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Size of nodal domains of the eigenvectors of a $G(n, p)$ graph

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Abstract

Consider an eigenvector of the adjacency matrix of a $G(n,p)$ graph. A nodal domain is a connected component of the set of vertices where this eigenvector has a constant sign. It is known that with high probability, there are exactly two nodal domains for each eigenvector corresponding to a nonleading eigenvalue. We prove that with high probability, the sizes of these nodal domains are approximately equal to each other.

KEYWORDS

Random matrices, Erdos-Renyi graph, nodal domains

1 | INTRODUCTION

Nodal domains of the eigenfunctions of the Laplacian on smooth manifolds have been studied for more than a century. We refer the readers to the book [26] for the details. If $f : M \to \mathbb{R}$ is such an eigenfunction on a manifold M, then the nodal domain is a connected component of the set M where the function f has a constant sign. The number and the geometry of nodal domains provide an important insight into the geometric structure of the manifold itself. A classical theorem of Courant states that the number of nodal domains of the eigenfunction corresponding to the kth smallest eigenvalue is upper bounded by k, and this number typically grows as k increases [8]. In [9] Dekel, Lee, and Linial pioneered the study of the nodal domains for graphs. This study was motivated by the usefulness of the eigenvectors of graphs in a number of partitioning and clustering algorithms, see [9] and the references therein. In the last 10 years, these eigenvectors have played a crucial role in many other computer science problems, including, for example, community detection [24, Section 5.5]. As the Laplacian of a graph is closely related to the adjacency matrix, Dekel, Lee, and Linial considered the eigenvectors of the latter matrix as an analog of the eigenfunctions of the Laplacian on a manifold. We will arrange the eigenvectors of the adjacency matrix in the order corresponding to the decreasing order of the eigenvalues. An easy variational argument shows that that the first, that is, the leading eigenvector has only one domain, so the study of nodal domains become nontrivial for the nonleading eigenvectors. In general, one has to distinguish between the strict and the nonstrict domains, where the former do not include vertices with zero coordinates.

The main result of [9] pertains to the $G(n, p)$ random graphs in the case when $p \in (0, 1)$ is a constant. Recall that an Erdős-Rényi Graph $G(n, p)$ is a random graph with n vertices and any two vertices are connected by an edge with probability p independently. In this case, the authors discovered a new phenomenon showing that the behavior of the number of nodal domains for a $G(n, p)$ graph is essentially different from that for a manifold. More precisely, they proved that with probability $1 - o(1)$, the two largest nonstrict nodal domains of any nonleading eigenvector contain all but $O_p(1)$ vertices, where the last quantity is uniform over the eigenvectors. Besides proving this striking result, [9] emphasized that the main approach to the study of nodal domains is through establishing delocalization properties of the eigenvectors of random matrices. At the time [9] was written, the study of delocalization was in its infancy. Indeed, their theorem relies on a partial case of [17, Theorem 3.3], which was the only result available at that time. As the information on the delocalization of the eigenvectors grew, so did the knowledge about the finer properties of the nodal domains. In [18], Nguyen, Tao, and Vu proved that, with probability $1 - o(1)$, any eigenvector does not have zero coordinates, which mean that the strong and the weak nodal domains of a $G(n, p)$ graph are the same with high probability. Also, Arora and Bhaskara [2] improved the main theorem of [9] by showing that if $p \geq n^{-1/19+o(1)}$ then with probability $1 - o(1)$, any nonleading eigenvector has exactly two nodal domains. We refer readers to the articles [10, 12, 13, 16, 23] on other recent developments of local statistics of eigenvalues or delocalization of eigenvectors for sparse Erdős-Rényi Graph $G(n, p)$.

After these results became available, Linial put forward a program of studying the geometry of nodal domains. Considering one of the domains as earth, and another one as water, one can investigate the length of the shoreline, which is the boundary of the domains, the distribution of heights and depths measured as distances to the shoreline, and so on. Unfortunately, this geometry turned out to be trivial in the case when $p \geq n^{-c}$ for some absolute constant $c \in (0, 1)$. More precisely, it was proved in [19] that with probability $1 - o(1)$, any vertex in the positive nodal domain is connected to the negative one, and the same is true for the vertices in the negative domain. Note that the case of very sparse graphs $p \leq n^{-c}$ is still open and may lead to a nontrivial geometry. The proof of this result relied on the combination of the no-gaps delocalization [21], and a more classical ℓ_∞ delocalization established by Erdős, Knowles, Yau, and Yin [12]. The no-gaps delocalization discussed in more detail below means that with high probability, any set S of vertices carries a nonnegligible proportion of the Euclidean norm of the eigenvector, and this proportion is bounded below by a function of $|S|/n$ only. The ℓ_∞ delocalization means that the maximal coordinate of any unit eigenvector does not exceed $n^{-1/2+o(1)}$ with high probability.

In this paper, we establish another natural property of nodal domains. Namely, we will show that with high probability, the nodal domains are balanced, that is, each one of them contains close to $n/2$ vertices with high probability. Unlike the previous ones, this property does not follow from the combination of the no-gaps and the ℓ_∞ delocalization. Indeed, the vector $u \in S^{n-1}$ with $n/3$ coordinates equal to $\sqrt{2}/\sqrt{n}$ and the rest $n/3$ coordinates equal to $-1/\sqrt{2n}$ satisfies both properties. Moreover, for such vector, $\sum_{j=1}^{n} u(j) = 0$, so it is orthogonal to the vector $(1/\sqrt{n}, \dots, 1/\sqrt{n})$ which is close to the leading eigenvector with high probability.

We prove that the nodal domains are roughly of the same size both for the bulk and for the edge eigenvectors. However, the methods of proof in these cases are different. Let us consider the bulk case first as the proof in this case is shorter. Let A be the adjacency matrix of $G(n, p)$. We denote eigenvalues of A by $\lambda_1 \geq \cdots \geq \lambda_n$ and the corresponding unit eigenvectors by $u_1, \dots, u_n$. With a slight abuse of terminology, we will call them the eigenvectors of the graph $G(n, p)$.

Theorem 1.1 (Bulk case). *There is $c \in (0, 1)$ such that the following holds. Let $\varepsilon, \kappa \in (0, 1)$. Let $G(n, p)$ be an Erdős-Rényi Graph with $p \in \left[n^{-c}, \frac{1}{2}\right]$. Let u_α be an eigenvector of $G(n, p)$ with $\alpha \in [\kappa n, n - \kappa n]$. Denote by P and N the nodal domains of this eigenvector. Then there exists $\eta = \eta(\varepsilon, \kappa) > 0$ such that, for a sufficiently large n,*

$$\mathbb{P}\left(|P| \vee |N| \geq \left(\frac{1}{2} + \varepsilon\right) n\right) \leq n^{-\eta}.$$

The proof relies on *quantum unique ergodicity theorem* for random matrices [6, Theorem 1.1] claiming that the distribution of the inner product of an eigenvector of A and any vector orthogonal to $(1, \ldots, 1)$ is asymptotically normal. Readers interested in quantum unique ergodicity are also referred to the articles [3,5,7]. For the edge case, that is, for the eigenvalues close to the edges of the spectrum, the bound similar to [6, Theorem 1.1] has been established only for the nonsparse regime, that is, for $p \in (0, 1)$ which does not depend of n, see [5]. On the other hand, the gaps between the eigenvalues near the edges of the spectrum are much larger. The eigenvalue gap is at least $n^{-2/3-o(1)}$ for edge eigenvalues while it is of order $n^{-1-o(1)}$ for bulk eigenvalues. Also, the edge eigenvalues enjoy stronger rigidity properties than the bulk ones. These facts allow to provide a stronger bound for the size of the nodal domains of an edge eigenvector.

Theorem 1.2 (Edge case). *Let $G(n, p)$ be an Erdős-Rényi Graph with $p \in (0, 1)$. There exists $\rho = \rho(p) > 0$ such that the following holds. Let u_α be a n on-leading eigenvector of $G(n, p)$ with $\min\{\alpha, n - \alpha\} \leq (\log n)^{\rho \log \log n}$. Denote by P and N the nodal domains of this eigenvector. Then, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ independent of n and p such that*

$$\mathbb{P}\left(|P| \vee |N| \geq \left(\frac{1}{2} + n^{-\frac{1}{6}+\varepsilon}\right) n\right) \leq n^{-\delta}.$$

for a sufficiently large n.

For a vector $u \in \mathbb{R}^n$, let $u(i)$ denote its ith component. Our goal in both Theorem 1.1 and 1.2 is to show that with high probability,

$$\sum_{i=1}^{n} \operatorname{sign}(u(i)) = o(n)$$

for an eigenvector u of A. This can be derived by Markov inequality if

$$\mathbb{E}\left(\sum_{i=1}^{n} \operatorname{sign}(u(i))\right)^2 = o\left(n^2\right).$$

The latter equation can be derived if for $i \neq j$,

$$\mathbb{E}\operatorname{sign}(u(i)\, u(j)) = o(1). \tag{1.1}$$

The proof in both the bulk and the edge case is aiming to show (1.1). Yet, the approaches are completely different. The proof in the bulk case relies on

Theorem 1.3 ([21, Theorem 1.5]). *Fix arbitrary constants $\delta, \kappa > 0$ Let A be an $n \times n$ be the adjacency matrix of a $G(n,p)$ graph with $n^{-c} \le p \le 1/2$ for some constant $c > 0$. For $\epsilon > c_1 n^{-1/7}$, every eigenvector v of A satisfies*

$$\left(\sum_{i \in I} |v(i)|^2 \right)^{1/2} \ge (c_2 \epsilon)^6 \|v\| .$$

for all $I \subset [n]$ with $|I| \ge \epsilon n$.

and

Theorem 1.4 ([6, Theorem 1.1]). *Fix arbitrary constants $\delta, \kappa > 0$ Let A be an $n \times n$ be the adjacency matrix of a $G(n,p)$ graph with $n^{-1+\delta} \le p \le 1/2$. Let $v_1, \ldots, v_n$ be its eigenvectors corresponding to the eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n$. For any polynomial $f : \mathbb{R} \to \mathbb{R}$ for any $n \ge n(f)$, $\alpha \in [\kappa n : n - \kappa n]$ and any $q \in S^{n-1}, q \perp (1, \ldots, 1)$, there exists an $\nu > 0$ such that*

$$|\mathbb{E} f(n\langle q, v_\alpha \rangle^2) - \mathbb{E} f(g^2)| \le n^{-\nu}.$$

The last theorem allows to estimate $\mathbb{E}\mathrm{sign}\,(u\,(i)\,u\,(j))$ by replacing $u\,(i)$ and $u\,(j)$ by independent normal random variables. Yet, this replacement is not straightforward. First, we have to transform the statement of Theorem 1.4 involving $\langle q, u \rangle^2$ into a one involving $u(i)$ and $u(j)$. Second, and more importantly, we have to approximate the function sign$(\cdot)$ by a polynomial. Since the polynomial function is unbounded on $\mathbb{R}$, we have to find an approximation which is close to the function sign$(\cdot)$ point-wise on the set $[-R, R] \setminus (-\delta, \delta)$ with some $0 < \delta < 1 < R$, and at the same time has a controlled growth at infinity. The latter property is needed to guarantee that the contribution of the values $u(i) \notin [-R, R]$ does not affect quality of the approximation. The contribution of the values $u(i) \in (-\delta, \delta)$ can be made small by choosing an appropriate δ due to the no-gaps delocalization.

For the edge case, we represent the adjacency matrix A in block form:

$$\begin{bmatrix} D & W^\top \\ W & B \end{bmatrix}$$

where B is $n-2$ by $n-2$, D is 2 by 2, and W is $n-2$ by 2. These matrices are independent. Moreover, using the results of [4, 11, 14], we show that with high probability, the matrix B has "typical" spectral properties. Relying on the independence of the blocks, it is possible to bound the expectation of sign$\,(u\,(1)\,u\,(2))$ conditioned on the event that B is typical. To use this approach for other pairs of coordinates, we have to show that with high probability, all $(n-2) \times (n-2)$ principal submatrices of A are typical. This cannot be derived from the union bound since one of the typical properties, namely the level repulsion, holds with probability $1 - O(n^{-\delta})$ for some $\delta > 0$. To overcome this problem, we condition on the event that the matrix A itself is typical, and show that on this event, with high probability, all $(n-2) \times (n-2)$ blocks are typical as well.

1.1 | Notation

First, c, c', C, C' will denote constants which may change from line to line. For a positive integer n, denote $[n] := \{1, 2, 3, \ldots, n\}$. For vectors $u, v \in \mathbb{R}^n$, let $\|u\|_2$ denote the Euclidean norm of u, $\|u\|_\infty$ denote the l_∞ norm of u, and $\langle u, v \rangle$ denote the standard inner product of u and v. The cardinality of a set S will be denoted by $|S|$. For $a, b \in \mathbb{R}$, the notation $a \wedge b$ and $a \vee b$ stands for the minimum and the maximum of a and b respectively.

For a random variable Z, we denote its ψ_2 norm by $\|Z\|_{\psi_2}$. The ψ_2 norm is defined by the equation

$$\mathbb{E}\exp\left(\left(\frac{|Z|}{\|Z\|_{\psi_2}}\right)^2\right) = 2.$$

We say Z is subgaussian if $\|Z\|_{\psi_2}$ exists. By subgaussian vector we mean a random vector with independent components whose ψ_2 norms are uniformly bounded.

Let $\mathbf{Mat}_{sym}(n)$ be the collection of all symmetric $n \times n$ matrices. For a symmetric $n \times n$ matrix $H = \{h_{ij}\}_{i,j=1}^n$, let $\|H\|$ denote its operater norm, $\|H\|_{HS}$ denotes its Hilbert-Schmidt norm. Precisely,

$$\|H\|_{HS}^2 = \sum_{i,j=1}^n h_{ij}^2 = \sum_{i=1}^n \lambda_i^2,$$

where $\{\lambda_i\}_{i=1}^n$ are eigenvalues of H. Furthermore, for $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$,

$$G(z) = \frac{1}{H - z}$$

denote the Green function of H, and define the Stieltjes Transform of H by

$$m(z) = \frac{1}{n}\operatorname{Tr}(G(z)) = \frac{1}{n}\sum_{i=1}^n \frac{1}{\lambda_i - z}$$

where $\{\lambda_i\}_{i=1}^n$ are eigenvalues of H.

Recall the semicircle-law

$$\rho_{sc}(x) = \frac{1}{2\pi}\sqrt{\left(4 - x^2\right)_+},$$

where $\left(4 - x^2\right)_+ = \max\{4 - x^2, 0\}$. The semicircle law proved in the classical paper of Wigner [25] is the limit distribution of the empirical distribution of eigenvalues of Wigner matrices, see for example, [1] for the precise formulation and extensions. The Stieltjes transform of ρ_{sc} is

$$m_{sc}(z) = \int_{\mathbb{R}} \frac{\rho_{sc}(x)}{x - z}\, \mathrm{d}x.$$

For a fixed n, let γ_i be the expected location of $i-$th eigenvalue (rearranged in a nonincreasing order) according to the semicircle law. That is, γ_i satisfies

$$\int_{\gamma_i}^2 \rho_{sc}(x)\, \mathrm{d}x = \frac{i}{n}.$$

Furthermore, it is easy to check that for $i = o(n)$, we have

$$\left(\pi\frac{i}{n}\right)^{2/3} \le 2 - \gamma_i \le \left(3\pi\frac{i}{n}\right)^{2/3}. \tag{1.2}$$

2 | BULK EIGENVECTOR

Consider a graph G with n vertices, and denote by A its adjacency matrix. Let $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$ be the eigenvalues of A and let v_α be the unit eigenvector corresponding to λ_α. In order to show that $\sum_{i=1}^n \text{sign}\big(v_\alpha(i)\big) = o(n)$, consider a random pair $(i,j) \subset [n]$ of distinct indices which is uniformly chosen among all such pairs. We will check below that if $\mathbb{E}\text{sign}(v_\alpha(i)\cdot v_\alpha(j)) = o(1)$, then this inequality holds, and the nodal domains are of the size close to $n/2$. We are going to establish this bound for the adjacency matrix of a typical $G(n,p)$ graph. Since sign is not a continuous function, it is hard to approach this task directly. Instead, we will approximate the function sign by a suitable polynomial f and show that $\mathbb{E}\big[f(v_\alpha(i)\cdot v_\alpha(j)) \mid A\big] = o(1)$ where the expectation is taken with respect to the random pair (k,l) and A is the adjacency matrix of a typical $G(n,p)$ graph, that is, it is chosen from some set of adjacency matrices whose probability is $1 - o(1)$. After that, we will have to estimate the error of this approximation. To implement the first step, we will use Theorem 1.4 to derive a similar bound for the expectation of an even polynomial of four random coordinates of the eigenvector. This will lead to a stronger bound for an even polynomial of two random coordinates. Finally, applying the latter bound to a one-variable polynomial of the product of two coordinates, we will get the desired estimate.

Let us formulate this statement precisely. Let $v_\alpha \in S^{n-1}$ be a bulk eigenvector of the $G(n,p)$ graph, and let $g_1, \dots, g_n \sim N(0,1)$ be independent standard normal random variables. Denote by $\mathbb{E}_{(i,j)}$ the expectation with respect to the random pair of coordinates (k,l), where the matrix A is regarded as fixed.

Lemma 2.1. *Let A, v_α be as in Theorem 1.4. Let (k,l) be a uniformly chosen random pair of elements of $[n]$. For any* even *polynomial $F : \mathbb{R}^2 \to \mathbb{R}$, there exists a $\nu > 0$ and a set $\mathcal{A}_F \in \mathbf{Mat}_{sym}(n)$ such that for all sufficiently large n,*

$$\mathbb{P}(A \in \mathcal{A}_F) \geq 1 - n^{-\nu},$$

and for any $A \in \mathcal{A}_F$,

$$|\mathbb{E}_{(k,l)}F(n^{1/2}v_\alpha(k), n^{1/2}v_\alpha(l)) - \mathbb{E}F(g_1, g_2)| \leq n^{-\nu}.$$

Proof. The proof breaks in two parts. First, we will show that the statement of Theorem 1.4 holds for any $q \in S^{n-1}$ such that $|\text{supp}(q)| \leq 4$. It is enough to prove the statement for $f(x) = x^d$. Without loss of generality, assume that $q = \sum_{j=1}^4 \alpha e_j$ with $\sum_{j=1}^4 \alpha_j^2 = 1$. Set $\beta := \langle \vec{1}, q\rangle = n^{-1/2}\sum_{j=1}^4 \alpha_j$. Then

$$|\beta| \leq \frac{4}{\sqrt{n}}, \quad q_0 := q - \beta\vec{1} \perp \vec{1} \quad \text{and} \quad \|q_0\|_2 = 1 + O(n^{-1/2}). \tag{2.1}$$

Recall that $w := \vec{1} - v_1$ satisfies

$$\|w\|_2 \leq 2\frac{\log n}{\sqrt{n}}, \tag{2.2}$$

see [21, Theorem 3].

Let us check that for any $d \in \mathbb{N}$,

$$\mathbb{E}(n\langle q, v_\alpha\rangle^2)^d \leq C(d)$$

for some function $C(d) > 0$. Indeed, since $\left\langle \vec{1}, v_\alpha \right\rangle = \langle w, v_\alpha \rangle$,

$$\mathbb{E}(n\langle q, v_\alpha\rangle^2)^d = \mathbb{E}(n\langle q_0 + \beta\sqrt{n}w, v_\alpha\rangle^2)^d \le 2^{2d}\left(\mathbb{E}(n\langle q_0, v_\alpha\rangle^2)^d + \beta^{2d}n^d\,\|w\|_2^{2d}\right)$$
$$\le 2^{2d}\left(\mathbb{E}(2g_1^2)^d + \left(16\frac{\log^2 n}{n}\right)^d\right) \le C(d).$$

where we used (2.1), (2.2) and Theorem 1.4 in the second inequality. By Cauchy-Schwarz inequality, this means that for any $k \in \mathbb{N}$,

$$\mathbb{E}|\sqrt{n}\,\langle q, v_\alpha\rangle\,|^k \le C'(k). \tag{2.3}$$

Therefore, for any $d \in \mathbb{N}$,

$$\left|\mathbb{E}(n\langle q, v_\alpha\rangle^2)^d - \mathbb{E}g^{2d}\right| \le \left|\mathbb{E}(n\langle q, v_\alpha\rangle^2)^d - \mathbb{E}(n\langle \frac{q_0}{\|q_0\|_2}, v_\alpha\rangle^2)^d\right| + \left|\mathbb{E}(n\langle \frac{q_0}{\|q_0\|_2}, v_\alpha\rangle^2)^d - \mathbb{E}g^{2d}\right|$$
$$\le \left|\mathbb{E}(n\langle q, v_\alpha\rangle^2)^d - \frac{1}{\|q_0\|_2^{2d}}\mathbb{E}(n\langle q-\beta\vec{1}, v_\alpha\rangle^2)^d\right| + n^{-\nu}$$
$$\le \left|\mathbb{E}(n\langle q, v_\alpha\rangle^2)^d - \mathbb{E}(n\langle q-\beta w, v_\alpha\rangle^2)^d\right| + 2n^{-\nu}$$
$$\le \sum_{j=1}^{n}\binom{2d}{j}\mathbb{E}|\sqrt{n}\langle q, v_\alpha\rangle|^{2d-j}\cdot\left(8\frac{\log n}{\sqrt{n}}\right)^j + 2n^{-\nu} \le n^{-\nu'}$$

for large n. Here, the third inequality follows from Theorem 1.4, the fourth one from (2.1) and (2.2), and the last one from (2.3). This shows that the conclusion of Theorem 1.4 holds for any $q \in S^{n-1}$ supported on four coordinates. The same argument can be used to prove this statement for any fixed number of coordinates, but we would not need it here.

