Entanglement, Renormalization and Effective Field Theory

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To my parents.

Abstract

We develop the idea that renormalization, decoupling of heavy particle effects from low energy physics and the construction of effective field theories are intimately linked to the entanglement of the low and high energy momentum modes. Using unitary transformations to decouple these modes we show in a scalar field theoretical model, how renormalization may be consistently implemented and how the low energy effective field theory can be constructed.

Contents

Chapter 1

Introduction

> We understand many things about particles and their interactions, but this and other mysteries make it very clear that we are nowhere close to a full understanding.
>
> — Martinus J. G. Veltman

Entanglement is ubiquitous in any quantum theory. In a free field theory the different momentum modes are not entangled. However, the introduction of interactions causes the entanglement, in particular, of the low momentum modes with the inaccessible high energy ones. In experiments only the low energy or larger wavelength modes are accessible and renormalization can be thought of as a procedure to separate the effects of the high energy modes from those of low energy. In the usual Wilsonian approach [1][2][3], the high energy modes are integrated out and in this way we arrive at a low energy effective action. An alternative viewpoint, that we discuss here, is to directly address the entanglement and by a series of unitary transformations decouple the low and high energy modes. The effective low energy Hamiltonian is then obtained by projecting onto the "high energy vacuum", i.e., the low energy subspace where there are no modes of heavy masses or of momenta larger than some cut-off scale which can appear as external states. In this thesis, we discuss renormalization, decoupling of heavy mass states [4] and the construction of effective field theories, [5][6] all from this perspective.

Our results are in agreement with those obtained by the standard methods and show that such a program can be successfully implemented, thereby providing another way to construct low energy effective theories.

The thesis is organized as follows.

In chapter 2 we discuss how the Hamiltonian of a theory decouples under a unitary transformation of the states.

In chapter 3 we consider a scalar field model with heavy and light fields and explicitly construct the unitary transformation that shows clearly how renormalization and decoupling works.

In chapter 4 we extend the previous construction to obtain an effective field theory of the light fields alone and make connection with previous work using the standard methods.

We conclude with a discussion of these results in chapter 5.

Chapter 2

Perturbative Decoupling, Renomalization and Matching of Hamiltonian Operators

2.1 Decoupling with Unitary Transformations

The subject of decoupling in Effective Field Theory has been studied extensively for the past many decades. The decoupling theorem states that if the low energy effective theory is renormalizable, and a physical renormalization scheme has been applied, then all effects due to heavy particles will appear as changes to couplings or are suppressed as $\frac{1}{M}$, where M is the mass of the heavy particle. In spite of the great success that Effective Field Theory has achieved in particle physics, seldom work has been done in a Hamiltonian framework. As we discussed in introduction, an alternative way to consider the decoupling is to introduce a series of unitary transformations to decouple high energy and low energy modes and then look at the low energy part of the spectrum.

Let's consider the Hamiltonian H of a full theory, and denote $H_{eff}(\mu)$ as the effective Hamiltonian defined at a scale μ. $H_{eff}(\mu)$ will generate the same physical results i.e. S-matrix elements for all the physical processes that do not involve momenta greater than μ. We can view $H_{eff}(\mu)$ as the projection of the full theory onto the low energy subspace:

$$H_{eff}(\mu) = P(\mu)HP(\mu), \tag{2.1}$$

where P(μ) is the projection operator at energy scale μ.

Our claim is that, at least perturbatively we can decouple low energy modes from high energy modes using a series of unitary transformations and therefore construct the high energy vacuum and obtain the "low energy subspace". Let $H_{decoupled}$ denote the decoupled Hamiltonian at low energy:

$$H_{decoupled} = \langle 0_{high}| \omega^\dagger H \omega |0_{high}\rangle\,, \tag{2.2}$$

where $|0_{high}\rangle$ denotes the high energy vacuum:

$$a_{high}\,|0_{high}\rangle = 0, \tag{2.3}$$

and a_{high} is the annihilation operator of high energy modes. The ω here is a product series of unitary transformations. Its jobs is to remove the terms in the full Hamiltonian which change the high energy vacuum structure. Those are terms containing only high energy creation operators. Since Hamiltonian operator is Hermitian, ω will inevitably cancel terms containing only high energy annihilation operators as well. Furthermore, we will normal order with respect to high energy vacuum. It's worth noting here that the $H_{decoupled}$ so calculated out is not a "perfectly physical" Hamiltonian operator like H or H_{eff}, rather than is an intermediate step. However, as we will see later its components have important physical meaning regarding renormalization and will also be involved in the matching process to get the physical H_{eff}.

Let's break ω into a product series:

$$\omega = \omega_0\omega_1\omega_2 ... \omega_n ... \tag{2.4}$$

Each ω_i partially diagonalizes the Hamiltonian to given order $\sim \frac{1}{\Lambda}$, Λ is the cut-off energy scale. Decompose the full Hamiltonian as:

$$H = H_1 + H_2 + H_A + H_B, \tag{2.5}$$

where H_1 only contains low energy modes, H_2 is the free part for high energy modes, H_A contains terms that only have high energy annihilation or creation operators and H_B is whatever left. For simplicity, we can set H_1 to be of energy order $\sim O(1)$, and the other three terms of order $\sim O(\Lambda)$.

Let's consider the following:

$$\begin{aligned} &\omega_0^\dagger \left(H_1 + H_2 + H_A + H_B\right) \omega_0 \\ =&e^{-i\Omega_0} \left(H_1 + H_2 + H_A + H_B\right) e^{i\Omega_0} \\ =&H_1 + H_2 + H_A + H_B + i[H_1, \Omega_0] + i[H_2, \Omega_0] + i[H_A, \Omega_0] + i[H_B, \Omega_0]... \end{aligned} \tag{2.6}$$

We want to eliminate H_A by choosing Ω_0 such that

$$i[H_2, \Omega_0] + H_A = 0. \tag{2.7}$$

This is our decoupling condition at order $\sim O(\Lambda)$, and since both H_2 and H_A are of order $\sim O(\Lambda)$, we can deduce that $\Omega_0 \sim O(1)$. Although we cancel out H_A, we create a new term $i[H_1, \Omega_0]$ of order $\sim O(1)$ that contains only annihilation or creation operators and in order to eliminate this new term, we need to introduce the next unitary operator $\omega_1 = e^{i\Omega_1}$ at order $\sim O(\frac{1}{\Lambda})$. Then the Hamiltonian becomes

$$\begin{aligned} &e^{-i\Omega_1} e^{-i\Omega_0} \left(H_1 + H_2 + H_A + H_B\right) e^{i\Omega_0} e^{i\Omega_1} \\ =&H_1 + H_2 + H_B + i[H_1, \Omega_0] + i[H_A, \Omega_0] + i[H_B, \Omega_0] + i[H_2, \Omega_1] + ... \end{aligned} \tag{2.8}$$

We now choose Ω_1 such that

$$i[H_1, \Omega_0] + i[H_2, \Omega_1] = 0, \tag{2.9}$$

and it is obvious that Ω_1 is of order $\sim O(\frac{1}{\Lambda})$. In general our decoupling condition will become:

$$i[H_1, \Omega_n] + i[H_2, \Omega_{n+1}] = 0, \tag{2.10}$$

where $\Omega_{n+1} \sim \frac{1}{\Lambda}\Omega_n$. Therefore, we can see the decoupling is indeed carried out in a perturbative fashion.

2.2 Decoupled Hamiltonian and S-matrix Elements

Going back to the calculation of $H_{decoupled}$ and using the decoupling conditions we have

$$\begin{aligned} H_{decoupled} = \langle 0_{high}| \, H_1 + H_2 + H_B + \frac{i}{2}[H_A, \Omega_0] + i[H_B, \Omega_0] - \frac{1}{3}[[H_A, \Omega_0], \Omega_0] \\ - \frac{1}{2}[[H_B, \Omega_0], \Omega_0] - \frac{1}{2}[[H_1, \Omega_0], \Omega_0] + i[H_B, \Omega_1] + O(\frac{1}{\Lambda}) \, |0_{high}\rangle \, . \end{aligned} \tag{2.11}$$

As we pointed out in the section 2.1, the $H_{decoupled}$ is not a "perfectly physical" Hamiltonian operator. $H_{decoupled}$ has two parts, the first part is $\langle 0_{high}| \, H_1 \, |0_{high}\rangle = H_1$ and the second part is from the normal ordering of H_B with respect to high energy vacuum and commutators in the expansion. As we will see later in the scalar filed theory example, each element in the second part can be understood as an S-matrix element in the full theory but expanded in terms of $\frac{1}{\Lambda}$. For instance, suppose we have the scattering process represented in Figure 2.1, the S-matrix element in the full theory is $-\lambda^2 \frac{i}{p^2 - M^2}$, and the corresponding term in $H_{decoupled}$ will be $-\lambda^2 (\frac{1}{M^2} + \frac{p^2}{M^4} + O(\frac{1}{M^6})) \frac{\Phi_L^4}{4!}$.

The last point to be addressed here is that, due to the connection between decoupled Hamiltonian and S-matrix elements, Feynman diagrams in the full theory will provide useful guidance in practical calculation, and this is the reason why we organize the calculation in chapter 3 by Feynman diagrams.

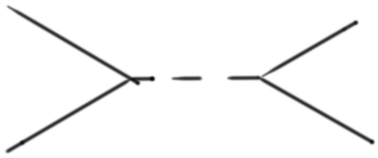


Figure 2.1 A two-to-two scattering in a scalar field theory with solid lines representing light fields Φ_L and the dashed line representing a heavy particle Φ_H. The coupling coefficient at the vertex is λ. The heavy particle mass M is much larger than the momentum p on the propagator.

2.3 Renormalization and Matching

Since the second part of $H_{decoupled}$ contains expansion of S-matrix elements, inevitably, there are UV divergences emerging from the loop calculation. Because only low energy modes can appear in $H_{decoupled}$, all the UV divergences should be canceled by the renormalization of H_1. In this way, we can determine the renormalization Z-factor of light field, light field mass and coupling constants of pure light interactions. It is shown in Section 3.3 that our results obtained from $H_{decoupled}$ indeed agree with the results from a traditional renormalization in the Lagrangian framework.

Effective Hamiltonian is obtained by matching order by order. First we decouple the full theory at tree level and match it onto the low energy physics to get the tree level effective Hamiltonian. Then we decouple, renormalize both full and effective theories to get $H_{decoupled}$ and $H^{eff}_{decoupled}$ at one loop order respectively. By matching $H_{decoupled}$ and $H^{eff}_{decoupled}$, we are able to get the effective Hamiltonian at one loop order and we can also proceed to higher orders iteratively in this fashion.

Chapter 3

Decoupling and Renormalization of A Scalar Field Theory

In this chapter we will work in the weak coupling regime of a scalar field theory with both heavy and light fields. Because we will be discussing renormalization in the Hamiltonian framework, we will consider mode expansions at a fixed time or effectively, we will be working in the Schrodinger picture.

