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NEIGHBORLINESS OF THE SYMMETRIC MOMENT CURVE

ALEXANDER BARVINOK, SEUNG JIN LEE AND ISABELLA NOVIK

Abstract. We consider the convex hull $\mathcal{B}_k$ of the symmetric moment curve $U_k(t) = (\cos t, \sin t, \cos 3t, \sin 3t, \ldots, \cos(2k-1)t, \sin(2k-1)t)$ in $\mathbb{R}^{2k}$, where t ranges over the unit circle $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$. The curve $U_k(t)$ is locally neighborly: as long as $t_1, \ldots, t_k$ lie in an open arc of $\mathbb{S}$ of a certain length $\phi_k > 0$, the convex hull of the points $U_k(t_1), \ldots, U_k(t_k)$ is a face of $\mathcal{B}_k$. We characterize the maximum possible length ϕ_k, proving, in particular, that $\phi_k > \pi/2$ for all k and that the limit of ϕ_k is $\pi/2$ as k grows. This allows us to construct centrally symmetric polytopes with a record number of faces.

§1. *Introduction and main results.* The main object of this paper is the *symmetric moment curve* that for a fixed k lies in $\mathbb{R}^{2k}$ and is defined by

$$U(t) = U_k(t) = (\cos t, \sin t, \cos 3t, \sin 3t, \ldots, \cos(2k-1)t, \sin(2k-1)t).$$

We note that

$$U(t+\pi) = -U(t) \quad \text{for all } t \in \mathbb{R}.$$

Since U is periodic, we consider U to be defined on the unit circle $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$. In particular, for every $t \in \mathbb{S}$, the points t and $t+\pi$ are antipodal points on the circle.

We define the convex body $\mathcal{B} \subset \mathbb{R}^{2k}$ as the convex hull of the symmetric moment curve

$$\mathcal{B} = \mathcal{B}_k = \operatorname{conv}(U(t) : t \in \mathbb{S}).$$

Hence $\mathcal{B}$ is symmetric about the origin, $\mathcal{B} = -\mathcal{B}$. We note that $\mathcal{B}_k$ has a non-empty interior in $\mathbb{R}^{2k}$ since $U_k(t)$ does not lie in an affine hyperplane.

We are interested in the facial structure of $\mathcal{B}$ (a *face* of a convex body is the intersection of the body with a supporting affine hyperplane; see, for example, [**1**, Ch. II]). The convex body $\mathcal{B}_k$ was introduced in [**3**] in the hope that an appropriate discretization of $\mathcal{B}_k$ produces centrally symmetric polytopes with many faces (although $\mathcal{B}_2$ was considered first by Smilansky [**14**] within a certain family of four-dimensional convex bodies). Besides being of intrinsic interest, such polytopes appear in problems of sparse signal reconstruction (see [**5, 13**]). An analogy with the (ordinary) trigonometric moment curve $M(t) \subset \mathbb{R}^{2k}$,

$$M(t) = (\cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos kt, \sin kt),$$

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provided the initial motivation. The convex hull of a set of arbitrary n points on the curve $M(t)$ is the *cyclic polytope*, which was first investigated by Carathéodory [4] and later by Motzkin [12] and Gale [7]. In the case of an odd dimension $2k+1$ one can define the cyclic polytope as the convex hull of n points on the curve $(t, t^2, \ldots, t^{2k+1})$. Cyclic polytopes maximize the number of faces of all dimensions in the class of polytopes of a given dimension and with a given number of vertices. This is the famous upper bound theorem conjectured by Motzkin [12] and proved by McMullen [11]. Such maximizers in the class of centrally symmetric polytopes are not known at present.

There are certain similarities between the curves $M(t)$ and $U(t)$. An affine hyperplane in $\mathbb{R}^{2k}$ intersects $M(t)$ in no more than $2k$ points and it is clear that the bound $2k$ cannot be made smaller. An affine hyperplane in $\mathbb{R}^{2k}$ intersects $U(t)$ in no more than $4k-2$ points (see Theorem 3.1 below) and it is again clear that the bound $4k-2$ cannot be made smaller in the class of centrally symmetric curves, since a hyperplane passing through the origin and some $2k-1$ points on the curve will necessarily intersect the curve also in the antipodal $2k-1$ points.

One crucial feature of the convex hull

$$\mathcal{C} = \mathcal{C}_k = \operatorname{conv}(M(t) : t \in \mathbb{S})$$

of the standard trigonometric moment curve is that it is *neighborly*, namely that for any $n \leqslant k$ distinct points $t_1, \ldots, t_n \in \mathbb{S}$ the convex hull $\operatorname{conv}(M(t_1), \ldots, M(t_n))$ is a face of $\mathcal{C}_k$ (see, for example, [1, Ch. II]). One of our main results is that the convex hull $\mathcal{B}_k$ of the symmetric moment curve is neighborly to a large extent.

THEOREM 1.1. *For every positive integer k there exists a number*

$$\frac{\pi}{2} < \phi_k < \pi$$

such that for an arbitrary open arc $\Gamma \subset \mathbb{S}$ of length ϕ_k and arbitrary distinct $n \leqslant k$ points $t_1, \ldots, t_n \in \Gamma$, the set

$$\operatorname{conv}(U(t_1), \ldots, U(t_n))$$

is a face of $\mathcal{B}_k$.

It is worth mentioning that Lemma 3.4 below implies that many (but not all) k-vertex faces of $\mathcal{B}_k$ are simplices. More precisely, if F is a face of $\mathcal{B}_k$ whose vertex set is $\{U(t_1), \ldots, U(t_k)\}$ where $t_1, \ldots, t_k \in \mathbb{S}$ lie in an open semicircle, then F is a $(k-1)$-dimensional simplex. Also, since the intersection of faces is a face, it is enough to verify Theorem 1.1 for $n = k$.

In what follows, we denote by ϕ_k the largest possible value that satisfies Theorem 1.1. We provide a characterization of ϕ_k, which, in principle, allows one to compute it, at least numerically and at least for moderate values of k. This characterization is, roughly, as follows. It is not hard to argue that if Γ is an open arc of length ϕ_k then there must be a way to move some of the points $t_1, \ldots, t_k \in \Gamma$ towards the endpoints of Γ, so that the limit position of the affine hyperplane supporting $\mathcal{B}_k$ at $U(t_1), \ldots, U(t_k)$ will touch the curve $U(t)$ somewhere else. We prove that this limiting configuration is as degenerate as it can possibly be: each point t_i collides with one of the endpoints a or b of the arc Γ so that if q such points collide with a and $k-q$ points with b, then the necessarily unique affine hyperplane tangent to the symmetric moment curve $U(t)$ at $t=a$ and $t=b$ with multiplicities $2q$ and $2k-2q$, respectively, is also tangent to $U(t)$ at some other point (cf. Theorem 5.1 below).

We have

$$\phi_2 = \frac{2\pi}{3} \approx 2.094\,395\,103$$

(this follows from results of Smilansky [14]), and we computed

$$\phi_3 = \pi - \arccos \frac{3-\sqrt{5}}{2} \approx 1.962\,719\,003$$

as well as

$$\begin{aligned}\phi_4 &= 2 \arccos\left(-\frac{1}{48}(91+336\sqrt{15})^{1/3} + \frac{119}{48(91+336\sqrt{15})^{1/3}} + \frac{29}{48}\right)\\ &\approx 1.870\,658\,532\end{aligned}$$

(cf. Example 5.2).

It is worth noting that in [3] we were only able to verify that $\phi_k > 0$, while in [8] the first explicit lower bound $\phi_k > \sqrt{6}k^{-3/2}$ was established.

We conjecture that for an even k the value of ϕ_k in Theorem 1.1 satisfies $\phi_k = 2\alpha_k$, where α_k is the smallest positive root of the equation

$$\cos\alpha + 1 + \sum_{j=1}^{k-1} (-1)^j \frac{(2j-1)!!}{(2j)!!} \tan^{2j}\alpha = 0.$$

Here $n!!$ is the product of positive integers not exceeding n and having the same parity as n. Some supporting evidence for the conjecture is provided in §7.

Theorem 1.1 allows one to construct $2k$-dimensional centrally symmetric polytopes with n vertices and

$$\left(4\cdot 2^{-k} - 4^{-k+1} + O\left(\frac{1}{n}\right)\right)\binom{n}{k}$$

faces of dimension $k-1$, a new record. Indeed, suppose that $n=4m$, and consider a centrally symmetric configuration $A = A_0 \cup A_1 \cup A_2 \cup A_3$ of $4m$ distinct points in $\mathbb{S}$, where each set A_j contains m distinct points in the vicinity of $j\pi/2$ for $j = 0, 1, 2, 3$ so that the length of any arc whose endpoints are in A_j and $A_{(j+1) \bmod 4}$ is less than ϕ_k. Letting

$$P = \operatorname{conv}(U(t) : t \in A),$$

we observe that P is a centrally symmetric polytope with $4m$ vertices and at least $4\binom{2m}{k} - 4\binom{m}{k}$ faces of dimension $k-1$. For example, if $k=2$ we obtain a four-dimensional centrally symmetric polytope with $4m$ vertices such that approximately 3/4 of all pairs of vertices are guaranteed to span edges of the polytope. Curiously, if we consider instead the convex hull of $4m$ points $U_2(t_i)$ for points t_i uniformly distributed on the circle $\mathbb{S}$ then only about $2/3$ of all pairs of vertices span edges of the resulting four-dimensional centrally symmetric polytope (see [3]).

In [2], we apply Theorem 1.1 to construct various families of centrally symmetric polytopes with many faces. Namely, we construct a d-dimensional centrally symmetric polytope P with about $3^{d/4} \approx (1.316)^d$ vertices such that every pair of non-antipodal vertices of P spans an edge of P. For $k \geqslant 1$, we construct a d-dimensional centrally symmetric polytope P with an arbitrarily large number n of vertices such that the number of k-dimensional faces of P is at least $(1-(\delta_k)^d)\binom{n}{k+1}$ for some $0 < \delta_k < 1$ (we have $\delta_1 \approx 3^{-1/4} \approx 0.77$ and $\delta_k \approx (1-5^{-k})^{1/(6k+4)}$ for $k > 1$). Finally, for an integer $k > 1$ and $\alpha > 0$, we construct a centrally symmetric polytope P with an arbitrarily large number n of vertices and of dimension $d = k^{1+o(1)}$ such that the number of k-dimensional faces of P is at least $(1-k^{-\alpha})\binom{n}{k+1}$.

