A GRAPH AND ITS COMPLEMENT WITH SPECIFIED PROPERTIES V: THE SELF-COMPLEMENT INDEX

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Abstract. The self-complement index $s(G)$ of a graph G is the maximum order of an induced subgraph of G whose complement is also induced in G. This new graphical invariant provides a measure of how close a given graph is to being self-complementary. We establish the existence of graphs G of order p having $s(G) = n$ for all positive integers $n < p$. We determine $s(G)$ for several families of graphs and find in particular that when G is a tree, $s(G) = 4$ unless G is a star for which $s(G) = 2$.

§1. *The self-complement index and the induced number.* Our purpose is to propose invariants which, in some sense, measure the degree to which a graph is self-complementary. To this end we define two related invariants which satisfy this requirement. We then show that the two are equivalent. Throughout we use the notation and terminology of [2]. In particular, all graphs are finite, without loops or multiple lines. The *order* of a graph G is the number p of points in it. And if X is a set of points in a graph G then we use $\langle X \rangle$ to denote the subgraph of G induced by X.

The *self-complement index* of a graph G, denoted $s(G)$, is defined as the order of the largest induced subgraph H of G, such that $\bar{H}$ is also induced in G. For a graph G of order p it is clear that $1 \leqslant s(G) \leqslant p$ as we do not include the null graph in the family of graphs; see Figure 1 in [3].

Now a related invariant of a graph G is defined. The *induced number* $m(G)$ is the minimum order of a graph which contains both G and $\bar{G}$ as induced subgraphs.

The first result indicates that $s(G)$ and $m(G)$ are essentially identical. We then proceed to show that $s(G)$ partitions the graphs of order p into p classes which are nonempty except when $s(G) = p$ and $p \equiv 2$ or $3 \pmod 4$. The number $s(G)$ is then derived for several important families of graphs.

§2. *The equivalence of the two invariants.*

THEOREM 1. *If G is a graph of order p with self-complement index $s(G)$ and induced number $m(G)$, then*

$$m(G) = 2p - s(G).$$

Proof. Let $s = s(G)$, $m = m(G)$, and let H of order s be a largest induced subgraph of G whose complement is also induced in G.

To prove the upper bound, we construct a graph F of order $2p-s$ in which both G and $\bar{G}$ are induced subgraphs. Consider disjoint copies of G and $\bar{G}$. Let $U \subset V(G)$

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induce H in G and let $W \subset V(\bar{G})$ induce H in $\bar{G}$. Let $\phi : U \to W$ be a bijection which gives an automorphism of H. Then F is obtained from $G \cup \bar{G}$ by identifying each $u \in V$ with $\phi(u)$ in W. The order of F is $2p-s$, and both G and $\bar{G}$ are induced subgraphs of F, so $m \leqslant 2p-s$.

To show the lower bound, suppose F' is a graph in which both G and $\bar{G}$ are induced subgraphs. Let X be a set of points of F' inducing G and let Y be a set of points inducing $\bar{G}$. Now if H' is the subgraph of F' induced by $X \cap Y$, then H' is also an induced subgraph of G since $X \cap Y \subset X$. Observe that $\overline{H'}$ is also an induced subgraph of G. Then because the order of H (a largest induced subgraph of G whose complement is also induced in G) is s, we have $|X \cap Y| \leqslant s$. Thus, $m \geqslant |X \cap Y| \geqslant 2p-s$, as required.

§3. *The partition of graphs of order p by s(G).* We shall develop an existence theorem for graphs with given self-complement index by showing that for all p and all $n < p$ there exists a graph G of order p with $s(G) = n$. Figure 1 illustrates such graphs for $p = 8$, using graph theoretic notation from [3, Ch. 2].