3.1 Preliminaries

Our subsequent analysis will apply to a scalar field theory with heavy and light fields (Φ_H and Φ_L respectively) with dynamics given by the following Hamiltonian:

$$\begin{aligned} H = \int d^3x \Bigg(&\frac{1}{2}\left((\partial\Phi_L)^2 + m^2\Phi_L^2\right) + \frac{1}{2}\left((\partial\Phi_H)^2 + M^2\Phi_H^2\right) \\ &+\frac{\lambda_0}{4!}\Phi_L(x)^4 + \frac{\lambda_1}{2}\Phi_H(x)\Phi_L(x)^2 + \frac{\lambda_2}{4}\Phi_L(x)^2\Phi_H(x)^2 + \frac{\lambda_3}{4!}\Phi_H(x)^4\Bigg). \end{aligned} \tag{3.1}$$

The fields have the usual mode expansions, however, we will need to consider light fields carefully. This is because the light fields contain two parts, one is the low frequency mode $\phi(x)$ and the other is the high frequency mode $\chi(x)$. In order to correctly project onto the low energy subspace, we want only low frequency fields $\phi(x)$ to appear in external lines. This can be taken into account in the usual expansion of all the fields (in Schrodinger picture) in the following manner:

$$\phi(x) = \sum_{p<M} \frac{1}{\sqrt{2V\epsilon_p}} \left(b_p e^{i\boldsymbol{px}} + b_p^\dagger e^{-i\boldsymbol{px}}\right), \tag{3.2a}$$

$$\chi(x) = \sum_{M<p} \frac{1}{\sqrt{2V\epsilon_p}} \left(b_p e^{i\boldsymbol{px}} + b_p^\dagger e^{-i\boldsymbol{px}}\right), \tag{3.2b}$$

$$\Phi_L(x) = \sum_{p} \frac{1}{\sqrt{2V\epsilon_p}} \left(b_p e^{i\boldsymbol{px}} + b_p^\dagger e^{-i\boldsymbol{px}}\right), \tag{3.2c}$$

$$\Phi_H(x) = \sum_{k} \frac{1}{\sqrt{2V\omega_k}} \left(a_k e^{i\boldsymbol{kx}} + a_k^\dagger e^{-i\boldsymbol{kx}}\right). \tag{3.2d}$$

From the expansion, we see that the ϕ and the χ fields are orthogonal, i.e., $\int d^3x\, \phi(x)\chi(x) = 0$. In the following we will not use the mode expansion of $\phi(x)$.

As discussed earlier, we want to split the total Hamiltonian into four parts: H_1 only contains low frequency modes of light particles; H_2 contains both the free part of heavy particles and high frequency modes of light particles; H_A contains only creation or annihilation operators, for example terms like aab, $a^\dagger a^\dagger b^\dagger$, etc; and H_B contains combinations of creation and annihilation operators, for example terms like, $b^\dagger a \phi(x)$ and $a^\dagger abb$, etc. Thus,

$$H = H_1 + H_2 + H_A + H_B. \tag{3.3}$$

For our case,

$$H_1 = \int d^3x \left(\frac{1}{2}\left((\partial\phi)^2 + m^2\phi^2\right) + \frac{\lambda_0}{4!}\phi^4(x)\right), \tag{3.4a}$$

$$H_2 = \int d^3x \sum_k \omega_k a_k^\dagger a_k + \sum_{M<p} \epsilon_p b_p b_p^\dagger. \tag{3.4b}$$

However, H_A and H_B are rather involved and will not be explicitly displayed here. As we proceed with the calculation, we will simply pick out the relevant terms in these by analyzing the coupling coefficient and the number of low energy light particles.

We argued earlier that $H_{decoupled} = \langle 0_{high}|\, \omega^\dagger H \omega\, |0_{high}\rangle$, where ω denotes a series of unitary transformations, $\omega = \omega_0\omega_1 \ldots \omega_n \ldots$. Our calculations will be limited to the first loop order and for these, we only need the first two terms in

Figure 3.1 One loop two point functions. Dashed lines represent heavy fields and solid lines represent light fields.

the unitary transformations:

$$\omega^\dagger H \omega = e^{-i\Omega_1} e^{-i\Omega_0} \left(H_1 + H_2 + H_A + H_B\right) e^{i\Omega_0} e^{i\Omega_1} \tag{3.5}$$

The right hand side of the above simplifies to

$$\begin{aligned} &H_1 + H_2 + H_B + \frac{i}{2}[H_A, \Omega_0] + i[H_B, \Omega_0] - \frac{1}{3}[[H_A, \Omega_0], \Omega_0] \\ &- \frac{1}{2}[[H_B, \Omega_0], \Omega_0] - \frac{1}{2}[[H_1, \Omega_0], \Omega_0] + i[H_B, \Omega_1] + O(\frac{1}{M}) \end{aligned} \tag{3.6}$$

In deriving the above we have set the cut-off energy scale to the heavy mass M and used the condition that the unitary transformations do not take us out of the high energy vacuum, i.e., $i\,[H_2, \Omega_0] + H_A = 0$, $i\,[H_1, \Omega_0] + i\,[H_2, \Omega_1] = 0$.

In the next two sections we will study decoupling and renormalization in this scalar field theory by calculating the decoupled Hamiltonian up to order $O(\frac{1}{M^2})$ at one loop level for the two and four point functions. The calculational techniques are far removed from the usual Feynman diagram methods, however, we have noticed that the Feynman diagrams provide a very good indication of which term in the expansion, eq.(3.6) contribute to the process of interest. Thus in the following, even though we are not using the usual Feynman-Dyson perturbative expansion, we will still refer to the corresponding diagrams in guiding us as to the choice of the relevant terms in eq.(3.6).

3.2 Decoupling

3.2.1 Two Point Function Calculation

There are four contributions to the two point function at one loop order which we choose to specify through ordinary Feynman diagrams. Consider the Figure 3.1(a) diagram which comes from the $\frac{\lambda_0}{4!}\Phi_L(x)^4$ term in the total Hamiltonian and make the mode expansion for the $\phi^2\chi^2$ piece:

$$6\frac{\lambda_0}{4!}\int d^3x \sum_{M<k}\sum_{M<p}\frac{1}{\sqrt{2V\epsilon_k}}\frac{1}{\sqrt{2V\epsilon_p}}e^{i(\boldsymbol{k}-\boldsymbol{p})\boldsymbol{x}}b_k b_p^\dagger \phi^2(x). \tag{3.7}$$

Normal ordering this term gives

$$6\frac{\lambda_0}{4!}\int d^3x \sum_{M<k}\sum_{M<p}\frac{1}{\sqrt{2V\epsilon_k}}\frac{1}{\sqrt{2V\epsilon_p}}e^{i(\boldsymbol{k}-\boldsymbol{p})\boldsymbol{x}}\left(\left[b_k, b_p^\dagger\right] + b_p^\dagger b_k\right)\phi^2(x), \tag{3.8}$$

and keeping the only commutator piece we get

$$\frac{\lambda_0}{8}\frac{1}{(2\pi)^3}\int d^3x\int d^3k\,\frac{\phi^2(x)}{\sqrt{k^2+m^2}} - \frac{\lambda_0}{8}\int d^3x\int_{k<M}\frac{d^3k}{(2\pi)^3}\frac{\phi(x)^2}{\sqrt{k^2+m^2}}. \tag{3.9}$$

Using dimensional regularization ($d = 3 - 2\epsilon$), the result is

$$\begin{aligned}
&-\int d^3x\,\frac{\phi^2(x)}{2}\frac{\lambda_0 m^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} - \ln\frac{m^2}{\mu^2} + 1\right) + \frac{\lambda_0}{4}C\int d^3x\,\frac{\phi(x)^2}{2},\\
&\frac{1}{\bar{\epsilon}} = \frac{1}{\epsilon} - \gamma + \ln 4\pi,\\
&C = -\int_{k<M}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{k^2+m^2}}.
\end{aligned} \tag{3.10}$$

The term proportional to C is from the restriction imposed on the momentum of high frequency light field $\chi(x)$.

Similarly, the Figure 3.1(b) arises from the term

$$\frac{\lambda_2}{4}\int d^3x \sum_{k}\sum_{p}\frac{1}{\sqrt{2V\omega_k}}\frac{1}{\sqrt{2V\omega_p}}e^{i(\boldsymbol{k}-\boldsymbol{p})\boldsymbol{x}}a_k a_p^\dagger \phi^2(x). \tag{3.11}$$

Normal ordering this term gives

$$\frac{\lambda_2}{4}\int d^3x \sum_k \sum_p \frac{1}{\sqrt{2V\omega_k}}\frac{1}{\sqrt{2V\omega_p}} e^{i(\boldsymbol{k}-\boldsymbol{p})\boldsymbol{x}} \left(\left[a_k, a_p^\dagger\right] + a_p^\dagger a_k\right) \phi^2(x). \tag{3.12}$$

Using dimensional regularization, we can get

$$-\int d^3x \frac{\phi^2(x)}{2}\frac{\lambda_2 M^2}{32\pi^2}\left(\frac{1}{\epsilon} - \ln\frac{M^2}{\mu^2} + 1\right). \tag{3.13}$$

Figure 3.1(c) is proportional to λ_1^2 and ϕ^2. Since it is second order in coupling constant, it must come from the term $\frac{i}{2}\left[H_A, \Omega_0\right]$ in eq.(3.6). To find the H_A in this case, call it $H_A^{2,1}$ and consider the mode expansion of $\frac{\lambda_1}{2}\Phi_H\Phi_L^2$

$$\begin{aligned}\frac{\lambda_1}{2}\Phi_H\Phi_L^2 = \frac{\lambda_1}{2}\int d^3x \sum_k \sum_{M<p}\sum_{M<q} \frac{1}{(2V)^{\frac{3}{2}}\sqrt{\omega_k\epsilon_p\epsilon_q}}\left(a_k e^{i\boldsymbol{kx}} + a_k^\dagger e^{-i\boldsymbol{kx}}\right) \\ \left(b_p e^{i\boldsymbol{px}} + b_p^\dagger e^{-i\boldsymbol{px}}\right)\left(b_q e^{i\boldsymbol{qx}} + b_q^\dagger e^{-i\boldsymbol{qx}}\right) + \frac{\lambda_1}{2}\int d^3x \sum_k \frac{1}{\sqrt{2V\omega_k}} \\ \phi^2(x)\left(a_k e^{i\boldsymbol{kx}} + a_k^\dagger e^{-i\boldsymbol{kx}}\right) + \ldots\end{aligned} \tag{3.14}$$

where dots represent terms proportional to $\phi(x)$.

This expansion has a piece proportional to $\frac{\lambda_1}{2}\int d^3x \sum_k \frac{1}{\sqrt{2V\omega_k}}\left(a_k e^{i\boldsymbol{kx}} + a_k^\dagger e^{-i\boldsymbol{kx}}\right)\phi(x)^2$ in the $H_A^{2,1}$. In addition, there's another term, which is of the form of H_B:

$$\frac{\lambda_1}{2}\int d^3x \sum_k \sum_{M<p}\sum_{M<q} \frac{1}{(2V)^{\frac{3}{2}}\sqrt{\omega_k\epsilon_p\epsilon_q}}\left(a_k e^{i\boldsymbol{kx}} + a_k^\dagger e^{-i\boldsymbol{kx}}\right) b_p e^{i\boldsymbol{px}} b_q^\dagger e^{-i\boldsymbol{qx}}. \tag{3.15}$$

Normal ordering this gives,

$$\frac{\lambda_1}{2}\int d^3x \sum_k \sum_{M<p}\sum_{M<q} \frac{1}{(2V)^{\frac{3}{2}}\sqrt{\omega_k\epsilon_p\epsilon_q}}\left(a_k e^{i\boldsymbol{kx}} + a_k^\dagger e^{-i\boldsymbol{kx}}\right) e^{i(\boldsymbol{k}-\boldsymbol{p})\boldsymbol{x}}\left(\left[b_p, b_q^\dagger\right] + b_q^\dagger b_p\right). \tag{3.16}$$

Finally, putting this together we get the net contribution of $H_A^{2,1}$ to be

$$H_A^{2,1} = \int d^3x \frac{\lambda_1}{2}\sum_k \frac{1}{\sqrt{2V\omega_k}}\left(a_k e^{i\boldsymbol{kx}} + a_k^\dagger e^{-i\boldsymbol{kx}}\right)\left(-\frac{m^2}{16\pi^2}\left(\frac{1}{\epsilon} - \ln\frac{m^2}{\mu^2} + 1\right) + \frac{C}{2} + \phi(x)^2\right), \tag{3.17}$$

where again $C = -\int_{k<M} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{k^2+m^2}}$ and is from the restriction on $\chi(x)$'s momentum. Denoting the contribution to Ω_0 in this case by $\Omega_0^{2,1}$ we get from the condition that $i\left[H_2, \Omega_0^{2,1}\right] + H_A^{2,1} = 0$,

$$\Omega_0^{2,1} = \int d^3y \, \frac{\lambda_1}{2} \sum_p \frac{-i}{\sqrt{2V\omega_p}\omega_p} \left(a_p e^{i\boldsymbol{py}} - a_p^\dagger e^{-i\boldsymbol{py}}\right) \left(-\frac{m^2}{16\pi^2}\left(\frac{1}{\epsilon} - \ln\frac{m^2}{\mu^2} + 1\right) + \frac{C}{2} + \phi(y)^2\right). \tag{3.18}$$