It is important to note that we are interested in asymptotics of the number of faces when the dimension is fixed and the number of vertices grows, in contrast to a number of results in the literature where both the dimension and the number of vertices grow (see [6, 10, 13]).

We prove that $\pi/2$ is the limit of neighborliness of $U_k(t)$ as k grows.

THEOREM 1.2. *Let ϕ_k be the largest number satisfying Theorem 1.1. Then*

$$\lim_{k \longrightarrow +\infty} \phi_k = \frac{\pi}{2} \approx 1.570\,796\,327.$$

Our computations strongly suggest that the values ϕ_k are monotone decreasing, but we were unable to prove that.

The complete facial structure of $\mathcal{B}_2$ was described by Smilansky [14]: the zero-dimensional faces are the points $U(t)$ for $t \in \mathbb{S}$, the one-dimensional faces are the intervals $[U(a), U(b)]$, where $a, b \in \mathbb{S}$ and the length of the shorter arc with the endpoints a and b is less than $2\pi/3$, and the two-dimensional faces are the triangles with the vertices $\{U(a), U(b), U(c)\}$, where $a, b, c \in \mathbb{S}$ are vertices of an equilateral triangle. There are no other faces of $\mathcal{B}_2$.

In [3, 15] the edges (one-dimensional faces) of $\mathcal{B}_k$ were completely characterized. Namely, it was shown in [3] that for $a \neq b$ the interval $[U(a), U(b)]$ is an edge of $\mathcal{B}_k$ if the length of the shorter arc with the endpoints $a, b \in \mathbb{S}$ is smaller than $\pi(2k-2)/(2k-1)$, and it was shown in [15] that there are no other edges.

Already for $\mathcal{B}_3$, the complete facial structure is not known; some faces of $\mathcal{B}_3$ were computed in [9].

We obtain the following partial result showing some "connectedness" of faces of $\mathcal{B}_k$.

THEOREM 1.3. *Let $\Gamma \subset \mathbb{S}$ be an open arc with the endpoints a and b and let $\overline{\Gamma}$ be the closure of Γ. Let $t_2, \ldots, t_k \in \mathbb{S} \backslash \overline{\Gamma}$ be distinct points such that the set $\overline{\Gamma} \cup \{t_2, \ldots, t_k\}$ lies in an open semicircle in $\mathbb{S}$. Suppose that* *the points* $U(a), U(t_2), \ldots, U(t_k)$ *lie in a face of* $\mathcal{B}_k$ *and that the points* $U(b), U(t_2), \ldots, U(t_k)$ *lie in a face of* $\mathcal{B}_k$. *Then for all* $t_1 \in \Gamma$ *the set*

$$\operatorname{conv}(U(t_1), \ldots, U(t_k))$$

is a face of $\mathcal{B}_k$.

Example 1.4. Let $k = 3$ and let $t_1, t_2, t_3 \in \mathbb{S}$ be any points such that the length of the arc with the endpoints t_1 and t_2 is less than $2\pi/5$, the length of the arc with the endpoints t_2 and t_3 is less than $2\pi/5$, and t_2 lies between t_1 and t_3. Then

$$\operatorname{conv}(U(t_1), U(t_2), U(t_3)) \tag{1.4.1}$$

is a face of $\mathcal{B}_3$. Indeed, without loss of generality, we may assume that $t_2 = 0$. If $t_1 = 2\pi/5$ and $t_3 = -2\pi/5$, the triangle (1.4.1) lies in the face of $\mathcal{B}_3$ determined by the equation $\cos 5t = 1$. If we move t_3 sufficiently close to $t_2 = 0$, then (1.4.1) is a face from our estimate of ϕ_3 in Theorem 1.1. Therefore, by Theorem 1.3, for $t_1 = 2\pi/5$, $t_2 = 0$, and all $-2\pi/5 < t_3 < 0$, the set (1.4.1) is a face of $\mathcal{B}_3$. Let us now fix some $-2\pi/5 < t_3 < 0$. If we move t_1 sufficiently close to $t_2 = 0$ then (1.4.1) is a face by Theorem 1.1. Applying Theorem 1.3 again we conclude that for all $0 < t_1 < 2\pi/5$, $-2\pi/5 < t_3 < 0$, and $t_2 = 0$, the set (1.4.1) is a face of $\mathcal{B}_3$.

In the rest of the paper, we prove Theorems 1.1–1.3. In §2, we introduce the analytic language of raked trigonometric polynomials which supplants the geometric language of affine hyperplanes in our proofs. We also outline the plan for the proofs.

§2. *Polynomials. The plan of the proofs.*

2.1. *Raked trigonometric and complex polynomials.* We consider *raked trigonometric polynomials* of degree at most $2k - 1$:

$$f(t) = c + \sum_{j=1}^{k} a_j \cos(2j-1)t + \sum_{j=1}^{k} b_j \sin(2j-1)t \quad \text{for } t \in \mathbb{S}, \tag{2.1.1}$$

where $c, a_j, b_j \in \mathbb{R}$. We say that $\deg f = 2k - 1$ if $a_k \neq 0$ or if $b_k \neq 0$. Equivalently, we can write

$$f(t) = c + \langle C, U(t) \rangle,$$

where $C = (a_1, b_1, \ldots, a_k, b_k) \in \mathbb{R}^{2k}$ and $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^{2k}$.

Writing

$$\cos nt = \frac{e^{int} + e^{-int}}{2} \quad \text{and} \quad \sin nt = \frac{e^{int} - e^{-int}}{2i}$$

and substituting $z = \mathrm{e}^{\mathrm{i}t}$, we associate with (2.1.1) a complex polynomial

$$\mathcal{P}(f)(z) = z^{2k-1}\left(c + \sum_{j=1}^{k} a_j \frac{z^{2j-1} + z^{1-2j}}{2} + \sum_{j=1}^{k} b_j \frac{z^{2j-1} - z^{1-2j}}{2i}\right). \tag{2.1.2}$$

Hence

$$\deg \mathcal{P}(f) \leqslant 4k - 2. \tag{2.1.3}$$

Moreover, if $\deg f = 2k - 1$ then for $p = \mathcal{P}(f)$ we have $\deg p = 4k - 2$ and $p(0) \neq 0$. Since

$$\cos(t + a) = \cos t \cos a - \sin t \sin a$$

and

$$\sin(t + a) = \sin t \cos a + \cos t \sin a,$$

for any fixed $a \in \mathbb{S}$ and any raked trigonometric polynomial $f(t)$, the function

$$h(t) = f(t + a) \quad \text{for } t \in \mathbb{S}$$

is also a raked trigonometric polynomial of the same degree.

Definition 2.2. We say that a point $t^* \in \mathbb{S}$ is a *root of multiplicity* m (where $m \geqslant 1$ is an integer) of a trigonometric polynomial f, if

$$f(t^*) = \cdots = f^{(m-1)}(t^*) = 0$$

and

$$f^{(m)}(t^*) \neq 0.$$

Similarly, we say that a number $z^* \in \mathbb{C}$ is a root of multiplicity m of a polynomial $p(z)$ if

$$p(z^*) = \cdots = p^{(m-1)}(z^*) = 0$$

and

$$p^{(m)}(z^*) \neq 0.$$

Remark 2.3. (1) We note that

$$\operatorname{conv}(U(t_1), \ldots, U(t_n))$$

for distinct $t_1, \ldots, t_n \in \mathbb{S}$ is a face of $\mathcal{B}_k$ if and only if there exists a raked trigonometric polynomial $f(t) \not\equiv 0$ of degree at most $2k - 1$ such that (i) each t_j, $j = 1, \ldots, n$, is a root of f of an even multiplicity, and (ii) f has no other roots. The roots of f should have even multiplicities as for f to determine a face, the values of f must not change sign on $\mathbb{S}$.

(2) We will often use the following observation: if f is a trigonometric polynomial with constant term 1 that does not change sign on $\mathbb{S}$ then $f(t) \geqslant 0$ for all $t \in \mathbb{S}$, since

$$\frac{1}{2\pi} \int_{\mathbb{S}} f(t)\, dt = 1.$$

2.4. *Plan of the proofs.* In §3, we prove basic facts about the roots of raked trigonometric polynomials. We bound their number and restrict possible positions in the circle $\mathbb{S}$ (Theorem 3.1). While the total number of roots of a raked trigonometric polynomial f with $\deg f \leqslant 2k-1$ does not exceed $4k-2$, we prove that the number of roots, counting multiplicities, in any open arc $\Gamma \subset \mathbb{S}$ of length less than π may not exceed $2k$; moreover, if that number is equal to $2k$ then any additional root of f has to lie in the opposite arc $\Gamma + \pi$.

In §4, we consider one-parameter families $f_s(t)$ of raked trigonometric polynomials with constant term 1, obtained by fixing some roots of $f_s(t)$ and moving one, possibly multiple, root, so that the total number of controlled roots is $2k$, counting multiplicities, and the roots remain in an open semicircle of $\mathbb{S}$. We prove that $(\partial/\partial s) f_s(t) \not\equiv 0$ (Theorem 4.2). In geometric terms, Theorem 4.2 implies that if we choose k distinct points $t_1, \dots, t_k$ lying in an open semicircle of $\mathbb{S}$, consider the (necessarily unique) affine hyperplane H tangent to the symmetric moment curve $U_k(t)$ at $t_1, \dots, t_k$ and then start moving the point t_1, while keeping the points $t_2, \dots, t_n$ intact, the velocity of the unit normal of H is never zero.

