Mapping Class Groups of Rational Maps

by

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This thesis is dedicated to mom and dad.

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ABSTRACT

Given a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree $d \geq 2$, it follows from work done in [25] by McMullen and Sullivan that in certain circumstances, the pure mapping class group PMCG(f) can be identified with a subgroup of the pure mapping class group of a Riemann surface. We investigate this identification and explore what types of subgroups of mapping class groups of surfaces arise in this way. We focus primarily on the case in which PMCG(f) can be viewed as a subgroup of a product of pure mapping class groups of punctured tori. A specific case of this setting — namely, when f is a generic quadratic rational map — was explored by Goldberg and Keen in [15]. The authors proved that for such a choice of f, PMCG(f) is an infinitely generated subgroup of PMCG($\Sigma_{1,2}$). We prove the analogous statement in the setting of cubic polynomials, and explicitly write down a collection of generators of PMCG(f) in terms of point-pushes and a Dehn twist. We then prove a general result that is independent of the degree of the map. Specifically, we prove that for f in an open subset of rational maps of degree d, PMCG(f) is an infinitely generated subgroup of a product of pure mapping class groups of punctured tori.

CHAPTER I

Introduction

In this thesis, we provide an account of a link between the topology of certain parameter spaces that arise in complex dynamics, and the mapping class groups of certain punctured surfaces.

1.1 A correspondence

This link begins with the association of a 1-dimensional complex surface to a rational self-map of the Riemann sphere with specific dynamical features. Given a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, the *grand orbit* $\mathrm{GO}(z)$ of a point $z \in \hat{\mathbb{C}}$ under f is the union of all points in $\hat{\mathbb{C}}$ whose orbit under f eventually intersects a point in the orbit of z. Under identification of points in the same grand orbit, rational maps with *attracting cycles* give rise to *quotient tori*, and rational maps with *parabolic cycles* give rise to *quotient spheres*. A common theme in the study of complex dynamics concerns the role of *critical points* of f — that is, points $z \in \hat{\mathbb{C}}$ with $f'(z) = 0$ — in understanding the dynamics of f. This theme arises in the setting of quotient surfaces as well: these quotient surfaces have punctures corresponding to grand orbits of critical points.

For much of this thesis, we will be working with the following definition.

Definition 1.1. A rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is called MCG-*generic* if

1. f is hyperbolic,
2. f has no acyclic critical points in its super-attracting basins, and
3. f has no critical orbit relations coming from critical points in attracting basins.

When we have a MCG-generic map f, we have a relationship between the mapping class group of f and the product of mapping class groups of finitely-punctured tori. Specifically, we can identify $\mathrm{MCG}(f)$ with a subgroup of $\prod_i \mathrm{MCG}(\Sigma_{1,n_i})$, where i indexes the attracting periodic cycles of f and and the number of punctures n_i is equal to the number of grand orbits containing critical points of f in the basin of the ith attracting cycle. Quotient surfaces, this identification, and all objects in question are discussed in more detail in Chapters II and III.

The mapping class group of a surface is a well-studied topic of great interest in geometric group theory[1], and a primary goal of this thesis is to leverage results in this field to say something about the dynamics of rational maps. In particular, we investigate the question:

Question 1.2. *Which subgroups of mapping class groups of surfaces arise from mapping class groups of rational maps?*

[1]in the literature, the mapping class group of a surface is sometimes called the modular group, and is often denoted by $\mathrm{Mod}(S)$.

To help answer this question, we devise the following definition.

Definition 1.3. An element γ of $\mathrm{MCG}(\Sigma_{0,m})$ or $\mathrm{MCG}(\Sigma_{1,n})$ is called a *dynamical mapping class* (or, a *pure dynamical mapping class* if working with $\mathrm{PMCG}(\Sigma_{i,n})$) if there exists some rational map f so that (the image of) γ is in $\mathrm{MCG}(f)$. An element of $\mathrm{MCG}(\Sigma_{0,m})$ or $\mathrm{MCG}(\Sigma_{1,n})$ that is not dynamical mapping classes is called a *non-dynamical mapping class.*

1.2 Outline and main results

The outline of the results in this document is as follows. Chapters II through V give preliminaries and background to complex dynamics, mapping class groups of surfaces, and a link between the two. In Chapter VI we work out an explicit example of the link between the mapping class group of a rational map and the mapping class group of a surface in the setting of quadratic polynomials. This setting serves as intuition for our first general result, proven in Chapter VII.

Theorem 1.4. *For every $n > 0$, there exists a non-dynamical mapping class in* $\mathrm{MCG}(\Sigma_{1,n})$.

We then focus on the specific setting of cubic polynomials. In this context, we prove the following.

Theorem 1.5. *The pure mapping class group of a* MCG-*generic cubic polynomial with a twice-punctured quotient torus is an infinitely generated subgroup of* $\mathrm{PMCG}(\Sigma_{1,2})$ *with generators explicitly given by point pushes and a Dehn twist.*

A specific case of this theorem, for the mapping class group of a perturbation of the cubic polynomial $z \mapsto z^3$, is proven in Chapter IX. The more general statement is proven in Chapter XII, along with a discussion of a parameter space picture for cubic polynomials that serves as intuition for a number of proofs in this document.

In Chapter XI, we prove the following general theorem, independent of degree.

Theorem 1.6. *Let f be a* MCG*-generic rational map. If f has an attracting cycle with at least two critical points in the basin of this cycle, then* $\mathrm{PMCG}(f)$ *is infinitely generated. Otherwise,* $\mathrm{PMCG}(f)$ *is finitely generated, with generators given by one Dehn twist for each attracting cycle of f.*

Finally, in Chapter XIII, we depart from the case of attracting cycles and quotient tori. We calculate the mapping class group of a bicritical rational map with a parabolic cycle, and relate this to the mapping class group of its quotient sphere. Specifically, we prove the following.

Theorem 1.7. *The pure mapping class group of a bicritical rational map with a parabolic cycle, where both critical points are attracted to the cycle with no critical orbit relations, is an infinitely generated free subgroup of* $\mathrm{PMCG}(\Sigma_{0,4}) \cong F_2$ *which is generated by*

- *One (based) loop enclosing each orbit relation, and*
- *One loop enclosing the connectedness locus.*

These generators correspond to point-pushes (and squares of point-pushes) around closed curves in $\Sigma_{0,4}$.

1.3 Notes and references

Much of the work done in establishing the relationship between the mapping class groups of rational maps and of surfaces was done in [25], and is summarized in Chapter III.

The exposition and result of the special case of Theorem 1.5 in Chapter IX mirror work done in [15] on the analogous question in the setting of quadratic rational maps. Specifically, as in [15], in this setting we construct a homeomorphism between a parameter space and a dynamical picture, and use the Birman exact sequence to interpret this homeomorphism in the context of mapping class groups. However, unlike in [15], the proof of Theorem 1.5 uses the idea of *spinning* as developed in [32]. This tool is discussed in Chapter VIII.

To prove Theorem 1.5 in generality, we make use of the notion of *mapping schemes* as developed in [29].

The notion of spinning arises again in Chapter XI as a way to leverage results about mapping class groups of surfaces to prove Theorem 1.6. In addition to spinning, the proof of Theorem 1.6 uses an analysis of the pure torus braid group — this background can be found in Chapter X.

Finally, the computation of the mapping class group for bicritical rational maps with a parabolic cycle makes use of the construction of global coordinates for this space in [27]. We use these results to draw and investigate pictures of a parameter space that allow us to explicitly calculate a class of dynamical mapping classes of a

4-times punctured sphere.

1.4 A remark on figures

One of the benefits of complex dynamics in a single variable is that one can draw many of the spaces and objects being studied. Often these images, in addition to being beautiful for their own sake, lend significant insight into the correct questions to ask and the best way to proceed. Embracing this valuable feature of the subject, in this thesis we include many figures to provide intuition, to illustrate ideas and methods of proof, and sometimes simply because the images are compelling in and of themselves. Many of the pictures were drawn using FractalStream (see `http://pi.math.cornell.edu/~noonan/fstream.html`). When more specific detail was needed, the pictures were created with Python. Scripts for each are available upon request.

CHAPTER II

Complex dynamics background

This chapter is written to give a gentle introduction and background to complex dynamics.

In a very general sense, dynamics is the study of long-term behavior. In complex dynamics, the objects whose long-term behavior is studied are holomorphic self-maps of a Riemann surface. The class of maps with the most interesting behavior, and therefore classically those most closely studied, are maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere. Such holomorphic self-maps of the Riemann sphere are called *rational maps*. The *degree* of a rational map is the degree of the map as viewed as topological map on a sphere. Every rational map can be written as a quotient of two complex polynomials, in which case the degree of the map is exactly the maximum of the degrees of the two polynomials when written in lowest terms.

Since degree 1 rational maps are Möbius transformations, which have no critical points and are therefore well-understood, throughout the course of this thesis, whenever we refer to a rational map will will assume it has degree $d \geq 2$.

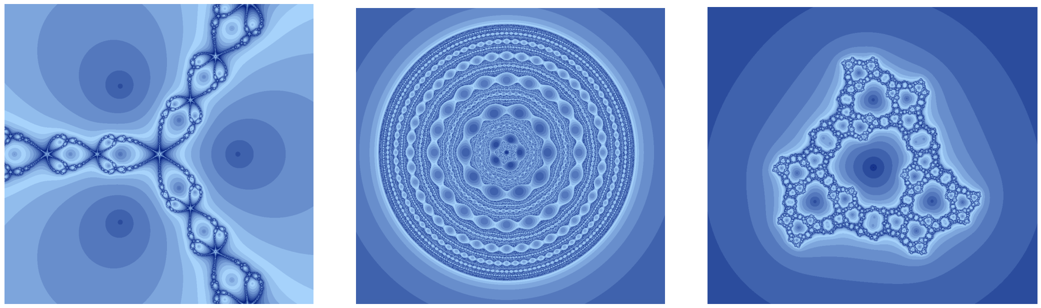


Figure 2.1: Julia sets for a number of rational maps. The blue region is the Fatou set, and the Julia set is its boundary.

2.1 Julia and Fatou sets

The notion of *long-term behavior* in this setting comes from the repeated composition of a map with itself. Specifically, given a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, we look at the sequence

$$\{f^n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}_{n \in \mathbb{N}}.$$

To understand this sequence of maps, we define the *Fatou* and the *Julia sets* associated to f, using the definition found in [28]. [1]

Definition 2.1. Given a rational map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, the *Fatou set* of f is the union of all open sets $U \subseteq \hat{\mathbb{C}}$ so that every sequence of iterates $f^{n_j}|_U$ contains a locally uniformly convergent subsequence. The *Julia set* of f is the complement of the Fatou set.

In the pursuit of understanding the dynamics of a map f, we will often look at the trajectory of a point $p \in \hat{\mathbb{C}}$ under iteration. In particular, the *orbit* of p is just the sequence $\{f^n(p)\}_{n \in \mathbb{N}}$. If $f(p) = p$, we say p is a fixed point, and if $f^k(p) = p$, with

[1] For more background on these definitions and associated properties of the sets defined, see [28] Chapter 5.

k the smallest integer satisfying this property, we say that $\{p, f(p), \ldots, f^{k-1}(p)\}$ is a periodic cycle of period k.

2.2 Fatou components

Throughout the course of this document, we will make a number of references to *local behavior* of certain maps. We highlight here a couple different types of components of the Fatou set, all of which are associated with a periodic cycle. Given a map f with a cycle $\mathbf{p} = \{p_0, \ldots, p_{k-1}\}$ of period k, we define the *multiplier* of the cycle to be

$$\lambda = (f^k)'(p_i),$$

which is the same quantity for each point p_i in the cycle, and therefore well-defined. We consider four cases.

1. If $0 < |\lambda| < 1$, we say $\mathbf{p}$ is *attracting*[2].
2. If $|\lambda| = 0$, we say $\mathbf{p}$ is *super-attracting*.
3. If $|\lambda|$ is a root of unity, we say $\mathbf{p}$ is *parabolic*.
4. If $|\lambda| > 1$, we say $\mathbf{p}$ is *repelling*.[3]

Attracting and super-attracting cycles are necessarily in the Fatou set of f, whereas parabolic and repelling cycles are in the Julia set. Each attracting, super-attracting, and parabolic cycle has an associated *basin* $\mathcal{B}_{\mathbf{p}}$, contained in the Fatou set. This basin

[2]Notice that in this thesis, when we refer to an attracting fixed point, we specifically mean an attracting fixed point that is not super-attracting.

[3]In this thesis we will mostly focus on attracting, super-attracting, and parabolic cycles. For more information on repelling cycles, the reader is referred to [28].

consists of all points in $\hat{\mathbb{C}}$ that converge to $\mathbf{p}$ under iteration. When the associated cycle $\mathbf{p}$ is understood, we will write $\mathcal{B}$ for the basin, suppressing the dependence on $\mathbf{p}$. In each of the attracting, super-attracting, and parabolic cases, the behavior of $f|_{\mathcal{B}_\mathbf{p}}$ under iteration can be understood via a local model: more detail is given in Chapters V and XIII.

2.3 Conjugacy

Given that we are studying the long-term behavior of maps, we often want to think of two maps as being "the same" if their long-term behavior is the same. Specifically, suppose we have some invertible map $\psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. We say that f is *conjugate* to g if

$$f = \psi^{-1} \circ g \circ \psi.$$

Notice that if this is the case, we also have

$$f^n = \psi^{-1} \circ g^n \circ \psi.$$

In particular, conjugation takes orbits of one map to orbits of the other, which provides a notion of dynamical equivalence. We will consider three types of conjugacy depending on the regularity of the conjugating map in question. If ψ is a homeomorphism, we say the f and g are *topologically conjugate.* If ψ is quasiconformal, we say f and g are *quasiconformally conjugate.* Finally, if ψ is a biholomorphic map, we say that f and g are *conformally conjugate.* In this last case, notice that ψ is necessarily a Möbius transformation and f and g are holomorphically equivalent. In particular, the Julia and Fatou sets of f and g differ only by a holomorphic change of coordinates.

Throughout the course of this thesis, we will often be restricting ourselves to looking at polynomial maps. Polynomial maps viewed as maps on the Riemann sphere have a fully ramified, super-attracting fixed point at ∞. In this setting, given a polynomial f, we define the *filled Julia set* of f as the set of all points in $\hat{\mathbb{C}}$ whose orbits remain bounded under iteration. We have that $J_f = \partial K_f$.

2.4 Parameter spaces

Given the notion of conformal equivalence, a natural question is to understand and classify the dynamics of *all* maps of, say, a given degree. The most well-understood example of this is in the family of quadratic polynomials. A quick computation shows that every quadratic polynomial is uniquely conjugate via an affine map (that is, a conformal map of $\hat{\mathbb{C}}$ that fixes the distinguished point ∞) to a polynomial of the form

$$f_c(z) = z^2 + c,$$

where $c \in \mathbb{C}$. In other words, the *moduli space* of quadratic polynomials is isomorphic to $\mathbb{C}$: given two polynomials $z \mapsto z^2 + c_1$ and $z \mapsto z^2 + c_2$, the polynomials are conformally conjugate if and only if $c_1 = c_2$. In the early 1900s, Pierre Fatou and Gaston Julia proved that there is a dichotomy: either the filled Julia set of a quadratic polynomial is connected, or it is a Cantor set. This partitions the parameter space

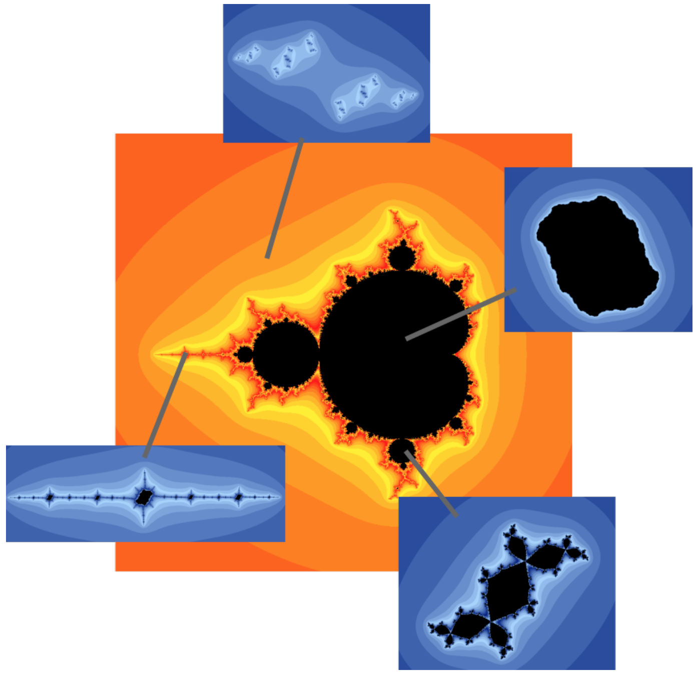


Figure 2.2: The Mandelbrot set, along with filled Julia sets for some parameters

into the *connectedness locus* $\mathcal{M}$, called the *Mandelbrot set*, given by

$$\mathcal{M} = \{c \in \mathbb{C} : K_c \text{ is connected}\},$$

and its complement $\mathbb{C} \setminus \mathcal{M}$, the Cantor locus.

The images in Figure 2.1 are examples of *dynamical* pictures — that is, each picture is associated with a single map f and every point in the image is colored based on its long-term behavior under iteration of f. On the other hand, the Mandelbrot set in Figure 2.2 is an example of a picture of a *parameter space* – each point in this space represents a quadratic polynomial, and the colors are chosen to distinguish maps with different dynamical properties. Throughout the rest of this document, we will have both dynamical and parameter pictures, and we will often work to draw parallels between the two. When possible, in this thesis we will draw dynamical space pictures in blue and parameter space pictures in orange to distinguish the two.

In the case of quadratic polynomials, the dichotomy that gave rise to the definition of $\mathcal{M}$ can be characterized entirely in terms of critical points. The filled Julia set of a quadratic polynomial f_c is connected if and only if $z_0 = 0$ (the unique critical point of f_c in $\mathbb{C}$) has bounded orbit. From this, we have

$$\mathcal{M} = \{c \in \mathbb{C} : \{f_c^n(0)\}_{n\in\mathbb{N}} \text{ is bounded.}\}$$

Remark 2.2. As a side note, this equivalent formulation is how all the images of the Mandelbrot set in this document are generated. It is much easier for a computer to check whether the orbit $\{f_c^n(0)\}$ escapes to ∞ as opposed to checking a topological property of some set.

CHAPTER III

Complex dynamics and geometric group theory: a link

The local dynamics of certain large classes of rational maps give rise to certain *quotient surfaces*. These surfaces will form the basis of most of the analysis in this thesis. We describe these surfaces and how they arise, and use them as motivation to define mapping class groups of both surfaces and maps, and to make connections between the two.

Throughout, we let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ denote a rational map of degree $d \geq 2$.

3.1 Quotient surfaces

As described in [25], the map f decomposes the Riemann sphere into a disjoint union

$$\hat{\mathbb{C}} = \hat{J} \sqcup \Omega^{\text{dis}} \sqcup \Omega^{\text{fol}}$$

where $\hat{J}$ is the union of the Julia set of f and the grand orbits of all of its periodic points and critical points, Ω^{dis} is the union of all attracting and parabolic basins (minus $\hat{J}$), and Ω^{fol} is the union of all super-attracting basins, Siegel disks, and

Herman rings[1] (minus $\hat{J}$). Under this subdivision, f is a covering map on Ω^{dis}.

Within $\Omega^{\text{dis}} \cup \Omega^{\text{fol}}$, we can define an equivalence relation that identifies points in the same grand orbit under f. That is, $z \sim f(z)$ for all $z \in \Omega^{\text{dis}} \cup \Omega^{\text{fol}}$. Then Ω^{dis} is the set where the equivalence relation is *discrete*, and Ω^{fol} is the set where the equivalence relation is indiscrete and *foliates* the set.

We will *mod out by the dynamics of f* (that is, take the quotient associated with this equivalence relation) on Ω^{dis}. The space Ω^{dis}/f is a disjoint union of punctured tori and punctured spheres. Here, the tori come from the basins of attracting periodic cycles of f, the spheres from the basins of parabolic periodic cycles of f, and the punctures correspond to the grand orbits of the critical points of f.

Definition 3.1. The Riemann surface Ω^{dis}/f is called the *quotient surface* associated to f.

Often we will mark a single attracting (respectively, parabolic) periodic cycle and consider the connected component of Ω^{dis}/f coming from the basin $\mathcal{B}$ of that periodic cycle. This is a torus (respectively, sphere) with finitely many punctures, each corresponding to the grand orbit of a critical point of f in $\mathcal{B}$. When the choice of cycle is understood, we will often refer to this Riemann surface as the *quotient torus* (respectively, *quotient sphere*). In this case, we denote the quotient surface as $\mathbb{T}_f$ or S^2_f respectively, and let $\mathcal{B}^* = \Omega^{\text{dis}} \cap \mathcal{B}$.

In this setting, we denote the covering map coming from the restriction of the

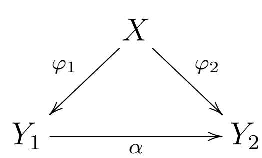
[1]Siegel disks and Herman rings are two other types of Fatou components. They will not be discussed in this thesis. For more information, the reader is referred to [28], specifically Chapters 11 and 15.

quotient map on Ω^{dis}/f by either $\Phi_f : \mathcal{B}^* \to \mathbb{T}_f$ in the case of a quotient torus, or $\mathrm{A}_f : \mathcal{B}^* \to S^2_f$ in the case of a quotient sphere.

3.2 Teichmüller space, moduli space, and mapping class groups

We will be discussing the relationship between the mapping class group of a rational map and that of a surface. We define both notions here.

For a 1-dimensional complex manifold X (potentially disconnected, and potentially with punctures), recall that $\mathrm{Teich}(X)$ consists of equivalence classes of pairs (φ, Y) where Y is a complex manifold and $\varphi : X \to Y$ is a quasiconformal homeomorphism[2]. Two elements (φ_1, Y_1) and (φ_2, Y_2) are equivalent in $\mathrm{Teich}(X)$ if we have a diagram



where $\alpha : Y_1 \to Y_2$ is conformal and the diagram commutes up to isotopy.

For a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $\mathrm{Teich}(f)$ consists of equivalence classes of pairs (h, g) where $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is quasiconformal, g is a rational map, and $h^{-1} \circ f \circ h = g$. Similarly to the classical theory, (h_1, g_1) and (h_2, g_2) are equivalent if we have a

[2]This definition, as well as those that follow, also make sense for a oriented topological surfaces. In the setting of topological surfaces, the quasiconformal homeomorphisms in these definitions are replaced by homeomorphisms or diffeomorphisms. This agrees with our definitions: for X, Y complex manifolds with no boundary components, a homeomorphism $\phi : X \to Y$ taking punctures to punctures can be promoted to a quasiconformal representative. For more details, see, for example, [14] or [24].

diagram

$$\begin{array}{ccc} & f & \\ {\scriptstyle c^{h_1}} \swarrow & & \searrow {\scriptstyle c^{h_2}} \\ g_1 & \xrightarrow[M]{} & g_2 \end{array}$$

where c^h denotes conjugation by h, M is a Möbius transformation, and the diagram commutes up to isotopy through quasiconformal maps that conjugate f to some rational map.

We have a similar parallel between the mapping class group of a surface and the mapping class group of a rational map. In particular, for such a 1-dimensional complex surface X, let $\mathrm{QC}(X)$ denote the space of quasiconformal homeomorphisms $\psi : X \to X$. We define

$$\mathrm{MCG}(X) = \mathrm{QC}(X)/\text{isotopy}.$$

Example 3.2. The Riemann sphere $\hat{\mathbb{C}}$ is trivial. To see this, note that any homeomorphism φ of the sphere has a representative in its isotopy class that fixes a point p. On $\mathbb{C} \cong \hat{\mathbb{C}} \setminus \{p\}$, we can then take the straight-line isotopy between φ and the identity. For non-trivial examples of mapping class groups, see Chapter IV.

Similarly, for f a rational map, let $\mathrm{QC}(f)$ denote the space of quasiconformal homeomorphisms $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ so that $h^{-1} \circ f \circ h = f$. Then

$$\mathrm{MCG}(f) = \mathrm{QC}(f)/\text{isotopy}.$$

The groups $\mathrm{MCG}(X)$ and $\mathrm{MCG}(f)$ act on $\mathrm{Teich}(X)$ and $\mathrm{Teich}(f)$ respectively:

For $\sigma \in \operatorname{MCG}(X)$, choose a representative $\psi \in \operatorname{QC}(X)$ of h. Then

$$\psi \cdot (\varphi, Y) = (\varphi \circ \psi^{-1}, Y).$$

For $\sigma \in \operatorname{MCG}(f)$, choose a representative $h \in \operatorname{QC}(f)$. Then

$$h \cdot (\tilde{h}, g) = (\tilde{h} \circ h^{-1}, g).$$

We use these actions to define the *moduli space* of a complex surface and of a rational map. In particular,

$$\mathcal{M}(X) = \operatorname{Teich}(X) / \operatorname{MCG}(X)$$

and

$$\mathcal{M}(f) = \operatorname{Teich}(f) / \operatorname{MCG}(f).$$

We think of $\mathcal{M}(X)$ as the space of surfaces homeomorphic to X up to biholomorphism. We think of $\mathcal{M}(f)$ as the space of rational maps quasiconformally conjugate to f up to conformal conjugacy.

3.3 Pure mapping class groups and moduli spaces

Throughout the course of this thesis, we will primarily refer to the *pure* mapping class group or *pure* moduli space of a map or a surface. In the setting of a rational map f of degree d, let $\mathcal{P}_f$ denote the set of $2d-2$ critical points of f (counted with multiplicity), and let X denote a complex surface with some punctures. Notice that in the definitions of the mapping class group and moduli space above, all maps in question must take critical points of f/punctures of X to critical points of f/punctures of X set-wise. These definitions can be modified to give definitions for PMCG(X), PMCG(f), $\mathcal{M}_*(X)$ and $\mathcal{M}_*(f)$, where all the notions are the same except that we require all maps in question to preserve $\mathcal{P}_X$ or $\mathcal{P}_f$ *pointwise* rel isotopy.

Let S_n denote the permutation group on n elements. If X has n punctures or $|\mathcal{P}_f| = n$, then S_n acts on MCG(X) or MCG(f) respectively by permuting the punctures or permuting the points $\mathcal{P}_f$, respectively. This gives rise to the following analogous relationships between the mapping class group and pure mapping class group of a surface, and the mapping class group and the pure mapping class group of a rational map via the following exact sequences:

$$1 \longrightarrow \text{PMCG}(X) \longrightarrow \text{MCG}(X) \longrightarrow S_n \longrightarrow 1$$

and

$$1 \longrightarrow \text{PMCG}(f) \longrightarrow \text{MCG}(f) \longrightarrow S_n \longrightarrow 1\ .$$

Remark 3.3. Given a holomorphic map, we can define the Teichmüller space, moduli space, and mapping class group of a map on a domain different from the Riemann sphere. In this case we will write, for example, PMCG(U, f) or Teich(U, f) where $f : U \to U$ is holomorphic and all quasiconformal maps in the definitions are also defined as maps from U to U.

Based on the similarity of the given definitions, it may not be surprising that there is a relationship between the mapping class groups of rational maps and the mapping class groups of Riemann surfaces. To best exploit this relationship, we restrict ourselves to looking at certain classes of rational maps, described in Definition 3.5.

Definition 3.4. A *critical orbit relation* of a rational map f is a relation of one of the two following forms.

1. There exist critical points $c_1 \neq c_2$ of f and $n, m \in \mathbb{N}$ so that $f^n(c_1) = f^m(c_2)$.
2. There exists a critical point c of f and $n, m \in \mathbb{N}$ with $n \neq m$ so that $f^n(c) = f^m(c)$.

We recall here the definition of an MCG-generic rational map as defined in the introduction, as this is the setting in which we will be working for much of the remainder of this thesis.

Definition 3.5. A rational map f is called MCG-*generic* if

1. f is hyperbolic,
2. f has no acyclic critical points in its super-attracting basins, and
3. f has no critical orbit relations coming from critical points in attracting basins.

Under these conditions, we can make a strong statement about the relationship between $\mathrm{MCG}(f)$ and the mapping class group of a certain Riemann surface.

Proposition 3.6. *If f is* MCG-*generic, then there is an inclusion* $\mathrm{MCG}(f) \hookrightarrow \mathrm{MCG}(\Omega^{\mathrm{dis}}/f)$.

Proof. From [25], we know that there is a natural isomorphism

$$\mathrm{Teich}(f) \cong M_1(J, f) \times \mathrm{Teich}(\Omega^{\mathrm{fol}}, f) \times \mathrm{Teich}(\Omega^{\mathrm{dis}}/f),$$

where $M_1(J, f)$ (the invariant Beltrami differentials on the Julia set of f) is trivial due to condition 1. of 3.5. Since f has no acyclic critical points in super-attracting basins (by condition 2. of 3.5) and no Siegel disks or Herman rings (by condition 1. of 3.5), $\mathrm{Teich}(\Omega^{\mathrm{fol}}, f)$ is trivial as well. Therefore, we have a natural isomorphism

$$\mathrm{Teich}(f) \cong \mathrm{Teich}(\Omega^{\mathrm{dis}}/f).$$

The actions of $\mathrm{MCG}(f)$ and $\mathrm{MCG}(\Omega^{\mathrm{dis}}/f)$ on $\mathrm{Teich}(f)$ and $\mathrm{Teich}(\Omega^{\mathrm{dis}}/f)$ are properly discontinuous, and so we have a diagram

$$\begin{array}{ccc} & \mathrm{Teich}(f) & \\ \swarrow & & \searrow \\ \mathcal{M}(f) & \dashrightarrow & \mathcal{M}(\Omega^{\mathrm{dis}}/f) \end{array}$$

where the maps from $\mathrm{Teich}(f)$ are covering maps. This gives a covering map $\mathcal{M}(f) \to \mathcal{M}(\Omega^{\mathrm{dis}}/f)$, which in turn induces an injection

$$\mathrm{MCG}(f) \hookrightarrow \mathrm{MCG}(\Omega^{\mathrm{dis}}/f).$$

□

Furthermore, the same statement holds if we don't allow for permutations of punctures or marked points. Specifically, we have the following.

Corollary 3.7. *If f is* MCG-*generic, then there is an inclusion* $\mathrm{PMCG}(f) \hookrightarrow \mathrm{PMCG}(\Omega^{\mathrm{dis}}/f)$.

Throughout this thesis, we will use this identification to view $\mathrm{MCG}(f)$ as a subgroup of $\mathrm{MCG}(\Omega^{\mathrm{dis}}/f)$ and $\mathrm{PMCG}(f)$ as a subgroup of $\mathrm{PMCG}(\Omega^{\mathrm{dis}}/f)$.

Finally, notice that Proposition 3.6 holds true for a larger class of rational maps — in particular, in the proof we needed only conditions 1 and 2 of Definition 3.5. As mentioned in the proof of 3.6, for hyperbolic rational maps, Ω^{dis}/f is a disjoint union of punctured tori. The benefit of the inclusion of condition 3 is that the grand orbits of critical points in Ω^{dis} are distinct, and so the number of punctures on the these tori correspond with the number of critical points in Ω^{dis}. Specifically, if f is MCG-generic, we have that

$$\mathrm{MCG}(\Omega^{\mathrm{dis}}/f) \cong \prod_i \mathrm{MCG}(\Sigma_{1,n_i}) \tag{3.1}$$

where n_i is the number of critical points in the ith attracting basin of f. That is, understanding the mapping class groups of punctured tori is key in understanding the mapping class groups of MCG-generic maps.