Let us extend the conclusion of Theorem 1.4 to even polynomials of four variables. Consider an even monomial $G(x_1, \dots, x_4) := x_1^{d_1}\cdot x_2^{d_2}\cdot x_3^{d_3}\cdot x_4^{d_4}$ with $d = d_1+d_2+d_3+d_4 \in 2\mathbb{N}$. Note that for this monomial, $G(\sqrt{n}v_\alpha(k_1), \dots, \sqrt{n}v_\alpha(k_4))$ can be represented as a finite linear combination of $(\sqrt{n}\,\langle q, v_\alpha\rangle)^d$ for different values of $q \in S^{n-1}$, $\mathrm{supp}(q) \subset \{k_1, \dots, k_4\}$. Hence,

$$\left|\mathbb{E}G(\sqrt{n}v_\alpha(k_1), \dots, \sqrt{n}v_\alpha(k_4)) - \mathbb{E}G(g_1, \dots, g_4)\right| \le n^{-\nu} \tag{2.4}$$

and this inequality can be extended to all even polynomials of four variables.

Now, let $F : \mathbb{R}^2 \to \mathbb{R}$ be an even polynomial. Let $s \in [\kappa n : n-\kappa n]$. For a pair $(i,j) \in \binom{[n]}{2}$, define a random variable

$$Y_{(i,j)} = F(\sqrt{n}v_\alpha(i), \sqrt{n}v_\alpha(j)) - \mathbb{E}F(g_i, g_j),$$

where $g_1, \dots, g_n$ are independent $N(0,1)$ random variables. Then for any distinct $i, j, k, l, \in [n]$,

$$|\mathbb{E}Y_{(i,j)}Y_{(k,l)}| = |\mathbb{E}F(\sqrt{n}v_\alpha(i), \sqrt{n}v_\alpha(j))F(\sqrt{n}v_\alpha(k), \sqrt{n}v_\alpha(l))$$
$$- \mathbb{E}F(\sqrt{n}v_\alpha(i), \sqrt{n}v_\alpha(j))\mathbb{E}F(g_k, g_l) - \mathbb{E}F(g_i, g_j)\mathbb{E}F(\sqrt{n}v_\alpha(k), \sqrt{n}v_\alpha(l))$$
$$+ \mathbb{E}F(g_i, g_j)\mathbb{E}F(g_k, g_l)|$$
$$\le |\mathbb{E}F(g_i, g_j)F(g_k, g_l) - 2\mathbb{E}F(g_i, g_j)\cdot\mathbb{E}F(g_k, g_l) + \mathbb{E}F(g_i, g_j)F(g_k, g_l)| + n^{-\nu}$$
$$= n^{-\nu},$$

where we used (2.4) with $G_1(x_1,x_2,x_3,x_4) = F(x_1,x_2)F(x_3,x_4)$, $G_2(x_1,x_2,x_3,x_4) = F(x_1,x_2)$, and $G_3(x_1,x_2,x_3,x_4) = F(x_3,x_4)$ to derive the inequality. A similar calculation shows that $|\mathbb{E}Y_{(i,j)}Y_{(k,l)}| = O(1)$ when i, j, k, l are not necessarily distinct. Hence,

$$\mathbb{E}\left(\frac{1}{\binom{n}{2}}\sum_{(i,j)\in\binom{[n]}{2}} Y_{(i,j)}\right)^2 \leq \frac{1}{\binom{n}{2}^2}\sum_{(i,j,k,l)\in\binom{[n]}{4}} \mathbb{E}Y_{(i,j)}Y_{(k,l)} + O(n^{-1}) \leq n^{-\nu}.$$

The Markov inequality implies that there exists a set $\mathcal{A}'_F \in \mathbf{Mat}_{sym}(n)$ such that for all sufficiently large n,

$$\mathbb{P}(A \in \mathcal{A}'_F) \geq 1 - n^{-\nu/2},$$

and for any $A \in \mathcal{A}'_F$,

$$\left|\frac{1}{\binom{n}{2}}\sum_{(i,j)\in\binom{[n]}{2}} F(\sqrt{n}v_\alpha(i), \sqrt{n}v_\alpha(j)) - \mathbb{E}F(g_1, g_2)\right| = \left|\frac{1}{\binom{n}{2}}\sum_{(i,j)\in\binom{[n]}{2}} Y_{(i,j)}\right| \leq n^{-\nu/4}.$$

The lemma is proved. ■

Applying the previous lemma to a polynomial $F(x, y) = f(x \cdot y)$ for a one-variable polynomial f, we derive the following corollary.

Corollary 2.2. *Let A, v_α be as in Theorem 1.4. Let (k, l) be a uniformly chosen random pair of elements of $[n]$. For any polynomial $f : \mathbb{R} \to \mathbb{R}$, there exists a $\nu > 0$ and a set $\mathcal{A}_f \subset \mathbf{Mat}_{sym}(n)$ such that for all sufficiently large n,*

$$\mathbb{P}(A \in \mathcal{A}_f) \geq 1 - n^{-\nu},$$

and for any $A \in \mathcal{A}_f$,

$$|\mathbb{E}_{(k,l)}f(nv_\alpha(k) \cdot v_\alpha(l)) - \mathbb{E}f(g_1g_2)| \leq n^{-\nu}.$$

To prove that the nodal domains are balanced, we will use Corollary 2.2 with f being an *odd* polynomial approximating sign(x) on some interval $[r, R]$. Since f is odd, $\mathbb{E}f(g_1g_2) = 0$. Hence, assuming that the nodal domains are unbalanced, it would be enough to show that $|\mathbb{E}_{(k,l)}f(nv_\alpha(k) \cdot v_\alpha(l))|$ is non-negligible to get a contradiction. The values of r and R will be chosen so that the absolute values of most of the coordinates will fall into this interval. A simple combinatorial calculation will show that if the nodal domains are unbalanced, then $\mathbb{E}_{(k,l)}\text{sign}(v_\alpha(k) \cdot v_\alpha(l)) = \Omega(1)$. Indeed, assume that for a given matrix A and vector v_j,

$$|P| \vee |N| \geq \left(\frac{1}{2} + \varepsilon\right).$$

Then

$$\mathbb{E}_{(k,l)}\text{sign}(v_\alpha(k) \cdot v_\alpha(l)) = \binom{n}{2}^{-1} \cdot \left[\binom{|P|}{2} + \binom{|N|}{2} - |P| \cdot |N|\right] \geq 4\varepsilon^2 + O(n^{-1}).$$

This reduces our task to the comparison between this quantity and $|\mathbb{E}_{(k,l)} f(nv_\alpha(k) \cdot v_\alpha(l))|$. To achieve it, we construct f approximating sign(x) pointwise on the set $[-R, -r] \cup [r, R]$ and show that the contribution of the coordinates falling outside of this set is negligible. For the interval $(-r, r)$, this will be done using the no-gaps delocalization. Handling the set $(-\infty, -R) \cup (R, \infty)$ is more delicate. Since the polynomial is unbounded on this set, we will control the L_2 norm of f and use the Markov inequality. This argument requires constructing the polynomial f which approximates sign(x) in two metrics simultaneously: uniformly on the set $[-R, -r] \cup [r, R]$ and in $L_2(\mu)$ norm on $\mathbb{R}$. The measure μ here will be the probability measure on $\mathbb{R}$ defined by

$$\mu(B) = \mathbb{P}(g_1 g_2 \in B).$$

Denote the density of the measure μ by ϕ. Instead of controlling two metrics at the same time, we will introduce one Sobolev norm which will be stronger than both metrics. Such norm can be chosen in many different ways. We will chose a particular way which makes the argument shorter.

Let $\eta : \mathbb{R} \setminus \{0\} \to (0, \infty)$ and $\psi : \mathbb{R} \to (0, \infty)$ be even functions such that

- $\eta \in C^1((0, \infty))$, $\psi \in C^1(\mathbb{R})$;
- $\eta(x), \psi(x) = \exp(-x/2)$ for all $x \geq 2$;
- $\eta(x) \geq \phi(x)$ for all $x > 0$, and $\eta \in L_1(\mathbb{R})$.

Consider a weighted Sobolev space H defined as the completion of the space of $C^1(\mathbb{R})$ functions for which the norm

$$\|f\|_H^2 := \int_{\mathbb{R}} f^2(x)\eta(x)\, dx + \int_{\mathbb{R}} (f'(x))^2 \psi(x)\, dx$$

is finite. Note that $H \subset C(\mathbb{R})$. Indeed, for any $M > 0$, $a < b$, $a, b \in [-M, M]$ and any $f \in C^1(\mathbb{R})$,

$$\begin{aligned} |f(b) - f(a)| &= \left| \int_a^b f'(a)\, dx \right| \leq \left(\min_{x \in [-M,M]} \psi(x) \right)^{-1} \cdot \int_a^b |f'(x)| \psi(x)\, dx \\ &\leq \left(\min_{x \in [-M,M]} \psi(x) \right)^{-1} \cdot \left(\int_a^b (f'(x))^2 \psi(x)\, dx \right)^{1/2} \left(\int_a^b \psi(x)\, dx \right)^{1/2} \\ &\leq \left(\min_{x \in [-M,M]} \psi(x) \right)^{-1} \cdot \|f\|_H \cdot \left(\max_{x \in [-M,M]} \psi(x) \right)^{1/2} \cdot (b-a)^{1/2}, \end{aligned} \tag{2.5}$$

and the same inequality holds for the completion.

We will need the following lemma.

Lemma 2.3. *Let $h \in C^1(\mathbb{R})$ be an odd function such that $\|h\|_\infty + \|h'\|_\infty < \infty$. Then for any $\delta > 0$, there exists an odd polynomial Q satisfying $\|Q - h\|_H < \delta$.*

Proof. Denote by $\mathscr{P}$ the set of all polynomials. Let E_{odd} be the set of all odd functions $h \in C^1(\mathbb{R})$ such that $\|h\|_\infty + \|h'\|_\infty < \infty$. It is enough to prove that $E_{odd} \subset \mathrm{Cl}_H(\mathscr{P})$. Indeed, if this is proved, then for any $\delta > 0$ there exists $q \in \mathscr{P}$ such that $\|h - q\|_H < \delta$. Setting $Q(x) = \frac{1}{2}(q(x) - q(-x))$ to make the polynomial odd would finish the proof.

Assume to the contrary that $E_{odd} \not\subset \mathrm{Cl}_H(\mathscr{P})$. Then there exists $h \in \mathrm{Cl}_H(E_{odd}) \setminus \{0\}$ such that $\langle h, x^n \rangle_H = 0$ for any $n \in \{0\} \cup \mathbb{N}$. We will prove that this assumption leads to a contradiction. To this end, set

$$F(z) = \int_{\mathbb{R}} h(x)e^{zx}\eta(x)\,dx + \int_{\mathbb{R}} h'(x)ze^{zx}\psi(x)\,dx.$$

Using the Cauchy-Schwarz inequality, one can check that the function F is analytic in $\{z : |\mathrm{Re}(z)| < 1/2\}$ and

$$F^{(n)}(0) = \int_{\mathbb{R}} h(x)x^n\eta(x)\,dx + \int_{\mathbb{R}} h'(x)nx^{n-1}\psi(x)\,dx = \langle h, x^n\rangle_H = 0.$$

Hence, $F(z) = 0$, and applying this conclusion to $z = it$, $t \in \mathbb{R}$, we see that h satisfies the equality

$$\big(h\eta - (h'\psi)'\big)^\wedge = 0 \text{ and thus } h\eta - (h'\psi)' = 0$$

in the sense of distributions where $(\cdot)^\wedge$ denotes the Fourier Transform. Since the function $h\eta$ is continuous on $(0, \infty)$, h satisfies the differential equation

$$h(x)\eta(x) - (h'(x)\psi(x))' = 0 \tag{2.6}$$

pointwise for all $x \in (0, \infty)$. This in turn means that h'' is well-defined on $(0, \infty)$. Actually, with a little effort, one can prove that this differential equation is satisfied for all $x \in \mathbb{R}$, but we would not need it for our proof.

Since $h \in \mathrm{Cl}_H(E_{odd})$, h is an odd continuous function. For $x \geq 2$, (2.6) reads

$$h(x) + \frac{1}{2}h'(x) - h''(x) = 0,$$

and so $h(x) = C_1 \exp(\lambda_1 x) + C_2 \exp(\lambda_2 x)$ with

$$\lambda_1 = \frac{1 - \sqrt{17}}{4}, \quad \lambda_2 = \frac{1 + \sqrt{17}}{4}$$

for all $x \geq 2$. Since $\lambda_2 > 1/2$ and $h \in H$, $C_2 = 0$. Without loss of generality, assume that $h(2) > 0$, that is, $C_1 > 0$. Then $h'(2) < 0$ and since $h(0) = 0$, $h(2) > 0$, there exists $x \in (0, 2)$ such that $h'(x) > 0$. Denote

$$x_0 = \sup\{x \in (0, 2) : h'(x) > 0\}.$$

Then $h'(x_0) = 0$ and since $h'(x) \leq 0$ for $x > x_0$, we have $h(x_0) > 0$. Hence, (2.6) implies that $h''(x_0) > 0$. Therefore $h'(x) > 0$ for some $x > x_0$, which contradicts the definition of x_0. This contradiction finishes the proof of the lemma. ∎

We are now ready to prove the main result of this section.

Proof of Theorem 1.1. Fix an $\varepsilon > 0$, and let Ω be the event that $|P| \vee |N| \geq (1/2 + \varepsilon)n$. Let (k, l) be a uniformly chosen random pair of distinct elements of $[n]$. Assume that Ω occurs. Then

$$\mathbb{P}(v(k)v(l) > 0 \mid A) \geq \frac{\binom{(1/2+\varepsilon)n}{2} + \binom{(1/2-\varepsilon)n}{2}}{\binom{n}{2}} = \frac{1}{2} + 2\varepsilon^2 + O(n^{-1}) \tag{2.7}$$

and

$$\mathbb{P}(v(k)v(l) < 0 \mid A) \le \frac{\left(\frac{1}{4} - \varepsilon^2\right) n^2}{\binom{n}{2}} = \frac{1}{2} - 2\varepsilon^2 + O(n^{-1}). \tag{2.8}$$

By the no-gap delocalization theorem [21, Theorem 1.5], for $r = c\varepsilon^{22}$,

$$\mathbb{P}\left(|\{j \in [n] : \ |v(j)| \le r^{1/2} n^{-1/2}\}| \ge (\varepsilon^2/8)n\right) \le \exp(-c\varepsilon n).$$

Let Ω_{large} be the event that $|\{j \in [n] : \ |v(j)| \le r^{1/2} n^{-1/2}\}| \le (\varepsilon^2/8)n$, and assume that $\Omega \cap \Omega_{large}$ occurs. Then

$$\mathbb{P}(n|v(k)| \cdot |v(l)| \le r \mid A) \le \mathbb{P}(|v(k)| \wedge |v(l)| < r^{1/2} n^{-1/2} \mid A) \le 1 - \frac{\binom{(1-(\varepsilon^2/8))n}{2}}{\binom{n}{2}} \le \frac{\varepsilon^2}{4}. \tag{2.9}$$

Let $R \ge (c_0\varepsilon)^{-4}$, where the constant $c_0 > 0$ will be chosen later. Since $\|v\|_2 = 1$,

$$|\{j \in [n] : \ |v(j)| \ge R^{1/2} n^{-1/2}\} \le \frac{n}{R} \le (c_0\varepsilon)^4 n,$$

so

$$\mathbb{P}(n|v(k)| \cdot |v(l)| \ge R \mid A) \le \mathbb{P}(|v(k)| \ge R^{1/2} n^{-1/2} \text{ or } |v(l)| \ge R^{1/2} n^{-1/2} \mid A) \le 2(c_0\varepsilon)^4. \tag{2.10}$$

Summarizing (2.7), (2.8), (2.9), and (2.10), and choosing c_0 small enough, we conclude that on the event $\Omega \cap \Omega_{large}$,

$$\mathbb{P}(nv(k)v(l) \in [r, R] \mid A) \ge \frac{1}{2} + \frac{3}{2}\varepsilon^2 + O(n^{-1})$$

and

$$\mathbb{P}(nv(k)v(l) \in [-r, -R] \mid A) \le \frac{1}{2} - \frac{3}{2}\varepsilon^2 + O(n^{-1}).$$

Let $h \in C^\infty(\mathbb{R})$ be an odd function such that $h(x) = \text{sign}(x)$ for any $x \notin (-r, r)$. Lemma 2.3 and inequality (2.5) imply that there exists an odd polynomial Q such that $\|h - Q\|_{L_2(\phi\, dx)} < \varepsilon$ and

$$\max_{x\in[-R,R]} |h(x) - Q(x)| \le \frac{\varepsilon^2}{2}.$$

By Corollary 2.2, there exists $\mathcal{A}_Q$ with $\mathbb{P}(A \in \mathcal{A}_Q) \ge 1 - n^{-\nu}$ such that for any $A \in \mathcal{A}_Q$,

$$\mathbb{E}_{(k,l)} Q(nv(k)v(l)) \le \mathbb{E}Q(g_1 g_2) + n^{-\nu} = n^{-\nu},$$

for sufficiently large n, since the polynomial Q is odd. We will provide a lower estimate of this expectation in terms of $\mathbb{P}(\Omega)$. We have

$$\mathbb{E}_{(k,l)} Q(v(k)v(l)) = \mathbb{E}_{(k,l)} Q(nv(k)v(l)) \cdot \mathbf{1}_{n|v(k)v(l)| \le R} + \mathbb{E}_{(k,l)} Q(nv(k)v(l)) \cdot \mathbf{1}_{n|v(k)v(l)| > R}.$$

Let us estimate these terms separately. On the event $\Omega \cap \Omega_{large}$,

$$\begin{aligned}\mathbb{E}[Q(nv(k)v(l)) \cdot \mathbf{1}_{n|v(k)v(l)|\le R} \mid A] \ge& \left(1 - \frac{\varepsilon^2}{2}\right)\mathbb{P}(nv(k)v(l) \in [r, R] \mid A)\\ &- \left(1 + \frac{\varepsilon^2}{2}\right)\mathbb{P}(nv(k)v(l) \in [-R, -r] \mid A)\\ &- \left(1 + \frac{\varepsilon^2}{2}\right)\mathbb{P}(nv(k)v(l) \in [-r, r] \mid A)\\ &\ge 2\varepsilon^2 + O(n^{-1}).\end{aligned}$$

If $A \in \mathcal{A}_{Q^2}$, then

$$\mathbb{E}[Q^2(nv(k)v(l)) \mid A] \le \mathbb{E}Q^2(g_1 g_2) + n^{-\nu} \le \left(\|h\|_{L_2(\phi\, dx)} + \varepsilon\right)^2 + n^{-\nu} \le C.$$

Hence, by (2.10) and Cauchy-Schwarz inequality, for any $A \in \mathcal{A}_{Q^2}$,

$$\begin{aligned}\mathbb{E}\left[Q(nv(k)v(l)) \cdot \mathbf{1}_{n|v(k)v(l)|>R} \mid A\right] &\le (\mathbb{P}[n|v(k)v(l)| \ge R \mid A])^{1/2} \cdot \left(E[Q^2(nv(k)v(l)) \mid A]\right)^{1/2}\\ &\le C(c_0\varepsilon)^2 \le \frac{\varepsilon^2}{2}\end{aligned}$$

if c_0 is chosen sufficiently small. Thus, if $A \in \mathcal{A}_{Q^2}$ and the event $\Omega \cap \Omega_{large}$ occurs and n is sufficiently large to absorb the $O(n^{-1})$ term, then

$$\mathbb{E}\left[Q(nv(k)v(l)) \mid A\right] \ge \frac{\varepsilon^2}{4},$$

and so, $A \notin \mathcal{A}_Q$. This means that $\Omega \cap \Omega_{large} \cap \{A \in \mathcal{A}_{Q^2} \cap \mathcal{A}_Q\} = \emptyset$, and so

$$\mathbb{P}(\Omega) \le \mathbb{P}(\Omega_{large}^c) + \mathbb{P}(A \in \mathcal{A}_{Q^2}^c) + \mathbb{P}(A \in \mathcal{A}_Q^c) \le n^{-\nu}.$$

The theorem is proved. ∎

3 | EDGE EIGENVECTOR

Let A be the adjacency matrix of a $G(n, p)$ graph with a fixed $p \in (0, 1)$. Denote by u a nonleading edge eigenvector. We are aiming to show that

$$\mathbb{E}\left(\text{sign}\left(u(1)\, u(2)\right)\right) \le n^{-1/3+\varepsilon} \tag{3.1}$$

for a sufficiently small $\varepsilon > 0$. If proved, it leads to

$$\mathbb{E}\left(\sum_i \text{sign} u(i)\right)^2 = n + \sum_{i \ne j} \mathbb{E}\text{sign}\left(u(i)\, u(j)\right) \le n + \binom{n}{2} n^{-1/3+\varepsilon} \le n^{5/3+\varepsilon},$$

because $u(i)u(j)$ has the same distribution as $u(1)u(2)$ for all $i \ne j$ due to the i.i.d. property of the entries the matrix.