Then we get,

$$\begin{aligned}
\frac{i}{2}\left[H_A^{2,1}, \Omega_0^{2,1}\right] &= -\frac{i}{2} \int\int d^3x d^3y \, \frac{\lambda_1^2}{4} \sum_k \sum_p \frac{-i}{2V\sqrt{\omega_k\omega_p}\omega_p} \left[a_k, -a_p^\dagger\right] e^{i\boldsymbol{kx}} e^{-i\boldsymbol{py}} \\
&\quad \left(\frac{m^2}{16\pi^2}\left(\frac{1}{\epsilon} - \ln\frac{m^2}{\mu^2} + 1\right) - \frac{C}{2}\right)\phi(y)^2 + \ldots \\
&= \int\int d^3x d^3y \, \frac{\lambda_1^2}{16} \sum_k \frac{1}{\omega_k^2} e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})} \left(\frac{m^2}{16\pi^2}\left(\frac{1}{\epsilon} - \ln\frac{m^2}{\mu^2} + 1\right) - \frac{C}{2}\right)\phi(y)^2 + \ldots,
\end{aligned} \tag{3.19}$$

where the dots denote three other similar terms arising from the commutator. Since momentum k is associated with external lines, we have $k \ll M$, and $\omega_k^2 \simeq M^2$. Including all these contributions we get

$$\begin{aligned}
\frac{i}{2}\left[H_A^{2,1}, \Omega_0^{2,1}\right] &= \int\int d^3x d^3y \, \frac{4\lambda_1^2}{16} \left(\frac{m^2}{16\pi^2}\left(\frac{1}{\epsilon} - \ln\frac{m^2}{\mu^2} + 1\right) - \frac{C}{2}\right)\phi(y)^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{M^2} e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})}, \\
&= \int d^3x \, \frac{\lambda_1^2 m^2}{32\pi^2 M^2} \left(\frac{1}{\epsilon} - \ln\frac{m^2}{\mu^2} + 1\right) \frac{\phi(x)^2}{2} - \frac{\lambda_1^2 C}{4M^2} \int d^3x \, \frac{\phi(x)^2}{2}.
\end{aligned} \tag{3.20}$$

Now let's consider Figure 3.1(d) which is the last contribution to the two point function at one loop order. Figure 3.1(d) has both light particle and heavy particle propagators. It arises also from the expansion of $\frac{\lambda_1}{2}\Phi_H\Phi_L^2$ and is not listed in eq.(3.14). This contribution is

$$H_A^{2,2} = 2\int d^3x \sum_k \sum_{M<p} \frac{\lambda_1}{2} \frac{\phi(x)}{2V\sqrt{\omega_k\epsilon_p}} \left(a_k e^{i\boldsymbol{kx}} b_p e^{i\boldsymbol{px}} + h.c.\right). \tag{3.21}$$

Similarly, from $i\left[H_2, \Omega_0^{2,2}\right] + H_A^{2,2} = 0$, we can get the $\Omega_0^{2,2}$ that corresponds to $H_A^{2,2}$,

$$\Omega_0^{2,2} = 2\int d^3y \sum_q \sum_{M<r} \frac{\lambda_1}{2} \left(\frac{-i\phi(x)}{2V\sqrt{\omega_q \epsilon_r}\left(\omega_q + \epsilon_r\right)} a_q e^{i\boldsymbol{kx}} b_r e^{i\boldsymbol{px}} + h.c. \right). \tag{3.22}$$

Thus, we need to calculate $\frac{i}{2}\left[H_A^{2,2}, \Omega_0^{2,2}\right]$. After a simple calculation, and normal ordering we can get:

$$\begin{aligned}\frac{i}{2}\left[H_A^{2,2}, \Omega_0^{2,2}\right] &= -\int\int d^3x d^3y \sum_k \sum_{p<M} \lambda_1^2 \frac{\phi(x)\phi(y)}{4V^2\omega_k\epsilon_p\left(\omega_k + \epsilon_p\right)} e^{i(\boldsymbol{k}+\boldsymbol{p})(\boldsymbol{x}-\boldsymbol{y})}, \\ &= -\int\int d^3x d^3y \sum_k \sum_{p<M} \lambda_1^2 \frac{\phi(x)\phi(y)e^{i(\boldsymbol{k}+\boldsymbol{p})(\boldsymbol{x}-\boldsymbol{y})}}{4V^2\sqrt{k^2+M^2}\sqrt{p^2+m^2}\left(\sqrt{k^2+M^2}+\sqrt{p^2+m^2}\right)}.\end{aligned} \tag{3.23}$$

In order to calculate this complicated integral, we need to split the forbidding fraction into two parts:

$$\begin{aligned}&\frac{1}{\sqrt{k^2+M^2}\sqrt{p^2+m^2}\left(\sqrt{k^2+M^2}+\sqrt{p^2+m^2}\right)}, \\ =&\frac{\sqrt{k^2+M^2}-\sqrt{p^2+m^2}}{\sqrt{k^2+M^2}\sqrt{p^2+m^2}\left(k^2+M^2-p^2-m^2\right)}, \\ =&\frac{1}{\sqrt{p^2+m^2}\left(k^2+M^2-p^2-m^2\right)} - \frac{1}{\sqrt{k^2+M^2}\left(k^2+M^2-p^2-m^2\right)}.\end{aligned} \tag{3.24}$$

Let $\boldsymbol{k}+\boldsymbol{p}=\boldsymbol{r}$. We know $\boldsymbol{r}$ is total external momentum, thus r is much smaller than M, which means $(k^2-p^2) \ll M^2$. Hence we can write:

$$\begin{aligned}&\frac{1}{\sqrt{p^2+m^2}\left(k^2+M^2-p^2-m^2\right)}, \\ =&\frac{1}{\sqrt{p^2+m^2}M^2\left(1-\frac{m^2}{M^2}+\frac{k^2-p^2}{M^2}\right)}, \\ \simeq&\frac{1}{\sqrt{p^2+m^2}M^2}\left(1+\frac{m^2}{M^2}-\frac{k^2-p^2}{M^2}+\left(\frac{k^2-p^2}{M^2}\right)^2\right).\end{aligned} \tag{3.25}$$

To the order $\frac{1}{M^2}$ this gives

$$\int d^3x \frac{m^2\lambda_1^2}{16\pi^2M^2}\left(\frac{1}{\bar{\epsilon}} - \ln\frac{m^2}{\mu^2} + 1\right)\frac{\phi(x)^2}{2} - \frac{\lambda_1^2 C}{2M^2}\int d^3x\, \frac{\phi(x)^2}{2}. \tag{3.26}$$

Similarly, the second term from eq.(3.24) gives

$$\frac{1}{\sqrt{k^2+M^2}M^2}\left(1+\frac{m^2}{M^2}-\frac{k^2-p^2}{M^2}+\left(\frac{k^2-p^2}{M^2}\right)^2\right). \tag{3.27}$$

Straightforwardly, the first two terms in eq.(3.27) gives

$$\begin{aligned}
&\int\int d^3xd^3y\,\lambda_1^2\phi(x)\phi(y)\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3r}{(2\pi)^3}\frac{1}{4\sqrt{k^2+M^2}M^2}e^{i\boldsymbol{r}(\boldsymbol{x}-\boldsymbol{y})}\left(1+\frac{m^2}{M^2}\right),\\
&=-\int d^3x\frac{\lambda_1^2}{16\pi^2}\left(\frac{1}{\bar{\epsilon}}-\ln\frac{M^2}{\mu^2}+1\right)\frac{\phi(x)^2}{2}\left(1+\frac{m^2}{M^2}\right).
\end{aligned} \tag{3.28}$$

The last two terms are a bit tricky to handle. First, we need to do some modification

$$\begin{aligned}
k^2-p^2&=(\boldsymbol{k}+\boldsymbol{p})(\boldsymbol{k}-\boldsymbol{p}),\\
&=2\boldsymbol{kr}-r^2.
\end{aligned} \tag{3.29}$$

Then we have from the $\frac{k^2-p^2}{M^2}$ term:

$$\begin{aligned}
&\int\int d^3xd^3y\,\lambda_1^2\phi(x)\phi(y)\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3r}{(2\pi)^3}\frac{1}{4\sqrt{k^2+M^2}M^2}e^{i\boldsymbol{r}(\boldsymbol{x}-\boldsymbol{y})}\left(-\frac{k^2-p^2}{M^2}\right),\\
&=\int\int d^3xd^3y\,\lambda_1^2\phi(x)\phi(y)\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3r}{(2\pi)^3}\frac{1}{4\sqrt{k^2+M^2}M^2}e^{i\boldsymbol{r}(\boldsymbol{x}-\boldsymbol{y})}\left(\frac{-2\boldsymbol{kr}+r^2}{M^2}\right),\\
&=\int\int d^3xd^3y\,\lambda_1^2\phi(x)\phi(y)\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3r}{(2\pi)^3}\frac{1}{4\sqrt{k^2+M^2}M^2}e^{i\boldsymbol{r}(\boldsymbol{x}-\boldsymbol{y})}\frac{r^2}{M^2},\\
&=\int\int d^3(x)d^3(y)\,\frac{\lambda_1^2}{4}\phi(x)\phi(y)\int\frac{d^3k}{(2\pi)^3}\frac{1}{M^2\sqrt{k^2+M^2}}\int\frac{d^3r}{(2\pi)^3}r^2e^{i\boldsymbol{r}(\boldsymbol{x}-\boldsymbol{y})},\\
&=\int\int d^3(x)d^3(y)\,\frac{\lambda_1^2}{4}\phi(x)\phi(y)\int\frac{d^3k}{(2\pi)^3}\frac{1}{M^2\sqrt{k^2+M^2}}\left(\delta^3(\boldsymbol{x}-\boldsymbol{y})\partial_y^2\right),\\
&=\int d^3(x)\,\frac{\lambda_1^2}{4}\phi(x)\partial^2\phi(x)\int\frac{d^3k}{(2\pi)^3}\frac{1}{M^2\sqrt{k^2+M^2}},\\
&=\int d^3(x)\,\frac{\lambda_1^2}{16\pi^2M^2}\left(\frac{1}{\bar{\epsilon}}-\ln\frac{M^2}{\mu^2}+1\right)\frac{1}{2}(\partial\phi(x))^2.
\end{aligned} \tag{3.30}$$

To the order $\frac{2}{M^2}$, we also need to consider the $\frac{4(\boldsymbol{kr})^2}{M^4}$ term from $\left(\frac{k^2-p^2}{M^2}\right)^2$:

$$\begin{aligned}
&\int\int d^3xd^3y\,\lambda_1^2\phi(x)\phi(y)\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3r}{(2\pi)^3}\frac{1}{4\sqrt{k^2+M^2}M^2}e^{i\boldsymbol{r}(\boldsymbol{x}-\boldsymbol{y})}\frac{4\left(\boldsymbol{kr}\right)^2}{M^4},\\
&=\int\int d^3xd^3y\,\lambda_1^2\phi(x)\phi(y)\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{k^2+M^2}M^6}\int\frac{d^3r}{(2\pi)^3}\left(\boldsymbol{kr}\right)^2e^{i\boldsymbol{r}(\boldsymbol{x}-\boldsymbol{y})},\\
&=\int\int d^3xd^3y\,\lambda_1^2\phi(x)\phi(y)\int\frac{d^3r}{(2\pi)^3}r^2e^{i\boldsymbol{r}(\boldsymbol{x}-\boldsymbol{y})}\int\frac{d^3k}{(2\pi)^3}\frac{k^2\cos^2\theta}{\sqrt{k^2+M^2}M^6},\\
&=\int\int d^3xd^3y\,\lambda_1^2\phi(x)\phi(y)\int\frac{d^3r}{(2\pi)^3}r^2e^{i\boldsymbol{r}(\boldsymbol{x}-\boldsymbol{y})}\frac{1}{32\pi^2M^2}\left(\frac{1}{\epsilon}-\ln\frac{M^2}{\mu^2}+\frac{3}{2}\right),\\
&=\int\int d^3xd^3y\,\lambda_1^2\phi(x)\delta^3(\boldsymbol{x}-\boldsymbol{y})\partial_y^2\phi(y)\frac{1}{32\pi^2M^2}\left(\frac{1}{\epsilon}-\ln\frac{M^2}{\mu^2}+\frac{3}{2}\right),\\
&=-\int d^3x\,\frac{\lambda_1^2}{16\pi^2M^2}\left(\frac{1}{\epsilon}-\ln\frac{M^2}{\mu^2}+\frac{3}{2}\right)\frac{1}{2}(\partial\phi(x))^2.
\end{aligned} \tag{3.31}$$