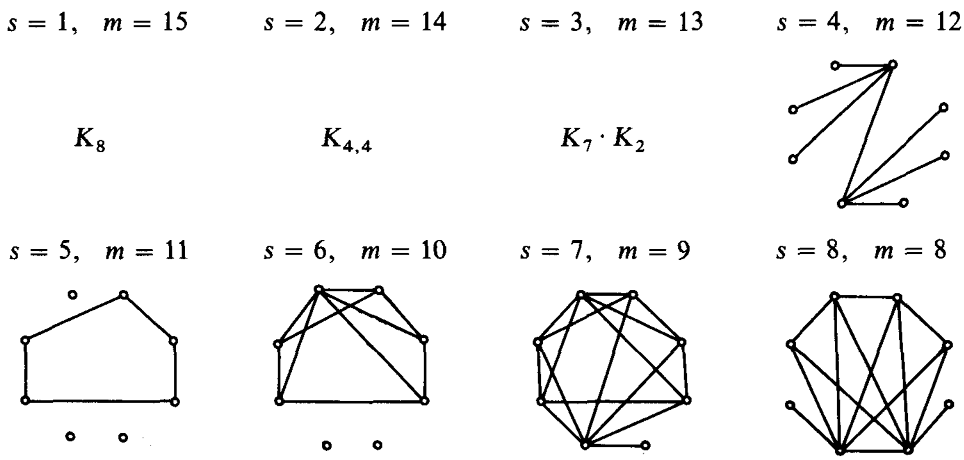

Fig. 1. Examples of 8-point graphs for each possible self-complement index and induced number.

It is well known that self-complementary graphs exist, if, and only if, $p \equiv 0$ or 1 (mod 4). Thus in our terminology there exist graphs of order p with self-complement index p, if, and only if, $p \equiv 0$ or 1 (mod 4).

Moreover the following properties follow immediately from the definition of self-complement index, and so the proofs are omitted.

THEOREM 2. *Let G be a graph of order p.*

(1) $s(G) = s(\bar{G})$.

(2) $s(G) = p$, *if, and only if, G is self-complementary.*

(3) *If H is a maximal induced self-complementary subgraph of G, then* $s(G) \geqslant |V(H)|$.

We shall find it convenient to use the following ternary and the quaternary operations which were introduced in [1] and [2]. Following the notation and terminology of [3], the *join* G_1+G_2 of two graphs is the union of G_1 and G_2 with the complete bigraph having point sets V_1 and V_2, and the *corona* $G_1 \circ G_2$ of two graphs G_1 with p points v_i and, G_2 is obtained from G_1 and p copies of G_2 by joining each point v_i of G_1 with all the points of the i-th copy of G_2. We shall require two related ternary operations denoted $G_1+G_2+G_3$ and $G_1+G_2 \circ G_3$. The ternary operation written $G_1+G_2+G_3$ on three disjoint graphs is defined as the union of the two joins G_1+G_2 and G_2+G_3. On the other hand, the ternary operation $G_1+G_2 \circ G_3$ is defined as the union of the join G_1+G_2 with the corona $G_2 \circ G_3$. Thus this resembles the composition of the path P_3, not with just one other graph, but with three graphs, one for each point of the path. Figure 2a illustrates the "random" graph $K_4-e = K_1+K_2+K_1$ and Figure 2b illustrates the graph $A = K_1+K_2 \circ K_1$. Of course, the quaternary operation $G_1+G_2+G_3+G_4$ is defined similarly and Figure 2c shows the graph $K_1+C_5+K_1+K_1$ which will occur in the proof of Theorem 3.

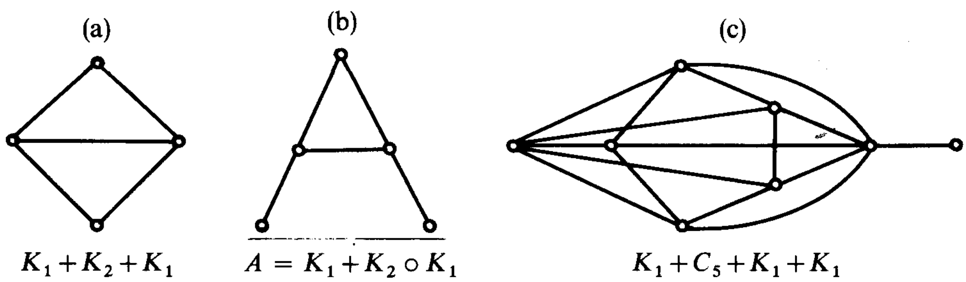

Fig. 2. Examples illustrating the ternary and quaternary operations.

THEOREM 3. *For all p and all positive integers $n < p$, there exists a graph G of order p with $s(G) = n$.*

Proof. We consider four cases according to whether $n \equiv 0, 1, 2$ or 3 (mod 4).

Case 0. $n \equiv 0 \pmod 4$. Let H be a self-complementary graph of order n. Then if $G = H \cup \bar{K}_{p-n}$, we have $s(G) = n$.

Case 1. $n \equiv 1 \pmod 4$. This case is handled exactly as Case 0.