In §5, we prove a characterization (see Theorem 5.1 and Lemma 5.7) of the value of ϕ_k introduced in Theorem 1.1. In analytic terms, we prove that if $\Gamma \subset \mathbb{S}$ is an open arc of a certain length and if a raked trigonometric polynomial $f(t)$ has $2k$ roots, counting multiplicities, in Γ, then f has no other roots in $\mathbb{S}$. Moreover, we prove that for the maximum possible length ϕ_k of such an arc Γ, there are positive even integers m_a and m_b such that $m_a + m_b = 2k$ and such that the unique, up to a non-zero multiple, raked trigonometric polynomial $f(t)$ of degree $2k-1$ that has a root of multiplicity m_a at one endpoint of Γ and a root of multiplicity m_b at the other endpoint of Γ also has a root of an even multiplicity in $\Gamma + \pi$ and does not change its sign on $\mathbb{S}$.

In §6, we prove Theorems 1.1 and 1.3. In particular, we prove that for every positive integer k there exists a number $\phi_k > \pi/2$ with the following property: if $f(t)$ is an arbitrary raked trigonometric polynomial of degree $2k-1$, with constant term 1, and such that $f(t)$ has $2k$ roots, counting multiplicities, in an open arc $\Gamma \subset \mathbb{S}$ of length ϕ_k and all roots in Γ have even multiplicities, then $f(t)$ is positive everywhere else in $\mathbb{S}$ (Theorem 6.1).

In §7, we prove Theorem 1.2. Also, for an even k, we deduce an equation for the value of $0 < \alpha < \pi/2$ such that the unique raked trigonometric polynomial of degree $2k-1$ with constant term 1 that has roots at $t = \pm\alpha$ of multiplicity k each also has a root of an even multiplicity at $t = \pi$ while remaining non-negative on $\mathbb{S}$. We conjecture that $\phi_k = 2\alpha$.

§3. *Roots and multiplicities.* We consider raked trigonometric polynomials $f(t)$ defined by (2.1.1). In this section we prove the following main result.

THEOREM 3.1. *Let $f(t) \not\equiv 0$ be a raked trigonometric polynomial of degree at most $2k-1$, let $t_1, \dots, t_n \in \mathbb{S}$ be distinct roots of f in $\mathbb{S}$, and let $m_1, \dots, m_n$ be their multiplicities.*

(1) *We have*

$$\sum_{i=1}^{n} m_i \leqslant 4k-2.$$

(2) *If the constant term of f is 0 and the set $\{t_1, \dots, t_n\}$ does not contain a pair of antipodal points, then*

$$\sum_{i=1}^{n} m_i \leqslant 2k - 1.$$

(3) *If $t_1, \dots, t_n$ lie in an open semicircle of $\mathbb{S}$, then*

$$\sum_{i=1}^{n} m_i \leqslant 2k.$$

(4) *Suppose that $t_1, \dots, t_n$ lie in an arc $\Gamma \subset \mathbb{S}$ of length less than π, that*

$$\sum_{i=1}^{n} m_i = 2k,$$

and that $t^* \in \mathbb{S}\backslash\Gamma$ is yet another root of f. Then $t^* \in \Gamma + \pi$.

To prove Theorem 3.1, we establish a correspondence between the roots of a trigonometric polynomial $f(t)$ and those of the corresponding complex polynomial $p(z) = \mathcal{P}(f)$ defined by (2.1.2).

LEMMA 3.2. *A point $t^* \in \mathbb{S}$ is a root of multiplicity m of $f(t)$ if and only if $z^* = \mathrm{e}^{\mathrm{i}t^*}$ is a root of multiplicity m of $\mathcal{P}(f)$.*

Proof. Let $p = \mathcal{P}(f)$. It follows from (2.1.2) that

$$p(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{(2k-1)\mathrm{i}t} f(t). \tag{3.2.1}$$

Differentiating (3.2.1), we infer by induction that

$$\mathrm{i}^r \sum_{j=1}^{r} d_{j,r} \mathrm{e}^{\mathrm{i}jt} p^{(j)}(\mathrm{e}^{\mathrm{i}t}) = \sum_{j=0}^{r} \mathrm{i}^{r-j} c_{j,r} \mathrm{e}^{(2k-1)\mathrm{i}t} \cdot f^{(j)}(t) \quad \text{for all } r \geqslant 1,$$

where the constants $c_{j,r}$, $d_{j,r}$ are positive integers. Thus, if $f^{(r)}(t^*)$ is zero for $r = 0, 1, \dots, m-1$ and non-zero for $r = m$, then so is $p^{(r)}(\mathrm{e}^{\mathrm{i}t^*})$, and vice versa. The statement now follows. □

3.3. *Proof of Theorem 3.1.* Part (1) follows from Lemma 3.2 and bound (2.1.3).

If f has a zero constant term, then f satisfies

$$f(t + \pi) = -f(t) \quad \text{for all } t \in \mathbb{S}.$$

Then $t_i + \pi$ is a root of $f(t)$ of multiplicity m_i and the proof of part (2) follows from part (1).

To prove part (3), let $g(t) = f'(t)$. Then g has a zero constant term. If

$$\sum_{i=1}^{n} m_i > 2k,$$

then by Rolle's theorem the total number of roots of $g(t)$ in the semicircle, counting multiplicities, is at least $2k$, and so $g(t) \equiv 0$ by part (2), which is a contradiction.

To prove part (4), we assume without loss of generality that $t_1, \ldots, t_n$ is the order of the roots on the arc Γ and let $\tilde{\Gamma}$ be the closed arc with the endpoints t_1 and t_n. By Rolle's theorem, the total number of roots of $g(t)$, counting multiplicities, in $\tilde{\Gamma}$ is at least $2k - 1$, and hence the total number of roots of $g(t)$, counting multiplicities, in $\tilde{\Gamma} \cup (\tilde{\Gamma} + \pi)$ is at least $4k - 2$. If $t^* \notin \tilde{\Gamma} \cup (\tilde{\Gamma} + \pi)$, then by Rolle's theorem there is a root of $g(t)$ outside $\tilde{\Gamma} \cup (\tilde{\Gamma} + \pi)$, and hence the total number of roots of $g(t)$ in $\mathbb{S}$, counting multiplicities, is at least $4k - 1$. Thus, by part (1), $g(t) \equiv 0$, which is a contradiction. □

We will utilize the following geometric corollary of Theorem 3.1.

LEMMA 3.4. *Let $t_1, \ldots, t_n \in \mathbb{S}$ be distinct points lying in an open semicircle and let $m_1, \ldots, m_n$ be positive integers such that*

$$\sum_{i=1}^{n} m_i = 2k.$$

Then the following $2k$ vectors,

$$\begin{aligned}
&U(t_i) - U(t_n) \quad \textit{for } i = 1, \ldots, n-1, \\
&\left.\frac{d^j}{dt^j} U(t)\right|_{t=t_i} \quad \textit{for } j = 1, \ldots, m_i - 1 \textit{ if } m_i > 1 \textit{ and } i = 1, \ldots, n, \\
&\left.\frac{d^{m_1}}{dt^{m_1}} U(t)\right|_{t=t_1},
\end{aligned}$$

are linearly independent in $\mathbb{R}^{2k}$.

Proof. Seeking a contradiction, we assume that the vectors are not linearly independent. Then there exists a non-zero vector $C \in \mathbb{R}^{2k}$ orthogonal to all these $2k$ vectors. Consider the raked trigonometric polynomial

$$f(t) = \langle C, U(t) - U(t_n) \rangle \quad \text{for } t \in \mathbb{S}.$$

Then $t_1, \ldots, t_n$ are roots of $f(t)$. Moreover, the multiplicity of t_i is at least m_i for $i > 1$ and at least $m_1 + 1$ for $i = 1$. It follows from part (3) of Theorem 3.1 that $f(t) \equiv 0$, which contradicts that $C \neq 0$. □

Finally, we prove that a raked trigonometric polynomial is determined, up to a constant factor, by its roots of the total multiplicity $2k$ provided those roots lie in an open semicircle.

COROLLARY 3.5. *Let $t_1, \ldots, t_n \in \mathbb{S}$ be distinct points lying in an open semicircle, and let $m_1, \ldots, m_n$ be positive integers such that*

$$\sum_{i=1}^{n} m_i = 2k.$$

Then there exists a unique raked trigonometric polynomial $f(t)$ of degree at most $2k-1$ and with constant term 1, such that t_i is a root of $f(t)$ of multiplicity m_i for all $i = 1, \ldots, n$. Moreover, f depends analytically on $t_1, \ldots, t_n$.

Proof. Such a polynomial $f(t)$ can be written as

$$f(t) = \langle C, U(t) - U(t_n)\rangle \quad \text{for } t \in \mathbb{S},$$

where $C \in \mathbb{R}^{2k}$ is orthogonal to the $2k-1$ vectors

$$\begin{aligned} &U(t_i) - U(t_n) && \text{for } i = 1, \ldots, n-1, \\ &\frac{d^j}{dt^j} U(t)\Big|_{t=t_i} && \text{for } j = 1, \ldots, m_i - 1 \text{ if } m_i > 1 \text{ and } i = 1, \ldots, n. \end{aligned}$$

By Lemma 3.4, these $2k-1$ vectors span a hyperplane in $\mathbb{R}^{2k-1}$ and hence, up to a scalar, there is a unique choice of C. By part (2) of Theorem 3.1, f has a non-zero constant term if $C \neq 0$. Therefore, there is a unique choice of C that makes the constant term of $f(t)$ equal 1. By part (3) of Theorem 3.1 the multiplicities of the roots t_i are exactly m_i for $i = 1, \ldots, n$. □

Note that in fact $\deg f = 2k-1$. This follows from part (3) of Theorem 3.1. We will also need the following "deformation construction".

LEMMA 3.6. *Let $f(t)$ be a raked trigonometric polynomial of degree $2k-1$ such that $f(-t) = f(t)$ for all $t \in \mathbb{S}$, and let $p = \mathcal{P}(f)$ be the corresponding complex polynomial associated with f via (2.1.2). Then $p(0) \neq 0$ and the multiset M of roots of p can be split into $2k-1$ unordered pairs $\{\zeta_j, \zeta_j^{-1}\}$ for $j = 1, \ldots, 2k-1$. Moreover, for any real $\lambda \neq 0$, the multiset M_λ consisting of $2k-1$ unordered pairs $\{\xi_j, \xi_j^{-1}\}$ defined by*

$$\xi_j + \xi_j^{-1} = \lambda(\zeta_j + \zeta_j^{-1}) \quad \textit{for } j = 1, \ldots, 2k-1$$

is the multiset of roots of a certain complex polynomial p_λ such that $p_\lambda = \mathcal{P}(f_\lambda)$ for a raked trigonometric polynomial $f_\lambda(t)$ of degree $2k-1$ satisfying $f_\lambda(-t) = f_\lambda(t)$.