Putting together all the previous results, we can get the total contribution from Figure 3.1(d) as

$$\begin{aligned}
&\frac{m^2\lambda_1^2}{16\pi^2M^2}\left(\frac{1}{\epsilon}-\ln\frac{m^2}{\mu^2}+1\right)\int d^3x\,\frac{\phi(x)^2}{2}-\frac{\lambda_1^2C}{2M^2}\int d^3x\,\frac{\phi(x)^2}{2}\\
&-\frac{\lambda_1^2}{16\pi^2}\left(\frac{1}{\epsilon}-\ln\frac{M^2}{\mu^2}+1\right)\left(1+\frac{m^2}{M^2}\right)\int d^3x\,\frac{\phi(x)^2}{2}-\frac{\lambda_1^2}{32\pi^2M^2}\int d^3x\,\frac{1}{2}(\partial\phi(x))^2.
\end{aligned} \tag{3.32}$$

Putting contributions from all the diagrams together and taking $\mu\approx M$ to get rid of log terms $\ln\frac{M^2}{\mu^2}$, we have the net result of two point functions as

$$\begin{aligned}
&-\frac{\lambda_0m^2}{32\pi^2}\left(\frac{1}{\epsilon}-\ln\frac{m^2}{M^2}+1\right)\int d^3x\,\frac{\phi(x)^2}{2}-\frac{\lambda_2M^2}{32\pi^2}\left(\frac{1}{\epsilon}+1\right)\int d^3x\,\frac{\phi(x)^2}{2}\\
&+\frac{3\lambda_1^2m^2}{32\pi^2M^2}\left(\frac{1}{\epsilon}-\ln\frac{m^2}{M^2}+1\right)\int d^3x\,\frac{\phi(x)^2}{2}-\frac{\lambda_1^2}{16\pi^2}\left(\frac{1}{\epsilon}+1\right)\left(1+\frac{m^2}{M^2}\right)\int d^3x\,\frac{\phi(x)^2}{2}\\
&-\frac{\lambda_1^2}{32\pi^2M^2}\int d^3x\,\frac{1}{2}(\partial\phi(x))^2+\left(\frac{\lambda_0C}{4}-\frac{3\lambda_1^2C}{2M^2}\right)\int d^3x\,\frac{\phi(x)^2}{2}.
\end{aligned} \tag{3.33}$$

There are several finite contributions proportional to $C=-\int_{k<M}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{k^2+m^2}}$, however they will not affect the renormalization of H_1 as we will show in section 3.3. In fact, they will only appear in $H_{decoupled}$ and get canceled out in the physical effective Hamiltonian H_{eff} during the matching which will be shown in chapter 4. Another way to think about the effects of these finite terms is that whenever we

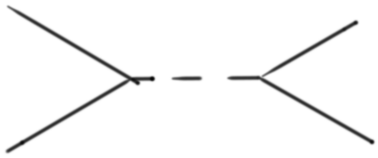
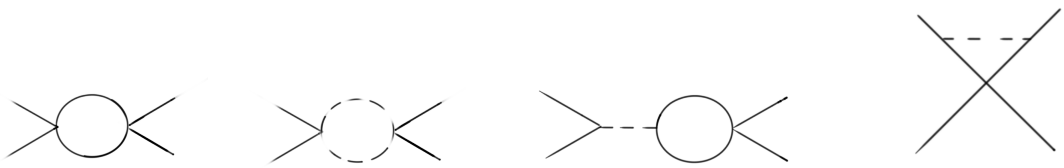


Figure 3.2 Four point function at tree level.



(a) (b) (c) (d)

Figure 3.3 Four point functions at one loop level.

have a light field in the loop, our calculation will produce these finite terms along with (in a linear way) the "troublesome" large log terms $\ln \frac{m^2}{M^2}$. Therefore, as long as the large log terms can be canceled during matching, we can persuade ourselves that these extra finite terms will get canceled and will not appear in H_{eff} as well.

3.2.2 Four Point Function Calculation

In the calculation of Figure 3.1(c) in section 3.2.1, there is one more term from the commutator $\frac{i}{2}\left[H_A^{2,1}, \Omega_0^{2,1}\right]$ left undiscussed that contributes to the four point function at tree level shown in Figure 3.2

$$\begin{aligned}\frac{i}{2}\left[H_A^{2,1}, \Omega_0^{2,1}\right]_{tree} &= i\int d^3x \int d^3y\, \frac{\lambda_1^2}{4}\sum_k\sum_p \frac{-i}{2V\sqrt{\omega_k\omega_p}\omega_p}\phi(x)^2\phi(y)^2[a_k, -a_p^\dagger]e^{i\boldsymbol{kx}-i\boldsymbol{py}},\\ &= -\int d^3x\int d^3y \int\frac{d^3k}{(2\pi)^3}\frac{\lambda_1^2}{8(k^2+M^2)}\phi(x)^2\phi(y)^2 e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})},\\ &\approx -\frac{3\lambda_1}{M^2}\int d^3x\,\frac{\phi(x)^4}{4!},\end{aligned} \tag{3.34}$$

where we have used the condition that k is associated with external momenta and therefore $k \ll M$.

In the loop calculation we will encounter extra finite terms similar to those we have discussed in section 3.2.1. However we will omit them in our calculation and check in chapter 4 that all the large log terms $\ln \frac{m^2}{M^2}$ produced in four point function calculation are canceled out, which implies these extra finite terms will be canceled too. Furthermore, we will set external momenta to 0, an assumption that will simplify calculation greatly but won't damage the physical essence of our theory.

Let's consider Figure 3.3(a). This diagram is of order λ_0^2, and it arises from the term $\frac{i}{2}\left[H_A^{4,1}, \Omega_0^{4,1}\right]$ where $H_A^{4,1}$ comes from the mode expansion of $\frac{\lambda_0}{4!}\Phi_L(x)^4$. In this expansion we must pick a term of form $\phi^2\chi^2$ which gives

$$H_A = \int d^3x\, 6\frac{\lambda_0}{4!}\phi(x)^2 \sum_k \sum_p \frac{1}{2V\sqrt{\epsilon_p \epsilon_k}} \left(b_p b_k e^{i(\boldsymbol{p}+\boldsymbol{k})\boldsymbol{x}} + b_p^\dagger b_k^\dagger e^{-i(\boldsymbol{p}+\boldsymbol{k})\boldsymbol{x}}\right). \tag{3.35}$$

From the equation $i\left[H_2, \Omega_0^{4,1}\right] + H_A^{4,1} = 0$, we can get the corresponding $\Omega_0^{4,1}$ to be

$$\Omega_0^{4,1} = \int d^3y\, \frac{\lambda_0}{4}\phi(y)^2 \sum_{k'} \sum_{p'} \left(\frac{-ie^{i\left(\boldsymbol{k}'+\boldsymbol{p}'\right)\boldsymbol{y}}}{2V\sqrt{\epsilon_{k'}\epsilon_{p'}}\left(\epsilon_{k'}+\epsilon_{p'}\right)} b_{k'} b_{p'} + \frac{ie^{-i\left(\boldsymbol{k}'+\boldsymbol{p}'\right)\boldsymbol{y}}}{2V\sqrt{\epsilon_{k'}\epsilon_{p'}}\left(\epsilon_{k'}+\epsilon_{p'}\right)} b_{k'}^\dagger b_{p'}^\dagger \right). \tag{3.36}$$

Then $\frac{i}{2}\left[H_A^{4,1}, \Omega_0^{4,1}\right]$ yields

$$-\frac{3\lambda_0^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} - \ln\frac{m^2}{\mu^2}\right) \int d^3x\, \frac{\phi(x)^4}{4!}. \tag{3.37}$$

In this calculation, we have used the approximation that total external momentum is zero, which means $\boldsymbol{k}+\boldsymbol{p}=0$ and $\epsilon_k = \epsilon_p = \sqrt{k^2+m^2}$.

The next contribution is from Figure 3.3(b). The calculation is similar to the first one except the corresponding H_A is different. In this case we have

$$H_A^{4,2} = \int d^3x\, \frac{\lambda_2}{4}\phi(x)^2 \sum_k \sum_p \frac{1}{2V\sqrt{\omega_p\omega_k}} \left(a_p a_k e^{i(\boldsymbol{p}+\boldsymbol{k})\boldsymbol{x}} + a_p^\dagger a_k^\dagger e^{-i(\boldsymbol{p}+\boldsymbol{k})\boldsymbol{x}}\right), \tag{3.38a}$$

$$\Omega_0^{4,2} = \int d^3y\, \frac{\lambda_2}{4}\phi(y)^2 \sum_{k'} \sum_{p'} \left(\frac{-ie^{i\left(\boldsymbol{k}'+\boldsymbol{p}'\right)\boldsymbol{y}}}{2V\sqrt{\omega_{k'}\omega_{p'}}\left(\omega_{k'}+\omega_{p'}\right)} a_{k'} a_{p'} + \frac{ie^{-i\left(\boldsymbol{k}'+\boldsymbol{p}'\right)\boldsymbol{y}}}{2V\sqrt{\omega_{k'}\omega_{p'}}\left(\omega_{k'}+\omega_{p'}\right)} a_{k'}^\dagger a_{p'}^\dagger \right). \tag{3.38b}$$

$H_A^{4,3}$	$\Omega_0^{4,3,1}$	$\Omega_0^{4,3,2}$
First combination (a)		
$\lambda_1 a\phi^2$	$\lambda_1 a^\dagger b^\dagger b^\dagger$	$\lambda_0 bb\phi^2$
$\lambda_1 a^\dagger \phi^2$	$\lambda_1 abb$	$\lambda_0 b^\dagger b^\dagger \phi^2$
Second combination (b)		
$\lambda_1 abb$	$\lambda_1 a^\dagger \phi^2$	$\lambda_0 b^\dagger b^\dagger \phi^2$
$\lambda_1 a^\dagger b^\dagger b^\dagger$	$\lambda_1 a\phi^2$	$\lambda_0 bb\phi^2$
Third combination (c)		
$\lambda_1 abb$	$\lambda_0 b^\dagger b^\dagger \phi^2$	$\lambda_1 a^\dagger \phi^2$
$\lambda_1 a^\dagger b^\dagger b^\dagger$	$\lambda_0 bb\phi^2$	$\lambda_1 a\phi^2$
Fourth combination (d)		
$\lambda_0 bb\phi^2$	$\lambda_1 a^\dagger b^\dagger b^\dagger$	$\lambda_1 a\phi^2$
$\lambda_0 b^\dagger b^\dagger \phi^2$	$\lambda_1 abb$	$\lambda_1 a^\dagger \phi^2$

Table 3.1: Combinations of $-\frac{1}{3}\left[\left[H_A^{4,3}, \Omega_0^{4,3,1}\right], \Omega_0^{4,3,2}\right]$ that contribute to Figure 3.3(c).

