and $b_2 = b_3 \approx 0.31578$ in (4.47) for the free fermion. In fact, fitting up to $O(1/n^{16})$, we find numerical evidence that $b_{2k} = b_{2k+1}$ for both the scalar and fermion. This indicates that a factor of $(n+1)/n$ can be factored out of the function in (4.47), so that

$$\sigma_n = \sigma_\infty \frac{n+1}{n}\Big(1 + \frac{b_2}{n^2} + \frac{b_4}{n^4} + \frac{b_6}{n^6} + \dots\Big)\,. \tag{4.48}$$

It is also interesting to study the ratios of the Rényi corner coefficients at large $n$: based on numerical fits in the range $n = 100$ to 2000 we find

$$\frac{\sigma_n^{(B)}}{\sigma_n^{(F)}} = \frac{3}{8}\Big[1 + \frac{\pi^2}{12\zeta(3)}\frac{1}{n^2} - 0.93871149\frac{1}{n^4} + O\Big(\frac{1}{n^5}\Big)\Big]\,. \tag{4.49}$$

The value of the $1/n^2$-coefficient is inferred from the numerics. Specifically, we fit to the function

$$\frac{3}{8}\Big(1 + \frac{d_1}{n} + \frac{d_2}{n^2} + \frac{d_3}{n^3} + \dots\Big)\,, \tag{4.50}$$

and find that $d_1 < 10^{-26}$, $\big|d_2 - \frac{\pi^2}{12\zeta(3)}\big| < 10^{-23}$, $d_3 < 10^{-19}$, $d_4 = -0.93871149\dots$, $d_5 < 10^{-13}$ etc. The vanishing of the odd powers in (4.50) is consistent with (4.48). Note also that we can now identify the number $b_2 = b_3 \approx 0.31578$ from the fit (4.47) of the free fermion Rényi entropy corner coefficient at large $n$ as $1 - \frac{\pi^2}{12\zeta(3)}$; this is the value given in the caption of figure 4.1.

Taking the Hurwitz zeta-function expressions for $\mathcal{F}_n^{(B/F)}$ from [123] and using (4.50) to perform a similar fit at large $n$ in the range 30 to 300, we find

$$\frac{\mathcal{F}_n^{(B)}}{\mathcal{F}_n^{(F)}} = \frac{3}{8}\Big[1 - \frac{\pi^2}{12\zeta(3)}\frac{1}{n^2} + 0.937106586\frac{1}{n^4} + O\Big(\frac{1}{n^5}\Big)\Big]\,. \tag{4.51}$$

Again, the value of the $1/n^2$-coefficient is inferred from the numerics which give $d_1 < 10^{-20}$, $\big|d_2 + \frac{\pi^2}{12\zeta(3)}\big| < 10^{-17}$, $d_3 < 10^{-14}$, $d_4 = 0.937106586\dots$, $d_5 < 10^{-12}$ etc. The behaviors of $\mathcal{F}_n^{(B/F)}$ individually is, however, very different that that of the Rényi corner coefficients. We find that $\mathcal{F}_n^{(B)} \sim n\mathcal{F}_\infty^{(B)}\big(1 + O(\frac{1}{n^4})\big)$ while $\mathcal{F}_n^{(F)} \sim n\mathcal{F}_\infty^{(F)}\big(1 + \frac{\pi^2}{12\zeta(3)}\frac{1}{n^2} + O(\frac{1}{n^4})\big)$.

It is not clear whether the similarities observed at large $n$ between $\sigma_n$ and $\mathcal{F}_n$ have any significance or if it is a coincidence. Perhaps future investigations will clarify this.

# APPENDIX A

# Metric Variation Formulae

We present here a list of formulae that are useful to computing the metric variations of various contractions of curvature tensors:

$$\int \mathrm{d}^d x \sqrt{\gamma}\, X \frac{\delta R}{\delta \gamma^{ij}(y)} = \sqrt{\gamma}\left(R_{ij}X + (\Box X)\gamma_{ij} - \nabla_i \nabla_j X\right), \tag{A.1}$$

$$\int \mathrm{d}^d x \sqrt{\gamma}\, X \frac{\delta (R_{kl}R^{kl})}{\delta \gamma^{ij}(y)} = \sqrt{\gamma}\left(2R_{ik}R^k{}_j X + \nabla_k \nabla_l (X R^{kl})\gamma_{ij} + \Box(X R_{ij}) - 2\nabla^k \nabla_i (X R_{kj})\right), \tag{A.2}$$

$$\int \mathrm{d}^d x \sqrt{\gamma}\, X \frac{\delta R^k{}_{mln}}{\delta \gamma^{ij}(y)} = \sqrt{\gamma}\left(-\frac{1}{2}\nabla_m \nabla_l X \gamma_{in}\delta^k_j - \frac{1}{2}\nabla_n \nabla_l X \gamma_{jm}\delta^k_i + \frac{1}{2}\nabla^k \nabla_l X \gamma_{im}\gamma_{jn}\right), \tag{A.3}$$

$$\begin{aligned}\int \mathrm{d}^d x \sqrt{\gamma}\, X \frac{\delta \Box Y}{\delta \gamma^{ij}(y)} &= \sqrt{\gamma}\left(X \nabla_i \nabla_j Y + \nabla_i (X \nabla_j Y) - \frac{1}{2}\nabla_k (X \nabla^k Y)\gamma_{ij}\right) \\ &\quad + \int \mathrm{d}^d x \sqrt{\gamma}\, \Box X \frac{\delta Y}{\delta \gamma^{ij}(y)}\,. \end{aligned} \tag{A.4}$$

All fields on the RHS of these equations depend on $y$.

# APPENDIX B

# Six-derivative Counterterms for Pure Gravity

In $d = 6$ dimensions one needs to consider counterterms with up to six derivatives. For the pure gravity case, the six-derivative Ansatz is given by equation (2.29). In this Ansatz, it is possible to include terms with contractions of two or three Riemann tensors, but it is easy to show that the coefficients of such terms will be zero.

The HJ equation at six-derivative order becomes

$$\mathcal{K}_{(6)} + 2\frac{\partial U_{(6)}}{\partial r} = 0\,. \tag{B.1}$$

The total derivatives of $Y_{(4)ij}$ that appear in $\mathcal{K}_{(6)}$ are now important because they are multiplied by the non-constant $Y_{(2)ij} = BR_{ij}$. In particular, we have that

$$Y_{(4)ij} = C_1\left(2R^{kl}R_{ikjl}+\frac{1}{2}\Box R\gamma_{ij}+\Box R_{ij}-\nabla_i\nabla_j R\right)+C_2\left(2RR_{ij}+2\Box R\gamma_{ij}-2\nabla_i\nabla_j R\right). \tag{B.2}$$

The coefficients $B$ and $C_{1,2}$ are those calculated in Section 2.3. Additionally, in the product $Y_{(2)ij}{Y_{(4)}}^{ij}$, terms proportional to $R^{ij}\nabla_i\nabla_j R$ can be changed to $R\nabla_i\nabla_j R^{ij} = \frac{1}{2}R\Box R$ by adding appropriate total derivatives and using the Bianchi identity. Finally, by using the variation rules of Appendix A, one realizes that $Y_{(6)} = 3U_{(6)}$ up to total derivative terms that can be ignored because $Y_{(6)}$ is only multiplied by the constant $U_{(0)}$. Putting everything together and demanding that the coefficient of each of the independent terms is zero gives differential equations for the coefficients $D_{1,2,3,4,5,6}$:

$$
\begin{aligned}
&R^3\text{-terms:} && \dot{D}_1 + \frac{d-6}{L} D_1 - \frac{dL^4}{16(d-1)^2(d-2)^3} = 0\,, \\
&RR_{ij}R^{ij}\text{-terms:} && \dot{D}_2 + \frac{d-6}{L} D_2 + \frac{L^4}{4(d-1)(d-2)^2(d-4)} = 0\,, \\
&R_i{}^j R_j{}^k R_k{}^i\text{-terms:} && \dot{D}_3 + \frac{d-6}{L} D_3 = 0\,, \\
&R^{ij}R^{kl}R_{ikjl}\text{-terms:} && \dot{D}_4 + \frac{d-6}{L} D_4 + \frac{2L^4}{(d-2)^3(d-4)} = 0\,, \\
&R\Box R\text{-terms:} && \dot{D}_5 + \frac{d-6}{L} D_5 - \frac{L^4}{4(d-1)(d-2)^3(d-4)} = 0\,, \\
&R_{ij}\Box R^{ij}\text{-terms:} && \dot{D}_6 + \frac{d-6}{L} D_6 + \frac{L^4}{(d-2)^3(d-4)} = 0\,.
\end{aligned}
\tag{B.3}
$$

Keeping only divergent contributions from the solutions of these equations, we obtain the result (for $d = 6$)

$$
U_{(6)} = -\frac{L^4 r}{128}\left( R_{ij}\Box R^{ij} - \frac{1}{20} R\Box R + 2R^{ij}R^{kl}R_{ikjl} + \frac{1}{5} RR_{ij}R^{ij} - \frac{3}{100} R^3 \right). \tag{B.4}
$$

# APPENDIX C

# One-point Functions

In this appendix we calculate the one-point functions for the quantum field theory operators dual to the fields of the FGPW model and explicitly check that the counterterm contributions cancel the divergences that come from the bulk action. One may consider three different one-point functions, $\langle O_\phi \rangle$, $\langle O_\psi \rangle$ and $\langle T_{ij} \rangle$, where the QFT operators $O_{\phi/\psi}$ are dual to the bulk fields $\phi/\psi$ respectively and the QFT energy-momentum tensor $T_{ij}$ is dual to the metric $\gamma_{ij}$.