Proof. This is [**3**, Lemma 5.1]. □

We call $f_\lambda(t)$ a λ-*deformation* of f.

§4. *Parametric families of trigonometric polynomials.*

4.1. *Parametric polynomials.* Let $\Gamma \subset \mathbb{S}$ be an open arc. We consider raked trigonometric polynomials

$$f_s(t) = 1 + \sum_{j=1}^{k} a_j(s)\cos(2j-1)t + \sum_{j=1}^{k} b_j(s)\sin(2j-1)t \quad \text{for } t \in \mathbb{S}, \tag{4.1.1}$$

where $a_j(s)$ and $b_j(s)$ are real analytic functions of $s \in \Gamma$. We define

$$g_s(t) = \frac{\partial}{\partial s} f_s(t),$$

and so

$$g_s(t) = \sum_{j=1}^{k} a_j'(s) \cos(2j-1)t + \sum_{j=1}^{k} b_j'(s) \sin(2j-1)t. \tag{4.1.2}$$

The goal of this section is to prove the following result.

THEOREM 4.2. *Let $\Gamma \subset \mathbb{S}$ be an open arc, let $t_2, \ldots, t_n \in \mathbb{S} \setminus \Gamma$ be distinct points such that the set $\Gamma \cup \{t_2, \ldots, t_n\}$ lies in an open semicircle, and let $m_1, \ldots, m_n$ be positive integers such that*

$$\sum_{i=1}^{n} m_i = 2k.$$

For every $s \in \Gamma$, let $f_s(t)$ be the unique raked trigonometric polynomial of degree $2k-1$ with constant term 1 such that for $i = 2, \ldots, n$ the point t_i is a root of $f_s(t)$ of multiplicity m_i and s is a root of $f_s(t)$ of multiplicity m_1 (cf. Corollary 3.5). Define

$$g_s(t) = \frac{\partial}{\partial s} f_s(t).$$

Then

$$g_s(t) \not\equiv 0 \quad \textit{for all } s \in \Gamma.$$

To prove Theorem 4.2, we use the notion of the wedge product.

4.3. *Wedge product.* Given linearly independent vectors $V_1, \ldots, V_{2k-1} \in \mathbb{R}^{2k}$ we define their wedge product

$$W = V_1 \wedge \cdots \wedge V_{2k-1}$$

as the unique vector W orthogonal to the hyperplane spanned by $V_1, \ldots, V_{2k-1}$ whose length is the volume of the $(2k-1)$-dimensional parallelepiped spanned by $V_1, \ldots, V_{2k-1}$ and such that the basis $V_1, \ldots, V_{2k-1}, W$ is co-oriented with the standard basis of $\mathbb{R}^{2k}$. If vectors $V_1, \ldots, V_{2k-1}$ are linearly dependent, we let

$$V_1 \wedge \cdots \wedge V_{2k-1} = 0.$$

Suppose that vectors $V_1(s), \ldots, V_{2k-1}(s)$ depend smoothly on a real parameter s. We will use the following standard fact:

$$\begin{aligned} &\frac{d}{ds}(V_1(s) \wedge \cdots \wedge V_{2k-1}(s)) \\ &\quad = \sum_{j=1}^{2k-1} V_1(s) \wedge \cdots \wedge V_{j-1}(s) \wedge \frac{d}{ds} V_j(s) \wedge V_{j+1}(s) \wedge \cdots \wedge V_{2k-1}(s). \end{aligned} \tag{4.3.1}$$

4.4. *Proof of Theorem 4.2.* For $s \in \Gamma$, consider the following ordered set of $2k-1$ vectors:

$$\begin{aligned} &U(t_i) - U(t_n) \quad \text{for } i = 2, \ldots, n-1, \\ &\left.\frac{d^j}{dt^j} U(t)\right|_{t=t_i} \quad \text{for } j = 1, \ldots, m_i - 1 \text{ if } m_i > 1 \text{ and } i = 2, \ldots, n, \\ &U(s) - U(t_n), \\ &\left.\frac{d^j}{dt^j} U(t)\right|_{t=s} \quad \text{for } j = 1, \ldots, m_1 - 1 \text{ if } m_1 > 1. \end{aligned} \tag{4.4.1}$$

Let $C(s)$ be the wedge product of vectors of (4.4.1). By Lemma 3.4, the vectors of (4.4.1) are linearly independent for all $s \in \Gamma$, and hence $C(s) \neq 0$ for all $s \in \Gamma$.

For $s \in \Gamma$, define a raked trigonometric polynomial

$$F_s(t) = \langle C(s), U(t) - U(t_n) \rangle. \tag{4.4.2}$$

We note that $F_s(t) \not\equiv 0$ for all $s \in \Gamma$. For $i = 2, \ldots, n$, the point t_i is a root of $F_s(t)$ of multiplicity at least m_i and s is a root of $F_s(t)$ of multiplicity at least m_1. By part (3) of Theorem 3.1 the multiplicities are exactly m_i. Let $\alpha(s)$ be the constant term of $F_s(t)$. Then

$$\alpha(s) = -\langle C(s), U(t_n) \rangle.$$

By part (2) of Theorem 3.1

$$\alpha(s) \neq 0 \quad \text{for all } s \in \Gamma.$$

Therefore,

$$f_s(t) = \frac{F_s(t)}{\alpha(s)}.$$

Seeking a contradiction, let us assume that $g_s(t) \equiv 0$ for some $s \in \Gamma$. We have

$$g_s(t) = \frac{\partial}{\partial s} f_s(t) = \frac{\alpha(s)(\partial/\partial s)F_s(t) - \alpha'(s)F_s(t)}{\alpha^2(s)}.$$

If $g_s(t) \equiv 0$, then

$$\alpha(s)\frac{\partial}{\partial s}F_s(t) - \alpha'(s)F_s(t) \equiv 0,$$

and (4.4.2) yields that

$$\alpha(s)C'(s) - \alpha'(s)C(s) = 0 \tag{4.4.3}$$

for some $s \in \Gamma$. Let us consider $C'(s)$, the derivative of the wedge product of (4.4.1). Applying formula (4.3.1) we note that all of the $2k-1$ terms of (4.3.1) except the last one are zeros since the corresponding wedge product either contains a zero vector or two identical vectors. Hence $C'(s)$ is the wedge product of the following ordered set of vectors:

$$\begin{aligned}
&U(t_i) - U(t_n) \quad \text{for } i = 2, \dots, n-1, \\
&\frac{d^j}{dt^j} U(t)\Big|_{t=t_i} \quad \text{for } j = 1, \dots, m_i - 1 \text{ if } m_i > 1 \text{ and } i = 2, \dots, n, \\
&U(s) - U(t_n), \\
&\frac{d^j}{dt^j} U(t)\Big|_{t=s} \quad \text{for } j = 1, \dots, m_1 - 2 \text{ if } m_1 > 2 \text{ and } j = m_1.
\end{aligned} \tag{4.4.4}$$

The wedge products (4.4.1) for $C(s)$ and (4.4.4) for $C'(s)$ differ in two vectors,

$$A(s) = \frac{d^{m_1-1}}{dt^{m_1-1}} U(t)\Big|_{t=s} \quad \text{and} \quad B(s) = \frac{d^{m_1}}{dt^{m_1}} U(t)\Big|_{t=s} \quad \text{if } m_1 > 1$$

and

$$A(s) = U(s) - U(t_n) \quad \text{and} \quad B(s) = \frac{d}{dt} U(t)\Big|_{t=s} \quad \text{if } m_1 = 1.$$

Vector $A(s)$ is present in (4.4.1) and absent in (4.4.4) while vector $B(s)$ is absent in (4.4.1) and present in (4.4.4). Therefore, (4.4.3) implies that the set consisting of the vector

$$\alpha(s)A(s) - \alpha'(s)B(s)$$

and the $2k-2$ vectors common to wedges (4.4.1) and (4.4.4) is linearly dependent. However, as $\alpha(s) \neq 0$, this contradicts Lemma 3.4. □

We will need the following result.

LEMMA 4.5. *Let $f_s(t)$ and $g_s(t)$ be trigonometric polynomials (4.1.1) and (4.1.2) respectively and let m be a positive integer.*

(1) *If $t^* \in \mathbb{S}$ is a root of $f_s(t)$ of multiplicity at least m for all $s \in \Gamma$, then t^* is a root of $g_s(t)$ of multiplicity at least m for all $s \in \Gamma$.*
(2) *If $m > 1$ and s is a root of $f_s(t)$ of multiplicity at least m for all $s \in \Gamma$, then s is a root of $g_s(t)$ of multiplicity at least $m-1$ for all $s \in \Gamma$.*

Proof. Suppose that

$$f_s(t^*) = \cdots = \frac{\partial^{m-1}}{\partial t^{m-1}} f_s(t)\Big|_{t=t^*} = 0.$$

Differentiating with respect to s yields part (1).

Suppose that

$$f_s(s) = \frac{\partial^j}{\partial t^j} f_s(t)\Big|_{t=s} = 0 \quad \text{for } j = 1, \dots, m-1.$$

Differentiating with respect to s we obtain

$$0 = \frac{\partial}{\partial s} f_s(t)\Big|_{t=s} + \frac{\partial}{\partial t} f_s(t)\Big|_{t=s} = \frac{\partial}{\partial s}\frac{\partial^j}{\partial t^j} f_s(t)\Big|_{t=s} + \frac{\partial^{j+1}}{\partial t^{j+1}} f_s(t)\Big|_{t=s}$$
for $j = 1, \dots, m-1$.

Therefore,

$$g_s(s) = \frac{\partial^j}{\partial t^j} g_s(t)\bigg|_{t=s} = 0 \quad \text{for } j = 1, \dots, m-2,$$

and the proof of part (2) follows. □

§5. *Critical arcs.* This section is devoted to verifying the following result.

THEOREM 5.1.

