CUBIC DIOPHANTINE INEQUALITIES

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§1. *Introduction.* Let $\lambda_1, \ldots, \lambda_s$ be nonzero real numbers and suppose that λ_1/λ_2 is irrational. In 1955, Davenport and Roth showed [**6**] that the values taken by

$$\lambda_1 x_1^3 + \ldots + \lambda_s x_s^3$$

at integer points $(x_1, \ldots, x_s)$ are dense on the real line, provided that $s \geqslant 8$. In the present paper we obtain the same result with seven variables.

THEOREM. *Let $\sigma = 10^{-4}$. Let $\lambda_1, \ldots, \lambda_7$ be non-zero real numbers with λ_1/λ_2 irrational. For each real μ, the inequality*

$$|\lambda_1 x_1^3 + \ldots + \lambda_7 x_7^3 + \mu| < \left(\max_j |x_j|\right)^{-\sigma}$$

has infinitely many solutions in integers.

The result is somewhat analogous with Linnik's theorem [7] that $R_{7,3}(n)$, the number of representations of n as the sum of seven cubes, is positive for large n. However, a proof of the above theorem only became a feasible prospect with the appearance of Vaughan's work [**10**, **11**, **12**]. Vaughan used the Hardy-Littlewood method to give a good lower bound for $R_{7,3}(n)$ [**10**], and then a lower bound of the expected order of magnitude [**12**], that is,

$$R_{7,3}(n) \gg n^{4/3}$$

for large n. The method of Davenport and Roth is a variant of the Hardy-Littlewood method. However, the bounds for even moments of smooth Weyl sums in Vaughan [**11**] 'just miss' what one would need to initiate a proof of the Theorem. Recently Wooley [**13**] gave sharper bounds for s-th power moments of such sums, for all real $s > 4$. These new moment estimates are the key element in our proof.

We introduce some notation concerning smooth Weyl sums. Denote by $\mathscr{A}(P, R)$ the set of R-smooth numbers of size at most P, that is,

$$\mathscr{A}(P, R) = \{n \in [1, P] \cap \mathbb{Z} : p \mid n \Rightarrow p \leqslant R\}.$$

Here and subsequently, $p, p_1, p_2, \ldots$ denote prime numbers. Let k be a fixed integer, $k \geqslant 3$. We define the smooth Weyl sum $f(\alpha) = f(\alpha; P, R)$ by

$$f(\alpha; P, R) = \sum_{x \in \mathscr{A}(P,R)} e(\alpha x^k)$$

where $e(z)$ denotes $e^{2\pi iz}$. Let

$$U_s(P, R)=\int_0^1 |f(\alpha; P, R)|^s d\alpha.$$

We shall say that an exponent $\mu_s=\mu_{s,k}$ is *permissible* whenever the exponent has the property that, for each $\varepsilon>0$, there exists a positive number $\eta=\eta(\varepsilon, s, k)$ such that whenever $R\leqslant P^\eta$, one has

$$U_s(P, R)\ll_{\varepsilon,s,k} P^{\mu_{s,k}+\varepsilon}.$$

In order to prove our Theorem we need the following results about moments.

PROPOSITION 1. *Let ξ be the positive root of the polynomial*

$$\xi^3+16\xi^2+28\xi-8$$

so that $\xi=0{\cdot}2495681\ldots.$ Then $\mu_{6,3}=3+\xi$ is a permissible exponent.

PROPOSITION 2. *The exponent $\mu_{20/3,3}=3{\cdot}7941603$ is permissible.*

For comparison, Vaughan [**11**, Theorem 4.4] showed that $\mu_{6,3}=3{\cdot}25$ is permissible; one would obtain the permissible exponent $\mu_{20/3,3}=3{\cdot}83$ on combining this with Hölder's inequality and Hua's inequality for the eighth moment.

Proposition 1 is a corollary of [**13**, Theorem 1.2]. We shall prove Proposition 2 in Section 2. Owing to the highly iterative nature of the method of [**13**], we are forced to calculate a number of intermediate moments. In compensation, the estimates provided below for each of these moments appear to be the best deriving from currently available arguments.