Then, by Markov's inequality, we can derive a bound for $\mathbb{P}\left(\left|\sum_i \text{sign}u\,(i)\right| \geq n^{5/6+\varepsilon}\right)$ and thus prove Theorem 1.2. Due to technical difficulties, we would not derive (3.1) directly. Instead, we find an event $\mathscr{A}$ so that

$$\mathbb{E}\,(\text{sign}\,(u\,(1)\,u\,(2))\ |\ \mathscr{A}\,) \leq n^{-1/3+\varepsilon}. \tag{3.2}$$

The event $\mathscr{A}$ will be constructed so that $\mathbb{P}\,(\mathscr{A}^{\,c}) \leq n^{-\delta}$ where $\delta > 0$ may depend on ε. In view of the estimate above, we have

$$\begin{aligned}\mathbb{P}\left(\left|\sum_i \text{sign}u\,(i)\right| \geq n^{5/6+\varepsilon/2}\right) &\leq \mathbb{P}\,(\mathscr{A}^{\,c}) + \mathbb{P}\left(\left|\sum_i \text{sign}u\,(i)\right| \geq n^{5/6+\varepsilon}\ |\ \mathscr{A}\right)\\ &\leq n^{-\delta} + n^{-\varepsilon} \leq n^{-\delta'},\end{aligned}$$

which finishes the proof of Theorem (1.2).

Up to a scaling, A is a Wigner matrix with two deterministic shifts:

$$\sqrt{\frac{1}{p(1-p)n}}A = H + \sqrt{\frac{pn}{1-p}}\vec{1}\,\vec{1}^{\top} - \sqrt{\frac{p}{(1-p)n}}I_n \tag{3.3}$$

where $H_{ij} = \left(h_{ij}\right)$ is a symmetric matrix with 0 diagonal, i.i.d entries h_{ij} with mean 0 and variance $1/n$ above the diagonal:

$$h_{ij} = \begin{cases}\sqrt{\frac{1-p}{p}}\frac{1}{\sqrt{n}} & \text{with probability } p,\\ -\sqrt{\frac{p}{1-p}}\frac{1}{\sqrt{n}} & \text{with probability } 1-p,\end{cases} \tag{3.4}$$

and $\vec{1} \in S^{n-1}$ is the vector such that every component equals $\frac{1}{\sqrt{n}}$. Notice that the last term in (3.3) does not affect the eigenvectors and the order of eigenvalues of $\sqrt{\frac{1}{p(1-p)n}}A$. Therefore, it is sufficient to prove (3.2) for the nonleading edge eigenvectors of

$$\tilde{A} := H + \sqrt{\frac{pn}{1-p}}\vec{1}\,\vec{1}^{\top}. \tag{3.5}$$

Furthermore, we will only prove the theorem for the eigenvectors belonging to the positive edge $\{u_\alpha :\ \alpha \leq \varphi_n^\rho\}$. The proof for eigenvectors $\{u_\alpha :\ n-\alpha \leq \varphi_n^\rho\}$ is essentially the same.

3.1 | Outline of the proof

To lighten the notation, assume that A is an $(n+2)\times(n+2)$ matrix.

It is convenient to break the matrix $\tilde{A}$ into the blocks:

$$\tilde{A} = \begin{bmatrix} D & W^{\top} \\ W & B \end{bmatrix}, \tag{3.6}$$

where B is of size $n \times n$ and D is of size 2×2. Let $G\,(z) := \frac{1}{B-z}$ be the Green function of B. We will write the eigenvalues of $\tilde{A}$ in terms of B, W and D:

Proposition 3.1. *Any $\lambda \in \mathbb{R}$ satisfying*

$$\det\left(W^{\top} G(\lambda) W - D + \lambda I_2\right) = 0 \tag{3.7}$$

is an eigenvalue of $\tilde{A}$. Furthermore, let $q \in \mathbb{R}^2$ be a nontrivial null vector of $W^{\top} G(\lambda) W - D + \lambda I_2$. Then, $\begin{bmatrix} q \\ -G(\lambda) Wq \end{bmatrix}$ is an eigenvector corresponding to λ.

Proof. Assume that

$$\det\left(W^{\top} G(\lambda) W - D + \lambda I_2\right) = 0.$$

Let $q \in \mathbb{R}^2$ be a nontrivial null vector of $W^{\top} G(\lambda) W - D + \lambda I_2$. Then, we have

$$\begin{bmatrix} D-\lambda & W^{\top} \\ W & B-\lambda \end{bmatrix} \begin{bmatrix} q \\ -G(\lambda) Wq \end{bmatrix} = \vec{0}.$$

Therefore, λ is an eigenvalue of $\tilde{A}$ and $u = \begin{pmatrix} q \\ -G(\lambda) Wq \end{pmatrix}$ is the corresponding eigenvector. ■

Up to a scaling, we have $q = \begin{bmatrix} 1 \\ -\frac{w_1^{\top} G(\lambda) w_1 - d_{11} + \lambda}{w_1^{\top} G(\lambda) w_2 - d_{12}} \end{bmatrix}$ where w_1, w_2 are the column vectors of W and $D = \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{bmatrix}$. Therefore,

$$\operatorname{sign}(u(1) u(2)) = \operatorname{sign}\left(-\frac{w_1^{\top} G(\lambda) w_1 - d_{11} + \lambda}{w_1^{\top} G(\lambda) w_2 - d_{12}}\right). \tag{3.8}$$

Our goal is to estimate $\mathbb{E}\operatorname{sign}\left(-\frac{w_1^{\top} G(\lambda) w_1 - d_{11} + \lambda}{w_1^{\top} G(\lambda) w_2 - d_{12}}\right)$. To this end, we would like to take advantage of independence of B, W, and D. However, the fact that λ depends on all these random quantities precludes us from using this independence straightforwardly. This forces us to consider

$$s(E) := \operatorname{sign}\left(-\frac{w_1^{\top} G(E) w_1 - d_{11} + E}{w_1^{\top} G(E) w_2 - d_{12}}\right)$$

for a constant E instead on dealing with λ directly. To analyze the behavior of the function s, it is necessary to know what the matrix B looks like.

Let $\{\mu_\alpha\}_{\alpha=1}^n$ be the eigenvalues of B arranged in a nonincreasing order and let $\{u_\alpha\}_{\alpha=1}^n$ be the corresponding unit eigenvectors. Observe that, up to a scaling factor $\sqrt{\frac{n+2}{n}}$, B is a Wigner matrix with a rank 1 shift:

$$B = M + \sqrt{\frac{p(n+2)}{(1-p)}}\, l l^{\top},$$

where M is the lower right n by n minor of H (from (3.3) and (3.4)), and $l \in \mathbb{R}^n$ is the vector with all its components equal to $\frac{1}{\sqrt{n+2}}$. Here, $\sqrt{\frac{n+2}{n}} M$ is a generalized Wigner matrix having nice spectral properties with high probability.

The proof of Theorem 1.2 breaks into 4 steps:

1. *Typical spectral properties of M.*
Here we are encountering the first obstacle. We want to fix a typical sample M to compute $s(E)$. In particular, we want this sample to have gaps between the eigenvalues close to the edge of order at least $n^{-2/3-\epsilon}$. Such property is called level repulsion in the edge:

Condition 3.2 (Level repulsion on edge)**.** *A random Hermitian matrix H is said to satisfy level repulsion at the edge, if for any $C_{LR} > 0$, and $\epsilon_{LR} > 0$, there exists $\delta_{LR} > 0$, with probability at least $1 - n^{-\delta_{LR}}$*

$$\max_{E \subseteq \left[2-n^{-2/3}\varphi_n^{C_{LR}},\, 2+n^{-2/3}\varphi_n^{C_{LR}}\right]} \mathcal{N}\left(E - n^{-2/3-\epsilon_{LR}},\, E + n^{-2/3-\epsilon_{LR}}\right) < 2. \tag{3.9}$$

We remark that it is known that a GOE (Gaussian orthogonal ensemble) matrix model satisfy this condition, and we will show in Appendix that our matrices H and M satisfy this condition as well.

Notice that such level repulsion is achievable with high probability for a single $n \times n$ principal minor M, but we need it for all minors simultaneously, and the probability estimate too weak to be combined with the union bound. Instead, we define $\mathscr{A}$ as the event that the $(n+2) \times (n+2)$ matrix H has the desired spectral properties. In this case, $\mathscr{A}$ is likely in a sense that $\mathbb{P}(\mathscr{A}^c) < n^{-\delta}$ for some $\delta > 0$. However, we cannot condition on $\mathscr{A}$ directly as in this way we will lose the independence of B, W, and D while estimating $s(E)$. Therefore, in the first step we will define the event $\mathscr{A}$ and show that

$$\mathbb{E}\left(|\mathbf{1}_{H \text{ is typical}} - \mathbf{1}_{M \text{ is typical}}|\right) \text{ is small enough.}$$

This would allow us to use independence while conditioning on the event that M is typical and avoid invoking the union bound while applying this argument to all $n \times n$ principal minors.

2. *From spectral properties of M to spectral properties of B.*
In the second step, we fix a typical M, and consider the spectral properties of its rank one perturbation B. We expect B to behave like M with an exceptional eigenvector almost parallel to l and the corresponding eigenvalue close to $\sqrt{\frac{p(n+2)}{1-p}}$. We will quantify these properties in Definition 3.10 in Section 3.3.

3. *Concentration of $w_i^\top G(E) w_j - d_{ij} + E$.*
The expression above is a key quantity in analyzing $s(E)$. To bound $s(\lambda)$ for λ being an edge eigenvalue of $\tilde{A}_p$, we have to understand the behavior of $s(E)$ for different E. To this end, we derive the concentration of $w_i^\top G(E) w_j$ for $i, j \in \{1, 2\}$. By definition,

$$w_i^\top G(E) w_j = \sum_{\alpha \in [n]} \frac{1}{\mu_\alpha - E} \langle w_i, u_\alpha \rangle \langle w_j, u_\alpha \rangle.$$

If E is much closer to an eigenvalue μ_{α_E} than any other eigenvalues, then, we expect $w_i^\top G(E) w_j$ to be dominated by the term $\frac{1}{\mu_{\alpha_E} - E} \langle w_i, u_{\alpha_E} \rangle \langle w_j, u_{\alpha_E} \rangle$. We will show that after conditioning on a typical B, with high probability in W and D we have

$$\forall i, j \in \{1, 2\} \;\; w_i^\top G(E) w_j \simeq -\delta_{ij} + \frac{\langle w_i, u_{\alpha_E} \rangle \langle w_j, u_{\alpha_E} \rangle}{\mu_{\alpha_E} - E} \tag{3.10}$$

4. *Completion of the proof.*

We combine the results obtained at previous steps to show that

$$\mathbb{E}\left(s\left(\lambda_{\alpha}\right)\mathbf{1}_{\text{H is typical}}\right)=n^{-1/3+C\varepsilon_{LR}}.$$

Once this estimate is proved, the main theorem follows immediately.

3.2 | A typical sample of M

Let M be an $n\times n$ principal submatrix of H. Let $\{\nu_{\alpha}\}_{\alpha=1}^{n}$ be the eigenvalues of M arranged in a non-increasing order and let $\{v_{\alpha}\}_{\alpha=1}^{n}$ be the corresponding unit eigenvectors. Let $G_M(z):=(M-z)^{-1}$ be the Green function of M and

$$m_M(z):=\frac{1}{n}\mathrm{Tr}\,(G(z))=\frac{1}{n}\sum_{\alpha=1}^{n}\frac{1}{\nu_{\alpha}-z}$$

be the Stieltjes transform of M.

A special role in the proof will be played by the level repulsion property, and the strength of the level repulsion has to be carefully chosen for matrices of different sizes. Let $t>0$. We will say that an $m\times m$ symmetric matrix B satisfies the level repulsion property with parameter t if for any two distinct eigenvalues $\nu,\ \nu'$ of A in $\left[2-n^{-2/3}\varphi_n^{3\rho},\ 2+n^{-2/3}\varphi_n^{3\rho}\right]$, we have

$$\left|\nu-\nu'\right|>t.$$

In the argument below, m takes values from $n-4$ to n. Denote the set of such matrices by $\mathscr{L}\mathscr{R}(n,t)$. Lemma 3.3 asserts that

$$\mathbb{P}(M\in\mathscr{L}\mathscr{R}(n,n^{-2/3-\varepsilon_{LR}}))\geq 1-n^{-\delta_{LR}}$$

for some $\delta_{LR}>0$. We start with a lemma showing that the parameter t in the definition of level repulsion can be adjusted without significantly changing this probability.

Lemma 3.3. *Let $C>0$. Let M be an $n\times n$ symmetric random matrix. There exists $\theta\in(1/2,1)$ which depends on the distribution of M such that*

$$\mathbb{P}\left(M\in\mathscr{L}\mathscr{R}\left(n,\theta n^{-2/3-\varepsilon_{LR}}-4\frac{\varphi_n^C}{n}\right)\right)-\mathbb{P}\left(M\in\mathscr{L}\mathscr{R}(n,\theta n^{-2/3-\varepsilon_{LR}})\right)\leq n^{-1/3+2\varepsilon_{LR}}.$$

Proof. For $k\geq 0$, denote

$$P_k:=\mathbb{P}\left(M\in\mathscr{L}\mathscr{R}\left(n,n^{-2/3-\varepsilon_{LR}}-k\frac{\varphi_n^C}{n}\right)\right).$$

Then $P_k\in(0,1)$ form an increasing sequence. Hence, there exists $k\leq 4n^{1/3-2\varepsilon_{LR}}$ such that

$$P_{k+4}-P_k\leq n^{-1/3+2\varepsilon_{LR}}.$$

This implies the lemma if we choose θ so that $\theta n^{-2/3-\varepsilon_{LR}}=n^{-2/3-\varepsilon_{LR}}-k\frac{\varphi_n^C}{n}$ and note that $\theta>1/2$. ∎

We will fix this value of θ for matrices H whose entries are distributed as in (3.4) for the rest of the proof.

Let us collect the properties of the $n \times n$ submatrices of H which we will use throughout the proof.

Definition 3.4. Fix $\varepsilon_{LR} > 0$ and $\rho > 1$, set

$$\eta = n^{-2/3-2\varepsilon_{LR}}. \tag{3.11}$$

Denote by $\mathscr{A}_{(n,k)}$ the set of symmetric $n \times n$ matrices M having the following properties:

- Isotropic local semicircular law:

$$\sup_{|E-2|\le n^{-2/3+3\varepsilon_{LR}}} \sup_{x,y\in\{e_i\}_{i=1}^n \cup\{l\}} |\langle x, G_M(E+i\eta)y\rangle - \langle x, y\rangle m_{sc}(E+i\eta)| < 3n^{-\frac{1}{3}+3\varepsilon_{LR}}, \tag{3.12}$$

- Rigidity of eigenvalues:

$$|\nu_\alpha - \gamma_\alpha| \le \varphi_n^{C_{re}} [\min(\alpha, n-\alpha+1)]^{-1/3} n^{-2/3}, \tag{3.13}$$

  where $C_{re} > 1$ is a universal constant, and γ_α satisfies $\int_{\gamma_\alpha}^2 \frac{2}{\pi}\sqrt{4-x^2}dx = \frac{\alpha}{n}$.
- l_∞-delocalization of eigenvectors:

$$\forall \alpha,\ \|v_\alpha\|_\infty \le \frac{\varphi_n^C}{\sqrt{n}}, \tag{3.14}$$

- Isotropic delocalization of eigenvectors:

$$\max_{\alpha\in[n]} |\langle v_\alpha, l\rangle|^2 < n^{\varepsilon_{LR}-1}, \tag{3.15}$$

- Level repulsion at the edge: $M \in \mathscr{LR}\left(n, \theta n^{-2/3-\varepsilon_{LR}} - k\frac{\varphi_n^C}{n}\right)$, that is, for any two distinct eigenvalues $\nu,\ \nu'$ of M in $\left[2-n^{-2/3}\varphi_n^{3\rho},\ 2+n^{-2/3}\varphi_n^{3\rho}\right]$, we have

$$\left|\nu - \nu'\right| > \theta n^{-2/3-\varepsilon_{LR}} - k\frac{\varphi_n^C}{n}. \tag{3.16}$$

  The value of θ is chosen to satisfy the condition of Lemma 3.3.

A typical Wigner matrix belongs to the set $\mathscr{A}_{(n,0)}$, see [11], [4]. However, we need this fact not for a single matrix M, but for all $n \times n$ principal submatrices of the $(n+2)\times(n+2)$ matrix H. Denote by $H^{(k)}$ the $(n+1)\times(n+1)$ principal submatrix of H with row and column k removed. Similarly, denote by $H^{(i,j)}$ the $n \times n$ principal submatrix of H with rows and columns i,j removed. The properties (3.12)-(3.14) hold with an overwhelming probability, which allows to use a union bound while establishing them. In contrast to it, property (3.16) holds only with probability $1 - n^{-\delta_{LR}}$ for some $\delta_{LR} > 0$, which is too weak to be combined with the union bound. To guarantee that the level repulsion holds with high probability for all principal submatrices, we show that the eigenvalues of these submatrices are located closely to the eigenvalues of the original matrix. To this end, we need the following lemma.

Lemma 3.5. *Let J be an $n \times n$ symmetric matrix satisfying conditions (3.13) and (3.14). Let $k \in [n]$, and let $J^{(k)}$ be the $(n-1)\times(n-1)$ principal submatrix of J with row and column k removed. Let $\mu \in \left[2 - n^{-2/3}\varphi_n^{3\rho}, 2 + n^{-2/3}\varphi_n^{3\rho}\right]$ be an eigenvalue of $J^{(k)}$. If J or $J^{(k)}$ satisfies (3.16), then there exists an eigenvalue λ of J such that*

$$0 \le \lambda - \mu \le \frac{\varphi_n^C}{n}. \tag{3.17}$$

Consequently, if one of the matrices J or $J^{(k)}$ satisfies condition (3.16), then the other one satisfies the same condition with a extra loss of $\frac{\varphi_n^C}{n}$.

Proof. Note that μ is an eigenvalue of the matrix $J - e_k e_k^\top J$ as well since the kth row of this matrix is 0. We will start with showing that there exists an eigenvalue λ of J satisfying (3.17). Let G_J be the Green function of J. By Sylvester's determinant identity, we have

$$\begin{aligned}0 &= \det\left(J - \mu - e_k e_k^\top J\right)\\ &= \det(J-\mu)\det\left(I_n - e_k e_k^\top J G_J(\mu)\right)\\ &= \det(J-\mu)\left(1 - e_k^\top J G_J(\mu) e_k\right).\end{aligned}$$

If $\det(J-\mu) = 0$, then we are done. Otherwise, $1 - e_k^\top J G_J(\lambda) e_k = 0$, which can be rewritten as

$$\sum_\alpha \frac{\lambda_\alpha}{\lambda_\alpha - \mu}\langle e_k, u_\alpha\rangle^2 = 1,$$

where $\lambda_1 \ge \cdots \ge \lambda_m$ are the eigenvalues of J, and $u_1, \dots, u_m$ are the corresponding unit eigenvectors.

For $\lambda_\alpha < 0$, we have $0 < \frac{\lambda_\alpha}{\lambda_\alpha - \mu} < \frac{2}{3}$ where the upper bound is due to $\lambda_\alpha > -3$ by (3.13). Then,

$$\sum_{\alpha,\lambda_\alpha<0} \frac{\lambda_\alpha}{\lambda_\alpha - \mu}\langle e_k, u_\alpha\rangle^2 \le \sum_{\alpha,\lambda_\alpha<0}\frac{2}{3}\langle e_k, u_\alpha\rangle^2 \le \frac{2}{3}.$$

Hence,

$$\sum_{\alpha,\lambda_\alpha>\mu} \frac{\lambda_\alpha}{\lambda_\alpha - \mu}\langle e_k, u_\alpha\rangle^2 \ge \sum_{\alpha,\lambda_\alpha\ge 0} \frac{\lambda_\alpha}{\lambda_\alpha - \mu}\langle e_k, u_\alpha\rangle^2 \ge \frac{1}{3}$$

as $\frac{\lambda_\alpha}{\lambda_\alpha - \mu} \le 0$ for all $\lambda_\alpha \in [0, \mu)$.