These one point functions can be calculated by variations of the renormalized action

$$S_{\rm ren} = \lim_{\rho\to 0} S_{\rm reg} = \lim_{\rho\to 0} (S_{\rm bulk} + S_{\rm GH} + S_{\rm ct})\,, \tag{C.1}$$

where the regularized action $S_{\rm reg}$ is the sum of the bulk action (2.57), the Gibbons-Hawking boundary term, and the counterterm action (2.82). In particular, the three correlation functions are given by:

$$\langle O_\phi \rangle = -\lim_{\rho\to 0} \frac{\log\rho}{\rho}\frac{1}{\sqrt{\gamma}}\frac{\delta S_{\rm reg}}{\delta\phi}\,, \quad \langle O_\psi \rangle = -\lim_{\rho\to 0} \frac{1}{\rho^{3/2}}\frac{1}{\sqrt{\gamma}}\frac{\delta S_{\rm reg}}{\delta\psi}\,, \quad \langle T_{ij} \rangle = -\lim_{\rho\to 0} \frac{1}{\rho}\frac{2}{\sqrt{\gamma}}\frac{\delta S_{\rm reg}}{\delta\gamma^{ij}}\,. \tag{C.2}$$

The variation of the bulk action gives only a boundary term since the rest of the contributions are set to zero by the equations of motion. Namely, one gets

$$\begin{aligned}
\frac{\delta S_{\text{bulk}}}{\delta\phi} &= \frac{1}{\kappa^2}\sqrt{\gamma}\Big( -\frac{2}{L}\rho\partial_\rho\phi\Big)\,,\\
\frac{\delta S_{\text{bulk}}}{\delta\psi} &= \frac{1}{\kappa^2}\sqrt{\gamma}\Big( -\frac{2}{L}\rho\partial_\rho\psi\Big)\,,\\
\frac{\delta S_{\text{bulk}}}{\delta\gamma^{ij}} &= \frac{1}{2\kappa^2}\sqrt{\gamma}\frac{\rho}{L}\big(\partial_\rho\gamma_{ij} - \gamma^{mn}\partial_\rho\gamma_{mn}\gamma_{ij}\big)\,.
\end{aligned} \tag{C.3}$$

On the other hand, the variation of the counterterm action has been already calculated during the renormalization process and it is related to the conjugate momenta of the fields:

$$\begin{aligned}
\frac{\delta S_{\text{ct}}}{\delta\phi} &= -\pi_\phi = -\frac{1}{\kappa^2}\sqrt{\gamma}p_\phi\,,\\
\frac{\delta S_{\text{ct}}}{\delta\psi} &= -\pi_\psi = -\frac{1}{\kappa^2}\sqrt{\gamma}p_\psi\,,\\
\frac{\delta S_{\text{ct}}}{\delta\gamma^{ij}} &= -\pi_{ij} = -\frac{1}{\kappa^2}\sqrt{\gamma}\left(Y_{ij} - \frac{1}{2}U\gamma_{ij}\right)\,.
\end{aligned} \tag{C.4}$$

After putting everything together, the following expressions are obtained:

$$\langle O_\phi\rangle = -\frac{1}{\kappa^2}\lim_{\rho\to 0}\frac{\log\rho}{\rho}\left[-\frac{2}{L}\rho\partial_\rho\phi + \frac{2}{L}\left(1+\frac{1}{\log\rho}\right)\phi\right], \tag{C.5}$$

$$\langle O_\psi\rangle = -\frac{1}{\kappa^2}\lim_{\rho\to 0}\frac{1}{\rho^{3/2}}\left[-\frac{2}{L}\rho\partial_\rho\psi + \frac{1}{L}\psi + \left(\frac{1}{3L}\left(1+3c\right)\psi^3 - \frac{L}{2}\left(\Box - \frac{1}{6}R\right)\psi\right)\log\rho\right], \tag{C.6}$$

$$\langle T_{ij}\rangle = -\frac{1}{\kappa^2}\frac{2}{\rho}\left[\frac{1}{2L}\rho\left(\partial_\rho\gamma_{ij} - \gamma_{ij}\gamma^{mn}\partial_\rho\gamma_{mn}\right) - Y_{ij} + \frac{1}{2}U\gamma_{ij}\right], \tag{C.7}$$

with

$$\begin{aligned}
Y_{ij} = \frac{L}{4}R_{ij} + \Bigg[&\frac{L}{24}(R_{ij}\psi^2 + 4\nabla_i\psi\nabla_j\psi - 2\psi\nabla_i\nabla_j\psi - (\nabla\psi)^2\gamma_{ij} - \psi\Box\psi\gamma_{ij})\\
&+ \frac{L^3}{96}(4RR_{ij} - 12R^{kl}R_{kilj} + \Box R\gamma_{ij} + 2\nabla_i\nabla_j R - 6\Box R_{ij})\Bigg]\log\rho\,,
\end{aligned} \tag{C.8}$$

and $U$ as calculated in Section 2.5.

To determine whether the above expressions are finite, one has to use the Fefferman-

Graham expansions for the metric and the scalar fields of the theory:

$$\gamma_{ij} = \frac{1}{\rho}\gamma_{(0)ij} + \big(\gamma_{(2)ij} + \gamma_{(2,1)ij}\log\rho\big) + \rho\big(\gamma_{(4)ij} + \gamma_{(4,1)ij}\log\rho + \gamma_{(4,2)ij}\log^2\rho\big) + \mathcal{O}(\rho^2) \quad \text{(C.9)}$$

$$\psi = \rho^{1/2}\psi_{(0)} + \rho^{3/2}\big(\psi_{(2)} + \psi_{(2,1)}\log\rho\big) + \mathcal{O}(\rho^{5/2}) \quad \text{(C.10)}$$

$$\phi = \rho\big(\phi_{(0)} + \phi_{(0,1)}\log\rho\big) + \mathcal{O}(\rho^2) \quad \text{(C.11)}$$

Notice that for the special case of the $\phi$-field there is a logarithmic term even in leading order in $\rho$. (This is generally true for all fields with scaling dimension $\Delta = d/2$.) All the coefficients of the above expansions can be determined in terms of $\gamma_{(0)ij}$, $\gamma_{(4)ij}$, $\phi_{(0)}$, $\phi_{(0,1)}$, $\psi_{(0)}$ and $\psi_{(2)}$ using the equations of motion for the fields and the metric. These undetermined coefficients encode information about the boundary QFT. Namely, the leading order coefficients $\phi_{(0,1)}$ and $\psi_{(0)}$ are related to the source of the respective QFT operators, while coefficients $\phi_{(0)}$ and $\psi_{(2)}$ are related to their vev rate. Additionally, the leading coefficient $\gamma_{(0)ij}$ in the expansion of $\gamma$ is the background metric of the boundary QFT. Finally, although $\gamma_{(4)ij}$ is not fully determined, its trace and covariant divergence can be related to the other expansion coefficients using Einstein's equation.

The substitution of the expansion (C.11) for $\phi$ into $\langle O_\phi \rangle$ directly leads to cancellation of all of the divergences, without using the equations of motion, and the result is

$$\langle O_\phi \rangle = -\frac{1}{\kappa^2}\frac{2}{L}\phi_{(0)} \,. \quad \text{(C.12)}$$

Plugging the expansion (C.10) for $\psi$ into $\langle O_\psi \rangle$ leads to direct cancellation of the divergent terms in leading order, i.e. those proportional to $1/\rho$, however, a logarithmic divergence remains:

$$\langle O_\psi \rangle = \frac{1}{\kappa^2}\left(\frac{2}{L}\psi_{(2)} + \frac{2}{L}\psi_{(2,1)}\right) + \frac{1}{\kappa^2}\lim_{\rho\to 0}\left[\frac{2}{L}\psi_{(2,1)} - \frac{1}{3L}(1+3c)\psi_{(0)}^3 + \frac{L}{2}\Big(\Box_{(0)} - \frac{1}{6}R_{(0)}\Big)\psi_{(0)}\right]\log\rho\,, \quad \text{(C.13)}$$

where $R_{(0)} \equiv R[\gamma_{(0)}]$ is the Ricci scalar obtained by the metric $\gamma_{(0)}$ and

$$\Box_{(0)}\psi_{(0)} \equiv \frac{1}{\sqrt{\gamma_{(0)}}}\partial_i\Big(\sqrt{\gamma_{(0)}}\gamma_{(0)}^{ij}\partial_j\psi_{(0)}\Big)\,. \quad \text{(C.14)}$$

In order to see the desired cancellations, one has to calculate the expansion coefficient $\psi_{(2,1)}$ via the equation of motion for the field $\psi$,

$$L^2\Box_\gamma\psi + 4\rho^2\partial_\rho^2\psi + 4\rho\partial_\rho\psi + 2\rho^2\partial_\rho\psi\,\mathrm{Tr}(\gamma^{-1}\partial_\rho\gamma) + 3\psi - 2c\psi^3 = 0\,. \quad \text{(C.15)}$$