Once we have these Propositions, and some further lemmata on Weyl sums which we assemble in Section 3, we are able to proceed with the proof of the Theorem in a relatively straightforward fashion. Section 4 reduces the problem to the estimation of an integral over the real line and deals with the major arc. This is familiar ground; compare Brüdern [**3**]. In Section 5, we discuss the minor arc. Proposition 1 yields some initial Diophantine approximation, and Proposition 2 can then be used to dispose of the most difficult case.

§2. *The* 20/3-*moment*. It is convenient to describe an exponent $\delta_s=\delta_{s,k}$ as *associated* if the exponent

$$\mu_s=s/2+\delta_s \tag{2.1}$$

is permissible in the notation of Section 1.

When $k=3$ and $s<5$ we are able to calculate the permissible exponents μ_s by using a simple corollary of [**13**, Theorem 1.1] which we summarize below in the form of a lemma.

LEMMA 1. *Let s be a real number with $s>2$, and suppose that $\delta_{s,3}$ and $\delta_{2s,3}$ are associated exponents. Then the exponent $\delta_{s+2,3}$ is associated, where*

$$\delta_{s+2}=\delta_s(1-\theta)+\tfrac{1}{2}s\theta$$

and

$$\theta=\frac{\delta_{2s}-2\delta_s}{4+\delta_{2s}-2\delta_s}.$$

Proof. We set $k=3$ and $t=1$ in [**13**, Theorem 1.1] and note equation (2.1) above.

When $k=3$ and $s>6$ one obtains superior permissible exponents by employing an estimate of [**13**, Lemma 3.4] within the argument of [**13**, §4].

LEMMA 2. *Define the integer $\tau=\tau(k)$ by*

$$\tau(k)=\begin{cases}1, & \text{when } k=3, \text{ and when } k\geqslant 8 \text{ and } k \text{ is even}\\ 0, & \text{otherwise.}\end{cases}$$

Let s and t be real numbers with $0<t\leqslant 1$ and $s+2t>4$. Write $v=s(1-t/4)^{-1}$. Suppose that $\mu_{s,k}$ and $\mu_{v,k}$ are permissible exponents. Then $\mu_{s+2t,k}$ is permissible, where

$$\mu_{s+2t}=\mu_s(1-\theta)+t+s\theta$$

and

$$\theta=\frac{t+(4-t)\mu_v-4\mu_s}{3tk+t(1-\tau)+(4-t)\mu_v-4\mu_s}.$$

Proof. Suppose that u and t are real numbers with $0<t\leqslant 1$ and $u+2t>4$, and write $s=u+2t$. Take φ to be a real number with $0\leqslant\varphi\leqslant 1/k$ to be chosen later, and write

$$M=P^{\varphi},\qquad H=PM^{-k}\qquad \text{and}\qquad Q=PM^{-1}. \tag{2.2}$$

We apply the argument of [**13**, §4], setting $v=u(1-t/4)^{-1}$ in the application of Lemma 3.4 of that paper. We may suppose that $\mu_{u,k}$ and $\mu_{v,k}$ are permissible exponents. Then as in [**13**, §4] (see in particular (4.2) and (4.3)), our choice for φ is

$$\varphi=\min(\theta, 1/k),$$

where

$$\theta=\frac{t+(4-t)\mu_v-4\mu_u}{3tk+t(1-\tau)+(4-t)\mu_v-4\mu_u}.$$

We may now mimic the argument of the proof of Theorem 1.1 in §4 of [**13**] to deduce that

$$\mu_s^* = \mu_u(1-\theta) + t + u\theta$$

is permissible, from which the lemma follows immediately.

In the remainder of the paper, we assume that $k=3$.

COROLLARY. *Let s be a real number with $s>2$, and suppose that $\delta_{s,3}$ and $\delta_{4s/3,3}$ are associated exponents. Then the exponent $\delta_{s+2,3}$ is associated, where*

$$\delta_{s+2} = \delta_s(1-\theta) + \tfrac{1}{2}s\theta$$

and

$$\theta = \frac{1+3\delta_{4s/3} - 4\delta_s}{9+3\delta_{4s/3} - 4\delta_s}.$$

Proof. We set $k=3$ and $t=1$ in Lemma 2, and note equation (2.1).