(1) *For every $k \geqslant 1$ there exists a non-empty open arc $\Gamma \subset \mathbb{S}$ with the following property: if $t_1, \dots, t_n \in \Gamma$ are distinct points and $m_1, \dots, m_n$ are positive even integers satisfying*

$$\sum_{i=1}^{n} m_i = 2k,$$

then the unique raked trigonometric polynomial $f(t)$ of degree $2k-1$ with constant term 1 that has each point t_i as a root of multiplicity m_i, has no other roots in $\mathbb{S}$. Moreover, $f(t) \geqslant 0$ for all $t \in \mathbb{S}$.

(2) *Let $\Gamma \subset \mathbb{S}$ be an open arc as in part* (1) *of the maximum possible length and let a and b be the endpoints of Γ. Then there are positive even integers m_a and m_b such that $m_a + m_b = 2k$ and such that the unique raked trigonometric polynomial $f(t)$ of degree $2k-1$ with constant term 1 that has a root at $t = a$ of multiplicity m_a and a root at $t = b$ of multiplicity m_b is non-negative on $\mathbb{S}$ and has a root (of necessarily even multiplicity) in the arc $\Gamma + \pi$.*

(3) *Fix positive even integers m_a and m_b such that $m_a + m_b = 2k$. Let $\Gamma \subset \mathbb{S}$ be an open arc of length less than π and let a be an endpoint of Γ. For $b \in \Gamma$ let $f_b(t)$ be the unique raked trigonometric polynomial of degree $2k-1$ with constant term 1 that has a root at $t = a$ of multiplicity m_a and a root at $t = b$ of multiplicity m_b. Let $x, y, z \in \Gamma$ be distinct points such that y lies between a and z and x lies between a and y. Suppose that $f_y(t) \geqslant 0$ for all $t \in \mathbb{S}$ and that f_y has a root (of necessarily even multiplicity) in the arc $\Gamma + \pi$. Then $f_x(t)$ is positive for all $t \in \mathbb{S} \backslash \{a, x\}$ while $f_z(t)$ is negative for some $t \in \mathbb{S}$.*

Let us denote for a moment the maximum possible length of an arc Γ satisfying part (1) of Theorem 5.1 by ψ_k. In Lemma 5.7 below we prove that $\psi_k = \phi_k$, the maximum length of an arc with the neighborliness property of Theorem 1.1.

Example 5.2. Suppose that $k = 2$. The only possible set of multiplicities in part (2) of Theorem 5.1 is $m_a = 2$ and $m_b = 2$. The polynomial $f(t) = 1 - \cos 3t$ has roots at $t = \pm 2\pi/3$ and a root at $t = 0$, all of multiplicity 2, while remaining non-negative on $\mathbb{S}$. Combining parts (3) and (2) of Theorem 5.1 we conclude that

$$\psi_2 = \frac{2\pi}{3} \approx 2.094\,395\,103.$$

Suppose that $k = 3$. The only possible set of multiplicities in part (2) of Theorem 5.1 is $m_a = 2$ and $m_b = 4$. The polynomial $f(t) = 1 - \cos 5t$ has roots at $t = 0$, $\pm 2\pi/5$, and $t = \pm 4\pi/5$, all of multiplicity 2, while remaining non-negative on $\mathbb{S}$. Applying to $f(t)$ the deformation of Lemma 3.6 with $\lambda = 1/\cos(\pi/5)$ results in the polynomial $f_\lambda(t)$ that has a root of multiplicity 4 at $t = \pi$, roots of multiplicity 2 at the points $\pm\alpha$ such that

$$\cos\alpha = \frac{\cos(2\pi/5)}{\cos(\pi/5)} = \frac{3-\sqrt{5}}{2},$$

and no other roots. Hence $f_\lambda(t)$ does not change its sign on $\mathbb{S}$. Scaling f_λ, if necessary, to make the constant term 1, we ensure that $f_\lambda(t)$ is non-negative on $\mathbb{S}$. It follows by Theorem 5.1 that

$$\psi_3 = \pi - \alpha = \pi - \arccos\frac{3-\sqrt{5}}{2} \approx 1.962\,719\,003.$$

Suppose that $k = 4$. There are two possibilities for multiplicities m_a and m_b in part (2) of Theorem 5.1. We have either $m_a = 2$ and $m_b = 6$ or $m_a = m_b = 4$. It turns out that the arc satisfying the latter conditions is shorter. As follows from Proposition 7.6 below, we have $\psi_4 = 2\alpha$, where $\alpha > 0$ is the smallest positive root of the equation

$$\cos\alpha + 1 - \tfrac{1}{2}\tan^2\alpha + \tfrac{3}{8}\tan^4\alpha - \tfrac{5}{16}\tan^6\alpha = 0.$$

Computations show that

$$\begin{aligned}\psi_4 &= 2\arccos\left(-\frac{1}{48}(91+336\sqrt{15})^{1/3} + \frac{119}{48(91+336\sqrt{15})^{1/3}} + \frac{29}{48}\right)\\ &\approx 1.870\,658\,532.\end{aligned}$$

In this case, the raked trigonometric polynomial f of degree 7 that has roots of multiplicity 4 at $t = \pm\psi_4/2$ also has a root of multiplicity 2 at $t = \pi$.

In general, our computations suggest that in part (2) of Theorem 5.1 one should always choose $m_a = m_b = k$ if k is even and $m_a = k+1$ and $m_b = k-1$ if k is odd, but we have been unable to prove that.

To prove Theorem 5.1, we need some technical results on convergence of trigonometric polynomials.

5.3. *Convergence of trigonometric polynomials.* All raked trigonometric polynomials (2.1.1) of degree at most $2k-1$ form a real $(2k+1)$-dimensional vector space, which we make into a normed space by letting

$$\|f\| = \max_{t\in\mathbb{S}} |f(t)|$$

for a trigonometric polynomial f. For a complex polynomial p of degree at most $4k-2$ we define

$$\|p\| = \max_{z:|z|=1} |p(z)| = \max_{z:\|z\|\leqslant 1} |p(z)|,$$

where the last equality follows by the maximum modulus principle for holomorphic functions. We note that

$$\|\mathcal{P}(f)\| = \|f\|$$

for any trigonometric polynomial f. We define the convergence of trigonometric and complex polynomials with respect to the norm $\|\cdot\|$.

LEMMA 5.4. *Fix a positive integer m. For a positive integer j, let $A_j \subset \mathbb{S}$ be a non-empty closed set and let $f_j(t)$ be a trigonometric polynomial of degree at most $2k-1$ that has at least m roots, counting multiplicities, in A_j. Suppose that $A_{j+1} \subset A_j$ for all j, and let*

$$B = \bigcap_{j=1}^{\infty} A_j.$$

Suppose further that for some trigonometric polynomial f we have

$$f = \lim_{j \longrightarrow +\infty} f_j.$$

Then f has at least m roots, counting multiplicities, in B.

Suppose, in addition, that $f \not\equiv 0$, $m = 2k$, B lies in an open semicircle, and that for every j the multiplicities of all roots of f_j in A_j are even. Then the multiplicities of all roots of f in B are even.

Proof. Let $p_j = \mathcal{P}(f_j)$. By Lemma 3.2, p_j is a complex polynomial that can be written as

$$p_j(z) = (z - z_{1j}) \cdots (z - z_{mj}) q_j(z), \tag{5.4.1}$$

where $q_j(z)$ is a complex polynomial of degree at most $4k - 2 - m$ and $z_{1j}, \ldots, z_{mj}$ are not necessarily distinct complex numbers of modulus 1 whose arguments lie in A_j. In addition,

$$\lim_{j \longrightarrow +\infty} p_j = p,$$

where $p(z) = \mathcal{P}(f)$. From this and (5.4.1) we infer that the numbers

$$\max_{z:|z|=\frac{1}{2}} |q_j(z)|$$

are uniformly bounded from above. Since all norms on the finite-dimensional space of complex polynomials of degree at most $4k - 2$ are equivalent and since $\max_{z:|z|=\frac{1}{2}} |q_j(z)|$ is also a norm, it follows that the norms $\|q_j\|$ are uniformly bounded from above. (We consider the circle $|z| = 1/2$ instead of $|z| = 1$ to make sure that the factors $z - z_{1j}, \ldots, z - z_{mj}$ in (5.4.1) are all separated from 0.) Hence we can find a subsequence $\{j_n\}$ such that

$$\lim_{n \longrightarrow +\infty} q_{j_n} = q$$

for some complex polynomial q and

$$\lim_{n \longrightarrow +\infty} z_{ij_n} = z_i^* \quad \text{where } z_i^* \in B \text{ for } i = 1, \ldots, m.$$

Then, necessarily

$$p(z) = (z - z_1^*) \cdots (z - z_m^*) q(z).$$

Hence by Lemma 3.2 the raked trigonometric polynomial $f(t)$ has at least m roots in B, counting multiplicities. If $m = 2k$ and $p \not\equiv 0$, part (3) of Theorem 3.1 implies that $z_1^*, \ldots, z_m^*$ are the only roots of $p(z)$ in B. The result follows. □

The following lemma plays the crucial role in our proof of Theorem 5.1.

LEMMA 5.5. *Let $\Gamma \subset \mathbb{S}$ be an open arc with the endpoints a and b and let $\overline{\Gamma}$ be its closure. Let $t_2, \ldots, t_n \in \mathbb{S} \setminus \overline{\Gamma}$ be distinct points such that the set $\overline{\Gamma} \cup \{t_2, \ldots, t_n\}$ lies in an open semicircle, and let $m_1, \ldots, m_n$ be positive even integers such that*

$$\sum_{i=1}^{n} m_i = 2k.$$

For $s \in \overline{\Gamma}$, let $f_s(t)$ be the unique raked trigonometric polynomial of degree $2k - 1$ with constant term 1 that has a root of multiplicity m_i at t_i for $i = 2, \ldots, n$ and a root of multiplicity m_1 at $t = s$. If both $f_a(t)$ and $f_b(t)$ are non-negative on $\mathbb{S}$, then for every $s \in \Gamma$, the trigonometric polynomial $f_s(t)$ is positive on $\mathbb{S} \setminus \{s, t_2, \ldots, t_n\}$.

