

ISOMORPHIC FACTORISATIONS V: DIRECTED GRAPHS

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Abstract. An isomorphic factorisation of a digraph D is a partition of its arcs into mutually isomorphic subgraphs. If such a factorisation of D into exactly t parts exists, then t must divide the number of arcs in D . This is called the divisibility condition. It is shown conversely that the divisibility condition ensures the existence of an isomorphic factorisation into t parts in the case of any complete digraph. The sufficiency of the divisibility condition is also investigated for complete m -partite digraphs. It is shown to suffice when $m = 2$ and t is odd, but counterexamples are provided when $m = 2$ and t is even, and when $m = 3$ and either $t = 2$ or t is odd.

§1. *Introduction.* An isomorphic factorisation of a digraph $D = (V, E)$ is a partition $\{E_1, E_2, \dots, E_t\}$ of the arc set E such that the spanning sub-digraphs $(V, E_1), (V, E_2), \dots, (V, E_t)$ are all isomorphic. If $H \simeq (V, E_1)$ we say H divides D and write either $H \mid D$ or $H \in D/t$. Also, if D has an isomorphic factorisation into exactly t isomorphic subgraphs, we say that D is divisible by t and write $t \mid D$. Similarly, an isomorphic factorisation of an ordinary (symmetric) graph G is defined in [4] to be a partition of its line set which induces isomorphic subgraphs, and the above notation is used in the analogous fashion for graphs.

For given t and a given digraph D having exactly q arcs, an obvious necessary condition for the divisibility of D by t is that t divide q . This is called the *Divisibility Condition* for D and t . To avoid triviality we always take $t > 1$. All basic graph theoretic notation can be found in [2].

In the first paper in this series [4] we showed that whenever G is a complete graph the Divisibility Condition is sufficient for the divisibility of G by t . The third paper [5] examined analogous questions for complete r -partite graphs when $r > 1$. It was found that when r is 2 the Divisibility Condition is sufficient, but when r is 3 and t is odd the Divisibility Condition is insufficient for the divisibility of a complete r -partite graph. Our present objective is to examine complete digraphs and complete multipartite digraphs in the same way.

Notation. If G is a graph, then DG is the digraph of G , where DG has a symmetric pair of arcs for each line of G .

In the next section we show the sufficiency of the Divisibility Condition for the complete digraph DK_p on p points, for all p . In fact, for given p and t , the Divisibility Condition implies the existence of a self-converse digraph in DK_p/t . Complete r -partite digraphs are examined in Section 3. We show that for $r = 2$ the Divisibility Condition is always sufficient when t is odd, but examples of insufficiency exist for all even t . When $r = 3$, however, we show that the Divisibility Condition is not sufficient for any odd t or for $t = 2$. This insufficiency when $t = 2$ is also shown to apply for any r which is divisible by 4. In the final section related problems, results and conjectures are considered.

§2. *Complete Directed Graphs.* Our purpose in this section is to prove that the Divisibility Condition is sufficient for the existence of a self-converse digraph in

DK_p/t for all p and t . When t is odd this result is a corollary of the Divisibility Theorem for complete graphs [4], whilst the case when t is even is handled by modifying the arguments used to prove the Divisibility Theorem.

THEOREM 1. *If t divides $p(p - 1)$ then the complete directed graph DK_p is divisible by t . In fact, DK_p/t contains a self-converse digraph.*

Proof. Five cases will be considered, mainly because the general constructions employed are not applicable for small values of p and t .

Case 1. t is odd. It follows that $t \mid p(p - 1)/2$, so by the Divisibility Theorem of [4], the complete graph K_p is divisible by t . If G is a graph in K_p/t , then the digraph of G , namely DG , is a self-converse member of DK_p/t .

Case 2. $t = 2$. Consider the complete directed graph DK_p with point set $\{1, 2, \dots, p\}$. Let D be the digraph containing exactly the arcs (i, j) for which $i < j$. Then D is self-converse and is in $DK_p/2$. (This D is the transitive tournament which is denoted by T_p in Moon [8].)

Case 3. $t = 4$. In this case the Divisibility Condition implies that $p = m$ or $m + 1$ for m divisible by 4. We give first the construction for $p = m$, then show how to modify it for $p = m + 1$.