CHAPTER IV

Mapping class groups of surfaces

The main results of this thesis involve understanding mapping class groups of rational maps in relation to the much more well-understood and studied mapping class groups of surfaces. In this chapter, we introduce some fundamentals of mapping class groups of surfaces that will be referenced and used throughout the rest of this thesis. The topics and examples introduced have been picked and pruned to tie into the narrative of rational maps — for a much more complete overview of the rich study of mapping class groups, the reader is referred to [14].

4.1 Dehn twists

We first introduce a general construction of a type of nontrivial mapping class element that can be defined locally around a curve on any underlying surface. It turns out that these types of elements, called Dehn twists, play a huge role in the study of mapping class groups.

We begin with an annulus $A = S^1 \times [0,1]$. We define a *twist map* $T : A \to A$ via

$$T(s,t) = (s + 2\pi t, t).$$

Let X be a Riemann surface, and let γ be a simple closed curve in X. Choose some annular neighborhood N of γ and a homeomorphism $\psi : A \to N$. We define the *Dehn twist about* γ to be $T_\gamma : X \to X$ given by

$$T_\gamma = \begin{cases} \text{id} & \text{on } X \setminus N \\ \psi \circ T \circ \psi^{-1} & \text{on } N \end{cases}$$

Notice that the twist map T fixes the boundary of A pointwise, and so the map T_γ is continuous. Furthermore, while T_γ depends on the choice of neighborhood N and homeomorphism ψ, the isotopy class of T_γ depends only on the curve γ, and so $T_\gamma \in \text{MCG}(X)$.

In fact, as long at γ is not trivial or peripheral (that is, as long as γ is not nullhomotopic or homotopic to a marked point in X), T_γ is a *nontrivial* element of the mapping class group (see, for example, [14], Proposition 3.1).

4.2 A first example: $\text{MCG}(\mathbb{T})$

The mapping class group of a torus can be explicitly calculated. This calculation will tie in to a number of results about mapping class groups of rational maps.

Proposition 4.1. *The mapping class group of the torus is*

$$\text{MCG}(\mathbb{T}) \cong \text{SL}_2(\mathbb{Z}).$$

Proof. The isomorphism comes from the action of a homeomorphism of $\mathbb{T}$ on homology. Specifically, let $\sigma : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving homeomorphism. This induces an automorphism σ_* on $H_1(\mathbb{T}; \mathbb{Z})$ which doesn't depend on the homotopy class of the original map σ (so in particular, we can promote σ to a quasiconformal homeomorphism in the same class). We consider the map from $\mathrm{MCG}(\mathbb{T})$ to $\mathrm{GL}_2(\mathbb{Z})$ given by

$$\sigma \mapsto \sigma_*.$$

View the torus $\mathbb{T}$ as the product $S^1 \times S^1$, and let $\{a, b\}$ be the basis of $H_1(\mathbb{T}; \mathbb{Z})$ given by $a = [S^1 \times 1]$ and $b = [1 \times S^2]$. Then we have that for a homeomorphism σ, we get that the matrix associated to σ_* is given by

$$M_\sigma = \begin{pmatrix} i(\sigma_*(a), b) & i(\sigma_*(b), b) \\ i(a, \sigma_*(a)) & i(a, \sigma_*(b)) \end{pmatrix}$$

where i denotes the algebraic intersection number. We can then calculate

$$\begin{aligned} \det(M_\sigma) &= i(\sigma_*(a), b) i(a, \sigma_*(b)) - i(\sigma_*(b), b) i(a, \sigma_*(a)) \\ &= i(\sigma_*(b), b) i(\sigma_*(a), a) - i(\sigma_*(a), b) i(\sigma_*(b) - a) \\ &= i(\sigma_*(a), \sigma_*(b)) \\ &= i(a, b) \\ &= 1 \end{aligned}$$

since f is orientation-preserving. Therefore, the map $\sigma \mapsto \sigma_*$ does in fact give a map

$$\mathrm{MCG}(\mathbb{T}) \to \mathrm{SL}_2(\mathbb{Z}).$$

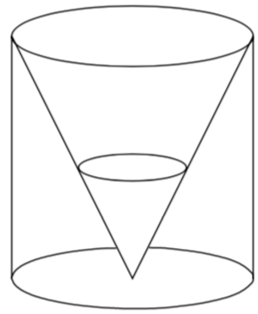


Figure 4.1: A picture proof (taken from [14]) that any homeomorphism of the disk that fixes the boundary is homotopic to the identity. Each horizontal slice in the image illustrates a slice of the homotopy. This is known as the *Alexander trick*.

To show that this map is an isomorphism, we show that it is both injective and surjective. Surjectivity follows from the fact that an element $M \in \mathrm{SL}_2(\mathbb{Z})$ induces a linear map on $\mathbb{R}^2$ that preserves $\mathbb{Z}^2$, and so M descends to a homeomorphism of $\mathbb{T} \cong \mathbb{R}^2/\mathbb{Z}^2$ whose image under our homomorphism is exactly M.

Let σ be a representative homeomorphism so that $M_\sigma = \mathrm{id} \in \mathrm{SL}_2(\mathbb{Z})$. Consider closed curves $\alpha, \beta \in \pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2$ so that $\alpha = (1,0)$ and $\beta = (0,1)$ in $\mathbb{Z}^2$ under this isomorphism. Since $\sigma_* = \mathrm{id}$, we must have that $\sigma|_\alpha$ and $\sigma|_\beta$ are both homotopic to the identity as maps. In particular, we can modify σ via homotopy to fix α and β pointwise. Now imaging cutting $\mathbb{T}$ along $\alpha \cup \beta$ — this yields a topological disk. Since σ fixes α and β, σ induces a homeomorphism of this disk that fixes the boundary pointwise. However, every homeomorphism of the disk that fixes the boundary is necessarily homotopic to the identity (see Figure 4.1). Therefore, σ is homotopic to the identity, and our homomorphism is an isomorphism. $\square$

In fact, the exact same proof goes through if we consider a once marked/punctured torus instead.

Proposition 4.2. *The mapping class group of the once punctured torus is*

$$\mathrm{MCG}(\Sigma_{1,1}) \cong \mathrm{SL}_2(\mathbb{Z}).$$

In both cases, we can take as generators the elements

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

in $\mathrm{SL}_2(\mathbb{Z})$. These correspond to Dehn twists around the $(1,0)$-curve and $(0,1)$-curve in $\mathbb{T}$ as in the proof of Proposition 4.1.

4.3 Point pushes

We will define the notion of a *point push*, another type of element of the mapping class group of a surface.

For a Riemann surface X, let $\hat{x} \in X$ be a marked point, and let γ be a simple closed curve in X, based at $\hat{x}$. Make an identification of an annular neighborhood N of γ with the annulus $A = S^1 \times [0,2]$ so that the point $\hat{x}$ corresponds to $(0,1)$ and γ corresponds to $S^1 \times \{1\}$ in A. We define an isotopy $H : A \times [0,1] \to A$ by

$$H((s,r),t) = \begin{cases} (s + 2\pi r t, r) & 0 \leq r \leq 1 \\ (s + 2\pi(2-r)t, r) & 1 \leq r \leq 2. \end{cases}$$

Notice that $H((s,0),t) = (s,0)$ and $H((s,2),t) = (s,2)$, and so H extends to an isotopy $H : X \times [0,1] \to X$ by $H(x,t) = x$ for all $x \in X \setminus N$. We define

$$\mathrm{Push}(\hat{x}, \gamma) \coloneqq H(x,1).$$

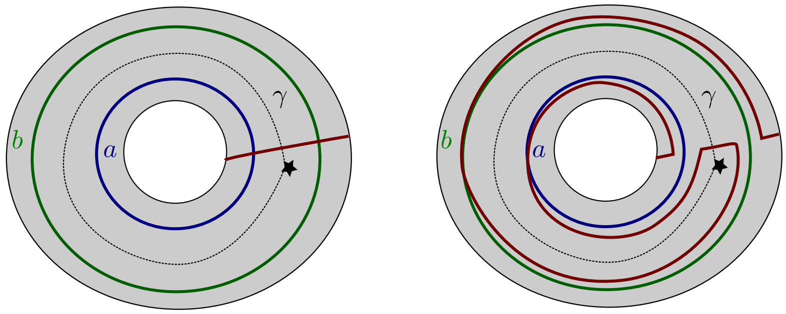


Figure 4.2: An illustration of the image of the red curve under the point pushing map around γ

We have that $H(\hat{x}, 1) = \hat{x}$, and so $\mathrm{Push}(\hat{x}, \gamma)$ is a homeomorphism of X fixing the marked point $\hat{x}$. The name comes from the fact that the isotopy H *pushes* the marked point $\hat{x}$ around the curve γ (see Figure 4.2). As in the case of the Dehn twist, while the homeomorphism depends on the choice of neighborhood N and identification with A, the isotopy class depends only on the homotopy class of γ.

We can write each point push as a product of two Dehn twists. Specifically, let a and b denote the isotopy classes of the simple closed curves gotten by pushing γ off to the left and to the right of $\hat{x}$ (via the standard orientation on $A = S^1 \times [0, 2]$). Then

$$\mathrm{Push}(\hat{x}, \gamma) = T_a T_b^{-1}.$$

4.4 The Birman exact sequence

let X be any (possibly punctured) oriented topological surface, and let $(X, \hat{x})$ denote the surface X with marked point $\hat{x} \in X$. It turns out point-pushing can be viewed as a well-defined map

$$\text{Push} : \pi_1(X, \hat{x}) \to \text{MCG}(X, \hat{x}),$$

and the image of this map has a simple characterization. Specifically, we get a homomorphism

$$\text{Forget} : \text{MCG}(X, \hat{x}) \to \text{MCG}(X)$$

that comes from forgetting the marked point $\hat{x}$. We have the following:

Theorem 4.3 (Birman exact sequence)**.** *The following sequence is exact:*

$$1 \longrightarrow \pi_1(X, \hat{x}) \xrightarrow{\text{Push}} \text{MCG}(X, \hat{x}) \xrightarrow{\text{Forget}} \text{MCG}(X) \longrightarrow 1$$

This result, originally proven in [2], can be shown via the long exact sequence of homotopy groups associated to the fiber bundle

$$\text{Homeo}^+(X, \hat{x}) \longrightarrow \text{Homeo}^+(X) \xrightarrow{\text{eval}_{\hat{x}}} X.$$

For more details, see Theorem 4.6 in [14].

The following fact about point-push elements of the mapping class group will be used a number of times. It says that the conjugate of a point-push is still a point-push.[1]

Lemma 4.4. *For any $\sigma \in \text{MCG}(X, \hat{x})$ and any $\gamma \in \pi_1(X, \hat{x})$, we have that*

$$\sigma \, \text{Push}(\hat{x}, \gamma)\sigma^{-1} = \text{Push}(\hat{x}, \sigma_*(\gamma)).$$

[1] In the following Lemma and the remainder of this thesis, to simplify notation, if γ_1 and γ_2 are two mapping classes, we write $\gamma_1\gamma_2 \coloneqq \gamma_2 \circ \gamma_1$. Notice that under this notation, mapping class elements are applied left-to-right.

CHAPTER V

Attracting basins and quotient tori

In this chapter, we give some background of the local dynamics in an attracting basin of a map. Many of the definitions and results follow [28], specifically Chapter 8.

5.1 Linearizing coordinates

If f has an attracting cycle $\mathbf{a}$ of multiplier λ with attracting basin $\mathcal{B}$, we will use two maps, ϕ and κ on $\mathcal{B}$ to track the convergence of points to the attracting cycle.

The map $\phi = \phi_f : \mathcal{B} \to \mathbb{C}^*$ is the usual *linearizing coordinate* on $\mathcal{B}$. That is, we can find biholomorphic ϕ in a neighborhood of the attracting cycle satisfying the functional equation

$$\phi \circ f = \lambda \cdot \phi. \tag{5.1}$$

This map can be extended radially on $\mathbb{C}^*$ via analytic continuation until we hit some critical point $c_0 \in \mathcal{B}$ of f. If a single critical point is hit in this way (as opposed to encountering two or more at the same radius) will be called the *preferred* critical point. Normalizing so that $\phi(\mathbf{a}) = 0$ and $\phi(c_0) = -1$ determines ϕ uniquely, and then ϕ can be extended to all of $\mathcal{B}$ to satisfying equation 5.1 as above.

This process implicitly gives the following classical result. For more detail see, for example, Lemma 8.5 in [28].

Lemma 5.1. *For every attracting cycle* $\mathbf{a}$ *of* f*, there is at least one critical point of* f *attracted to* $\mathbf{a}$*.*

In the context of quotient surface, this translates to the following.

Corollary 5.2. *If* f *has a quotient torus, this torus has at least one puncture.*

Notice here that under this procedure, the quotient torus of a map comes equipped with some extra dynamical information. In particular, the quotient torus $\mathbb{T}_f$ has a *distinguished homology class*. This class comes from a simple counter-clockwise closed curve in $\mathcal{B}$ that is a circle in the linearizing coordinates and surrounds the attractor (see Figure 5.1). We refer to any representative of this curve on $\mathbb{T}_f$ as the *distinguished curve* on $\mathbb{T}_f$. Such a curve will play a large role in describing the mapping class groups of different maps.

We also define here the *filled potential function* (as introduced in [31]), closely related to the linearizing coordinate ϕ, to be

$$\hat{\kappa} : \mathcal{B} \to [-\infty, \infty)$$

given by

$$\hat{\kappa}(z) = \frac{\log|\phi(z)|}{\log\frac{1}{|\lambda|}}.$$

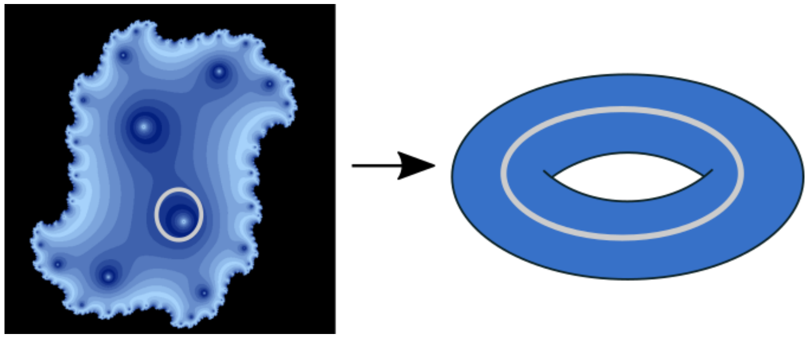


Figure 5.1: A representative of the distinguished curve on a quotient torus

Notice that

$$\hat{\kappa}(f(z)) = \hat{\kappa}(z) - 1.$$

We define, for $s \in \mathbb{R}$, $U(s)$ to be the connected component of $\hat{\kappa}^{-1}([-\infty, s))$ containing the attracting cycle $\mathbf{a}$ and $L(s)$ to be the connected component of $\hat{\kappa}^{-1}([-\infty, s])$ containing the attracting cycle.

Then we have that $U(s)$ is always a Jordan domain with $U(s) \subseteq L(s)$. Furthermore, $L(s) = \overline{U(s)}$ when there is no critical point of ϕ (or, equivalently, no backwards image of a critical point of f) in $\partial U(s)$. Otherwise, $L(s)$ is a multiply pinched disk — that is, a closed topological disk with a finite number of pairs of boundary points identified (see Figure 5.2).

We define

$$\kappa(z) = \inf\{s : z \in U(s)\}.$$

The following fact about κ will be useful later.

Lemma 5.3. *For any $s \in \mathbb{R}$, κ is locally constant on int$(L(s) \setminus U(s))$.*

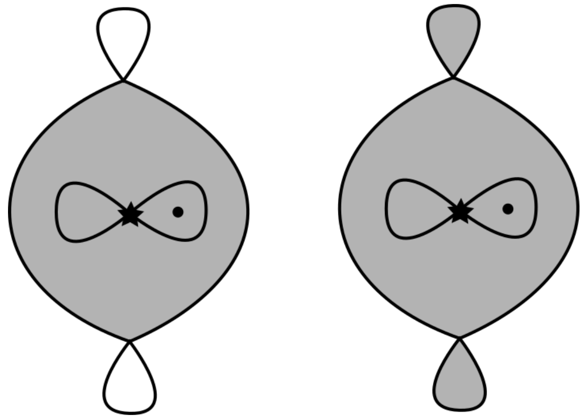


Figure 5.2: An example of the difference between U and L. The image shows some level curves of $|\phi|$. If ϕ is normalized so that $\phi(c_0) = -1$, then the (open) grey region on the left is $U(0)$ and the (closed) grey region on the right is $L(0)$.

Proof. Notice first that this holds vacuously for most s; in particular, if $L(s) = \overline{U(s)}$. On the other hand, if we choose s so that there exists some critical point of ϕ, then $L(s) \setminus U(s)$ is a union of disks and pinched disks. By definition of κ, for all $z \in L(s) \setminus U(s)$ we have $\kappa(z) = s$. □

We will use both ϕ and κ to compare relative positions of critical points in $\mathcal{B}$. In some very general sense, we think of $\phi(z)$ as measuring how long z takes to converge to the attracting cycle under f, and $\kappa(z)$ as measuring how far z is from the cycle when extending ϕ radially.

CHAPTER VI

Quadratic polynomials

As we saw in Chapter V, if f has a quotient torus, this torus must have at least one puncture. We provide an explicit calculation in the simplest case where the quotient torus has *exactly* one puncture. We first work through the example of the mapping class group of a quadratic polynomial with an attracting fixed point. This setting is especially nice, both because we can explicitly see the mapping class elements that we obtain, and also because this example exhibits some behavior that is important for understanding the general case.

Notice that the space of quadratic polynomials with an attracting fixed point is given by exactly those polynomials corresponding to parameters in the main cardioid of the Mandelbrot set. However, the parameter $c = 0$, corresponding to the map $z \mapsto z^2$ has a *super-attracting* fixed point. Here, Ω_f^{dis} is empty and f has no quotient torus. In other words, if f is a quadratic polynomial with an attracting fixed point, f is *not* globally quasiconformally conjugate to $z \mapsto z^2$. Therefore, we want to work in the punctured main cardioid. Denote by $\mathcal{C}^*$ the punctured main cardioid $\mathcal{C} \setminus \{0\}$.

Proposition 6.1. *Let $f_c : z \mapsto z^2 + c$ where $c \in \mathcal{C}^*$. Let a be the attracting fixed point of f and U a linearizing neighborhood of a. Then*

$$\mathbb{T}_c = U/f_c \cong \mathbb{C}/\Lambda$$

where $\Lambda = 2\pi i\mathbb{Z} \oplus \log(\lambda)\mathbb{Z}$ with λ the multiplier of the fixed point.

Proof. Since $c \neq 0$, the fixed point a of f_c is attracting with multiplier

$$\lambda_c := f'(0) = 1 - \sqrt{1 - 4c}.$$

For each such $c \in \mathcal{C}^*$, the attracting fixed point attracts the critical point $z = 0$ by Lemma 5.1.

Since U is linearizing neighborhood of f around this fixed point, there is some disk $\mathbb{D}_r$ with $\phi_f : U \to \mathbb{D}_r$ where $\phi_f(a) = 0$. Let $\lambda = f'(a)$, and let

$$M_\lambda : z \mapsto \lambda z$$

denote the map given by multiplication by λ. Then f is conformally conjugate to M_λ on U.

We have the identification $z \sim f(z)$ under the quotient map Φ_f, and since ϕ_f conjugates f to M_λ,

$$\mathbb{T}_c \cong \mathbb{D}_r/M_\lambda.$$

That is, we can consider the quotient of the disk $\mathbb{D}_r$ by the equivalence $z \sim \lambda z$. Let A_λ denote the annulus in $\mathbb{D}_r$ bounded by concentric circles of radius 1 and $|\lambda|$. Then A_λ is a fundamental domain for the quotient $z \sim \lambda z$. Notice also that, taking a path

γ from 1 to λ in A_λ, there is a unique branch of the log function defined along this curve satisfying $\log(1) = 0$. Under this branch, the exponential map takes the region

$$B = [0, \log \lambda) \times [0, 2\pi)$$

conformally onto A_{r_0}. Therefore, $\mathbb{T}_c$ is conformally isomorphic to $\mathbb{C}/\Lambda$ with $\Lambda = 2\pi i\mathbb{Z} \oplus (\log \lambda_c)\mathbb{Z}$ (see Figure 6.1). □

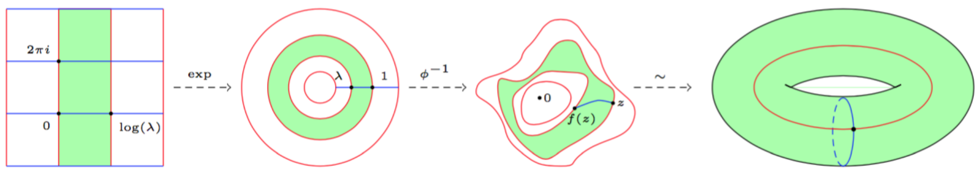


Figure 6.1: The identification torus coming from a map with an attracting fixed point

Now consider the quotient torus Ω^{dis}/f_c. This torus is exactly $\mathbb{T}_c$ punctured at the image of the critical point 0 of f_c under the quotient map. By Proposition 4.2, we have that $\text{MCG}(f) \hookrightarrow \text{MCG}(\Sigma_{1,1}) \cong \text{SL}_2(\mathbb{Z})$.

To make the connection between $\text{MCG}(f_c)$ and $\text{MCG}(\Sigma_{1,1})$, we use the following.

Proposition 6.2. *Any two quadratic polynomials in the same hyperbolic component of the Mandelbrot set are quasiconformally conjugate.*

For a discussion of this result, including a proof, see, for example, section 4.1 in [5].

This gives the following.

Corollary 6.3. *For $f_c \in \mathcal{C} \setminus \{0\}$, we have that*

$$\mathcal{M}(f_c) = \mathcal{C} \setminus \{0\}$$

and

$$\mathrm{MCG}(f_c) = \pi_1(\mathcal{C} \setminus \{0\}, f_c) \cong \mathbb{Z}$$

.

Proof. Certainly f_c cannot be quasiconformally conjugate to any map without an attracting fixed point, and so by Proposition 6.2, $\mathcal{M}(f_c) = \mathcal{C} \setminus \{0\}$. But therefore,

$$\mathrm{MCG}(f_c) \cong \pi_1(\mathcal{M}(f_c), f_c) \cong \mathbb{Z}.$$

□

Choosing a base point $f_{c_0} \in \mathcal{C} \setminus \{0\}$, we then see that

$$\mathrm{MCG}(f_{c_0}) \cong \mathbb{Z} \hookrightarrow \mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{MCG}(\mathbb{T}_{c_0}).$$

In particular, notice that not every element of the mapping class group of a punctured torus is induced via conjugacies of quadratic polynomials. Using Proposition 6.1, we can calculate explicitly what generator of $\mathbb{Z} \hookrightarrow \mathrm{SL}_2(\mathbb{Z})$ we get from the quadratic parameter space.

Proposition 6.4. *For f a quadratic polynomial with an attracting cycle,*

$$\mathrm{PMCG}(f) = \mathrm{MCG}(f) \cong \mathbb{Z}$$

with generator coming from a Dehn twist around the distinguished curve.

Proof. First suppose f has an attracting fixed point. Since there is only a single critical point in the basin of this attracting fixed point, fixing the punctures as a set and fixing the punctures pointwise are equivalent, and so $\mathrm{MCG}(f) = \mathrm{PMCG}(f)$.

Fix a base point $f_{c_0} \in \mathcal{C} \setminus \{0\}$, so that $\mathrm{MCG}(f_{c_0}) \cong \mathbb{Z}$ with a generator coming from the induced mapping class h obtained from moving around a loop in $\mathcal{C}$ around 0. To calculate which mapping class we get as a generator, let us fix our base point c_0 such that $\lambda_{c_0} = 0.5$, so that $\mathbb{T}_{c_0}$ comes from the lattice $2\pi i\mathbb{Z} \oplus \log(0.5)\mathbb{Z}$. Now, take the loop $\gamma : [0,1] \to \mathcal{C}$ such that the multiplier $\lambda_{\gamma(t)}$ of the unique attracting fixed point at each $\gamma(t)$ is $0.5e^{2\pi it}$. Note that γ is a simple closed curve in the c-plane winding once around 0, and therefore $[\gamma]$ generates $\pi_1(\mathcal{C} \setminus \{0\}, c_0)$ and induces h.

By Proposition 4.2, $\mathrm{MCG}(\mathbb{T}_{c_0}) \cong \mathrm{SL}_2(\mathbb{Z})$. Let β be the distinguished curve on $\mathbb{T}_{c_0}$, and choose simple closed curves α in $\mathbb{T}_{c_0}$ as below, so that Dehn twists T_α and T_β around α and β, respectively, get sent to

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

generators of $\mathrm{MCG}(\mathbb{T}_{c_0})$ (see Figure 6.2). Notice that here, our choice of α can also be described explicitly from a dynamical standpoint — specifically, α is the image of a curve in $\mathcal{B}$ with $\arg(\phi(z))$ constant.

To determine which element of the mapping class group we get from h, we look at what happens to α and β under this map. As t ranges from 0 to 1, we see that $\log \lambda_{\gamma(t)} = \log \gamma + 2\pi it$. On the other hand, clearly β is fixed under h. Therefore, under the identification of $\mathrm{MCG}(\Sigma_{1,1})$ with $\mathrm{SL}_2(\mathbb{Z})$, the map h corresponds to the map fixing $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and sending $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 11 \end{pmatrix}$. But this is exactly B, and therefore $h = T_\beta$.

The proof for when f has an attracting cycle that is not a fixed point goes through in the same way, by looking at a loop in the corresponding hyperbolic component around the center. $\square$

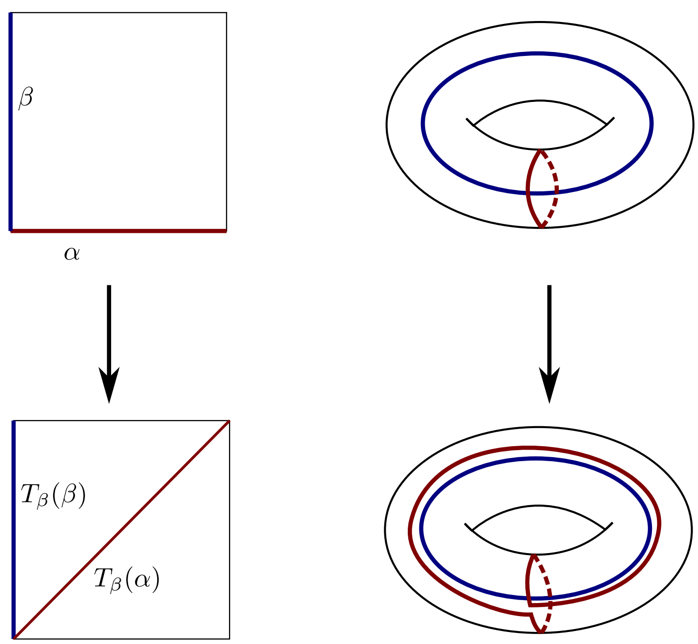


Figure 6.2: Two views of the Dehn twist T_β

CHAPTER VII

Non-dynamical mapping classes

A natural question that arises is whether we can find a loop in some other parameter space whose induced mapping class with the above notation is T_α. Towards this end, we first develop a necessary condition for an element $\sigma \in \mathrm{MCG}(\Sigma_{1,k})$ to be a dynamical mapping class. This condition will be useful in Chapter XI as well.

7.1 Factoring the quotient map

Often in this thesis, when looking at families of rational maps, we will want to *mark* the critical points of the maps. To this end, we first define the space of *critically marked maps* of degree d, following the definition given in [26].

Definition 7.1. Let Rat_d^{cm} denote the space of critically marked rational maps of degree d, so that Rat_d^{cm} consists of elements $(f, c_1, \ldots, c_{2d-2})$ where $f \in \mathrm{Rat}_d$ and $c_1, \ldots, c_{2d-2}$ is an ordered list of the critical points of f.

Let $(f, c_1, \ldots, c_{2d-2}) \in \mathrm{Rat}_d^{cm}$ be a critically marked MCG-generic map of degree $d \geq 2$ with a marked quotient torus. Without loss of generality, assume the marked cycle is a fixed point a with basin $\mathcal{B}$, and let λ denote the multiplier of a. Choosing a different marking if necessary, let $c_1, \ldots, c_k$ denote the critical points attracted to a, so that the quotient torus $\mathbb{T}_f$ under the quotient map Φ_f has k punctures $p_1, \ldots, p_k$. Notice that any pure mapping class $h \in \mathrm{PMCG}(f)$ must fix critical points of f pointwise.

Let β be the distinguished curve on $\mathbb{T}_f$. We can factor the quotient map Φ_f through a space that comes from "unrolling" $\mathbb{T}_f$ along β as follows. Consider the space

$$\mathcal{L}^* \coloneqq \phi_f(\mathcal{B}^*) \subseteq \mathbb{C}^*$$

where ϕ_f is the linearizing map on $\mathcal{B}$. Let $M_\lambda(z) \coloneqq \lambda z$, and $\Psi_\lambda : \mathcal{L}^* \to \mathbb{T}_f$ be the quotient map coming from identifying grand orbits under M_λ. Then

$$\Phi_f = \Psi_\lambda \circ \phi_f.$$

Notice further that while $\phi_f(c_i)$ and $\Psi_\lambda(\phi_f(c_i))$ are all well-defined (that is, we can view Φ_f as a map from all of $\mathcal{B}$), the map Φ_f is only a *covering map* on $\mathcal{B}^* \subseteq \mathcal{B}$. We will often switch between thinking of the images of the critical points of f under ϕ_f and Ψ as either marked points or punctures. When the distinction is important, we will make it.

First note that $h \in \mathrm{PMCG}(f)$ induces an isotopy class of homeomorphisms $\tilde{h} \in \mathrm{PMCG}(\mathcal{L}^*)$ that commutes with multiplication by λ via the following diagram.