Following the similar calculation we are able to get the result:

$$\frac{i}{2}\left[H_A^{4,2}, \Omega_0^{4,2}\right] = -\frac{3\lambda_2^2}{32\pi^2}\left(\frac{1}{\epsilon} - \ln\frac{M^2}{\mu^2}\right)\int d^3x\,\frac{\phi(x)^4}{4!}. \tag{3.39}$$

Let's now consider the contributions that correspond to Figure 3.3(c). Notice that, in this diagram, we have the product of coupling constants as $\lambda_0\lambda_1^2$, which means we need $-\frac{1}{3}\left[\left[H_A, \Omega_0\right], \Omega_0\right] - \frac{1}{2}\left[\left[H_B, \Omega_0\right], \Omega_0\right]$ from eq.(3.6). For clarity, we label H_A in the commutator as $H_A^{4,3}$, the first Ω_0 next to $H_A^{4,3}$ as $\Omega_0^{4,3,1}$, the second Ω_0 as $\Omega_0^{4,3,2}$ and H_B as $H_B^{4,3}$. There are many ways to pick $H_A^{4,3}$, $\Omega_0^{4,3,1}$, and $\Omega_0^{4,3,2}$ in $-\frac{1}{3}\left[\left[H_A^{4,3}, \Omega_0^{4,3,1}\right], \Omega_0^{4,3,2}\right]$, and we split them into four kinds of combinations shown in Table 3.1. We will show the explicit calculation of the first combination Table 3.1(a) and put the calculation of other combinations in the appendix A.

Considering the following terms from the first combination Table 3.1(a):

$$H_A^{4,3} = \frac{\lambda_1}{2}\int d^3x \sum_k \frac{\phi(x)^2}{\sqrt{2V\omega_k}} a_k e^{i\boldsymbol{kx}}, \tag{3.40a}$$

$$\Omega_0^{4,3,1} = \frac{\lambda_1}{2}\int d^3y \sum_{k'}\sum_p\sum_q \frac{ie^{-i\left(\boldsymbol{k}'+\boldsymbol{p}+\boldsymbol{q}\right)\boldsymbol{y}}}{(2V)^{3/2}\sqrt{\omega_{k'}\epsilon_p\epsilon_q}\left(\omega_{k'}+\epsilon_p+\epsilon_q\right)} a_{k'}^\dagger b_p^\dagger b_q^\dagger, \tag{3.40b}$$

$$\Omega_0^{4,3,2} = \frac{\lambda_0}{4}\int d^3z \sum_{p'}\sum_{q'} \phi(z)^2 \frac{-ie^{i\left(\boldsymbol{p}'+\boldsymbol{q}'\right)\boldsymbol{z}}}{2V\sqrt{\epsilon_{p'}\epsilon_{q'}}\left(\epsilon_{p'}+\epsilon_{q'}\right)} b_{p'} b_{q'}. \tag{3.40c}$$

We first calculate $\left[H_A^{4,3}, \Omega_0^{4,3,1}\right]$

$$\begin{aligned}\left[H_A^{4,3}, \Omega_0^{4,3,1}\right] &= \frac{\lambda_1^2}{4}\int d^3x \int d^3y\, \phi(x)^2 \sum_k \sum_{k'} \sum_p \sum_q \frac{i e^{i\boldsymbol{kx}} e^{-i\left(\boldsymbol{k}'+\boldsymbol{p}+\boldsymbol{q}\right)\boldsymbol{y}}}{(2V)^2 \sqrt{\omega_k \omega_{k'} \epsilon_p \epsilon_q}\left(\omega_{k'}+\epsilon_p+\epsilon_q\right)} \\ &\quad \left[a_k, a_{k'}^\dagger b_p^\dagger b_q^\dagger\right], \\ &= \frac{\lambda_1^2}{4}\int d^3x \int d^3y\, \phi(x)^2 \sum_k \sum_p \sum_q \frac{i e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})} e^{-i(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{y}}}{(2V)^2 \omega_k \sqrt{\epsilon_p \epsilon_q}\left(\omega_k+\epsilon_p+\epsilon_q\right)} b_p^\dagger b_q^\dagger, \\ &= \frac{i\lambda_1^2}{16}\int d^3x\, \phi(x)^2 \sum_p \sum_q \frac{e^{-i(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{x}}}{VM\sqrt{\epsilon_p\epsilon_q}\left(M+\epsilon_p+\epsilon_q\right)} b_p^\dagger b_q^\dagger.\end{aligned} \tag{3.41}$$

In the above calculation, we also use the assumption that total external momentum is 0, which implies the momentum k associated with heavy particle is therefore 0 and $w_k = M$. Then we have

$$\begin{aligned}-\frac{1}{3}\left[\left[H_A^{4,3}, \Omega_0^{4,3,1}\right], \Omega_0^{4,3,2}\right] &= -\frac{\lambda_1^2\lambda_0}{192}\int d^3x \int d^3z \sum_p \sum_q \sum_{p'} \sum_{q'} \phi(x)^2\phi(z)^2 \\ &\quad \frac{e^{i\left(\boldsymbol{p}'+\boldsymbol{q}'\right)\boldsymbol{z}-i(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{x}}}{2V^2M\sqrt{\epsilon_p\epsilon_q\epsilon_{p'}\epsilon_{q'}}\left(\epsilon_{p'}+\epsilon_{q'}\right)\left(M+\epsilon_p+\epsilon_q\right)}\left[b_p^\dagger b_q^\dagger, b_{p'} b_{q'}\right], \\ &= \frac{\lambda_1^2\lambda_0}{192}\int d^3x \int d^3z \sum_p \sum_q \phi(x)^2\phi(z)^2 \frac{e^{i(\boldsymbol{p}+\boldsymbol{q})(\boldsymbol{x}-\boldsymbol{z})}}{2V^2M\epsilon_p^3\left(M+2\epsilon p\right)}, \\ &= \frac{\lambda_1^2\lambda_0}{192}\int d^3x\, \phi(x)^4 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2V^2M\epsilon_p^3\left(M+2\epsilon_p\right)},\end{aligned} \tag{3.42}$$

where we have used the fact that $\boldsymbol{p}+\boldsymbol{q} = \boldsymbol{k} = 0$, and thus $\epsilon_p = \epsilon_q$. Finally, taking the Hermitian conjugation into consideration, we have the result of combination Table 3.1(a) to be

$$\frac{\lambda_1^2\lambda_0}{96}\int d^3x\, \phi(x)^4 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2V^2M\epsilon_p^3\left(M+2\epsilon_p\right)}. \tag{3.43}$$

Combinations Table 3.1(b) and Table 3.1(c) will yield the same result:

$$\frac{\lambda_1^2\lambda_0}{96}\int d^3x\, \phi(x)^4 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2M^2\epsilon_p^3}. \tag{3.44}$$

$H_B^{4,3}$	$\Omega_0^{4,3,1}$	$\Omega_0^{4,3,2}$
$\lambda_1 b^\dagger b^\dagger a$	$\lambda_0 bb\phi^2$	$\lambda_1 a^\dagger \phi$
$\lambda_1 b^\dagger b^\dagger a$	$\lambda_1 a^\dagger \phi^2$	$\lambda_0 bb\phi^2$
$\lambda_1 a^\dagger bb$	$\lambda_1 a\phi^2$	$\lambda_0 b^\dagger b^\dagger \phi^2$
$\lambda_1 a^\dagger bb$	$\lambda_0 b^\dagger b^\dagger \phi^2$	$\lambda_1 a\phi^2$

Table 3.2: Combinations of $-\frac{1}{2}\left[\left[H_B^{4,3}, \Omega_0^{4,3,1}\right], \Omega_0^{4,3,2}\right]$ that contribute to Figure 3.3(c).

Table 3.1(d) will give

$$\frac{\lambda_1^2\lambda_0}{96}\int d^3x\,\phi(x)^4\int\frac{d^3p}{(2\pi)^3}\frac{1}{M^2\epsilon_p^2(M+2\epsilon_p)}. \tag{3.45}$$

Adding all four kinds of combinations together, we will get

$$\begin{aligned}
&\frac{\lambda_1^2\lambda_0}{96}\int d^3x\,\phi(x)^4\int\frac{d^3p}{(2\pi)^3}\left(\frac{1}{M^2\epsilon_p^2\left(M+2\epsilon_p\right)}+\frac{1}{2\epsilon_p^3 M\left(M+2\epsilon_p\right)}+\frac{1}{M^2\epsilon_p^3}\right),\\
&=\frac{\lambda_1^2\lambda_0}{64}\int d^3x\phi(x)^4\int\frac{d^3p}{(2\pi)^3}\frac{1}{M^2\epsilon_p^3},\\
&=\frac{3\lambda_1^2\lambda_0}{32\pi^2M^2}\int d^3x\,\frac{\phi(x)^4}{4!}\left(\frac{1}{\bar\epsilon}-\ln\frac{m^2}{\mu^2}\right).
\end{aligned} \tag{3.46}$$

The combinations of $-\frac{1}{2}\left[\left[H_B^{4,3}, \Omega_0^{4,3,1}\right], \Omega_0^{4,3,2}\right]$ are shown in Table 3.2, and the result is

$$\frac{3\lambda_1^2\lambda_0}{32\pi^2M^2}\int d^3x\,\frac{\phi(x)^4}{4!}\left(\frac{1}{\bar\epsilon}-\ln\frac{m^2}{\mu^2}\right). \tag{3.47}$$

Therefore, the total contribution from Figure 3.3(c) is

$$\frac{3\lambda_1^2\lambda_0}{16\pi^2M^2}\int d^3x\,\frac{\phi(x)^4}{4!}\left(\frac{1}{\bar\epsilon}-\ln\frac{m^2}{\mu^2}\right). \tag{3.48}$$

As for the Figure 3.3(d), only the term $-\frac{1}{2}\left[\left[H_B^{4,4}, \Omega_0^{4,4,1}\right], \Omega_0^{4,4,2}\right]$ will contribute. Similarly, we divide the whole commutator into several kinds of combinations shown in Table 3.3 and calculate them respectively. We will give the resul of each kind of combination directly and put the explicit calculation in appendix A.

Table 3.3(a) will give

$$\frac{\lambda_0\lambda_1^2}{16}\int d^3x\,\phi(x)^4\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k\epsilon_k^2\left(\omega_k+\epsilon_k\right)^2}. \tag{3.49}$$

$H_B^{4,4}$	$\Omega_0^{4,4,1}$	$\Omega_0^{4,4,2}$
First combination (a)		
$\lambda_0 b^\dagger b^\dagger \phi^2$	$\lambda_1 a^\dagger b^\dagger \phi$	$\lambda_1 ab\phi$
$\lambda_0 b^\dagger b \phi^2$	$\lambda_1 ab\phi$	$\lambda_1 a^\dagger b^\dagger \phi$
Second combination (b)		
$\lambda_1 b^\dagger a \phi$	$\lambda_1 a^\dagger b^\dagger \phi$	$\lambda_0 bb\phi^2$
$\lambda_1 a^\dagger b \phi$	$\lambda_1 ab\phi$	$\lambda_0 b^\dagger b^\dagger \phi^2$
Third combination (c)		
$\lambda_1 b^\dagger a \phi$	$\lambda_0 bb\phi^2$	$\lambda_1 a^\dagger b^\dagger \phi$
$\lambda_1 a^\dagger b^\dagger \phi$	$\lambda_0 b^\dagger b^\dagger \phi^2$	$\lambda_1 ab\phi$

Table 3.3: Combinations of $-\frac{1}{2}\left[\left[H_B^{4,4}, \Omega_0^{4,4,1}\right], \Omega_0^{4,4,2}\right]$ that contribute to Figure 3.3(c).