Given $m = 4s$ let A_1, A_2, A_3 and A_4 be four disjoint point sets each of cardinality s , and consider the complete directed graph DK_m with point set

$$V = A_1 \cup A_2 \cup A_3 \cup A_4.$$

Let D_0 be the digraph with point set V containing just those arcs (u, v) for which $u \in A_1$ and $v \in A_2, u \in A_1$ and $v \in A_3, u \in A_3$ and $v \in A_2$, or $u, v \in A_4$. Then D_0 is self-converse and is a member of $DK_m/4$. A factorisation of DK_m into four digraphs isomorphic to D_0 can be obtained as follows. Let ϕ be an automorphism of the point

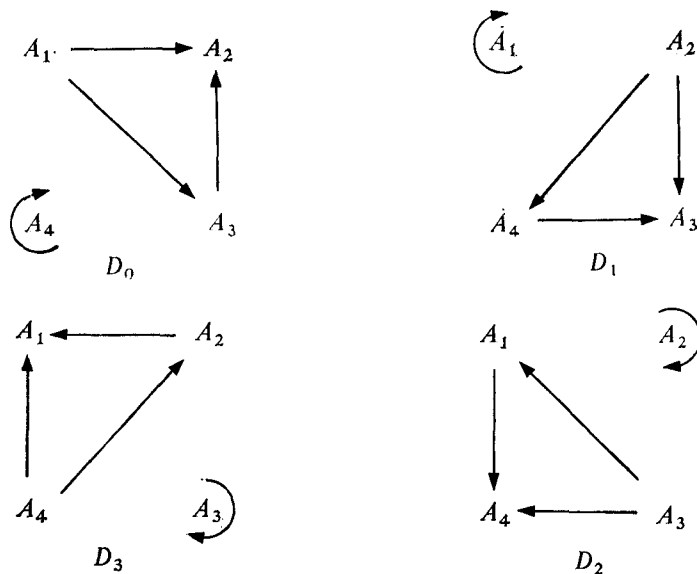


Fig 1. An isomorphic factorisation of $DK_m/4$ when $m = 4s$

set of DK_m such that $\phi(A_1) = A_2, \phi(A_2) = A_3, \phi(A_3) = A_4$ and $\phi(A_4) = A_1$. Then for $i = 1, 2$ and 3 , the map ϕ^i provides an isomorphism between D_0 and another digraph, say D_i . The digraphs D_0, D_1, D_2 and D_3 provide the required factorisation, as illustrated in Figure 1.

Next, let w be a point not in V and consider the complete directed graph DK_{m+1} with point set $V' = V \cup \{w\}$. Let D_0' be the digraph with point set V' containing those arcs in D_0 together with all arcs (u, w) and (w, u) for which $u \in A_4$. It follows that D_0' is self-converse, and is an element of $DK_{m+1}/4$. An isomorphic factorisation of DK_{m+1} can be found in the same way as for DK_m , this time using the automorphism of V' which agrees with ϕ on V and fixes w .

Case 4. $t > 4$ is even and $p \leq 4$. It follows that $(p, t) = (3, 6), (4, 6)$ or $(4, 12)$. It is easily seen that in these cases DK_p/t contains respectively a single arc, a symmetric pair of arcs, and a single arc (plus isolates).

Case 5. $t > 4$ is even and $p \geq 5$. We use induction on p . First, suppose $p \leq t - 1$. Then since $t \mid p(p - 1)$ and $t > 2$ we have $p \leq t - 2$. It follows that

$$\frac{p(p - 1)}{2t} + \frac{1}{2} < \frac{p - 1}{2}.$$

So if $m = p(p - 1)/t$, we see that $(m + 1)/2 < (p - 1)/2$. Therefore, by Lemma 2 of [4], the complete graph K_p can be line-labelled so that any $\frac{1}{2}(m + 1)/2$ consecutive lines are disjoint, where x denotes the largest integer not exceeding x . Using this line-labelling of K_p , we label the arcs of DK_p with the integers $1, 2, \dots, p(p - 1)$ by assigning the numbers $2i - 1$ and $2i$ to the two arcs which correspond to the line labelled i . Any m consecutive arcs will then induce a spanning sub-digraph of DK_p isomorphic to $(m/2)DK_2$ when m is even, and $\frac{1}{2}(m - 1)DK_2$ together with an isolated arc when m is odd. Hence DK_p is divisible by t , and in both cases the factor digraph is self-converse.