Let β be the largest positive integer so that $\lambda_\beta > \mu$. Together with (3.13), we have

$$2 - n^{-2/3}\varphi_n^{3\rho} \le \mu \le \lambda_\beta \le \gamma_1 + n^{-2/3}\varphi_n^{3\rho} \le 2 + n^{-2/3}\varphi_n^{3\rho}$$

and hence

$$|2 - \gamma_\beta| \le |2 - \lambda_\beta| + |\lambda_\beta - \gamma_\beta| \le 2n^{-2/3}\varphi_n^{3\rho}.$$

With the estimate of γ_β in (1.2), we conclude that

$$\beta \le \varphi_n^{C\rho}. \tag{3.18}$$

Assume that $\beta > 1$, and let $\alpha < \beta$. If J satisfies (3.16), then

$$\lambda_\alpha - \mu \geq \lambda_{\beta-1} - \lambda_\beta \geq n^{-2/3-\varepsilon_{LR}}.$$

On the other hand, assume that $J^{(k)}$ satisfies (3.16), and let μ' be the smallest eigenvalue of $J^{(k)}$ which is greater than μ. Due to the Cauchy interlacing theorem, we know that

$$\mu < \lambda_\beta < \mu' < \lambda_\alpha.$$

Then,

$$\lambda_\alpha - \mu \geq \mu' - \mu \geq n^{-2/3-\varepsilon_{LR}}.$$

In both cases, (3.18), (3.14) and (3.13) applied with $\alpha = 1$ imply

$$\sum_{\alpha<\beta} \frac{\lambda_\alpha}{\lambda_\alpha - \mu} \langle e_k, u_\alpha \rangle^2 \leq \beta \frac{\lambda_1}{n^{-2/3-\varepsilon_{LR}}} \max_\alpha \|u_\alpha\|_\infty^2 = O\left(n^{-1/3+C\varepsilon_{LR}}\right).$$

If $\beta = 1$, the inequality above is vacuous. Thus, in both cases,

$$\frac{\lambda_\beta}{\lambda_\beta - \mu} \langle e_k, u_\alpha \rangle^2 \geq \frac{1}{3} + O\left(n^{-1/3+C\varepsilon_{LR}}\right)$$

which in combination with (3.13), (3.14) leads to

$$\frac{\varphi_n^C}{n} \geq \lambda_\beta - \mu > 0$$

establishing (3.17). Since (3.17) holds for all $\mu \in \left[2 - n^{-2/3}\varphi_n^{3\rho},\, 2 + n^{-2/3}\varphi_n^{3\rho}\right]$, the second part of the lemma follows from (3.16) for one of the matrices J or $J^{(k)}$ and interlacing of their eigenvalues. ∎

Equipped with Lemma 3.5, we derive the desired result about the typical behavior of the principal submatrices. We remind the reader that for convenience, we consider graphs with $n + 2$ vertices.

Theorem 3.6. *Let A be the adjacency matrix of a $G(n + 2, p)$ graph, and let*

$$H = \frac{1}{\sqrt{p\,(1-p)\,(n+2)}} A - \sqrt{\frac{p(n+2)}{1-p}} \vec{1}\,\vec{1}^\top - \sqrt{\frac{p}{(1-p)(n+2)}} I_n,$$

where $\vec{1} \in S^{n+1}$ is the vector such that every component equals $\frac{1}{\sqrt{n+2}}$. Let $\mathscr{A}$ be the set of $(n+2)\times(n+2)$ symmetric matrices H such that the matrix itself belongs to $\mathscr{A}_{(n+2,2)}$, all its principal $(n + 1) \times (n + 1)$ submatrices belong to $\mathscr{A}_{(n+1,3)}$, and all its principal $n \times n$ submatrices belong to $\mathscr{A}_{(n,4)}$.

Then

$$\mathbb{P}(H \in \mathscr{A}) \geq 1 - n^{-\delta}$$

for some $\delta = \delta(p,\ \rho,\ \varepsilon_{LR}) > 0$. Moreover, for any $i, j \in [n]$,

$$\mathbb{E}\left|\mathbf{1}_{\mathscr{A}_{(n,0)}}(H^{(i,j)}) - \mathbf{1}_{\mathscr{A}}(H)\right| \leq n^{-1/3+2\varepsilon_{LR}}.$$

Proof. For (3.12) and (3.15), we use the probability estimate in [4, Theorem 2.12, 2.16]. For (3.13) and (3.14), we use the probability estimate in [11, Theorem 2.1, 2.2]. Combining them, we conclude that (3.12)-(3.15) hold for the matrix H itself, as well as for all its $(n+1)\times(n+1)$ and $n\times n$ principal submatrices with probability at least $1-n^{-1}$.

In addition to it, (3.16) holds for H with $k=2$ with probability at least $1-n^{-\delta}$. Then Lemma 3.5, together with the properties (3.12)–(3.15) allow us to extend (3.16) with $k=3$ to all its $(n+1)\times(n+1)$ principal minors. As these minors possess the same properties, (3.16) further extends with $k=4$ to all $n\times n$ principal minors. Let us prove the second inequality. Denote by $\mathscr{B}$ the set of all $(n+2)\times(n+2)$ symmetric matrices satisfying conditions (3.12)-(3.15). Then

$$\mathbb{P}\left(H^{(i,j)}\in\mathscr{A}_{(n,0)}\text{ and }H\notin\mathscr{A}\right)\leq\mathbb{P}\left(H^{(i,j)}\in\mathscr{A}_{(n,0)}\text{ and }H\notin\mathscr{A}\text{ and }H\in\mathscr{B}\right)+\mathbb{P}(H\notin\mathscr{B})\leq n^{-1}$$

since by Lemma 3.5, $\mathscr{A}_{(n,0)}\cap\mathscr{A}^c\cap\mathscr{B}\subset\mathscr{A}_{(n,0)}\cap\mathscr{A}^c_{(n+2,2)}\cap\mathscr{B}=\emptyset$. Also, notice that all the minors $H^{(i,j)}$ have the same distribution, so the value of θ is the same for all i,j. Hence,

$$\mathbb{P}\left(H^{(i,j)}\notin\mathscr{A}_{(n,0)}\text{ and }H\in\mathscr{A}\right)\leq\mathbb{P}\left(H^{(i,j)}\notin\mathscr{A}_{(n,0)}\text{ and }H^{(i,j)}\in\mathscr{A}_{(n,4)}\right)\leq n^{-1/3+2\epsilon_{LR}}$$

by Lemma 3.3. The result follows. ■

3.3 | Introduction of the shift

In this section, we will derive the typical properties of all $n\times n$ principal submatrices of $\tilde{A}$. Recall that we denoted such submatrix by B, and

$$B=M+\sqrt{\frac{p\,(n+2)}{(1-p)}}ll^{\top}\tag{3.19}$$

where M is an $n\times n$ principal submatrix of H, and $l=\left(\frac{1}{\sqrt{n+2}},\ldots,\frac{1}{\sqrt{n+2}}\right)$ is almost a unit vector. We expect B to behave close to M in a sense that its nonleading eigenvalues and eigenvectors possess similar properties. The argument at this stage is deterministic. We fix the matrix $M\in\mathscr{A}_{(n,0)}$ and treat B as its rank one perturbation.

We start with showing that the nonleading edge eigenvalues of B are very close to that of M.

Lemma 3.7. *Let $M\in\mathscr{A}_{(n,0)}$ be an $n\times n$ symmetric matrix with eigenvalues $\nu_1\geq\cdots\geq\nu_n$, and let B be as in (3.19). Let $\mu_1\geq\cdots\geq\mu_n$ be the eigenvalues of B. If β is such that $\left|\nu_\beta-2\right|\leq n^{-2/3}\varphi_n^{2\rho}$, then*

$$\left|\nu_\beta-\mu_{\beta+1}\right|\leq n^{-1+C\epsilon_{LR}}\tag{3.20}$$

for some universal constant $C>0$. Furthermore, $\mu_{\beta+1}$ is an eigenvalue of M if and only if $\langle l,\,v_\beta\rangle=0$, where v_β is a unit eigenvector of M corresponding to ν_β. In the case $\mu_{\beta+1}$ is not an eigenvalue of M, we have

$$\frac{\langle l,\,v_{\beta+1}\rangle^2}{\nu_\beta-\mu_{\beta+1}}\geq 1-o(1)\tag{3.21}$$

We remark that (3.20) is a simple case of [15, Theorem 2.7], which deals with a deterministic finite rank shift.

Proof. Suppose that μ is an eigenvalue of B. By Sylvester's determinant identity we have

$$0 = \det\left(M - \mu I_n + \sqrt{\frac{p(n+2)}{1-p}}ll^{\top}\right)$$
$$= \det(M - \mu I_n)\det\left(I_n + G_M(\mu)\sqrt{\frac{p(n+2)}{1-p}}ll^{\top}\right)$$
$$= \det(M - \mu I_n)\left(1 + l^{\top}G_M(\mu)\sqrt{\frac{p(n+2)}{1-p}}l\right),$$

and $\left(1 + l^{\top}G_M(\mu)\sqrt{\frac{p(n+2)}{1-p}}l\right) = 0$ if

$$\sum_{\alpha\in[n]} \frac{\langle l, v_\alpha\rangle^2}{v_\alpha - \mu} = -\frac{1}{\sqrt{\frac{p(n+2)}{1-p}}}. \tag{3.22}$$

The matrix B is a rank one positive semidefinite perturbation of M, so the eigenvalues of M and B are interlacing:

$$\mu_1 \geq v_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq v_n. \tag{3.23}$$

For the leading eigenvalue, $\mu_1 \geq \frac{1}{2}\sqrt{\frac{p(n+2)}{1-p}}$ due to the fact that $\|M\| = O(1)$ by (3.13).

Let β be such that $|v_\beta - 2| < n^{-2/3}\varphi_n^{2\rho}$. We consider two cases. First, assume that $\langle l, v_\alpha\rangle \neq 0$ for $\alpha \in \{\beta, \beta+1\}$. Then $\mu_{\beta+1} \notin \{v_\beta, v_{\beta+1}\}$, so $\det\left(M - \mu_{\beta+1}I_n\right) \neq 0$, and (3.22) holds.

We claim that

$$\sum_{\alpha\neq\beta} \frac{\langle l, v_\alpha\rangle^2}{v_\alpha - E} \leq -1 + o(1). \tag{3.24}$$

for all $E \in \left(v_{\beta+1}, v_\beta\right)$.

If the claim is proved, then, by (3.22),

$$\frac{\langle l, v_{\beta+1}\rangle^2}{v_\beta - \mu_{\beta+1}} = -\frac{1}{\sqrt{\frac{p(n+2)}{1-p}}} - \sum_{\alpha\neq\beta}\frac{\langle l, v_\alpha\rangle^2}{v_\alpha - \mu_{\beta+1}} \geq 1 - o(1)$$

By (3.15), we have $\langle l, v_{\beta+1}\rangle^2 < n^{\varepsilon_{LR}-1}$, which allows to conclude that

$$0 < v_\beta - \mu_{\beta+1} \leq n^{2\varepsilon_{LR}-1}$$

as required.

Assume now that $\langle l, v_\alpha\rangle = 0$ for some $\alpha \in \{\beta, \beta+1\}$. Considering an infinitesimally small perturbation $M^{(\varepsilon)} = \sqrt{1-\varepsilon^2}M + \varepsilon G$ with a GOE matrix G, we can guarantee that $\langle l, v_\alpha\rangle \neq 0$ a.s. In this case, the perturbed eigenvalue $\mu_{\beta+1}^{(\varepsilon)}$ of $M^{(\varepsilon)}$ satisfies the inequality above. Letting $\varepsilon \to 0$ and using the stability of eigenvalues, we conclude that $\mu_{\beta+1} = v_\beta$ completing the proof of (3.20). This argument also shows that $\mu_{\beta+1}$ is an eigenvalue of M if and only if $\langle l, v_\beta\rangle = 0$.

It remains to verify (3.24). This will be done by comparing the right hand side of (3.24) with

$$\operatorname{Re}\langle l,\, G_M(E+\mathrm{i}\eta)\, l\rangle = \sum_{\alpha\in[n]} \frac{\nu_\alpha - E}{(\nu_\alpha - E)^2+\eta^2}\langle l,\, v_\alpha\rangle^2.$$

Assume first that $\frac{1}{2}\nu_\beta + \frac{1}{2}\nu_{\beta+1} \le E \le \nu_\beta$. In view of (3.16),

$$\nu_{\beta+1} + \frac{1}{2}n^{-2/3-\varepsilon_{LR}} < \frac{\nu_{\beta+1}+\nu_\beta}{2} < E < \nu_\beta < \nu_{\beta-1} - n^{-2/3-\varepsilon_{LR}}.$$

(we omit the last inequality if $\beta = 1$.) Hence, for $\alpha \neq \beta$, we have

$$|E-\nu_\alpha| > \frac{1}{2}n^{-2/3-\varepsilon_{LR}} = \frac{1}{2}\eta n^{\varepsilon_{LR}}$$

(recall that $\eta = n^{-2/3-2\varepsilon_{LR}}$) and so

$$\frac{1}{\nu_\alpha - E} = \left(1+O\left(n^{-2\varepsilon_{LR}}\right)\right)\frac{\nu_\alpha - E}{(\nu_\alpha - E)^2+\eta^2}.$$

Therefore,

$$\begin{aligned}\sum_{\alpha>\beta}\frac{1}{\nu_\alpha - E}\langle l,\, v_\alpha\rangle^2 &= \left(1+O\left(n^{-2\varepsilon_{LR}}\right)\right)\sum_{\alpha>\beta}\frac{\nu_\alpha - E}{(\nu_\alpha - E)^2+\eta^2}\langle l,\, v_\alpha\rangle^2\\ &= \left(1+O\left(n^{-2\varepsilon_{LR}}\right)\right)\left(\operatorname{Re}\langle l,\, G_M(E+\mathrm{i}\eta)\, l\rangle - \sum_{\alpha\le\beta}\frac{\nu_\alpha - E}{(\nu_\alpha - E)^2+\eta^2}\langle l,\, v_\alpha\rangle^2\right).\end{aligned}$$

since all the summands have the same sign. Now we will evaluate the two terms in the brackets. The first one can be approximated using the local semicircular law, and the second one is negligible, because the sum consists of a few terms, and each term is small. Indeed, using (3.13) and (1.2), we have

$$\beta \le \varphi_n^{C\rho}.$$

(The argument is the same as that for (3.18).)

With the trivial bound $|\nu_\alpha - E| < 2n^{-2/3}\varphi_n^{2\rho}$, we get

$$\left|\sum_{\alpha\le\beta}\frac{\nu_\alpha - E}{(\nu_\alpha - E)^2+\eta^2}\langle l,\, v_\alpha\rangle^2\right| \le \beta\frac{n^{-2/3}\varphi_n^{3\rho}}{\eta^2}n^{-1+\varepsilon_{LR}} \le n^{-1/3+6\varepsilon_{LR}}$$

if n is sufficiently large. The isotropic local semicircular law (3.12) yields

$$\operatorname{Re}\langle l,\, G_M(E+\mathrm{i}\eta)\, l\rangle = \langle l,\, l\rangle \operatorname{Re} m_{sc}(E+\mathrm{i}\eta) + O\left(n^{-1/3+3\varepsilon_{LR}}\right).$$

Using the fact that $m_{sc}(z) = \frac{-z+\sqrt{z^2-4}}{2}$ with the branch cut at $[-2, 2]$, for $|z-2| < s < 1$ we have

$$m_{sc}(z) = -1 + O(\sqrt{s}).$$

Thus,

$$\mathrm{Re}\,\langle l,\, G_M(E+\mathrm{i}\eta)\, l\rangle = -1 + O\left(n^{-1/3+3\varepsilon_{LR}}\right)$$

and we conclude that

$$\sum_{\alpha>\beta} \frac{1}{\nu_\alpha - E}\langle l,\, v_\alpha\rangle^2 \le -1 + o(1)$$

for all $E \in \left(\frac{1}{2}\nu_\beta + \frac{1}{2}\nu_{\beta+1},\, \nu_\beta\right)$. Since $E \mapsto \sum_{\alpha>\beta} \frac{1}{\nu_\alpha - E}\langle l,\, v_\alpha\rangle^2$ is increasing for $E > \nu_{\beta+1}$, the inequality above extends to all $E \in \left(\nu_{\beta+1},\, \nu_\beta\right)$. Together with

$$\sum_{\alpha<\beta} \frac{1}{\nu_\alpha - E}\langle l,\, v_\alpha\rangle^2 \le \beta \frac{1}{n^{-2/3-\varepsilon_{LR}}} n^{-1+\varepsilon_{LR}} = o(1)$$

for $E \in \left(\nu_{\beta+1},\, \nu_\beta\right)$, we conclude that all $E \in \left(\nu_{\beta+1},\, \nu_\beta\right)$ satisfy

$$\sum_{\alpha\neq\beta} \frac{1}{\nu_\alpha - E}\langle l,\, v_\alpha\rangle^2 \le -1 + o(1),$$

completing the proof of the lemma. ∎

Our next aim is comparing the Stieltjes transform of B to that of the semicircular law. This will be done via the comparison of the former to the Stieltjes transform of M.

Lemma 3.8. *Let $M \in \mathscr{A}_{(n,0)}$ be an $n \times n$ symmetric matrix, and let B be as in (3.19). Then*

$$\sup_{E:\, |E-2|\le\varphi_n^{2\rho}} \left| m_B(E+\mathrm{i}\eta) - m_{sc}(E+\mathrm{i}\eta)\right| \le n^{-1/3+C\varepsilon_{LR}},$$

where

$$m_B(z) := \frac{1}{n}\sum_{\alpha=1}^{n} \frac{1}{\mu_\alpha - z}$$

is the Stieltjes transform of B and $\eta = n^{-2/3-2\varepsilon_{LR}}$.

Proof. Fix E such that $|E-2| \le \varphi_n^{2\rho}$. We estimate the real part and imaginary of the Stieltjes transform part separately. Let us start with the real part.

$$\mathrm{Re}\, m_B(E+\mathrm{i}\eta) = \frac{1}{n}\sum_\alpha \frac{\mu_\alpha - E}{(\mu_\alpha - E)^2 + \eta^2}.$$

Let β be the smallest integer such that $\nu_\beta < E - \eta$. Recall that we have the interlacing property:

$$E - \eta > \nu_\beta \ge \mu_{\beta+1} \ge \nu_{\beta+1} \ge \mu_{\beta+2} \cdots \ge \mu_n \ge \nu_n.$$

The function $x \to \frac{x}{x^2+\eta^2}$ is decreasing when $|x| > \eta$. Based on this fact, we obtain

$$\sum_{\alpha=\beta}^{n-1} \frac{\nu_\alpha - E}{(\nu_\alpha - E)^2 + \eta^2} \le \sum_{\alpha=\beta+1}^{n} \frac{\mu_\alpha - E}{(\mu_\alpha - E)^2 + \eta^2} \le \sum_{\alpha=\beta+1}^{n} \frac{\nu_\alpha - E}{(\nu_\alpha - E)^2 + \eta^2}.$$

Furthermore, as $\frac{x}{x^2+\eta^2}$ lies in $\left[-\frac{1}{2\eta}, \frac{1}{2\eta}\right]$ for all $x \in \mathbb{R}$, we have

$$\operatorname{Re} m_M(E+\mathrm{i}\eta) - \frac{\beta}{n\eta} \leq \operatorname{Re} m_B(E+\mathrm{i}\eta) \leq \operatorname{Re} m_M(E+\mathrm{i}\eta) + \frac{\beta}{n\eta},$$

and the bound for the real part follows.

For the imaginary part we have

$$\operatorname{Im} m_B(E+\mathrm{i}\eta) = \frac{1}{n}\sum_{\alpha} \frac{\eta}{(\lambda_B - E)^2 + \eta^2}.$$

The function $x \to \frac{\eta}{x^2+\eta^2}$ is increasing if $x < 0$, hence

$$\sum_{\alpha=\beta+1}^{n-1} \frac{\eta}{(\nu_\alpha - E)^2 + \eta^2} \leq \sum_{\alpha=\beta+1}^{n} \frac{\eta}{(\mu_\alpha - E)^2 + \eta^2} \leq \sum_{\alpha=\beta}^{n} \frac{\eta}{(\nu_\alpha - E)^2 + \eta^2}.$$

Since $\frac{\eta}{x^2+\eta^2} \in \left[0, \frac{1}{\eta}\right]$ for all x, we conclude that

$$\operatorname{Im} m_M(E+\mathrm{i}\eta) - \frac{2\beta}{n\eta} \leq \operatorname{Im} m_B(E+\mathrm{i}\eta) \geq \operatorname{Im} m_M(E+\mathrm{i}\eta) + \frac{2\beta}{n\eta}.$$

Similar to how we derive (3.18), using (3.13) and (1.2), we have

$$\beta \leq \varphi_n^{C\rho}.$$

We conclude that

$$|m_M(E+\mathrm{i}\eta) - m_B(E+\mathrm{i}\eta)| \leq \varphi_n^{C\rho} n^{-1/3+2\epsilon_{LR}}.$$

In view of (3.12),

$$|m_M(E+\mathrm{i}\eta) - m_{sc}(E+\mathrm{i}\eta)| = \left|\frac{1}{n}\sum_i \langle e_i, G(E+\mathrm{i}\eta) e_i\rangle - m_{sc}(E+\mathrm{i}\eta)\right| \leq 3n^{-\frac{1}{3}+3\epsilon_{LR}}$$

which in combination with the previous inequality finishes the proof. ■

Next, we will derive the delocalization properties of edge eigenvectors of B.