Proof of Proposition 2. We estimate the 20/3-moment in steps (a)–(h) below.

(a) $0<s\leqslant 4$. By combining [**13**, Lemma 2.1] with (2.1), we find that the exponent $\delta_s=0$ is associated.

(b) $s=5, 6$. The proof of [**13**, Theorem 1.2] shows that the exponents $\delta_6 = 0{\cdot}2495682$ and $\delta_5 = 0{\cdot}0880919$ are associated.

(c) $s=9/2$. We apply Lemma 1 with $s=5/2$. Thus we deduce from (a) that whenever δ_5 is an associated exponent, then so is

$$\delta_{9/2} = \frac{5\delta_5}{4(4+\delta_5)}.$$

Consequently, from (b), we deduce that $\delta_{9/2} = 0{\cdot}0269356$ is an associated exponent.

(d) $s=16/3$. We apply Hölder's inequality to interpolate linearly between available associated exponents δ_5 and δ_6. Thus, whenever δ_5 and δ_6 are associated exponents, then so is

$$\delta_{16/3} = \tfrac{2}{3}\delta_5 + \tfrac{1}{3}\delta_6.$$

It therefore follows from (b) that $\delta_{16/3} = 0{\cdot}1419174$ is an associated exponent.

(e) $s=14/3$. We apply Lemma 1 with $s=8/3$. Thus we deduce from (a) that whenever $\delta_{16/3}$ is an associated exponent, then so is

$$\delta_{14/3} = \frac{4\delta_{16/3}}{3(4+\delta_{16/3})}.$$

Thus, in view of (d), we find that $\delta_{14/3} = 0{\cdot}0456850$ is an associated exponent.

(f) $s=13/2$. We apply the Corollary with $s=9/2$. Then we deduce that whenever $\delta_{9/2}$ and δ_6 are associated exponents, then so is

$$\delta_{13/2}=\delta_{9/2}(1-\theta)+\frac{9\theta}{4}$$

where

$$\theta=\frac{1+3\delta_6-4\delta_{9/2}}{9+3\delta_6-4\delta_{9/2}}.$$

Consequently, it follows from (b) and (c) that $\delta_{13/2}=0{\cdot}4053175$ is an associated exponent.

(g) $s=56/9$. We aply Hölder's inequality to interpolate linearly between available associated exponents δ_6 and $\delta_{13/2}$. Thus, whenever δ_6 and $\delta_{13/2}$ are associated exponents, then so is

$$\delta_{56/9}=\tfrac{5}{9}\delta_6+\tfrac{4}{9}\delta_{13/2}.$$

Thus we find from (b) and (f) that $\delta_{56/9}=0{\cdot}3187902$ is an associated exponent.

(h) $s=20/3$. We apply the Corollary with $s=14/3$. Thus we deduce that whenever $\delta_{14/3}$ and $\delta_{56/9}$ are associated exponents, then so is

$$\delta_{20/3}=\delta_{14/3}(1-\theta)+\tfrac{7}{3}\theta$$

where

$$\theta=\frac{1+3\delta_{56/9}-4\delta_{14/3}}{9+3\delta_{56/9}-4\delta_{14/3}}.$$

Consequently, from (e) and (g), it follows that $\delta_{20/3}=0{\cdot}4608269$ is an associated exponent.

Proposition 2 now follows from (h) on recalling (2.1).

§3. *Lemmata on Weyl sums.* Let ε be a sufficiently small positive constant and let η be sufficiently small as a function of ε. Constants implicit in '$\ll$' or 'O' notation will depend at most on ε. If

$$A\ll B\ll A,$$

we write $a\asymp B$.