Proof. Let us consider

$$g_s(t) = \frac{\partial}{\partial s} f_s(t)$$

as in Theorem 4.2. By Lemma 4.5, for all $s \in \Gamma$, the point t_i is a root of $g_s(t)$ of multiplicity at least m_i for $i = 2, \ldots, n$ and s is a root of $g_s(t)$ of multiplicity at least $m_1 - 1$. Let $\mathbb{S}_+$ be an open semicircle containing $\overline{\Gamma}$ and the points $t_2, \ldots, t_n$.

Seeking a contradiction, let us assume that $f_{t_1}(t^*) = 0$ for some $t_1 \in \Gamma$ and some $t^* \in \mathbb{S} \setminus \{t_1, t_2, \ldots, t_n\}$. By part (4) of Theorem 3.1, $t^* \in \mathbb{S}_+ + \pi$. We have $f_a(t^*) \geqslant 0$ and $f_b(t^*) \geqslant 0$. Therefore, the function

$$s \longmapsto f_s(t^*)$$

attains a local minimum in Γ at some point s^*. Then

$$g_{s^*}(t^*) = 0 \quad \text{and} \quad f_{s^*}(t^*) \leqslant 0.$$

Since $f_{s^*}(t)$ has a constant term of 1, we obtain

$$f_{s^*}(t) + f_{s^*}(t + \pi) = 2 \quad \text{for all } t \in \mathbb{S},$$

and hence

$$t^* + \pi \neq s^*, t_2, \ldots, t_n.$$

Since the constant term of $g_{s^*}(t)$ is 0, part (2) of Theorem 3.1 implies that $g_{s^*}(t) \equiv 0$. This however contradicts Theorem 4.2.

Hence for every $s \in \Gamma$ the trigonometric polynomial $f_s(t)$ has no roots other than $s, t_2, \ldots, t_n$. By Remark 2.3(2), we have $f_s(t) > 0$ for all $t \in \mathbb{S} \setminus \{s, t_2, \ldots, t_n\}$. □

5.6. *Proof of Theorem 5.1.* To prove part (1), let us choose a point $t^* \in \mathbb{S}$ and let us assume, seeking a contradiction, that there is a nested sequence of open arcs

$$\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_i \supset \cdots \tag{5.6.1}$$

such that

$$\bigcap_{j=1}^{\infty} \Gamma_j = \{t^*\},$$

and such that for every j there is a raked trigonometric polynomial $f_j(t)$ of degree $2k-1$, with constant term 1, with $2k$ roots, counting multiplicities, in Γ_j and a root somewhere else on the circle. By part (4) of Theorem 3.1, that additional root must lie in $\Gamma_j + \pi$. Let $h_j(t)$ be the scaling of f_j to a trigonometric polynomial of norm 1. Then there is a subsequence of the sequence $h_j(t)$ converging to a raked trigonometric polynomial h. In particular, $\|h\| = 1$, and hence $h(t) \not\equiv 0$. It follows from Lemma 5.4 that t^* is a root of h of multiplicity at least $2k$ and that $t^* + \pi$ is a root of h. Since both t^* and $t^* + \pi$ are roots of $h(t)$, we obtain that $h(t)$ has a zero constant term and that $t^* + \pi$ is, in fact, a root of $h(t)$ of multiplicity at least $2k$. Hence part (1) of Theorem 3.1 implies that $h(t) \equiv 0$, which is a contradiction.

By Remark 2.3(2), a trigonometric polynomial with constant term 1 that does not change its sign on $\mathbb{S}$ is non-negative on $\mathbb{S}$. Finally, the example of polynomial $1 - \cos(2k-1)t$ shows that the length of an arc Γ in part (1) is less than π.

To prove part (2), we construct a nested sequence of open arcs (5.6.1) such that

$$\bigcap_{j=1}^{\infty} \Gamma_j = \overline{\Gamma},$$

where $\overline{\Gamma}$ is the closure of Γ. By our assumption, for every j there is a raked trigonometric polynomial $f_j(t)$ of degree at most $2k-1$ that has $2k$ roots counting multiplicity in Γ_j and a root elsewhere, necessarily in $\Gamma_j + \pi$. As in the proof of part (1), let us scale $f_j(t)$ to a trigonometric polynomial $h_j(t)$ such that $\|h_j\| = 1$ and construct the limit trigonometric polynomial h. Then $h \not\equiv 0$, and, by Lemma 5.4, h has roots $t_1, \ldots, t_n \in \overline{\Gamma}$ of even multiplicities $m_1, \ldots, m_n$ such that $m_1 + \cdots + m_n = 2k$, and a root $t^* \in \overline{\Gamma} + \pi$. By part (2) of Theorem 3.1, h has a non-zero constant term.

We rescale h to a raked trigonometric polynomial $f(t)$ with constant term 1. Then each t_i is a root of $f(t)$ of multiplicity m_i and $f(t^*) = 0$.

Our assumption that Γ is of maximum possible length implies that the endpoints a and b of Γ are roots of $f(t)$. Our goal is to show that for every $i = 1, \ldots, n$ we have either $t_i = a$ or $t_i = b$; that is, that there are no roots inside Γ.

Seeking a contradiction, let us assume that $t_1 \in \Gamma$. We choose a closed arc $A \subset \Gamma$ with the endpoints x and y, containing t_1 in its interior and such that $t_i \notin A$ for $i = 2, \ldots, n$. For $s \in A$, let $f_s(t)$ be the raked trigonometric polynomial of Theorem 4.2 that has a root at $t = s$ of multiplicity m_1 and a root at t_i of multiplicity m_i for $i = 2, \ldots, n$. In particular,

$$f_s = f \quad \text{if } s = t_1.$$

We observe that

$$f_s(t) \geqslant 0 \quad \text{for all } t \in \mathbb{S} \text{ and all } s \in A.$$

Indeed, if $f_s(t_0) < 0$ for some $t_0 \in \mathbb{S}$ then a trigonometric polynomial $\hat{f}$ with constant term 1 that has a root of multiplicity m_1 at s and roots of multiplicity m_i at some points $\hat{t}_i \in \Gamma$ sufficiently close to t_i will also satisfy $\hat{f}(t_0) < 0$, which contradicts the definition of Γ. Hence $f_x(t) \geqslant 0$ for all $t \in \mathbb{S}$ and $f_y(t) \geqslant 0$ for all $t \in \mathbb{S}$. Lemma 5.5 then implies that $f(t^*) = f_{t_1}(t^*) > 0$, which is contradiction.

To prove part (3), we note that for any $b \in \Gamma$ sufficiently close to a, by part (1) of the theorem we have $f_b(t) > 0$ for all $t \in \mathbb{S}\setminus\{a, b\}$. We can choose such a point b so that x lies between b and y and then $f_x(t) > 0$ for all $t \in \mathbb{S}\setminus\{a, x\}$ by Lemma 5.5. Assume now that $f_z(t) \geqslant 0$ for all $t \in \mathbb{S}$. Then by Lemma 5.5 we have $f_y(t) > 0$ for all $t \in \mathbb{S}\setminus\{a, y\}$, which is a contradiction. □

LEMMA 5.7. *Let ψ_k be the maximum length of an open arc Γ in Theorem 5.1 and let ϕ_k be the maximum length of an open arc Γ in Theorem 1.1. Then $\psi_k = \phi_k$.*

Proof. From Remark 2.3(1) it follows immediately that $\phi_k \geqslant \psi_k$.

Let $\Gamma \subset \mathbb{S}$ be an open arc of length ψ_k with the endpoints a and b and let $\tilde{\Gamma} \supset \Gamma$ be a closed arc with the endpoints a and c strictly containing Γ and lying in an open semicircle. By part (2) of Theorem 5.1 there exist positive even integers m_a and m_b such that $m_a + m_b = 2k$ and a raked trigonometric polynomial $f(t)$ of degree $2k - 1$ and with constant term 1 that has a root at $t = a$ of multiplicity m_a, a root at $t = b$ of multiplicity m_b, and some other root $t^* \in \Gamma + \pi$. For $s \in \tilde{\Gamma}$ let $f_s(t)$ be the unique raked trigonometric polynomial of degree $2k - 1$ with constant term 1 that has a root of multiplicity m_a at $t = a$ and a root of multiplicity m_b at $t = s$. Seeking a contradiction, let us assume that for any distinct $t_1, \ldots, t_k \in \tilde{\Gamma}$, the unique raked trigonometric polynomial of degree $2k - 1$ and with constant term 1 that has roots of multiplicity two at $t_1, \ldots, t_k$ remains non-negative on the entire circle $\mathbb{S}$. As in §5.6, using the limit argument, we conclude that $f_c(t) \geqslant 0$ for all $t \in \mathbb{S}$. This, however, contradicts part (3) of Theorem 5.1 since f_b has a root in $\Gamma + \pi$.

In view of Remark 2.3(1), it follows that for some distinct $t_1, \ldots, t_k \in \tilde{\Gamma}$, the convex hull

$$\operatorname{conv}(U(t_1), \ldots, U(t_k))$$

is not a face of $\mathcal{B}_k$. Hence $\phi_k \leqslant \psi_k$. □

§6. *Neighborliness of the symmetric moment curve.* In this section we prove Theorems 1.1 and 1.3. Our proofs are based on the following main result.

THEOREM 6.1. *For every positive integer k there exists a number $\pi > \phi_k > \pi/2$ such that if $\Gamma \subset \mathbb{S}$ is an open arc of length ϕ_k, if $t_1, \ldots, t_n \in \Gamma$ are distinct points, and $m_1, \ldots, m_n$ are positive even integers such that*

$$\sum_{i=1}^{n} m_i = 2k,$$

then the unique raked trigonometric polynomial $f(t)$ of degree $2k-1$ with constant term 1 that has a root of multiplicity m_i at t_i for $i = 1, \ldots, n$ is positive everywhere else on the circle $\mathbb{S}$.

The proof is based on Theorem 5.1 and the following lemma.