By the asymptotic expansions for $\psi$ and the metric, the terms proportional to $\rho^{3/2}$ give

$$\psi_{(2,1)} = -\frac{1}{4}\Big(L^2\Box_{(0)} + \mathrm{Tr}(\gamma_{(0)}^{-1}\gamma_{(2)}) - 2c\psi_{(0)}^2\Big)\,\psi_{(0)}\,. \quad \text{(C.16)}$$

Finally, $\gamma_{(2)}$ is determined using Einstein's equation:

$$\mathcal{R}_{\mu\nu}[g] = \partial_\mu\phi\partial_\nu\phi + \partial_\mu\psi\partial_\nu\psi + \frac{1}{3L^2}V(\phi,\psi)g_{\mu\nu}\,. \quad \text{(C.17)}$$

The $ij$ component of this equation is

$$\begin{aligned} L^2R_{ij}[\gamma] = {} & 2\rho^2\partial_\rho^2\gamma_{ij} + 2\rho\partial_\rho\gamma_{ij} + \rho^2\,\mathrm{Tr}(\gamma^{-1}\partial_\rho\gamma)\partial_\rho\gamma_{ij} - 2\rho^2\gamma^{mn}\partial_\rho\gamma_{mi}\partial_\rho\gamma_{nj} \\ & - \frac{1}{2}\rho^2\,\mathrm{Tr}(\gamma^{-1}\partial_\rho\gamma^{-1}\partial_\rho\gamma)\gamma_{ij} + \rho^2\,\mathrm{Tr}(\gamma^{-1}\partial_\rho^2\gamma)\gamma_{ij} + \rho\,\mathrm{Tr}(\gamma^{-1}\partial_\rho\gamma)\gamma_{ij} \\ & + L^2\partial_i\phi\partial_j\phi + L^2\partial_i\psi\partial_j\psi + 2\rho^2(\partial_\rho\psi)^2\gamma_{ij} + 2\rho^2(\partial_\rho\psi)^2\gamma_{ij} + \frac{2}{L^2}V(\phi,\psi)\gamma_{ij}\,. \end{aligned} \quad \text{(C.18)}$$

Expanding it and keeping terms up to $\mathcal{O}(1)$ one finds

$$\gamma_{(2)ij} = -\frac{L^2}{2}\left(R_{(0)ij} - \frac{1}{6}R_{(0)}\gamma_{(0)ij}\right) - \frac{1}{6}\psi^2_{(0)}\gamma_{(0)ij}. \tag{C.19}$$

Now using these results for $\psi_{(2,1)}$ and $\gamma_{(2)}$ in $\langle O_\psi\rangle$ exactly cancels the logarithmic term and gives the following finite result for the one-point function:

$$\langle O_\psi\rangle = \frac{1}{\kappa^2}\left[\frac{2}{L}\psi_{(2)} - \frac{L}{2}\left(\Box_{(0)} - \frac{1}{6}R_{(0)}\right)\psi_{(0)} + \frac{1}{3L}(1+3c)\psi^3_{(0)}\right]. \tag{C.20}$$

A similar approach leads to the renormalized one-point function of the energy-momentum tensor. A direct substitution of the asymptotic expansions in equation (C.7) leads to the cancellation of the leading $\mathcal{O}(\rho^{-2})$ divergences. However, the remaining divergences can be canceled only after solving Einstein's equation for $\gamma_{(4,1)}$ and $\gamma_{(4,2)}$. Terms proportional to $\rho\log\rho$ give

$$\gamma_{(4,2)ij} = -\frac{1}{6}\phi^2_{(0,1)}\gamma_{(0)ij}\,, \tag{C.21}$$

while terms proportional to $\rho$ give

$$\begin{aligned}
\gamma_{(4,1)ij} =& \frac{L^4}{8}\left(R^{kl}_{(0)}R_{(0)ikjl} - \frac{1}{3}R_{(0)}R_{(0)ij}\right) - \frac{L^4}{32}\left(R^{kl}_{(0)}R_{(0)kl} - \frac{1}{3}R^2_{(0)}\right)\gamma_{(0)ij}\\
&+ \frac{L^4}{16}\left(\Box_{(0)}R_{(0)ij} - \frac{1}{3}\nabla_i\nabla_j R_{(0)} - \frac{1}{6}\Box_{(0)}R_{(0)}\gamma_{(0)ij}\right)\\
&+ \frac{L^2}{4}\psi_{(0)}\left(\frac{1}{3}\nabla_i\nabla_j + \frac{1}{6}\gamma_{(0)ij}\Box_{(0)} - \frac{1}{6}R_{(0)ij}\right)\psi_{(0)}\\
&- \frac{L^2}{6}\left(\nabla_i\psi_{(0)}\nabla_j\psi_{(0)} - \frac{1}{4}\gamma^{kl}_{(0)}\nabla_k\psi_{(0)}\nabla_l\psi_{(0)}\gamma_{(0)ij}\right)\\
&- \frac{1}{24}(1+3c)\,\psi^4_{(0)}\gamma_{(0)ij} - \frac{1}{3}\phi_{(0)}\phi_{(0,1)}\gamma_{(0)ij}\,.
\end{aligned} \tag{C.22}$$

Then, the renormalized energy momentum tensor will be given by:

$$
\begin{aligned}
\langle T_{ij}\rangle = &-\frac{2}{L}\gamma_{(4)ij} - \frac{1}{L}\left(\frac{1}{3}\phi_{(0)}^2 - \phi_{(0)}\phi_{(0,1)} + \frac{2}{3}\phi_{(0,1)}^2 - \frac{1}{72}\left(1-3c\right)\psi_{(0)}^4 + \psi_{(0)}\psi_{(2)}\right)\gamma_{(0)ij} \\
&+ \frac{L}{8}\left(\gamma_{(0)}^{kl}\nabla_k\psi_{(0)}\nabla_l\psi_{(0)} + \psi_{(0)}\left(\Box_{(0)} - \frac{1}{9}R_{(0)}\right)\psi_{(0)}\right)\gamma_{(0)ij} \\
&- \frac{L}{4}\psi_{(0)}\left(\nabla_i\nabla_j - \frac{1}{2}R_{(0)ij}\right)\psi_{(0)} + \frac{L^3}{32}\left(R_{(0)kl}R_{(0)}^{kl} + \frac{1}{9}R_{(0)}^2 + \Box_{(0)}R_{(0)}\right)\gamma_{(0)ij} \\
&+ \frac{L^3}{4}\left(R_{(0)i}{}^{k}R_{(0)kj} - \frac{3}{2}R_{(0)}^{kl}R_{(0)ikjl} + \frac{1}{4}\nabla_i\nabla_j R_{(0)} - \frac{3}{4}\Box_{(0)}R_{(0)ij}\right) .
\end{aligned}
\tag{C.23}
$$

The trace of the stress-tensor one-point function gives a much simpler expression, since the trace $\mathrm{Tr}(\gamma_{(0)}^{-1}\gamma_{(4)})$ can be obtained from the $\rho\rho$ component of Einstein's equation, which gives

$$
\rho^2\,\mathrm{Tr}(\gamma^{-1}\partial_\rho\gamma\gamma^{-1}\partial_\rho\gamma) - 2\rho^2\,\mathrm{Tr}(\gamma^{-1}\partial_\rho^2\gamma) - 2\rho\,\mathrm{Tr}(\gamma^{-1}\partial_\rho\gamma) = (2\rho\partial_\rho\phi)^2 + (2\rho\partial_\rho\psi)^2 + \frac{L^2}{3}V(\phi,\psi)\,.
\tag{C.24}
$$

Keeping only terms of order $\mathcal{O}(\rho^2)$ in this yields

$$
\begin{aligned}
\mathrm{Tr}(\gamma_{(0)}^{-1}\gamma_{(4)}) = &\frac{L^4}{16}\left(R_{(0)ij}R_{(0)}^{ij} - \frac{2}{9}R_{(0)}^2\right) - \frac{L^2}{8}\psi_{(0)}\left(\Box_{(0)} - \frac{5}{18}R_{(0)}\right)\psi_{(0)} \\
&- \frac{1}{3}(2\phi_{(0)}^2 + \phi_{(0,1)}^2) + \frac{1}{9}\left(1 + \frac{3}{2}c\right)\psi_{(0)}^4 - \psi_{(0)}\psi_{(2)}\,.
\end{aligned}
\tag{C.25}
$$

After plugging in the above result the trace anomaly becomes

$$
\begin{aligned}
\langle T^i{}_i\rangle = &\frac{1}{L}\left(4\phi_{(0)}\phi_{(0,1)} - 2\phi_{(0,1)}^2 - \frac{1}{6}\left(1+3c\right)\psi_{(0)}^4 - 2\psi_{(0)}\psi_{(2)}\right) \\
&+ \frac{L}{2}\left(\psi_{(0)}\Box_{(0)}\psi_{(0)} + \gamma_{(0)}^{ij}\partial_i\psi_{(0)}\partial_j\psi_{(0)}\right) - \frac{L^3}{8}\left(R_{(0)ij}R_{(0)}^{ij} - \frac{1}{3}R_{(0)}^2\right) .
\end{aligned}
\tag{C.26}
$$

We must emphasize that the above results for the one-point functions are true only up to contributions from finite counterterms in the action.