LEMMA 3. *Let* $N=465767$,

$$\theta_1=38416N^{-1},\qquad \theta_2=41160N^{-1},\qquad \theta_3=45668N^{-1},$$
$$\theta_4=51506N^{-1},\qquad \theta_5=58521N^{-1},$$

and define $g(\alpha)=g(P;\alpha)$ *by*

$$g(P;\alpha)=\sum_{\substack{P^{\theta_i}<p_i\leqslant 2P^{\theta_i}\\(1\leqslant i\leqslant 5)}}\ \sum_{P/(p_1\dots p_5)<y\leqslant 2P/(p_1\dots p_5)} e(\alpha(p_1\dots p_5y)^3).$$

Then

$$\int_0^1 |g(P;\alpha)|^6 d\alpha \ll P^{3+2\theta_5+\varepsilon}.$$

Proof. This is (a trivial variant of) the case $\ell=5$ of Brüdern [**4**].

Notice that the exponent in Lemma 3 is a little worse than Vaughan's value $\mu_{6,3}=3.25$ mentioned in Section 1. In compensation we have Lemma 5 below, where we make good use of the long inner summation over y. Lemma 4 is the stronger counterpart for an 'ordinary' Weyl sum.

LEMMA 4. *Let* $A\geqslant P^{3/4+\varepsilon}$. *Suppose that*

$$\left|\sum_{P<x\leqslant 2P} e(\beta x^3)\right|\geqslant A.$$

Then there are coprime integers b, r *such that*

$$1\leqslant r\ll P^3A^{-3}$$

$$|\beta-b/r|\ll r^{-1/3}A^{-1}P^{-2}.$$

Proof. See for example the Lemma in [**1**], §4.

LEMMA 5. *Let* $B\gg P^{8/9}$. *Let* g *be as in Lemma* 3. *If* $|g(P;\beta)|\geqslant B$ *then there are coprime* a, q *with*

$$1\leqslant q\ll P^3B^{-3}$$

$$|\beta-a/q|\ll q^{-1/3}B^{-1}P^{-2}.$$

Proof. Let

$$\mathcal{M}=\{p_1\dots p_5\colon P^{\theta_i}<p_i\leqslant 2P^{\theta_i}\},$$

$$k(\alpha,Z)=\sum_{Z<z\leqslant 2Z} e(\alpha z^3).$$

We have

$$g(\alpha)=\sum_{m\in\mathcal{M}} k(\alpha m^3,P/m).$$

We write $\theta_0=1-\theta_1-\theta_2-\dots-\theta_5=0{\cdot}494\dots$ and $Y=P^{\theta_0}$. For $m\in\mathcal{M}$ we have $P/m\asymp Y$.

Given $m \in \mathcal{M}$, we may choose coprime b, r with $1 \leqslant r \leqslant 24Y^2$,

$$|am^3 - b/r| \leqslant (24r)^{-1} Y^{-2}.$$

If $r > Y$ then, by Weyl's inequality,

$$k(am^3, P/m) \ll Y^{3/4+\varepsilon}$$

while for $r \leqslant Y$,

$$k(am^3, P/m) \ll r^{-1/3} Y \left(1 + Y^3 \left| am^3 - \frac{b}{r} \right| \right)^{-1} + r^{1/2+\varepsilon}.$$

This follows at once from [**8**, Lemma 6.1]. We see that

$$k(am^3, P/m) \ll Y^{3/4+\varepsilon}$$

unless

$$1 \leqslant r \leqslant Y^{3/4}; \qquad |am^3 - b/r| \leqslant \tfrac{1}{2} r^{-1} Y^{-9/4}. \tag{3.1}$$

Consequently,

$$B \leqslant |g(\alpha)| \ll Y^{3/4+\varepsilon} P^{\theta_1 + \dots + \theta_5} + \sum_{m \in \mathcal{M}'} r^{-1/3} Y \left(1 + Y^3 \left| am^3 - \frac{b}{r} \right| \right)^{-1}$$

where $\mathcal{M}'$ is the subset of $\mathcal{M}$ such that (3.1) holds. But

$$Y^{3/4+\varepsilon} P^{\theta_1 + \dots + \theta_5} \ll P^{1+\varepsilon} Y^{-1/4} \ll P^{8/9-\varepsilon},$$

so that

$$B \ll \sum_{m \in \mathcal{M}'} r^{-1/3} Y \left(1 + Y^3 \left| am^3 - \frac{b}{r} \right| \right)^{-1}.$$