Since Table 3.3(b) and Table 3.3(c) only differ from an exchange of $\Omega_0^{4,4,1}$ and $\Omega_0^{4,4,2}$, they will give the same result:

$$\frac{\lambda_1^2 \lambda_0}{32} \int d^3x\, \phi(x)^4 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_p \epsilon_p^3 (\omega_p + \epsilon_p)}. \tag{3.50}$$

Adding all three results together we will have

$$-\frac{1}{2}\left[\left[H_B^{4,4}, \Omega_0^{4,4,1}\right], \Omega_0^{4,4,2}\right] = \frac{\lambda_1^2 \lambda_0}{16} \int d^3x\, \phi(x)^4 \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{\omega_k \epsilon_k^2 (\omega_k + \epsilon_k)^2} + \frac{1}{\omega_k \epsilon_k^3 (\omega_k + \epsilon_k)}\right]. \tag{3.51}$$

Let's consider $\int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k \epsilon_k^3 (\omega_k + \epsilon_k)}$ first,

$$\begin{aligned} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k \epsilon_k^3 (\omega_k + \epsilon_k)} &= \int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{\epsilon_k^3 (M^2 - m^2)} - \frac{1}{\omega_k \epsilon_k^2 (\omega_k + \epsilon_k)(\omega_k - \epsilon_k)}\right), \\ &\approx \int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{\epsilon_k^3 M^2} - \frac{1}{\omega_k \epsilon_k^2 (\omega_k + \epsilon_k)(\omega_k - \epsilon_k)}\right). \end{aligned} \tag{3.52}$$

Calculation of $\int \frac{d^3k}{(2\pi)^3} \frac{1}{\epsilon_k^3 M^2}$ is straightforward, and we save it for later. Now consider

$$\begin{aligned} &\int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{\omega_k \epsilon_k^2 (\omega_k + \epsilon_k)^2} - \frac{1}{\omega_k \epsilon_k^2 (\omega_k + \epsilon_k)(\omega_k - \epsilon_k)}\right), \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{-2\epsilon_k}{\omega_k \epsilon_k^2 (\omega_k + \epsilon_k)^2 (\omega_k - \epsilon_k)}, \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{-2\epsilon_k (\omega_k - \epsilon_k)}{\omega_k \epsilon_k^2 (\omega_k + \epsilon_k)^2 (\omega_k - \epsilon_k)^2}, \\ &\approx \int \frac{d^3k}{(2\pi)^3} \left(\frac{-2}{\epsilon_k M^4} + \frac{2}{\omega_k M^4}\right). \end{aligned} \tag{3.53}$$

To the order $O(\frac{1}{M^2})$, we only need to keep $\int \frac{d^3k}{(2\pi)^3}\frac{-2}{\omega_k M^4}$. In total we have the contribution from Figure 3.3(d) to be

$$\begin{aligned}
&\frac{\lambda_1^2\lambda_0}{16}\int d^3x\,\phi(x)^4\int\frac{d^3k}{(2\pi)^3}\left(\frac{2}{\omega_k M^4}+\frac{1}{\epsilon_k^3 M^2}\right),\\
&=-\frac{\lambda_1^2\lambda_0}{64\pi^2M^2}\int d^3x\,\phi(x)^4\left(\left(\frac{1}{\epsilon}+1-\ln\frac{M^2}{\mu^2}\right)-\left(\frac{1}{\epsilon}-\ln\frac{m^2}{\mu^2}\right)\right)m=, \qquad (3.54)\\
&=-\frac{3\lambda_0\lambda_1^2}{8\pi^2M^2}\left(1-\ln\frac{M^2}{\mu^2}+\ln\frac{m^2}{\mu^2}\right)\int d^3x\,\frac{\phi(x)^4}{4!}.
\end{aligned}$$

Taking $\mu \approx M$ and summing the contributions from the four diagrams together, we get the final result of four point functions at one loop level, up to order $O(\frac{1}{M^2})$:

$$\begin{aligned}
&-\frac{3\lambda_0^2}{32\pi^2}\left(\frac{1}{\epsilon}-\ln\frac{m^2}{M^2}\right)\int d^3x\,\frac{\phi(x)^4}{4!}-\frac{3\lambda_2^2}{32\pi^2}\frac{1}{\epsilon}\int d^3x\,\frac{\phi(x)^4}{4!}\\
&+\frac{3\lambda_1^2\lambda_0}{16\pi^2M^2}\left(\frac{1}{\epsilon}-\ln\frac{m^2}{M^2}\right)\int d^3x\,\frac{\phi(x)^4}{4!}-\frac{3\lambda_1^2\lambda_0}{8\pi^2M^2}\left(1+\ln\frac{m^2}{M^2}\right)\int d^3x\,\frac{\phi(x)^4}{4!}
\end{aligned} \qquad (3.55)$$

3.3 Renormlization

We argued in section 2.3 that the UV divergences in the calculation of $H_{decoupled}$ contain information regarding the renormalization of H_1. Let's first write H_1 as

$$H_1=\int d^3x\,\frac{1}{2}(\partial\phi_{bare}(x))^2+\frac{1}{2}m_{bare}^2\phi_{bare}^2(x)+\frac{\lambda_0^{bare}}{4!}\phi_{bare}^4(x), \qquad (3.56)$$

and then introduce the renormalization Z factor such that

$$\phi_{bare}=\sqrt{Z_\phi}\phi, \qquad (3.57a)$$

$$m_{bare}=\sqrt{Z_m}m, \qquad (3.57b)$$

$$\lambda_0^{bare}=Z_{\lambda_0}\lambda_0. \qquad (3.57c)$$

Expanding these Z factors in terms of $\frac{1}{\epsilon}$, we have

$$Z_\phi = 1 + \delta^1_\phi(\frac{1}{\epsilon}) + O(\frac{1}{\epsilon^2}), \tag{3.58a}$$
$$Z_m = 1 + \delta^1_m(\frac{1}{\epsilon}) + O(\frac{1}{\epsilon^2}), \tag{3.58b}$$
$$Z_{\lambda_0} = 1 + \delta^1_{\lambda_0}(\frac{1}{\epsilon}) + O(\frac{1}{\epsilon^2}). \tag{3.58c}$$

Implementing these expansion, we can rewrite H_1 as

$$\begin{aligned} H_1 = \int d^3x\, &\frac{1}{2}(\partial\phi(x))^2 + \frac{1}{2}m^2\phi^2(x) + \frac{\lambda_0}{4!}\phi^2(x) \\ &\frac{1}{2}\delta^1_\phi(\partial\phi(x))^2 + \frac{1}{2}(\delta^1_\phi + \delta^1_m)m^2\phi^2(x) + \frac{\lambda_0}{4!}(2\delta^1_\phi + \delta^1_{\lambda_0})\phi^4(x) + O(\frac{1}{\epsilon^2}). \end{aligned} \tag{3.59}$$

Using $\overline{MS}$ scheme, we can cancel the UV divergences in $H_{decoupled}$ by counterterms in H_1. From two point calculation in section 3.2.1, we have UV divergence:

$$\left(-\frac{\lambda_0 m^2}{32\pi^2} - \frac{\lambda_2 M^2}{32\pi^2} + \frac{\lambda_1^2 m^2}{32\pi^2 M^2} - \frac{\lambda_1^2}{16\pi^2}\right)\frac{1}{\epsilon}\int d^3x\,\frac{\phi^2(x)}{2}. \tag{3.60}$$

From four point calculation in section 3.2.2, we have UV divergence:

$$\left(-\frac{3\lambda_0^2}{32\pi^2} - \frac{3\lambda_2^2}{32\pi^2} + \frac{3\lambda_1^2\lambda_0}{16\pi^2 M^2}\right)\frac{1}{\epsilon}\int d^3x\,\frac{\phi^4(x)}{4!}. \tag{3.61}$$

Since, there's no term proportional to the momentum, we know $\delta^1_\phi = 0$. Then we can use the following renormalization conditions:

$$\left(\delta^1_m m^2 + \left(-\frac{\lambda_0 m^2}{32\pi^2} - \frac{\lambda_2 M^2}{32\pi^2} + \frac{\lambda_1^2 m^2}{32\pi^2 M^2} - \frac{\lambda_1^2}{16\pi^2}\right)\frac{1}{\epsilon}\right)\int d^3x\,\frac{\phi^2(x)}{2} = 0, \tag{3.62a}$$
$$\left(\delta^1_{\lambda_0}\lambda_0 + \left(-\frac{3\lambda_0^2}{32\pi^2} - \frac{3\lambda_2^2}{32\pi^2} + \frac{3\lambda_1^2\lambda_0}{16\pi^2 M^2}\right)\frac{1}{\epsilon}\right)\int d^3x\,\frac{\phi^4(x)}{4!} = 0, \tag{3.62b}$$

to obtain

$$\delta^1_\phi = 0, \tag{3.63a}$$
$$\delta^1_m = \frac{1}{m^2}\left(\frac{\lambda_0 m^2}{32\pi^2} + \frac{\lambda_2 M^2}{32\pi^2} - \frac{\lambda_1^2 m^2}{32\pi^2 M^2} + \frac{\lambda_1^2}{16\pi^2}\right)\frac{1}{\epsilon}, \tag{3.63b}$$
$$\delta^1_{\lambda_0} = \frac{1}{\lambda_0}\left(\frac{3\lambda_0^2}{32\pi^2} + \frac{3\lambda_2^2}{32\pi^2} - \frac{3\lambda_1^2\lambda_0}{16\pi^2 M^2}\right)\frac{1}{\epsilon}, \tag{3.63c}$$

which agree with the results from the traditional renormalization in a Lagrangian framework.

Chapter 4

Construction of Effective Field Theories

In this chapter we will use the results from chapter 3 to construct the effective field theory up to order $O(\frac{1}{M^2})$, and show the extra finite terms in $H_{decoupled}$ are indeed canceled during matching.

In $H_{decoupled}$, at tree level up to order $O(\frac{1}{M^2})$, we have $(\lambda_0 - \frac{3\lambda_1}{M^2}) \int d^3x \, \frac{\phi^4(x)}{4!}$ that corresponds to Figure 4.1(a) and Figure 4.1(b). First, matching the decoupled Hamiltonian at tree level onto the low energy subspace, we have

$$H_{eff}^{tree} = \int d^3x \, \frac{1}{2}(\partial\phi(x))^2 + \frac{1}{2}m^2\phi^2(x) + (\lambda_0 - \frac{3\lambda_1}{M^2})\frac{\phi^4(x)}{4!} + O(\frac{1}{M^4}). \tag{4.1}$$

As we claimed in section 2.3, to proceed to one loop level, we need to decouple and renormalize H_{eff}^{tree} as well. To do this, we first change the low frequency modes ϕ's in H_{eff}^{tree} back to the full light field Φ_L. Then we can follow the similar process to decouple the high frequency and low frequency modes by unitary transformations.

$$H'_{decoupled} = \langle 0_{high}| \, \omega'^{\dagger} H_{eff}^{tree} \omega' \, |0_{high}\rangle \,, \tag{4.2}$$

where $H'_{decoupled}$ is the decoupled Hamiltonian corresponding to H_{eff}^{tree} and ω' is the corresponding unitary transformation.

First, we have a contribution from a two point function shown in Figure 4.2. The calculation here is similar to the calculation of Figure 3.1(a), and we only

(a) (b)

Figure 4.1 Tree level contributions in $H_{decoupled}$.

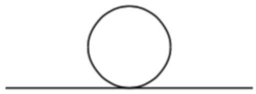


Figure 4.2 One loop two point function contribution in $H'_{decoupled}$

need to substitute λ_0 with $\lambda_0 - \frac{3\lambda_1^2}{M^2}$. Therefore, the result is

$$-\frac{m^2}{32\pi^2}(\lambda_0 - \frac{3\lambda_1^2}{M^2})(\frac{1}{\epsilon} - \ln\frac{m^2}{\mu^2} + 1)\int d^3x\,\frac{\phi^2(x)}{2} + \frac{\lambda_0 - \frac{3\lambda_1^2}{M^2}}{4}C\int d^3x\,\frac{\phi^2(x)}{2}, \quad (4.3)$$

where $C = -\int_{k<M}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{k^2+m^2}}$ is the same as in section 3.2.1. Then we do the renormalization in $\overline{MS}$ scheme to cancel out the divergent part and take $\mu \approx M$. The finite terms left are

$$-\frac{m^2}{32\pi^2}(\lambda_0 - \frac{3\lambda_1^2}{M^2})(-\ln\frac{m^2}{M^2} + 1)\int d^3x\,\frac{\phi^2(x)}{2} + \frac{\lambda_0 - \frac{3\lambda_1^2}{M^2}}{4}C\int d^3x\,\frac{\phi^2(x)}{2}. \quad (4.4)$$

To do the matching, we use the finite terms in eq.(3.33) from section 3.2.1 to subtract eq.(4.4) to get the two point interactions to be added in $H_{eff}^{one\,loop}$:

$$\left(-\frac{\lambda_2 M^2}{32\pi^2} - \frac{\lambda_1^2}{16\pi^2} - \frac{\lambda_1^2 m^2}{16\pi^2 M^2}\right)\int d^3x\,\frac{\phi^2(x)}{2} - \frac{\lambda_1^2}{32\pi^2 M^2}\int d^3x\,\frac{1}{2}(\partial\phi(x))^2. \quad (4.5)$$

As we expect the extra finite terms are all canceled out along with the large log terms during matching and will not appear in the effective Hamiltonian.