It can now be assumed that $p \geq t$ and that DK_{p-t}/t contains a self-converse digraph D . In the event that $p = t$, the empty digraph will suffice for D . By using similar methods to the proof of Lemma 3 in [4], we will construct a digraph in DK_p/t with point set $\{1, 2, \dots, p\}$. First, take an isomorphic factorisation D_1, D_2, \dots, D_t of DK_{p-t} on the point set $\{t + 1, t + 2, \dots, p\}$ with $D_i \cong D$ for all i . To D_1 add the points $\{1, 2, \dots, t\}$, add all arcs between the points of D_1 and the point 1, and add the symmetric pairs of arcs with endpoints $\{1, 2\}, \{1, 3\}, \dots, \{1, (t/2) + 1\}, \{1, (t/2) + 2\}$ together with the arc $(t/2, t)$. The resultant digraph H_1 is self-converse. This is because $t \geq 6$, so the arc $(t/2, t)$ is disjoint from all other arcs, and because D_1 is self-converse. Furthermore, H_1 is an element of DK_p/t , the i -th digraph in an appropriate factorisation of DK_p being obtained by permuting the points of D_1 so as to form D_i and simultaneously applying the permutation $(1\ 2 \dots t)^{i-1}$ to the points $\{1, 2, \dots, t\}$.

§3. Complete r -partite Directed Graphs. The complete r -partite directed graph of type (n_1, n_2, \dots, n_r) is the complement of the disjoint union $DK(n_1) \cup \dots \cup DK(n_r)$ of complete directed graphs and is denoted by $DK(n_1, n_2, \dots, n_r)$. As usual, the complete 2-partite directed graphs are called *bipartite* while the 3-partite graphs are called *tripartite*. In this section we examine the question: For which r and t are all complete r -partite directed graphs, satisfying the Divisibility Condition, divisible by t ? We demonstrate that for $r = 2$, this is true, if, and only if, t is odd. For $r = 3$, on

the other hand, we know of no value of t such that all complete tripartite directed graphs satisfying the Divisibility Condition are divisible by t . For $t = 2$ and for all odd t we give examples of complete tripartite digraphs satisfying the Divisibility Condition but not divisible by t .

THEOREM 2. *The Divisibility Condition is sufficient for the existence of a digraph in $DK(m, n)/t$ for all m and n , if, and only if, t is odd.*

Proof. Suppose t is odd. Then $t \mid 2mn$ implies $t \mid mn$, so by Theorem 1 of [5], the complete bipartite graph $K_{m, n}$ is divisible by t . Thus $G \in K_{m, n}/t$ for some graph G , and its digraph DG is therefore a member of $DK(m, n)/t$.

Suppose on the other hand that t is even. We shall show that if $m = st/2$ where $s \geq 3$ is odd, the digraph $DK(m, 1)$ is not divisible by t even though the Divisibility Condition is satisfied. Suppose to the contrary that there is some $D \in DK(m, 1)/t$. It follows that D has precisely s arcs. As a consequence of the structure of $DK(m, 1)$, these arcs must all be adjacent at one point, u say. As $s \geq 3$, the point u either has in-degree greater than 1 or out-degree greater than 1. Therefore, in any subdigraph of $DK(m, 1)$ isomorphic to D , the point u of D must coincide with the point of in degree m in $DK(m, 1)$. So if u has in-degree k in D , it follows that $m = tk$. This contradicts the choice of $m = st/2$ where s is odd.

We next consider complete tripartite directed graphs. It was shown in [5] that if $t > 1$ is odd and $m \geq t(t + 1)$, the complete tripartite graph $K(1, 1, m)$ is not divisible by 3. This was done by considering the degrees of the points in a hypothetical member of $K(1, 1, m)/t$. If out-degrees are considered instead, the same argument suffices to prove the following.