For convenience write $m = m_1 m_2$ with $m_1 = p_1 p_2 p_3$, $m_2 = p_4 p_5$. For any $m = m_1 m_2 \in \mathcal{M}'$ we may choose coprime c, s (depending only on m_1) such that

$$\left| am_1^3 - \frac{c}{s} \right| \leqslant \frac{1}{sY^{9/4}}, \qquad s \leqslant Y^{9/4}.$$

Then

$$\left| am^3 - \frac{cm_2^3}{s} \right| \leqslant \frac{m_2^3}{sY^{9/4}},$$

so that

$$\left| \frac{cm_2^3}{s} - \frac{b}{r} \right| \leqslant \frac{m_2^3}{sY^{9/4}} + \tfrac{1}{2} r^{-1} Y^{-9/4},$$

$$\begin{aligned} |cm_2^3 r - bs| &\leqslant m_2^3 r Y^{-9/4} + \tfrac{1}{2} s Y^{-9/4} \\ &\leqslant 64 P^{3(\theta_4 + \theta_5) - 3\theta_0/2} + \tfrac{1}{2} < 1. \end{aligned}$$

It follows that $r=s(s,m_2^3)^{-1}$, and hence

$$B\ll Y\sum_{m_1m_2\in\mathcal{M}'}s^{-1/3}(s,m_2^3)^{1/3}(1+Y^3m_2^3|am_1^3-c/s|)^{-1}$$

$$\ll YP^{\theta_4+\theta_5}\sum_{m_1}s^{-1/3}(1+Y^3P^{3(\theta_4+\theta_5)}|am_1^3-c/s|)^{-1}. \tag{3.2}$$

The last sum extends over $m_1=p_1p_2p_3(P^{\theta_i}<p_i\leqslant 2P^{\theta_i})$.

We now repeat this line of argument. The sum in (3.2) may be restricted to those m_1 for which

$$1\leqslant s\ll P^{1/3}, \tag{3.3}$$

$$|am_1^3-c/s|<P^{1/9-3(\theta_4+\theta_5)+\varepsilon}Y^{-3}. \tag{3.4}$$

We now choose coprime a, q,

$$1\leqslant q\leqslant P^{3/2},\qquad \left|\alpha-\frac{a}{q}\right|\leqslant\frac{1}{qP^{3/2}}.$$

For those m_1 satisfying (3.3), (3.4),

$$\left|\frac{c}{s}-\frac{am_1^3}{q}\right|<P^{1/9-3(\theta_4+\theta_5)+\varepsilon}Y^{-3}+\frac{m_1^3}{qP^{3/2}},$$

$$|cq-am_1^3s|<sP^{3/2+1/9-3(\theta_4+\theta_5)+\varepsilon}Y^{-3}+sm_1^3P^{-3/2}<1$$

after a short computation. Thus $s=q/(q,m_1^3)$ and

$$B\ll YP^{\theta_4+\theta_5}\sum_{m_1}q^{-1/3}(q,m_1^3)^{1/3}\left(1+P^3\left|\alpha-\frac{a}{q}\right|\right)^{-1}$$

$$\ll Pq^{-1/3}\left(1+P^3\left|\alpha-\frac{a}{q}\right|\right)^{-1}.$$

The lemma follows at once.

LEMMA 6. *Let*

$$f(\alpha)=\sum_{P<x\leqslant 2P}e(\alpha x^3).$$

Let

$$\mathcal{F}=\{\alpha\in[0,1]:|f(\alpha)|>P^{3/4+\varepsilon}\}.$$

Then

$$\int_{\mathcal{F}}|f(\alpha)|^4d\alpha\ll P^{1+\varepsilon}.$$

Proof. See for example Brüdern [2], proof of (4.6).