## APPENDIX D

# Derivation of Manifestly Local Soft-Subtracted Recursion Relation

In this appendix, we derive the manifestly local form (3.19) of the subtracted recursion relations. For a given factorization channel, consider from the recursion relations (3.18) the expression

$$\frac{\hat{\mathcal{A}}_L^{(I)}(z_I^\pm)\hat{\mathcal{A}}_R^{(I)}(z_I^\pm)}{F(z_I^\pm)P_I^2(1-z_I^\pm/z_I^\mp)} = \sum_{z_I=z_I^\pm} \text{Res}_{z=z_I} \frac{\hat{\mathcal{A}}_L^{(I)}(z)\hat{\mathcal{A}}_R^{(I)}(z)}{z\,F(z)\,\hat{P}_I^2} = \oint_{\mathcal{C}} dz\, \frac{\hat{\mathcal{A}}_L^{(I)}(z)\hat{\mathcal{A}}_R^{(I)}(z)}{z\,F(z)\,\hat{P}_I^2}\,, \tag{D.1}$$

where the contour surrounds only the two poles $z_I^\pm$. The second equality is non-trivial and deserves clarification. In the second expression, the subamplitudes $\hat{\mathcal{A}}_L^{(I)}(z)$ and $\hat{\mathcal{A}}_R^{(I)}(z)$ are only defined precisely on the residue values $z = z_I^\pm$ for which the internal momentum $\hat{P}_I$ is on-shell; in general one cannot just think of $\hat{\mathcal{A}}_{L,R}^{(I)}(z)$ as functions of $z$. However, in the product $\hat{\mathcal{A}}_L^{(I)}(z)\hat{\mathcal{A}}_R^{(I)}(z)$, one can eliminate the internal momentum $\hat{P}_I$ in favor of the $n$ shifted external momenta by using momentum conservation. Then the resulting expression can be analytically continued in $z$ away from the residue value. This is implicitly what has been done in performing the second step in (D.1).

Let us assess the large-$z$ behavior of the integrand in (D.1). The L and R subamplitudes have couplings $g_L$ and $g_R$ such that $g_L g_R = g_n$, with $g_n$ the coupling of $\mathcal{A}_n$. Their mass-dimensions are related as $[g_L] + [g_R] = [g_n]$. Hence, using $n_L + n_R = n + 2$ and (3.26), we find that the numerator behaves at large $z$ as

$$\hat{\mathcal{A}}_L^{(I)}(z)\hat{\mathcal{A}}_R^{(I)}(z) \to z^{D_L} z^{D_R} = z^{6-n-[g_n]-\sum_{i=1}^n s_i - 2s_P} = z^{D+2-2s_P}\,, \tag{D.2}$$

where $s_P$ denotes the spin of the particle exchanged on the internal line and $D$ is the large $z$ behavior of the $\mathcal{A}_n$ which we know satisfies $D - \sum_{i=1}^n \sigma_i < 0$, by the assumption that the amplitude $\mathcal{A}_n$ is recursively constructible by the criterion (3.20). We therefore conclude that the integrand in (D.1) behaves as $z^{D-1-\sum_{i=1}^n \sigma_i - 2s_P}$, i.e. it goes to zero as $1/z^2$ or faster. Hence, there is no simple pole at $z \to \infty$.

If we deform the contour, we get the sum over all poles $z \neq z_I^\pm$ in $\frac{\hat{\mathcal{A}}_L^{(I)}(z)\hat{\mathcal{A}}_R^{(I)}(z)}{z\,F(z)\,\hat{P}_I^2}$. Let us *assume that* $\mathcal{A}_L^{(I)}$ *and* $\mathcal{A}_R^{(I)}$ *are both local*: they have no poles and hence we pick up exactly the simple poles at $z = 0$ and $z = 1/a_i$ for $i = 1, 2, \ldots, n$. We then conclude that the soft recursion relations take the form

$$\mathcal{A}_n = \sum_I \sum_{z'=0,\frac{1}{a_1},\ldots,\frac{1}{a_n}} \sum_{|\psi^{(I)}\rangle} \mathrm{Res}_{z=z'} \frac{\hat{\mathcal{A}}_L^{(I)}(z)\hat{\mathcal{A}}_R^{(I)}(z)}{z\,F(z)\,\hat{P}_I^2}\,, \tag{D.3}$$

where $F(z) = \prod_{i=1}^n (1 - a_i z)^{\sigma_i}$. This form of the recursion relation is manifestly rational in the momenta.

Note that only the $z = 0$ residues give pole terms in $\mathcal{A}_n$. Therefore the sum of the $1/a_i$ residues over all channels must be a local polynomial in the momenta.

# APPENDIX E

# Explicit Expressions for Amplitudes

In this appendix, we present expressions for the 4- and 6-point amplitudes of the theories discussed in the main text. The 6-point amplitudes were reconstructed with the 4-point ones as input, by means of the subtracted recursion relations and the the supersymmetry Ward identities also discussed in the main text.

## E.1 Supersymmetric $\mathbb{CP}^1$ NLSM

Below, we list the amplitudes for the $\mathbb{CP}^1$ $\mathcal{N} = 1$ supersymmetric NLSM. This model is discussed in Section 3.6 as an illustration of our methods.

The 4-point amplitudes are:

$$\mathcal{A}_4(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}}) = \frac{1}{\Lambda^2} s_{13}\,, \tag{E.1}$$

$$\mathcal{A}_4(1_Z 2_{\bar{Z}} 3^+_\psi 4^-_\psi) = -\frac{1}{\Lambda^2}[23]\langle 24\rangle = \frac{1}{2\Lambda^2}\langle 4|p_1 - p_2|3]\,, \tag{E.2}$$

$$\mathcal{A}_4(1^+_\psi 2^-_\psi 3^+_\psi 4^-_\psi) = -\frac{1}{\Lambda^2}[13]\langle 24\rangle\,. \tag{E.3}$$

They serve as the input for computing the 6-point amplitudes recursively:

$$\begin{aligned}
&\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}} 5_Z 6_{\bar{Z}}) \\
&\quad = \frac{1}{\Lambda^4}\left[\left(\frac{s_{13}s_{46}}{p_{123}^2} + (1 \leftrightarrow 5) + (3 \leftrightarrow 5)\right) + (2 \leftrightarrow 4) + (2 \leftrightarrow 6) + 3p_{135}^2\right], && \text{(E.4)} \\
&\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}} 5_\psi^+ 6_\psi^-) \\
&\quad = \frac{1}{\Lambda^4}\left[\left(\frac{s_{13}[54]\langle 46\rangle}{p_{123}^2} + (2 \leftrightarrow 4)\right) - \left(\frac{s_{24}[51]\langle 16\rangle}{p_{156}^2} + (1 \leftrightarrow 3)\right)\right. \\
&\qquad \left. - \left(\left(\frac{[54]\langle 4|p_{126}|2]\langle 26\rangle}{p_{126}^2} + (1 \leftrightarrow 3)\right) + (2 \leftrightarrow 4)\right) + \langle 6|p_{135}|5]\right], && \text{(E.5)} \\
&\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_\psi^+ 4_\psi^- 5_\psi^+ 6_\psi^-) \\
&\quad = \frac{1}{\Lambda^4}\left[-\left(\frac{[31]\langle 1|p_{123}|5]\langle 46\rangle}{p_{123}^2} - (3 \leftrightarrow 5)\right) + \left(\frac{[35]\langle 4|p_{126}|2]\langle 26\rangle}{p_{126}^2} - (4 \leftrightarrow 6)\right)\right. \\
&\qquad \left. - \left(\left(\frac{[51]\langle 16\rangle[32]\langle 24\rangle}{p_{156}^2} - (3 \leftrightarrow 5)\right) - (4 \leftrightarrow 6)\right)\right], && \text{(E.6)} \\
&\mathcal{A}_6(1_\psi^+ 2_\psi^- 3_\psi^+ 4_\psi^- 5_\psi^+ 6_\psi^-) \\
&\quad = \frac{1}{\Lambda^4}\left[\left(\frac{[13]\langle 2|p_{123}|5]\langle 46\rangle}{p_{123}^2} - (1 \leftrightarrow 5) - (3 \leftrightarrow 5)\right) - (2 \leftrightarrow 4) - (2 \leftrightarrow 6)\right]. && \text{(E.7)}
\end{aligned}$$

Note that only the pure scalar amplitudes and the 2-fermion amplitudes have local terms. The 6-point amplitudes satisfy the NMHV supersymmetry Ward identities in (3.68)-(3.70).

