The Unactualized Certainty-Actuality Correspondence

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Abstract

This paper investigates the correspondence between unactualized certainties, interpreted ontologically, and actualities. It does this first through the lens of a recently proposed enrichment of axiomatic probability which makes it possible to distinguish between actualities and unactualized possibilities, including those which are certain. Two kinds of unactualized certainties are considered: those due to the sample space being a singleton, and those involving a sample space with more than one element. After comparing standard axiomatic probability with the enrichment in regards to how they represent the distinction, attention is then focused on quantum mechanics. There, the correspondence will be examined through the lens of a recently proposed modification of the standard formalism, the *Heisenberg Interpretation* which, unlike the standard quantum formalism but like the enrichment of probability, also permits distinctions between unactualized possibilities and actualities. Two situations are found to exemplify the correspondence there: one involving partially measured entangled systems and the other involving the Born rule. Finally, it will be shown how the correspondence clarifies the very concept of an unactualized possibility.

1 Introduction

One sometimes hears an assertion like: "Since event X just happened, its probability is 1". Under a conception of possibilities as things which have not actualized yet, the assertion is false: a probability is a measure over *possibilities*, and an event that just happened is not a possibility but an *actuality*. Contrast this with the situation in which event X has not happened yet, but is *certain* to occur. In *this* case, the probability of that event is indeed 1.

Certainty is usually regarded as an epistemic property of belief[1], but a basic presupposition of this paper is that it can be given an ontological interpretation in the same way-*and, indeed, for the same situations*-as probability itself. For example, a hypothetical loaded coin with a frequency of 1 of yielding heads in the limit of an infinite sequence of coin flips has an outcome which can be interpreted to be ontologically certain. Similarly, if we think of the probability of getting any one of the six possible outcomes of the roll of a six-sided die in terms of its disposition (rather than our knowledge or belief), then the certainty can again be interpreted in ontological terms. In the rest of this paper, "unactualized certainty" will refer to ontic certainty whenever probability is interpreted in ontological terms[1].

This paper will investigate the correspondence between unactualized certainties and actualities, first through the lens of axiomatic probability, and then through that of quantum mechanics. It will do this by comparing in each case the standard formalism to recently proposed modifications which enable in each formalism previously unaccommodatable distinctions between actualities and unactualized possibilities, including those which are certain. It will be shown that in each case, making the distinction leads to greater conceptual clarity, and that in the case of quantum mechanics, it may help with the conceptualization of quantum entanglement and the Born rule. The paper will conclude by showing that an unactualized possibility can be considered an unactualized certainty diminished by the availability of alternatives, rather than by any intrinsic distinction.

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[1]Unactualized certainty is reminiscent of the modal concept of *necessity*. Though there are similarities, one can see the difference already by the fact that something can be both actually *and* necessarily the case, but not actually the case and also be characterized by an unactualized certainty. If something is actually the case, then its associated certainty has actualized.

2 Enriched Axiomatic Probability

Probability, as axiomatized by Kolmogorov[2], does not distinguish between unactualized possibilities and actualities. As such, it fails to capture the *concept* of probability, at least to the extent that possibilities are not considered to be merely characteristic of actualities since, in a trivial sense, every actuality can be considered to be possible.
Here one should keep in mind that the concept of probability is independent of its *interpretation* (e.g. frequentist, Bayesian, dispositionalist, etc.): under any given interpretation, the concept of probability remains the same, namely, as a unit measure over possibilities. In standard axiomatic probability, there is no inherent difference between a probability and a unit measure over things which are not possibilities, such as a unit length.
Recently, I proposed an enrichment of the standard axiomatization meant to represent the probability concept in the formalism by adding some additional mathematical structure[3]:

> Let $\Omega = \bigcup_{i=1}^{N} E_i$ be a set where N is either finite or countably infinite, $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ a set of its mutually exclusive subsets E_i, and call the pair $(\Omega, \mathcal{A})$ a measurable space. Let $\Gamma = \{\gamma | \gamma = f(\omega)\}$ be a set where $f : \Omega \to \Gamma$ is a bijection and let $g : \Omega \to \Gamma$ be a surjection. A real-valued function $P : \mathcal{A} \to \mathbb{R}$ satisfying
> - Axiom 0: $P(g^{-1}(\gamma)) = P(\Omega)$
> - Axiom 1: $P(\Omega) = 1$
> - Axiom 2: $0 \leq P(E_i) \leq 1$
> - Axiom 3: $P \bigcup_{i=1}^{N} E_i = \sum_{i=1}^{N} P(E_i)$
>
> is called a *probability*.