LEMMA 7. *Let* $1 \leqslant B \leqslant P^{1/4-\varepsilon}$. *Let* $\mathscr{E}(B)=\{\alpha\in[0,1]:|f(\alpha)|\geqslant PB^{-1}\}$. *Then*

$$\int_{\mathscr{E}(B)} |f(\alpha)|^3 d\alpha \ll P^{\varepsilon} B.$$

Proof. By Lemma 6,

$$\int_{\mathscr{E}(B)} |f|^3 d\alpha \leqslant P^{-1} B \int_{\mathscr{F}} |f|^4 d\alpha \ll P^{\varepsilon} B.$$

§4. *The Davenport-Heilbronn method.* According to Davenport [5] for every integer r there is a function $K\colon \mathbb{R}\to\mathbb{R}$ with

$$K(-\alpha)=K(\alpha), \qquad K(\alpha)<C(r)\min(1,|\alpha|^{-r})$$

whose Fourier transform satisfies

$$\hat{K}(\alpha)=\begin{cases}1, & |\alpha|\leqslant 1/3\\ 0, & |\alpha|\geqslant 1\end{cases}$$

and $0\leqslant \hat{K}(\alpha)\leqslant 1$ for $1/3<|\alpha|\leqslant 1$.

In proving our Theorem we may assume $\lambda_1>0$, $\lambda_2<0$ (change x_j into $-x_j$ if necessary). Let

$$f_1(\alpha)=\sum_{P\lambda_1^{-1/3}<x\leqslant 2P\lambda_1^{-1/3}} e(\alpha\lambda_1 x^3),$$

$$g_2(\alpha)=g(P|\lambda_2|^{-1/3},\lambda_2)$$

(in the notation of Lemma 3), and

$$h_j(\alpha)=\sum_{y\in\mathscr{A}(X,X^{\eta})} e(\alpha\lambda_j y^3),$$

where $X=P/\log P$. Now consider

$$\mathscr{N}(P)=\int_{-\infty}^{\infty} f_1(\alpha)g_2(\alpha)h_3(\alpha)\dots h_7(\alpha)e(\alpha\mu)K(\alpha P^{-\sigma})d\alpha. \tag{4.1}$$

By a familiar argument (see *e.g.* [3], §3), it suffices to prove that $\mathscr{N}(P)\to\infty$ as P runs through some sequence of positive numbers tending to infinity.

Let

$$\mathfrak{M}=\{\alpha:|\alpha|\leqslant P^{-2-\varepsilon}\},$$

$$\mathfrak{m}=\{\alpha:P^{-2-\varepsilon}<|\alpha|\leqslant P^{\sigma+\varepsilon}\},$$

$$\mathscr{T}=\{\alpha:|\alpha|>P^{\sigma+\varepsilon}\}.$$

It is easy to see that for a suitable choice of $r=r(\varepsilon)$, the contribution of $\mathscr{T}$ to $\mathscr{N}(P)$ is $O(1)$; compare [3], p. 54.

On the 'major arc' $\mathscr{M}$ we can give satisfactory approximations to f_1 and g_2. Let

$$v(\beta, P)=\int_B^{2B} e(\beta\gamma^3)d\gamma.$$

Then, for $\alpha\in\mathscr{M}$,

$$\begin{aligned} f_1(\alpha)&=v(\lambda_1\alpha, P\lambda_1^{-1/3})+O(1)\\ &=\lambda_1^{-1/3}v(\alpha, P)+O(1), \end{aligned} \tag{4.2}$$

by Theorem 4.1 of Vaughan [8]. Similarly, for

$$|\alpha|\leqslant P^{-5/2},$$

Theorem 2 of Vaughan [9] yields

$$\begin{aligned} g_2(\alpha)&=\sum_{\substack{P^{\theta i}<p_i\leqslant 2P^{\theta i}\\ (1\leqslant i\leqslant 5)}} (v(\lambda_2p_1^3\dots p_5^3\alpha, P|\lambda_2|^{-1/3}(p_1\dots p_5)^{-1})+O(1+P^{3/2}|\alpha|^{1/2}))\\ &=|\lambda_2|^{-1/3}\Gamma v(-\alpha, P)+O(P^{8/9}). \end{aligned} \tag{4.3}$$

Here

$$\Gamma=\prod_{1\leqslant i\leqslant 5}\sum_{P^{\theta i}<p_i\leqslant 2P^{\theta i}}\frac{1}{p_i}\asymp(\log P)^{-5}. \tag{4.4}$$

