A FURTHER GENERALIZATION OF HILBERT'S INEQUALITY

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§1. *Introduction.* Hilbert's inequality asserts that

$$\left|\sum_{\substack{r,s\\ r\neq s}} \frac{a_r\bar{a}_s}{r-s}\right| \leqslant \pi \sum_r |a_r|^2,$$

for arbitrary complex numbers a_r. The constant π was first obtained by Schur [**5**], and is best possible. Following a suggestion of Selberg, Montgomery and Vaughan [**4**] showed that

$$\left|\sum_{\substack{r,s\\ r\neq s}} \frac{a_r\bar{a}_s}{\gamma_r-\gamma_s}\right| \leqslant \pi\delta^{-1} \sum_r |a_r|^2, \tag{1}$$

where the γ_r are distinct real numbers and

$$\delta = \min_{\substack{r,s\\ r\neq s}} |\gamma_r-\gamma_s|. \tag{2}$$

Still more generally, they showed also that

$$\left|\sum_{\substack{r,s\\ r\neq s}} \frac{a_r\bar{a}_s}{\gamma_r-\gamma_s}\right| < \tfrac{3}{2}\pi \sum_r |a_r|^2\delta_r^{-1}, \tag{3}$$

where

$$\delta_r = \min_{\substack{s\\ r\neq s}} |\gamma_r-\gamma_s|. \tag{4}$$

This latter inequality is considerably more delicate than (1), and it contains (1) apart from the larger constant. (It remains unknown whether (3) holds with the constant π.) We now formulate a still more general inequality which includes (3) apart from a further imprecision in the constant.

THEOREM. *Let $\rho_r=\beta_r+i\gamma_r$ be complex numbers with $\beta_r\geqslant 0$, and let δ_r be given by* (4). *Then*

$$\left|\sum_{\substack{r,s\\ r\neq s}} \frac{a_r\bar{a}_s}{\rho_r+\bar{\rho}_s}\right| < 84 \sum_r |a_r|^2\delta_r^{-1}. \tag{5}$$

[MATHEMATIKA, **45** (1998), 35–39]

Since the possibility that this inequality might hold was proposed by researchers considering the distribution of zeros of L-functions on the one hand, and those considering questions of metric Diophantine approximation on the other, it may be hoped that this inequality will be of use in a variety of investigations. It is not difficult to construct examples from which it may be seen that the weaker hypotheses $\beta_r \geqslant 0$, $|\rho_r - \rho_s| \geqslant \delta_r$ do not imply an inequality of the above sort. On the other hand, Graham and Vaaler have established an inequality intermediate to (1) and (5) in which δ is given by (2) and all the β's are equal but with the best possible constant (see [**1**], equation (5.11)).

COROLLARY. *Under the above hypotheses, for any* $U > 0$,

$$\int_0^U \left| \sum_{r=1}^R a_r e^{-\rho_r u} \right|^2 du = \sum_{r=1}^R |a_r|^2 \frac{1-e^{2\beta_r U}}{2\beta_r} + 168\theta \sum_{r=1}^R |a_r|^2 \delta_r^{-1}$$

for some θ, $-1 \leqslant \theta \leqslant 1$.

If $\beta_r > 0$ for all r then we can let $U \to \infty$ in the above.

§2. *Proof of the Theorem.* Let $\rho_r' = \delta_r + \rho_r$. We note that

$$\frac{1}{\rho_r + \bar{\rho}_s} - \frac{1}{\rho_r' + \bar{\rho}_s'} = (\delta_r + \delta_s)(\rho_r + \bar{\rho}_s)^{-1}(\rho_r' + \bar{\rho}_s')^{-1}.$$

Since $|\rho_r + \bar{\rho}_s| \geqslant |\gamma_r - \gamma_s|$ and $|\rho_r' + \bar{\rho}_s'| \geqslant |\gamma_r - \gamma_s|$, it follows that

$$\left| \sum_{\substack{r,s \\ r \neq s}} \frac{a_r \bar{a}_s}{\rho_r + \bar{\rho}_s} - \sum_{\substack{r,s \\ r \neq s}} \frac{a_r \bar{a}_s}{\rho_r' + \bar{\rho}_s'} \right| \leqslant \sum_{\substack{r,s \\ r \neq s}} (\delta_r + \delta_s) \frac{|a_r \bar{a}_s|}{(\gamma_r - \gamma_s)^2}.$$

However, Montgomery and Vaughan [**4**] have shown (see the estimate of T_6 on pp. 80–81) that the expression on the right above is at most

$$17 \sum_r |a_r|^2 \delta_r^{-1}. \tag{6}$$

Here the constant 17 is not optimal, and it would be interesting to know what the best constant is. By taking $\gamma_r = r$, $a_r = 1$ for all r, it is evident that the best constant is at least as large as $2\pi^2/3$.