We can then consider the contribution from the four point function shown in Figure 4.3. Again the calculation is similar to the one of Figure 3.3(a), and we

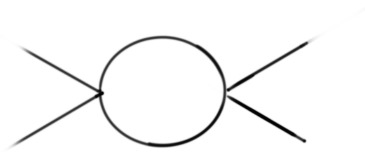


Figure 4.3 One loop four point function contribution in $H'_{decoupled}$

only need to substitute λ_0 with $\lambda_0 - \frac{3\lambda_1^2}{M^2}$. Hence, we have

$$-\frac{3\left(\lambda_0 - \frac{3\lambda_1^2}{M^2}\right)^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} - \ln\frac{m^2}{M^2}\right)\int d^3x\,\frac{\phi^4(x)}{4!}. \tag{4.6}$$

After renormalization, up to order $O(\frac{1}{M^2})$, we are left with

$$\left(\frac{3\lambda_0^2}{32\pi^2} - \frac{9\lambda_1^2\lambda_0}{16\pi^2 M^2}\right)\ln\frac{m^2}{M^2}\int d^3x\,\frac{\phi^4(x)}{4!}. \tag{4.7}$$

Using the finite terms in eq.(3.55) from section 3.2.2 to subtract eq.(4.7), we get the four point interactions to be added in $H_{eff}^{oneloop}$ as

$$-\frac{3\lambda_0\lambda_1^2}{8\pi^2 M^2}\int d^3x\,\frac{\phi^4(x)}{4!}. \tag{4.8}$$

Large log terms are again canceled during matching as we expect. In conclusion, up to order $O(\frac{1}{M^2})$, the effective Hamiltonian at one loop level so constructed based on our theory is

$$\begin{aligned} H_{eff}^{oneloop} = \int d^3x\,\frac{1}{2}(\partial\phi(x))^2 + \frac{1}{2}m^2\phi^2(x) - \left(\frac{\lambda_2 M^2}{32\pi^2} + \frac{\lambda_1^2}{16\pi^2}(1 + \frac{m^2}{M^2})\right)\frac{\phi^2(x)}{2} \\ + \left(\lambda_0 - \frac{3\lambda_1^2}{M^2} - \frac{3\lambda_0\lambda_1^2}{8\pi^2 M^2}\right)\frac{\phi^4(x)}{4!} - \frac{\lambda_1^2}{32\pi^2 M^2}\frac{1}{2}(\partial\phi(x))^2. \end{aligned} \tag{4.9}$$

Chapter 5

Discussion

In this thesis we have shown the consistency of the approach to Hamiltonian renormalization which emphasizes its basic origin as due to the entanglement between the low and high energy modes of the theory. Using unitary transformations on states to decouple the high energy modes from the low energy ones and projecting the transformed Hamiltonian to the low energy subspace, correctly accounts for renormalization effects and the property of decoupling in quantum field theories. We have also shown how the same approach can be consistently used in the construction of effective field theories. The next step would be to understand how different measures of entanglement like entanglement entropy and mutual information may be used to analyse the properties of decoupling and to shed light on another striking property of quantum field theories, namely the insensitivity of the low energy physics to the details of the short distance structure.

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Appendix A

Appendix for Chapter 3

A.1 Explicit Calculation of Figure 3.3(c)

Since combination Table 3.1(b) and Table 3.1(c) only differ from an exchange of $\Omega_0^{4,3,1}$ and $\Omega_0^{4,3,2}$, they will yield the same result. Let's consider the following from Table 3.1(b):

$$H_A^{4,3} = \frac{\lambda_1}{2} \int d^3x \sum_k \sum_p \sum_q \frac{e^{i(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\boldsymbol{x}}}{(2V)^{3/2}\sqrt{\omega_k \epsilon_p \epsilon_q}} a_k b_p b_q, \tag{A.1a}$$

$$\Omega_0^{4,3,1} = \frac{\lambda_1}{2} \int d^3y \sum_{k'} \frac{i\phi(y)^2}{\sqrt{2V\omega_{k'}}\omega_{k'}} a_{k'}^\dagger e^{-i\boldsymbol{k}'\boldsymbol{y}}, \tag{A.1b}$$

$$\Omega_0^{4,3,2} = \frac{\lambda_0}{4} \int d^3z \sum_{p'} \sum_{q'} \phi(z)^2 \frac{ie^{-i\left(\boldsymbol{p}'+\boldsymbol{q}'\right)\boldsymbol{z}}}{2V\sqrt{\epsilon_{p'}\epsilon_{q'}}\left(\epsilon_{p'}+\epsilon_{q'}\right)} b_{p'}^\dagger b_{q'}^\dagger. \tag{A.1c}$$

The first commutator gives

$$\begin{aligned}
\left[H_A^{4,3}, \Omega_0^{4,3,1}\right] &= \frac{i\lambda_1^2}{4} \int d^3x \int d^3y \sum_k \sum_{k'} \sum_p \sum_q \frac{\phi(y)^2}{(2V)^2} \frac{e^{i(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\boldsymbol{x}-i\boldsymbol{k}'\boldsymbol{y}}}{\sqrt{\omega_k\omega_{k'}\omega_q\omega_p}\omega_{k'}} \left[a_k b_p b_q, a_{k'}^\dagger\right], \\
&= \frac{i\lambda_1^2}{16} \int d^3x \int d^3y \sum_p \sum_q \frac{e^{i(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{x}}}{V\sqrt{\epsilon_p}\epsilon_q M^2} \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})} \phi(y)^2 b_p b_q, \\
&= \frac{i\lambda_1^2}{16} \int d^3x\, \phi(x)^2 \sum_p \sum_q \frac{e^{i(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{x}}}{VM^2\sqrt{\epsilon_p\epsilon_q}} b_p b_q.
\end{aligned} \tag{A.2}$$

The second commutator is

$$\begin{aligned}\left[\left[H_A^{4,3},\Omega_0^{4,3,1}\right],\Omega_0^{4,3,2}\right] &= -\frac{\lambda_1^2\lambda_0}{64}\int d^3x\int d^3z\,\phi(x)^2\phi(z)^2, \\ &\sum_p\sum_q\sum_{p'}\sum_{q'}\frac{e^{i(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{x}-i\left(\boldsymbol{p}'+\boldsymbol{q}'\right)\boldsymbol{z}}}{2V^2M^2\sqrt{\epsilon_p\epsilon_q\epsilon_{p'}\epsilon_{q'}}\left(\epsilon_{p'}+\epsilon_{q'}\right)}\left[b_pb_q,b^\dagger_{p'}b^\dagger_{q'}\right], \\ &= -\frac{\lambda_1^2\lambda_0}{64}\int d^3x\,\phi(x)^4\int\frac{d^3p}{(2\pi)^3}\frac{1}{2M^2\epsilon_p^3}.\end{aligned} \tag{A.3}$$

As a result,

$$-\frac{1}{3}\left[\left[H_A^{4,3},\Omega_0^{4,3,1}\right],\Omega_0^{4,3,2}\right] = \frac{\lambda_1^2\lambda_0}{192}\int d^3x\,\phi(x)^4\int\frac{d^3p}{(2\pi)^3}\frac{1}{2M^2\epsilon_p^3} \tag{A.4}$$

and consider the Hermitian conjugation we have the final result for both Table 3.1(b) and Table 3.1(c) to be

$$\frac{\lambda_1^2\lambda_0}{96}\int d^3x\,\phi(x)^4\int\frac{d^3p}{(2\pi)^3}\frac{1}{2M^2\epsilon_p^3}. \tag{A.5}$$

For the combination Table 3.1(d), we have

$$H_A^{4,3} = \int d^3x\,\frac{\lambda_0}{4}\phi(x)^2\sum_p\sum_q\frac{e^{i(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{x}}}{2V\sqrt{\epsilon_p\epsilon_q}}b_pb_q, \tag{A.6a}$$

$$\Omega_0^{4,3,1} = \int d^3y\,\frac{\lambda_1}{2}\sum_k\sum_{p'}\sum_{q'}\frac{ie^{-i\left(\boldsymbol{k}+\boldsymbol{p}'+\boldsymbol{q}'\right)\boldsymbol{y}}}{(2V)^{3/2}\sqrt{\omega_k\epsilon_{p'}\epsilon_{q'}}\left(\omega_k+\epsilon_{p'}+\epsilon_{q'}\right)}a_k^\dagger b^\dagger_{p'}b^\dagger_{q'}, \tag{A.6b}$$

$$\Omega_0^{4,3,2} = \int d^3z\,\frac{\lambda_1}{2}\phi(z)^2\sum_{k'}\frac{-ie^{\boldsymbol{k}'\boldsymbol{z}}}{\sqrt{2V\omega_{k'}}\omega_{k'}}a_{k'}. \tag{A.6c}$$

Following similar steps, we first calculate the commutator inside:

$$\begin{aligned}\left[H_A^{4,3},\Omega_0^{4,3,1}\right] &= \frac{i\lambda_0\lambda_1}{8}\int d^3x\int d^3y\,\phi(x)^2\sum_p\sum_q\sum_k\sum_{p'}\sum_{q'}\frac{e^{i(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{x}-i\left(\boldsymbol{k}+\boldsymbol{p}'+\boldsymbol{q}'\right)\boldsymbol{y}}}{(2V)^{5/2}\left(\omega_K+\epsilon_{p'}+\epsilon_{q'}\right)}\\
&\left[b_pb_q,a_k^\dagger b_{p'}^\dagger b_{q'}^\dagger\right],\\
&= \frac{i\lambda_1\lambda_0}{8}\int d^3x\int d^3y\,\phi(x)^2\sum_k\sum_p\sum_q\frac{2e^{i(\boldsymbol{p}+\boldsymbol{q})(\boldsymbol{x}-\boldsymbol{y})-i\boldsymbol{k}\boldsymbol{y}}}{(2V)^{5/2}\sqrt{\omega_k}\epsilon_p\epsilon_q\left(\omega_k+\epsilon_p+\epsilon_q\right)}a_k^\dagger,\\
&= \frac{i\lambda_1\lambda_0}{8}\int d^3x\int d^3y\,\phi(x)^2\sum_k\sum_p\frac{e^{-i\boldsymbol{k}\boldsymbol{y}}}{(2V)^{3/2}\epsilon_p^2\sqrt{\omega_k}\left(\omega_k+2\epsilon_p\right)}a_k^\dagger\\
&\int\frac{d^3(p+q)}{(2\pi)^3}e^{i(\boldsymbol{p}+\boldsymbol{q})(\boldsymbol{x}-\boldsymbol{y})},\\
&= \frac{i\lambda_1\lambda_0}{8}\int d^3x\,\phi(x)^2\sum_k\sum_p\frac{e^{-i\boldsymbol{k}\boldsymbol{x}}}{(2V)^{3/2}\epsilon_p^2\sqrt{\omega_k}\left(\omega_k+2\epsilon_p\right)}a_k^\dagger.\end{aligned} \tag{A.7}$$

The total commutator is

$$\begin{aligned}\left[\left[H_A^{4,3},\Omega_0^{4,3,1}\right],\Omega_0^{4,3,2}\right] &= \frac{\lambda_1^2\lambda_0}{16}\int d^3x\int d^3z\,\phi(z)^2\phi(x)^2\sum_k\sum_p\sum_{k'}\frac{e^{i\boldsymbol{k}'\boldsymbol{z}-i\boldsymbol{k}\boldsymbol{x}}}{(2V)^2\epsilon_p^2\sqrt{\omega_k\omega_{k'}}\omega_{k'}\left(\omega_k+2\epsilon_p\right)}\\
&\left[a_k^\dagger,a_{k'}\right],\\
&= -\frac{\lambda_1^2\lambda_0}{64}\int d^3x\,\phi(x)^4\int\frac{d^3p}{(2\pi)^3}\frac{1}{M^2\epsilon_p^2\left(M+2\epsilon_p\right)}.\end{aligned} \tag{A.8}$$

Therefore,

$$-\frac{1}{3}\left[\left[H_A^{4,3},\Omega_0^{4,3,1}\right],\Omega_0^{4,3,2}\right] = \frac{\lambda_1^2\lambda_0}{192}\int d^3x\,\phi(x)^4\int\frac{d^3p}{(2\pi)^3}\frac{1}{M^2\epsilon_p^2\left(M+2\epsilon_p\right)}. \tag{A.9}$$

Considering the Hermitian conjugation, the final result from the combination Table 3.1(d) is

$$\frac{\lambda_1^2\lambda_0}{96}\int d^3x\,\phi(x)^4\int\frac{d^3p}{(2\pi)^3}\frac{1}{M^2\epsilon_p^2\left(M+2\epsilon_p\right)}. \tag{A.10}$$

Calculation of the contributions from $-\frac{1}{2}\left[\left[H_B^{4,3},\Omega_0^{4,3,1}\right],\Omega_0^{4,3,2}\right]$ is very similar to the calculation of Table 3.1(b) and Table 3.1(c).