## E.2 Supersymmetric Dirac-Born-Infeld Theory

The amplitudes of $\mathcal{N} = 1$ supersymmetric Dirac-Born-Infeld theory are all recursively constructible. The 4-point amplitudes are

$$\mathcal{A}_4(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}}) = \frac{1}{\Lambda^4} s_{13}^2 \,, \tag{E.8}$$

$$\mathcal{A}_4(1_Z 2_{\bar{Z}} 3_\psi^+ 4_\psi^-) = \frac{1}{\Lambda^4} s_{13}[32]\langle 24\rangle = \frac{1}{2\Lambda^4} s_{13}\langle 4|p_1 - p_2|3] \,, \tag{E.9}$$

$$\mathcal{A}_4(1_\psi^+ 2_\psi^- 3_\psi^+ 4_\psi^-) = -\frac{1}{\Lambda^4} s_{13}[13]\langle 24\rangle \,. \tag{E.10}$$

and the results of soft subtracted recursion for the 6-point amplitudes are

$$\begin{aligned}
&\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}} 5_Z 6_{\bar{Z}}) \\
&= \frac{1}{\Lambda^8}\left[\left(\frac{s_{13}^2 s_{46}^2}{p_{123}^2} + (1 \leftrightarrow 5) + (3 \leftrightarrow 5)\right) + (2 \leftrightarrow 4) + (2 \leftrightarrow 6) - p_{135}^6\right],
\end{aligned} \tag{E.11}$$

$$\begin{aligned}
&\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}} 5_\psi^+ 6_\psi^-) \\
&= \frac{1}{\Lambda^8}\Bigg[\left(\left(\frac{s_{26}s_{35}[54]\langle 4|p_{126}|1]\langle 16\rangle}{p_{126}^2} + (1 \leftrightarrow 3)\right) + (2 \leftrightarrow 4)\right) + \left(\frac{s_{13}^2 s_{46}[54]\langle 46\rangle}{p_{123}^2} + (2 \leftrightarrow 4)\right) \\
&\qquad - \left(\frac{s_{15}s_{24}^2[51]\langle 16\rangle}{p_{156}^2} + (1 \leftrightarrow 3)\right) + \left(s_{13}s_{24} - (s_{13}+s_{24})\,p_{135}^2\right)\langle 6|p_{24}|5]\Bigg],
\end{aligned} \tag{E.12}$$

$$\begin{aligned}
&\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_\psi^+ 4_\psi^- 5_\psi^+ 6_\psi^-) \\
&= \frac{1}{\Lambda^8}\Bigg[(s_{24}+s_{26})\,p_{135}^2[35]\langle 46\rangle - \left(\left(\frac{s_{15}s_{24}[51]\langle 16\rangle[32]\langle 24\rangle}{p_{156}^2} - (3 \leftrightarrow 5)\right) - (4 \leftrightarrow 6)\right) \\
&\qquad - \left(\frac{s_{13}s_{46}[32]\langle 2|p_{123}|5]\langle 46\rangle}{p_{123}^2} - (3 \leftrightarrow 5)\right) + \left(\frac{s_{26}s_{35}[35]\langle 4|p_{126}|2]\langle 26\rangle}{p_{126}^2} - (4 \leftrightarrow 6)\right)\Bigg],
\end{aligned} \tag{E.13}$$

$$\begin{aligned}
&\mathcal{A}_6(1_\psi^+ 2_\psi^- 3_\psi^+ 4_\psi^- 5_\psi^+ 6_\psi^-) \\
&= \frac{1}{\Lambda^8}\left[\left(\frac{s_{13}s_{46}[13]\langle 2|p_{123}|5]\langle 46\rangle}{p_{123}^2} - (1 \leftrightarrow 5) - (3 \leftrightarrow 5)\right) - (2 \leftrightarrow 4) - (2 \leftrightarrow 6)\right].
\end{aligned} \tag{E.14}$$

The 6-point amplitudes satisfy the NMHV supersymmetry Ward identities in (3.68)-(3.70). As in the case of the NLSM, only the pure scalar amplitudes and the 2-fermion amplitudes have local terms.

## E.3 Supersymmetric Born-Infeld Theory

In this subsection, we list the amplitudes of Born-Infeld theory. This theory is the leading order contribution to the effective field theory of a Goldstone $\mathcal{N}=1$ vector multiplet. The

4-point amplitudes are

$$\mathcal{A}_4(1^+_\psi 2^-_\psi 3^+_\psi 4^-_\psi) = -\frac{1}{\Lambda^4}[13]\langle 24\rangle s_{13}\,, \tag{E.15}$$

$$\mathcal{A}_4(1^+_\psi 2^-_\psi 3^+_\gamma 4^-_\gamma) = \frac{1}{\Lambda^4}[13][23]\langle 24\rangle^2 = -\frac{1}{2\Lambda^4}[13]\langle 4|p_1-p_2|3]\langle 24\rangle\,, \tag{E.16}$$

$$\mathcal{A}_4(1^+_\gamma 2^-_\gamma 3^+_\gamma 4^-_\gamma) = \frac{1}{\Lambda^4}[13]^2\langle 24\rangle^2\,. \tag{E.17}$$

Except for the all-vector amplitudes, all amplitudes are constructible with soft subtracted recursion. The all-vector amplitudes are the amplitudes of Born-Infeld theory, and they are fixed in terms of the other amplitudes using the supersymmetry Ward identities. In particular, at 6-points, we use (3.70) and the remaining five identities in (3.68)-(3.70) are used as checks. The results are

$$\begin{aligned}&\mathcal{A}_6(1^+_\psi 2^-_\psi 3^+_\psi 4^-_\psi 5^+_\psi 6^-_\psi)\\&= \frac{1}{\Lambda^8}\left[\left(\frac{s_{13}s_{46}[13]\langle 2|p_{123}|5]\langle 46\rangle}{p_{123}^2} - (1\leftrightarrow 5) - (3\leftrightarrow 5)\right) - (2\leftrightarrow 4) - (2\leftrightarrow 6)\right],\end{aligned} \tag{E.18}$$

$$\begin{aligned}&\mathcal{A}_6(1^+_\gamma 2^-_\gamma 3^+_\psi 4^-_\psi 5^+_\psi 6^-_\psi)\\&= \frac{1}{\Lambda^8}\Bigg[\left(\frac{s_{46}[13]^2\langle 2|p_{123}|5]\langle 23\rangle\langle 46\rangle}{p_{123}^2} - (3\leftrightarrow 5)\right) + \left(\frac{s_{35}[14][35]\langle 6|p_{124}|1]\langle 24\rangle^2}{p_{124}^2} - (4\leftrightarrow 6)\right)\\&\quad - \left(\left(\frac{[13][14]\langle 4|p_{134}|5]^2\langle 52\rangle\langle 26\rangle}{p_{134}^2} - [13]\langle 2|p_{35}|1]\langle 6|p_{46}|5]\langle 24\rangle - (3\leftrightarrow 5)\right) - (4\leftrightarrow 6)\right)\Bigg],\end{aligned} \tag{E.19}$$

$$\begin{aligned}&\mathcal{A}_6(1^+_\gamma 2^-_\gamma 3^+_\gamma 4^-_\gamma 5^+_\psi 6^-_\psi)\\&= \frac{1}{\Lambda^8}\Bigg[\left(\frac{[13]^2\langle 2|p_{123}|5]^2\langle 54\rangle\langle 46\rangle}{p_{123}^2} + (2\leftrightarrow 4)\right) + \left(\frac{[35][36]\langle 6|p_{124}|1]^2\langle 24\rangle^2}{p_{124}^2} + (1\leftrightarrow 3)\right)\\&\quad + \left(\left(\frac{[15]^2[36]\langle 2|p_{125}|3]\langle 25\rangle\langle 46\rangle^2}{p_{125}^2} + (1\leftrightarrow 3)\right) + (2\leftrightarrow 4)\right) + [13]^2\langle 6|p_{24}|5]\langle 24\rangle^2\Bigg],\end{aligned} \tag{E.20}$$

$$\begin{aligned}&\mathcal{A}_6(1^+_\gamma 2^-_\gamma 3^+_\gamma 4^-_\gamma 5^+_\gamma 6^-_\gamma)\\&= \frac{1}{\Lambda^8}\left[\left(\frac{[13]^2\langle 2|p_{123}|5]^2\langle 46\rangle^2}{p_{123}^2} + (1\leftrightarrow 5) + (3\leftrightarrow 5)\right) + (2\leftrightarrow 4) + (2\leftrightarrow 6)\right].\end{aligned} \tag{E.21}$$

In this case, only $\mathcal{A}_6(1_\gamma^+ 2_\gamma^- 3_\gamma^+ 4_\gamma^- 5_\psi^+ 6_\psi^-)$ and $\mathcal{A}_6(1_\gamma^+ 2_\gamma^- 3_\psi^+ 4_\psi^- 5_\psi^+ 6_\psi^-)$ have local terms.