The additional structure here consists of:

1. a set Γ, called the *outcome set*, which represents the collection of actualities, to be distinguished from possibilities which are represented by the sample space Ω.
2. a map f which brings each element of the sample space into a one-to-one correspondence with an element of the outcome set.
3. an axiom which says that probability is a measure over a *fiber* g^{-1} on each element of the outcome set such that this fiber is just what we call the sample space.

It will be useful to give the two new maps specific names:

- The bijective map f will be called the *Possibility-actuality correspondence*. The name is self-explanatory; it ensures that there is never a mismatch between the sample space and the outcome set.
- The surjective map g will be called the *Unactualized certainty-actuality(UC-A) correspondence*. The name is meant to indicate that g always puts a set of possibilities over which the probability measure is unity into correspondence with an actuality. *The concept behind this map is the main focus of this paper.*

The axiomatic enrichment takes "possibility" to mean *unactualized possibility*, which is to say that the trivial sense (as mentioned above) is separated out and completely subsumed under the meaning of "actuality". The word "chance" seems closest in meaning to this, but for the sake of avoiding any ambiguity, I will continue to mention "unactualized possibility" below.
In probability, there are two types of situations in which the $UC - A$ correspondence can be considered: a type of situation in which the sample space contains only a single element, and one in which it contains more than one element. We consider the first type next.

3 Pro-actuality

There is one specific kind of situation in which f and g become indistinguishable, namely when the sample space is a singleton (in which case, of course, the outcome set is as well). In this type of situation, in other words, a single unactualized possibility becomes a certainty. We can now see how the assertion at the beginning of this paper went wrong: an actuality is trivially a possibility, but also a certainty. But so is the element of a singleton sample space! The difference, that the first is actualized and the second is not, cannot be represented in standard probability. Yet, as we will see, the distinction is sufficiently important that it deserves its own terminology:

- A *pro-actuality* is defined as an unactualized possibility which is a certainty.

For instance, consider a hypothetical coin which is rigged to always return heads upon a flip. In other words, the probability of obtaining heads here is 1. However, one has to actually flip the coin before the certainty becomes actualized. The relation between the actual outcome and the certainty preceding it is given by the *UC-A* correspondence: the lone element of its domain is heads as a pro-actuality (however, see section 10.), and the lone element of its co-domain is heads as an actuality. Under the axiomatic enrichement, flipping such a coin changes which of the two we take to represent the situation, as in figure 1.:

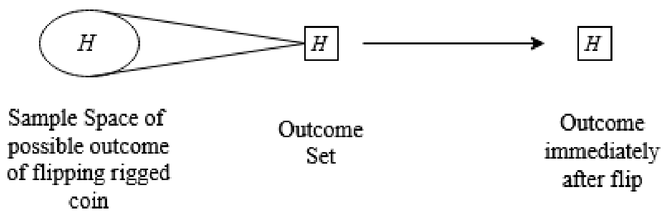

Figure 1: A transition from pro-actuality to actuality visualized in the axiomatic enrichment. Pro-actualities are characterized by $f = g$. The convention will be to omit the sample space in the visualization of an actual outcome.

Left of the arrow, the situation is represented by the element of the fiber on H (the reason behind this circumspect phrasing will be given in section 10.), while right of the arrow, it is represented by H itself. The convention will be to omit the sample space when actual outcomes are visualized. Standard axiomatic probability, on the other hand, does not have the resources to make a distinction between a pro-actuality and an actual outcome.
Now, let us consider that it is possible to decide to never flip the coin. This suggests a distinction between two kinds of pro-actuality:

- A *conditionally actualizable pro-actuality* is defined as a pro-actuality which depends on some triggering event for its actualization .
- An *unconditionally actualizable pro-actuality* is defined as a pro-actuality which does not depend upon any triggering event for its actualization.

A triggering event is simply any kind of event or process-*other than the passage of time alone*-which brings about the transformation of a pro-actuality into an actuality. In the previous example, the flipping of the coin is a triggering event, so in this situation, the pro-actuality prior to the coin flip is conditionally actualizable.
An unconditionally actualizable pro-actuality, in contrast, not requiring any triggering event, conjures up perhaps most readily the spontaneous actualization of an unactualized certainty, the probability of occurrence of which over a finite time interval is one. There is much more to be said about this, but as it lies outside the scope of this work, it will be reserved for future treatment.

4 From Probability to the Quantum

When the sample space contains more than one element, the unactualized certainty can be thought to be distributed over its elements. To get a better feel for how the $UC-A$ correspondence captures a key distinction absent in the standard axiomatization of probability in such cases, let us visualize a concrete example. Consider the throw of a six-sided die in three stages:

1. The die is about to be thrown.
2. The die is thrown and an outcome is obtained, say, three.
3. The die is picked up and about to be thrown again.