A.2 Explicit Calculation of Figure 3.3(d)

In terms of Table 3.3(a), let's consider the following:

$$H_B^{4,4} = \int d^3x\, \frac{\lambda_0}{2}\phi(x)^2 \sum_p \sum_q \frac{e^{i(\boldsymbol{p}-\boldsymbol{q})\boldsymbol{x}}}{2V\sqrt{\epsilon_p\epsilon_q}} b_p^\dagger b_q, \tag{A.11a}$$

$$\Omega_0^{4,4,1} = \int d^3y\, \lambda_1\phi(y) \sum_k \sum_{p'} \frac{ie^{-i\left(\boldsymbol{k}+\boldsymbol{p}'\right)\boldsymbol{y}}}{2V\sqrt{\omega_k\epsilon_{p'}}\left(\omega_k+\epsilon_{p'}\right)} a_k^\dagger b_{p'}^\dagger, \tag{A.11b}$$

$$\Omega_0^{4,4,2} = \int d^3z\, \lambda_1\phi(z) \sum_{k'} \sum_{q'} \frac{-ie^{i\left(\boldsymbol{k}'+\boldsymbol{q}'\right)\boldsymbol{z}}}{2V\sqrt{\omega_{k'}\epsilon_{q'}}\left(\omega_{k'}+\epsilon_{q'}\right)} a_{k'} b_{q'}. \tag{A.11c}$$

The first commutator is

$$\begin{aligned}
\left[H_B^{4,4}, \Omega_0^{4,4,1}\right] &= \frac{i\lambda_0\lambda_1}{2}\int d^3x \int d^3z\, \phi(x)^2\phi(y) \sum_p\sum_q\sum_k\sum_{p'} \frac{e^{-i(\boldsymbol{p}-\boldsymbol{q})\boldsymbol{x}-i\left(\boldsymbol{k}+\boldsymbol{p}'\right)\boldsymbol{y}}}{4V^2\sqrt{\epsilon_p\epsilon_q\omega_k\epsilon_{p'}}\left(\omega_k+\epsilon_{p'}\right)} \\
&\quad \left[b_p^\dagger b_q, a_k^\dagger b_{p'}^\dagger\right], \\
&= \frac{i\lambda_0\lambda_1}{8}\int d^3x \int d^3y\, \phi(x)^2\phi(y) \sum_p\sum_q\sum_k \frac{e^{i\boldsymbol{q}(\boldsymbol{x}-\boldsymbol{y})-i\boldsymbol{p}\boldsymbol{x}-i\boldsymbol{k}\boldsymbol{y}}}{V^2\sqrt{\epsilon_p\omega_k}\epsilon_q\left(\omega_k+\epsilon_q\right)} b_p^\dagger a_k^\dagger, \\
&= \frac{i\lambda_0\lambda_1}{8}\int d^3x \int d^3y\, \phi(x)^2\phi(y) \sum_p\sum_k \frac{e^{-i\boldsymbol{p}\boldsymbol{x}-i\boldsymbol{k}\boldsymbol{y}}}{V\sqrt{\epsilon_p\omega_k}\epsilon_p\left(\omega_k+\epsilon_p\right)} b_p^\dagger a_k^\dagger \int \frac{d^3q}{(2\pi)^3}\, e^{i\boldsymbol{q}(\boldsymbol{x}-\boldsymbol{y})}, \\
&= \frac{i\lambda_0\lambda_1}{8}\int d^3x\, \phi(x)^3 \sum_p\sum_k \frac{e^{-i\boldsymbol{p}\boldsymbol{x}-i\boldsymbol{k}\boldsymbol{y}}}{V\sqrt{\epsilon_p\omega_k}\epsilon_p\left(\omega_k+\epsilon_p\right)} b_p^\dagger a_k^\dagger,
\end{aligned} \tag{A.12}$$

where we have used the condition that external momenta are zero, and therefore $|k| = |p| = |q|$, which gives $\epsilon_p = \epsilon_q$. The second commutator is

$$\begin{aligned}
\left[\left[H_B^{4,4}, \Omega_0^{4,4,1}\right], \Omega_0^{4,4,2}\right] &= \frac{\lambda_0\lambda_1^2}{16}\int d^3x \int d^3z\, \phi(x)^3\phi(z) \sum_p\sum_k\sum_{k'}\sum_{q'} \\
&\quad \frac{e^{i\left(\boldsymbol{k}'+\boldsymbol{q}'\right)\boldsymbol{z}-i(\boldsymbol{k}+\boldsymbol{p})\boldsymbol{x}}}{V^2\sqrt{\omega_k\omega_{k'}\epsilon_p\epsilon_{q'}}\left(\omega_k+\epsilon_p\right)\left(\omega_{k'}+\epsilon_{q'}\right)\epsilon_p} \left[b_p^\dagger a_k^\dagger, a_{k'} b_{q'}\right], \\
&= \frac{-\lambda_0\lambda_1^2}{16}\int d^3x \int d^3z\, \phi(x)^3\phi(z) \sum_k\sum_q \frac{e^{i(\boldsymbol{k}+\boldsymbol{p})(\boldsymbol{z}-\boldsymbol{x})}}{V^2\omega_k\epsilon_p^2\left(\omega_k+\epsilon_p\right)^2}, \\
&= \frac{-\lambda_0\lambda_1^2}{16}\int d^3x\, \phi(x)^4 \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k\epsilon_k^2\left(\omega_k+\epsilon_k\right)^2}.
\end{aligned} \tag{A.13}$$

Considering the Hermitian conjugation, we will get the final result for Tabel 3.3(a) as:

$$\frac{\lambda_0\lambda_1^2}{16}\int d^3x\,\phi(x)^4\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k\epsilon_k^2\left(\omega_k+\epsilon_k\right)^2}. \tag{A.14}$$

Table 3.3(b) and Table 3.3(c) will give the same result, and we consider the following:

$$H_B^{4,4}=\lambda_1\int d^3x\,\phi(x)\sum_k\sum_p\frac{e^{i(\boldsymbol{k}-\boldsymbol{p})\boldsymbol{x}}}{2V\sqrt{\omega_k\epsilon_p}}b_p^\dagger a_k, \tag{A.15a}$$

$$\Omega_0^{4,4,1}=\lambda_1\int d^3y\,\phi(y)\sum_{k'}\sum_q\frac{ie^{-i\left(\boldsymbol{k}'+\boldsymbol{q}\right)\boldsymbol{y}}}{2V\sqrt{\omega_{k'}\epsilon_q}\left(\omega_{k'}+\epsilon_q\right)}a_{k'}^\dagger b_q^\dagger, \tag{A.15b}$$

$$\Omega_0^{4,4,2}=\frac{\lambda_0}{4}\int d^3z\,\phi(z)^2\sum_{p'}\sum_{q'}\frac{-ie^{i\left(\boldsymbol{p}'+\boldsymbol{q}'\right)\boldsymbol{z}}}{2V\sqrt{\epsilon_{p'}\epsilon_{q'}}\left(\epsilon_{p'}+\epsilon_{q'}\right)}b_{p'}b_{q'}. \tag{A.15c}$$

First commutator is

$$\begin{aligned}\left[H_B^{4,4},\Omega_0^{4,4,1}\right]&=i\lambda_1^2\int d^3x\int d^3y\,\phi(x)\phi(y)\sum_k\sum_p\sum_{k'}\sum_q\frac{e^{i(\boldsymbol{k}-\boldsymbol{p})\boldsymbol{x}-i\left(\boldsymbol{k}'+\boldsymbol{q}\right)\boldsymbol{y}}}{4V^2\sqrt{\omega_k\omega_{k'}\epsilon_p\epsilon_q}\left(\omega_{k'}+\epsilon_q\right)}\\&\quad\left[a_kb_p^\dagger,ak'^\dagger b_q^\dagger\right],\\&=\frac{i\lambda_1^2}{4}\int d^3x\int d^3y\,\phi(x)\phi(y)\sum_k\sum_p\sum_q\frac{e^{i(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{k}-i\boldsymbol{p}\boldsymbol{x}-i\boldsymbol{q}\boldsymbol{y}}}{V^2\omega_k\sqrt{\epsilon_p\epsilon_q}\left(\omega_k+\epsilon_q\right)}b_p^\dagger b_q^\dagger,\\&=\frac{i\lambda_1^2}{4}\int d^3x\,\phi(x)^2\sum_p\sum_q\frac{e^{-i(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{x}}}{V^2\omega_p\sqrt{\epsilon_p\epsilon_q}\left(\omega_p+\epsilon_q\right)}b_p^\dagger b_q^\dagger.\end{aligned} \tag{A.16}$$

Again, we use the condition that external momenta are zero, and rewrite ω_k as ω_p, since $|k|=|p|$. The second commutator gives:

$$\begin{aligned}\left[\left[H_B^{4,4},\Omega_0^{4,4,1}\right],\Omega_0^{4,4,2}\right]&=\frac{\lambda_1^2\lambda_0}{16}\int d^3x\int d^3z\,\phi(x)^2\phi(z)^2\sum_p\sum_q\sum_{p'}\sum_{q'}\\&\quad\frac{e^{-i(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{x}+i\left(\boldsymbol{p}'+\boldsymbol{q}'\right)\boldsymbol{z}}}{2V^2\omega_p\sqrt{\epsilon_p\epsilon_q\epsilon_{p'}\epsilon_{q'}}\left(\epsilon_{p'}+\epsilon_{q'}\right)\left(\omega_p+\epsilon_q\right)}\left[b_p^\dagger b_q^\dagger,b_{p'}b_{q'}\right],\\&=\frac{-\lambda_1^2\lambda_0}{16}\int d^3x\int d^3z\,\phi(x)^2\phi(z)^2\sum_p\sum_q\frac{e^{i(\boldsymbol{p}+\boldsymbol{q})(\boldsymbol{z}-\boldsymbol{x})}}{V^2\omega_p\epsilon_p\epsilon_q\left(\epsilon_p+\epsilon_q\right)\left(\omega_p+\epsilon_q\right)},\\&=\frac{-\lambda_1^2\lambda_0}{32}\int d^3x\,\phi(x)^4\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega_p\epsilon_p^3\left(\omega_p+\epsilon_p\right)}.\end{aligned} \tag{A.17}$$

Considering the Hermitian conjugation, we will get the final result for both Table 3.3(b) and Table 3.3(c):

$$\frac{\lambda_1^2\lambda_0}{32}\int d^3x\,\phi(x)^4\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\omega_p\epsilon_p^3\,(\omega_p+\epsilon_p)}. \tag{A.18}$$