For

$$P^{-5/2}<|\alpha|\leqslant P^{-2-\varepsilon} \tag{4.5}$$

we claim that

$$|g_2(\alpha)|<P^{8/9}. \tag{4.6}$$

For suppose the contrary. By Lemma 5 there are coprime integers a, q with

$$\begin{aligned} &q\ll P^{1/3},\\ &|\lambda_2\alpha-a/q|\ll q^{-1/3}P^{-26/9}\ll q^{-1}P^{-2}. \end{aligned} \tag{4.7}$$

We cannot have $q=1$, since then $a=0$ and

$$\alpha\ll P^{-26/9},$$

which contradicts (4.5). Consequently we may deduce from (4.7) that

$$|\lambda_2\alpha|>1/2q\gg P^{-1/3}.$$

Again, this contradicts (4.5) and we have established (4.6).

Combining (4.2), (4.3)

$$f_1(\alpha)g_2(\alpha)=|\lambda_1\lambda_2|^{-1/3}\Gamma|v(\alpha, P)|^2+O(P^{8/9}(1+|v(\alpha, P)|)),$$

for $|\alpha|<P^{-5/2}$, while

$$f_1(\alpha)g_2(\alpha)=O(P^{8/9}(1+|v(\alpha, P)|)),$$

when (4.5) holds. Thus, abbreviating the integrand in (4.1),

$$\int_{\mathcal{M}} f_1 g_2 h_3 \ldots h_7 e(\alpha\mu) K d\alpha$$

$$= |\lambda_1\lambda_2|^{-1/3}\Gamma \int_{|\alpha|<P^{-5/2}} |v(\alpha, P)|^2 h_3 \ldots h_7 e(\alpha\mu) K d\alpha$$

$$+ O\left(P^{8/9} \int_{-1}^{1} (1+|v(\alpha, P)|)|h_3 \ldots h_7| d\alpha\right). \tag{4.8}$$

Integration by parts yields

$$v(\alpha, P) \ll P(1+P^3|\alpha|)^{-1}.$$

With the trivial bound $h_j = O(P)$ we get

$$\int_{-1}^{1} |v(\alpha, P)|\,|h_3 \ldots h_7| d\alpha \ll P^3 \log P. \tag{4.9}$$

From the bound $h_3 = O(P)$ and

$$\int_0^1 |h_j|^4 d\alpha \ll P^{2+\varepsilon}$$

we get

$$\int_{-1}^{1} |h_3 \ldots h_7| d\alpha \ll P^{3+\varepsilon}. \tag{4.10}$$

Combining (4.9) and (4.10), the O term in (4.8) is

$$\ll P^{4-1/10}. \tag{4.11}$$

By following through the argument on p. 55 of [**3**], we find that

$$\int_{-P^{-5/2}}^{P^{-5/2}} |v(\alpha, P)|^2 h_3 \ldots h_7 K e(\alpha\mu) d\alpha \gg P^{4-\varepsilon}. \tag{4.12}$$

We may now obtain

$$\int_{\mathcal{M}} f_1 g_2 h_3 \ldots h_7 e(\alpha\mu) K d\alpha \gg P^{4-\varepsilon} \tag{4.13}$$

on assembling (4.8), (4.11) and (4.12).

§5. *The minor arc.* Since $K(\alpha) \ll 1$, our Theorem will follow from (4.13) once we show that

$$\int_{\mathfrak{m}} |f_1 g_2 h_3 \dots h_7| d\alpha \ll P^{4-2\varepsilon}. \tag{5.1}$$

Let

$$E = \{\alpha \in \mathfrak{m}: |f_1(\alpha)| < P^{3/4+\varepsilon}\}.$$

By Hölder's inequality,

$$\int_E |f_1 g_2 h_3 \dots h_7| d\alpha \leqslant P^{3/4+\varepsilon} \left(\int_{\mathfrak{m}} |g_2|^6 d\alpha \right)^{1/6} \prod_{j=3}^{7} \left(\int_{\mathfrak{m}} |h_j|^6 d\alpha \right)^{1/6}.$$