In the standard axiomatization, the first stage is represented by a sample space containing 6 elements, the second stage by a sample space containing a single element, and the third stage, once again, by a sample space containing 6 elements, as visualized in figure 2.

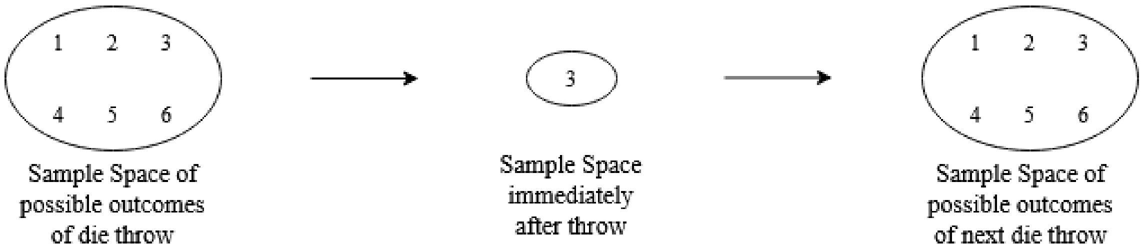

Figure 2: In the standard axiomatization, immediately upon the throw of the die, the sample space updates to only include the outcome of the throw of the die. The third stage is represented by another update. Thus, standard axiomatic probability conflates actuality with unactualized certainty.

As the second stage is represented by a sample space containing the single outcome, *axiomatic probability conflates actuality with unactualized certainty.* As mentioned, this may at least partly explain assertions like that at the beginning of this paper[2].

In the enriched axiomatization, the first stage is represented by the outcome set such that the fiber on each of its elements is the sample space. In this stage, the outcome set is essentially ignored because we are interested in the probability. The second stage is represented by a single element of the outcome set and now it is the fiber which is ignored because we are interested in the actual outcome of the throw. Probability is only defined on the fiber, not on the outcome set. Thus, the enriched axiomatization does *not* license assertions as in the beginning of this paper. The third stage is represented once again by an outcome set containing all possible outcomes as actualities and the fiber on each has "reappeared" because the die is about to be thrown again. All this is as visualized in figure 3.

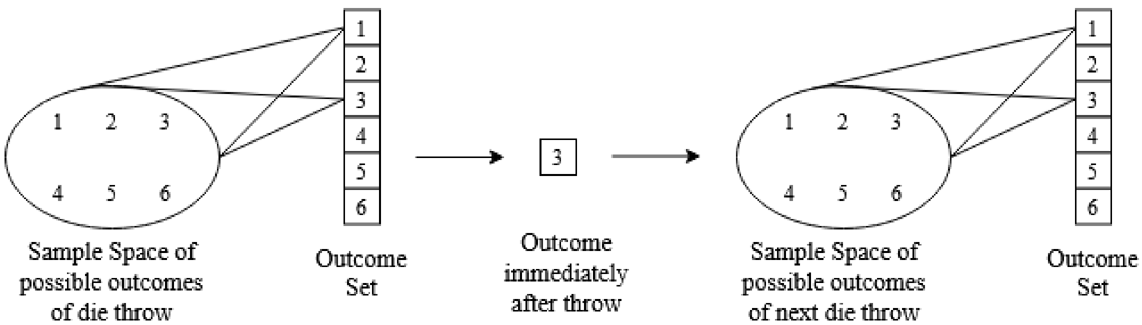

Figure 3: In the enriched axiomatization, the outcome in the second stage is represented by an element of the outcome set and the fiber which represents the sample space is ignored or "disappears". Thus, the outcome is correctly represented as an actuality. The fiber "reappears" in the next stage, when unactualized possibilities become available again. In this diagram, a depiction of the map f was omitted, and in order to avoid clutter, only the fiber on the outcomes 1 and 3 is represented visually, though, of course, there is a fiber on every element of the outcome set.

Consider, again, figure 2. Because the process of a die throw is transparently observable to us, we have no problem interpreting it in a way that compensates for the error in the mathematical representation, i.e. we may psychologically substitute an actuality for the outcome, even though the mathematical formalism represents it as an unactualized certainty.

But let us now imagine how we would interpret the standard axiomatic representation if the observation of a die throw (or analogous events) was not (and never had been) available to us. How would we interpret the mathematical representation then?