Lemma 3.9. *Let $M \in \mathscr{A}_{(n,0)}$ be an $n\times n$ symmetric matrix, and let B be as in (3.19). Let $\mu_1 \geq \cdots \geq \mu_n$ be the eigenvalues of B, and let $u_1, \ldots, u_n$ be the corresponding unit eigenvectors. If β is such that $\left|\mu_{\beta+1} - 2\right| \leq n^{-2/3}\varphi_n^{2\rho}$, then*

$$\left|\langle u_{\beta+1}, l\rangle\right| \leq n^{-1+C\epsilon_{LR}}. \tag{3.25}$$

and

$$\left\|u_{\beta+1}\right\|_\infty \leq \frac{n^{1/6+6\epsilon_{LR}}}{\sqrt{n}}. \tag{3.26}$$

Proof. As pointed out in Lemma 3.7, $\mu_{\beta+1}$ is an eigenvalue of M if and only if $\langle l, v_\beta\rangle = 0$. In this case, we have $v_\beta = u_{\beta+1}$ so the statement follows trivially.

Now we assume $\mu_{\beta+1}$ is not an eigenvalue of M, in which case, it satisfies (3.22). Using this equality, one can directly check that

$$u = \sum_{\alpha\in[n]} \frac{\langle l, v_\alpha\rangle}{\nu_\alpha - \mu_{\beta+1}} v_\alpha$$

is an eigenvector of B corresponding to eigenvalue $\mu_{\beta+1}$.

First, we provide a lower bound for $\|u\|_2$. By Lemma 3.7, we have $\frac{\langle l, v_{\beta+1}\rangle^2}{|\nu_\beta - \mu_{\beta+1}|} \geq \frac{1}{2}$ and $|\nu_\beta - \mu_{\beta+1}| \leq n^{-1+C\varepsilon_{LR}}$. This allows to bound the norm of u by one of the coefficients:

$$\|u\|_2^2 \geq \frac{\langle l, v_{\beta+1}\rangle^2}{|\nu_\beta - \mu_{\beta+1}|^2} \geq \frac{1}{4} n^{1-C\varepsilon_{LR}}. \tag{3.27}$$

Recall that by (3.22),

$$\langle u, l\rangle = \sum_{\alpha\in[n]} \frac{\langle l, v_\alpha\rangle^2}{\nu_\alpha - \mu_{\beta+1}} = -\frac{1}{\sqrt{\frac{p(n+2)}{1-p}}}.$$

This yields

$$|\langle u_{\beta+1}, l\rangle| = \frac{|\langle u, l\rangle|}{\|u\|_2} \leq n^{-1+C\varepsilon_{LR}}$$

if n is sufficiently large.

Now we will estimate $\|u\|_\infty = \max_{i\in[n]} \left|\sum_{\alpha\in[n]} \frac{\langle l, v_\alpha\rangle\langle e_i, v_\alpha\rangle}{\nu_\alpha - \mu_{\beta+1}}\right|$. We break the sum isolating the main term:

$$\begin{aligned}
|\langle u, e_i\rangle| &\leq \left|\frac{\langle l, v_\beta\rangle}{\nu_\beta - \mu_{\beta+1}}\right| \|v_\beta\|_\infty + \left|\sum_{\alpha\neq\beta} \frac{\langle l, v_\alpha\rangle\langle e_i, v_\alpha\rangle}{\nu_\alpha - \mu_{\beta+1}}\right| \\
&\leq \left|\frac{\langle l, v_\beta\rangle}{\nu_\beta - \mu_{\beta+1}}\right| \|v_\beta\|_\infty + \sqrt{\sum_{\alpha\neq\beta} \frac{\langle l, v_\alpha\rangle^2}{(\nu_\alpha - \mu_{\beta+1})^2}} \sqrt{\sum_{\alpha\neq\beta} \langle e_i, v_\alpha\rangle^2} \\
&\leq \left|\frac{\langle l, v_\beta\rangle}{\nu_\beta - \mu_{\beta+1}}\right| \|v_\beta\|_\infty + \sqrt{\sum_{\alpha\neq\beta} \frac{\langle l, v_\alpha\rangle^2}{(\nu_\alpha - \mu_{\beta+1})^2}}.
\end{aligned}$$

We will show below that

$$\sqrt{\sum_{\alpha\neq\beta} \frac{\langle l, v_\alpha\rangle^2}{(\nu_\alpha - \mu_{\beta+1})^2}} \leq n^{1/6+2\varepsilon_{LR}}. \tag{3.28}$$

If this inequality holds, (3.27) implies

$$\left\|u_{\beta+1}\right\|_\infty = \frac{\|u\|_\infty}{\|u\|_2} \le \frac{\left|\frac{\langle l, v_\beta\rangle}{v_\beta-\mu_{\beta+1}}\right| \left\|v_\beta\right\|_\infty}{\|u\|_2} + \frac{n^{1/6+2\epsilon_{LR}}}{\|u\|_2}$$
$$\le \frac{\left|\frac{\langle l, v_\beta\rangle}{v_\beta-\mu_{\beta+1}}\right| \left\|v_\beta\right\|_\infty}{\left|\frac{\langle l, v_\beta\rangle}{v_\beta-\mu_{\beta+1}}\right|} + 4n^{1/6-1/2+3\epsilon_{LR}} \le n^{-1/3+4\epsilon_{LR}},$$

where we used $\left\|v_\beta\right\|_\infty \le \frac{\varphi_n^C}{\sqrt{n}}$ from (3.14) in the last inequality. This completes the proof of the lemma modulus (3.28).

In the rest of the proof, we focus on establishing (3.28) by comparing $\sum_{\alpha\neq\beta} \frac{\langle l, v_\alpha\rangle^2}{(v_\alpha-E)^2}$ with

$$\frac{1}{\eta}\mathrm{Im}\,\langle l,\, G_M\,(E+\mathrm{i}\eta)\,l\rangle = \frac{1}{\eta}\mathrm{Im}\sum_{\alpha\in[n]} \frac{\langle l,\, v_\alpha\rangle^2}{v_\alpha - E - \mathrm{i}\eta} = \sum_{\alpha\in[n]} \frac{\langle l,\, v_\alpha\rangle^2}{(v_\alpha - E)^2 + \eta^2}$$

for any $E \in \left(\frac{v_\beta+v_{\beta+1}}{2},\, v_\beta\right)$ which includes $\mu_{\beta+1}$. The approach is basically the same as in approximation of $\sum_{\alpha\neq\beta} \frac{\langle l, v_\alpha\rangle^2}{v_\alpha-E}$ by $\mathrm{Re}\,\langle l,\, G\,(E+\mathrm{i}\eta)\,l\rangle$ in Lemma 3.7. As in this lemma, we use $|v_\alpha - E| > \frac{1}{2}\eta n^{\epsilon_{LR}}$ for $\alpha \neq \beta$ to derive

$$\frac{\eta}{(v_\alpha - E)^2} = \left(1 + O\left(n^{-2\epsilon_{LR}}\right)\right) \frac{\eta}{(v_\alpha - E)^2 + \eta^2}.$$

Thus,

$$\sum_{\alpha\neq\beta} \frac{\langle l,\, v_\alpha\rangle^2}{\left(v_\alpha - \mu_{\beta+1}\right)^2} = \left(1 + O\left(n^{-2\epsilon_{LR}}\right)\right) \left[\frac{1}{\eta}\mathrm{Im}\,\langle l,\, G_M\,(E+\mathrm{i}\eta)\,l\rangle - \frac{\langle l,\, v_\beta\rangle^2}{\left(v_\beta - \mu_{\beta+1}\right)^2 + \eta^2}\right]$$
$$\le \left(1 + O\left(n^{-2\epsilon_{LR}}\right)\right) \frac{1}{\eta}\mathrm{Im}\,\langle l,\, G_M\,(E+\mathrm{i}\eta)\,l\rangle.$$

By (3.12) we have

$$\mathrm{Im}\,\langle l,\, G\,(E+\mathrm{i}\eta)\,l\rangle = \mathrm{Im}\, m_{sc}\,(E+\mathrm{i}\eta) + O\left(n^{-1/3+3\epsilon_{LR}}\right).$$

As $|E-2| < n^{-2/3}\varphi_n^{3\rho}$ and $\eta = n^{-2/3-2\epsilon_{LR}}$, a direct estimate yields $\mathrm{Im}\, m_{sc}\,(E+\mathrm{i}\eta) = O\left(n^{-1/3}\varphi_n^{3\rho}\right)$. Therefore,

$$\sum_{\alpha\neq\beta} \frac{\langle l,\, v_\alpha\rangle^2}{\left(v_\alpha - \mu_{\beta+1}\right)^2} \le n^{1/3+4\epsilon_{LR}}$$

proving (3.28) and finishing the proof of the lemma. ■

We have shown that if $M \in \mathscr{A}_{(n,0)}$, then the matrix B shares the spectral properties of M. Let us summarize these properties.

Definition 3.10. Denote by $\mathscr{T}_{(n,k)}$ the set of $n \times n$ symmetric matrix B with eigenvalues $\mu_1 \geq \cdots \geq \mu_n$ and unit eigenvectors $u_1, \ldots, u_n$ possessing the following properties.

- Eigenvalue properties:
  - Local semicircular law:

$$\sup_{E:\, |E-2| \leq \varphi_n^{2\rho}} |m_B(E + i\eta) - m_{sc}(E + i\eta)| \leq n^{-1/3 + C\varepsilon_{LR}}, \tag{3.29}$$

where $m_B(z) := \frac{1}{n} \sum_{\alpha=1}^{n} \frac{1}{u_\alpha - z}$ is the Stieltjes transform of B and $\eta = n^{-2/3-2\varepsilon_{LR}}$.
  - Rigidity of the eigenvalues:

$$\forall \alpha = 1, \ldots, n-1 \; |\mu_{\alpha+1} - \gamma_\alpha| \leq \varphi_n^{2C_{re}} [\min(\alpha, n - \alpha + 1)]^{-1/3} n^{-2/3}, \tag{3.30}$$

  - Leading eigenvalue:

$$\mu_1 \geq \frac{1}{2} \sqrt{\frac{p}{1-p} n}. \tag{3.31}$$

- Edge eigenvector properties:
  - Isotropic delocalization:
    for β such that $\left|\mu_\beta - 2\right| \leq n^{-2/3} \varphi_n^{2\rho}$, we have

$$\langle u_\beta, l \rangle = O\left(n^{-1+c\varepsilon_{LR}}\right). \tag{3.32}$$

  - ℓ_∞ delocalization:
    for β such that $\left|\mu_\beta - 2\right| \leq n^{-2/3} \varphi_n^{2\rho}$,

$$\left\|u_\beta\right\|_\infty \leq \frac{n^{1/6+4\varepsilon_{LR}}}{\sqrt{n}}. \tag{3.33}$$

- Level repulsion at the edge: $B \in \mathscr{LR}\left(n, \theta n^{-2/3-\varepsilon_{LR}} - k\frac{\varphi_n^C}{n}\right)$, that is,
  for any two distinct eigenvalues ν, ν' of B in $\left[2 - n^{-2/3}\varphi_n^{3\rho}, 2 + n^{-2/3}\varphi_n^{3\rho}\right]$, we have

$$\left|\nu - \nu'\right| > \theta n^{-2/3-\varepsilon_{LR}} - k\frac{\varphi_n^C}{n}. \tag{3.34}$$

The matrices $B \in \mathscr{T}_{(n,1)}$ will be called typical below. In particular, we've shown that $M \in \mathscr{A}_{(n,0)}$ implies $B \in \mathscr{T}_{(n,1)}$.

Theorem 3.6 implies that probability close to 1, the normalized adjacency matrix of a $G(n,p)$ graph is typical along with its principal submatrices. We will formulate it as a corollary.

Corollary 3.11. *Let A be the adjacency matrix of a $G(n+2,p)$ graph, and let*

$$\tilde{A} = \frac{1}{\sqrt{p(1-p)(n+2)}} A - \sqrt{\frac{p}{(1-p)(n+2)}} I_{n+2}$$

and

$$H = \tilde{A} - \sqrt{\frac{p(n+2)}{1-p}} \vec{1}\,\vec{1}^{\top}.$$

Let $\mathscr{T}$ be the set of all matrices $\tilde{A}$ such that $H \in \mathscr{A}$. Then

$$\mathbb{P}(\tilde{A} \in \mathscr{T}) \geq 1 - n^{-\delta}$$

for some $\delta = \delta(p, \rho, \varepsilon_{LR}) > 0$. Moreover, for any $i, j \in [n]$,

$$\mathbb{E}\left|\mathbf{1}_{\mathscr{T}_{(n,1)}}(\tilde{A}^{(i,j)}) - \mathbf{1}_{\mathscr{T}}(\tilde{A})\right| \leq n^{-1/3+2\varepsilon_{LR}}.$$

Proof. Except for (3.30) and (3.31), these conditions have been derived from the corresponding conditions on H above. Condition (3.30) follows from the interlacing of the eigenvalues of $\tilde{A}_p$ and its principal submatrices. Finally, (3.31), follows from (3.13) for $\alpha = 1$ since

$$\mu_1 \geq \langle l, Bl\rangle \geq \sqrt{\frac{p\,(n+2)}{1-p}}\, \|l\|_2^4 - \lambda_1(M)\, \|l\|_2^2 \geq \frac{1}{2}\sqrt{\frac{p}{1-p}} n.$$

Both probability estimates follow now from Theorem 3.6. ∎

3.4 | Concentration of $w_i^{\top} G(E)\, w_j - d_{ij} + E$

In this section, we fix an $n \times n$ matrix $B \in \mathscr{T}_{(n,1)}$. Let E be a constant such that $|E - 2| \leq n^{-2/3}\varphi_n^{2\rho}$. Let $\{\mu_\alpha\}_{\alpha=1}^n$ be eigenvalues of B arranged in the nonincreasing order and let $\{u_\alpha\}_{\alpha=1}^n$ be the corresponding unit eigenvectors. Let $G(E) = \sum_\alpha \frac{1}{\mu_\alpha - E} u_\alpha u_\alpha^{\top}$ be the Green function of B.

Denote by α_E the integer such that

$$\left|\mu_{\alpha_E} - E\right| = \min_\alpha |\mu_\alpha - E|.$$

In this section we will prove the following lemma:

Lemma 3.12. *Let $B \in \mathscr{T}_{(n,1)}$. With probability greater than $1 - \exp(-c(p)\,\varphi_n)$ ($\varphi_n := (\log n)^{\log\log n}$) in w_1 and w_2, we have*

$$\forall i, j \in \{1, 2\}\;\; w_i^{\top} G(E)\, w_j = -\left(1 + O\left(n^{-2\varepsilon_{LR}}\right)\right)\delta_{ij} + \frac{\langle w_i, u_{\alpha_E}\rangle\langle w_j, u_{\alpha_E}\rangle}{\mu_{\alpha_E} - E} + O\left(n^{-1/3+C\varepsilon_{LR}}\right) \quad (3.35)$$

for all $E \in \left[2 - n^{-2/3}\varphi_n^{2\rho}, 2 + n^{-2/3}\varphi_n^{2\rho}\right]$ and $\alpha_E \in [n]$ is the integer so that $\left|\mu_{\alpha_E} - E\right| \leq$ $\min_{\alpha\in[n]} |\mu_\alpha - E|$.

By level repulsion (3.34), we have

$$|\mu_\alpha - E| > \frac{1}{8} n^{-2/3-\varepsilon_{LR}} \quad (3.36)$$

for $\alpha \neq \alpha_E$. Decompose G to separate the main term:

$$G(E) = \sum_{\alpha\in[n]} \frac{1}{\mu_\alpha - E} u_\alpha u_\alpha^\top = \sum_{\alpha\neq\alpha_E} \frac{1}{\mu_\alpha - E} u_\alpha u_\alpha^\top + \frac{1}{\mu_{\alpha_E} - E} u_{\alpha_E} u_{\alpha_E}^\top := L(E) + \frac{1}{\mu_{\alpha_E} - E} u_{\alpha_E} u_{\alpha_E}^\top.$$

For $i = 1, 2$, we express w_i as

$$w_i = \tilde{w}_i + \sqrt{\frac{p}{1-p}}\, l,$$

where $\tilde{w}_i$ has i.i.d. components with the same distribution as in (3.4). In particular, one can treat $\sqrt{n+2}\,\tilde{w}_i$ as an isotropic subgaussian vector whose entries have ψ_2-norms bounded by $K(p)$.

Our goal is to show that $w_i^\top L(E) w_j$ is concentrated about $-\delta_{i,j}$. To achieve that, we represent it as

$$w_i^\top L(E) w_j = \tilde{w}_i^\top L(E) \tilde{w}_j + \sqrt{\frac{p}{1-p}}\, l^\top L(E) \tilde{w}_j + \sqrt{\frac{p}{1-p}}\, \tilde{w}_i^\top L(E)\, l + \frac{p}{1-p}\, l^\top L(E)\, l \tag{3.37}$$

and estimate each summand separately. We start with the bilinear term.

Lemma 3.13. *Fix an $n \times n$ matrix $B \in \mathcal{T}_{(n,1)}$. With probability greater than $1 - \exp(-c(p)\varphi_n)$ ($\varphi_n := (\log n)^{\log\log n}$) in w_1 and w_2, we have*

$$\tilde{w}_i^\top L(E) \tilde{w}_j = -\left(1 + O\left(n^{-2\epsilon_{LR}}\right)\right)\delta_{ij} + O\left(n^{-1/3 + C\epsilon_{LR}}\right) \tag{3.38}$$

for $E \in \left[2 - n^{-2/3}\varphi_n^{2\rho},\, 2 + n^{-2/3}\varphi_n^{2\rho}\right]$. Here, $O\left(n^{-2\epsilon_{LR}}\right)$ and $O\left(n^{-1/3+C\epsilon_{LR}}\right)$ mean some deterministic functions of n with the prescribed asymptotic, and $c(p)$ is a constant that depends only on p.

Proof of Lemma 3.13. Fix $E \in \left[2 - n^{-2/3}\varphi_n^{2\rho},\, 2 + n^{-2/3}\varphi_n^{2\rho}\right]$. We will first estimate the expectation of $\tilde{w}_1^\top L(E) \tilde{w}_1$ and then use the Hanson-Wright inequality to derive the concentration.

First, we will estimate the expectation.

Since $\mathbb{E}_{\tilde{w}_1,\tilde{w}_2} \tilde{w}_1^\top L(E) \tilde{w}_2 = 0$ by independence of $\tilde{w}_1$ and $\tilde{w}_2$, and since $\mathbb{E}_{\tilde{w}_2} \tilde{w}_2^\top L(E) \tilde{w}_2 = \mathbb{E}_{\tilde{w}_1} \tilde{w}_1^\top L(E) \tilde{w}_1$, we have to evaluate only the last quantity. Using the fact that $\tilde{w}_1$ has independent entries with mean 0 and variance $\frac{1}{n+2}$, we obtain

$$\mathbb{E}_{\tilde{w}_1} \tilde{w}_1^\top L(E) \tilde{w}_1 = \mathbb{E}_{\tilde{w}_1} \sum_{\alpha\neq\alpha_E} \frac{1}{\mu_\alpha - E} \langle u_\alpha, \tilde{w}_1 \rangle^2 = \sum_{\alpha\neq\alpha_E} \frac{1}{\mu_\alpha - E} \frac{\sum_{i\in[n]} u_\alpha^2(i)}{n+2} = \frac{1}{n+2} \sum_{\alpha\neq\alpha_E} \frac{1}{\mu_\alpha - E}.$$

Recall that for all $\alpha \in [n-1]$, we have rigidity of eigenvalues (3.30):

$$|\mu_{\alpha+1} - \gamma_\alpha| \le 2\varphi_n^{A_{sls}} \left[\min(\alpha, n - \alpha + 1)\right]^{-1/3} n^{-2/3}.$$

Hence, $|\{\alpha : \mu_\alpha > E, \,\&\, \alpha \neq \alpha_E\}| \le \varphi_n^{C\rho}$, and

$$\sum_{\alpha:\, \mu_\alpha > E \,\&\, \alpha\neq\alpha_E} \frac{1}{\mu_\alpha - E} \le |\{\alpha : \mu_\alpha > E, \,\&\, \alpha \neq \alpha_E\}| \cdot \frac{1}{4} n^{2/3+\epsilon_{LR}} \le n^{2/3+2\epsilon_{LR}} \tag{3.39}$$

We write

$$\frac{1}{\mu_\alpha - E} = \left(1 + \frac{\eta^2}{(\mu_\alpha - E)^2}\right) \frac{\mu_\alpha - E}{(\mu_\alpha - E)^2 + \eta^2},$$

and set $\eta := n^{-2/3-2\epsilon_{LR}}$. With this choice of η, we have $|\mu_\alpha - E| > \frac{1}{4} n^{\epsilon_{LR}} \eta$ from (3.36), and so $\left(1 + \frac{\eta^2}{(\mu_\alpha - E)^2}\right) = 1 + O\left(n^{-2\epsilon_{LR}}\right)$. Therefore,

$$\begin{aligned} \frac{1}{n} \sum_{\alpha:\, \mu_\alpha < E \,\&\, \alpha \neq \alpha_E} \frac{1}{\mu_\alpha - E} &= \left(1 + O\left(n^{-2\epsilon_{LR}}\right)\right) \sum_{\alpha:\, \lambda_\alpha < E \,\&\, \alpha \neq \alpha_E} \frac{1}{n} \frac{\mu_\alpha - E}{(\mu_\alpha - E)^2 + \eta^2} \\ &= \left(1 + O\left(n^{-2\epsilon_{LR}}\right)\right) \left[\operatorname{Re} m_B (E + i\eta) - \frac{1}{n} \sum_{\alpha:\, \mu_\alpha > E \text{ or } \alpha = \alpha_E} \frac{\mu_\alpha - E}{(\mu_\alpha - E)^2 + \eta^2}\right] \\ &= \left(1 + O\left(n^{-2\epsilon_{LR}}\right)\right) \operatorname{Re} m_B (E + i\eta) + O\left(n^{-1/3+3\epsilon_{LR}}\right), \end{aligned} \tag{3.40}$$

where the last equality relies on (3.39). Combining (3.39) and (3.40), we get

$$\frac{1}{n} \sum_{\alpha \neq \alpha_E} \frac{1}{\mu_\alpha - E} = \left(1 + O\left(n^{-2\epsilon_{LR}}\right)\right) \operatorname{Re} m_B (E + i\eta) + O\left(n^{-1/3+3\epsilon_{LR}}\right).$$

We have $\operatorname{Re} m_B (E + i\eta) = \operatorname{Re} m_{sc} (E + \mathrm{i}\eta) + O\left(n^{-1/3+C\epsilon_{LR}}\right) = -1 + O\left(n^{-1/3+C\epsilon_{LR}}\right)$ by (3.29). Thus, if ϵ_{LR} is small enough, then

$$\frac{1}{n} \sum_{\alpha \neq \alpha_E} \frac{1}{\mu_\alpha - E} = -1 + O\left(n^{-C\epsilon_{LR}}\right).$$

We conclude that

$$\mathbb{E}_{\tilde{w}_1} \tilde{w}_1^\top L(E) \tilde{w}_1 = -1 + O\left(n^{-C\epsilon_{LR}}\right).$$

Now we are ready to derive concentration via Hanson-Wright inequality [20] by the second author and Vershynin.