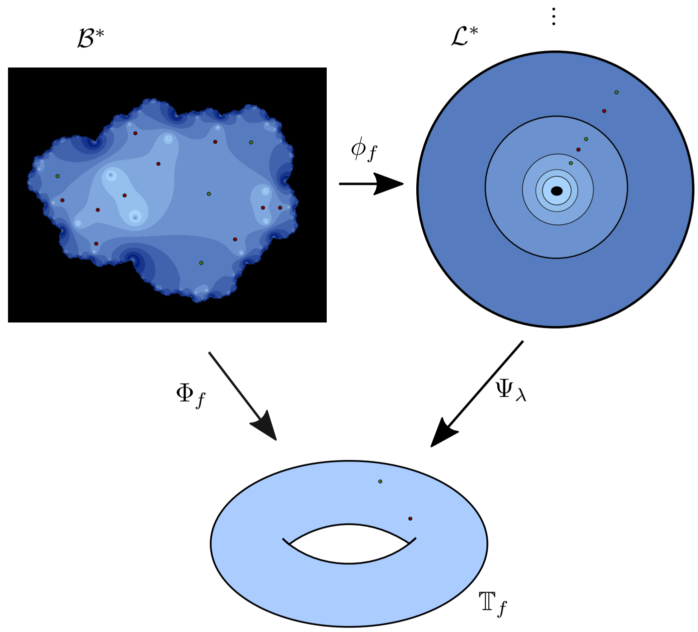
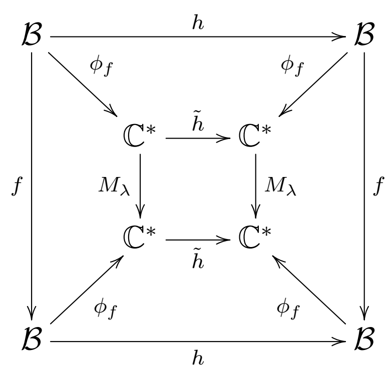


Figure 7.1: Factoring the quotient map



The map $\tilde{h}$ is defined in a neighborhood of $0 \in \mathbb{C}$ by

$$\tilde{h} = \phi_f \circ h \circ \phi_f^{-1}$$

on a neighborhood where ϕ_f is bijective, and then extended to all of $\mathcal{L}$ by

$$\tilde{h}(z) = \lambda^{-k} \cdot \tilde{h}(\lambda^k \cdot z).$$

In particular, we have that for all critical points c_i of f,

$$\tilde{h}(\phi_f(c_i)) = \phi_f(c_i)$$

On the other hand, if $\zeta \in \mathrm{PMCG}(\mathbb{T}_f)$, we can lift ζ to a homeomorphism class $\tilde{\zeta} \in \mathrm{MCG}(\mathcal{L}^*)$ under the covering map Ψ_λ. This lift will not be unique, but the different choices of lifts must differ by multiplication by λ. In particular, depending on the choice of lift, it is not true that $\tilde{\zeta} \in \mathrm{PMCG}(\mathcal{L}^*)$, but this failure can be easily understood (that is, all marked points can be shifted "up" or "down" in $\mathcal{L}^*$). Let $\mathrm{MCG}_\lambda(\mathcal{L}^*)$ denote the subgroup of $\mathrm{MCG}(\mathcal{L}^*)$ made up of classes of maps that send each marked point z of $\mathcal{L}^*$ to $\lambda^n z$ for $n \in \mathbb{Z}$

Lemma 7.2. *If $\tilde{\zeta} \notin \mathrm{MCG}_\lambda(\mathcal{L}^*)$, then ζ is not a dynamical mapping class.*

Proof. If ζ were a dynamical mapping class, then we must have $\tilde{\zeta} = \tilde{h}$ for some f and some $h \in \mathrm{PMCG}(f)$. However, as we saw above, $\tilde{h} \in \mathrm{MCG}_\lambda(\mathcal{L}^*)$. □

Now, let $\mathcal{A}^* \subseteq \mathrm{Rat}_d^{cm}$ be the space of critically marked maps of degree d with a marked quotient torus (that is, a marked attracting cycle). We have a map

$$\mathrm{Forget} : \mathcal{A}^* \to \mathcal{M}(\Sigma_1)$$

which fills in the punctures on the quotient torus. If $f \in \mathcal{A}^*$, we get an induced injection

$$\mathrm{Forget}_* : \mathrm{PMCG}(f) \to \mathrm{MCG}(\Sigma_1) \cong \mathrm{SL}_2(\mathbb{Z}).$$

We will prove the following general proposition.

Proposition 7.3. *For any* $f \in \mathcal{A}^*$,

$$\text{Forget}_*(\text{PMCG}(f)) \cong \mathbb{Z}$$

generated by T_β.

Proof. Let $\zeta \in \text{PMCG}(\Sigma_1)$, and suppose ζ is represented by a matrix $M_\zeta \in \text{SL}_2(\mathbb{Z})$. Suppose that $\zeta \in \text{Forget}_*(\text{PMCG}(f))$ for some f. Then there exists some $h \in \text{PMCG}(f)$ so that $\text{Forget}_*(h) = \zeta$. Consider the ramified cover $\Psi_\lambda : \mathbb{C}^* \to \text{Forget}(\mathbb{T}_f)$ and let

$$p = \Psi_\lambda(c_1) \in \mathcal{F}(\mathbb{T}_f)$$

be a marked point corresponding to a critical point c_1 of f.

Recall that $\text{PMCG}(\text{Forget}(\mathbb{T}_f))$ is generated by Dehn twists T_α and T_β corresponding to matrices A and B in $\text{SL}_2(\mathbb{Z})$. Let β_p be a curve homotopic to β in $\text{Forget}(\mathbb{T}_f)$ that goes through p.

Let

$$M_\zeta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The image $\zeta(\beta_p)$ corresponds to

$$M_\zeta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}.$$

But now, consider the lift $\tilde{\beta}_p \in \mathbb{C}^*$ so that $\Psi_\lambda(\tilde{\beta}_p) = \beta_p$ and so that $\tilde{\beta}_p$ goes through $\phi(c_1)$. The image $\tilde{\zeta}(\tilde{\beta}_p)$ then necessarily goes through $\lambda^b \phi(c_i)$. In particular, $\tilde{\zeta}(\phi(c_1)) = \phi(c_1)$ if and only if $b = 0$.

So if ζ is a dynamical mapping class, we must have that

$$M_\zeta = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}.$$

But since $M_\zeta \in \mathrm{SL}_2(\mathbb{Z})$, we therefore must have that

$$M_\zeta = \begin{pmatrix} \pm 1 & 0 \\ n & \pm 1 \end{pmatrix} = \pm B^n.$$

Therefore,

$$\mathrm{Forget}_*(\mathrm{PMCG}(f)) \hookrightarrow \mathbb{Z}.$$

To finish the proof, we show that the generator $B = T_\beta$ is in fact an element of $\mathrm{Forget}_*(\mathrm{PMCG}(f))$. The strategy is as follows: first, we construct an element of $\mathrm{PMCG}(f)$ by quasiconformal deformation of the multiplier of f. This is a standard procedure, and further details can be found in, for example, [5]. Having done that, we show that the image of this element in $\mathrm{Forget}_*(\mathrm{MCG}(f))$ is exactly T_β. This is simply a generalization of Proposition 6.4.

Let

$$\lambda(t) = \lambda \exp(2\pi i t)$$

and let $\mathbb{T}_\lambda(t)$ be the torus corresponding to the lattice $2\pi i\mathbb{Z} \oplus (\log \lambda(t))\mathbb{Z}$. Here, we make sure to choose our branch of logarithm so that $\log \lambda(t)$ varies continuously in t, so that, in particular, $\lambda = \lambda(0) = \lambda(1)$, but $\log(\lambda(1)) = 2\pi i + \log(\lambda(0))$. Notice that $\mathrm{Forget}(\mathbb{T}_f) = \mathbb{T}_{\lambda(0)}$.

Let $q_t : \mathbb{T}_{\lambda(0)} \to \mathbb{T}_{\lambda(t)}$ be a family of quasiconformal homeomorphisms. For each q_t we define an f-invariant conformal structure on $\mathbb{C}$ in a few steps. First, let σ_0 be the standard conformal structure on $\hat{\mathbb{C}}$. We define a conformal structure σ_t^* on $\mathcal{B}$ by pulling back the dilatation of q_t to all of $\mathcal{B}$ under the map Ψ_λ. We set $\sigma_t = \sigma_t^*$ on $\mathcal{B}$, and $\sigma = \sigma_0$ on $\mathbb{C} \setminus \mathcal{B}$. By the Measurable Riemann Mapping Theorem, there exists a quasiconformal map

$$h_t : \mathbb{C} \to \mathbb{C},$$

unique up to postcomposition with a Möbius transformation, so that $h_t \circ f \circ h_t^{-1}$ preserves the standard conformal structure, and is therefore a rational map f_t. By construction, f_t has a corresponding attracting fixed point with multiplier $\lambda(t)$, and the f_t vary holomorphically in t.

Notice further that $f_1 = f_0 = f$ up to conjugation by a Möbius transformation. This is the analogue of "moving around the super-attracting puncture" as in the example of the main cardioid. The class h_1 is an element of $\mathrm{PMCG}(f)$, and its image in $\mathrm{PMCG}(\Sigma_{1,1})$ is exactly the Dehn twist T_β.

□

From this, we immediately see that for each $n \geq 1$, not every mapping class of an n-punctured torus can be realized as a mapping class of a rational map. As a particular example, we get the following corollary.

Corollary 7.4. *The element $T_\alpha \in \mathrm{PMCG}(\Sigma_{1,1})$ is not a dynamical mapping class.*

We conclude this chapter by using Proposition 7.3 to prove one component of Theorem 1.6.

Corollary 7.5. *Let f be an* MCG*-generic rational map, and suppose that every attracting cycle of f contains exactly one grand orbit containing a critical point in its associated basin. Then*

$$\mathrm{PMCG}(f) \cong \mathbb{Z}^k,$$

where k is the number of attracting cycles of f.

Proof. Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ denote the attracting cycles of f. By Equation 3.1, we see that we can view

$$\mathrm{PMCG}(f) \hookrightarrow \prod_{j=1}^{k} \mathrm{PMCG}(\Sigma_{1,1}) \cong \prod_{j=1}^{k} \mathrm{SL}_2(\mathbb{Z}).$$

More specifically,

$$\mathrm{PMCG}(f) \cong \prod_{j=1}^{k} \mathrm{PMCG}(\mathcal{B}_{\mathbf{a}_j}, f|_{\mathcal{B}_{\mathbf{a}_j}}) \subseteq \prod_{j=1}^{k} \mathrm{PMCG}(\Sigma_{1,1}).$$

For each cycle $\mathbf{a}_j$ with multiplier λ_j, we can construct a representative element $q_1 \in \mathrm{PMCG}(\mathcal{B}_{\mathbf{a}_j}, f|_{\mathcal{B}_{\mathbf{a}_j}})$ by quasiconformally deforming the multiplier λ_j inside the basin.

By Proposition 7.3,

$$\mathrm{PMCG}(\mathcal{B}_{\mathbf{a}_j}, f|_{\mathcal{B}_{\mathbf{a}_j}}) \cong \mathrm{Forget}_*(\mathrm{PMCG}(\mathcal{B}_{\mathbf{a}_j}, f|_{\mathcal{B}_{\mathbf{a}_j}})) \cong \mathbb{Z}$$

and the result follows. □

CHAPTER VIII

Spinning

In the case of a once-punctured torus, we understand the mapping class group well enough to be able to explicitly calculate dynamical mapping classes. However, for higher-degree rational maps with quotient tori with more of punctures, we need more sophisticated tools.

Throughout the paper, we will make use of the technique of deforming a rational map via *spinning*, as developed in [32]. We describe the process here.

8.1 Definition of spinning

Fix a rational map $(f_0, c_1, \dots, c_{2d-2}) \in \mathrm{Rat}_d^{cm}$ with a marked attracting cycle $\mathbf{a}$ with basin $\mathcal{B}$. Let $\mathbb{T}_{f_0}$ denote the quotient torus associated with $\mathbf{a}$, with projection

$$\Phi_0 : \mathcal{B}^* \to \mathbb{T}_{f_0}.$$

Choose a critical point $\tilde{c} \in \mathcal{B}$ of f_0, and let $c = \Phi_0(\tilde{c})$ be its image puncture in $\mathbb{T}_0$. Denote

$$\mathbb{T}_{f_0}^{\#} = \mathbb{T}_{f_0} \cup \{c\}.$$

Choose a simple closed curve $\gamma : [0,1] \to \mathbb{T}_0^{\#}$ passing through c and parametrized so that $\gamma(0) = \gamma(1) = c$. We will "spin" c around γ. To do so, we modify the complex structure of $\mathbb{T}_0^{\#}$ in an annular neighborhood of γ by a translation, spread that structure to $\hat{\mathbb{C}}$ under Φ_0, and then apply the Measurable Riemann Mapping Theorem to obtain a new rational map.

More specifically, choose an annular neighborhood A of γ in $\mathbb{T}_{f_0}^{\#}$ with core curve γ, making A small enough so that it does not contain any punctures on $\mathbb{T}_{f_0}^{\#}$. We can choose a unique universal cover $\tilde{A}$ given by a strip, so that

$$\tilde{A} = \{z \in \mathbb{C} : -2k < \Im(z) < 2k\}$$

and so that the projection $p : \tilde{A} \to A$ has $p^{-1}(c) = \mathbb{Z}$ and $p(\mathbb{R}) = \gamma$. We spin the critical point around γ by defining

$$\tilde{h}_t : \tilde{A} \to \tilde{A}$$

to interpolate between the identity on $\partial\tilde{A}$ and translation to the right by t on the strip $-k \leq \Im(z) \leq k$.

As described in [32], $\tilde{h}_t$ is quasiconformal and descends to a homeomorphism

$$h_t : A \to A.$$

Furthermore, h_t extends to a quasiconformal homeomorphism on $\mathbb{T}_{f_0}^{\#}$, where $h_t = \text{id}$ on the complement of A.

We now use the map $h_t : \mathbb{T}_{f_0}^{\#} \to \mathbb{T}_{f_0}^{\#}$ to create a new rational map. In particular, pull back the dilatation of h_t on $\mathbb{T}_{f_0}$ to the basin $\mathcal{B}$ under the projection Φ_0. Then, invoking the Measurable Riemann Mapping Theorem, there exists a quasiconformal homeomorphism

$$H_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

that has the same dilatation. Therefore, the map

$$f_t = H_t \circ f_0 \circ H_t^{-1} \tag{8.1}$$

preserves the standard structure on $\hat{\mathbb{C}}$, and thus f_t is a rational map. Furthermore, H_t is unique up to postcomposition with a Möbius transformation, and different choices of H_t lead to Möbius conjugate maps f_t. Therefore, f_t defines a point in $\mathrm{Rat}_d \,/\, \mathrm{PSL}_2(\mathbb{C})$. Even more specifically, $f_t \in \mathcal{M}_*(f_0)$, and so we get a map

$$\sigma : \mathbb{R} \to \mathcal{M}_*(f_0)$$

given by

$$t \mapsto f_t$$

as defined above. Then $\sigma(\mathbb{R}) \subseteq \mathcal{M}_*(f_0)$ is some path in pure moduli space.

Definition 8.1. This path $\sigma(\mathbb{R})$ coming from spinning will be called the *spinning path.*

We will often be looking just at the segment $\sigma([0,1])$.

We write

$$f_t = \mathrm{Spin}_{c,t}(f_0, \gamma)$$

to denote the map f_t at time t coming from spinning the image of critical point c of the base map f_0 around the curve $\gamma \in \mathbb{T}_{f_0}$ based at $\Phi_0(c)$.

Extension to non-simple curves. Notice that in the construction of spinning, it is important that γ is a simple closed curve on the torus. However, if γ can be written as a concatenation of simple closed curves $\gamma_0, \ldots, \gamma_{m-1}$, we may extend the definition of $\mathrm{Spin}_{c,t}(f_0, \gamma)$ via spinning around each of the γ_j in turn. That is, we define

$$\mathrm{Spin}_{c,t}(f_0, \gamma) \coloneqq \mathrm{Spin}_{c,t-k}(f_0, \gamma_k)$$

for the appropriate value of $k \in \{0, \ldots, m-1\}$ so that $0 \leq mt - k \leq 1$.

Spinning and mapping classes. In the following work, we will use spinning to construct specific mapping classes of punctured tori. To do this, we distinguish between the map Spin, whose image is a rational map in $\mathcal{M}_*(f_0)$, and the map $\mathscr{S}$, whose image is a quasiconformal homeomorphism. In particular, define

$$\mathscr{S}_c(f_0, -) : \pi_1(\mathbb{T}_0^{\#}, c) \to \mathrm{QC}(f_0)$$

so that $\mathscr{S}_c(f_0, \gamma) = H_1$, where H_1 is homeomorphism with

$$f_1 = H_1 \circ f_0 \circ H_1^{-1}$$

as in equation (8.1). This image is not guaranteed to give an element of $\mathrm{PMCG}(f_0)$ — in fact, $\mathscr{S}_c(f_0, \gamma) \in \mathrm{PMCG}(f_0)$ exactly when $f_1 = f_0$ in $\mathcal{M}_*(f_0)$. This is summarized in the proposition below.

Proposition 8.2. *The image* $\mathscr{S}_c(f_0, -) : \pi_1(\mathbb{T}_{f_0}^{\#}, c) \to \mathrm{QC}(f_0)$ *is an element of* $\mathrm{PMCG}(f_0) \hookrightarrow \mathrm{PMCG}(\mathbb{T}_{f_0})$ *exactly when the associated spinning path satisfies*

$$\sigma(1) = \sigma(0) = f_0.$$

In this case,

$$\mathscr{S}_c(f_0, \gamma) = \text{Push}(c, \gamma) \subseteq \text{PMCG}(\mathbb{T}_{f_0}).$$

Proof. If $\sigma(0) = \sigma(1)$, the spinning path $\sigma = \sigma([0, 1])$ will be a loop in $\mathcal{M}_*(f_0)$. That is,

$$H_1 \circ f \circ H_1^{-1} = f$$

in $\mathcal{M}_*(f_0)$ and so we get a representative

$$H_1 \in \text{PMCG}(f_0).$$

On the level of the quotient torus, $h_1 : \mathbb{T}^{\#}_{f_0} \to \mathbb{T}^{\#}_{f_0}$ is exactly the point-push $\text{Push}(c, \gamma)$ (see figure 8.1).

□

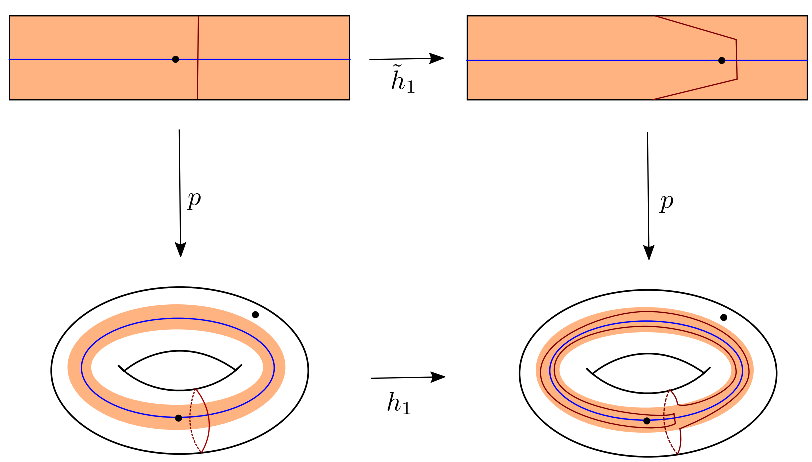


Figure 8.1: Spinning induces a point pushing map

Notice that we can find a spinning path based at f_0 that induces *any* point-push in $\text{PMCG}(\Sigma_{1,n})$ by simply pushing c around around a corresponding representative curve in $\pi_1(\mathbb{T}^{\#}_{f_0})$. However, we are not necessarily guaranteed to get a *dynamical* mapping class in this fashion, since the spinning path inducing this element might not be a loop in $\mathcal{M}_*(f_0)$. In other words, we are not guaranteed that the spinning construction on the chosen curve will result in $\sigma(1) = f_0$. Much of the remainder of this document explores the following question.

Question 8.3. *For a fixed base map with quotient torus, which point pushes on the torus yield dynamical mapping classes?*

Formulating the question in this way allows us to leverage results about the well-studied mapping class groups of surfaces to explore the comparatively less well understood mapping class group of rational maps.

We also remark that the formulation of Question 8.3 does not rely on working in a space of maps with a fixed degree. In some sense, it is only the local behavior near the attracting cycle of the map that governs the relationship between the mapping class group of the map and of the surface. This is what allows us to prove the statement in Theorem 1.6 for MCG-generic maps of arbitrary degree.

8.2 Spinning spaces

We conclude this chapter with a number of facts about spinning, and associated notation. Specifically, we build up a space of maps where the dynamics of all but a single critical point are fixed. For more detail, the reader is referred to [32]. Many of the results of this paper are built up from dynamical mapping classes that come from loops in this space.

As in [32], we will work in the space GRat_d, consisting of a cover of all conjugacy classes of *generic* critically marked rational maps of degree d (that is, those with simple critical points), where the locations of the critical points are globally defined functions. We will focus further on the open subspace of GRat_d consisting of maps with an attracting cycle with at least two critical points attracted to this cycle. Without loss of generality, for a map f_0 in this subspace, we may assume this cycle is a fixed point, by replacing f_0 with a sufficient power of itself. Finally, we can conjugate the map to normalize so that f_0 has a marked attracting fixed point at 0 with marked critical points $c_1 = 1, c_2 = \infty, c_3, \ldots, c_n$ attracted to 0. For the remainder of this chapter, we assume that any base map chosen is equipped with such a marking and normalized accordingly.

Fix now a base map f_0, equipped with a marking. We further impose the condition that $c_1, \ldots, c_{n-1}$ are attracted to 0 with no critical orbit relations. We will build up a submanifold of GRat_d containing f_0 in which the dynamics of $c_1, \ldots, c_{n-1}$ are fixed, but the dynamics of c_n is allowed to vary. We have a map $\Lambda : \mathrm{GRat} \to \mathbb{C}$ given by

$$\Lambda(f) = f'(0)$$

that sends each map to its multiplier at 0. We define $Y(f_0)$ to be the connected component of $\Lambda^{-1}(\Lambda(f))$ that contains f_0.

Lemma 8.4 (Corollary 2.1, [32])**.** *The space $Y(f_0)$ is a closed submanifold of GRat_d of dimension $2d - 2 - n$.*

We have another map $\Phi : \mathrm{GRat}_d \to \mathbb{C}^{n-1}$ given in terms of the linearizing coordinate of the map. Specifically, for $f \in \mathrm{GRat}_d$, we can normalize ϕ_f, multiplying by a scalar so that $\phi_f(0) = 0$ and $\phi_f(c_1) = -1$. With this constraint, the map ϕ_f is unique. Then, define

$$\Phi(f) = (\phi_f(c_1), \ldots, \phi_f(c_{n-1})).$$

Let $X(f_0)$ be the connected component of $\left(\Phi^{-1}|_{Y(f_0)}\right)(\Phi(f_0))$ containing f_0. We once again get that $X(f_0)$ is a closed submanifold of GRat_d.

Remark 8.5. In our setting, the space $X(f_0)$ is one complex dimensional, and the map $f \mapsto \phi_f(c_n)$ gives a local coordinate on $X(f_0)$.

In [32], the authors prove that the spinning path σ that comes from spinning c_n around a curve on $\mathbb{T}_{f_0}$ lies in $X(f_0)$. Specifically, we will make use of the following facts.

Lemma 8.6 (Lemma 2.1, [32]). *Spinning does not change the multiplier of the marked attracting cycle.*

Notice further that for all $c_1, \ldots, c_n$ critical points of f_0, for each

$$f_t := \mathrm{Spin}_{c_n,t}(\gamma, f_0),$$

we inherit a marking $c_1^t, \ldots, c_n^t$ on critical points of f_t since there is a unique critical point c_i^t of f_t so that $\{c_i^t\}$ varies continuously in t. This gives the following (compare with Proposition 2.4, VIII).

Lemma 8.7. *The linearizing coordinates of critical points of the maps on the spinning path satisfy*

$$\phi_t(c_i^t) = \phi_0(c_i)$$

and

$$\kappa_t(c_i^t) = \kappa_0(c_i)$$

for all $1 \leq i < n$.

CHAPTER IX

Cubic polynomials

Looking at the special case of cubic polynomials with an attracting fixed point is very informative, since in this case we can again leverage the low dimension to explicitly calculate mapping class groups. The method of proof closely follows that of [15] in which the authors prove an analogous theorem for quadratic rational maps. In addition, a number of the results needed for the main theorem of this chapter were proven in [31]. However, the use of spinning in the proof is novel, and serves to illustrate the main way in which we get our hands on mapping classes in higher degrees.

In this chapter, we prove the following special case of Theorem 1.5 where the base map is taken to be in the hyperbolic component containing $z \mapsto z^3$.

Theorem 9.1. *Let* f_0 *be a* MCG*-generic cubic polynomial with an attracting fixed point with both critical points in the basin. Then* $\mathrm{PMCG}(f_0)$ *is an infinitely generated subgroup of* $\mathrm{PMCG}(\Sigma_{1,2})$ *with generators given explicitly by a Dehn twist and an infinite collection of point pushes.*

9.1 Coordinates and parameter slices

Throughout, we use the general coordinates $f_{c,\lambda} : \mathbb{C} \to \mathbb{C}$ where

$$f_{c,\lambda}(z) = \frac{\lambda}{3(c^2-4)} z^3 - \frac{\lambda c}{c^2-4} z^2 + \lambda z$$

where the map has a fixed point at 0 with multiplier λ and critical points at

$$c_+ = c + 2$$

and

$$c_- = c - 2.$$

We will consider parameters in $(c, \lambda) \in \mathbb{C} \times \mathbb{D}^*$ (where $0 < |\lambda| < 1$ for $\lambda \in \mathbb{D}^*$, so that the fixed point at 0 is attracting).

In this space, there is a symmetry coming from the marking of the critical points.

Lemma 9.2. *The map $f_{c,\lambda}$ is affine conjugate to the map $f_{c',\lambda'}$ if and only if $\lambda = \lambda'$ and $c = \pm c'$.*

Proof. Any conjugacy between two such maps must fix 0 and therefore be of the form $z \mapsto \alpha z$. Furthermore, such a conjugacy will preserve the multiplier at that fixed point, and therefore necessarily $\lambda = \lambda'$. It is easy to check that then we must have $\alpha = \pm 1$ and therefore $c' = \pm c$. □

Let $X \subseteq \mathbb{C} \times \mathbb{D}^*$ denote the space of pairs (c, λ) associated cubic polynomials $f_{c,\lambda}$ that we are interested in — that is, those with both finite critical points in the immediate basin of 0. We want to say something about the mapping class group of a polynomial in f, and therefore we're interested in which maps $f_{c,\lambda}$ parametrized by $\mathbb{C} \times \mathbb{D}^*$ are quasiconformally conjugate. In particular, note that any two maps in X with no critical orbit relations so that both critical points are in the same immediate basin are quasiconformally conjugate (see, for example, [25]). Therefore $\mathbb{C} \times \mathbb{D}^*$ breaks up into the following subsets:

1. The set X that we are interested in.
2. The set E where $f_{c,\lambda}$ coming from a point in E has only *one* critical point attracted to 0.
3. The set A where $f_{c,\lambda}$ coming from a point in A has both critical points attracted to 0, but only one critical point in the *immediate* basin. The set A is discussed in much greater detail in Chapter XII.
4. The set P where a critical point of $f_{c,\lambda}$ is a preimage of 0.
5. The set O where the two critical points of $f_{c,\lambda}$ have a critical orbit relation of the form $f^n_{c,\lambda}(c+2) = f^m_{c,\lambda}(c-2)$.

Let $\mathscr{P}^3_0$ be the space of cubic polynomials that is parametrized by X – that is, parametrized by $(\mathbb{C} \times \mathbb{D}^*) \setminus (E \cup A \cup P \cup O)$. Fix a map $f_0 \in \mathscr{P}^3_0$. We have the following.

Lemma 9.3. *The projection map p sending a map $g \in \mathscr{P}^3_0$ to its Möbius conjugacy class gives a degree 2 cover $\mathscr{P}^3_0 \to \mathcal{M}(f_0)$ that is ramified over the curve $\{0\} \times \mathbb{D}^*$.*

Proof. This follows from the fact that any two maps in $\mathscr{P}^3_0$ are quasiconformally conjugate, as well as the analysis of $\mathrm{PSL}_2(\mathbb{C})$ conjugacy classes in Lemma 9.2. □

In the coordinates of $\mathscr{P}_0^3$, the two critical points are marked, and the identification of $f_{c,\lambda}$ with $f_{-c,\lambda}$ in $\mathcal{M}(f_0)$ comes from interchanging the marking. From this, we see that the image of $\pi_1(\mathscr{P}_0^3, f_0)$ under the map p_* is exactly $\mathrm{PMCG}(f_0)$ — the pure mapping class group of f_0 coming from those elements of $\mathrm{MCG}(f_0)$ which fix the critical points pointwise. Using this, we will study the fundamental group of the space $\mathscr{P}_0^3$ to understand $\mathrm{PMCG}(f_0)$.

We will often restrict to looking at only the left or right half-plane for a fixed $\lambda \in \mathbb{D}^*$, which we denote $\mathbb{H}_L$ and $\mathbb{H}_R$, respectively. (Here we suppress the dependence on λ since, as we will see, these slices for different λ all have a similar structure.)

Lemma 9.4. *Any two maps $f, g \in \mathbb{H}_L$ (respectively $\mathbb{H}_R$) are in different Möbius conjugacy classes.*

Proof. This follows immediately from Lemma 9.2. □

A picture of the slice $F_{1/2} = \overline{\mathbb{H}_L \cup \mathbb{H}_R}$ of this space with $\lambda = 1/2$ is shown in Figure 9.1. The region X we are focusing on is the unbounded orange region. The set E is the black region. We can also begin to see the set P here – this is the subset of points at the center of the lighter orange regions accumulating on the boundary of E. Some components of A might be seen if the reader looks very closely; they are smaller orange regions inside the black region.

We first make a couple more remarks about this parameter slice, since it is one that will play a key role in the results. In this slice, we write $f_c := f_{1/2,c}$ for simplicity. Recall that the critical points of the map f_c are at $c + 2$ and $c - 2$. We make

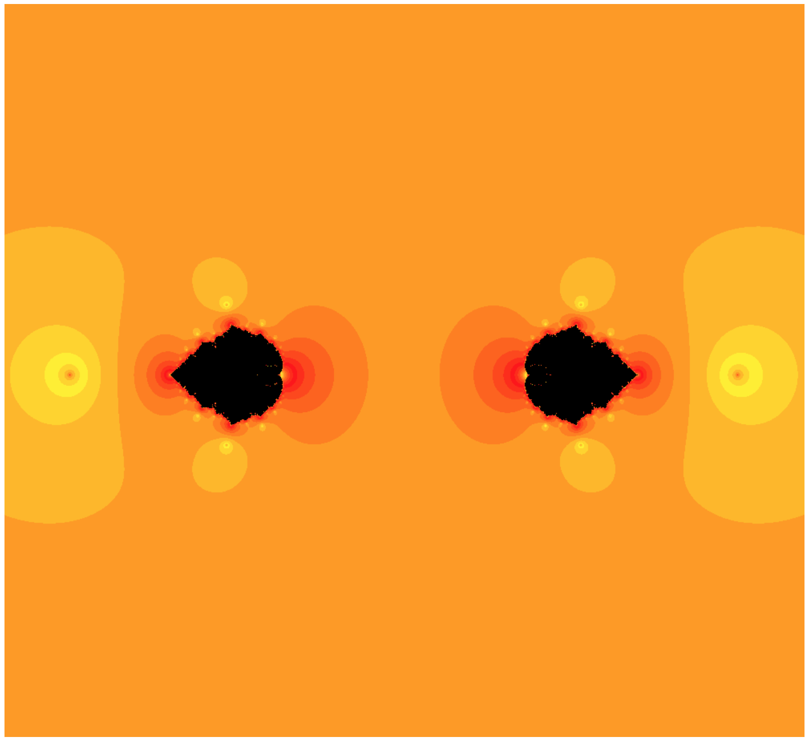


Figure 9.1: The c-parameter slice with fixed $\lambda = 1/2$

observations about the half-plane $\mathbb{H}_R$ with the understanding that by symmetry, analogous statements are true about $\mathbb{H}_L$.