[2]Such assertions may also be partly due to conflating, in a subtle manner, ontic probability with epistemic probability. In epistemic probability, *there is no distinct state of actuality*: the best one can do is to express certainty in one's credence. Consequently, from a purely epistemic perspective, there is *no intrinsic difference* between the certainty in one's credence before and after an event with probability one has occurred. So, in a worldview according to which all probabilities are epistemic, such a distinction would likely never even suggest itself as, in the final analysis, the distinction is *ontic* in nature.

First, since the formalism fails to distinguish between unactualized possibilities and actualities, we might take each element of the sample space to represent an actuality. We would then be faced with having to make sense out of a system characterized by a combination of mutually incompatible descriptions. We might even choose the word *superposition* for the combination and *state* for each way this object we call a "die" would have to be in order to give a particular outcome.
Then, we would notice that upon the triggering event in the second stage, the superposition of states mysteriously "collapses" to a single state. We might wonder how and why the triggering event could possibly annihilate the probability associated with all but one state of a system.
Finally, we would notice that shortly after collapse following the triggering event, the overall state of the die "spreads out" to once again include all six outcome states as part of its overall state.
Of course, this caricature was framed suggestively to recall the perplexities that have plagued physicists, philosophers and mathematicians for generations when trying to understand what quantum mechanics actually tells us about reality[4][5][6].
The similarity between probability and quantum mechanics with respect to the distinction between unactualized possibilities and actualities motivates the interpretation of quantum states as unactualized possibilities, unless they are undergoing a physical process which has been given the unfortunately anthropocentric label "measurement". If quantum states are thought of in this way, then the ideas discussed up until now, and specifically the concept of a $UC - A$ correspondence, turn out to apply to quantum mechanics as well.

5 Potentiality vs. Actualizability

One may consider the unactualized possibilities in probability as "potentialities". In the context of mathematics, it makes little difference what they are called (as long as the label is used consistently), but as we wish to consider now an analogous situation in physics (i.e. quantum mechanics), their meaning, particularly with respect to their ontological character, gains importance because the referents are no longer abstract objects of our mind but concrete things "out there in the world". Thus, it pays to choose a name with care.
Potentiality is not currently a concept in physics, but philosophy. For the sake of clarifying its meaning and contrasting it with the term that will be used in the quantum context, it it is worthwhile to give a brief philosophical analysis of this term within the context of physics.
After considering dispositions of several kinds of physical systems (e.g. electrical charge can be thought of in terms of the disposition to create an electric field, radioactivity in terms of the disposition of a particle to decay etc.), Kistler argues that potentiality and dispositionality refer semantically to the same concept and only differ with respect to pragmatic constraints, i.e. they differ with respect to the kind of situations in which the use of one term or the other seems particularly appropriate [7]. He identifies two criteria which determine when it seems more appropriate to call something a potentiality than a dispositionality:

1. A potentiality ceases after it has actualized (whereas a dispositionality may not), and
2. The probability for actualization of a potentiality is less than one (whereas for a dispositionality it can be one).

Kistler then applies this conception of potentiality to Heisenberg's writings in his book *Physics and Philosophy* [4] in which Heisenberg distinguished between "possibilities or potentialities" and "things or facts" to arrive at a conception of the measurement problem which will turn out to express a basic assumption of the interpretation to be discussed shortly[3]. Kistler identifies a difficulty:

> "Heisenberg's claim according to which an a quantum system undergoes, at the moment of measurement, a transition from possibility to actuality, cannot mean that, when the system is measured, it goes from a state of possible existence into a state of real existence, simply because at a time at which it only has possible existence, it has no existence at all, and a non-existent system, quantum mechanical or not, cannot enter into any interactions and cannot in particular undergo any measurement. So no

[3]Heisenberg was not the first physicist to seriously consider the potentiality concept in quantum mechanics. For example, David Bohm, in his celebrated textbook[8], written 7 years earlier, frequently refers to "quantum potentialities" (see e.g. p. 132).

measurement could bring "it" into actual existence if it had not been actual before the measurement."

This criticism is perceptive, but there is a conceivable way around it, namely if *a quantum system emerges out of something else which in an essential way is neither that quantum system nor anything that could be considered to be a "collection of its parts with a disposition to assemble into the system"* , and that when we say that we are "measuring a quantum system" that this could just be a way of referring (due to our ignorance of what is "really" going on) to certain kinds of interactions which cause a quantum system to emerge (as an actuality) out of that "something else". This will, in fact, be an assumption underlying the interpretation presented here, but discussed explicitly in a future paper.
The kind of unactualized possibility that arises due to the fact that a physical system could emerge from something that cannot even be considered a "a collection of parts with a disposition to assemble into it" seems altogether different from a dispositionality (and therefore also potentiality, as explicated here), and that makes it appropriate to select a different term than a potentiality for this phenomenon. So, to increase conceptual precision, I will make the following distinction:

- A *potentiality* associated with a physical system is defined as an unactualized possibility arising out of the system's disposition to transform itself or its environment in accord with the two criteria above.
- An *actualizability* associated with a physical system is defined as an unactualized possibility arising out of a disposition for a system to come into actual existence in spacetime.