LEMMA 6.2. *Let $f(t)$ be the raked trigonometric polynomial of degree $2k-1$ with constant term 1 that has a root of multiplicity $2m$ at $t = 0$ and a root of multiplicity $2n$ at $t = \pi/2$, where m and n are positive integers such that $m + n = k$. Then $f(t)$ has no other roots in the circle $\mathbb{S}$.*

Proof. We have

$$f(t) = 1 + \sum_{j=1}^{k} a_j \cos(2j-1)t + \sum_{j=1}^{k} b_j \sin(2j-1)t$$

for some real a_j and b_j. In addition,

$$f'(0) = \cdots = f^{(2m-1)}(0) = 0 \quad \text{and} \quad f'(\pi/2) = \cdots = f^{(2n-1)}(\pi/2) = 0. \tag{6.2.1}$$

Let

$$a(t) = \sum_{j=1}^{k} a_j \cos(2j-1)t \quad \text{and} \quad b(t) = \sum_{j=1}^{k} b_j \sin(2j-1)t,$$

so that

$$f(t) = 1 + a(t) + b(t) \quad \text{and} \quad f'(t) = a'(t) + b'(t). \tag{6.2.2}$$

Observe that

$$\frac{d^{2r-1}}{dt^{2r-1}} a(t)\bigg|_{t=0} = 0 \quad \text{and} \quad \frac{d^{2r}}{dt^{2r}} b(t)\bigg|_{t=0} = 0 \tag{6.2.3}$$

for any positive integer r, and

$$\frac{d^{2r}}{dt^{2r}} a(t)\bigg|_{t=\pi/2} = 0 \quad \text{and} \quad \frac{d^{2r-1}}{dt^{2r-1}} b(t)\bigg|_{t=\pi/2} = 0 \tag{6.2.4}$$

for any positive integer r.

Combining (6.2.1)–(6.2.4) we conclude that $t = 0$ is a root of $a'(t)$ of multiplicity at least $2m - 1$ and a root of $b'(t)$ of multiplicity at least $2m$.

Similarly, $t = \pi/2$ is a root of $a'(t)$ of multiplicity at least $2n$ and a root of $b'(t)$ of multiplicity at least $2n-1$. Since $f(0)=0$, we obtain that $a(t) \not\equiv 0$, and hence $a'(t) \not\equiv 0$. Also since $f(\pi/2)=0$, it follows that $b(t) \not\equiv 0$, and hence $b'(t) \not\equiv 0$. By part (2) of Theorem 3.1, the trigonometric polynomial $a'(t)$ has a root of multiplicity $2m-1$ at $t=0$, a root of multiplicity $2n$ at $t=\pi/2$ and no other roots in the circle $\mathbb{S}$, while the trigonometric polynomial $b'(t)$ has a root of multiplicity $2m$ at $t=0$, a root of multiplicity $2n-1$ at $t=\pi/2$ and no other roots in the circle.

We conclude that the functions $a(t)$ and $b(t)$ are monotone on the interval $0 < t < \pi/2$. Since $a(0) = -1$ and $a(\pi/2) = 0$, we infer that $a(t)$ is monotone increasing for $0 < t < \pi/2$, and hence $a(t) < 0$ for all $0 < t < \pi/2$. Since $b(0) = 0$ and $b(\pi/2) = -1$, we obtain that $b(t)$ is monotone decreasing for $0 < t < \pi/2$, and therefore $b(t) < 0$ for all $0 < t < \pi/2$. As

$$a(t+\pi) = -a(t) \quad \text{and} \quad b(t+\pi) = -b(t),$$

it follows that $a(t) > 0$ for $\pi < t < 3\pi/2$ and $b(t) > 0$ for $\pi < t < 3\pi/2$. Therefore,

$$f(t) \geqslant 1 \quad \text{for all } \pi \leqslant t \leqslant 3\pi/2.$$

The latter equation yields the result, as by part (4) of Theorem 3.1, a root t^* of $f(t)$ distinct from 0 and $\pi/2$, if exists, must satisfy $\pi \leqslant t^* \leqslant 3\pi/2$. □

6.3. *Proof of Theorem 6.1.* By part (1) of Theorem 5.1, there exists a number $\eta_k > 0$ such that if a raked trigonometric polynomial $f(t)$ of degree $2k-1$ and with a constant term 1 has roots at $t=0$ and $t=\eta_k$ with positive even multiplicities summing up to $2k$, then $f(t)$ is positive everywhere else. It follows from Lemmas 6.2 and 5.5 that the same remains true for all $0 < \eta_k \leqslant \pi/2$. Using the shift $f(t) \longmapsto f(t+a)$ of raked trigonometric polynomials, we conclude that for every arc $\Gamma \subset \mathbb{S}$ of length not exceeding $\pi/2$, a raked trigonometric polynomial $f(t)$ of degree $2k-1$ with constant term 1 that has roots of even multiplicities summing up to $2k$ at the endpoints of Γ remains positive everywhere else in $\mathbb{S}$. The proof now follows from part (2) of Theorem 5.1. □

6.4. *Proofs of Theorems 1.1 and 1.3.* Theorem 1.1 follows from Theorem 6.1 and Remark 2.3(1), while Theorem 1.3 follows from Remark 2.3(1) and Lemma 5.5. □

§7. *The limit of neighborliness.* In this section, we prove Theorem 1.2. Our goal is to construct a raked trigonometric polynomial $f_k(t)$ of degree $2k-1$ such that $f_k(t)$ has a root of multiplicity $2k-2$ at $t=0$, roots of multiplicity 2 each at $t = \pm\beta_k$ for some $\pi/2 < \beta_k < \pi$, and such that $f_k(t) \geqslant 0$ for all $t \in \mathbb{S}$. It then follows from Theorem 5.1 and Lemma 5.7 that $\phi_k \leqslant \beta_k$, and, establishing that $\beta_k \longrightarrow \pi/2$ as k grows, we complete the proof.

LEMMA 7.1. *The function*

$$f(t) = \sin^{2k-1} t$$

is a raked trigonometric polynomial of degree $2k-1$.

Proof. We have

$$\begin{aligned}
\sin^{2k-1} t &= \left(\frac{e^{it} - e^{-it}}{2i}\right)^{2k-1} = \frac{1}{(-4)^{k-1}} \frac{1}{2i} \sum_{j=0}^{2k-1} \binom{2k-1}{j} (-1)^j e^{i(2k-2j-1)t} \\
&= \frac{1}{(-4)^{k-1}} \sum_{j=0}^{k-1} \binom{2k-1}{j} \\
&\quad \times \left(\frac{(-1)^j e^{i(2k-2j-1)t} + (-1)^{2k-1-j} e^{i(2j-2k+1)t}}{2i}\right) \\
&= \frac{1}{(-4)^{k-1}} \sum_{j=0}^{k-1} \binom{2k-1}{j} (-1)^j \sin(2k-2j-1)t. \qquad \square
\end{aligned}$$

LEMMA 7.2. *For* $k \geqslant 1$ *let*

$$h_k(t) = \int_0^t \sin^{2k-1}(\tau)\, d\tau.$$

Then $h_k(t)$ *is a raked trigonometric polynomial of degree* $2k-1$ *and* $t = 0$ *is a root of* $h_k(t)$ *of multiplicity* $2k$. *Moreover,*

$$h_k(t) = \frac{(2k-2)!!}{(2k-1)!!}\left(1 - (\cos t) \sum_{j=0}^{k-1} \frac{(2j-1)!!}{(2j)!!} \sin^{2j} t\right),$$

where we agree that $0!! = (-1)!! = 1$.

Proof. From Lemma 7.1, $h_k(t)$ is a raked trigonometric polynomial of degree $2k-1$. Moreover, $h_k(0) = 0$ and $h_k'(t) = \sin^{2k-1} t$, from which it follows that $t = 0$ is a root of $h_k(t)$ of multiplicity $2k$. Since

$$h_1(t) = \int_0^t \sin\tau\, d\tau = 1 - \cos t,$$

and since for $n > 1$

$$\int_0^t \sin^n \tau\, d\tau = -\frac{1}{n}(\sin^{n-1} t)(\cos t) + \frac{n-1}{n} \int_0^t \sin^{n-2} \tau\, d\tau,$$

we obtain by induction that

$$\int_0^t \sin^{2k-1} \tau\, d\tau = \frac{(2k-2)!!}{(2k-1)!!}\left(1 - \cos t \sum_{j=0}^{k-1} \frac{(2j-1)!!}{(2j)!!} \sin^{2j} t\right),$$

as claimed. $\square$

LEMMA 7.3. *Let* $h_k(t)$ *be the trigonometric polynomial defined in Lemma 7.2 and let*

$$F_k(t) = \sin^2(t) h_{k-1}(t) - h_k(t).$$

Then there exists a unique

$$\frac{\pi}{2} < \beta_k < \pi$$

such that

$$F_k(\beta_k) = 0.$$

In addition,

$$\lim_{k \longrightarrow +\infty} \beta_k = \frac{\pi}{2}.$$

Proof. From Lemma 7.2, we deduce

$$F_k\left(\frac{\pi}{2}\right) = h_{k-1}\left(\frac{\pi}{2}\right) - h_k\left(\frac{\pi}{2}\right) = \frac{(2k-4)!!}{(2k-3)!!} - \frac{(2k-2)!!}{(2k-1)!!} > 0$$

and

$$F_k(\pi) = -h_k(\pi) = -2\frac{(2k-2)!!}{(2k-1)!!} < 0.$$

Moreover,

$$F_k'(t) = 2(\sin t)(\cos t)h_{k-1}(t) - h_k'(t) + \sin^2(t)h_{k-1}'(t) = 2(\sin t)(\cos t)h_{k-1}(t).$$

In particular, $F_k'(t) < 0$ for $\pi/2 < t < \pi$, and hence $F_k(t)$ is decreasing on the interval $\pi/2 < t < \pi$. Since $F_k(\pi/2) > 0$ and $F_k(\pi) < 0$, there is a unique $\pi/2 < \beta_k < \pi$ such that $F_k(\beta_k) = 0$.