In view of (6), it is enough to show that

$$\left| \sum_{\substack{r,s \\ r \neq s}} \frac{a_r \bar{a}_s}{\rho_r' + \bar{\rho}_s'} \right| \leqslant 67 \sum_r |a_r|^2 \delta_r^{-1}.$$

To simplify notation, from this point on we write ρ_r for ρ_r', and assume that $\beta_r \geqslant \delta_r$. Clearly

$$\sum_{r,s} \frac{a_r \bar{a}_s}{\rho_r + \bar{\rho}_s} = \int_0^\infty \left| \sum_{r=1}^R a_r e^{-\rho_r u} \right|^2 du. \tag{7}$$

Here the right-hand side is non-negative, and the terms $r=s$ on the left contribute an amount $\frac{1}{2}\sum_r |a_r|^2\beta_r^{-1} \leqslant \frac{1}{2}\sum_r |a_r|^2\delta_r^{-1}$. Hence

$$\sum_{\substack{r,s\\ r\neq s}} \frac{a_r\bar{a}_s}{\rho_r+\bar{\rho}_s} \geqslant -\tfrac{1}{2}\sum_r |a_r|^2\delta_1^{-1},$$

and to complete the proof it suffices to show that

$$\int_0^\infty \left|\sum_{r=1}^R a_r e^{-\rho_r u}\right|^2 du \leqslant 67 \sum_r |a_r|^2 \delta_r^{-1}.$$

By the basic duality principle, as expressed for example by taking $p=q=2$ in Theorem 286 of Hardy, Littlewood and Pólya [2], the above is equivalent to the assertion that

$$\sum_{r=1}^R \delta_r \left|\int_0^\infty f(u)e^{-\rho_r u}du\right|^2 \leqslant 67 \int_0^\infty |f(u)|^2 du \tag{8}$$

for all $f \in L^2_{[0,\infty)}$. Write $s=\sigma+it$, and for $\sigma>0$ put

$$F(s)=\int_0^\infty f(u)e^{-su}du.$$

This function is analytic for $\sigma>0$, and is in the Hardy class H^2 on the half-plane $\sigma \geqslant 0$. From the basic properties of such functions, as discussed in Chapter 8 of Hoffman [3], for example, we know that $\lim_{\sigma\to 0^+} F(s)$ exists for almost all t; we call its value $F(it)$. Moreover, $F(it) \in L^2(\mathbb{R})$, and

$$\int_{-\infty}^\infty |F(it)|^2 dt = 2\pi \int_0^\infty |f(u)|^2 du. \tag{9}$$

For $\sigma>0$ we may express $F(s)$ in terms of $F(it)$ by means of the Poisson kernel:

$$F(s)=\frac{\sigma}{\pi}\int_{-\infty}^\infty \frac{F(iv)}{\sigma^2+(v-t)^2}dv. \tag{10}$$

Let

$$\theta(x)=\sup_{\substack{\xi\\ \xi\neq x}} \frac{1}{\xi-x}\int_x^\xi |F(iv)|dv$$

be the Hardy–Littlewood maximal function of $F(iv)$. On integrating by parts in (10) we find that

$$|F(s)| \leqslant \frac{2\sigma}{\pi}\,\theta(x) \int_{-\infty}^{\infty} \frac{|v-x|\,|v-t|}{(\sigma^2+(v-t)^2)^2}\,dv.$$

As $|v-x|\,|v-t| \leqslant |x-t|\,|v-t| + (v-t)^2$, we find that the above is at most

$$\theta(x)\left(\frac{2|t-x|}{\pi\sigma}+1\right).$$

In this relation we take $s=\rho_r$, divide both sides by the expression in parentheses, square both sides, and integrate with respect to x, $\gamma_r - \delta_r/2 \leqslant x \leqslant \gamma_r + \delta_r/2$. This gives

$$\delta_r |F(\rho_r)|^2 \leqslant \left(1+\frac{\delta_r}{\pi\beta_r}\right) \int_{\gamma_r-\delta_r/2}^{\gamma_r+\delta_r/2} |\theta(x)|^2 dx.$$

Here $\beta_r \geqslant \delta_r$ and the intervals of integration are disjoint for distinct r. Hence it follows that the left-hand side of (8) is

$$\sum_{r=1}^{R} \delta_r |F(\rho_r)|^2 \leqslant \left(1+\frac{1}{\pi}\right) \int_{-\infty}^{\infty} \theta(x)^2 dx.$$

By the Hardy–Littlewood inequality (see p. 33 of Zygmund [6]), this latter integral is less than or equal to $8\int_{-\infty}^{\infty} |F(it)|^2 dt$. Hence by (9) we see that (8) holds with constant $16\pi(1+1/\pi) = 66{\cdot}265\ldots < 67$. This completes the proof.

To derive the Corollary it suffices to square out, integrate term-by-term, and apply the Theorem twice.

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