## E.4 Supersymmetric Quartic Galileon Theory

Below, we list the amplitudes of an $\mathcal{N} = 1$ supersymmetric quartic Galileon. This model was discussed in detail in [60] and reviewed in Section 3.8. The 4-point amplitudes are

$$\mathcal{A}_4(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}}) = \frac{1}{\Lambda^6} s_{12} s_{13} s_{23}\,, \tag{E.22}$$

$$\mathcal{A}_4(1_Z 2_{\bar{Z}} 3_\psi^+ 4_\psi^-) = \frac{1}{\Lambda^6} s_{12} s_{23} [32]\langle 24\rangle = \frac{1}{2\Lambda^6} s_{12} s_{23} \langle 4|p_1 - p_2|3]\,, \tag{E.23}$$

$$\mathcal{A}_4(1_\psi^+ 2_\psi^- 3_\psi^+ 4_\psi^-) = -\frac{1}{\Lambda^6} [13]\langle 24\rangle s_{12} s_{23}\,. \tag{E.24}$$

At 6-point, only the amplitudes with at most two fermions are constructible with soft subtracted recursion relations. The remaining ones are fixed by the supersymmetry Ward identities (3.68)-(3.70), and we find

$$\begin{aligned}
&\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}} 5_Z 6_{\bar{Z}}) \\
&= \frac{1}{\Lambda^{12}} \left[ \left( \frac{s_{12} s_{13} s_{23} s_{45} s_{46} s_{56}}{p_{123}^2} + (1 \leftrightarrow 5) + (3 \leftrightarrow 5) \right) + (2 \leftrightarrow 4) + (2 \leftrightarrow 6) \right],
\end{aligned} \tag{E.25}$$

$$\begin{aligned}
&\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_Z 4_{\bar{Z}} 5_\psi^+ 6_\psi^-) \\
&= \frac{1}{\Lambda^{12}} \left[ \left( \frac{s_{12} s_{13} s_{23} s_{45} s_{56} [54]\langle 46\rangle}{p_{123}^2} + (2 \leftrightarrow 4) \right) - \left( \frac{s_{16} s_{23} s_{24} s_{34} s_{56} [51]\langle 16\rangle}{p_{156}^2} + (1 \leftrightarrow 3) \right) \right. \\
&\qquad \left. + \left( \left( \frac{s_{12} s_{16} s_{34} s_{45} [53]\langle 3|p_{126}|2]\langle 26\rangle}{p_{126}^2} + (1 \leftrightarrow 3) \right) + (2 \leftrightarrow 4) \right) \right],
\end{aligned} \tag{E.26}$$

$$\begin{aligned}
&\mathcal{A}_6(1_Z 2_{\bar{Z}} 3_\psi^+ 4_\psi^- 5_\psi^+ 6_\psi^-) \\
&= \frac{1}{\Lambda^{12}} \left[ \left( \frac{s_{12} s_{23} s_{45} s_{56} [31]\langle 1|p_{46}|5]\langle 46\rangle}{p_{123}^2} - (3 \leftrightarrow 5) \right) \right. \\
&\qquad + \left( \frac{s_{12} s_{16} s_{34} s_{45} [35]\langle 4|p_{16}|2]\langle 26\rangle}{p_{126}^2} - (4 \leftrightarrow 6) \right) \\
&\qquad \left. - \left( \left( \frac{s_{16} s_{23} s_{34} s_{56} [32]\langle 24\rangle [51]\langle 16\rangle}{p_{156}^2} - (3 \leftrightarrow 5) \right) - (4 \leftrightarrow 6) \right) \right],
\end{aligned} \tag{E.27}$$

$$\begin{aligned}&\mathcal{A}_6(1_\psi^+ 2_\psi^- 3_\psi^+ 4_\psi^- 5_\psi^+ 6_\psi^-)\\&=\frac{1}{\Lambda^{12}}\left[\left(\frac{s_{12}s_{23}s_{45}s_{56}[13]\langle 2|p_{13}|5]\langle 46\rangle}{p_{123}^2}-(1\leftrightarrow 5)-(3\leftrightarrow 5)\right)-(2\leftrightarrow 4)-(2\leftrightarrow 6)\right].\end{aligned} \tag{E.28}$$

None of the amplitudes have local terms.

## E.5 Chiral Perturbation Theory

Below, we list the color-ordered amplitudes of the $\frac{U(N)\times U(N)}{U(N)}$ sigma model, with higher derivative corrections, referred to as chiral perturbation theory in the main text. Different color orderings are related to the ones listed by momentum relabelling. At 4-point we have

$$\mathcal{A}_4[1,2,3,4]=\frac{g_2}{\Lambda^2}t+\frac{g_6}{\Lambda^6}t\left(s^2+t^2+u^2\right)+\frac{g_8}{\Lambda^8}st^2u+\mathcal{O}(\Lambda^{-10}) \tag{E.29}$$

and at 6-point

$$
\begin{aligned}
&\mathcal{A}_6[1,2,3,4,5,6] \\
&= \frac{g_2^2}{\Lambda^4}\left[\frac{s_{13}s_{46}}{p_{123}^2} + \frac{s_{24}s_{15}}{p_{234}^2} + \frac{s_{35}s_{26}}{p_{345}^2} - s_{24} - s_{26} - s_{46}\right] \\
&+ \frac{g_2 g_6}{\Lambda^8}\left[\frac{s_{13}s_{46}}{p_{123}^2}\left(s_{12}^2 + s_{13}^2 + s_{23}^2 + s_{45}^2 + s_{46}^2 + s_{56}^2\right)\right. \\
&\quad + \frac{s_{24}s_{15}}{p_{234}^2}\left(s_{23}^2 + s_{24}^2 + s_{34}^2 + s_{56}^2 + s_{15}^2 + s_{16}^2\right) + \frac{s_{35}s_{26}}{p_{345}^2}\left(s_{34}^2 + s_{35}^2 + s_{45}^2 + s_{16}^2 + s_{26}^2 + s_{12}^2\right) \\
&\quad - 2\big(s_{26}^3 + s_{23}s_{26}^2 + s_{25}s_{26}^2 + s_{34}s_{26}^2 + s_{45}s_{26}^2 + s_{23}^2 s_{26} + s_{25}^2 s_{26} + s_{34}^2 s_{26} + s_{35}^2 s_{26} + s_{45}^2 s_{26} \\
&\qquad + s_{23}s_{34}s_{26} + s_{23}s_{35}s_{26} + s_{25}s_{35}s_{26} + s_{34}s_{36}s_{26} + s_{23}s_{45}s_{26} + s_{34}s_{45}s_{26} + s_{36}s_{45}s_{26} \\
&\qquad + s_{46}^3 + s_{24}s_{25}^2 + s_{24}s_{35}^2 + s_{24}s_{45}^2 + s_{23}s_{46}^2 + s_{25}s_{46}^2 + s_{34}s_{46}^2 + s_{35}s_{46}^2 + s_{36}s_{46}^2 \\
&\qquad + s_{45}s_{46}^2 + s_{24}s_{35}s_{36} + s_{25}^2 s_{46} + s_{34}^2 s_{46} + s_{35}^2 s_{46} + s_{36}^2 s_{46} + s_{45}^2 s_{46} + s_{23}s_{25}s_{46} \\
&\qquad + s_{25}s_{34}s_{46} + s_{23}s_{45}s_{46} + s_{34}s_{45}s_{46} + s_{35}s_{45}s_{46} + s_{36}s_{45}s_{46}\big) \\
&\quad - 4\big(s_{24}^3 + s_{25}s_{24}^2 + s_{35}s_{24}^2 + s_{45}s_{24}^2 + s_{23}^2 s_{24} + s_{34}^2 s_{24} + s_{36}^2 s_{24} + s_{23}s_{25}s_{24} + s_{25}s_{34}s_{24} \\
&\qquad + s_{23}s_{35}s_{24} + s_{25}s_{35}s_{24} + s_{34}s_{35}s_{24} + s_{26}s_{36}s_{24} + s_{23}s_{45}s_{24} + s_{25}s_{45}s_{24} + s_{34}s_{45}s_{24} \\
&\qquad + s_{35}s_{45}s_{24} + s_{36}s_{45}s_{24} + s_{23}s_{25}s_{26} + s_{25}s_{26}s_{34} + s_{25}s_{26}s_{45} + s_{23}^2 s_{46} + s_{25}s_{26}s_{46} \\
&\qquad + s_{23}s_{34}s_{46} + s_{23}s_{35}s_{46} + s_{34}s_{35}s_{46} + s_{23}s_{36}s_{46} + s_{25}s_{36}s_{46} + s_{26}s_{36}s_{46} + s_{34}s_{36}s_{46} \\
&\qquad + s_{35}s_{36}s_{46} + s_{25}s_{45}s_{46} + s_{26}s_{45}s_{46}\big) \\
&\quad - 6\big(s_{23}s_{24}^2 + s_{34}s_{24}^2 + s_{36}s_{24}^2 + s_{26}^2 s_{24} + s_{46}^2 s_{24} + s_{23}s_{26}s_{24} + s_{25}s_{26}s_{24} + s_{23}s_{34}s_{24} \\
&\qquad + s_{26}s_{34}s_{24} + s_{23}s_{36}s_{24} + s_{25}s_{36}s_{24} + s_{26}s_{45}s_{24} + s_{25}s_{46}s_{24} + s_{35}s_{46}s_{24} + s_{45}s_{46}s_{24} \\
&\qquad + s_{26}s_{46}^2 + s_{25}s_{34}s_{36} + s_{25}s_{36}s_{45} + s_{26}^2 s_{46} + s_{23}s_{26}s_{46} + s_{26}s_{34}s_{46}\big) \\
&\quad \left. - 8s_{24}\big(s_{24}s_{26} + s_{34}s_{36} + s_{23}s_{46} + s_{24}s_{46} + s_{34}s_{46} + s_{36}s_{46}\big) - 12 s_{24}s_{26}s_{46}\right] + \mathcal{O}(\Lambda^{-10})\,.
\end{aligned}
\tag{E.30}
$$

These amplitudes are discussed in further detail in Section 3.8.4.

## APPENDIX F

# Recursion Relations and Ward Identities

We show that if the seed amplitudes of a recursive theory satisfy a set of Ward identities, then all recursively constructible $n$-point amplitudes also satisfy them. For Abelian groups, this follows from two features:

(a) additive charges have Ward identities that simply state that the sum of charges of the states in an amplitude must vanish.