Theorem 3.14 ([20]). *Let $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$ be a random vector with independent components X_i with satisfy $\mathbb{E} X_i = 0$, and $\|X_i\|_{\psi_2} \leq K$. Let A be an $n \times n$ matrix. Then, for every $t \geq 0$,*

$$\mathbb{P}\left(\left|X^\top A X - \mathbb{E} X^\top A X\right| > t\right) \leq 2 \exp\left(-c \min\left(\frac{t^2}{K^4 \|A\|_{HS}^2}, \frac{t}{K^2 \|A\|}\right)\right) \tag{3.41}$$

To this end, we need to estimate the operator norm and Hilbert Schmidt norm of $L(E)$. The operator norm can be estimated directly:

$$\|L(E)\| \leq \max_{\alpha \neq \alpha_E} \frac{1}{\mu_\alpha - E} \leq \frac{1}{4} n^{2/3+\epsilon_{LR}}.$$

For the Hilbert Schmidt norm, a derivation similar to (3.40) yields

$$\|L(E)\|_{HS}^2 = \sum_{\alpha \neq \alpha_E} \frac{1}{(\mu_\alpha - E)^2}$$

$$
\begin{aligned}
&= (1+o(1)) \sum_{\alpha\neq\alpha_E} \frac{1}{(\mu_\alpha - E)^2+\eta^2} = (1+o(1)) \frac{n}{\eta} \sum_{\alpha\neq\alpha_E} \frac{\eta}{n} \frac{1}{(\mu_\alpha - E)^2+\eta^2} \\
&= (1+o(1)) \frac{n}{\eta} \left[\operatorname{Im} m_B(E+\mathrm{i}\eta) - \frac{\eta}{n} \frac{1}{\left(\mu_{\alpha_E} - E\right)^2+\eta^2} \right] \\
&= (1+o(1)) \frac{n}{\eta} \left(\operatorname{Im} m_{sc}(E+\mathrm{i}\eta) + O\left(n^{-1/3+C\varepsilon_{LR}}\right) - \frac{\eta}{n} \frac{1}{\left(\mu_{\alpha_E} - E\right)^2+\eta^2} \right),
\end{aligned} \tag{3.42}
$$

where we used $|m_{sc}(E+\mathrm{i}\eta) - m(E+\mathrm{i}\eta)| \leq O\left(n^{-1/3+C\varepsilon_{LR}}\right)$ from (3.29). A direct computation shows that $\operatorname{Im}\left(m_{sc}(E+\mathrm{i}\eta)\right) = O\left(n^{-1/3+C\varepsilon_{LR}}\right)$ and $\frac{\eta}{n} \frac{1}{\left(\mu_{\alpha_E}-E\right)^2+\eta^2} = O\left(\frac{1}{n\eta}\right) = O\left(n^{-1/3+2\varepsilon_{LR}}\right)$. Hence,

$$
\|L(E)\|_{HS}^2 = \sum_{\alpha\neq\alpha_E} \frac{1}{(\mu_\alpha - E)^2} = (1+o(1)) \frac{n}{\eta} O\left(n^{-1/3+C\varepsilon_{LR}}\right) = O\left(n^{4/3+C\varepsilon_{LR}}\right).
$$

One can easily show that $\left\| \sqrt{n+2}\tilde{w}_1(i) \right\|_{\psi_2} \leq C\sqrt{\frac{1-p}{p}}$. An application Hanson-Wright inequality with $X = \sqrt{n+2}\tilde{w}_1$ and $A = L(E)$ yields

$$
\mathbb{P}\left(\left| \tilde{w}_1^\top L(E)\tilde{w}_1 - \mathbb{E}_{w_1} \tilde{w}_1^\top L(E)\tilde{w}_1 \right| \geq \frac{t}{n+2} \right) \leq 2\exp\left(-c(p) \frac{t}{n^{2/3+C\varepsilon_{LR}}} \right)
$$

for any $t > 1$. Taking $t = n^{2/3+2C\varepsilon_{LR}}$, we get

$$
\tilde{w}_1^\top L(E)\tilde{w}_1 = \underbrace{-1 + O\left(n^{-2\varepsilon_{LR}}\right)}_{\mathbb{E}\tilde{w}_1^\top L(E)\tilde{w}_1} + O\left(n^{-1/3+2C\varepsilon_{LR}}\right)
$$

with probability at least $1 - \exp(-c(p)\varphi_n)$. (Recall that $\varphi_n = \log n^{\log\log n}$.)

Notice that, the same estimate works for $\tilde{w}_2$ and $\tilde{w}_1 + \tilde{w}_2$ as well: with probability at least $1 - \exp(-c(p)\varphi_n)$,

$$
(\tilde{w}_1 + \tilde{w}_2)^\top L(E)(\tilde{w}_1 + \tilde{w}_2) = \underbrace{\mathbb{E}\tilde{w}_1^\top L(E)\tilde{w}_1 + \mathbb{E}\tilde{w}_2^\top L(E)\tilde{w}_2}_{\mathbb{E}(\tilde{w}_1+\tilde{w}_2)^\top L(E)(\tilde{w}_1+\tilde{w}_2)} + O\left(n^{-1/3+2C\varepsilon_{LR}}\right).
$$

Therefore, by the linearity, adjusting the constant C appropriately we have

$$
\tilde{w}_1^\top L(E)\tilde{w}_2 = O\left(n^{-1/3+C\varepsilon_{LR}}\right),
$$

with probability at least $1 - \exp(-c(p)\varphi_n)$, thus obtaining (3.38) for a fixed E.

To extend this to all $E \in \left[2 - n^{-2/3}\varphi_n^{2\rho}, 2 + n^{-2/3}\varphi_n^{2\rho}\right]$, we will use a net argument. Let $\mathcal{N}$ be a κ-net in $\left[2 - n^{-2/3}\varphi_n^{2\rho}, 2 + n^{-2/3}\varphi_n^{2\rho}\right]$ with $\kappa = n^{-100}$ and assume that (3.38) holds for all $E \in \mathcal{N}$. Since $|\mathcal{N}|$ is polynomial in n, this event has probability bounded by $\exp(-c(p)\varphi_n)$.

Recall that the coordinates of $\sqrt{n+2}\tilde{w}_i$ are independent, centered, subgaussian random variables with $\left\| \sqrt{n+2}\tilde{w}_1(k) \right\|_{\psi_2} \leq C\sqrt{\frac{1-p}{p}}$. By Hoeffding's inequality,

$$\sqrt{n+2}\langle \tilde{w}_i, u_\alpha \rangle = \sum_{k=1}^{n} \sqrt{n+2}\tilde{w}_i(k)\, u_\alpha(k)$$

is also subgaussian since $\|u_\alpha\|_2 = 1$. Similarly, $(n+2)\|\tilde{w}_i\|_2^2$, being a sum of subexponential random variables, satisfies Bernstein's inequality. Together with a union bound, these two facts imply

$$\mathbb{P}\left(\exists \alpha \in [n], i \in \{1, 2\}\ |\langle \tilde{w}_i, u_\alpha \rangle| \geq \frac{\varphi_n}{\sqrt{n+2}}\ \&\ \|w_i\|_2 \leq \varphi_n \right) \leq \exp(-c(p)n).$$

Assume that these two events occur in addition to the assumption that (3.38) holds for all $E \in \mathcal{N}$ which we already made. Let $E \in \left[2 - n^{-2/3}\varphi_n^{2\rho}, 2 + n^{-2/3}\varphi_n^{2\rho}\right]$, and choose $E' \in \mathcal{N}$ such that $|E - E'| < \kappa$. Suppose that $\alpha_E \neq \alpha_{E'}$, then

$$\begin{aligned}
&\left|\tilde{w}_i^\top L(E)\tilde{w}_j - \tilde{w}_i^\top L(E')\tilde{w}_j\right| \\
&\leq \|\tilde{w}_i\|_2 \|\tilde{w}_j\|_2 \sum_{\alpha \neq \alpha_E, \alpha_{E'}} \left| \frac{1}{\mu_\alpha - E} - \frac{1}{\mu_\alpha - E'} \right| + \left| \frac{\langle \tilde{w}_i, u_{\alpha_{E'}} \rangle \langle \tilde{w}_j, u_{\alpha_{E'}} \rangle}{\mu_{\alpha_{E'}} - E} \right| + \left| \frac{\langle \tilde{w}_i, u_{\alpha_E} \rangle \langle \tilde{w}_j, u_{\alpha_E} \rangle}{\mu_{\alpha_E} - E'} \right| \\
&\leq \|\tilde{w}_i\|_2 \|\tilde{w}_j\|_2 \sum_{\alpha \neq \alpha_E, \alpha_{E'}} \frac{4\kappa}{\eta^2} + \left| \frac{\langle \tilde{w}_i, u_{\alpha_{E'}} \rangle \langle \tilde{w}_j, u_{\alpha_{E'}} \rangle}{\mu_{\alpha_{E'}} - E} \right| + \left| \frac{\langle \tilde{w}_i, u_{\alpha_E} \rangle \langle \tilde{w}_j, u_{\alpha_E} \rangle}{\mu_{\alpha_E} - E'} \right| \\
&\leq \|\tilde{w}_i\|_2 \|\tilde{w}_j\|_2 \frac{4n}{\eta^2}\kappa + \left| \frac{\langle \tilde{w}_i, u_{\alpha_{E'}} \rangle \langle \tilde{w}_j, u_{\alpha_{E'}} \rangle}{\mu_{\alpha_{E'}} - E} \right| + \left| \frac{\langle \tilde{w}_i, u_{\alpha_E} \rangle \langle \tilde{w}_j, u_{\alpha_E} \rangle}{\mu_{\alpha_E} - E'} \right|
\end{aligned}$$

Since $\alpha_E \neq \alpha_{E'}$, we have $\min\left\{ \left|\mu_{\alpha_{E'}} - E\right|, \left|\mu_{\alpha_E} - E'\right| \right\} \geq \frac{1}{8} n^{-2/3-\varepsilon_{LR}}$. Together with $|\langle \tilde{w}_i, u_\alpha \rangle| \leq \frac{\varphi_n}{\sqrt{n+2}}$, this yields

$$\left| \frac{\langle \tilde{w}_i, u_{\alpha_{E'}} \rangle \langle \tilde{w}_j, u_{\alpha_{E'}} \rangle}{\mu_{\alpha_{E'}} - E} \right| + \left| \frac{\langle \tilde{w}_i, u_{\alpha_E} \rangle \langle \tilde{w}_j, u_{\alpha_E} \rangle}{\mu_{\alpha_E} - E'} \right| = O\left(n^{-1/3+2\varepsilon_{LR}}\right).$$

Thus,

$$\left|\tilde{w}_i^\top L(E)\tilde{w}_j - \tilde{w}_i^\top L(E')\tilde{w}_j\right| \leq \|\tilde{w}_i\|_2 \|\tilde{w}_j\|_2 \frac{4n}{\eta^2}\kappa + O\left(n^{-1/3+2\varepsilon_{LR}}\right)$$

As $\kappa = n^{-100}$, the difference is bounded by $O\left(n^{-1/3+2\varepsilon_{LR}}\right)$. The same bound holds for the case $\alpha_E = \alpha_{E'}$, and the proof is simpler, since the last two terms do not appear. Therefore, (3.38) holds for E as well if constant C is appropriately adjusted. ∎

Next, we bound the linear and constant terms in (3.37).

Lemma 3.15. *Fix an $n \times n$ matrix $B \in \mathcal{T}_{(n,1)}$. With probability greater than $1 - \exp\left(c(p)\varphi_n^C n\right)$, for any E such that $|E - 2| \leq n^{-2/3}\varphi_n^{2\rho}$,*

$$l^\top L(E)\, l = O\left(n^{-1/3+C\varepsilon_{LR}}\right), \text{ and } \tilde{w}_1^\top L(E)\, l = O\left(n^{-1/3+C\varepsilon_{LR}}\right). \tag{3.43}$$

Here, $c(p)$ is a constant that depends only on p.

Proof. Application of Hoeffding's inequality to $\langle \tilde{w}_i, u_\alpha \rangle$ yields

$$\mathbb{P}\left(\langle \tilde{w}_i, u_\alpha \rangle^2 \geq \frac{\varphi_n}{n+2}\right) \leq \exp\left(-c(p)\,\varphi_n\right),$$

and so

$$\max_{\alpha, i} \langle \tilde{w}_i, u_\alpha \rangle^2 \leq \frac{\varphi_n}{n}$$

with probability greater than $1 - \exp(-c(p)\,\varphi_n)$. In view of this inequality and the fact that $\left(\sum_{\alpha \neq 1} \langle l, u_\alpha \rangle^2\right)^{\frac{1}{2}} = \left|P_{u_1^\perp} l\right| = O\left(n^{-1/2+c\varepsilon_{LR}}\right)$,

$$\left|\sum_{\alpha \neq 1, \alpha_E} \frac{\langle \tilde{w}_i, u_\alpha \rangle \langle l, u_\alpha \rangle}{\mu_\alpha - E}\right| \leq \left(\sum_{\alpha \neq 1, \alpha_E} \langle l, u_\alpha \rangle^2\right)^{\frac{1}{2}} \left(\sum_{\alpha \neq 1, \alpha_E} \frac{\langle \tilde{w}_i, u_\alpha \rangle^2}{(\mu_\alpha - E)^2}\right)^{\frac{1}{2}}$$
$$= O\left(n^{-1+c'\varepsilon_{LR}}\right) \sqrt{\sum_{\alpha \neq 1, \alpha_E} \frac{1}{(\mu_\alpha - E)^2}}.$$

Again, one can approximate $\sum_{\alpha \neq 1, \alpha_E} \frac{1}{(\mu_\alpha - E)^2}$ by $\frac{n}{\eta} \operatorname{Im} m_{sc}(E + \mathrm{i}\eta)$ as before and obtain

$$\sum_{\alpha \neq 1, \alpha_E} \frac{1}{(\mu_\alpha - E)^2} = O\left(n^{4/3+C\varepsilon_{LR}}\right).$$

This shows that

$$\left|\sum_{\alpha \neq 1, \alpha_E} \frac{\langle \tilde{w}_i, u_\alpha \rangle \langle l, u_\alpha \rangle}{\mu_\alpha - E}\right| = O\left(n^{-1/3+C\varepsilon_{LR}}\right).$$

with probability greater than $1 - \exp(-c(p)\,\varphi_n)$.

Furthermore, recall that by (3.31), $\mu_1 \geq \frac{1}{2}\sqrt{\frac{p(n+2)}{1-p}}$. Thus $\left|\frac{\langle \tilde{w}_i, u_1 \rangle \langle l, u_1 \rangle}{\mu_1 - E}\right| = o\left(\frac{1}{\sqrt{pn}}\right)$, and

$$|l^\top L(E)\, l| = \left|\sum_{\alpha \neq \alpha_E} \frac{\langle l, u_\alpha \rangle^2}{\mu_\alpha - E}\right| \leq \left(\frac{1}{4} n^{2/3+\varepsilon_{LR}} \sum_{\alpha \neq 1, \alpha_E} \langle l, \tilde{u}_\alpha \rangle^2\right) + \frac{1}{\mu_1 - E} \leq n^{-1/3+C\varepsilon_{LR}}.$$

Again, this result can extend easily for all $E \in \left[2 - n^{-2/3}\varphi_n^{2\rho}, 2 + n^{-2/3}\varphi_n^{2\rho}\right]$ by a net argument. We omit the proof here since it is the same as the net argument in Lemma 3.13. ■

Combining Lemmas 3.13 and 3.15, we obtain Lemma 3.12.

3.5 | Estimate of $s(\lambda)$

Recall that in Corollary 3.11, we denoted by $\mathcal{T}$ be the set of $(n+2) \times (n+2)$ symmetric matrices all whose $n \times n$ principal submatrices are typical in a sense that they satisfy the conditions in $\mathcal{T}_{(n,5)}$. Suppose that λ_α is an eigenvalue of $\tilde{A}$ and $v_\alpha \in \mathbb{R}^{n+2}$ is the corresponding unit corresponding eigenvector. As in (3.8),

$$\operatorname{sign}\left(v_\alpha(1)\,v_\alpha(2)\right)=s\left(\lambda_\alpha\right)=\operatorname{sign}\left(-\frac{w_1^\top G\left(\lambda_\alpha\right)w_1-d_{11}+\lambda_\alpha}{w_1^\top G\left(\lambda_\alpha\right)w_2-d_{12}}\right).$$

In this section, we will prove the following:

Lemma 3.16. *Let A be the adjacency matrix of a G(n,p) graph, and let $\lambda_1\geq\cdots\geq\lambda_n$ be the eigenvalues of the matrix*

$$\tilde{A}=\frac{1}{\sqrt{p\left(1-p\right)\left(n+2\right)}}A-\sqrt{\frac{p}{(1-p)(n+2)}}I_{n+2}$$

Fix $2\leq\alpha\leq\varphi_n^\rho$. Then

$$\mathbb{E}\left(s\left(\lambda_\alpha\right)\cdot\mathbf{1}_{\mathcal{T}}\left(A\right)\right)=O\left(n^{-1/3+C\varepsilon_{LR}}\right).$$

As $\mathcal{T}$ pertains to all $n\times n$ principal submatrices, the same bound holds for $\mathbb{E}\left(\operatorname{sign}\left(v_\alpha\left(i\right)v_\alpha\left(j\right)\right)\cdot\mathbf{1}_{\mathcal{T}}\left(\tilde{A}\right)\right)$ for any $i\neq j$.

Once this lemma is proved, Theorem 1.2 follows easily:

Proof. For $2\leq\alpha\leq\varphi_n^\rho$, we have $\mathbb{E}(\operatorname{sign}(u_\alpha(i)u_\alpha(j)\mid\mathcal{T})=O\left(n^{-1/3+C\varepsilon_{LR}}\right)$ for all $i\neq j$. Hence,

$$\mathbb{E}\left(\left(\sum_{i=1}^{n+2}\operatorname{sign}(u_\alpha(i))\right)^2\,|\mathcal{T}\right)=O(n^{5/3+C\varepsilon_{LR}}).$$

Applying Markov's inequality we get

$$\mathbb{P}\left(\left|\sum_{i=1}^{n+2}\operatorname{sign}(u_\alpha(i))\right|>n^{5/3+C'\varepsilon}\right)<n^{-\delta_{LR}}+n^{-\varepsilon_{LR}}.$$

∎

The proof of this lemma will be based on the concentration we get from Lemma 3.12. Let B be the $n\times n$ principal submatrix containing the last n rows and columns. If $\tilde{A}\in\mathcal{T}$, then $B\in\mathcal{T}_{(n,1)}$.