Notice that for $f_c \in \mathbb{H}_R$, the critical point $c_- = c - 2$ is always attracted to 0 and is the *preferred* critical point (as defined in Chapter V).

To use the properties of this 1-dimensional slice to say something about the mapping class group, we relate $F_{1/2}$ to $\mathscr{P}_0^3$. Specifically, as in [15], we get a fibration

$$F_{1/2} \longrightarrow \mathscr{P}_0^3 \xrightarrow{\Lambda} \mathbb{D}^*. \tag{9.1}$$

where the map $\Lambda : \mathscr{P}_0^3 \to \mathbb{D}^*$ sends $f \mapsto \lambda = f'(0)$, the multiplier of its attracting fixed point. From the long exact homotopy sequence of fibrations, we get the short exact sequence

$$(9.2) \qquad 1 \longrightarrow \pi_1(F_{1/2}, f_0) \longrightarrow \pi_1(\mathscr{P}_0^3, f_0) \longrightarrow \pi_1(\mathbb{D}^*, \lambda) \longrightarrow 1$$

Lemma 9.5. *The short exact sequence given in (9.2) splits on the right.*

Proof. We want to show that there exists a section $\iota : \pi_1(\mathbb{D}^*, \lambda) \to \pi_1(\mathscr{P}_0^3, f_0)$. But since $\pi_1(\mathbb{D}^*, \lambda) \cong \mathbb{Z}$, we can simply send a generator of $\pi_1(\mathbb{D}^*, \lambda)$ to some preimage in $\pi_1(\mathscr{P}_0^3, f_0)$ under the map $\pi_1(\mathscr{P}_0^3) \to \pi_1(\mathbb{D}^*, \lambda)$, giving the necessary section. □

By Lemma 9.5, we then have that

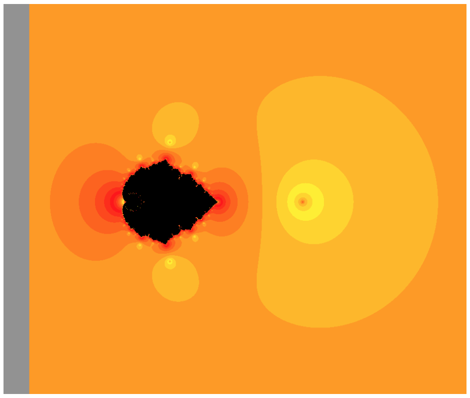
$$\text{PMCG}(f_0) \cong \pi_1(F_{1/2}, f_0) \rtimes \mathbb{Z}.$$

This decomposition lets us focus primarily on an analysis of the slice $F_{1/2}$.

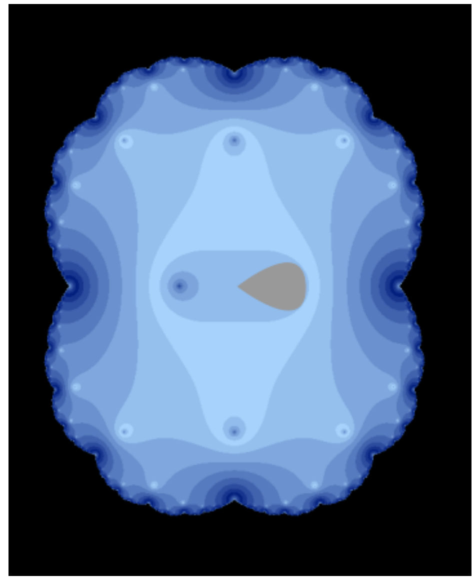
To calculate $\pi_1(F_{1/2}, f_0)$, we relate the parameter plane $F_{1/2}$ to the dynamical plane of a quadratic polynomial. The quadratic polynomial we choose is $Q(z) = z^2 + 0.5z$. Its filled Julia set is pictured below for reference, alongside the right half-plane $\mathbb{H}_R$ from the slice $F_{1/2}$.

We make this connection between these two spaces by proving the analogue of Theorem 3.3 in [15]. In particular, we will construct a homeomorphism from $\mathbb{H}_R$ to a subset of the filled Julia set K_Q.

Note that Q has a single critical point at $c_Q = -1/4$, and an attracting fixed point of multiplier $1/2$ at 0. Let ϕ_Q be the linearizing map around 0 for Q, normalized so that $\phi_Q(c_Q) = 1$, and κ_Q the filled potential function associated with ϕ_Q, as defined as in Chapter V. Recall that for each $s \in \mathbb{R}$, we have open and closed sets $U_Q(s)$



(a) Parameter slice for cubic polynomials



(b) Dynamical plane for quadratic polynomial Q

Figure 9.2: Similarities between dynamical and parameter spaces. We define a homeomorphism mapping the imaginary axis on the left to the boundary of the gray region on the right.

and $L_Q(s)$ respectively. The set $U_Q(0)$ is an open topological disk containing the attracting fixed point at 0, with c_Q on its boundary. Let $\mathcal{K} = \operatorname{int}(K) \setminus \overline{U_Q(0)}$ (see figure 9.2b — $\mathcal{K}$ is the interior of the filled Julia set minus the closed gray disk in the center).

9.2 Defining the homeomorphism

We define a homeomorphism $H : \mathbb{H}_R \setminus (E \cup A) \to \mathcal{K}$. That is, this is a map from the large orange region in Figure 9.2a to the blue region in Figure 9.2b. Let $f_c \in \mathbb{H}_R \setminus (E \cup A)$. Then since $f_c \notin E \cup A$, it has both critical points contained in the *immediate* basin of the attracting fixed point at 0. Furthermore, since $f_c \in \mathbb{H}_R$, the preferred critical point of f_c is given by $c_- = c - 2$.

In defining the homeomorphism, we mirror work done in [31]. However, we restate this work here in our more specialized environment, for the ideas will be extended in Chapter XI to prove Theorem 1.6.

We normalize the linearizing map $\phi_c \coloneqq \phi_{f_c}$ so that $\phi_c(c_-) = \phi_Q(c_Q) = 1$. The neighborhoods $U_c(0) \coloneqq U_{f_c}(0) \subseteq K_{f_c}$ and $U_Q(0) \subseteq K_Q$ satisfy $c_- \in \partial U_c(0)$ and $c_Q \in \partial U_Q(0)$. Furthermore, ϕ_c and ϕ_Q are biholomorphic on $U_c(0)$ and $U_Q(0)$ respectively. Define

$$\xi_c \coloneqq \phi_Q^{-1} \circ \phi_c.$$

Then since both Q and f_c have multiplier $\lambda = 1/2$ at 0, $\xi_c : U_c(0) \to U_Q(0)$ conjugates f_c to Q on a neighborhood of 0.

Since ξ is analytic, we can extend ξ to a homeomorphism on the boundaries, with $\xi_c(c_-) = c_Q$. We now want to extend ξ to larger subsets of K_{f_c}. To do so, we look at the topological structure of the sets of $U_c(s)$, $L_c(s)$, $U_Q(s)$, and $L_Q(s)$.

Let $T = \hat{\kappa}(c_+)$ and let $s \in \mathbb{N}$. When $s < T$, we have the following.

1. Each $U_c(s)$ and $U_Q(s)$ is a Jordan domain with 2^s preimages of c_- and c_Q on the boundary, respectively.

2. Each $L_c(s)$ and $L_Q(s)$ is a pinched disk with 2^s pinch points at the same preimages of c_- and c_Q, respectively.

3. The sets satisfy $\overline{U_c(s)} \subseteq L_c(s)$ and $\overline{U_Q(s)} \subseteq L_Q(s)$.

4. Furthermore, $f(U_c(s)) = U_c(s-1)$, $f(L_c(s)) = L_c(s-1)$, $Q(U_Q(s)) = U_Q(s-1)$, and $Q(L_Q(s)) = L_Q(s-1)$.

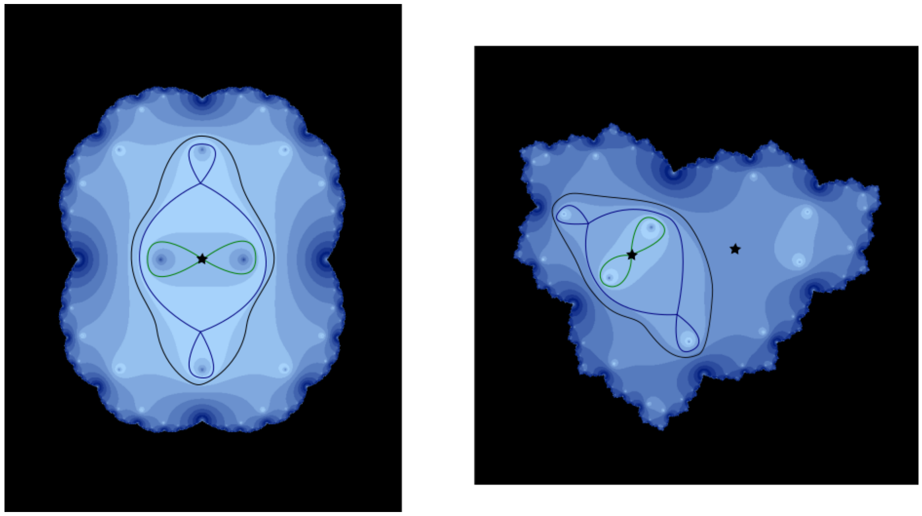


Figure 9.3: An image of $\partial L_Q(s)$ (on the left) and $\partial L_c(s)$ (on the right) for different values of s and one choice of c. The green curve is $\partial L_*(0)$, the blue is $\partial L_*(1)$, and the black is partial $L_*(s)$ for $1 < s < 2$.

For an example of some of these regions, see Figure 9.3.

Let $S \in \mathbb{N}$ be such that $S < T \leq S+1$. We extend ξ_c sequentially to $U_c(s)$ and $L_c(s)$ for $s \in \{0, \ldots, S\}$. In particular,

$$f_c(L_c(0)) \subseteq U_c(0)$$

and so we can lift

$$\xi_c \circ f_c : L_c(0) \to U_Q(0)$$

under Q so that the following diagram commutes.

$$\begin{array}{ccc} L_c(0) & \overset{\xi_c}{\dashrightarrow} & L_Q(0) \\ {\scriptstyle f_c}\downarrow & & \downarrow{\scriptstyle Q} \\ U_c(0) & \underset{\xi_c}{\longrightarrow} & U_Q(0) \end{array}$$

We can continue this sequentially, extending to $U_c(s)$ and $L_c(s)$ using property (4) above, whenever $s \leq S$.

However, we really want to extend ξ_c to the critical point c_+. This extension of ξ_c splits into two cases. In particular, we know that $c_+ \in L_c(S+1)$. The extension depends on whether or not $c_+ \in U_c(S+1)$.

Case 1: $c_+ \in U_c(S+1)$

Define

$$\Omega := \{z \in U_c(S+1) : |\phi_f(z)| < |\phi_f(c_+)|\} \subseteq U_c(S+1).$$

This is a Jordan domain with c_+ on the boundary. Furthermore, since $f(\Omega) \subseteq U_c(S)$, we have ξ_c defined on $f(\Omega)$ and we can take the correct lift to extend ξ_c to Ω.

Notice that this extension is necessarily biholomorphic, and therefore there is a further homeomorphic extension to $\partial\Omega$, and $\xi_c(c_+)$ is well-defined.

Case 2: $c_+ \in L_c(S+1) \setminus U_c(S+1)$

Here, the extension is slightly more delicate. The set $L_c(S+1) \setminus U_c(S+1)$ has 2^{S+1} components, all but one of which are topological disks. Label these disks $\mathbb{D}_1, \ldots, \mathbb{D}_{2^{S+1}-1}$. The final component $\mathbb{D}_0$ is the one that contains the critical point c_+. Topologically, it is a pinched disk. We can then find an open topological disk $D_+ \subseteq \mathbb{D}_0$ so that $c_+ \in \partial D_+$ and so that every point $z \in D_+$ satisfies

$$|\phi_f(z)| < |\phi_f(c_+)|$$

In fact, there are two disks satisfying this property — choose D_+ to be one of them.

Then

$$f|_{D_+} : D_+ \to L_c(S)$$

is biholomorphic. Additionally, there is an analogous disk $D_Q \subseteq L_Q(S+1)$ so that the restriction $Q|_{D_Q} : D_Q \to L_Q(S)$ is biholomorphic, and $\phi_Q(D_Q) = \phi_f(D_+)$. Then we can extend ξ_c to $U_c(S+1) \cup D_+$ by

$$\xi_c = Q|_{D_Q}^{-1} \circ f|_{D_+}$$

on D_+.

Furthermore, the same logic holds for each of the disks $\mathbb{D}_1, \ldots, \mathbb{D}_{2^{S+1}-1}$. In particular, there are analogous disks in K_Q and since each $\mathbb{D}_m$ has exactly one point on $\partial L_c(S)$ (where ξ_c is defined), it is clear how to extend ξ_c to $U_c(S+1) \cup \mathbb{D}_m$. Thus, in this case we can extend the domain

$$\Omega \coloneqq U_c(S+1) \cup \mathbb{D}_1 \cup \cdots \cup \mathbb{D}_{2^{S+1}-1} \cup D_+$$

and extend the map $\xi_c : \Omega \to K_Q$ biholomorphically so that, once again, ξ_c conjugates f to Q on this subset. Finally, we can extend ξ_c homeomorphically to the boundary. In particular, we get that $\xi_c(c_+)$ is well-defined. Notice that choosing D_+ differently would give ξ_c defined on a different subset of K_f, but the value $\xi_c(c_+)$ of its extension would not change.

We define the map

$$H(f_c) = \xi_c(c_+).$$

The construction of ξ_c immediately gives us a number of properties of H.

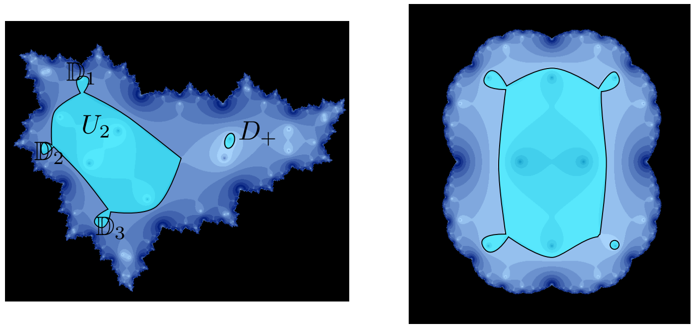


Figure 9.4: Extending the conjugacy ξ_c

Lemma 9.6. *The map* $H : \mathbb{H}_R \setminus (E \cup A) \to \mathcal{K}$ *satisfies the following properties.*

1. *If* $f_c^n(c_+) = 0$*, then* $Q^n(H(f_c)) = 0$*.*

2. *If* $f_c^m(c_+) = f_c^n(c_-)$*, then* $Q^m(H(f_c)) = Q^n(c_Q)$*.*

In other words, H *takes cubic polynomials with critical orbit relations to points in* $\mathcal{K}$ *in the critical orbit of* Q*.*

To prove injectivity of H, we will make use of the following lemma. We prove the lemma in greater generality than is necessary for this section, so that we can reference it later. The proof of the lemma, as well as the proof of injectivity of H, follow from work done in [31] (see, for example, section 5). However, we write down the proofs in our context since the ideas will be essential in proving Theorem 1.6 is Chapter XI.

Lemma 9.7. *Let* f_1 *and* f_2 *be* MCG*-generic, each with a marked fixed point. Suppose we have connected, disjoint open sets* U_1*,* V_1*,* U_2 *and* V_2 *so that*

1. $f_i(U_i) \subseteq U_i$ *and* $f_i(V_i) \subseteq V_i$*,*

2. all critical points of f_i *are in* $U_i \cup V_i$*, and*

3. there exist conformal maps $\varphi_U : U_1 \to U_2$ *and* $\varphi_V : V_1 \to V_2$ *that conjugate* f_1 *to* f_2 *on these domains.*

Then $f_1 = f_2$ *in* $\mathcal{M}_*(f_1) = \mathcal{M}_*(f_2)$*.*

Proof. Since the immediate basin of any periodic attracting point must have a critical point, and since the U_i and V_i are forward invariant, all such periodic points of f_i must be in $U_i \cup V_i$. Since f_i is MCG-generic, every point in the Fatou set of f_i must eventually be attracted to one of these periodic points. Therefore,

$$\bigcup_{n>0} f_i^{-n}(U_i \cup V_i) \subseteq \hat{\mathbb{C}} \setminus J_{f_i}.$$

We define a map $\eta_0 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by setting $\eta_0 = \varphi_U$ on U_1 and $\eta_0 = \varphi_V$ on V_1. We extend η_0 to all of $\hat{\mathbb{C}}$ in such a way that η_0 is globally quasiconformal. Specifically, we define η_0 as a $\mathcal{C}^1$ interpolation on the closed annulus between U_1 and V_1 as in Lemma 2.22 in [5]. Then η_0 is a quasiconformal conjugacy between f_1 and f_2 that is conformal on the sets U_1 and V_1.

If a_1 is the marked fixed point of f_1 and a_2 the marked fixed point of f_2, notice that we must have that $\eta_0(a_1) = a_2$.

We now iteratively lift η_0 to maps that are conformal on larger and larger unions of disks. To do so, we find a sequence of maps $\eta_k : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ so that the following diagram commutes:

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
\hat{\mathbb{C}} & \xrightarrow{\eta_k} & \hat{\mathbb{C}} \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\hat{\mathbb{C}} & \xrightarrow{\eta_1} & \hat{\mathbb{C}} \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\
\hat{\mathbb{C}} & \xrightarrow{\eta_0} & \hat{\mathbb{C}}
\end{array}$$

That is, let η_1 be the lift of $\eta_0 \circ f_1$ with $\eta_1(a_1) = a_2$. This lift is guaranteed to exist because f_1 and f_2 are covering maps away from their critical sets Crit_1, and Crit_2 respectively, and η_0 takes critical points of f_1 to critical points of f_2. Therefore,

$$(h_0 \circ f_1)_*(\pi_1(\hat{\mathbb{C}} \setminus \mathrm{Crit}_1, a_1)) \subseteq (f_2)_*(\pi_1(\hat{\mathbb{C}} \setminus \mathrm{Crit}_2, a_2))$$

and the lift η_1 exists and is unique.

Notice further that η_1 is a conformal conjugacy on $f_1^{-1}(U_1) \cup f_1^{-1}(V_1)$, and that $\eta_1 = \eta_0$ on U_1.

We can then iterate this procedure to generate a family of maps $\{\eta_k\}$. Taking a subsequence, we get a limit map η_∞ that is conformal on all of $\hat{\mathbb{C}}$ except, maybe, the Julia set J_{f_1}. However, since f_1 is MCG-generic, it is hyperbolic, and therefore J_{f_1} has Lebesgue measure zero. Thus, η_∞ is conformal and, by construction, conjugates f_1 to f_2. □

Lemma 9.8. *The map $H : \mathbb{H}_R \setminus (E \cup A) \to \mathcal{K}$ is injective.*

Proof. We prove that H is injective on $\mathbb{H}_R \setminus (O \cup P \cup E \cup A)$. Since H is continuous and $O \cup P$ is made up of isolated points, the result will follow.

Suppose we have two maps f_{c_1} and f_{c_2} in $\mathbb{H}_R \setminus (O \cup P)$, so that $H(f_{c_1}) = H(f_{c_2})$. We will construct a conjugacy between the two maps, and since every map in $\mathbb{H}_R$ has a unique conjugacy class representative in this space, this will give us that $c_1 = c_2$.

Let ϕ_1 and ϕ_2, respectively, denote the linearizing map for f_{c_1} and f_{c_2}. As in the construction of the map H, we have sets Ω_1 and Ω_2 with maps $\xi_1 : \Omega_1 \to K_Q$ and $\xi_2 : \Omega_2 \to K_Q$ that conjugate $f_{c_1}|_{\Omega_1}$ and $f_{c_2}|_{\Omega_2}$, respectively, to Q.

Consider the map $\eta : \Omega_1 \to \Omega_2$ given by

$$\eta = \xi_{c_2}^{-1} \circ \xi_{c_1}.$$

This map can be extended to a homeomorphism that satisfies

$$\eta(c_+^1) = c_+^2.$$

We now want to extend η even further — in particular, we want a biholomorphic extension of η to a Jordan domain in K_{f_1} that contains c_+^1 in its interior. To do so, let

$$S_i = \kappa(c_+^i)$$

so that η is defined from $\overline{U(S_1)} \to \overline{U(S_2)}$. We can then extend η to $L(S_1)$, since f_{c_1} maps each component of $L(S_1) \setminus U(S_1)$ either homeomorphically or 2-to-1 into $U(S_1)$, where η is defined.

Finally, we extend η to $U(S_1+1)$. To do so, consider the annuli $A_i = U(S_i)\setminus L(S_i-1)$. We have that η maps $\overline{A}_1$ to $\overline{A}_2$. We take the larger annuli $B_i = U(S_i+1)\setminus L(S_i)$. Then $f_{c_i} : B_i \to A_i$ is a 3-to-1 covering, and we can lift $\eta \circ f_{c_1}$ by f_{c_2} to extend η.

Take $\tilde{s}$ where $S_1 < \tilde{s} < S_1 + 1$, so that η is defined and biholomorphic on $U(\tilde{s})$ and c_+^1 is in its interior.

On the other hand, since each of the f_{c_i} are cubic polynomials, we also have Böttcher maps

$$\psi_i : V_i \to \hat{\mathbb{C}} \setminus \mathbb{D}$$

conjugating f_{c_i} to $z \mapsto z^3$ on a neighborhood V_i of infinity.

Therefore, by Lemma 9.7, $f_{c_1} = f_{c_2}$ in $\mathcal{M}_*(f_{c_1})$. □

Intuitively, this lemma establishes a correspondence between the set $O \cup P$ in our parameter slice and preimages of the fixed point and the critical value in K_Q.

9.3 An inverse via spinning

To show that $H : \mathbb{H}_R \setminus (E \cup A) \to \mathcal{K}$ is a homeomorphism, we just need to show that it is surjective. Here is where our proof differs significantly from that in [15]. In particular, we construct a right inverse to H using spinning. We will define a map

$$\mathscr{S} : \mathcal{K} \setminus \mathrm{GO}(c_Q) \to \mathbb{H}_R \setminus (A \cup E \cup O)$$

so that

$$H \circ \mathscr{S} = \mathrm{id}$$

where defined.

Once we have $\mathscr{S}$ as above, we can then extend the map continuously to the countable number of points in $\mathrm{GO}(c_Q)$, giving $\mathscr{S} : \mathcal{K} \to \mathbb{H}_R \setminus (A \cup E)$. Since H injective, we then get that H is a homeomorphism.

Let $f_0 \in \mathbb{H}_R$ be the base cubic map and Q be as above. Recall that we have quotient tori $\mathbb{T}_0$ and $\mathbb{T}_Q$ for f_0 and Q, respectively. Let Φ_0 and Φ_Q denote the respective quotient maps. There are punctures on $\mathbb{T}_0$ coming from $\Phi_0(c_-)$ and $\Phi_0(c_+)$. Let $\mathbb{T}_0^\#$ denote the filled-in torus $\mathbb{T}_0 \cup \{\Phi(c_-)\}$ on which spinning is defined.

Proposition 9.9. *The two tori $\mathbb{T}_0^\#$ and $\mathbb{T}_Q$ are isomorphic.*

Proof. We explicitly get the isomorphism by considering $\phi_Q^{-1} \circ \phi_0 : U_0(0) \to U_Q(0)$. This is a biholomorphism that descends to a map $\zeta : \mathbb{T}_0 \to \mathbb{T}_Q$. Furthermore, because of our choices of ϕ_Q and ϕ_0, we see that $\Phi_0(c_-) \mapsto \Phi_Q(c_Q)$ under ζ. Therefore, $\mathbb{T}_0^\# \cong \mathbb{T}_Q$. □

Let $\gamma : [0, 1] \to \mathbb{T}_0^\#$ be a path with $\gamma(0) = \gamma(1) = \Phi_0(c_+)$. We additionally require that γ can be decomposed into a finite concatenation of simple closed paths based at $\gamma(0)$ (so that spinning around γ is well-defined).

We denote $\mathrm{Spin}_t(\gamma) \coloneqq \mathrm{Spin}_{\Phi_0(c_+),t}(\gamma, f_0)$, the cubic polynomial obtained from spinning $\Phi_0(c_+)$ around γ from $\gamma(0)$ to $\gamma(t)$.

Proposition 9.10. *For any choice of γ and t as above, the cubic map $f_{\gamma,t} = \mathrm{Spin}_t(\gamma)$ is in $\mathbb{H}_R$.*

Proof. Spinning does not change the multiplier of the attractor, and therefore the new map has a unique conjugacy class representative in $\mathbb{H}_R$. □

For any $f_{\gamma,t} = \text{Spin}_t(\gamma)$ with critical points $c_-^{\gamma,t}$ and $c_+^{\gamma,t}$, we therefore have an associated $\xi_{\gamma,t}$ biholomorphic on a neighborhood around the attractor of $f_{\gamma,t}$ that conjugates $f_{\gamma,t}$ to Q, and $\xi_{\gamma,t}(c_+^{\gamma,t})$ is well-defined. Define

$$p_{\gamma,t} = \xi_{\gamma,t}(c_1(\gamma,t)) \in K_Q.$$

We define the map

$$\mathscr{S} : \mathcal{K} \to \mathbb{H}_{\mathbb{R}}$$

as follows. Let $p_0 = H(f_0) \in K_Q$. For a point $p \in K_Q \setminus (U_Q \cup \text{GO}(c_Q))$, we choose a path $\tilde{\gamma} : [0,1] \to K_Q \setminus U_Q$ that projects to a "nice" path on the quotient torus $\mathbb{T}_Q$. In particular, for most p, we will want the path to satisfy the following conditions.

Condition 9.11. (a) $\tilde{\gamma}([0,1]) \subseteq \mathcal{K} \setminus \text{GO}(c_Q)$,

(b) $\tilde{\gamma}(0) = \tilde{\gamma}(1) = p_0$,

(c) $\tilde{\gamma}(1/2) = p$,

(d) $\tilde{\gamma}$ can be written as a finite union of concatenated sub-paths $\tilde{\gamma}_1 \ldots \tilde{\gamma}_r$ so that the endpoints of each sub-path are contained in $\text{GO}(p_0)$, and for each $\tilde{\gamma}_i$, *either*

   i. $\arg(\phi(\tilde{\gamma}_i(t)))$ is constant across all t in the domain of $\tilde{\gamma}_i$, or

   ii. $|\phi(\tilde{\gamma}_i(t))|$ is constant across all t in the domain of $\tilde{\gamma}_i$.

Here, condition (a) guarantees that the projection $\Phi_Q \circ \tilde{\gamma}$ to the quotient torus $\mathbb{T}_Q$ avoids the puncture $\Phi_Q(c_Q)$ on this torus. Condition (d) guarantees that the projection traces out a path in the fundamental group of $\mathbb{T}_Q$ as a word in two standard generators, defined in detail later in this chapter. This in turn guarantees that the spinning path on $\mathbb{T}_Q$ can be written as a concatenation of simple closed curves. Recall from Chapter VIII that this means that spinning around this path is well-defined.

Specifically, we have the following.

Lemma 9.12. *If $\tilde{\gamma}$ satisfies Condition 9.11, its projection $\gamma : [0,1] \to \mathbb{T}_Q$ defined by*

$$\gamma = \Phi_Q \circ \tilde{\gamma}$$

is a concatenation of simple closed curves on $\mathbb{T}_Q$.

Proof. Let

$$Z_{mod} = \{z \in K_Q : |\phi(z)| = |\phi(p_0)|\},$$

and

$$Z_{arg} = \{z \in K_Q : \arg(z) = \arg(p_0)\}.$$

Then the images $\beta = \Phi_Q(Z_{mod})$ and $\alpha = \Phi_Q(Z_{arg})$ are two curves on $\mathbb{T}_Q$ that generate $\pi_1(\mathbb{T}_Q, \Phi_Q(p_0)) \cong F_2$. Furthermore, by definition of Φ_Q, every point in the grand orbit of p_0 gets mapped to the same point that p_0 does — namely the base point of the generating curves α and β.

Therefore, we see that the image

$$\Phi_Q(\tilde{\gamma}([0,1])$$

is completely made up of a concatenation of powers of α and powers of β. In particular, the image is a concatenation of simple closed curves on $\mathbb{T}_Q$. $\square$

We now construct a spinning path.

Lemma 9.13. *For each point $p \in \mathcal{K} \setminus \mathrm{GO}(c_Q)$, we can find a path $\tilde{\gamma} : [0,1] \to \mathcal{K} \setminus \mathrm{GO}(c_Q)$ with $\tilde{\gamma}(0) = \tilde{\gamma}(1) = p_0$ and $\tilde{\gamma}(1/2) = p$, so that the projection $\Phi_Q \circ \tilde{\gamma}$ is a concatenation of simple closed curves on $\mathbb{T}_Q$.*

Proof. For most points $p \in \mathcal{K} \setminus \mathrm{GO}(c_Q)$, this lemma is proven by constructing a path $\tilde{\gamma}$ satisfying condition 9.11, and then appealing to Lemma 9.12.

Throughout this proof, to ease the notational burden, we write $\phi = \phi_Q$ for the normalized linearizing coordinate for Q.

To do so, consider the loci

$$\Gamma_p = \{z : \arg(\phi(z)) = \arg(\phi(p))\}$$

and

$$\Gamma_{p_0} = \{z : \arg(\phi(z)) = \arg(\phi(p_0))\},$$

and let γ_p and γ_{p_0} be the connected components of Γ_p and Γ_{p_0}, respectively, containing p and p_0, respectively. Choose some $K > 1$ together with a point q^K on γ_p so that

1. $|\phi(q^K)| = K$, and
2. $|\phi(q^K)| = 2^m |\phi(p)|$ for some m (notice that this, together with the argument condition along the path γ_p guarantees that q^K and p are in the same grand orbit).

Choose q_0^K on γ_{p_0} according to the same constraints, with $|\phi(q_0^K)| = |\phi(q^K)|$. We then parametrize γ_p as a path from p to q^K and γ_{p_0} as a path from p_0 to q_0^K.

Finally, consider the set $U_K \subseteq K_Q$ given by

$$U_K = \{z \in K_Q : |\phi(z)| < K\}.$$

Since $K > 1$ (that is since $K > \phi(c)$), the domain U_K is connected. In particular, letting $\delta = \partial U_K$, we see that it must be the case that $q^K \in \delta$ and $q_0^K \in \delta$. Parametrize δ as a path from q^K to q_0^K, and notice further that for all points z in the image of this path, we have that

$$|\phi(z)| = K = |\phi(p_0)|.$$

Let δ' be the parametrized path from q_0^K to q^K so that the concatenation $\delta\delta'$ is the entire boundary of U_K.

The path given by the concatenation

$$\gamma_{p_0}\delta\gamma_p^{-1}$$

is now a path from p_0 to p that can be broken up into component paths as in 9.11 (d).