To find the limit behavior of β_k, we use the expansion

$$(1-x)^{-1/2} = \sum_{j=0}^{\infty} \frac{(2j-1)!!}{(2j)!!} x^j \quad \text{for real } -1 < x < 1.$$

Substituting $x = \sin^2 t$ we obtain

$$\sum_{j=0}^{\infty} \frac{(2j-1)!!}{(2j)!!} \sin^{2j} t = -\frac{1}{\cos t} \quad \text{provided } \pi/2 < t < \pi.$$

Hence from Lemma 7.2, for $\pi/2 < t < \pi$ we have

$$\begin{aligned} h_k(t) &= \frac{(2k-2)!!}{(2k-1)!!}\left(1 - (\cos t)\left(\sum_{j=0}^{\infty} \frac{(2j-1)!!}{(2j)!!} \sin^{2j} t \right.\right. \\ &\qquad \left.\left. - \sum_{j=k}^{\infty} \frac{(2j-1)!!}{(2j)!!} \sin^{2j} t\right)\right) \\ &= \frac{(2k-2)!!}{(2k-1)!!}\left(2 + (\cos t) \sum_{j=k}^{\infty} \frac{(2j-1)!!}{(2j)!!} \sin^{2j} t\right) \end{aligned}$$

and

$$\begin{aligned} \frac{(2k-3)!!}{(2k-4)!!} F_k(t) &= 2\sin^2 t - 2\frac{2k-2}{2k-1} \\ &\qquad + (\cos t) \sum_{j=k}^{\infty} \left(\frac{(2j-3)!!}{(2j-2)!!} - \frac{2k-2}{2k-1}\frac{(2j-1)!!}{(2j)!!}\right) \sin^{2j} t. \end{aligned}$$

It follows that $F_k(t) < 0$ for every $\pi/2 < t < \pi$ such that $\sin^2 t \leqslant (2k-2)/(2k-1)$. Since $F_k(t)$ is decreasing for $\pi/2 < t < \pi$, we conclude that

$$\sin^2 \beta_k > \frac{2k-2}{2k-1}, \tag{7.3.1}$$

and hence

$$\lim_{k \longrightarrow +\infty} \beta_k = \frac{\pi}{2},$$

as desired. □

LEMMA 7.4. *Let $h_k(t)$ be the trigonometric polynomial defined in Lemma 7.2 and let β_k be the number defined in Lemma 7.3. Let*

$$f_k(t) = \sin^2(\beta_k) h_{k-1}(t) - h_k(t).$$

Then $f_k(t)$ is a raked trigonometric polynomial of degree $2k-1$ such that $t = 0$ is a root of $f_k(t)$ of multiplicity $2k-2$, $t = \pm\beta_k$ are the roots of multiplicity 2 each and $f_k(t) \geqslant 0$ for all $t \in \mathbb{S}$.

Proof. It follows by Lemma 7.2 that $f_k(t)$ is a raked trigonometric polynomial of degree $2k-1$ and that $t = 0$ is a root of $f_k(t)$ of multiplicity at least $2k-2$. From the definition of β_k in Lemma 7.3, we conclude that $t = \beta_k$ is a root of $f_k(t)$. Moreover, since

$$f_k'(t) = \sin^{2k-1} t - (\sin^2 \beta_k) \sin^{2k-3} t,$$

we have $f'(\beta_k) = 0$, so the multiplicity of the root at $t = \beta_k$ is at least 2. By part (3) of Theorem 3.1, the multiplicities of the roots at $t = 0$ and $t = \beta_k$ are $2k-2$ and 2 respectively and there are no other roots of $f_k(t)$ in the open arc $0 < t < \pi$. Also, by Lemma 7.2 and (7.3.1), we have

$$f_k(\pi) = 2 \sin^2(\beta_k) \frac{(2k-4)!!}{(2k-3)!!} - 2 \frac{(2k-2)!!}{(2k-1)!!} > 0.$$

Since $f_k(-t) = f_k(t)$, we conclude that $t = -\beta_k$ is a root of $f_k(t) \geqslant 0$ of multiplicity 2 and that $f_k(t) > 0$ for all $t \neq 0, \pm\beta_k$. □

7.5. *Proof of Theorem 1.2.* Let ψ_k be the maximum length of an open arc satisfying part (1) of Theorem 5.1. It follows from Lemma 7.4 that $\psi_k \leqslant \beta_k$, and hence from Lemma 5.7 that $\phi_k \leqslant \beta_k$. Lemma 7.3 then yields the proof. □

All available computational evidence suggests that for even k the smallest length of the arc in part (2) of Theorem 5.1 is achieved when the multiplicities m_a and m_b are equal: $m_a = m_b = k$. The following results provides an explicit equation for the length of such an arc.

PROPOSITION 7.6. *Suppose that k is even. Let $\alpha_k > 0$ be the smallest number such that the necessarily unique raked trigonometric polynomial $f(t)$ of degree $2k-1$ with constant term 1 that has roots at $t = \alpha_k$ and $t = -\alpha_k$ of* *multiplicity k each also has a root t^* elsewhere in $\mathbb{S}$. Then $t^* = \pi$ and α_k is the smallest positive root of the equation $F(\alpha) = 0$ where*

$$F(\alpha) = \cos\alpha + 1 + \sum_{j=1}^{k-1} (-1)^j \frac{(2j-1)!!}{(2j)!!} \tan^{2j}\alpha. \tag{7.6.1}$$

Proof. We note that the raked trigonometric polynomial $\tilde{f}(t) = f(-t)$ also has a root of multiplicity k at $t = \alpha_k$ and a root of multiplicity k at $t = -\alpha_k$. By Corollary 3.5, we must have $\tilde{f}(t) = f(t)$, and hence

$$f(t) = 1 + \sum_{j=1}^{k} a_j \cos(2j-1)t$$

for some real $a_1, \dots, a_k$. Then the raked trigonometric polynomial $f'(t)$ has roots at $t = \alpha_k, -\alpha_k, \alpha_k + \pi$, and $-\alpha_k + \pi$ of multiplicity $k-1$ each as well as roots at $t = 0$ and $t = \pi$. By part (1) of Theorem 3.1, $f'(t)$ has no other roots and $a_k \neq 0$. By part (4) of Theorem 3.1, the root t^* must lie in an open arc Γ with the endpoints $\alpha_k + \pi$ and $-\alpha_k + \pi$. From the definition of α_k, it follows that $f(t) \geqslant 0$ for all $t \in \Gamma$, and hence t^* is a local minimum of $f(t)$. Thus $f'(t^*) = 0$, and so $t^* = \pi$. Moreover, $t^* = \pi$ is a root of $f(t)$ of multiplicity 2.

We choose

$$\lambda = \frac{1}{\cos\alpha_k}$$

in Lemma 3.6 and consider the λ-deformation $f_\lambda(t)$ of $f(t)$. Let

$$p = \mathcal{P}(f) \quad \text{and} \quad p_\lambda = \mathcal{P}(f_\lambda).$$

Since α_k and $-\alpha_k$ are roots of $f(t)$ of multiplicity k each, the complex numbers $\mathrm{e}^{\mathrm{i}\alpha_k}$ and $\mathrm{e}^{-\mathrm{i}\alpha_k}$ are roots of p of multiplicity k each. Then $z = 1$ is a root of $p_\lambda(z)$ of multiplicity $2k$, and hence $t = 0$ is a root of f_λ of multiplicity $2k$.

As $t = \pi$ is a root of f of multiplicity 2, it follows that $z = -1$ is a root of $p(z)$ of multiplicity 2. Thus,

$$\frac{-1 + \sin\alpha_k}{\cos\alpha_k} \quad \text{and} \quad \frac{-1 - \sin\alpha_k}{\cos\alpha_k} \tag{7.6.2}$$

are roots of $p_\lambda(z)$.

Since $t = 0$ is a root of multiplicity $2k$ of $f_\lambda(t)$ the trigonometric polynomial $f_\lambda(t)$ should be proportional to the trigonometric polynomial $h_k(t)$ of Lemma 7.2. Therefore,

$$p_\lambda(z) = \gamma z^{2k-1}\left(1 - \left(\frac{z+z^{-1}}{2}\right)\right)\left(1 + \sum_{j=1}^{k-1} \frac{(2j-1)!!}{(2j)!!}\left(\frac{z-z^{-1}}{2i}\right)^{2j}\right)\right)$$
$$\text{for some } \gamma \neq 0. \tag{7.6.3}$$

Substituting either of the roots of (7.6.2) in (7.6.3), we obtain the desired equation

$$F(\alpha_k) = 0.$$

Suppose now that some number $0<\alpha<\pi/2$ also satisfies the equation $F(\alpha)=0$. Then

$$\frac{-1+\sin\alpha}{\cos\alpha} \quad \text{and} \quad \frac{-1-\sin\alpha}{\cos\alpha} \tag{7.6.4}$$

are roots of polynomial $q=\mathcal{P}(h_k)$, where $h_k(t)$ is the trigonometric polynomial of Lemma 7.2. Let us choose $\lambda=\cos\alpha$ and let $g_\lambda(t)$ be the λ-deformation of $h_k(t)$ as in Lemma 3.6. Let $q_\lambda=\mathcal{P}(g_\lambda)$. Since the numbers introduced in (7.6.4) are roots of q, we conclude that $z=-1$ is a root of multiplicity 2 of $q_\lambda(z)$, and hence $t=\pi$ is a root of multiplicity 2 of $g_\lambda(t)$. Similarly, since $t=0$ is a root of multiplicity $2k$ of $h_k(t)$, we conclude that $z=1$ is a root of multiplicity $2k$ of $q(z)$, and hence the numbers $\mathrm{e}^{\mathrm{i}\alpha}$ and $\mathrm{e}^{-\mathrm{i}\alpha}$ are roots of q_λ, each of multiplicity k. Therefore, $t=\alpha$ and $t=-\alpha$ are roots of $g_\lambda(t)$, each of multiplicity k. It then follows, by minimality of α_k, that $\alpha\geqslant\alpha_k$, which completes the proof. □

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Alexander Barvinok,
Department of Mathematics,
University of Michigan,
Ann Arbor, MI 48109-1043,
U.S.A.
E-mail: barvinok@umich.edu

Seung Jin Lee,
Department of Mathematics,
University of Michigan,
Ann Arbor, MI 48109-1043,
U.S.A.
E-mail: lsjin@umich.edu

Isabella Novik,
Department of Mathematics,
University of Washington,
Seattle, WA 98195-4350,
U.S.A.
E-mail: novik@math.washington.edu