Consider $\alpha=2$ first. Let $\mu_1'\geq\mu_{n+1}'$ be the eigenvalues of the $(n+1)\times(n+1)$ matrix containing the last $(n+1)$ rows and columns of $\tilde{A}$. Per (3.30) for $\tilde{A}$, $\lambda_2\in\left[2-n^{-2/3}\varphi_n^{2\rho},\,2+n^{-2/3}\varphi_n^{2\rho}\right]$, so interlacing and Lemma 3.5 imply that

$$\mu_2'\leq\lambda_2\leq\mu_2'+\frac{\varphi_n^C}{n}<\mu_1'$$

where μ_1' satisfies (3.31). Repeating this argument for B, in view of (3.34) and (3.31), we conclude that $\lambda_2\in[\mu_2,\mu_1]$. For $2<\alpha\leq\varphi_n^\rho$, (3.34) similarly yields $\lambda_\alpha\in[\mu_\alpha,\mu_{\alpha-1}]$.

Condition on the submatrix B. Since $\alpha\leq\varphi_n^\rho$, by the estimate that $\int_{2-t}^{2}\frac{1}{2\pi}\sqrt{4-x^2}\,\mathrm{d}x\geq\frac{1}{2\pi}t^{3/2}$, we have $2-\gamma_\alpha\leq n^{-2/3}\varphi_n^\rho$ and thus $2-\mu_\alpha\leq n^{-2/3}\varphi_n^{2\rho}$ due to rigidity of eigenvalues (3.30).

Let $\mathcal{A}_{wGw}$ be the set of $n\times2$ matrices W such that (3.35) in Lemma 3.12 holds. Specifically, $\mathcal{A}_{wGw}$ is defined by the condition

$$\forall i,j\in\{1,2\}\;\; w_i^\top G\left(E\right)w_j=-\left(1+O\left(n^{-2\varepsilon_{LR}}\right)\right)\delta_{ij}+\frac{\langle w_i,\,u_{\alpha_E}\rangle\langle w_j,\,u_{\alpha_E}\rangle}{\mu_{\alpha_E}-E}+O\left(n^{-1/3+C_1\varepsilon_{LR}}\right)\quad(3.44)$$

for all $E \in \left[2 - n^{-2/3}\varphi_n^{2\rho},\ 2 + n^{-2/3}\varphi_n^{2\rho}\right]$ and a universal constant $C_1 > 0$. Here, $\alpha_E \in [n]$ is the integer so that $\left|\mu_{\alpha_E} - E\right| \le \min_{\alpha\in[n]} |\mu_\alpha - E|$.

Before we move on to the proof directly, let us introduce another set. Let $\mathscr{A}_W$ be a set of W such that for $i \in \{1, 2\}$

$$n^{-1/3+\kappa\varepsilon_{LR}} \le \sqrt{n}\,|\langle \tilde{w}_i, u_\alpha\rangle| \le \log^2 n \tag{3.45}$$

where $\kappa \ge \max\{2C_1, 8\}$ and

$$\tilde{w}_i = w_i - \sqrt{\frac{p}{1-p}}\,l.$$

Lemma 3.17. *Let the W be the $n\times 2$ block W of $\tilde{A}$ defined in (3.6). With the notation above, we have*

$$\mathbb{P}(W \in \mathscr{A}_W) \ge 1 - n^{-1/3+2\kappa\varepsilon_{LR}},$$

and

$$\mathbb{P}(\langle \tilde{w}_i, u_\alpha\rangle > 0) = \frac{1}{2} + O\left(n^{-1/3+5\varepsilon_{LR}}\right) \quad \textit{for } i = 1, 2. \tag{3.46}$$

Proof. The upper bound in (3.45) holds with the desired probability due to Hoeffding's inequality. We will estimate the probability that the lower bound holds and prove (3.46) at the same time. Let $X_k := \sqrt{n+2}\,\tilde{w}_1(k)\,u_\alpha(k)$. Since $\tilde{w}_1(k)$ has mean 0 and variance $\frac{1}{n+2}$, we set

$$S_n = \frac{\sum_{k\in[n]} X_k}{\sum_{k\in[n]} \mathbb{E}X_k^2} = \sqrt{n+2}\langle \tilde{w}_1, u_\alpha\rangle$$

Observe that $\mathbb{E}X_k^2 = u_\alpha(k)^2$ and $\mathbb{E}X_k^3 \le c(p)\,|u_\alpha(k)|^3$ where $c(p) > 0$ is a constant depends on p. Let F_n and Φ be the cumulative distributions of S_n and the standard normal random variable respectively. By the Berry-Esseen Theorem (see, e.g., [22, Theorem 2.2.17]) we have

$$\sup_{x\in\mathbb{R}} |F_n(x) - \Phi(x)| \le C\left(\sum_{i=1}^n \mathbb{E}X_i^2\right)^{-1/2} \cdot \max_i \frac{\mathbb{E}|X_i|^3}{\mathbb{E}X_i^2} \le c(p)\,\frac{\|u_\alpha\|_\infty}{\|u_\alpha\|_2}.$$

Recall that from (3.26) in the definition of $\mathscr{T}_{(n,1)}$, we have the l_∞-norm bound: $\|u_\alpha\|_\infty \le n^{-1/3+4\varepsilon_{LR}}$. Together with $\|u_\alpha\|_2 = 1$ it yields

$$\sup_{x\in\mathbb{R}} |F_n(x) - \Phi(x)| \le n^{-1/3+5\varepsilon_{LR}}$$

if n is large enough. Thus,

$$\mathbb{P}\left(\sqrt{n}\,|\langle \tilde{w}_1, u_\alpha\rangle| \le n^{-1/3+\kappa\varepsilon_{LR}}\right) \le \mathbb{P}\left(\sqrt{n}\,|g| \le n^{-1/3+\kappa\varepsilon_{LR}}\right) + 2n^{-1/3+5\varepsilon_{LR}} \le n^{-1/3+1.5\kappa\varepsilon_{LR}},$$

where $g \sim N(0, 1)$ is a normal random variable. Furthermore, we also obtain (3.46) by comparing Φ and F_n. ■

Proof of Lemma 3.16. By (3.7), if $\lambda \in \mathbb{R}$ is an eigenvalue of $\tilde{A}$, then $\det\left(W^\top G(\lambda) W - D + \lambda I_2\right) = 0$. Let

$$f(E) := \frac{\left(w_1^\top G(E) w_1 - d_{11} + E\right)\left(w_2^\top G(E) w_2 - d_{22} + E\right)}{\left(w_1^\top G(E) w_2 - d_{12}\right)^2}.$$

Thus, λ is an eigenvalue whenever $f(\lambda) = 1$. We will use the function $f(E)$ to determine the location of the eigenvalues.

Let $\mathscr{A}_D$ be the set of all 2×2 symmetric matrices D such that $\max_{i,j \in \{1,2\}} \left|d_{ij}\right| = O\left(c(p) n^{-1/2}\right)$. Recall the definitions of A_{wGw} and A_W from (3.44) and (3.45), respectively. Assume that $W \in A_{wGw} \cap A_W$ and $D \in \mathscr{A}_D$. We will see below that this is a likely event.

Under these conditions, the argument becomes deterministic. By (3.26) from the definition of $\mathscr{T}_{(n,1)}$, we have $|\langle u_\alpha, l\rangle| \le n^{-1+2\varepsilon_{LR}}$. Hence,

$$\langle w_i, u_\alpha\rangle = (1 + o(1))\langle \tilde{w}_i, u_\alpha\rangle$$

and in particular $\langle w_i, u_\alpha\rangle$ and $\langle \tilde{w}_i, u_\alpha\rangle$ have the same sign.

Observe that $E \mapsto w_1^\top G(E) w_1 - d_{11} + E$ is a strictly increasing function on $(\mu_\alpha, \mu_{\alpha-1})$. It tends to $-\infty$ as $E \to \mu_\alpha^+$ and $+\infty$ as $E \to \mu_{\alpha-1}^-$. Thus, it crosses 0 only once. Let E_0 be maximum of the roots of $w_1^\top G(E) w_1 - d_{11} + E$ and $w_2^\top G(E) w_2 - d_{22} + E$ on $(\mu_\alpha, \mu_{\alpha-1})$. Then by (3.44) and $\left|d_{ij}\right| = O\left(c(p) n^{-1/2}\right)$,

$$-\left(1 + O\left(n^{-2\varepsilon_{LR}}\right)\right) + \frac{\langle w_i, u_{\alpha_{E_0}}\rangle^2}{\mu_{\alpha_{E_0}} - E_0} + E_0 = 0$$

for some $i \in \{1,2\}$. As $\mu_{\alpha-1} > E_0 > \mu_\alpha \ge 2 - n^{-2/3}\varphi_n^{2\rho}$, this implies that $E_0 > \mu_{\alpha_{E_0}}$, and thus $\alpha_{E_0} = \alpha$. Moreover, $E_0 - 1 = 1 + O\left(n^{-2\varepsilon_{LR}}\right)$, and so

$$E_0 = \left(1 + O\left(n^{-2\varepsilon_{LR}}\right)\right)\max\left\{\langle w_1, u_\alpha\rangle^2, \langle w_2, u_\alpha\rangle^2\right\} + \mu_\alpha.$$

For $E > E_0$, both $w_1^\top G(E) w_1 - d_{11} + E$ and $w_2^\top G(E) w_2 - d_{22} + E$ are positive. Setting

$$E_1 = 2\max\left\{\langle w_1, u_\alpha\rangle^2, \langle w_2, u_\alpha\rangle^2\right\} + \mu_\alpha,$$

for $E \in [\mu_\alpha, E_1]$, we also have $\alpha_E = \alpha$, and

$$\left|\frac{\langle w_1, u_\alpha\rangle\langle w_2, u_\alpha\rangle}{\mu_\alpha - E}\right| \ge \left|\frac{\langle w_1, u_\alpha\rangle\langle w_2, u_\alpha\rangle}{\mu_\alpha - E_1}\right| = \frac{1}{2}\min\left\{\left|\frac{\langle w_1, u_{\alpha_E}\rangle}{\langle w_2, u_\alpha\rangle}\right|, \left|\frac{\langle w_2, u_\alpha\rangle}{\langle w_1, u_\alpha\rangle}\right|\right\} > \log^{-2} n \cdot n^{-1/3+\kappa\varepsilon_{LR}}. \tag{3.47}$$

by (3.45). Hence, $w_1^\top G(E) w_2 - d_{12}$ has no zeros in the interval $[\lambda_\alpha, E_1]$. Furthermore, because

$$\min\left\{\left|\frac{\langle w_1, u_\alpha\rangle}{\langle w_2, u_\alpha\rangle}\right|, \left|\frac{\langle w_2, u_\alpha\rangle}{\langle w_1, u_\alpha\rangle}\right|\right\} \le 1,$$

using (3.44) and $\left|d_{ij}\right| = O\left(c(p) n^{-1/2}\right)$ again, we get

$$\left(w_1^\top G(E_1) w_2 - d_{12}\right)^2 = \left(\frac{\langle w_1, u_\alpha\rangle\langle w_2, u_\alpha\rangle}{\mu_\alpha - E_1} + \mathrm{O}\left(n^{-1/3+C_1\varepsilon}\right)\right)^2$$

$$= \left(\frac{1}{2}\min\left\{\left|\frac{\langle w_1, u_\alpha\rangle}{\langle w_2, u_\alpha\rangle}\right|, \left|\frac{\langle w_2, u_\alpha\rangle}{\langle w_1, u_\alpha\rangle}\right|\right\} + O\left(n^{-1/3+C_1\varepsilon_{LR}}\right)\right)^2$$
$$\le \frac{1}{4} + o(1) \le \frac{1}{2}.$$

Together with

$$\left(w_1^\top G(E_1) w_1 - d_{11} + E_1\right)\left(w_2^\top G(E_1) w_2 - d_{22} + E_1\right) = 1 + o(1)$$

this yields $f(E_1) > 1$. Since $f(E_0) = 0$, there exists $\lambda \in (E_0, E_1)$ such that $f(\lambda) = 1$, which shows that $\lambda_\alpha \in (E_0, E_1)$.

Now we will focus on $s(\lambda_\alpha)$. Since $\lambda_\alpha > E_0$, the $w_1^\top G(\lambda_\alpha) w_1 - d_{11} + \lambda_\alpha$ is positive. Also,

$$w_1^\top G(\lambda_\alpha) w_2 - d_{12} = \frac{\langle w_1, u_\alpha\rangle\langle w_2, u_\alpha\rangle}{\mu_\alpha - \lambda_\alpha} + O\left(n^{-1/3+C\varepsilon_{LR}}\right),$$

and the magnitude of the leading term is significantly greater than $O\left(n^{-1/3+C\varepsilon_{LR}}\right)$ by (3.47). Since $\mu_\alpha - \lambda_\alpha < 0$, the expression above has the same sign as $-\langle w_1, u_\alpha\rangle\langle w_2, u_\alpha\rangle$. Therefore, we conclude that

$$s(\lambda_\alpha) = \operatorname{sign}\left(-\frac{w_1^\top G(\lambda_\alpha) w_1 - d_{11} + \lambda}{w_1^\top G(\lambda_\alpha) w_2 - d_{12}}\right) = \operatorname{sign}\left(\langle w_1, u_\alpha\rangle\langle w_2, u_\alpha\rangle\right) = \operatorname{sign}\left(\langle \tilde{w}_1, u_\alpha\rangle\langle \tilde{w}_2, u_\alpha\rangle\right)$$

for any $\tilde{A} \in \mathscr{T}$, $W \in \mathscr{A}_{wGw} \cap \mathscr{A}_W$, and $D \in \mathscr{A}_D$.

It remains to estimate the expectation of $s(\lambda_\alpha)$. Recall that we conditioned on the block B, and W and D are independent of B. Denote this conditional expectation and probability by $\mathbb{E}_{W,D}$ and $\mathbb{P}_{W,D}$. We have

$$\begin{aligned}\left|\mathbb{E}_{W,D}\left(s(\lambda_\alpha)\mathbf{1}_{\mathscr{T}}(A)\right)\right| &\le \left|\mathbb{E}_{W,D}\left(s(\lambda_\alpha)\mathbf{1}_{\mathscr{T}}(A)\mathbf{1}_{\mathscr{A}_W}(W)\mathbf{1}_{\mathscr{A}_{wGw}}(W)\mathbf{1}_{\mathscr{A}_D}(D)\right)\right| \\ &\quad + \mathbb{P}_{W,D}\left(W \notin \mathscr{A}_{wGw} \cup \mathscr{A}_W\right) + \mathbb{P}_{W,D}\left(D \notin \mathscr{A}_D\right) \\ &= \left|\mathbb{E}_{W,D}\left(\operatorname{sign}\left(\langle \tilde{w}_1, u_\alpha\rangle\langle \tilde{w}_2, u_\alpha\rangle\right)\mathbf{1}_{\mathscr{T}}(A)\mathbf{1}_{\mathscr{A}_W}(W)\mathbf{1}_{\mathscr{A}_{wGw}}(W)\mathbf{1}_{\mathscr{A}_D}(D)\right)\right| \\ &\quad + O\left(n^{-1/3+C'\varepsilon_{LR}}\right).\end{aligned}$$

We can get rid of the indicators in the leading term in a similar way:

$$\begin{aligned}&\left|\mathbb{E}_{W,D}\left(\operatorname{sign}\left(\langle \tilde{w}_1, u_\alpha\rangle\langle \tilde{w}_2, u_\alpha\rangle\right)\mathbf{1}_{\mathscr{T}}(A)\mathbf{1}_{\mathscr{A}_W}(W)\mathbf{1}_{\mathscr{A}_{wGw}}(W)\mathbf{1}_{\mathscr{A}_D}(D)\right)\right| \\ &\quad\le \left|\mathbb{E}_{W,D}\left(\operatorname{sign}\left(\langle \tilde{w}_1, u_\alpha\rangle\langle \tilde{w}_2, u_\alpha\rangle\right)\mathbf{1}_{\mathscr{T}}(A)\right)\right| + \mathbb{P}_{W,D}\left(W \notin \mathscr{A}_{wGw} \cup \mathscr{A}_W\right) + \mathbb{P}_{W,D}\left(D \notin \mathscr{A}_D\right) \\ &\quad\le \left|\mathbb{E}_{W,D}\left(\operatorname{sign}\left(\langle \tilde{w}_1, u_\alpha\rangle\langle \tilde{w}_2, u_\alpha\rangle\right)\mathbf{1}_{\mathscr{T}}(A)\right)\right| + O\left(n^{-1/3+C'\varepsilon_{LR}}\right).\end{aligned}$$

Removing the conditioning over B, we get

$$\begin{aligned}&\left|\mathbb{E}\left(s(\lambda_\alpha)\mathbf{1}_{\mathscr{T}}(\tilde{A})\right)\right| \\ &\quad\le \left|\mathbb{E}\left(\operatorname{sign}\left(\langle \tilde{w}_1, u_\alpha\rangle\langle \tilde{w}_2, u_\alpha\rangle\right)\mathbf{1}_{\mathscr{T}}(A)\right)\right| + O\left(n^{-1/3+C'\varepsilon_{LR}}\right) \\ &\quad\le \left|\mathbb{E}\left(\operatorname{sign}\left(\langle \tilde{w}_1, u_\alpha\rangle\langle \tilde{w}_2, u_\alpha\rangle\right)\mathbf{1}_{\mathscr{T}_{(n,1)}}(A^{(1,2)})\right)\right| + \mathbb{E}\left|\mathbf{1}_{\mathscr{T}}(A) - \mathbf{1}_{\mathscr{T}_{(n,1)}}(\tilde{A}^{(1,2)})\right| + O\left(n^{-1/3+C'\varepsilon_{LR}}\right)\end{aligned}$$

In view of Corollary 3.11, the second term does not exceed $n^{-1/3+2\varepsilon_{LR}}$. To bound the first term, we condition again on the block $B = \tilde{A}^{(1,2)}$ such that $\tilde{A}^{(1,2)} \in \mathscr{T}_{(n,1)}$ and apply (3.46). By this inequality,

$$P_i := \mathbb{P}\left[\langle \tilde{w}_1, u_a\rangle \geq 0 \mid \tilde{A}^{(1,2)}\right] := \frac{1}{2} + p_i$$

where $p_i = O\left(n^{-1/3+5\varepsilon_{LR}}\right)$. Using the independence of $\tilde{w}_1$ and $\tilde{w}_2$, we get

$$\begin{aligned}\mathbb{E}\left[\text{sign}\left(\langle \tilde{w}_1, u_a\rangle\langle \tilde{w}_2, u_\alpha\rangle\right) \mid \tilde{A}^{(1,2)}\right] &= P_1P_2 + (1-P_1)(1-P_2) - P_1(1-P_2) - (1-P_1)P_2 \\ &= 4p_1p_2 = O\left(n^{-2/3+10\varepsilon_{LR}}\right).\end{aligned}$$

Removing the conditioning completes the proof of Lemma 3.16. ∎

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A | APPENDIX

In this section we establish the spectral properties of symmetric random matrices appearing in Definition 3.4. Namely, we prove the following lemma:

Lemma A.1. *Fix $p \in (0, 1)$, $D > 0$. Let H_p be a symmetric $n \times n$ matrix with zero diagonal and i.i.d entries above the diagonal. The nondiagonal entries have the distribution:*

$$h_{ij} = \begin{cases} \sqrt{\frac{1-p}{p}} \frac{1}{\sqrt{n}} & \text{with probability } p, \\ -\sqrt{\frac{p}{1-p}} \frac{1}{\sqrt{n}} & \text{with probability } 1-p. \end{cases}$$

Then, H_p satisfies (3.12) – (3.14) with probability greater than $1 - n^{-D}$. Furthermore, for a sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ such that $H_p \in \mathscr{L}\mathscr{R}\left(n, n^{-2/3-\varepsilon}\right)$ with probability greater than $1 - n^{-\delta}$.

Note that condition (3.16) involving θ and k can be derived from the second part of this lemma by appropriately adjusting ε.

Conditions (3.12) and (3.14) were derived in [11, Theorem 2.1, 2.2] and Conditions (3.13) and (3.14) were proved in [4, Theorem 2.12, 2.16].

Condition (3.16) was proved in [14]. However, the matrix model is slighly different from ours. To show that H_p satisfies level repulsion at the edge, we rely on the fact that GOE satisfies this condition and apply Green Function Comparison Theorem. This strategy is stated as Proposition 2.4 in [14]:

Proposition A.2. *Let H^v and H^w be $n \times n$ symmetric random matrices with independent entries h^v_{ij} and h^w_{ij} such that the $\mathbb{E}h^v_{ij} = \mathbb{E}h^w_{ij} = 0$ and $\mathbb{E}(h^v_{ij})^2 = \mathbb{E}(h^w_{ij})^2 = \sigma^2_{ij}$. Assume that $\Sigma = (\sigma_{ij})$ satisfies the following conditions*

1. *For $j \in [n]$, $\sum_{i=1}^n \sigma^2_{ij} = 1$.*
2. *There exists $\delta_W > 0$ such that 1 is a simple eigenvalue of Σ and* Spec(″) ⊆ [−1+ffi$_W$, 1−ffi$_W$]∪{1}.
3. *There is a constant C_W, independent of n, such that $\max_{ij}\{\sigma^2_{ij}\} \leq \frac{C_W}{n}$.*

Also, assume that h_{ij} have a uniformly subexponential decay. Namely, there exists a constant $\nu > 0$, independent of n, such that for any $x \geq 1$ and $1 \leq i, j \leq n$ we have

$$\mathbb{P}\left(|h_{ij}| > x\sigma_{ij}\right) \leq \nu^{-1}\exp(-x^\nu).$$

Assume that H^v satisfies the Level Repulsion Condition*, that is, for a sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ such that $H^v \in \mathscr{L}\mathscr{R}\left(n, n^{-2/3-\varepsilon}\right)$ with probability greater than $1 - n^{-\delta}$. Then the same holds for H^w with a different $\delta = \delta(\varepsilon)$.*

The level repulsion condition has been proved for the GOE ensemble, see, for example, [1]. By GOE we mean that a $n \times n$ symmetric random matrix W with independent centered gaussian entries (up to symmetry) where the off-diagonal entries have variance $1/n$ and the diagonal entries have variance $2/n$. We would like to apply Proposition A.2 with $H^v = W$ and $H^w = H_p$. The first two moments of the off-diagonal entries of these two ensembles are the same. The variances of the diagonal entries differ, but since there are only n of them, it will be possible to show that they do not affect the level repulsion significantly.