We define the entire path $\tilde{\gamma}$ as the concatenation

$$\gamma_{p_0}\delta\gamma_p^{-1}\gamma_p\delta'\gamma_{p_0}^{-1},$$

parametrized so that $\tilde{\gamma}(1/2) = p$. This path then satisfies conditions (b)-(d). Furthermore, if $\arg(p_0) \neq \arg(c)$ and $\arg(p) \neq \arg(c)$, we are also guaranteed that this path satisfies condition (a). However, depending on our choice of base point and the fixed point p, it is possible that the constructed path $\tilde{\gamma}$ may contain a point in the grand orbit of the critical point.

In this case, we must make a small perturbation of the constructed path. In order to guarantee that we end up with a path on which spinning is well-defined, we make this perturbation on the level of the projection $\Phi_Q(\tilde{\gamma})$ and lift it back to $K_Q \setminus U_Q$. In particular, consider $\gamma = \Phi_Q(\tilde{\gamma})$, whose image is a path contained in $\mathbb{T}_Q^{\#}$. In the case that the marked point $\Phi_Q(c_Q)$ is on this path, we perturb γ slightly to avoid $\Phi_Q(c_Q)$, but so that γ is still a concatenation of simple closed curves. In doing so, we make sure not to modify γ in a neighborhood of $\Phi_Q(p)$. This is possible, since $p \notin \mathrm{GO}(c_Q)$, and therefore $\Phi_Q(p) \neq \Phi_Q(c_Q)$.

We then lift the modified γ under the covering map Φ_Q to a modified $\tilde{\gamma}$, so that $\tilde{\gamma}(0) = p_0$. This modified $\tilde{\gamma}$ now satisfies the conditions of the lemma. □

Choose $\tilde{\gamma}$ as above with projection $\gamma : [0,1] \to \mathbb{T}_Q$ with $\gamma = \Phi_Q \circ \tilde{\gamma}$, so that

$$\gamma(0) = \gamma(1) = \Phi_Q(p_0).$$

Under the isomorphism $\mathbb{T}_Q \cong \mathbb{T}_0^{\#}$, we can view γ as a curve $\gamma : [0,1] \to \mathbb{T}_0^{\#}$ based at $\Phi_0(c_+)$.

Then, we define

$$\mathscr{S}(p) = \mathrm{Spin}_{\Phi(c_+),1/2}(\gamma, f_0).$$

First, notice a few facts about the maps $\mathrm{Spin}_{\Phi(c_+),1/2}(\gamma, f_0)$.

Lemma 9.14. *If γ is a curve on $\mathbb{T}_0^{\#}$ and $f_t = \mathrm{Spin}_{\Phi(c_+),t}(\gamma, f_0)$, then*

$$\gamma(t) = \phi_{f_t}(c_+^{\gamma,t}) = \phi_Q(p_{\gamma,t}).$$

Proof. The first equality follows from the definition of spinning.

By the assumptions on critical orbit relations, $p_{\gamma,t}$ is not a critical point of ϕ_Q, and therefore ϕ_Q is locally invertible around $p_{\gamma,t}$. We've also seen that therefore $\xi_{\gamma,t} = \phi_Q^{-1} \circ \phi_{f_t}$ for the correct choice of branch of ϕ_Q^{-1}. In particular, we have that

$$\begin{aligned}\phi_Q(p_{\gamma,t}) &= \phi_Q(\xi_{\gamma,t}(c_+^{\gamma,t})) \\ &= \phi_{f_t}(c_+^{\gamma,t})\end{aligned}$$

□

Proposition 9.15. *The map $\mathscr{S}$ is well-defined. That is, this construction depends only on the point $p \in \mathcal{K}$.*

Proof. Let $p \in \mathcal{K}$. Suppose we choose two loops γ_1 and γ_2 as described above. Notably, we have that $\gamma_1(1/2) = \gamma_2(1/2) = p$. Let

$$f_{\gamma_1} = \mathrm{Spin}_{1/2}(\gamma_1) \text{ and } f_{\gamma_2} = \mathrm{Spin}_{1/2}(\gamma_2).$$

But note that since $\gamma_1(1/2) = \gamma_2(1/2)$,

$$\xi_{f_{\gamma_1}}(c_+^1) = \xi_{f_{\gamma_2}}(c_+^2)$$

and since H is injective, by Lemma 9.8, f_{γ_1} and f_{γ_2} are conjugate.

□

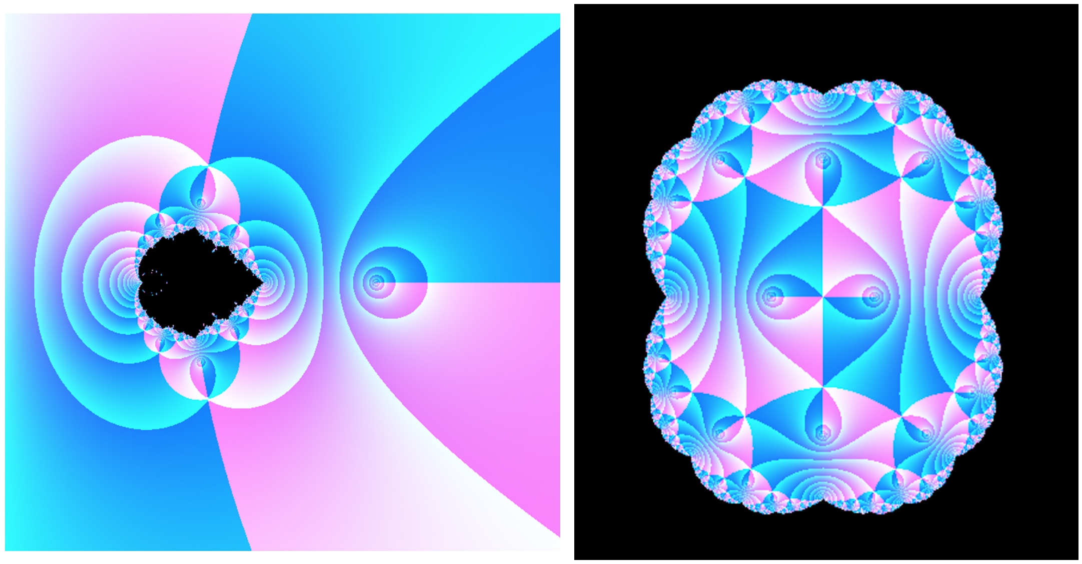


Figure 9.5: The homeomorphism between the parameter plane (left) and the dynamical plane (right).

Notice that we constructed $\mathscr{S}$ so that $\mathscr{S}(p)$ is exactly the map satisfying $\xi_{\mathscr{S}(p)}(c_+) = p$, so that $H(\mathscr{S}(p)) = p$. This gives the following.

Proposition 9.16. *The map $\mathscr{S} : \mathcal{K} \to \mathbb{H}_R \setminus (E \cup A)$ is an inverse to H.*

9.4 Remark: a connection with translation surfaces

Figure 9.5 gives an illustration of the homeomorphism H. (In the coordinates chosen for the parameter space, this homeomorphism turns the filled Julia set "inside out".) Notice that we can endow $\mathcal{B}_Q$ with the structure of a *square-tiled surface* — that is, $\Phi_Q : \mathcal{B}_Q \to \mathbb{T}_Q$ is a branched cover (branched over the image of the critical point c_Q). The image on the left shows such a tiling. From this point of view, $\mathcal{B}_Q$ is an *infinite translation surface*.

9.5 Understanding the mapping class group

Understanding the dynamics of a single quadratic polynomial is relatively simple in comparison to understanding a parameter slice. As such, we will follow the famous mantra of Adrien Douady to "sow the seeds in the dynamical plane and harvest in the parameter plane" to use the structure of $\mathcal{K}$ to say something about the slice $F_{1/2}$ and then, in turn, about $\mathrm{MCG}(f_0)$.

We remark that the remainder of the results in this section can be proved via techniques in [15]. For details on how these techniques are applied in this setting, the reader is referred to Chapter XII, where the discussion turns to the mapping class group of a map with multiple attracting basin components. However, we opt for a different analysis that is more indicative of techniques used in higher-degree cases. The idea behind this proof is to consider *all* point-pushes on the torus and, via tools from spinning, consider which ones come from dynamical mapping classes.

9.5.1 Calculation of dynamical mapping classes

Recall that we defined the set $\mathcal{K} = \mathrm{int}(K_Q) \setminus U_Q(0)$. Let

$$O^{n,m} = \{z \in K_Q : Q^n(z) = Q^m(c_Q)\}$$

and

$$P^n = \{z \in K_Q : Q^n(z) = 0\}.$$

That is, $O^{n,m}$ is the set of points in K_Q whose nth forward image agree with the mth forward image of the critical point of Q, and P^n is the set of nth preimages of

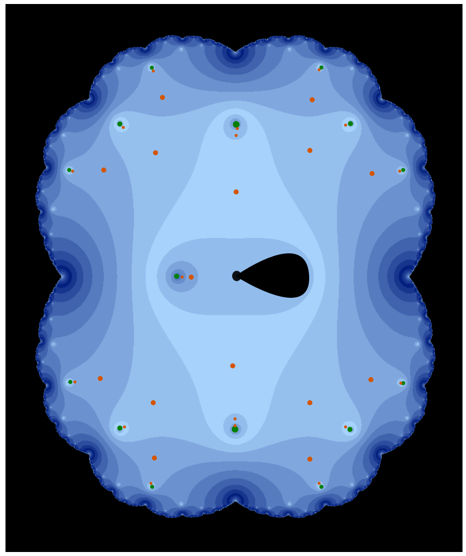


Figure 9.6: The space $\mathcal{B}^*$. A subset of the points P^n is drawn in green, and of $Q^{n,m}$ in orange.

0 under Q.

Let

$$\mathcal{K}^* = \mathcal{K} \setminus \left(\bigcup_{n,m=0}^{\infty} (O^{n,m} \cup P^n) \right).$$

The set P^n is made up of 2^n points in $\mathcal{K}$, and as $n \to \infty$, the sets P^n accumulate on the boundary of K_Q. Similarly, $O^{n,m}$ is made up of 2^n points, one "close to" each element of P^n. We also have that $\bigcup_{n,m=0}^{\infty} O^{n,m}$ accumulate on each point in $\bigcup_{n=0}^{\infty} P_n$. We see that

$$\pi_1(\mathcal{K}^*, p) \cong F_\infty,$$

the free group on countably many generators (see Figure 9.6).

We have that, by Lemma 9.6,

$$\mathcal{K}^* \cong F_{1/2}.$$

For any choice of base map $f_0 \in F_{1/2}$, let $v_0 := H(f_0) \in \mathcal{K}^*$. We then have that

$$\pi_1(F_{1/2}, f_0) \cong \pi_1(\mathcal{K}^*, v_0)$$

with the isomorphism coming from H_*. So

$$\pi_1(F_{1/2}, f_0) \cong F_\infty.$$

We can lift any closed curve γ based at $v = \Phi_Q(v_0) \in \mathbb{T}_Q$ to a curve $\tilde{\gamma}$ based at $v_0 \in K_Q$ under Φ_Q. If this curve is a loop, invoking the isomorphism coming from $\mathscr{S}_*$, we get a dynamical mapping class. We will enumerate an infinite set of curves $\gamma_n \in \pi_1(\mathbb{T}_Q, v)$ whose lifts $\tilde{\gamma}_n$ generate $\pi_1(\mathcal{K}^*, v_0)$, which will in turn imply that the collection $\mathrm{Spin}(\gamma_n)$ generates $\pi_1(F_{1/2}, f_0)$.

Let $\tilde{\beta}_0$ be the connected component of $\phi_Q^{-1}(|\phi_Q(v_0)|)$ containing v_0, and let $\beta \in \mathbb{T}_Q$ be given by

$$\beta = \Phi_Q(\tilde{\beta}_0)$$

(so that β is a representative of the distinguished curve of $\mathbb{T}_Q$). Similarly, consider the curve

$$\tilde{\alpha} = \phi_Q^{-1}(\arg(\phi_Q(v_0))) \subseteq K_Q.$$

Let

$$\alpha = \Phi_Q(\tilde{\alpha}).$$

Then $\pi_1(\mathbb{T}_Q, v) \cong F_2$ with free generators given by $[\alpha]$ and $[\beta]$.

We consider the structure of

$$\Phi_Q^{-1}(\alpha) \cup \Phi_Q^{-1}(\beta).$$

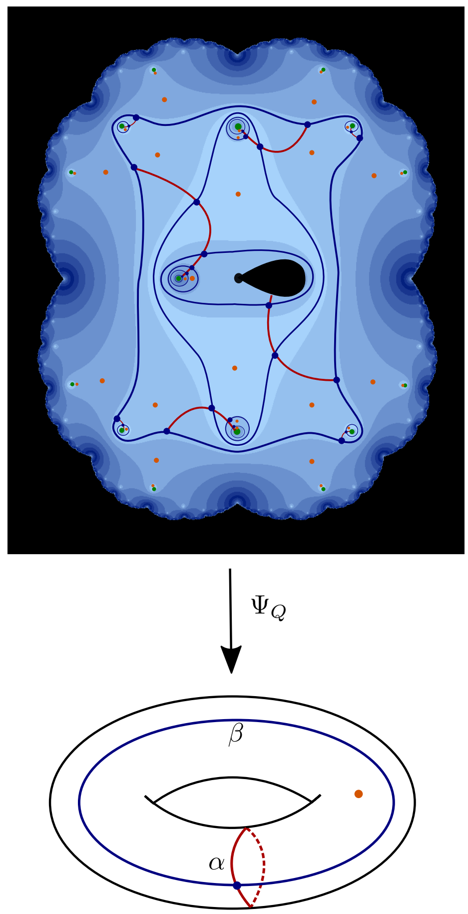


Figure 9.7: A subset of the lift of the curves α and β to $\mathcal{K}^*$

The lifts of α and β under Φ_Q partition $\mathcal{K}^*$ into components such that

1. Each component contains exactly one puncture in $O^{n,m}$, and
2. Each component is bounded by some set of lifts of α and β.

From here, we can enumerate a generating set for $\pi_1(\mathcal{K}^*, v_0)$ in terms of the curves in $\pi_1(\mathbb{T}_Q, v)$ that lift to them (though notice that the exact curves of course depend on our choice of base map f_0). For example, if we choose f_0 so that $H(f_0) = v_0$ with $-0.5 < \phi_Q(v_0) < -1$ and $\kappa(v_0) = \kappa(-0.25)$, we get that the subset of the loops that generate coming from punctures $O^{1,m}$ accumulation on P^1 come from the lift of curves of the form

$$\{\alpha^n \beta \alpha^{-n}\}_{n \geq 0}.$$

In particular, we can find a map f_0 so that

$$\{\text{Spin}(\alpha^n \beta \alpha^{-n})\}_{n \geq 0}$$

are dynamical mapping classes. Classes of this form will play an important role in Chapter XI. In fact, using the structure of the punctures P^n and $O^{n,m}$, we get that $\pi_1(F_{1,2}, f_0)$ are generated by $\text{Spin}(v, \gamma)$ where γ is in one of the following classes:

$$\{\alpha^n \beta \alpha^{-n}\}_{n \geq 0}$$

$$\{(\alpha^{-k} \beta^{2m+1})(\alpha^n \beta \alpha^{-n})(\alpha^{-k} \beta^m)^{-1}\}_{k>1, n>0, 1 \leq 2m+1 < 2^k}$$

$$\{\alpha^{-k} \beta^{2^k} \alpha^k\}_{k \geq 1}$$

$$\{(\alpha^{-k} \beta^{2m} \alpha \beta^{-m} \alpha^{k-1})\}_{k>1, 1<2m<2^k}$$

Let $\mathcal{G}_{1/2,v}$ denote this set of generators.

We are now ready to prove a special case of Theorem 1.5.

Theorem 9.17. *The pure mapping class group* $\mathrm{PMCG}(f_0)$ *is an infinitely generated subgroup of* $\mathrm{PMCG}(\Sigma_{1,2})$ *with generators explicitly given by point pushes and a Dehn twist.*

Proof. We saw in Lemma 9.5 that

$$\mathrm{PMCG}(f_0) \cong \pi_1(F_{1/2}, f_0) \rtimes \mathbb{Z}.$$

We saw that $\pi_1(F_{1/2}, f_0)$ is generated by $\mathcal{G}_{1/2,v}$, and by construction, the images of these generators in $\mathrm{PMCG}(\mathbb{T}_0)$ are exactly point pushes. Finally, the generator coming from the copy of $\mathbb{Z} \cong \pi_1(\mathbb{D}_*, \lambda)$ is a Dehn twist as calculated in Proposition 7.3 coming from varying the multiplier of the base map. □

Notice that in this setting of cubic polynomials, we recover a very similar result to the main result of [15] in the setting of quadratic rational maps.

CHAPTER X

Pure torus braid groups

The progress on understanding the mapping class group of a rational map — between the classical case of quadratic polynomials, the case of quadratic rational maps in [15], and now the setting of cubic polynomials — have all relied heavily on an explicit description of the space of MCG-generic maps. Attempts to promote these proofs to higher-degree settings run against a number of challenges as the spaces in question increase in dimension and complexity. Therefore, in order to begin to understand higher-degree rational maps, we need to develop different tools.

In particular, for f a general critically marked MCG-generic map of degree d, we will rely heavily on the idea of spinning marked points on the identification torus in order to construct elements of $\mathrm{PMCG}(f)$. As we saw in Chapters VIII and IX, the elements of $\mathrm{PMCG}(\Sigma_{1,n})$ that come from this spinning construction can often be written in the form of point-pushes, and we think about following trajectories of critical points as we move around in $\mathcal{M}_*(f)$. This idea motivates the introduction of *surface braid groups*, in which we consider these elements from spinning as inducing a trajectory of the punctures on the identification torus.

Definition 10.1. For a Riemann surface M[1] and $n \in \mathbb{N}$, the *nth configuration space of M* is defined as

$$F_n(M) = \{p_1, \ldots, p_n \in M^n : p_i \neq p_j \text{ for } i \neq j\}.$$

Definition 10.2. Given M, n as above, we define the *pure surface braid group* to be

$$P_n(M) = \pi_1(F_n(M)).$$

The configuration space $F_n(M)$ is connected, and so the choice of base point is suppressed in $P_n(M)$. There are many ways to define $P_n(M)$, and while we take Definition 10.2 to be our definition for this document, we will often think of an element of $P_n(M)$ as a trajectory of n non-colliding particles on M that return to their starting locations.

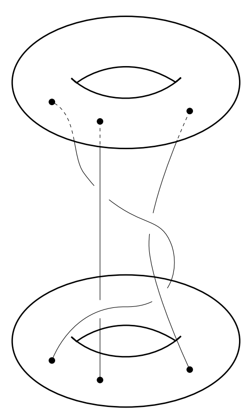


Figure 10.1: An element of $P_3(\Sigma_1)$

[1]As in Chapter II, this definition makes sense for any oriented topological surface. However, in our setting all of our surfaces are Riemann surfaces.

There is a natural way to view the pure surface braid group (or potentially a quotient of the surface braid group) as a subset of the pure mapping class group of a surface M with punctures. We will mostly be interested in the case where $M = \Sigma_1$ is a torus, and the rest of this chapter will focus on this special case.

Proposition 10.3 (c.f. [17], Section 2.4)**.** *We have the following short exact sequence:*

$$1 \longrightarrow P_n(\Sigma_1)/Z(P_n(\Sigma_1)) \longrightarrow \mathrm{PMCG}(\Sigma_{1,n}) \longrightarrow \mathrm{MCG}(\Sigma_1) \longrightarrow 1$$

Proof. A more general proof for surface braid groups of higher genus can be found in [17]. We recreate the proof for our case of a torus here.

Let $n \geq 1$ and fix a base point $\mathcal{E}_n$ of n distinct points on the torus Σ_1. It is true that the map

$$\Psi : \mathrm{Homeo}^+(\Sigma_1) \to F_n(\Sigma_1)$$

given by the images of $\mathcal{E}_n$ under the homeomorphism is a locally trivial fiber bundle with fiber $\mathrm{Homeo}_+(\Sigma_1, \mathcal{E}_n)$ — that is, orientation-preserving homeomorphisms of Σ_1 that preserve $\mathcal{E}_n$ pointwise. But note that

$$\pi_0(\mathrm{Homeo}^+(\Sigma_1, \mathcal{E}_n)) = \mathrm{PMCG}(\Sigma_{1,n}).$$

Therefore, taking the long exact homotopy sequence of the fibration

$$\mathrm{Homeo}^+(\Sigma_1, \mathcal{E}_n) \to \mathrm{Homeo}^+(\Sigma_1) \to F_n(\Sigma_1)$$

we get

$$\ldots \longrightarrow \pi_1(\mathrm{Homeo}^+(\Sigma_1, \mathcal{E}_n)) \longrightarrow \pi_1(\mathrm{Homeo}^+(\Sigma_1)) \longrightarrow \pi_1(F_n(\Sigma_1))$$

$$\longrightarrow \pi_0(\mathrm{Homeo}^+(\Sigma_1, \mathcal{E}_n)) \longrightarrow \pi_0(\mathrm{Homeo}^+(\Sigma_1)) \longrightarrow 1.$$

We have that

$$\pi_1(\mathrm{Homeo}^+(\Sigma_1, \mathcal{E}_n)) = 1$$

(see, for example, [18]). Furthermore,

$$\pi_1(\mathrm{Homeo}^+(\Sigma_1)) \cong \mathbb{Z}^2$$

and the inclusion in $P_n(\Sigma_1)$ is exactly the center $Z(P_n(\Sigma_1)) \cong \mathbb{Z}^2$ (c.f. [1]). $\square$

We use this relationship to move the problem of proving properties of $\mathrm{PMCG}(f) \subseteq \mathrm{PMCG}(\Sigma_{1,n})$ into the setting of the pure braid group $P_n(\Sigma)$.

In particular, we leverage the following explicit presentation for $P_n(\Sigma_1)$ (see [16]).

Theorem 10.4 (González-Meneses)**.** *There is a finite presentation of $P_n(\Sigma_1)$ with generators given by*

$$\mathcal{G}_B = \{a_i\}_{1 \le i \le n} \cup \{b_i\}_{1 \le i \le n} \cup \{T_{i,j}\}_{1 \le i < j \le n}$$

and relations $\mathcal{R}_B$. The relations are of the form

1. $a_n^{-1} b_n^{-1} a_n b_n = \Pi_{i=1}^{n-1} T_{i,n-1}^{-1} T_{i,n}$
2. $a_i a_j = a_j a_i$, $1 \le i < j \le n$
3. $b_i b_j = b_j b_i$, $1 \le i < j \le n$

4. $a_i b_j^{-1} a_i^{-1} b_j = T_{i,j} T_{i,j-1}^{-1}$, $1 \leq i < j \leq n$

5. $a_i b_i a_j b_i^{-1} a_i^{-1} a_j^{-1} = T_{i,j} T_{i,j-1}^{-1}$, $1 \leq i < j \leq n$

6. $T_{i,j} T_{k,\ell} = T_{k,\ell} T_{i,j}$, $1 \leq i < j < k < \ell \leq n$ *or* $1 \leq i < k < \ell \leq j \leq n$

7. $T_{k,\ell} T_{i,j} T_{k,\ell}^{-1} = T_{i,k-1} T_{i,k}^{-1} T_{i,j} T_{i,\ell}^{-1} T_{i,k} T_{i,k-1}^{-1} T_{i,\ell}$, $1 \leq i < k \leq j < \ell \leq n$

8. $a_i T_{j,k} = T_{j,k} a_i$, $1 \leq i < j < k \leq n$ *or* $1 \leq j < k < i \leq n$

9. $b_i T_{j,k} = T_{j,k} b_i$, $1 \leq i < j < k \leq n$ *or* $1 \leq j < k < i \leq n$

10. $a_i (b_j^{-1} a_j^{-1} T_{j,k} b_j a_j) = (b_j^{-1} a_j^{-1} T_{j,k} b_j a_j) a_i$, $1 \leq j < i < k \leq n$

11. $b_i (b_j^{-1} a_j^{-1} T_{j,k} b_j a_j) = (b_j^{-1} a_j^{-1} T_{j,k} b_j a_j) b_i$, $1 \leq j < i < k \leq n$

12. $T_{j,n} = \left(\Pi_{i=1}^{j-1} b_i^{-1} a_i^{-1} T_{i,j-1} T_{i,j}^{-1} a_i b_i \right) a_j b_j a_j^{-1} b_j^{-1}$, $1 \leq i < j \leq n$.

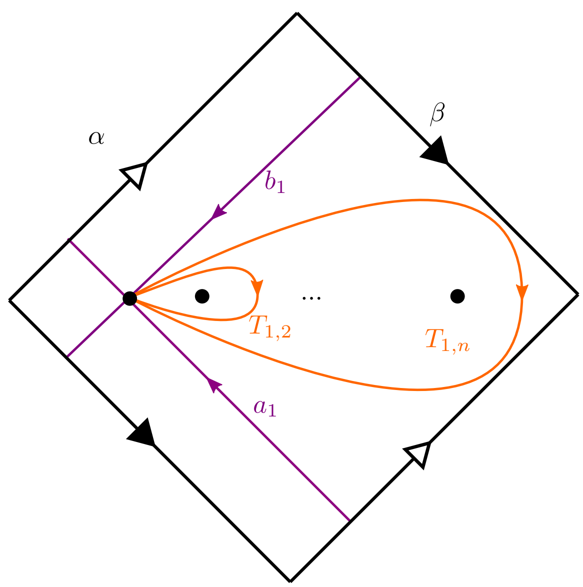


Figure 10.2: Generators of the pure torus braid group

Remark 10.5. For the sake of readability, we abuse notion to use b_i, T_{ij}, or a_i to represent both braids in $P_n(\Sigma_1)$, as well as based curves on the torus (in which case we think about the corresponding braid as coming from pushing the base point around that curve).

As we saw in Proposition 10.3, to relate the torus braid group to the mapping class group, we will need to understand the center $Z(P_n(\Sigma_1))$. The following is a result of [1].

Proposition 10.6 (c.f. [1])**.** *In terms of the presentation given in 10.4, the center satisfies*

$$Z := Z(P_n(\Sigma_1)) \cong \mathbb{Z}^2$$

with generators given by

$$\{a_1 a_2 \dots a_n,\ b_1 b_2 \dots b_n\}.$$

We will use the presentation in 10.4 to come up with a general criterion that all braids arising from dynamics must satisfy. To this end, the generators given by a_i play a special role.

Definition 10.7. For a word $w \in P_n(\Sigma_1)$, the *total a_i-degree*, denoted $\deg_{a_i}(w)$ is the sum of the exponents of a_i in w.

Looking at the relations coming from Theorem 10.4, we immediately get the following.

Proposition 10.8. *The total a_i-degree of a braid in $P_n(\Sigma_1)$ is well-defined.*

Notice that Proposition 10.6 implies that the total a_i-degree of a word in $P_n(\Sigma_1)/Z$ is *not* well-defined. However, we will make use of the following notion.

Definition 10.9. The *a-degree tuple* of a word w is defined to be

$$\deg_a(w) = (\deg_{a_1}(w), \deg_{a_2}(w), \ldots, \deg_{a_n}(w))$$

With this notion of, in some sense, "total a-degree" of a word, we have the following.

Proposition 10.10. *The a-degree tuple of a braid in $P_n(\Sigma_1)/Z$ is well-defined as an element of $\mathbb{Z}^n/\Delta$, where $\Delta \subseteq \mathbb{Z}^n$ is the diagonal.*

Proof. The relations for $P_n(\Sigma_1)$ preserve a_i-degree, and the relations coming from modding out by Z preserve the a-degree tuple up to addition by an element in the diagonal. □

CHAPTER XI

Mapping class groups of rational maps of higher degree

We now tie the surface braid group back to the mapping class group.

Lemma 11.1. *We have a commutative diagram*

$$\begin{array}{ccc} \mathrm{PMCG}(f) & \xrightarrow{\Lambda_*} & \pi_1(\mathbb{D}^*) \\ {\scriptstyle \Phi_*}\big\downarrow & & \big\downarrow{\scriptstyle m_*} \\ \mathrm{PMCG}(\Sigma_{1,n}) & \xrightarrow{\mathcal{F}_*} & \mathrm{MCG}(\Sigma_1) \end{array}$$

where the vertical maps are inclusions.

Proof. This comes from the diagram

$$\begin{array}{ccc} \mathcal{M}_*(f) & \xrightarrow{\Lambda} & \mathbb{D}^* \\ {\scriptstyle \Phi}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ \mathcal{M}_*(\Sigma_{1,n}) & \xrightarrow{\mathcal{F}} & \mathcal{M}(\Sigma_1) \end{array}$$

where m is the modulus map sending $\lambda \in \mathbb{D}^*$ to the torus $\mathbb{T}_\lambda$ with lattice

$$\Lambda = 2\pi i \oplus \log \lambda.$$

□

For a critically marked MCG-generic map $(f, c_1, \ldots, c_{2d-2})$ with marked attracting cycle $\mathbf{a}$, let $\mathrm{PDyn}_n(f)$ denote the pullback that makes the following diagram commute:

(11.1)
$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{PDyn}_n(f) & \longrightarrow & \mathrm{PMCG}(f) & \xrightarrow{\Lambda_*} & \pi_1(\mathbb{D}^*) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \Phi_* & & \downarrow & & \\ 1 & \longrightarrow & P_n(\Sigma_1)/Z & \longrightarrow & \mathrm{PMCG}(\Sigma_{1,n}) & \xrightarrow{\mathcal{F}_*} & \mathrm{MCG}(\Sigma_1) & \longrightarrow & 1 \end{array}$$

so that $\mathrm{PDyn}_n(f) = \ker(\Lambda_*)$.

Note that since $\pi_1(\mathbb{D}_*) \cong \mathbb{Z}$, the top sequence splits on the left, and therefore we can realize

$$\mathrm{PMCG}(f) \cong \mathrm{PDyn}_n(f) \rtimes \mathbb{Z}.$$

Consider now the quotient torus $\mathbb{T}_f$ associated with f. We choose a critical point c of f and set of generators of $\pi_1(\mathbb{T}_f, \Phi_f(c))$ with dynamical significance. Specifically, recall from Chapter VII that we have dynamically distinguished curves $\alpha, \beta \in \mathrm{Forget}(\mathbb{T}_f)$, where β is the image of the distinguished curve in K_f around the attractor. We will work with a set of generators $\{\alpha, \beta_1, \ldots, \beta_n\}$ of the fundamental group of the n-times punctured torus $\mathbb{T}_f$ so that $\beta_1, \ldots, \beta_n$ are homotopic to β in $\mathrm{Forget}(\mathbb{T}_f)$ (see Figure 11.1).

Consider the elements of $\mathrm{PMCG}(\mathbb{T}_f)$ given by

$$h_m = \mathrm{Push}(p_1, \alpha)^m \, \mathrm{Push}(p_1, \beta_1) \, \mathrm{Push}(p_1, \alpha)^{-m}$$

where $m \geq 0$ and α and β_1 are as in Figure 11.1.