It is clear that

$$\int_{\mathfrak{m}} |g_2(\alpha)|^6 d\alpha \ll P^{\sigma} \int_0^1 |g_2(\alpha)|^6 d\alpha \tag{5.2}$$

and similarly for h_j. It follows from Proposition 1 and Lemma 3 that

$$\int_E |f_1 g_2 h_3 \dots h_7| d\alpha \ll P^{3/4+\varepsilon+\sigma} (P^{3+0\cdot 25129})^{1/6} (P^{3+0\cdot 24957})^{5/6}$$

$$\ll P^{4-2\varepsilon}.$$

Next, we treat the set

$$F = \{\alpha \in \mathfrak{m}: P^{3/4+\varepsilon} \leqslant |f_1(\alpha)| \leqslant P^{4/5}\}.$$

By Hölder's inequality

$$\int_F |f_1 g_2 h_3 \dots h_7| d\alpha \ll \left(\sup_{\alpha \in F} |f_1(\alpha)|^{1/2} \right) \left(\int_F |f_1|^4 d\alpha \right)^{1/8}$$

$$\times \left(\int_{\mathfrak{m}} |g_2|^8 d\alpha \right)^{1/8} \prod_{j=3}^{7} \left(\int_{\mathfrak{m}} |h_j|^{20/3} d\alpha \right)^{3/20}$$

$$\ll P^{\tau},$$

where

$$\tau = \tfrac{2}{5} + \tfrac{1}{8} + \tfrac{5}{8} + 5 \times \tfrac{3}{20} \times 3\cdot 79417 + \sigma < 4 - 2\varepsilon.$$

Here we have used the definition of F, the analogue of (5.2), Lemma 6, Hua's inequality and Proposition 2.

It remains to treat those α with $|f_1(\alpha)|>P^{4/5}$. We begin with

$$G=\{\alpha\in\mathfrak{m}: |f_1(\alpha)|>P^{4/5}, |g_2(\alpha)|\leqslant P^{8/9}\}.$$

By Hölder's inequality,

$$\int_G |f_1g_2h_3\dots h_7|\,d\alpha \ll \left(\sup_{\alpha\in G}|g_2(\alpha)|^{2/3}\right)\left(\int_G |f_1|^3 d\alpha\right)^{1/3} \times\left(\int_{\mathfrak{m}} |g_2|^8 d\alpha\right)^{1/24}\prod_{j=3}^{7}\left(\int_{\mathfrak{m}}|h_j|^8 d\alpha\right)^{1/8} \ll P^{\nu}$$

where

$$\nu=\tfrac{16}{27}+\tfrac{1}{15}+(\tfrac{1}{24}+\tfrac{5}{8})\times 5+\sigma+\varepsilon<4-2\varepsilon.$$

Here we have used the definition of G, the analogue of (5.2), Lemma 7 and Hua's inequality.

The set which remains is

$$H=\{\alpha\in\mathfrak{m}: |f_1(\alpha)|>P^{4/5}, |g_2(\alpha)|>P^{8/9}\}$$

and it is at this stage that we must restrict P to the values

$$P=q^{2/3}$$

(q a denominator of a convergent to the irrational number λ_1/λ_2). We split H into $O((\log P)^2)$ sets

$$H(A,B)=\{\alpha: A<|f_1(\alpha)|\leqslant 2A, B<|g_2(\alpha)|\leqslant 2B\}$$

with $P^{4/5}\leqslant A\leqslant P$, $P^{8/9}\leqslant B\leqslant P$. Because of the Diophantine approximation provided by Lemmata 4 and 5, we may follow the argument in Baker [**1**], pp. 89–90 to bound the Lebesgue measure of $H(A,B)$ by

$$\ll P^{9/4+7\sigma/6+\varepsilon}(AB)^{-7/2}.$$

Consequently, with trivial bounds on h_j,

$$\int_H |f_1g_2h_3\dots h_7|\,d\alpha \ll P^{5+9/4+2\sigma}(AB)^{-5/2}\ll P^{7/2}.$$

We have established (5.1), and the Theorem follows.

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