THEOREM 3. *If $t > 1$ is odd and $m \geq t(t + 1)$, then $DK(1, 1, m)/t$ is empty.*

For $r = 3$ and t even, we do not have a result as broad as Theorem 3; all we have is the following counterexample to the sufficiency of the Divisibility Condition when $t = 2$.

THEOREM 4. *If $m > 5$ is odd, then $DK(1, 2, m)/2$ is empty.*

Proof. Assume $m > 5$ is odd and suppose $DK(1, 2, m)/2$ is nonempty. In $D = DK(1, 2, m)$ let a be the point of degree $m + 2$, b_1 and b_2 the points of degree $m + 1$, and c_1, \dots, c_m the points of degree 3. Let D' and D'' denote the two spanning subdigraphs of an isomorphic factorisation of D into two parts. For any point u of D , let u' denote the out degree of u in D' , and u'' the out-degree of u in D'' . Also, let ϕ be a permutation of the points of D which maps D' to D'' .

We first note that $a' + a'' = m + 2$. Since m is odd, this means either

$$a' > (m + 2)/2 \quad \text{or} \quad a'' > (m + 2)/2;$$

we can assume $a' > (m + 2)/2$, if necessary interchanging D' and D'' . Now, $\phi(a)$ cannot be a because $a' \neq a''$, and $\phi(a)$ cannot be c_i for any i because $c_i'' \leq 3$ whereas $a' > 3$ since $m > 5$. Thus $\phi(a) = b_1$ or b_2 ; take $\phi(a) = b_1$, if necessary interchanging b_1 and b_2 . It follows that $b_1'' = a'$ and $b_1' = m + 1 - a' < m/2$.

Now suppose $b_2' > (m + 1)/2$. It follows that $\phi(b_2)' > (m + 1)/2 > 3$, so $\phi(b_2) = a$ or b_2 . But $a'' = m + 2 - a' < (m + 2)/2$, and thus $a'' \leq (m + 1)/2$, which disallows $\phi(b_2) = a$. Hence $\phi(b_2) = b_2$, so $b_2'' = b_2'$, which contradicts

$b_2' > (m + 1)/2$. On the other hand, if $b_2' < (m + 1)/2$, it follows that $b_2'' > (m + 1)/2$, and it can be seen that $\phi^{-1}(b_2)$ must consequently be b_1 . Thus $b_1' = b_2'' > (m + 1)/2$, which contradicts $b_1' < m/2$. We must conclude that $b_2' = b_2'' = (m + 1)/2$. Moreover, this means that $\phi^{-1}(b_2)$ can only be b_2 , as $a' > (m + 2)/2$ and $b_1' < m/2$, while $c_1' \leq 3 < (m + 1)/2$. Since $\phi(b_1) \neq a$, we therefore have $\phi^{-1}(a) = c_j$ and $\phi(b_1) = c_k$ for some j and k . As ϕ fixes b_2 , $\phi(c_k)$ is either a (in which case $j = k$) or c_l for some $l \neq k$. We consider these two possibilities separately, and reach a contradiction in each case.

Case 1. $\phi(c_k) = a$. It follows that the permutation ϕ contains the 3-cycle $(a b_1 c_k)$. If the arc (a, b_1) is in D' , it follows that $(b_1, c_k) \in D''$, so $(b_1, c_k) \notin D'$, and following on from this we have $(c_k, a) \notin D''$, $(c_k, a) \in D'$, $(a, b_1) \in D''$ and thus $(a, b_1) \notin D'$. A similar contradiction is reached if $(a, b_1) \in D''$.

Case 2. $\phi(c_k) = c_l$. It follows that $\phi^{-1}(c_j) = c_h$ for some $h \neq j$. Then D does not contain the arc (c_h, c_j) , so in particular, neither does D' . Using the same type of argument as in Case 1, we find therefore that D' contains (c_j, a) , D'' contains (a, b_1) , D' contains (b_1, c_k) and hence D'' contains (c_k, c_l) which is a contradiction.

We conclude that $D/2$ is empty for odd $m > 5$, which proves the theorem.