(b) CPT conjugate states sitting on either end of a factorization channel have equal and opposite charges.

Hence recursion will result in amplitudes that respect the Abelian symmetry so long as the seed amplitudes do.

Now consider Ward identities generated by elements of a semi-simple Lie algebra. In the root space decomposition of the algebra, we can choose a triplet of generators: raising operators $\mathcal{T}_+$, lowering operators $\mathcal{T}_-$, and "diagonal" $\mathcal{T}_0$ generators, for each positive root that satisfy the algebra

$$[\mathcal{T}_+, \mathcal{T}_-] = \mathcal{T}_0\,, \qquad [\mathcal{T}_+, \mathcal{T}_0] = -2\mathcal{T}_+\,, \qquad [\mathcal{T}_-, \mathcal{T}_0] = 2\mathcal{T}_-\,. \tag{F.1}$$

In order for representations of this algebra to be physical, CPT must be an algebra automorphism. The CPT charge conjugation generator $\mathcal{C}$ must also flip the sign of the additive

$\mathcal{T}_0$-charge. So we determine the action of $\mathcal{C}$ to be

$$\begin{aligned}
\mathcal{C} \cdot \mathcal{T}_0 \cdot X &= -\mathcal{T}_0 \cdot \mathcal{C} \cdot X = -\mathcal{T}_0 \cdot \tilde{X}\,, \\
\mathcal{C} \cdot \mathcal{T}_+ \cdot X &= -\mathcal{T}_- \cdot \mathcal{C} \cdot X = -\mathcal{T}_- \cdot \tilde{X}\,, \\
\mathcal{C} \cdot \mathcal{T}_- \cdot X &= -\mathcal{T}_+ \cdot \mathcal{C} \cdot X = -\mathcal{T}_+ \cdot \tilde{X}\,,
\end{aligned} \tag{F.2}$$

where $X$ is a physical state and we have defined the conjugate state $\tilde{X}$ to be the charge conjugate of $X$, i.e. $\tilde{X} = \mathcal{C} \cdot X$.

If the S-matix is recursively constructible (at some order in the derivative expansion) then each $n$-point amplitude is given as a sum over factorization singularities with residues given in terms of a product of amplitudes with fewer external states

$$\mathcal{A}_n(1, \cdots, n) = \sum_I \sum_X \operatorname*{Res}_{z=z_I^\pm} \frac{\hat{\mathcal{A}}_L^{(I)}(z)\hat{\mathcal{A}}_R^{(I)}(z)}{z\hat{P}_I(z)^2 F(z)}\,, \tag{F.3}$$

where $I$ labels all possible factorization channels and $X$ the exchanged internal states. Since $\mathcal{T}_0$ is diagonal, the Ward identity generated by $\mathcal{T}_0$ works just like in the Abelian case – charges can be assigned to the physical states and recursion preserves this charge in any $n$-point amplitude. More complicated are the non-diagonal generators $\mathcal{T}_\pm$. For simplicity, we present the argument explicitly for $SU(2)_R$ Ward identities as they apply to the $\mathcal{N} = 2$ NLSM described in Section 3.6.2. For $SU(2)_R$, the action of $\mathcal{T}_+$ on the fermion helicity states is given in (3.93). The scalar and vectors are singlets under $SU(2)_R$.

The statement of the $SU(2)_R$ Ward identity is that $\mathcal{T}_+ \cdot \mathcal{A}_n(1, ..., n) = 0$. The inductive assumption is that this holds true for the lower-point amplitudes in the recursive expression for $\mathcal{A}_n(1, ..., n)$. We already know from Section 3.6.2 that $SU(2)_R$ is a symmetry of the 3- and 4-point amplitudes, so that provides the basis of induction.

The action of $\mathcal{T}_+$ on the recursive expression for an $n$-point amplitude is

$$\begin{aligned}\mathcal{T}_+ \cdot \mathcal{A}_n(1, ..., n) &\equiv \sum_{i=1}^{n} (-1)^{P_i} \mathcal{A}_n(1, ..., \mathcal{T}_+ \cdot i, ..., n) \qquad \text{(F.4)} \\ &= \sum_I \sum_X \operatorname*{Res}_{z=z_I^\pm} \Bigg[ \sum_{i\in I} (-1)^{P_i} \frac{\hat{\mathcal{A}}_L^{(I)}(\dots, \mathcal{T}_+ \cdot i, \dots, X)\hat{\mathcal{A}}_R^{(I)}(\dots)}{z\hat{P}_I(z)^2 F(z)} \\ &\quad + \sum_{i\notin I} (-1)^{P_i} \frac{\hat{\mathcal{A}}_L^{(I)}(\dots)\hat{\mathcal{A}}_R^{(I)}(\tilde{X}, \dots, \mathcal{T}_+ \cdot i, \dots)}{z\hat{P}_I(z)^2 F(z)} \Bigg], \qquad \text{(F.5)}\end{aligned}$$

where $P_i = 0$ or $1$ corresponds to the additional signs in the prefactors for the action of $\mathcal{T}_+$ as given in Table 3.93. We now prove that this expression vanishes channel by channel. Without loss of generality, we will show that the contribution from the $(1 \dots k)^\pm$ channel vanishes independently, where $+$ means the contribution from the $z^\pm$ residue. The argument follows for all other factorization channels by replacing $(1 \dots k)^\pm$ by $I^\pm$. For the $(1 \dots k)$-channel, the relevant part of (F.4) that we want to show vanishes is

$$\begin{aligned}\sum_X \Bigg[ \Bigg( \sum_{i=1}^{k} (-1)^{P_i} \hat{\mathcal{A}}_L(1, \dots, \mathcal{T}_+ \cdot i, \dots, k, X) \Bigg) \hat{\mathcal{A}}_R(\tilde{X}, k+1, \dots, n) \\ + \hat{\mathcal{A}}_L(1, \dots, k, X) \Bigg( \sum_{i=k+1}^{n} (-1)^{P_i} \hat{\mathcal{A}}_R(\tilde{X}, k+1, \dots, \mathcal{T}_+ \cdot i, \dots, n) \Bigg) \Bigg]. \qquad \text{(F.6)}\end{aligned}$$

By the inductive assumption, the lower-point amplitudes respect the $\mathcal{T}_+$ Ward identities

$$\sum_{i=1}^{k} (-1)^{P_i} \hat{\mathcal{A}}_L(1, \dots, \mathcal{T}_+ \cdot i, \dots, k, X) = (-1)^{P_X+1} \hat{\mathcal{A}}_L(1, \dots, k, \mathcal{T}_+ \cdot X), \qquad \text{(F.7)}$$

and similarly for $\hat{\mathcal{A}}_R$. Using this relation and splitting the sum over particles $X$ allows us to rewrite (F.6) as

$$\begin{aligned}&- \sum_X (-1)^{P_X} \Big[ \hat{\mathcal{A}}_L(1, \dots, k, \mathcal{T}_+ \cdot X) \hat{\mathcal{A}}_R(\tilde{X}, k+1, \dots, n) \Big] \\ &- \sum_{X'} (-1)^{P_{\tilde{X}'}} \Big[ \hat{\mathcal{A}}_L(1, \dots, k, X') \hat{\mathcal{A}}_R(\mathcal{T}_+ \cdot \tilde{X}', k+1, \dots, n) \Big]. \end{aligned} \qquad \text{(F.8)}$$

In the second line we have made a change of dummy summation variable that we now exploit further.

It is non-trivial, but turns out to be true for $SU(2)_R$ as we have explicitly checked, that if we define $X' = \mathcal{T}_+ \cdot X$ and sum over $X$ instead of $X'$, the second line of (F.8) gives exactly the same result. We can then write (F.8) as

$$-\sum_X \Big[(-1)^{P_X} \hat{\mathcal{A}}_L(1,\ldots,k,\mathcal{T}_+ \cdot X)\hat{\mathcal{A}}_R(\tilde{X},k+1,\ldots,n) \\ + (-1)^{P_{\tilde{X}'}} \hat{\mathcal{A}}_L(1,\ldots,k,\mathcal{T}_+ \cdot X)\hat{\mathcal{A}}_R(\mathcal{T}_+ \cdot C \cdot \mathcal{T}_+ \cdot X, k+1,\ldots,n)\Big]\,. \tag{F.9}$$