We proceed in two steps. First, we prove the level repulsion condition for a $n \times n$ matrix $\tilde{W}$ whose off diagonal entries are the same as for W and the diagonal entries are 0. Then, we apply Proposition A.2 to $H^v = \tilde{W}$ and $H^w = H_p$.

Thus, it is sufficient to prove

Proposition A.3. *The level repulsion estimates hold for $\tilde{W}$.*

The proof of this proposition is standard and is included it for the reader's convenience. It follows the proof of A.2 which relies on Lemma 2.6 (Green Function Comparison Theorem) and Lemma 2.7 in [14].

Since the second moments of the diagonal entries of W and $\tilde{W}$ differ, we need a substitute for Green Function Comparison Theorem. The rest of the proof will be exactly the same as of Proposition A.2.

Before stating the result precisely, we will sketch the idea behind the comparison. Consider the Stieltjes Transform of a symmetric matrix H is $m(z) = \frac{1}{n}\mathrm{Tr}\left(\frac{1}{H-z}\right)$. Suppose $\lambda_1, \ldots, \lambda_n$ are eigenvalues of H. Then,

$$\frac{n}{\pi}\mathrm{Im}\, m(E + \mathrm{i}\eta) = \sum_{i \in [n]} \frac{1}{\pi} \frac{\eta}{(\lambda_i - E)^2 + \eta^2}.$$

If we choose η to be sufficiently small, then each summand is an approximation of the delta function at each eigenvalue. On one hand, this provides a way to estimate number of eigenvalues in an interval. Taking η to be sufficiently small, we should have

$$\sum_{\alpha}^{n} \mathbf{1}_{(a,b)}(\lambda_\alpha) \simeq n \int_a^b \frac{1}{\pi} \operatorname{Im} m\,(E + \mathrm{i}\eta)\ \mathrm{d}E.$$

On the other hand, $\operatorname{Im} m\,(E + \mathrm{i}\eta)$ can be expressed in terms of the Green Function $G\,(z) := \frac{1}{H-z}$.

$$\operatorname{Im} m\,(E + \mathrm{i}\eta) = \frac{1}{n} \sum_i \operatorname{Im} G_{ii}\,(E + \mathrm{i}\eta)\,.$$

We will use Lindeberg's method to replace the diagonal entries of W by those of $\tilde{W}$ one by one and estimate the expectation of the difference of Green functions.

Now we state the substitute for Lemma 2.6 in [14]:

Lemma A.4 (Green Function Comparison Theorem). *Let $F : \mathbb{R} \mapsto \mathbb{R}$ be a bounded smooth function whose first and second derivatives are bounded as well. There exists a constant $\varepsilon_0 > 0$ and for such $\varepsilon < \varepsilon_0$ and for any real numbers $E_1, E_2 \in \left[2 - n^{-2/3+\varepsilon}, 2 + n^{2/3+\varepsilon}\right]$, setting $\eta = n^{-2/3-\varepsilon}$ we have*

$$\left|\left(\mathbb{E}^W - \mathbb{E}^{\tilde{W}}\right) F\left(n \int_{E_1}^{E_2} \operatorname{Im} m\,(y + i\eta)\ \mathrm{d}y\right)\right| \le cn^{-1/3+c\varepsilon}.$$

Lindeberg's method is based on replacing the entries one by one. Yet, our proof uses the strong local semicircle law, see Theorem A.6 below. Application of this law requires scaling of the matrix so that the variance matrix will be doubly stochastic. However, replacing diagonal entries of W by 0 appearing in $\tilde{W}$ results in two essentially different scalings of the variance matrix to the doubly stochastic form. To deal with this obstacle, we perform replacement in smaller steps which will require n^2 steps instead of n.

Define n^2 symmetric random matrices $\left\{W_{\beta,\gamma}\right\}_{\beta,\gamma=0}^{n}$ whose off-diagonal entries are the same as of W and $\tilde{W}$. Let $\left\{h_{i,j}\right\}_{i,j=1}^{n}$ be i.i.d $N\left(0, \frac{2}{n^2}\right)$ random variables. The diagonal entries of $W_{\beta,0}$ are

$$\left(W_{\beta,0}\right)_{i,i} = \sum_{j=1}^{\beta} h_{j,i}.$$

In particular, the diagonal entries of $W_{\beta,0}$ are centered gaussian variables with variance $\frac{2\beta}{n^2}$. Thus, the variance matrix of $W_{\beta,0}$ is doubly stochastic if we scale it by a factor $1+O(n^{-1})$. Furthermore, $W_{0,0} = \tilde{W}$ and $W_{n,0} = W$.

Now we define the diagonal entries of $W_{\beta,\gamma}$:

$$\left(W_{\beta,\gamma}\right)_{ii} = \begin{cases} \sum_{j=1}^{\beta} h_{j,i} & \text{if } i > \gamma, \\ \sum_{j=1}^{\beta} h_{j,i} + h_{\beta+1,i} & \text{if } i \le \gamma. \end{cases}$$

In other words, we have

$$W_{\beta,\gamma+1} = W_{\beta,\gamma} + h_{\beta+1,\gamma+1} e_{\gamma+1} e_{\gamma+1}^{\top}$$

and

$$W_{\beta,n} = W_{\beta+1,0}.$$

Our goal is to show that

$$\left| \left(\mathbb{E}^{W_{\beta,\gamma}} - \mathbb{E}^{W_{\beta,\gamma+1}} \right) F \left(n \int_{E_1}^{E_2} \operatorname{Im} m\,(y + i\eta)\ \mathrm{d}y \right) \right| \le n^{-2} n^{-1/3+c\varepsilon}$$

for each $k = 0, \dots, n-1$ and $\gamma = 0, \dots, n-1$. Then the statement of the theorem will follow immediately. Before we move on to the proof, we need the following proposition.

Proposition A.5. *Fix a sufficiently small $\varepsilon > 0$. Let $\mathscr{I} := \left\{ E + \mathrm{i}\eta \ :\ |E-2| < n^{-2/3+\varepsilon} \right\}$ and $\eta = n^{-2/3-2\varepsilon}$.Then, for any $D > 0$, if n is sufficiently large, we have*

$$\mathbb{P} \left(\max_{\beta,\gamma} \sup_{z \in \mathscr{I}} \left| \left(G_{\beta,\gamma}(z) \right)_{ij} - \delta_{ij} \right| > n^{-1/3+4\varepsilon} \right) < n^{-D}$$

where $G_{\beta,\gamma}(z) = \frac{1}{W_{\beta,\gamma} - z}$ is the Green function of $W_{\beta,\gamma}$.

Let's recall a theorem in [11, Theorem 2.1].

Theorem A.6 (Strong local semicircular law)**.** *Suppose that H satisfies the assumption of Proposition A.2. Then, for every s, $D > 0$ and $0 < \varepsilon < 1/3$, we have*

$$\mathbb{P} \left(\sup_{|E-2| \le n^{-2/3+\varepsilon}} \max_{i,j \in [n]} \left| (G(E + \mathrm{i}\eta))_{ij} - 1 \right| < 4n^{-\frac{1}{3}+s+\varepsilon} \right) \ge 1 - n^{-D} \tag{A1}$$

where $\eta = n^{-2/3-\varepsilon}$ and $n \ge n(s, D, \varepsilon)$.

This theorem implies that $\max_\beta \sup_{z \in \mathscr{I}} \left| \left(G_{\beta,0}(z) \right)_{ij} - \delta_{ij} \right| \le 4n^{-\frac{1}{3}+3\varepsilon}$ with probability at least $1 - n^{-D}$. We extend this properties to $W_{\beta,\gamma}$ by comparison.

Proof of Proposition A.5. Fix β. Fix a sample of $W_{\beta,0}$ such that

$$\sup_{|E-2| \le n^{-2/3+\varepsilon}} \max_{i,j \in [n]} \left| (G(E + \mathrm{i}\eta))_{ij} - 1 \right| < 4n^{-\frac{1}{3}+3\varepsilon}$$

for $|E-2| \le n^{-2/3+\varepsilon}$ and the samples of $\left\{ h_{\beta+1,\gamma} \right\}_{\gamma=1}^{n}$ such that $\max_\gamma \left| h_{\beta+1,\gamma} \right| \le \frac{\varphi_n}{n}$ where $\varphi_n = (\log n)^{\log\log n}$. Notice that both conditions hold with probability at least $1 - n^{-D}$.

Define $s_0 = 4n^{-\frac{1}{3}+3\varepsilon}$ and $s_{\gamma+1} = s_\gamma \left(1 + \frac{1}{\varphi_n n} \right)$. We claim that

$$\left| \left(G_{\beta,\gamma}(E + \mathrm{i}\eta) \right)_{i,j} - \delta_{ij} \right| \le \phi(i,j,\gamma) s_\gamma \tag{A.2}$$

where

$$\phi(i,j,\gamma) := 1 + \mathbf{1}_{i \ge \gamma} + \mathbf{1}_{j \ge \gamma}.$$

If it is true, then we have

$$\max_{\beta,\gamma}\sup_{z\in\mathcal{J}}\left|\left(G_{\beta,\gamma}\left(E+\mathrm{i}\eta\right)\right)_{ij}-\delta_{ij}\right|\leq 3s_n\leq 3s_0(1+\frac{1}{\varphi_n n})^n\leq n^{-1/3+4\epsilon}.$$

If the matrices A and $A+B$ are invertible, then the following resolvent identity holds:

$$\frac{1}{A+B}=\frac{1}{A}-\frac{1}{A}B\frac{1}{A+B}.$$

Applying the equality repeatedly we get

$$\frac{1}{A+B}=\frac{1}{A}-\frac{1}{A}B\frac{1}{A}+\frac{1}{A}B\frac{1}{A}B\frac{1}{A}-\left(\frac{1}{A}B\right)^3\frac{1}{A}\cdots\pm\left(\frac{1}{A}B\right)^k\frac{1}{A+B}.$$

Suppose that (A.2) holds up to $\gamma-1$. Let $A=W_{\beta,\gamma-1}-(E+\mathrm{i}\eta)I_n$ and $B=h_{\beta+1,\gamma}e_\gamma e_\gamma^\top$. For simplicity, we write

$$h=h_{\beta+1,\gamma},\ P=e_\gamma e_\gamma^\top,\ R=\frac{1}{A}=G_{\beta,\gamma-1}\left(E+\mathrm{i}\eta\right),\text{ and }S=\frac{1}{A+B}=G_{\beta,\gamma}\left(E+\mathrm{i}\eta\right).$$

The equality above can be written as

$$S=\frac{1}{A+B}=R-hRPR+h^2\left(RP\right)^2R+\dots h^k\left(RP\right)^kS.$$

Entry-wise, we have

$$\begin{aligned}S_{ij}=&R_{ij}-hR_{i\gamma}R_{\gamma j}+h^2R_{i\gamma}R_{\gamma\gamma}R_{\gamma j}\dots(-1)^kh^kR_{i\gamma}R_{\gamma\gamma}^{k-1}S_{\gamma j}\\=&R_{ij}-hR_{i\gamma}R_{\gamma j}\left(\sum_{l=0}^{k}\left(-hR_{\gamma\gamma}\right)^l\right)+(-1)^kh^kR_{i\gamma}R_{\gamma\gamma}^{k-1}S_{\gamma j}\end{aligned}\tag{A.3}$$

We will use the following uniform bound of the entries of S:

$$|S_{\gamma j}|\leq\|S\|=\left\|\frac{1}{W_{\beta,\gamma}(E+\mathrm{i}\eta)}\right\|\leq\frac{1}{\eta}\leq n^{2/3+\epsilon}.$$

Together with $|h|<\frac{\varphi_n}{n}$ and $\max\left\{|R_{i\gamma}|,\ |R_{\gamma j}|\right\}\leq 1+s_\gamma\leq 2$, this means that the last summand in (A.3) is less than $\frac{1}{n^3}$ if we pick $k=5$. From now on we will fix $k=5$. Then,

$$\left|\sum_{l=0}^{k}\left(-hR_{\gamma\gamma}\right)^l\right|\leq C.$$

for some absolute constant $C>0$. Therefore,

$$\begin{aligned}\left|S_{ij}-\delta_{ij}\right|&\leq\left|R_{ij}-\delta_{ij}\right|+C\left|hR_{i\gamma}R_{j\gamma}\right|+\frac{1}{n^3}\\&\leq\phi\left(i,j,\gamma-1\right)s_{\gamma-1}+C\left|hR_{i\gamma}R_{\gamma j}\right|+\frac{1}{n^3}.\end{aligned}$$

It remains to show

$$\phi(i,j,\gamma-1)\,s_{\gamma-1}+C\left|hR_{i\gamma}R_{j\gamma}\right|+\frac{1}{n^3}\leq\phi(i,j,\gamma)\,s_\gamma.$$

Consider $\gamma\notin\{i,j\}$. We use the bound $\left|R_{i\gamma}\right|\leq 3s_{\gamma-1}$ and $\left|R_{\gamma j}\right|\leq 3s_{\gamma-1}<\frac{1}{\varphi_n^3}$ to get

$$C\left|hR_{i\gamma}R_{j\gamma}\right|+\frac{1}{n^3}\leq s_{\gamma-1}\frac{C}{n\varphi_n^2}+\frac{1}{n^3}\leq s_{\gamma-1}\frac{1}{n\varphi_n}.$$

Therefore, we have

$$\begin{aligned}\left|S_{ij}-\delta_{ij}\right|\leq&\phi(i,j,\gamma-1)\,s_{\gamma-1}+\frac{1}{n\varphi_n}s_{\gamma-1}\\\leq&\phi(i,j,\gamma-1)\,s_{\gamma-1}\left(1+\frac{1}{\varphi_n n}\right)\\\leq&\phi(i,j,\gamma)\,s_\gamma\end{aligned}$$

In the case $\gamma\in\{i,j\}$, we use the trivial bounds that $\max\left\{\left|R_{i\gamma}\right|,\left|R_{j\gamma}\right|\right\}\leq 1+3s_{\gamma-1}\leq 2$. Thus, we have

$$C\left|hR_{i\gamma}R_{j\gamma}\right|+\frac{1}{n^3}\leq\frac{4\varphi_n}{n}+\frac{1}{n^3}\leq s_0.$$

Notice that $\phi(i,j,\gamma)-\phi(i,j,\gamma-1)\geq 1$ since $\gamma\in\{i,j\}$.

$$\left|S_{ij}-\delta_{ij}\right|\leq\phi(i,j,\gamma-1)\,s_{\gamma-1}+s_0\leq\phi(i,j,\gamma)\,s_\gamma.$$

The result follows. ■

Now we are ready to prove Lemma A.4.

Proof. Recall that our goal is to show that

$$\left|\left(\mathbb{E}^{W_{\beta,\gamma}}-\mathbb{E}^{W_{\beta,\gamma+1}}\right)F\left(n\int_{E_1}^{E_2}\operatorname{Im}m(y+i\eta)\,\mathrm{d}y\right)\right|\leq n^{-2}n^{-1/3+c\varepsilon}$$

With probability greater than $1-n^{-D}$, we have

$$\sup_{|E-2|\leq n^{-2/3+\varepsilon}}\left|\left(G_{\beta,\gamma}(E+\mathrm{i}\eta)\right)_{ij}-\delta_{ij}\right|\leq n^{-1/3+\varepsilon}.$$

for $\beta=0,\dots,n-1$ and $\gamma=0,\dots,n-1$. Now, we fix β and γ. Fix a sample of $W_{\beta,\gamma-1}$ such that the above inequality holds.

We recycle the notation from the proof of Proposition A.5. Let $A=W_{\beta,\gamma-1}-(E+\mathrm{i}\eta)I_n$ and $B=h_{\beta+1,\gamma}e_\gamma e_\gamma^\top$. For simplicity, we write

$$h=h_{\beta+1,\gamma},\ P=e_\gamma e_\gamma^\top,\ R=\frac{1}{A}=G_{\beta,\gamma-1}(E+\mathrm{i}\eta),\ \text{and } S=\frac{1}{A+B}=G_{\beta,\gamma}(E+\mathrm{i}\eta).$$

Then,

$$S_{ij} = R_{ij} + hR_{i\gamma}R_{j\gamma} + h^2 R_{i\gamma}R_{\gamma\gamma}R_{j\gamma} + h^3 R_{i\gamma}R_{\gamma\gamma}^2 S_{j\gamma},$$

where, as before, $|S_{j\gamma}| \leq \|S\| \leq n^{2/3+\varepsilon}$. Taking expectation with respect to h and using $|R_{i\gamma}| \leq n^{-1/3+\varepsilon} + \delta_{i\gamma}$, we get

$$|\mathbb{E}_h S_{ii} - R_{ii}| \leq \frac{2}{n^2} n^{-2/3+2\varepsilon} + \frac{C}{n^3} n^{1/3+2\varepsilon} + \delta_{i\gamma}\frac{C}{n^2}.$$

Furthermore, if $|h| \leq \frac{\varphi_n}{n}$, then by (A.3)

$$|S_{ii} - R_{ii}| \leq C\left|hR_{i\gamma}R_{\gamma i}\right| + \frac{1}{n^3} \leq \varphi_n n^{-5/3+3\varepsilon} + \delta_{i\gamma}\frac{\varphi_n}{n}.$$

Therefore,

$$\left|\sum_{i=1}^{n}(S_{ii} - R_{ii})\right| \leq n^{-2/3+4\varepsilon} \quad \text{when } |h| \leq \frac{\varphi_n}{n}, \tag{A.4}$$

and

$$\left|\mathbb{E}_h \sum_{i=1}^{n}(S_{ii} - R_{ii})\right| \leq n^{-5/3+3\varepsilon}. \tag{A.5}$$

Now we examine the difference:

$$\begin{aligned} &F\left(\int_{E_1}^{E_2}\sum_i S_{ii}(y+\mathrm{i}\eta)\,\mathrm{d}y\right) - F\left(\int_{E_1}^{E_2}\sum_i R_{ii}(y+\mathrm{i}\eta)\,\mathrm{d}y\right) \\ &= F'\left(\int_{E_1}^{E_2}\sum_i R_{ii}(y+\mathrm{i}\eta)\,\mathrm{d}y\right)\left(\int_{E_1}^{E_2}\sum_i (S_{ii}(y+\mathrm{i}\eta) - R_{ii}(y+\mathrm{i}\eta))\,\mathrm{d}y\right) \\ &\quad + O\left(\left(\int_{E_1}^{E_2}\sum_i (S_{ii}(y+\mathrm{i}\eta) - R_{ii}(y+\mathrm{i}\eta))\,\mathrm{d}y\right)^2\right) \end{aligned}$$

where we rely on the fact that F'' is bounded. Since $|E_2 - E_1| \leq 2n^{-2/3+\varepsilon}$, by (A.4) we have

$$\left(\int_{E_1}^{E_2}\sum_i (S_{ii}(y+\mathrm{i}\eta) - R_{ii}(y+\mathrm{i}\eta))\,\mathrm{d}y\right)^2 \leq \left(2n^{-2/3+\varepsilon}n^{-2/3+4\varepsilon}\right)^2 \leq n^{-8/3+C\varepsilon}$$

if $|h| \leq \frac{\varphi_n}{n}$. Furthermore, if we take the expectation with respect to h and $W_{\beta,\gamma}$, the same bound still holds. Indeed, we can apply this bound conditioning on $|h| \leq \frac{\varphi_n}{n}$, and use a trivial bound

$$\left(\int_{E_1}^{E_2}\sum_i S_{ii}(y+\mathrm{i}\eta) - R_{ii}(y+\mathrm{i}\eta)\,\mathrm{d}y\right)^2 \leq n^C$$

valid with some fixed constant $C > 0$ for other h. Similarly, (A.5) yields

$$\left| \mathbb{E}_{W_{\beta,\gamma-1}} \mathbb{E}_h \left(\int_{E_1}^{E_2} \sum_i (S_{ii}(y + \mathrm{i}\eta) - R_{ii}(y + \mathrm{i}\eta)) \, \mathrm{d}y \right) \right| \leq n^{-7/3+C\epsilon}.$$

Therefore, we conclude that

$$\left(\mathbb{E}^{W_{\beta,\gamma-1}} - \mathbb{E}^{W_{\beta,\gamma}} \right) \operatorname{Im} m (E + \mathrm{i}\eta) \leq n^{-2} n^{-1/3+c\epsilon}$$

finishing the proof. ■