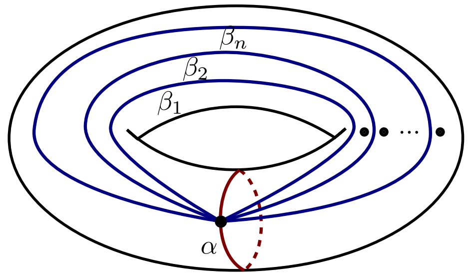


Figure 11.1: Generators of the fundamental group of an n-times punctured torus

We want to show that the family $\{h_m\}_{m\geq 0}$ is in fact made up of *dynamical* mapping classes — that is, that with an appropriate choice of base map f, there exist elements of $\mathrm{PMCG}(f)$ that induce each h_m in $\mathrm{PMCG}(\mathbb{T}_f)$.

11.1 A local homeomorphism

Throughout the rest of this subsection, we choose a critically marked MCG-generic base map f_0 of degree d satisfying the following conditions:

1. The map f_0 has an attracting cycle $\mathbf{a}$ with multiplier λ_0, with critical points $c_1, \ldots, c_n$ in the basin of $\mathbf{a}$, where $n \geq 2$.
2. The critical point c_n satisfies
   (a) $\phi_{f_0}(c_n) < \phi_{f_0}(c_i)$ for $i < n$, and
   (b) $\kappa_{f_0}(c_n) > \kappa_{f_0}(c_i)$ for $i < n$.

Note that by Lemma 5.3, $\kappa_{f_0}(c_n)$ is locally constant in a neighborhood of f_0.

Let $k_i = \phi_{f_0}(c_i)$. Let $D_\phi \subseteq \mathbb{C}$ be an open disk centered at 0 so that $k_n \in D_\phi$ but $k_i \notin D_\phi$ for $i < n$, and let $D^*_\phi = D_\phi \setminus \{0\}$. As in Chapter VII, the quotient map $\Phi_{f_0} : \mathcal{B}^* \to \mathbb{T}_0$ factors through $\mathcal{L}^* \supseteq D^*_\phi$, with quotient map $\Psi : \mathcal{L}^* \to \mathbb{T}_0$ coming

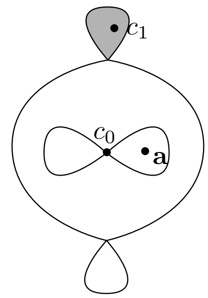


Figure 11.2: Some ϕ-level curves of the map f_0 when $n = 2$. Here, κ_{f_0} is constant in the grey shaded region.

from the identification $z \sim \lambda z$. Let $c = \Phi_{f_0}(c_n) \in \mathbb{T}_0$, the image of the spun critical point on the quotient torus.

Let $\tilde{\gamma}$ be the simple closed curve $t \mapsto k_n e^{2\pi i t}$ in D^*_ϕ. Then $\tilde{\gamma}$ projects to a simple closed curve $\gamma = \Psi(\tilde{\gamma})$ in $\mathbb{T}_0 \cup \{c\}$. Notice that once again, γ is a representative of the dynamically distinguished curve on the quotient torus.

We consider the spinning path

$$\{f_t = \text{Spin}(\gamma, t)\}_{t \in [0,1]}.$$

In this section, we work toward the following theorem.

Theorem 11.2. *We have that $f_0 = f_1$ in $\mathcal{M}_*(f_0)$.*

Notice that this is exactly what it means for the spinning path $\sigma[0,1]$ to be a closed loop — that is, for the induced quasiconformal conjugacy between f_0 and f_1 to be in PMCG(f_0).

To prove this theorem, we reveal a homeomorphism between a subset of $\mathcal{L}^*$ and subset of parameter space.

11.2 A disk-like subset of parameter space

Recall from Chapter VIII that we have a one-complex-dimensional submanifold $X(f_0) \subseteq \mathrm{GRat}_d$ in which spinning paths coming from spinning c_n lie. We define the subset $W(f_0) \subseteq X(f_0)$ to be the connected component of maps $f \in X(f_0)$ satisfying condition (2) above that contains the base map f_0.

We will construct a large number of maps $f \in W(f_0)$ coming from spinning along certain paths. In particular, for fixed $m \geq 0$ and $\ell \in \mathbb{Z}$, let $\gamma \in \mathbb{T}_0$ be given by the curve $\alpha^m \beta^\ell \alpha^{-m}$ and consider the family of maps f_t coming from

$$f_t = \mathrm{Spin}_{c_n,t}(f_0, \gamma).$$

Lemma 11.3. *The family of maps* $\{f_t\}$ *is in* $W(f_0)$.

Proof. It suffices to show that condition (2) is satisfied for all f_t, since then each f_t is in the same path component as f_0 in $X(f_0)$.

To show (2a), recall that if we are spinning c_n, then $\phi_{f_t}(c_i^t)$ stays constant for all $i < n$, and $\phi_{f_t}(c_n^t) = \tilde{\gamma}(t)$ for the lift $\tilde{\gamma} \in \mathcal{L}^*$ of the spinning curve $\gamma \in \mathbb{T}_0$. Since $\tilde{\gamma} \subseteq \mathbb{D}_{k_n}$, condition (2a) follows.

To show (2b) we use the following fact: if c_1 is the preferred critical point of f_0 and $\phi_{f_0}(c_n) < \phi_{f_0}(c_1)$, then there exists some neighborhood $N_n \subseteq K_{f_0}$ containing c_n so that $\kappa_{f_0} \equiv \kappa_{f_0}(c_n)$ is constant on N_n — specifically, we can take N_n be any open neighborhood of c_n in $L(s) \setminus U(s)$ (see Figure 11.2).

Now recall that for a map f_t, we defined

$$\hat{\kappa}_{f_t}(z) = \frac{\log|\phi_{f_t}(z)|}{\log\frac{1}{|\lambda|}},$$

and

$$\kappa_{f_t}(z) = \inf\{s : z \in U(s)\}$$

where $U(s) = \hat{\kappa}_{f_t}[-\infty, s)$. Furthermore, since $|\phi_{f_t}(c_n^t)| \leq |\phi_{f_0}(c_n)|$, we have that

$$\hat{\kappa}_{f_t}(c_n^t) \leq \hat{\kappa}_{f_0}(c_n)$$

for all choices of spinning parameter t. Then, since $|\phi_t(c_n^t)| < |\phi_{f_t}(c_i^t)|$ for $i < n$, we know that $\kappa_{f_t}(c_n^t) = |\lambda^\ell k_i|$. In particular, $\kappa_{f_t}(c_n^t)$ is a constant function in t for our choice of spinning path. Therefore, $\kappa_{f_t}(c_n^t) > \kappa_{f_t}(c_i)$ for $i < n$, and so each spinning image f_t is in the subset W_0 of parameter space. □

Lemma 11.4. *There exists $g \in W(f_0)$ with $\phi_g(c_n) = 0$.*

Proof. For each m, consider $h_m = \mathrm{Spin}_1(f_0, \alpha^m)$. We will show that the limit $G = \lim_{m\to\infty} h_m$ exists, and that this limit is exactly the map g we are looking for. This follows almost immediately from Theorem 1.3 in [32]. The only difference is that we are spinning along a curve γ in the *opposite direction* as the authors in [32], and the proof of Theorem 1.3 goes through under this modification. That is, we get that if G exists,

1. G has an attracting fixed point with multiplier λ, and

2. There exist embeddings $J_m : \mathcal{U} \to \hat{\mathbb{C}}$ converging to an embedding J, where $\mathcal{U}$ is a forward-invariant neighborhood in the attracting basin of f_0 that contains $c_1, \dots, c_{n-1}$, so that $J_m \circ f_0 = h_m \circ J_m$, and $J \circ f_0 = G \circ J$.

In particular, G has an attracting fixed point with critical points $c'_1, \dots, c'_n$ attracted, and

$$\phi_G(c'_i) = \phi(c_i) \text{ and } \kappa_G(c'_i) = \kappa(c_i)$$

for $i < n$. Furthermore, since $\phi_{h_m}(c_n) = \lambda^m \phi_{f_0}(c_n)$ and the latter converges to 0 as $n \to \infty$, we see that

$$\phi_G(c'_n) = 0.$$

Finally, as we saw in Lemma 11.3, $\kappa_{h_m}(c_n^{h_m}) = \kappa_{f_0}(c_n)$. Therefore, $G \in W(f_0)$, as desired. □

Since $f \mapsto \phi_f(c_n)$ provides local coordinates on $X(f_0)$ in a neighborhood of g, we have a neighborhood $U_g \subseteq W(f_0)$ centered at g where $f \mapsto \phi_f(c_n)$ is a homeomorphism $U_g \to \mathbb{D}$. We can extend this homeomorphism radially to a neighborhood of $W(f_0)$ that contains the maps f_t.

Corollary 11.5. *For γ as above, where*

$$f_1 = \mathrm{Spin}_{c_n,1}(\gamma, f_0),$$

we have that $f_0 = f_1 \in W(f_0)$.

Proof. The images satisfy $\phi_{f_0}(c_n) = \phi_{f_1}(c_n)$ and the map $f \mapsto \phi_f(c_n)$ is injective. □

Recall that we have identified potential dynamical mapping classes

$$h_m = \mathrm{Push}(p_1, \alpha)^m \,\mathrm{Push}(p_1, \beta_1)\,\mathrm{Push}(p_1, \alpha)^{-m} \in \mathrm{PMCG}(\mathbb{T}_{f_0})$$

Corollary 11.6. *The element h_m is an element of* $\mathrm{PMCG}(f_0)$.

Proof. With f_0 as described above, consider the curve $\gamma = \alpha^m \beta_1 \alpha^{-m}$. By Theorem 11.2, we get that $f_1 = \mathrm{Spin}_{c_n,1}(f_0, \gamma)$ induces an element $\tilde{h}_m \in \mathrm{PMCG}(f_0)$. Furthermore, by Proposition 8.2, $\tilde{h}_m$ is exactly equal to h_m on the level of the quotient torus. □

Let $H \cong F_\infty$ be the free group generated by $\{h_m\}_{m>0}$. In particular,

$$H \subseteq \mathrm{PMCG}(f_0) \hookrightarrow \mathrm{PMCG}(\mathbb{T}_{f_0}).$$

Lemma 11.7. *The subgroup $H \subseteq \mathrm{PMCG}(f_0)$ is contained in* $\mathrm{PDyn}_n(f_0)$.

Proof. The map $\mathrm{Spin}_{c_n,t}(f_0, \gamma)$ fixes the multiplier of the attracting cycle, and so for all $h \in H$, $\Lambda(h) = c$ for some constant $c \in \mathbb{D}^*$ (in fact, $c = \lambda_0$). Therefore,

$$\Lambda_*(H) = \mathrm{id}$$

and so $H \subseteq \ker(\Lambda_*)$. □

Lemma 11.8. *The group H is a subgroup of $P_n(\Sigma_1)/Z$ with generators given by*

$$\{h_m \coloneqq a_1^m b_1 a_1^{-m}\}_{m\geq 0}$$

Proof. Consider the map $\mathrm{PDyn}_n(f) \to P_n(\Sigma_1)/Z$ given by the diagram

$$\begin{array}{ccc} \mathrm{PDyn}_n(f) & \xhookrightarrow{\iota_f} & \mathrm{PMCG}(f) \\ \downarrow & & \downarrow \Phi_* \\ P_n(\Sigma_1)/Z & \xhookrightarrow[\iota_\Sigma]{} & \mathrm{PMCG}(\Sigma_{1,n}) \end{array}$$

as in Equation 11.1. We have seen that

$$\Phi_*(\mathrm{Spin}_{c_n,1}(\alpha, f_0)) = \mathrm{Push}(p_1, \alpha)$$

and

$$\Phi_*(\mathrm{Spin}_{c_n,1}(\beta, f_0)) = \mathrm{Push}(p_1, \beta_1).$$

Furthermore, the braid a_1 is sent under the inclusion ι_Σ to the point-push $\mathrm{Push}(p_1, \alpha)$ and b_1 is sent to $\mathrm{Push}(p_1, \beta_1)$ under our labeling conventions (see, for example, Figure 11.3), so the result follows. $\square$

Recall that we defined the notion of a-degree tuple of a braid $\sigma \in \mathbb{P}_n(\Sigma_1)/Z$, and that $\mathrm{PDyn}_n(f_0) \subseteq \mathbb{P}_n(\mathbb{T}_{f_0})/Z$.

Proposition 11.9. *Every element of* PDyn_n *must have a-degree tuple* $(0, \ldots, 0) \in \mathbb{Z}^n/\Delta$.

Proof. Let b be a braid in $\mathrm{PDyn}_n(f_0)$, realized as a word in the generators of P_n. We can lift this braid under the quotient map Ψ to the space $\mathcal{L}^*$. Suppose by means of contradiction that b does not have a-degree tuple $(0, \ldots, 0)$. That is, there are two indices (without loss of generality, say 1 and 2), so that the total degree of a_1 is different than the total degree of a_2. Notice that under the lift via Ψ, each of the associated trajectories of generators $\{b_i\} \cup \{T_{i,j}\}$ lift to *closed* curves in $\mathcal{L}^*$, whereas any lift trajectory associated with each a_i is *not* a closed curve. Therefore, we see that the critical images $\phi_{f_t}(c_1^t),\ \phi_{f_t}(c_2^t) \in \mathcal{L}^*$ associated with the marked points p_1 and p_2 on the torus, vary over the continuous deformation along the parameter space path, in such a way that

$$\phi_{f_0}(c_1^0)/\phi_{f_0}(c_2^0) \neq \phi_{f_1}(c_1^1)/\phi_{f_1}(c_2^1).$$

Therefore, even after accounting for the normalization of the linearizing functions ϕ_{f_t}, the dynamical properties of the critical points of the maps f_0 and f_1 differ. Specifically, there is no normalization of ϕ_{f_0} and ϕ_{f_1} so that $\phi_{f_0}(c_1^0) = \phi_{f_1}(c_1^1)$ and $\phi_{f_0}(c_2^0) = \phi_{f_1}(c_2^1)$. In particular, this means we can't have $f_0 = f_1$ up to conformal conjugacy, contradicting the assumption that b is a dynamical braid. $\square$

For $\sigma \in \mathrm{PDyn}_n(f)$, let $\deg_a(\sigma) \in \mathbb{Z}^n/\Delta$ denote the a-degree tuple of σ, and let $\mathbf{0} \coloneqq (0, \ldots, 0) \in \mathbb{Z}^n/\Delta$.

Recall that

$$\mathrm{PMCG}(f) \cong \mathrm{PDyn}_n(f) \rtimes \mathbb{Z}.$$

We will prove, as the main theorem in this chapter, the following.

Theorem 11.10. *The group* $\mathrm{PMCG}(f)$ *is an infinitely generated subgroup of* $\mathrm{PMCG}(\Sigma_{1,n})$.

Over the course of the proof, it will become clear that the dynamical braid group $\mathrm{PDyn}_n(f)$ is infinitely generated. However, is it *not* true in general that the semidirect product of an infinitely generated group with $\mathbb{Z}$ is itself infinitely generated, so we need to dive deeper into how $\mathbb{Z}$ acts on $\mathrm{PDyn}_n(f)$.

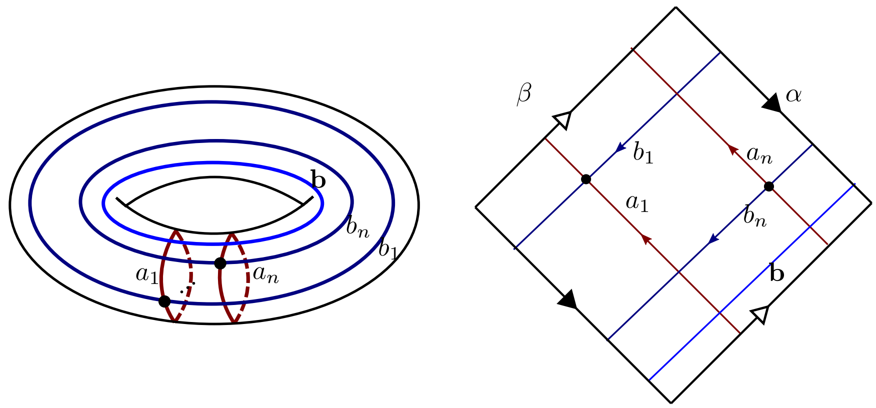


Figure 11.3: The action of $\mathbb{Z}$ on PDyn_n via the lift $\mathbf{b}$

Recall that $\mathbb{Z} \cong \pi_1(\mathbb{D}^*) \subseteq \mathrm{MCG}(\Sigma_{1,1})$ is generated by the Dehn twist T_β. We define a group homomorphism $\varphi : \mathbb{Z} \to \mathrm{Aut}(\mathrm{PDyn}_n(f))$ by

$$\varphi(T_\beta) \mapsto (h \mapsto T_{\mathbf{b}} h T_{\mathbf{b}}^{-1})$$

where $\mathbf{b}$ is the curve (with respect to the labeling of punctures) as shown in figure 11.3. Notice that $\mathrm{Forget}_*(T_{\mathbf{b}}) = T_\beta$, and so this homomorphism does in fact realize

$$\mathrm{PMCG}(f) = \mathrm{PDyn}_n(f) \rtimes_\varphi \mathbb{Z}.$$

We write

$$\varphi_\beta \coloneqq \varphi(T_\beta).$$

Let $\mathcal{G}_1$ be a generating set of $\mathrm{PDyn}_n(f)$, and $\mathcal{R}_1$ the associated relations. Then we can realize $\mathrm{PMCG}(f)$ as having generators $\mathcal{G}_1 \cup \{T_\beta\}$ and relations $\mathcal{R}_1 \cup \mathcal{R}_2$, where elements of $\mathcal{R}_2$ are of the form

$$T_\beta g T_\beta^{-1} = \varphi_\beta(g)$$

for $g \in \mathcal{G}$.

That is, we can write

$$\mathrm{PMCG}(f) = \langle \mathcal{G}_1 \cup \{T_\beta\} | \mathcal{R}_1 \cup \mathcal{R}_2 \rangle.$$

As we have seen previously, the degree of the a_i-generators of the torus braid group play a significant role in the discussion of dynamical braids. While that a_1-degree of an element of P_n/Z is not well-defined, we can make the following definition with the a-degree tuple in mind.

Definition 11.11. The *relative a_1-degree* of an a-degree tuple $(d_1, \ldots, d_n)$ is defined to be

$$\deg_{a_1}^{rel} = \max_i \{d_1 - d_i\}.$$

For example,

$$\deg_{a_1}^{rel}(1, 0, 0, 0) = 1$$

and

$$\deg_{a_1}^{rel}(-1, 1, 0, 0) = -1$$

Lemma 11.12. *The relative a_1-degree $\deg_{a_1}^{rel}$ is well-defined in $\mathrm{PMCG}(f)$, and*

$$\deg_{a_1}^{rel}(w) = 0$$

for all $w \in \mathcal{G}_1 \cup \{T_\beta\} \cup \mathcal{R}_1 \cup \mathcal{R}_2$.

Proof. Certainly $\deg_{a_1}^{rel}$ is well-defined in $\mathrm{PDyn}_n(f)$, since the a-degree tuple is well-defined in this setting. To pass from $\mathrm{PDyn}_n(f)$ to $\mathrm{PMCG}(f)$, we first calculate $\varphi_\beta(h)$ for the generators of $P_n(\Sigma_{1,n})$.

We do this in three parts, one for generators of the form b_i, one for generators of the form T_{ij}, and one for generators of the form a_i.

The first two calculations are straightforward. Notice that the curve $\mathbf{b}$ does not intersect any of the curves b_i or $T_{i,j}$, and therefore

$$T_{\mathbf{b}} h T_{\mathbf{b}}^{-1} = h$$

for each such element. Thus, we have

$$\varphi(T_\beta)(b_i) = b_i$$

and

$$\varphi(T_\beta)(T_{i,j}) = T_{i,j}.$$

On the other hand, the curves a_i intersect the curve $\mathbf{b}$ for each i. To calculate $\varphi(T_\beta)(a_i)$, notice that by Lemma 4.4, we have that

$$T_{\mathbf{b}} a_i T_{\mathbf{b}}^{-1} = \mathrm{Push}(T_{\mathbf{b}}(a_i)).$$

But we also have that

$$T_{\mathbf{b}}(a_i) = a_i b_i T_{i,n}$$

(see Figure 11.4), so that

$$\varphi(T_\beta)(a_i) = a_i b_i T_{i,n}$$

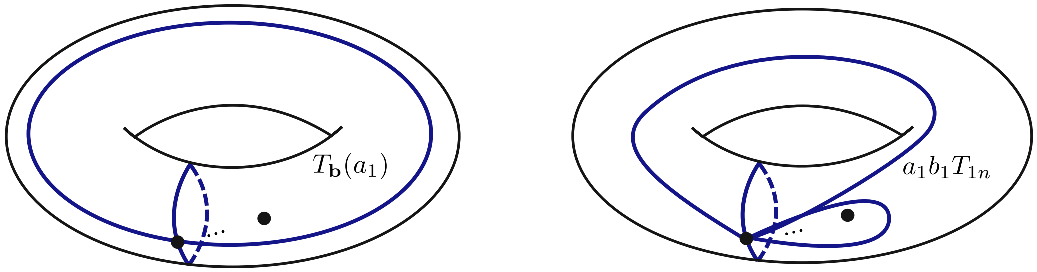


Figure 11.4: The action of $\phi(T_\beta)$ on a_i

Since $\mathrm{PDyn}_n(f) \subseteq P_n(\Sigma_{1,n})$, we can write any word $g \in \mathrm{PDyn}_n(f)$ as some finite product of generators $g = h_1 \dots h_k$ with $h_i \in P_n(\Sigma_{1,n})$. Then

$$T_{\mathbf{b}} g T_{\mathbf{b}}^{-1} = T_{\mathbf{b}} h_1 T_{\mathbf{b}}^{-1} T_{\mathbf{b}} h_2 T_{\mathbf{b}}^{-1} \dots T_{\mathbf{b}} h_k T_{\mathbf{b}}^{-1},$$

which preserves relative a_1-degree.

We know already that the same is true for elements of $\mathcal{R}_1$, since elements of $\mathcal{R}_1$ are products of elements of the braid relations in P_n, which also preserve relative a_1-degree. This shows that $\deg_{a_1}^{rel}$ is well-defined. Finally, we've seen that the generators of $\mathrm{PDyn}_n(f)$ must have relative a_1-degree equal to 0, and the single extra generator T_β clearly satisfies the same. □

Lemma 11.13. *The relative a_1-degree is subadditive. That is, for w, x,*

$$\deg_{a_1}^{rel}(wx) \leq \deg_{a_1}^{rel}(w) + \deg_{a_1}^{rel}(x).$$

Proof. Certainly we have that

$$\deg_a(wx) = \deg_a(w) + \deg_a(x).$$

Suppose that

$$\deg_a(w) = (d_1^w, \ldots, d_n^w) \text{ and } \deg_a(x) = (d_1^x, \ldots, d_n^x)$$

so that

$$\deg_a(wx) = (d_1^w + d_1^x, \ldots, d_n^w + d_n^x).$$

Consider

$$(d_1^w + d_1^x) - (d_j^w + d_j^x).$$

We have that $d_1^w - d_j^w \leq \deg_{a_1}^{rel}(w)$ and $d_1^x - d_j^x \leq \deg_{a_1}^{rel}(x)$, and therefore

$$(d_1^w + d_1^x) - (d_j^w + d_j^x) \leq \deg_{a_1}^{rel}(w) + \deg_{a_1}^{rel}(x)$$

and the subadditivity follows. □

Definition 11.14. The *leading a_1-degree* of a word $w \in F_{\mathcal{G}_B}$ (the free group in the generators of the braid group) is defined to be

$$\deg_{a_1}^{lead} = \max_{w_1 : w = w_1 w_2} \deg_{a_1}^{rel}(w_1).$$

Lemma 11.15. *If $w, x \in F_{\mathcal{G}_B}$ with $\deg_{a_1}^{rel}(w) = \deg_{a_1}^{rel}(x) = 0$ and $\deg_{a_1}^{lead}(w) > \deg_{a_1}^{lead}(x) > 0$, then the leading a_1-degree satisfies*

$$\deg_{a_1}^{lead}(wx) = \deg_{a_1}^{lead}(w).$$

Proof. Let $L_w = \deg_{a_1}^{lead}(w)$ and $L_x = deg_{a_1}^{lead}(x)$. Choose w_1 and x_1 so that $w = w_1 w_2$, $x = x_1 x_2$ and $\deg_{a_1}^{rel}(w_1) = L_w$ and $\deg_{a_1}^{rel}(x_1) = L_x$. Then $\deg_{a_1}^{rel}(w_2) = -L_w$ and $\deg_{a_1}^{rel}(x_2) = -L_x$. We can write

$$wx = w_1 w_2 x_1 x_2$$

and note that since $L_x < L_w$, we must have that

$$\deg_{a_1}^{rel}(w_2 w_1) \leq -L_w + L_x < 0$$

by Lemma 11.13. Now consider a combination

$$wx = w_1' w_2'$$

so that $\deg_{a_1}^{lead}(wx) = \deg_{a_1}^{rel}(w_1')$. Certainly we cannot have that w_1' is a subword of w_1, since we could always expand w_1' to w_1 and increase the relative a_1-degree, contradicting maximality of $\deg_{a_1}^{rel}(w_1')$. On the other hand, since $\deg_{a_1}^{rel}(w_2 w_1) < 0$, if w_1' contained w_1 as a subword, we would have

$$\deg_{a_1}^{rel}(w_1') < \deg_{a_1}^{rel}(w_1) = L_w.$$

Again, this contradicts maximality. Therefore, we must have $w_1 = w_1'$ and

$$\deg_{a_1}^{lead}(wx) = L_w.$$

□

Lemma 11.16. *The leading a_1-degree of words that generate the relators $\mathcal{R}_1$ and $\mathcal{R}_2$ are bounded.*

Proof. Since $\mathcal{R}_1$ and $\mathcal{R}_2$ are finitely generated by words with relative a_1-degree 0, this follows from Lemma 11.15. □

Proof of Theorem 11.10. By Lemma 11.16 there exists some $M < \infty$ so that

$$\deg_{a_1}^{lead}(r) \leq M$$

for all $r \in \mathcal{R}$. Let $m_1, m_2 \geq M$ with $m_1 \neq m_2$. A relation between h_{m_1} and h_{m_2} would look like

$$h_{m_1} r_1 = h_{m_2} r_2$$

with $r_1, r_2 \in \mathcal{R}$. But then by 11.15,

$$\deg_{a_1}^{lead}(h_{m_1} r_1) = m_1 \neq m_2 = \deg_{a_1}^{lead}(h_{m_2} r_2).$$

Therefore, there are infinitely many elements of H with no relations in PMCG(f), and so PMCG(f) is infinitely generated. □

This now gives us the tools needed to prove Theorem 1.6.

Proof of Theorem 1.6. First, suppose that f has an attracting cycle $\mathbf{a}$ with at least two grand orbits containing critical points in the basin. We choose a base map $f_0 \in \mathcal{M}_*(f)$ and a labeling of critical points satisfying the conditions in Subsection 11.1, so that PMCG(f_0) $\cong$ PMCG(f). Then by Theorem 11.10, PMCG(f_0) is infinitely generated.

On the other hand, suppose f has attracting cycles $\mathbf{a_1}, \ldots \mathbf{a_k}$, each with a single grand orbit containing a critical point in the corresponding basin $\mathcal{B}_{\mathbf{a_1}}, \ldots \mathcal{B}_{\mathbf{a_k}}$. Then by Corollary 7.5, PMCG(f) is finitely generated by Dehn twists, with one Dehn twist corresponding to each attracting cycle. □

CHAPTER XII

Other cubic components

Recall that in Chapter IX we completely characterized the mapping class group of a generic cubic polynomial with an attracting fixed point in the setting where *both critical points were in the immediate basin of the attracting fixed point.* Intuitively, many of the same ideas should hold for cubic polynomials a fixed point with multiple attracting basin components, and we extend this calculation here.

Notice that this extension corresponds to choosing a base map in one of the components A of our parameter slice as classified in Chapter IX.

In fact, in a lot of ways, the calculations in this setting are easier, since we avoid the ambiguity in choosing a marking when both critical points are on the same linearizing potential in the immediate basin of an attracting fixed point. In particular, a MCG-generic cubic polynomial with an attracting fixed point with multiple basin components comes with a natural marking of critical points, via the following simple result.

Lemma 12.1. *Let f be a* MCG-*generic cubic polynomial with attracting fixed point* *a so that the attracting basin $\mathcal{A}$ is disconnected. Then there is exactly one critical point in the immediate basin of f.*

Proof. We have seen in Lemma 5.1 that the immediate basin A_0 of an attracting cycle must contain at least one critical point. On the other hand, suppose that both critical points of f are contained in A_0. Then $f(A_0) \subseteq A_0$ and $f : A_0 \to A_0$ would have degree 3. Since f is a degree 3 map, we must have that $f^{-1}(A_0) \subseteq A_0$. Therefore, every point attracted to a must be in the immediate basin A_0 — that is, $\mathcal{A} = A_0$. □

We will refer to the critical point in A_0 as the *preferred* critical point.

The extra machinery introduced in this chapter, much of it adapted from [29], is applied to be able to calculate the mapping class in full generality, independent of the dynamics with which the non-preferred critical point maps into A_0.

As in Chapter IX, we establish a connection between the hyperbolic components of our parameter space and the parameter space of a quadratic polynomial. To do this in full generality for every hyperbolic component, we use the notion of mapping schemes developed in [29].

12.1 Mapping schemes

Following [29], the *full mapping scheme* S_f associated to a hyperbolic rational map f is

1. A set of vertices $\{s_U\}_{U \in \mathcal{U}}$ where $\mathcal{U}$ is the set of Fatou components containing a

critical or post-critical point,

2. A set of weights $\{w(s_U)\}_{U \in \mathcal{U}}$ where $w(s_U)$ is the number of critical points in U, counted with multiplicity, and

3. A map $F_f : |\mathcal{U}| \to |\mathcal{U}|$ that takes s_U to $s_{f(U)}$.

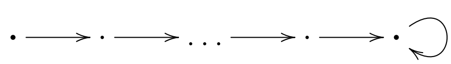
The *reduced mapping scheme* $\overline{S}_f$ is closely related, and is given by

1. A set of vertices $\{s_U\}_{U \in \overline{\mathcal{U}}}$ where $\overline{\mathcal{U}}$ is the set of Fatou components containing a critical point,

2. Weights $\{w(s_U)\}_{U \in \overline{\mathcal{U}}}$ where $w(s_U)$ is the number of critical points in U, counted with multiplicity, and

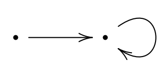
3. A map $\overline{F}_f : |\overline{\mathcal{U}}| \to |\overline{\mathcal{U}}|$ that takes s_U to s_V, where $V = f^k(U)$ is the first Fatou component to contain a critical point of f.

Let $\mathcal{H}$ be a cubic hyperbolic component that *does not* contain the map $z \mapsto z^3$. In this special case, each $f \in \mathcal{H}$ has a full mapping scheme of the following form:



where the first and last vertex have weight 1 and the rest have weight 0.

Furthermore, the *reduced* mapping scheme of each such component is identical, namely



where both vertices have weight 1.

12.2 Blaschke products

Recall that for each point $a \in \mathbb{D}$, there is exactly one map $\beta_a : \mathbb{D} \to \mathbb{D}$ so that

1. β_a is a Möbius transformation of the Riemann sphere $\hat{\mathbb{C}}$ that maps $\mathbb{D}$ onto itself,
2. $\beta_a(a) = 0$, and
3. $\beta_a(1) = 1$.