Since $\mathcal{T}_+ \cdot C \cdot \mathcal{T}_+ \cdot X = \mathcal{T}_+ \cdot \mathcal{T}_- \cdot \tilde{X}$, this becomes

$$-\sum_X \Big[(-1)^{P_X} \hat{\mathcal{A}}_L(1,\ldots,k,\mathcal{T}_+ \cdot X)\hat{\mathcal{A}}_R(\tilde{X},k+1,\ldots,n) \\ + (-1)^{P_{\mathcal{T}_- \cdot \tilde{X}} + Q_{\tilde{X}} + 1} \hat{\mathcal{A}}_L(1,\ldots,k,\mathcal{T}_+ \cdot X)\hat{\mathcal{A}}_R(\mathcal{T}_+ \cdot \mathcal{T}_- \cdot \tilde{X}, k+1,\ldots,n)\Big]\,. \tag{F.10}$$

where $Q_X$ refers to the prefactors for the action of $\mathcal{T}_-$ as given in Table 3.93. This vanishes when $\mathcal{T}_+ \cdot \mathcal{T}_- \cdot \tilde{X} = \tilde{X}$ and $P_{\mathcal{T}_- \cdot \tilde{X}} + Q_{\tilde{X}} = 0$ for any state $X$ such that $\mathcal{T}_+ \cdot X \neq 0$. For $SU(2)_R$, we can check explicitly that these conditions are satisfied. The only states for which $\mathcal{T}_+ \cdot X \neq 0$ are $X = \psi^{2+}$ and $\psi_1^-$. Their conjugates are $\tilde{X} = \psi_2^-$ and $\psi^{2+}$, respectively, and by (3.93) we have

$$\mathcal{T}_+ \cdot \mathcal{T}_- \cdot \psi^{1+} = \mathcal{T}_+ \cdot \psi^{2+} = \psi^{1+} \qquad \mathcal{T}_+ \cdot \mathcal{T}_- \cdot \psi_2^- = \mathcal{T}_+ \cdot \psi_1^- = \psi_2^-\,, \tag{F.11}$$

$$P_{\mathcal{T}_- \cdot \psi^{1+}} + Q_{\psi^{1+}} = 0 + 0 = 0 \qquad P_{\mathcal{T}_- \cdot \psi_2^-} + Q_{\psi_2^-} = 1 + 1 = 0 \ (\mathrm{mod}\ 2)\,. \tag{F.12}$$

If follows that from the inductive step that all amplitudes satisfy the $SU(2)_R$ Ward identities when the seed amplitudes do.

## APPENDIX G

# Amplitude Relations in $\mathcal{N} = 2$ $\mathbb{CP}^1$ NLSM

Below are explicit formulae, derived from $\mathcal{N} = 2$ supersymmetry Ward identities, for all amplitudes in this model with total spin $\leq 1$ expressed as linear combinations of amplitudes with strictly greater total spin. Collectively these formulae allow us to construct every tree-level amplitude in the $\mathcal{N} = 2$ $\mathbb{CP}^1$ sigma model using unsubtracted recursion. The needed relations are:

$$\begin{aligned}&\mathcal{A}_{2n}\left(1_Z, 2_{\bar{Z}}, 3_Z, 4_{\bar{Z}} ..., (2n)_{\bar{Z}}\right)\\&= \sum_{k=1}^{n-1} \frac{\langle 1, 2k+1\rangle}{\langle 12\rangle} \mathcal{A}_{2n}\left(1_Z, 2^-_{\psi_1}, 3_Z, 4_{\bar{Z}}, ..., (2k+1)^+_{\psi^1}, ..., (2n)_{\bar{Z}}\right)\end{aligned} \tag{G.1}$$

$$\begin{aligned}&\mathcal{A}_{2n}\left(1^+_{\psi^1}, 2^-_{\psi_1}, 3_Z, 4_{\bar{Z}}, ..., (2n)_{\bar{Z}}\right)\\&= \sum_{k=1}^{n-1} \frac{[2, 2k+2]}{[21]} \mathcal{A}_{2n}\left(1^+_{\gamma}, 2^-_{\psi_1}, 3_Z, 4_{\bar{Z}}, ..., (2k+2)^-_{\psi_2}, ..., (2n)_{\bar{Z}}\right)\end{aligned} \tag{G.2}$$

$$\begin{aligned}&\mathcal{A}_{2n+1}\left(1^+_{\gamma}, 2^+_{\gamma}, 3_Z, 4_{\bar{Z}}, ..., (2n+1)_Z\right)\\&= \sum_{k=1}^{n-2} \frac{\langle 3, 2k+3\rangle}{\langle 34\rangle} \mathcal{A}_{2n+1}\left(1^+_{\gamma}, 2^+_{\gamma}, 3_Z, 4^-_{\psi_2}, 5_Z, ..., (2k+3)^+_{\psi^2}, ..., (2n+1)_Z\right)\end{aligned} \tag{G.3}$$

$$\begin{aligned}&\mathcal{A}_{2n}\left(1^+_{\psi^1}, 2^-_{\gamma}, 3^+_{\psi^2}, 4_{\bar{Z}}, 5_Z, ..., (2n)_{\bar{Z}}\right)\\&= \sum_{k=1}^{n-1} \frac{[3, 2k+2]}{[31]} \mathcal{A}_{2n}\left(1^+_{\gamma}, 2^-_{\gamma}, 3^+_{\psi^2}, 4_{\bar{Z}}, ..., (2k+2)^-_{\psi_2}, ..., (2n)_{\bar{Z}}\right)\end{aligned} \tag{G.4}$$

$$\mathcal{A}_{2n}\left(1_\gamma^+, 2_\gamma^-, 3_Z, 4_{\bar{Z}}, 5_Z, ..., (2n)_{\bar{Z}}\right)$$
$$= \sum_{k=1}^{n-1} \frac{[1, 2k+2]}{[13]} \mathcal{A}_{2n}\left(1_\gamma^+, 2_\gamma^-, 3_{\psi^2}^+, 4_{\bar{Z}}, ..., (2k+2)_{\psi_2}^-, ..., (2n)_{\bar{Z}}\right) \tag{G.5}$$

$$\mathcal{A}_{2n+1}\left(1_\gamma^+, 2_{\psi^1}^+, 3_{\psi^2}^+, 4_Z, 5_{\bar{Z}}, ..., (2n+1)_{\bar{Z}}\right)$$
$$= -\frac{\langle 42\rangle}{\langle 45\rangle} \mathcal{A}_{2n+1}\left(1_\gamma^+, 2_\gamma^+, 3_{\psi^2}^+, 4_Z, 5_{\psi_2}^-, 6_Z, ..., (2n+1)_{\bar{Z}}\right)$$
$$+ \sum_{k=1}^{n-2} \frac{\langle 4, 2k+4\rangle}{\langle 45\rangle} \mathcal{A}_{2n+1}\left(1_\gamma^+, 2_{\psi^1}^+, 3_{\psi^3}^+, 4_Z, 5_{\psi_2}^-, 6_Z, ..., (2k+4)_{\psi^2}^+, ..., (2n+1)_{\bar{Z}}\right) \tag{G.6}$$

$$\mathcal{A}_{2n}\left(1_{\psi^1}^+, 2_{\psi_1}^-, 3_{\psi^2}^+, 4_{\psi_2}^-, 5_Z, 6_{\bar{Z}}, ..., (2n)_{\bar{Z}}\right)$$
$$= \frac{[32]}{[31]} \mathcal{A}_{2n}\left(1_\gamma^+, 2_\gamma^-, 3_{\psi^2}^+, 4_{\psi_2}^-, 5_Z, 6_{\bar{Z}}, ..., (2n)_{\bar{Z}}\right)$$
$$+ \sum_{k=1}^{n-2} \frac{[3, 2k+4]}{[31]} \mathcal{A}_{2n}\left(1_\gamma^+, 2_{\psi_1}^-, 3_{\psi^2}^+, 4_{\psi_2}^-, 5_Z, ..., (2k+4)_{\psi_2}^-, ..., (2n)_{\bar{Z}}\right) \tag{G.7}$$

$$\mathcal{A}_{2n}\left(1_{\psi^1}^+, 2_{\psi_1}^-, 3_{\psi^1}^+, 4_{\psi_1}^-, 5_Z, 6_{\bar{Z}}, ..., (2n)_{\bar{Z}}\right)$$
$$= \frac{[42]}{[41]} \mathcal{A}_{2n}\left(1_\gamma^+, 2_\gamma^-, 3_{\psi^1}^+, 4_{\psi_1}^-, 5_Z, 6_{\bar{Z}}, ..., (2n)_{\bar{Z}}\right)$$
$$+ \sum_{k=1}^{n-2} \frac{[4, 2k+4]}{[41]} \mathcal{A}_{2n}\left(1_\gamma^+, 2_{\psi_1}^-, 3_{\psi^1}^+, 4_{\psi_1}^-, 5_Z, ..., (2k+4)_{\psi_2}^-, ..., (2n)_{\bar{Z}}\right) \tag{G.8}$$

$$\mathcal{A}_{2n+1}\left(1_{\psi^1}^+, 2_{\psi^1}^+, 3_{\psi^2}^+, 4_{\psi^2}^+, 5_{\bar{Z}}, 6_Z, ..., (2n+1)_{\bar{Z}}\right)$$
$$= -\frac{\langle 21\rangle}{\langle 25\rangle} \mathcal{A}_{2n+1}\left(1_\gamma^+, 2_{\psi^1}^+, 3_{\psi^2}^+, 4_{\psi^2}^+, 5_{\psi_2}^-, 6_Z, 7_{\bar{Z}}, ..., (2n+1)_{\bar{Z}}\right)$$
$$+ \sum_{k=1}^{n-2} \frac{\langle 2, 2k+4\rangle}{\langle 25\rangle} \mathcal{A}_{2n+1}\left(1_{\psi^1}^+, 2_{\psi^1}^+, 3_{\psi^2}^+, 4_{\psi^2}^+, 5_{\psi_2}^-, 6_Z, ..., (2k+4)_{\psi^2}^+, ..., (2n+1)_{\bar{Z}}\right). \tag{G.9}$$

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