Case 2. $n \equiv 2 \pmod 4$. Let H be a self-complementary graph of order $n-1$ and let $G = (H+K_1) \cup \bar{K}_{p-n}$. Then $s(G) = n$ since $H+K_1$ and $H \cup K_1 = \overline{H+K_1}$ are maximal complementary induced subgraphs of G. The maximality follows, since any larger induced subgraph H' contains either two isolated points or is $(H+K_1) \cup K_1$. But, if H' contained two isolated points, then $\overline{H'}$ would have two points of degree $d \geqslant n-1$. But G does not have two such points. On the other hand, if $H' = (H+K_1) \cup K_1$, it is easy to verify that $\overline{H'}$ is not induced in G.

Case 3. $n \equiv 3 \pmod 4$. Let H be a self-complementary graph of order $n-2$. Define the graph G to be $(K_1+H+K_1+K_1) \cup \bar{K}_{p-n-1}$. It now follows that $s(G) = n$

since $H+K_1+K_1$ and $(H+K_1) \cup K_1 = \overline{H+K_1+K_1}$ are maximal complementary induced subgraphs of G. The maximality of the induced subgraph $H+K_1+K_1$ of order n is easily verified as in the preceding case.

§4. *The self-complement index of some families of graphs.* We now derive the self-complement index for several important families of graphs. It is convenient to write $G > H$ when H is an induced subgraph of G.

THEOREM 4. *The self-complement indexes of complete graphs, complete bigraphs, complete graphs plus one endline, cycles, and complete graphs plus two independent endlines, all of order p, are given by:*

(1) $s(K_p) = 1$;

(2) $s(K_{m,n}) = 2$, $m+n = p$ *and* $\max\{m, n\} \geqslant 2$;

(3) $s(K_{p-1} \cdot K_2) = 3$;

(4) $s(C_p) = 4$;

(5) $s(K_{p-4}+K_2 \circ K_1) = 5, p \geqslant 5$.

Proof. (1) By Theorem 2(3), $s(G) \geqslant 1$ for all graphs G since K_1 is trivially an induced subgraph of any graph.

Conversely, if $s(G) \geqslant 2$, then G must have two non-adjacent points, and since any pair of points of K_p are adjacent, $s(K_p) = 1$.

(2) Let G be a complete bigraph $K_{m,n}$. Since G contains both K_2 and $\bar{K}_2$ as induced subgraphs, $s(G) \geqslant 2$. Let H be a graph of order 3 and assume that $G > H$ and $G > \bar{H}$. Since H or $\bar{H}$ contains P_3 or K_3 as an induced subgraph, $G > P_3$ and $G > \bar{P}_3 = K_2 \cup K_1$, or $G > K_3$ and $G > \bar{K}_3$. However, both cases are impossible for $K_{m,n}$.

(3) Let G be a graph $K_{p-1} \cdot K_2$. Since G contains P_3 and $\bar{P}_3 = K_2 \cup K_1$ as induced subgraphs, $s(G) \geqslant 3$. Let H be a graph of order 4 and assume that $G > H$ and $G > \bar{H}$. Either H or $\bar{H}$ must contain as an induced subgraph one of the following: $\bar{K}_4$, $K_2 \cup \bar{K}_2$, $P_3 \cup K_1$, $2K_2$, $K_{1,3}$ or P_4. Therefore, G must also contain one of them as an induced subgraph, which is impossible for $G = K_{p-1} \cdot K_2$ since G contains neither $\bar{K}_3$ nor $2K_2$ as an induced subgraph.

(4) If $p \geqslant 6$, the cycle C_p contains P_4 as an induced subgraph and so $s(C_p) \geqslant 4$. For any 5 points W of C_p, the graph $\overline{\langle W \rangle}$ contains C_3, a contradiction as no larger cycle contains a triangle.

(5) Let G be the graph $K_{p-4}+K_2 \circ K_1$, $p \geqslant 5$. Since G contains a self-complementary graph $A = K_1+K_2 \circ K_1$ of order 5 as an induced subgraph, we have $s(G) \geqslant 5$. If G contains an induced subgraph H of order 6 such that $\bar{H}$ is also induced in G, then either H or $\bar{H}$ contains at most seven lines since $\binom{6}{2} = 15$. But the subgraph induced by any 6 points of G contains at least 8 lines. Thus $s(G) = 5$.

By Theorem 4(2), every star has self-complement index 2. We now determine this index for all other trees.

THEOREM 5. *For any tree T other than a star,*

$$s(T) = 4.$$

Proof. Note that all trees with at most 3 points are stars. Unless T is a star, T contains P_4 as an induced subgraph, so that $s(T) \geqslant 4$. Now we show the reverse inequality. Assume that $s(T) \geqslant 5$, that is, there exists a subtree H of G of order 5 such that $T > H$ and $T > \bar{H}$. This is impossible since $\bar{H}$ must then contain a cycle.

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