It can be seen that β_a has the form

$$\beta_a(z) = \left(\frac{1-\overline{a}}{1-a}\right)\frac{z-a}{1-\overline{a}z}.$$

A *Blaschke product* is a map $\beta : \mathbb{D} \to \mathbb{D}$ of the form

$$\beta(z) = \xi\beta_{a_1}(z)\ldots\beta_{a_d}(z),$$

where $|\xi| = 1$. Notice that in this form, $\xi = \beta(1)$. Blaschke products are exactly the holomorphic self-maps of the disk. In particular, we have the following.

Lemma 12.2 (c.f. [29])**.** *Every proper holomorphic map from $\mathbb{D}$ to $\mathbb{D}$ can be written uniquely as a Blaschke product. Furthermore, these maps extend continuously to $\overline{\mathbb{D}}$.*

Again following [29], we associate to each mapping scheme a model space of Blaschke products. But we work in a much more specific setting, and use slightly different normalizations. We first discuss these normalizations.

Definition 12.3. A proper holomorphic map $\beta : \mathbb{D} \to \mathbb{D}$ is *1-anchored* if $\beta(1) = 1$, is *fixed point centered* if $\beta(0) = 0$, and is *critically centered* if the critical points

$c_1, \ldots, c_{d-1}$ satisfy

$$c_1 + \cdots + c_{d-1} = 0.$$

In the setting we are working in, all our Blaschke products will be either degree 1 or degree 2. Notice that the only 1-anchored, fixed point centered Blaschke product of degree 1 is the identity. Also, every Blaschke product has exactly one fixed point in $\mathbb{D}$, and so a fixed point centered map has that unique fixed point at 0. Finally, a degree 2 Blaschke product has exactly one critical point in $\mathbb{D}$, and therefore a degree 2 critically centered Blaschke product is exactly one whose unique critical point is at 0.

Definition 12.4. We associate to the mapping scheme S with $|S| = n$ the model space $\boldsymbol{\mathcal{B}}^{S,1/2}$ consisting of proper holomorphic maps

$$\boldsymbol{\beta} : \{0, \ldots, n\} \times \mathbb{D} \to \{0, \ldots, n\} \times \mathbb{D}$$

where

$$\boldsymbol{\beta}(i, z) = x(F(i), \beta_i(z))$$

where $F(i) = i + 1$ if $i < n$ and $F(n) = n$. We require that each β_i is 1-anchored. Furthermore, if vertex i has weight 0, we require that β_i is the identity map. If vertex i is periodic, (that is, if $i = n$), then β_i is fixed point centered and that fixed point has multiplier $1/2$, and if vertex i is preperiodic with weight $w > 0$ (that is, if $i = 0$), then β_i is critically centered.

These definitions are constructed to mirror cubic polynomials in the parameter slice $F_{1/2}$ as constructed in Chapter IX. Recall that in Chapter IX, we arbitrarily chose our favorite multiplier $\lambda = 1/2 \in \mathbb{D}^*$ with the understanding that analogous statements hold for other choices of $\lambda \in \mathbb{D}^*$ due to the fibration in 9.1. We do the same in this setting, choosing to define and investigate the model space $\mathcal{B}^{S,1/2}$, with the understanding that statements will hold for an analogous slice $\mathcal{B}^{S,\lambda}$ for $\lambda \in \mathbb{D}^*$. In fact, we will show that maps in the model space $\mathcal{B}^{S,1/2}$ exactly correspond to cubic polynomials in $F_{1/2}$ with both critical points attracted to the fixed point at 0.

Lemma 12.5. *There exists a unique* 1*-anchored, fixed point centered degree* 2 *Blaschke product* β_λ *with multiplier* $\lambda = 1/2$ *at* 0*, and every Blaschke product with fixed point with multiplier* 1/2 *is uniquely conjugate to* β_λ*.*

Proof. Any degree 2 1-anchored, fixed point centered Blaschke product has the form

$$\beta(z) = \xi z \frac{z-a}{1-\bar{a}z}$$

where $a \in \mathbb{D}$ and $\xi = \frac{1-\bar{a}}{1-a}$. It is easy to check that there is then a unique value of a so that $\beta'(0) = 1/2$. This map is given by

$$\beta_\lambda(z) = z\frac{z+\frac{1}{2}}{1+\frac{1}{2}z}.$$

Now let β be a degree 2 Blaschke product with multiplier 1/2 at its fixed point. Clearly conjugating by a Möbius automorphism of the disk will not change that multiplier. Furthermore, since β has degree 2, there is a *unique* Möbius automorphism sending the fixed point $z_0 \in \mathbb{D}$ to 0 and the (unique) fixed point on the boundary $\partial\mathbb{D}$ to 1. Therefore, β must be uniquely conjugate to β_λ. $\square$

We now discuss degree 2 critically centered Blaschke products.

Lemma 12.6. *Let β be a degree 2 Blaschke product and let $z_1 \in \partial\mathbb{D}$ with $\beta(z_1) = 1$ (note that there are two choices of such z_1). Then there exists a unique Möbius automorphism h of $\mathbb{D}$ so that $h(1) = z_1$ and $\beta \circ h$ is critically centered.*

Proof. The map h must send critical points of $\beta \circ h$ to those of β. Since β has degree 2, it has exactly one critical point, say at c. Then we require $h(0) = c$ and $h(1) = z_1$, uniquely determining the map h. □

Definition 12.7. A *boundary marking* q for a map $\boldsymbol{\beta} \in \mathcal{B}^{S,1/2}$ is a function $\{0, \ldots, n\}\times \to \{0, \ldots, n\} \times \partial\mathbb{D}$ so that

$$q(F(i)) = \boldsymbol{\beta}(q(i))$$

for all $i \in \{0, \ldots, n\}$.

Note that this is coming directly from Definition 4.12 in [29]. However, in our greatly restricted setting, we can simplify the notion of a boundary marking drastically. For the periodic vertex n of degree 2, $q(n)$ must be a fixed point of β_n on $\partial\mathbb{D}$, and therefore $q(n) = 1$. This in turn means that for the preceding weight zero vertices where β_i is the identity, we also must have $q(i) = 1$. So in our case, a boundary marking will be a choice $z_1 \in \partial\mathbb{D}$ so that $\beta_0(z_1) = 1$. There are two such choices.

From this observation and the proceeding two lemmas, we immediately get the following theorem.

Theorem 12.8. *Let $\boldsymbol{\beta} : \{0, \ldots, n\} \times \mathbb{D} \to \{0, \ldots, n\} \times \mathbb{D}$ such that $(i \times \mathbb{D}) \to (F(i) \times \mathbb{D})$ by a degree 2 Blaschke product if $i = 0, n$ and a degree 1 Blaschke product otherwise. Let q be a boundary marking for $\boldsymbol{\beta}$. Then we can find a unique automorphism $\boldsymbol{h} : \{0, \ldots, n\} \times \mathbb{D} \to \{0, \ldots, n\} \times \mathbb{D}$ so that $\boldsymbol{h}^{-1} \circ \boldsymbol{\beta} \circ \boldsymbol{h} \in \mathcal{B}^{S,1/2}$ with $\boldsymbol{h}(i, 1) = q(i)$.*

We also want to understand the topology of $\mathcal{B}^{S,1/2}$. To do so, we need the following lemma.

Lemma 12.9. *For every $v \in D$, there exists a unique degree 2 critically centered Blaschke product β so that $\beta(1) = 1$ and v is the critical value of β.*

Proof. The existence of such a map comes from the result (see, for example, Theorem 16 in [30]) that given $n-1$ points counted with multiplicity, there exists a Blaschke product of degree n realizing those points as its critical values. The uniqueness then follows from the fact that two maps β_1 and β_2 satisfying the conditions of the lemma agree at 3 points. □

From this, we get the following.

Theorem 12.10 (c.f. [29], Lemma 4.11)**.** *The space $\mathcal{B}^{S,1/2}$ is homeomorphic to an open cell of real dimension 2*

Proof. We've seen that there is only one possible map for vertices $\{1, \ldots, n\}$ with our normalization conditions, so the dimension of $\mathcal{B}^{S,1/2}$ is determined completely by the dimension of degree 2, 1-anchored, critically centered Blaschke products —

that is, by β_0. But this space has dimension 2 by Lemma 12.9, parametrized by the critical value $\beta_0(0)$. □

To begin to establish the relationship between our parameter slice component $\mathcal{H}$ and the space $\mathcal{B}^S$, we notice that restriction of a map $f \in \mathcal{H}$ to the relevant Fatou components, after conjugation, gives us a map in $\mathcal{B}^{S,1/2}$. To talk more about this map, we need a notion of "critically minimal" in the slices in which we are working. Both maps $f \in H$ and maps $\boldsymbol{\beta} \in \mathcal{B}^{S,1/2}$ can't be critically finite, since they have a critical point of multiplier 1/2, which must attract a critical point with no critical orbit relations. However, each such map has a second critical point, which we will refer to as the *flexible* critical point.

Definition 12.11. A map $f \in H$ or $\boldsymbol{\beta} \in \mathcal{B}^{S,1/2}$ is called *critically minimal* if the size of the forward orbit of the flexible critical point of the map is minimal over all the maps in the space.

Lemma 12.12. *The space $\mathcal{B}^{S,1/2}$ has a unique critically minimal map, given by $\beta_0(z) = z^2$. The size of the orbit is exactly equal to the number of preperiodic vertices in the mapping scheme.*

Proof. The flexible critical point c of $\boldsymbol{\beta}$ can have finite forward orbit exactly if $\beta_0(c)$ is equal to a preimage of 0 under β_λ (since each β_i is equal to the identity). The minimal way in which this can happen is, of course, if $\beta_0(c) = 0$. But β_0 is critically centered, and so $c = 0$. This implies 0 is a critical point of β_0 with multiplicity 2, and therefore $\beta_0(z) = z^2$. □

Let $\mathcal{H}^{1/2}$ be the locus of $\mathcal{H}$ where each map has multiplier 1/2 at the attracting fixed point. That is, $\mathcal{H}^{1/2}$ is one of the smaller orange connected components in Figure 9.1.

We now establish a homeomorphism between $\mathcal{H}^{1/2}$ and $\mathcal{B}^{S,1/2}$. This will immediately give us that $\mathcal{H}$ is a topological 2-cell. More importantly, though, we will use this map to establish a homeomorphism between $\mathcal{H}^{1/2}$ and the filled Julia set of a quadratic polynomial, which will in turn let us talk about mapping classes.

The discussion that follows in a specific application of a number of the results in Section 5 of [29].

We have a map $\mathbf{restr} : \mathcal{H}^{1/2} \to \mathcal{B}^{S,1/2}$ given by restricting $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ to its Fatou components containing a critical point (and their forward images). We let

$$\boldsymbol{\beta}_f := \mathbf{restr}(f).$$

We get the following.

Theorem 12.13 (c.f. [29], Theorem 5.1)**.** *The map* $\mathbf{restr} : \mathcal{H}^{1/2} \to \mathcal{B}^{S,1/2}$ *is a diffeomorphism that sends* f *to the map* $\boldsymbol{\beta}_f$ *that is conformally conjugate to* f *restricted to the relevant Fatou components.*

We use Theorem 12.13 to make the correspondence between parameter and dynamical space.

Theorem 12.14. *The parameter component* $\mathcal{H}^{1/2}$ *is homeomorphic to* $\mathcal{B}_Q$*, the interior of the filled Julia set* K_Q *for the quadratic polynomial* $Q(z) = z^2 + \frac{1}{2}z$*.*

Proof. Recall that by Theorem 12.10, $\mathcal{B}^{S,1/2}$ has coordinates given by the critical value $\beta_0(0)$ of β_0. In particular, we have a homeomorphism $\mathcal{B}^{S,1/2} \to \mathbb{D}$ given by $\boldsymbol{\beta} \mapsto \beta_0(0)$. Finally, let $R : \mathbb{D} \to K_Q$ denote the map that conjugates β_λ to $Q|_{K_Q}$. Thus, we construct the map $h : \mathcal{H}^{1/2} \to \mathcal{B}_Q$ via the composition

$$\mathcal{H}^{1/2} \xrightarrow{\textbf{restr}} \mathcal{B}^{S,1/2} \xrightarrow{\boldsymbol{\beta} \mapsto \beta_0(0)} \mathbb{D} \xrightarrow{R} \mathcal{B}_Q$$

and see that this gives a homeomorphism. □

To use this theorem to say something about the mapping class of some $f \in \mathcal{H}^{1/2}$, we need some properties of this homeomorphism. In particular, as before, we relate the sets P and O in $\mathcal{H}^{1/2}$ to preimages of 0 and $-1/4$ (the fixed point and the critical point) in K_Q (compare with the statement of Lemma 9.6).

Theorem 12.15. *Let v_f denote the flexible critical value of a map $f \in \mathcal{H}^{1/2}$, and let v_0 denote the critical value in the periodic component. Recall that Q has critical point c_Q at $-1/4$ and critical value v_Q at $-1/8$. The homeomorphism $h : \mathcal{H}^{1/2} \to \mathcal{B}_Q$ has the following properties:*

1. *If $f^n(v_f) = 0$, then $Q^n(h(f)) = 0$.*
2. *If $f^n(v_0) = f^m(v_f)$, then $Q^n(-1/8) = Q^m(h(f))$.*

In other words, h takes cubic polynomials with critical orbit relations to points in K_Q in the critical orbit of Q.

Proof. This follows via the fact that all conjugacies involved preserve critical points and orbits. □

To better understand h, we prove the equivalent of Lemma 4.5 in [15]. Recall that we have quotient maps $\Psi_Q : \mathcal{B}_Q^* \to \mathbb{T}_Q$ with $\mathbb{T}_Q$ a once-punctured torus and $\Psi_f : \mathcal{B}_f^* \to \mathbb{T}_f$ with $\mathbb{T}_f$ a twice-punctured torus.

Theorem 12.16. *The following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{B}_Q & \xleftarrow{\;h\;} & \mathcal{H}^{1/2} & \longrightarrow & X \\ & \searrow^{\Psi_Q} & & & \downarrow{\scriptstyle g \mapsto \mathbb{T}_g} \\ & & \mathbb{T}_Q & \longrightarrow & \mathcal{M}_*(\mathbb{T}_f) \end{array}$$

Proof. Fix $f \in \mathcal{H}^{1/2}$. Let $\mathcal{A}_0$ be the immediate basin of 0, and let $p \in \mathcal{A}_0$ be the first image of the flexible critical point that lands in $\mathcal{A}_0$. Let v denote the restricted critical value of f, and recall that $-1/8$ is the unique critical value of Q. In constructing the map h, we get a conformal isomorphism

$$\xi : \mathcal{A}_0 \to \mathcal{B}_Q$$

given by restriction of f to $\mathcal{A}_0$ which gives a degree 2 Blaschke product, followed by the Riemann map R. Notice that this map sends v to $-1/8$, and sends p to $h(f)$.

Associated to $\mathcal{A}_0$ we get the twice-punctured torus S_f coming from modding out by the dynamics. But now, the isomorphism ξ descends to an isomorphism

$$\mathbb{T}_f \to \mathbb{T}_Q \setminus \Phi_Q(h(p))$$

which gives us the commutative diagram above. $\square$

Again following [15], we denote

$$\nu = \Psi_Q \circ h.$$

The map ν lets us relate representatives of $\pi_1(\mathcal{H}^{1/2}, f)$ to curves on the punctured torus $\mathbb{T}_Q$ by using what we know about the structure of the filled Julia set K_Q.

Now that the correspondence has been established, the calculations of dynamical mapping classes proceed directly as in Section 4 in [15].

Specifically, we see that $\pi_1(\mathcal{H}^{1/2}, f)$ is generated by

1. One based loop enclosing each puncture coming from an orbit relation between the two critical points, and
2. One based loop enclosing each puncture coming from one of the critical points (specifically, the flexible critical point) mapping onto the attracting fixed point.

For an illustration of this, see Figure 12.1

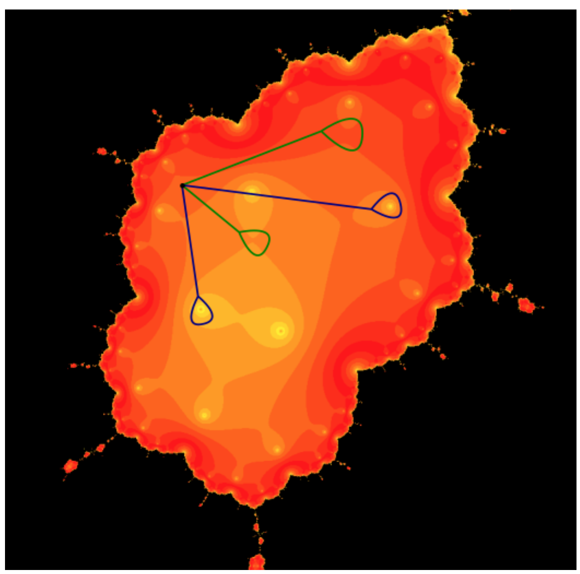


Figure 12.1: Loops of type 1 (in green) and type 2 (in blue) in a component $\mathcal{H}^{1/2}$.

We can then follow the analysis in [15] to get a similar result. We summarize the procedure here, and refer the reader to the original paper for more detail.

By choosing representatives of these generators in $\mathcal{H}^{1/2}$ and looking at their images under the map $\nu : \mathcal{H}^{1/2} \to \mathbb{T}_f$, we see that the images of each of these generators is either a point-push or the square of a point-push.

We can then use the fiber structure

$$\mathcal{H}^{1/2} \xrightarrow{\iota} \mathcal{H} \xrightarrow{f \mapsto \lambda_{\mathbf{a}}} \mathbb{D}^*$$

to get the analogue of Theorem 4.7 in [15].

Theorem 12.17. *For $f \in \mathcal{H}$, the pure mapping class group* $\mathrm{PMCG}(f) \hookrightarrow \mathrm{PMCG}(\Sigma_{1,2})$ *is infinitely generated, and can be expressed as a product*

$$\mathrm{PMCG}(f) = \pi_1(\mathcal{H}^{1/2}, f) \rtimes \mathbb{Z}$$

with generators given by

1. *A Dehn twist around a simple closed curve β in $\Sigma_{1,2}$ (for the $\mathbb{Z}$ factor), and*
2. *Countably many generators (for the $\pi_1(\mathcal{H}^{1/2}, f)$) given by*
   - *(a) Loops around orbit relations of the form $f^n(v_f) = v_0$, which correspond to the square of a point-push in* $\mathrm{PMCG}(\Sigma_{1,2})$*,*
   - *(b) Loops around orbit relations of the form $f^n(v_f) = f^m(v_0)$ with $n, m \geq 1$, and loops around orbit relations of the form $f^n(v_f) = f^m(v_f)$ with $n \neq m$, both of which correspond to a point-push in* $\mathrm{PMCG}(\Sigma_{1,2})$*.*

As in Chapter IX, generators in terms of the curves around which the point-pushes are defined could be explicitly characterized as elements in $\pi_1(\Sigma_{1,2})$ if desired.

12.3 A map of parameter space

Consider the parameter space slice $F_{1/2}$ as described in Chapter IX. We have seen

Figure 12.2: A closeup of a baby Mandelbrot set with attached components in a slice of cubic parameter space.

that the orange components in Figure 12.2 correspond to the intersection of the locus $f'(0) = 1/2$ with the set of cubic maps with *both* critical points attracted to 0. We have also seen that the unbounded component in this figure corresponds to a slice of the component $\mathcal{H}_0$ that contains the map $z \mapsto z^3$ — that is, the cubic hyperbolic component with both critical points in the same immediate basin of the attracting fixed point. Each of the other components has a full mapping scheme of the form

$$\bullet \longrightarrow \cdot \longrightarrow \dots \longrightarrow \cdot \longrightarrow \bullet \circlearrowleft$$

Therefore, to each such component, we can associate an integer $n \geq 1$ representing the number of *non-periodic vertices* of its mapping scheme. In particular, n encodes the number of iterations before the flexible critical point (necessarily not in the immediate basin) gets mapped into the immediate basin of 0. For a component $\mathcal{H}$, let $\chi(\mathcal{H}) = n$ be this association.

Moreover, notice that in Figure 12.2, there are a number of mini Mandelbrot sets, each with hyperbolic components "budding off" them at various angles. We make this precise, and calculate $\chi(\mathcal{H})$ for each such component.

To do so, we first discuss a parameterization of the boundary of a component $\mathcal{H}^{1/2}$.

Recall that given a slice $\mathcal{H}^{1/2}$, we have a homeomorphism $h : \mathcal{H}^{1/2} \to \mathring{K}_Q$ (or to $\mathring{K}_Q \setminus U_Q$ if we are working with the component $\mathcal{H}_0^{1/2}$).

12.3.1 External rays

The boundary $J_Q = \partial K_Q$ has a natural parameterization via $\mathbb{R}/\mathbb{Z}$ coming from the *external angles* of the polynomial. That is, recall that Q is given by $Q(z) = z^2 + 1/2z$. There is a Böttcher coordinate $\varphi : \hat{\mathbb{C}} \setminus \mathbb{D} \to \hat{\mathbb{C}} \setminus K_Q$ that conjugates Q to the map $z \mapsto z^2$ on the basin of ∞. Radial rays in $\hat{\mathbb{C}} \setminus \mathbb{D}$ can be pulled back under the

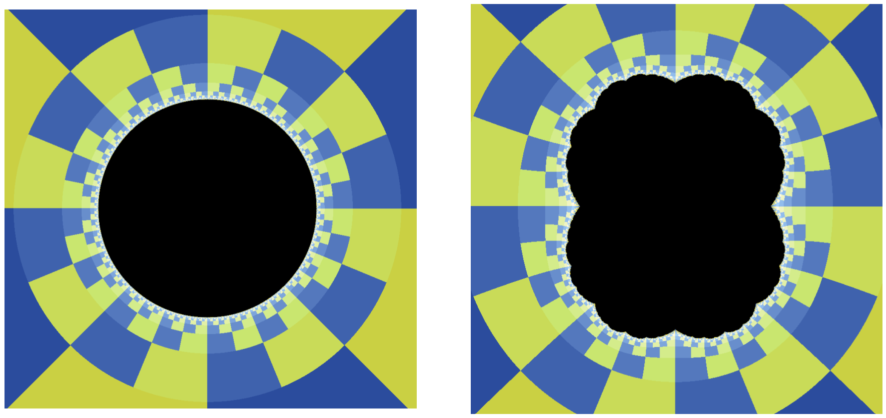


Figure 12.3: The Böttcher map and external rays

biholomorphic map φ to rays

$$\mathcal{R}_t = \{\varphi^{-1}\left(re^{2\pi it}\right) : r \in (1, \infty))\}.$$

Furthermore, in this setting the limit

$$\lim_{r \searrow 1} \varphi^{-1}\left(re^{2\pi it}\right) \in K_Q$$

exists (the ray *lands*) for every $t \in \mathbb{R}/\mathbb{Z}$, and furthermore, the map $\mathbb{R}/\mathbb{Z}$ that sends each angle to its landing point on K_Q is a homeomorphism.

Notice that this correspondence allows us to parameterize the boundary of any component $\mathcal{H}^{1/2}$ by angles in $\mathbb{R}/\mathbb{Z}$.

Lemma 12.18. *If $P_t \in \partial\mathcal{H}^{1/2}$ has a parabolic cycle, then t is periodic under the doubling map on $\mathbb{R}/\mathbb{Z}$.*

Proof. Choose a polynomial $P \in \partial\mathcal{H}^{1/2}$ for some component $\mathcal{H}^{1/2}$ such that P has a parabolic cycle. Let $\{p_i\}$ be some sequence of polynomials so that each $p_i \in \text{int}(\mathcal{H}^{1/2})$, and so that the *points* in the slice $F_{1/2}$ corresponding to the polynomials p_i converge to the point corresponding to P. Since each $p_i \in \text{int}(\mathcal{H}^{1/2})$, under the homeomorphism $h : \mathcal{H}^{1/2} \to K_Q$, each p_i gets sent to $h(p_i) \in K_Q$, and the sequence $\{h(p_i)\}$ converges to the point $h(P)$ on the boundary of K_Q. To show that $h(P)$ corresponds to an angle that is periodic under the doubling map, we have to show that $h(P) \in J_Q$ is periodic under Q. To see this, let $\mathbf{b} = \{b_1, \ldots, b_k\}$ denote the parabolic cycle of P. Any perturbation of P that lies inside $\mathcal{H}^{1/2}$ has a single attracting cycle with both critical points attracted to it, and so under this perturbation, the parabolic cycle $\mathbf{b}$ must split into two repelling cycles. We see that the flexible critical point c_+^i of the polynomial p_i tends to this associated repelling cycle. Therefore, the images $h(p_i)$ converge to a point in J_Q that is a cycle. In other words, $h(P)$ is periodic. □

We will use this to prove the following.

Theorem 12.19. *For each $P_t \in \partial\mathcal{H}^{1/2}$ with a parabolic cycle, the component $\mathcal{H}^{1/2}$ is attached to a baby Mandelbrot set $\mathcal{M}_t$ at P_t.*

For the proof, we will need to make use the straightening theorem, which in turn requires the notion of a polynomial-like map. These classical results are presented here.

12.3.2 Polynomial-like maps and straightening

The results in this subsection can be originally attributed to [12]. The exposition and definitions of objects involved are adapted from [13].

Definition 12.20. Let $U, V \subseteq \mathbb{C}$ be topological disks satisfying $\overline{U} \subseteq V$. A holomorphic map $f : U \to V$ is called *polynomial-like* of degree d if f is proper, and every point in V has d preimages under f (counted with multiplicity).

A polynomial-like map has a notion of a filled Julia set, much like that of a polynomial.

Definition 12.21. A polynomial-like map f has *filled Julia set* K_f given by

$$K_f = \{z \in U : f^n(z) \in U \text{ for all } n\} = \bigcup_{n \geq 0} \{f^{-n}(U)\}$$

At its core, the straightening theorem says that every polynomial-like map of degree d can be "straightened" via conjugacy to a polynomial of degree d. The notion of straightening comes from the idea of *hybrid equivalence.*

Definition 12.22. Two polynomial-like maps $f : U \to V$ and $g : U' \to V'$ are *hybrid equivalent* if the are quasiconformally conjugate via a map φ satisfying $\overline{\partial}\varphi = 0$ almost everywhere on ∂K_f.

Notice that if the Julia set $J_f = \partial K_f$ has measure 0, this notion just requires that the conjugacy be holomorphic on the interior of K_f.

Theorem 12.23 (The Straightening Theorem, [12])**.** *If f is a polynomial-like map of degree d, there exists a polynomial P of degree d so that f is hybrid equivalent to P. If K_f is connected, then P is unique up to affine conjugation.*

The straightening theorem can be applied to *families* of quadratic-like maps to understand why parameter spaces of higher-degree maps contain copies of the Mandelbrot set in them.

We first define the notion of an analytic family of polynomial-like maps.

Definition 12.24. Consider a collection $\mathcal{F}$ of polynomial-like maps $\{f_\lambda : U_\lambda \to V_\lambda\}_{\lambda\in\Lambda}$, where Λ is a Riemann surface. Define

$$U = \{(\lambda, z) : z \in U_\lambda\}$$

$$V = \{(\lambda, z) : z \in V_\lambda\},$$

and a map $f : U \to V$ given by

$$f(\lambda, z) = f_\lambda(z).$$

Then $\mathcal{F}$ is an *analytic family* if

1. Both U and V are homeomorphic to $\Lambda \times \mathbb{D}$,
2. The projection $\overline{U} \to \Lambda$ given by $(\lambda, z) \mapsto \lambda$ is proper, and
3. The map $f : U \to V$ is both proper and holomorphic.

If we have such an analytic family $\mathcal{F}$, we can define

$$\mathcal{M}_\mathcal{F} = \{\lambda \in \Lambda : K_f \text{ is connected}\}.$$

If the family consists of quadratic-like maps, we get the following.

Theorem 12.25. *Let $\mathcal{F}$ be a family of quadratic-like maps, and let $W \subseteq \Lambda$ be homeomorphic to a disk, with $\mathcal{M}_\mathcal{F} \subseteq W$. Let ω_λ be the critical point of f_λ, and assume that*

1. for each $\lambda \in \Lambda \setminus W$, we have that $f_\lambda(\omega_\lambda) \in V_\lambda \setminus U_\lambda$, and

2. as λ winds around ∂W, $f(\omega_\lambda) - \omega_\lambda$ winds once around 0.

Then $\mathcal{M}_\mathcal{F}$ is homeomorphic to $\mathcal{M}$ via a map that is analytic on its interior.

12.3.3 Mandelbrot sets in $F_{1/2}$

We now turn back to our parameter space roadmap. Specifically, we prove Theorem 12.19.

Proof. By Lemma 12.18, the map P_t has a parabolic cycle. Recall that every map in this slice has an attracting fixed point at 0 that attracts the preferred critical point c_-; however the second critical point c_+ may exhibit other behavior. Take Λ to be the connected component of the set $\{c \in F_{1/2} : c_+ \text{ is not attracted to } 0 \text{ or } \infty\}$ that contains the parameter corresponding to P_t. Since the maps in this slice vary holomorphically in c, the subset of cubic polynomials $\{f_\lambda\}_{\lambda \in \Lambda}$ forms an analytic family of quadratic-like maps (restricting the polynomials to the basins of this second cycle). The straightening theorem then gives that $\mathcal{M}_t$ is homeomorphic to $\mathcal{M}$. □

Next, we give a procedure for finding Blaschke components in $F_{1/2}$ off of a baby Mandelbrot set $\mathcal{M}_t$.

Much as in the way we parametrized the boundary of K_Q using dynamical external rays, we can catalogue and locate specific points of $\partial\mathcal{M}$ in terms of parameter external rays. Specifically, it is true that

$$\hat{\mathbb{C}} \setminus \mathcal{M} \cong \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}.$$

Let $\varphi^{\mathcal{M}} : \hat{\mathbb{C}} \setminus \mathcal{M} \cong \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ be the uniformizing map. Then once again, we can pull back radial external rays on $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ to $\hat{\mathbb{C}} \setminus \mathcal{M}$.

Specifically, we define

$$\varphi^{\mathcal{M}}(c) = \varphi_c(c)$$

and

$$\mathcal{R}_t^{\mathcal{M}} = \varphi^{\mathcal{M}}\left(re^{2\pi i t}\right).$$

If $\lim_{r \searrow 1} R_t^{\mathcal{M}}$ exists, we say that the parameter ray of angle t *lands* at this limit point in $\mathcal{M}$. It is not known if all rays land. However, we do have the following classical result (see, for example, [11]).

Proposition 12.26. *If t is periodic under the doubling map on $\mathbb{R}/\mathbb{Z}$, then the parameter ray $\mathcal{R}_t^{\mathcal{M}}$ lands on $\mathcal{M}$.*

We will be focusing specifically on the *dyadic* angles — that is, elements $\theta \in \mathbb{R}/\mathbb{Z}$ of the form

$$\theta = \frac{a}{2^k}$$

for some k. Notice that these are exactly the angles that are pre-fixed under doubling. Proposition 12.26 guarantees that if θ is dyadic, then $\mathcal{R}_\theta^{\mathcal{M}}$ lands at a point $c_\theta \in \mathcal{M}$.