Another result shown in [5] was that, if $4 \mid (r + 1)$ and $m > r + 1$, then the complete $(r + 1)$ -partite graph $K(r : 1, m)$ is not divisible by 2, where $K(r : 1, m)$ has r parts consisting of a single point and one part containing m points. As in the case of Theorem 3, the following corresponding result for directed graphs is obtained by considering out-degrees rather than degrees in the proof.

THEOREM 5. *Let $DK(r : 1, m)$ be the complete $(r + 1)$ -partite directed graph with r parts of cardinality 1 and one part of cardinality m . If $4 \mid (r + 1)$ and $m > r + 1$, then $DK(r : 1, m)$ is not divisible by 2.*

§4. *Related Problems.* We have been studying various complete r -partite directed graphs and different values of t in an attempt to find precisely when the Divisibility Condition is sufficient to ensure the existence of an isomorphic factorisation. Sufficiency was shown for all complete directed graphs with any t , and for all complete bipartite directed graphs with any odd t . Counterexamples to sufficiency were found with $r = 2$ for any even t , and with $r = 3$ for any odd $t > 1$ and for $t = 2$. We conjecture that for any $t > 1$, there is a complete tripartite directed graph D such that D and t satisfy the Divisibility Condition but D/t is empty.

One can of course consider the sufficiency of the Divisibility Condition for more general classes of digraphs, as for example the symmetric digraphs. Each symmetric digraph is of the form DG for some graph G . The number of arcs in DG is even, and we can ask for which graphs G does 2 actually divide DG ? A survey of the graphs on $p \leq 5$ points reveals that $DG/2$ is nonempty for all such graphs except when G is isomorphic to $K_{1,3}$ or to the graph H in Figure 2. Of course $DK_{1,3}$ is one of the

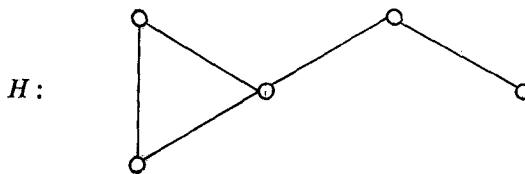


Fig. 2 A graph H with $DH/2$ empty

counterexamples provided in the proof of Theorem 2, but the other graph, H , is not. The general question appears to be difficult.

A number of authors have approached isomorphic factorisations of complete digraphs and complete bipartite digraphs as problems in combinatorial designs. A survey of recent results is given by Bermond and Sotteau [1], and a detailed translation of the language of block designs into the language of graphs and their factors is given in [6]. The most general results are due to Wilson [9], who showed that for every digraph D there is a lower bound $N_0(D)$ such that $D \mid DK_p$ for all $p \geq N_0(D)$ satisfying certain obvious necessary conditions. In special cases more is known about $N_0(D)$. For example $N_0(D)$ is determined in [3] to be 0 when D is any oriented 4-cycle, and Köhler [7] has shown for the directed 6-cycle Z_6 that when $p \neq 12$, $Z_6 \mid DK_p$, if, and only if, $p > 6$ and $p \equiv 0, 1, 3$ or $4 \pmod{6}$. In the latter, it is also shown that $Z_{2k} \mid DK_p$ if $p \equiv 1 \pmod{2k}$ and that $Z_{2k} \mid DK_{m,n}$ if $k \mid m$ and $n \geq k$.

What distinguishes the factorisation problem considered in the present series of papers is that the number of parts t for the factorisation is held constant. Studies motivated by design theory generally hold the factor itself constant. There is also an intermediate class of problems in which the number of factors increases linearly with the number of points in the graph to be factored. Such is the case with factorisations of complete graphs into spanning subgraphs which are unions of cycles; when the cycles are all of the same length such a factorisation is called a *resolvable* design (see [6]). The tantalising open problem of P. J. Kelly also falls into this intermediate class. He conjectured [8, p. 7] that if T is any regular tournament on p points then $Z_p \mid T$. This requires a factorisation of T into exactly $(p-1)/2$ copies of Z_p .

We conclude by conjecturing that the Divisibility Condition does suffice for the family of complete multipartite digraphs of the form $DK(m, m, \dots, m)$, analogous to the conjecture for graphs in [5].

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