Now for each baby Mandelbrot set $\mathcal{M}_t$, the homeomorphism between $\mathcal{M}_t$ and $\mathcal{M}$ gives distinguished points $f_t^\theta \in \partial\mathcal{M}_t$ corresponding to the dyadic angle landing points on $\mathcal{M}$.

Theorem 12.27. *If $\mathcal{M}_t$ is a baby Mandelbrot set whose center of its main cardioid is a map with superattracting cycle of period p, then for each dyadic angle $\theta = \frac{a}{2^k}$, the*

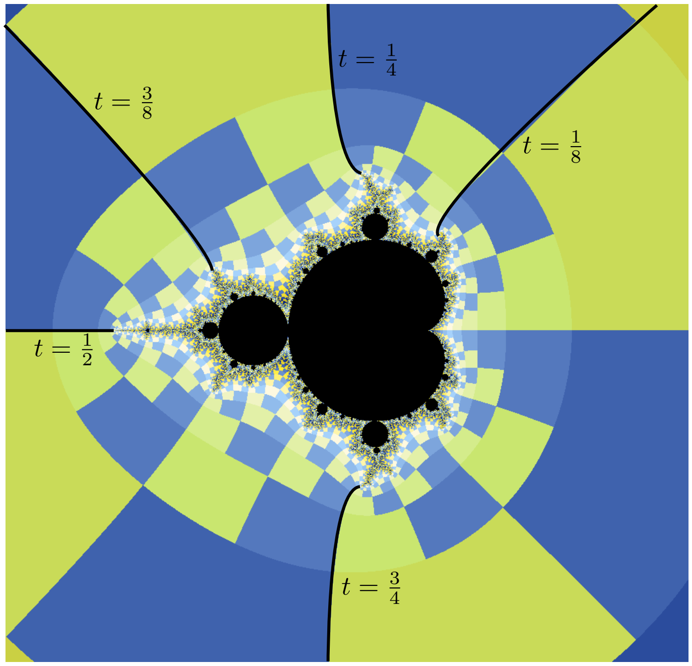


Figure 12.4: External rays for $\mathcal{M}$, with a number of dyadic rays and their landing points highlighted.

flexible critical point of f_t^θ *has orbit type* $(pk+1, p)$, *and* f_t^θ *is the point of attachment of a component* $\mathcal{H}$ *with* $\chi(\mathcal{H}) = pk + 1$.

Proof. If the main cardioid of $\mathcal{M}_t$ has a superattracting cycle of period p, the straightening theorem tells us that for each polynomial $f_{\mathcal{M}_t}$ in this baby Mandelbrot set, $f_{\mathcal{M}_t}^p$ is quadratic-like on the correct restriction of domain. Specifically, the homeomorphism $h^{-1} : \mathcal{M} \to \mathcal{M}_t$ takes maps with periodic critical orbit of period q to maps with periodic critical orbit of period pq.

For $\theta = \frac{a}{2^k}$, the map $f_c(\theta) \in \partial\mathcal{M}$ — that is, the landing point of the external ray $\mathcal{R}_\theta^{\mathcal{M}}$ on the boundary of $\mathcal{M}$ — has a pre-fixed critical orbit of pre-period $k+1$. Therefore, under h^{-1} this gets mapped to $f_t^\theta \in \partial M_t$ with pre-periodic (flexible) critical orbit of period p and pre-period $pk+1$.

Notice that the flexible critical orbit is therefore eventually mapped onto a repelling cycle, which necessarily lies on the boundary of the basin of the attracting fixed point of f_t^θ at 0. Therefore, f_t^θ lies on the boundary of a hyperbolic component $\mathcal{H}_{t,k}$ with both critical points attracted to 0. Moreover, making a perturbation of f_t^θ to land in the interior of $\mathcal{H}_{t,k}$ gives that the flexible critical point takes $pk+1$ steps to land in the immediate basin of 0. Therefore, $\chi(\mathcal{H}_{t,k}) = pk+1$. □

CHAPTER XIII

Parabolic fixed points

As mentioned in Chapter III, if f is a rational map with a parabolic cycle $\mathbf{a}$, then $\mathrm{MCG}(\mathcal{B}_{\mathbf{a}}, f)$ is a subset of $\mathrm{MCG}(\Sigma_{0,n+2})$, the mapping class group a sphere with $n+2$ punctures, where n is the number of grand orbits containing critical points in the basin $\mathcal{B}_{\mathbf{a}}$.

We remark that there is a lot to be explored in the general theory of the calculation of $\mathrm{MCG}(f)$ when f has a parabolic cycle, and we expect that many of the tools developed for the case of an attracting cycle will be able to be applied for parabolic cycles as well.

Here, we illustrate a first example in which f has a parabolic cycle and $\mathrm{MCG}(f)$ is non-trivial. To do so, we work in the setting of bicritical rational maps. This setting has the benefit that in [27], the author shows that the moduli space of all such maps of a fixed degree is biholomorphic to $\mathbb{C}^2$, and constructs explicit coordinates. The ability to work in these coordinates allows for explicit calculations of the mapping class group.

13.1 Coordinates

We parametrize a certain subset of bicritical rational maps by pairs $(\lambda, b) \in \mathbb{D}^* \times \mathbb{C}$. We first describe these maps combining ideas from [15] and [27]. In particular, fix a degree $n \geq 2$ and let f be a bicritical rational map of degree n so that the following are satisfied:

- f has critical points $c_+ = +1$ and $c_- = -1$, each of local degree n
- f has a fixed point at ∞ with multiplier λ.

Proposition 13.1. *Every bicritical rational map with a fixed point can be written in such a form, and in this form the maps will have corresponding critical values* $v_+ = f(c_+) = \frac{1}{\lambda}(n + b)$ *and* $v_- = f = \frac{1}{\lambda}(-n + b)$ *for some* $b \in \mathbb{C}$.

For such a map f, we write $f = f_{\lambda,b}$.

Proof. In [27], the author shows that every bicritical rational map with a fixed point can be put in the normal form

$$g_{\mu,\xi}(z) = \frac{(1 + \mu + \xi)z^n + (1 - \mu - \xi)}{(1 - \mu + \xi)z^n + (1 + \mu - \xi)}$$

where in this normal form the map has critical points at 0 and ∞ with local degree n and a fixed point at 1 (c.f. [27], Theorem 2.1). Furthermore, the multiplier of the fixed point is given by $\lambda = n\mu$. Conjugating by a Möbius transformation sending

$$0 \mapsto 1$$

$$\infty \mapsto -1$$

$$1 \mapsto \infty$$

gives us the map

$$f_{\mu,\xi}(z) = \frac{1}{\mu}\left(\frac{(1+z)^n + (-1-z)^n + \xi(-(1+z)^n + (-1-z)^n)}{(1+z)^n - (-1-z)^n}\right)$$

with a fixed point at ∞ with multiplier λ, and critical points at ± 1. Making the substitution $\lambda = n\mu$ and $b = -n\xi$, we get the required $f_{\lambda,b}$, and computation shows that

$$f_{\lambda,b}(1) = \frac{1}{\lambda}(n+b)$$

and

$$f_{\lambda,b}(-1) = \frac{1}{\lambda}(-n+b).$$

□

We can explicitly calculate the coordinates (X, Y) in the moduli space of degree n bicritical rational maps $\mathcal{M}_n \cong \mathbb{C}^2$ by using Milnor Lemma 1.7 and Corollary 2.2. given λ and b. In particular, since X is given by the negative of the cross-ratio $\xi(c_+ : c_- : v_+ : v_-)$, we have that for $f_{\lambda,b}$,

$$X = \frac{(1-v_+)(-1-v_-)}{2(v_+ - v_-)} = -\frac{(b+\lambda-n)(b-\lambda+n)}{4\lambda n}$$

and Y is given as a polynomial of X depending on λ whenever $\lambda \neq 0$.

Notice that this tells us that *the conjugacy class of $f_{\lambda,b}$ depends only on X.*

For reference, explicit formulas for $f_{\lambda,b}$ for $n \leq 4$ are listed below.

$n = 2$	$f_{\lambda,b} = \frac{1}{\lambda}(z + b + 1/z)$
$n = 3$	$f_{\lambda,b} = \frac{1}{\lambda}\left(\frac{3z^3 + 3bz^2 + 9z + b}{3z^2 + 1}\right)$
$n = 4$	$f_{\lambda,b} = \frac{1}{\lambda}\left(\frac{z^4 + bz^3 + 6z^2 + bz + 1}{z^3 + z}\right)$

Table 13.1: Some explicit examples of the parameterizations of the $f_{\lambda,b}$ for small degree

Notice that the case where $n = 2$ is exactly the setting in which [15] work, and the coordinates they choose are exactly those in the table. In fact, the results in their paper go through for general bicritical rational maps with almost no changes using the general coordinates as above.

For such a map $f_{\lambda,b}$, we have a dichotomy. Let $K_{\lambda,b}$ be the filled Julia set of $f_{\lambda,b}$, in the sense that $K_{\lambda,b}$ is the complement of the basin of infinity.

Theorem 13.2 (Theorem 3.1, [27]). *Either $K_{\lambda,b}$ is a Cantor set and $f_{\lambda,b}|_{K_{\lambda,b}}$ is topologically conjugate to the shift map, or $K_{\lambda,b}$ is connected.*

13.2 Parabolic fixed points

In this section, we focus on maps $f_{\lambda,b}$ with $\lambda = 1$. This is the space $\mathrm{Per}_1(1)$ — the maps (of degree n) with a fixed point of multiplier 1. The following result is proven in [27] and will be very useful for our purposes.

Theorem 13.3 (Theorem 4.2, [27])**.** *The space* $\mathrm{Per}_1(1)$ *is isomorphic to* $\mathbb{C}$. *The connectedness locus* $\mathcal{C}_{par} := \mathcal{C} \cap \mathrm{Per}_1(1)$ *is compact, connected, and full in* $\mathrm{Per}_1(1)$, *and the shift locus* $\mathcal{S}_{par} = \mathrm{Per}_1(1) \setminus \mathcal{C}_{par}$ *is conformally isomorphic to a punctured disk.*

We will work with generic maps in the shift locus. Notice that for all $f \in \mathcal{S}_{\mathrm{par}}$, since $\infty \in J_f$, it cannot be the case that $\infty \in \mathrm{GO}(c_+) \cup \mathrm{GO}(c_-)$. In other words, the only critical orbit relations coming from maps in $\mathcal{S}_{\mathrm{par}}$ arise from a collision in the orbits of the two critical points. We define

$$\mathcal{S}^*_{\mathrm{par}} := \mathcal{S}_{\mathrm{par}} \setminus \{f : f^n(c_-) = f^m(c_+) \text{ for some } m, n\}.$$

Fix $f \in \mathcal{S}^*_{\mathrm{par}}$. Then f has a parabolic fixed point at ∞ with multiplier 1, and with both critical points of f in $\mathcal{B}_\infty$, the basin of infinity.

Recall that $\mathcal{B}^*_\infty$ is defined to be $\mathcal{B}_\infty \setminus (\mathrm{GO}(\infty) \cup \mathrm{GO}(c_+) \cup \mathrm{GO}(c_-))$ — that is, in this setting $\mathcal{B}_\infty = \Omega^{\mathrm{dis}}$.

We then have a covering map

$$\Psi_f : \mathcal{B}^*_\infty \to \Omega^{\mathrm{dis}}/f \cong S_f$$

coming from modding out by the dynamics of f, where S_f is a four-times punctured sphere. We in turn get an induced inclusion

$$\Psi_* : \mathrm{PMCG}(f) \cong \pi_1(\mathcal{S}^*_{\mathrm{par}}, f) \to \mathrm{PMCG}(\Sigma_{0,4}).$$

So we see that to understand the mapping class group of a base map f, we will need to understand the mapping class group of a 4-times punctured sphere.

Remark 13.4. Much as how in the case of an attracting cycle, the quotient torus has a dynamically significant homology class of curve, in the case of a parabolic cycle the punctures on the quotient sphere $\Sigma_{0,n}$ have dynamical distinctions. In particular, there are two distinguished punctures in $\Sigma_{0,n}$ coming from the image of the parabolic cycle under the quotient map. In fact, dynamical mapping class elements in $\Sigma_{0,n}$ must fix the distinguished punctures (and the non-distinguished punctures) set-wise. It might make more sense to think about the dynamical mapping classes here in the setting of a *noded torus* (see Figure 13.1).

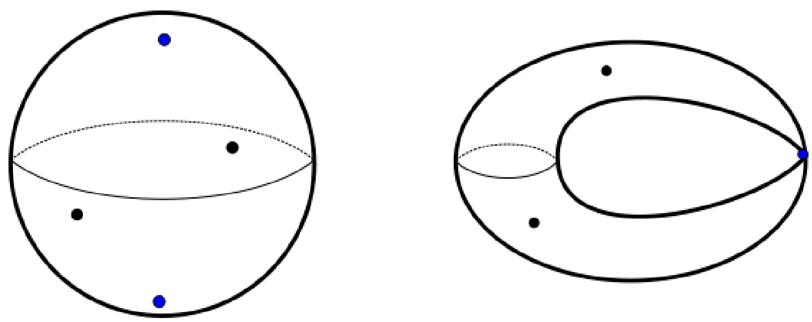


Figure 13.1: A sphere with two dynamically distinguished punctures (in blue), versus a noded torus.

13.3 The mapping class group of $\Sigma_{0,4}$

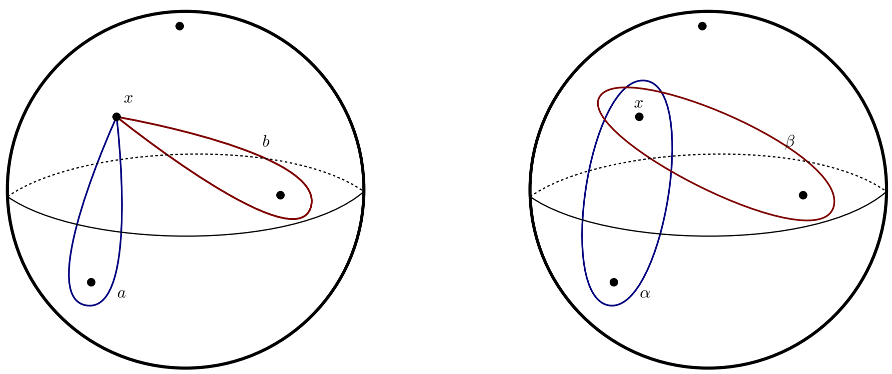
The Birman exact sequence from Theorem 4.3 applied to the four-times punctured sphere gives us the following.

$$1 \longrightarrow \pi_1(\Sigma_{0,3}, x) \xrightarrow{\text{Push}} \mathrm{PMCG}(\Sigma_{0,4}) \xrightarrow{\text{Forget}} \mathrm{PMCG}(\Sigma_{0,3}) \longrightarrow 1$$

Note that $\mathrm{PMCG}(\Sigma_{0,3})$ is trivial and so we get an isomorphism

$$\pi_1(\Sigma_{0,3}, x) \cong F_2 \to \mathrm{PMCG}(\Sigma_{0,4}).$$

If we choose generators for $\pi_1(\Sigma_{0,3}, x)$ as on the left in in Figure 13.2, under this isomorphism they get sent to point pushes around the corresponding curves in $\Sigma_{0,4}$. However, notice that writing these point pushes as a product of Dehn twists, one of the twists is trivial. Therefore, we take as generators for $\mathrm{PMCG}(\Sigma_{0,4})$ the Dehn twists T_α and T_β around the curves α and β in the right of Figure 13.2.



Generators of $\pi_1(\Sigma_{0,3})$ Dehn twists as generators of $\mathrm{PMCG}(\Sigma_{0,4})$

Figure 13.2: The isomorphism between $\pi_1(\Sigma_{0,3})$ and $\mathrm{PMCG}(\Sigma_{0,4})$

13.4 The topology of $\mathcal{S}^*_{\mathrm{par}}$

Fix a base map $f_0 \in \mathcal{S}^*_{\mathrm{par}}$. Notice that

$$\pi_1(\mathcal{S}^*_{\mathrm{par}}, f_0) \cong F_\infty,$$

the free group on countably many generators, where we can take generators to be loops based at f and winding around each of the countably many isolated punctures in $\mathcal{S}^*_{\text{par}}$, plus one loop around the connectedness locus.

As in Chapter IX, we relate the parameter space $\mathcal{S}^*_{\text{par}}$ to $\text{MCG}(\Sigma_{0,4})$ using the Birman exact sequence. In particular, let F_n be the unique degree n polynomial with a parabolic fixed point at 0 and with a single critical point with multiplicity $n-1$. Let $\mathcal{B}_n$ be the parabolic basin of P_n, and let $\mathcal{P}_n$ be the maximal attracting petal in $\mathcal{B}_n$ with Fatou coordinate ϕ_n so that ϕ_n takes $\mathcal{P}_n$ homeomorphically onto a right half-plane.

In [27], in the author's alternate proof of Theorem 4.2, he describes a map

$$h : \mathcal{S}_{\text{par}} \to \mathcal{B}_n \setminus \mathcal{P}_n.$$

This map is defined much as the map from the parameter space to the basin of a quadratic polynomial as in Chapter IX. Essentially, for a map $g \in \mathcal{S}_{\text{par}}$, a conjugacy is defined between a petal in the parabolic basin for g and a petal in $\mathcal{B}_n$. The map h is then given by where this conjugacy takes the critical value of g.

Lemma 13.5 (Theorem 4.2, [27])**.** *The map $h : \mathcal{S}_{par} \to \mathcal{B}_n \setminus \mathcal{P}_n$ is a conformal isomorphism.*

Proof. The proof is exactly that found in the alternate proof of Theorem 4.2 in [27], with the small difference that our model space is affine conjugate to the one given in the paper. □

Lemma 13.6. *The map h has the following properties.*

1. *The map $g \in \mathcal{S}_{par}$ with critical values v_+ and v_- has a critical orbit relation $g^n(v_+) = g^m(v_-)$ if and only if $F^n(w_+) = F^n(w_-)$, where w_+ is the critical value of F and $w_- = h(g)$.*

2. *As g tends to $\partial\mathcal{S}_{par}$, $w = h(b)$ tends to the boundary of the petal $\partial\mathcal{P}_n$.*

Proof. Statement (1) of the lemma comes directly from the construction of the map h. Statement (2) is proven in [27], in the alternate proof of 4.2. □

Compare this result to the corresponding result, Lemma 9.6, in Chapter IX. In particular, we again have a correspondence between our parameter space and a model space given by the dynamics of a specific map, and this correspondence preserves critical orbit relations. However, notice that in the setting of a parabolic fixed point, the open subset of parameter space in which we have MCG-generic maps does not contain punctures corresponding to maps where one of the critical points is pre-fixed.

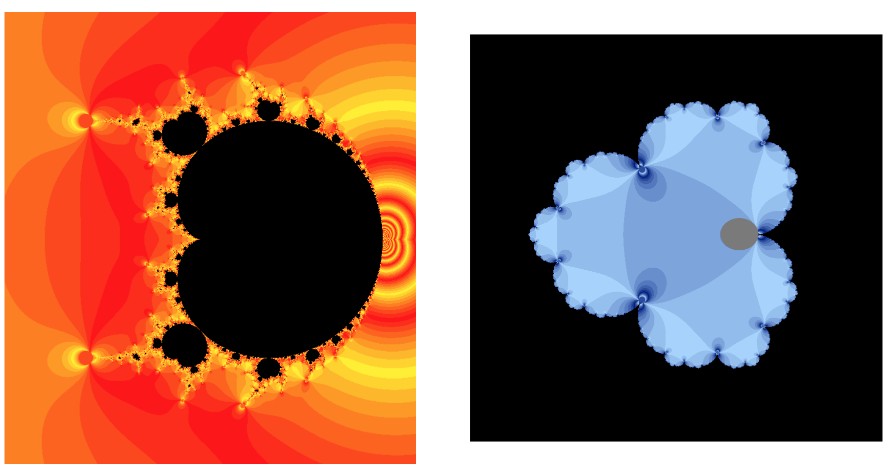


Figure 13.3: The correspondence between parameter (left) and dynamical (right) planes when $n = 3$. The petal image $\mathcal{P}_n$ in the dynamical plane is in grey.

Let $\mathcal{B}_n^* \subseteq \mathcal{B}_n$ be given by

$$\mathcal{B}_n^* \setminus \{\mathrm{GO}(c)\}.$$

That is, $\mathcal{B}_n^*$ is the subset of points in the basin of 0 whose orbit never intersects the orbit of the critical point of P_n. Then by Lemma 13.6, we see that h restricts to

$$h : \mathcal{S}^*_{\mathrm{par}} \to \mathcal{B}_n^* \setminus \mathcal{P}_n.$$

The construction of h gives the following commutative diagram

$$\begin{array}{ccc} \mathcal{B}_n^* \setminus \mathcal{P}_n & \xrightarrow{h^{-1}} & \mathcal{S}^*_{\mathrm{par}} \\ \downarrow{\scriptstyle \psi_{P_n}} & & \downarrow{\scriptstyle \Psi} \\ \Sigma_{0,3} & \xrightarrow{\text{puncture}} & \mathcal{M}_*(\Sigma_{0,4}) \end{array}$$

where the vertical maps are covering maps coming from modding out by the dynamics on the basin of the map in question.

This gives us an induced diagram on fundamental groups that lets us work in the model (dynamical) space $\mathcal{B}_n \setminus \mathcal{P}_n$. In particular, we have that

$$\pi_1(\mathcal{S}^*_{\mathrm{par}}, f_0) \cong \pi_1(\mathcal{B}_n^* \setminus \mathcal{P}_n, x)$$

where elements of $\pi_1(\mathcal{S}^*_{\mathrm{par}}, f_0) \cong \mathrm{PMCG}(f_0) \hookrightarrow \mathrm{PMCG}(\Sigma_{0,4})$ can be given by images of point-pushes

$$\pi_1(\Sigma_{0,3}, \Psi_{P_n}(x)) \to \mathrm{PMCG}(\Sigma_{0,4}).$$

From this, we get the following main result.

Theorem 13.7. *The pure mapping class group* $\mathrm{PMCG}(f_0)$ *is an infinitely generated free subgroup of* $\mathrm{PMCG}(\Sigma_{0,4}) \cong F_2$ *which is generated by*

- *One (based) loop enclosing each orbit relation, and*
- *One loop enclosing the connectedness locus.*

These generators correspond to point-pushes (and squares of point-pushes) around closed curves in $\Sigma_{0,4}$.

Proof. The idea of the proof again follows that of Theorem 4.7 in [15]. As with so many of the results of this flavor, the setup has been done to enable us to prove this result by understanding generators of the fundamental group of a single, simpler dynamical picture. Consider the space $\mathcal{B}_n^* \setminus \mathcal{P}_n$. The punctures O_n in $\mathcal{B}_n^*$ come from points whose orbits intersect the orbit of the critical point c of P_n — these punctures form an infinite discrete set. Our choice of base point $f_0 \in \mathcal{S}_{\text{par}}^*$ gives us a corresponding base point $q_0 = h(f_0) \in \mathcal{B}_n^* \setminus \mathcal{P}_n$, and we can choose generators of $\pi_1(\mathcal{B}_{\text{par}}^* \setminus \mathcal{P}_n, q_0)$, one surrounding each of the punctures in O_n. We want to understand the image of the projection $\psi_{P_n}(\gamma)$ of these generators $\{\gamma\}$.

To do so, first choose a puncture $q_i \in O_n$ satisfying $P_n^{m_1}(q_i) = P_n^{m_2}(c)$ for some m_1, m_2 with $m_2 > 0$. Since $m_2 > 0$, q_i is *not* a critical point of the coordinate ϕ for P_n, and there is a punctured neighborhood $\mathcal{N}_i$ of q_i that maps homeomorphically under ψ_{P_n} to a neighborhood of $\Sigma_{0,3}$, and $\psi_{P_n}(\mathcal{N}_i)$ is a punctured neighborhood around the marked point $\psi_{P_n}(c) \in \Sigma_{0,3}$.

We can choose the generator γ_i around puncture q_i so that γ_i projects under ψ_{P_n} to a simple closed curve in $\Sigma_{0,3}$, and parametrized as a segment from q_0 to a point in $\mathcal{N}_i$ traversed forward, a homotopically nontrivial loop contained completely in $\mathcal{N}_i$, and then the same segment traversed backward. The image $\psi_{P_n}(\gamma_i)$ is then conjugate to a loop in $\Sigma_{0,3}$ enclosing only the puncture $\psi_{P_n}(c)$, and by Lemma 4.4 corresponds to a point-push in $\mathrm{PMCG}(\Sigma_{0,4})$.

If on the other hand, the puncture q_j corresponds to a point with $P_n^{m_1}(q_j) = c$ for some $m_1 > 0$, the situation is similar with a slight complication: the map $\psi_{P_n} : \mathcal{N}_j \to \Sigma_{0,3}$ is now a 2-to-1 cover branched at q_j. The same analysis shows that a generator γ_j constructed as above corresponding to the *square* of a point-push in $\mathrm{PMCG}(\Sigma_{0,4})$.

Finally, we have a single generator in $\pi_1(\mathcal{B}_n^* \setminus \mathcal{P}_n, q_0)$ coming from a loop around the petal $\mathcal{P}_n$, enclosing no other punctures in $\mathcal{B}_n^*$. In $\mathcal{S}_{\mathrm{par}}^*$, this corresponds to a loop around the connectedness locus $\mathcal{C}$. Again, a choice of loop γ_* corresponds to a point-push around the projection $\psi_{P_n}(\gamma_*) \in \Sigma_{0,3}$. $\square$

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] J. S. Birman. Mapping class groups and their relations to braid groups. *Communications on Pure and Applied Mathematics*, 22:213–238, 1969.

[2] J. S. Birman. The algebraic structure of surface mapping class groups. *Discrete groups and automorphic functions*, pages 163 – 198, 1977.

[3] J. S. Birman and J. Cannon. *Braids, Links, and Mapping Class Groups.* Princeton University Press, 1974.

[4] P. Blanchard, R. L. Devaney, and L. Keen. The dynamics of complex polynomials and automorphisms of the shift. *Inventiones mathematicae*, 104:545–580, 1991.

[5] B. Branner and N. Fagella. *Quasiconformal Surgery in Holomorphic Dynamics.* Cambridge University Press, 2014.

[6] S. Bullett. Lecture notes in holomorphic dynamics and hyperbolic geometry, February 2013.

[7] A. Cattabriga and M. Mulazzani. (1, 1)-knots via the mapping class group of the twice punctured torus. *Advances in Geometry*, 4(2):263 – 277, 2004.

[8] B. Cavallo, J. Delgado, D. Kahrobaei, and E. Ventura. Algorithmic recognition of infinite cyclic extensions. *Journal of Pure and Applied Algebra*, 221(9):2157 – 2179, 2017.

[9] L. DeMarco. Combinatorics and topology of the shift locus. In F. Bonahon, R. Devaney, F. Gardiner, and D. Saric, editors, *Conformal Dynamics and Hyperbolic Geometry*, Contemporary Mathematics, pages 35–48. American Mathematical Society, 2012.

[10] L. DeMarco and K. M. Pilgrim. Polynomial basins of infinity. *Geometric and Functional Analysis*, 21, 2011.

[11] A. Douady and J. H. Hubbard. Etude dynamique des polynômes complexes. *Publications mathématiques d'Orsay*, 84, 1984.

[12] A. Douady and J. H. Hubbard. On the dynamics of polynomial-like mappings. *Ann. Scient., Ec. Norm. Sup* 4^e *series*, 18:287–343, 1985.

[13] N. Fagella. The theory of polynomial-like mappings – the importance of quadratic polynomials. *Proceedings of the 7th EWM meeting*, pages 57–70, 1995.

[14] B. Farb and D. Margalit. *A Primer on Mapping Class Groups.* Princeton University Press, 2012.

[15] L. R. Goldberg and L. Keen. The mapping class group of a generic quadratic rational map and automorphisms of the 2-shift. *Inventiones Mathematicae*, 101:335–372, 1990.

[16] J. Gonzalez-Meneses. New presentations of surface braid groups. *Journal of Knot Theory and Its Ramifications*, 10:431–451, 2001.

[17] J. Guaschi and D. Juan-Pineda. A survey of surface braid groups and the lower algebraic k-theory of their group rings. *Handbook of Group Actions*, 2:23–76, 2015.

[18] M.-E. Hamstrom. Homotopy groups of the space of homeomorphisms on a 2-manifold. *Illinois J. Math.*, page 563573, 1966.

[19] I. Kra. On the nielsen-thurston-bers type of some self-maps of riemann surfaces. *Acta Math*, 146:231270, 1981.

[20] R. Mañé, P. Sad, and D. Sullivan. On the dynamics of rational maps. *Annales scientifiques de l'École Normale Supérieure*, Ser. 4, 16(2):193–217, 1983.

[21] C. McMullen. Braiding of the attractor and the failure of iterative algorithms. *Inventiones mathematicae*, 91(2):259–272, 1988.

[22] C. T. McMullen. Automorphisms of rational maps. In D. Drasin, C. J. Earle, F. W. Gehring, I. Kra, and A. Marden, editors, *Holomorphic Functions and Moduli I*, volume 10 of *Mathematical Sciences Research Institute Publications*, pages 31–60. Springer, 1988.

[23] C. T. McMullen. *Complex Dynamics and Renormalization.* Princeton University Press, 1994.

[24] C. T. McMullen. Riemann surfaces, dynamics and geometry. Lecture notes, 2018.

[25] C. T. McMullen and D. Sullivan. Quasiconformal homeomorphisms and dynamics iii. the teichmller space of a holomorphic dynamical system. *Advances in Mathematics*, 135, 04 1998.

[26] J. Milnor. Geometry and dynamics of quadratic rational maps, with an appendix by the author and lei tan. *Experiment. Math.*, 2(1):37–83, 1993.

[27] J. Milnor. On rational maps with two critical points. *Experimental Mathematics*, 9:481–522, 2000.

[28] J. Milnor. *Dynamics in One Complex Variable.* Princeton University Press, 2006.

[29] J. Milnor. Hyperbolic components, with an appendix by a. poirier. In F. Bonahon, R. Devaney, F. Gardiner, and D. Saric, editors, *Conformal Dynamics and Hyperbolic Geometry*, Contemporary Mathematics, pages 183–232. American Mathematical Society, 2012.

[30] T. W. Ng and C. Y. Tsang. Polynomials versus finite blaschke products. In *Blaschke Products and Their Applications. Fields Institute Communications, vol 65.*, pages 249–273. Springer, Boston, MA, 2013.

[31] C. Petersen and L. Tan. *Analytic Coordinates Recording Cubic Dynamics*, pages 413–449. A K Peters, USA, 2009.

[32] K. M. Pilgrim and T. Lei. Spinning deformations of rational maps. *Conformal Geometry and Dynamics*, 8:52–86, 2004.

[33] F. Przytycki. Iterations of rational functions: which hyperbolic components contain polynomials? *Fundamenta Mathematicae*, 149:95–118, 1996.

[34] B. Wittner. *On the Bifurcation Loci of Rational Maps of Degree Two*. PhD thesis, Cornell University, 1987.

[35] S. Zakeri. On Critical Points of Proper Holomorphic Maps on The Unit Disk. *Bulletin of the London Mathematical Society*, 30(1):62–66, 01 1998.