The difference between these two types of possibilities is, put starkly, that prior to the triggering event, in the case of a potentiality, a physical system exists already at least in some sense, even if only in the form of a disassembled collection of parts, whereas in the case of an actualizability it does not. The concept of an actualizability is forced upon us in quantum mechanics if we presume that

1. A quantum state represents everything about a physical system there is to be represented, and
2. The physical system represented by a quantum state not under 'measurement' is a kind of ontic unactualized possibility.

Thus, in the die example of the previous section, the elements of the sample space have to be considered potentialities *unless* the triggering event is a *de novo* creation of a die in one of six possible configurations corresponding to each outcome. Only in that case could the unactualized possibilities be more properly considered to be actualizabilities.
One can imagine an imperfect analogy in everyday life: orient a funnel over a pan containing six die molds corresponding to each outcome configuration. The "triggering event" would then consist of the random alignment of the funnel with one of the molds, the pouring of a suitable fluid into it and its subsequent freezing or other solidification.
The analogy is imperfect because it is possible to consider the creation of the die in terms of the disposition of the atoms of the fluid to arrange themselves accordingly. The imperfection illustrates that *no classical phenomenon can exactly exemplify the actualizability concept*, but if one is willing to merely consider the analogy in terms of the absence of, "solid die parts" prior to the triggering event (i.e. keeping it at the level of thermodynamics, rather than statistical mechanics[4]), then that provides at least some sense of it: when it comes to material systems, potentiality can be thought of in terms of a disposition for undergoing a rearrangement of pre-existing parts, but actualizability has to be thought of in terms of a disposition for undergoing something akin to a *phase change*, leading to the *emergence* of something that did not exist before.

6 The Heisenberg Interpretation of Quantum Mechanics

The mathematical structure of quantum mechanics is obviously different from that of axiomatic probability, so in that context reference to $UC - A$ correspondence cannot mean reference to the map g, but rather to a quantum analog of it.

[4]the disposition of the fluid to turn solid is just the disposition of its atoms to rearrange themselves accordingly

In previous work, it was shown that the standard quantum formalism cannot coherently accommodate the distinction between, as Heisenberg called them, "things or facts" and "potentialities or possibilities"[3]. This motivated the proposal for a mathematically (but not conceptually!) slight modification of the standard formalism which does accommodate it (and which leaves all predictions of QM tested so far the same). The modified formalism is called the *Heisenberg Interpretation of quantum mechanics*[9]. In a way, the Heisenberg interpretation is to standard quantum mechanics what the axiomatic enrichment of probability is to standard axiomatic probability.
Rather than listing the postulates of the modified formalism, I will simply highlight how they differ from those of the standard formalism to which it is, apart from the differences below, identical[5]:

- Hilbert space $\mathcal{H}$ is explicitly postulated to be a space of actualizabilities.
- An unstructured set $\mathcal{C}$, called the *classical states* set is postulated which contains a classical or actual counterpart to every eigenstate in the Hilbert space in every measurement basis (this is the analog of the outcome set Γ). From it, one can define subsets $\mathcal{B}_\alpha \subsetneq \mathcal{C}$ which contain the actual counterpart to each eigenstate in measurement bases α. These subsets will be called *classical basis sets.*
- The non-unitary transformation $|\Psi\rangle \longrightarrow |\psi\rangle$, i.e. state reduction, of the standard formalism for any given quantum state is considered an element of the converse of partial map $\mathfrak{e} : \mathcal{H} \rightharpoonup \mathcal{H}$ with range$\{|\Psi\rangle\}$ which maps its eigenstates in all possible measurement bases to it and will be called the *eigenstate map.* The converse will be called the *collapse relation.*
- By forming the Cartesian product $\mathcal{H} \times \mathcal{C}$, it is possible to define a variety of new relations, in particular those which permit the eigenstate map to be expressible as a composition

$$\mathfrak{g} \circ \mathfrak{f} = \mathfrak{e} \tag{1}$$

  where the partial map $\mathfrak{g} : \mathcal{C} \rightharpoonup \mathcal{H}$ with range $\{|\Psi\rangle\}$ will be called the *classical-quantum correspondence* and the partial map $f : \mathcal{H} \rightharpoonup \mathcal{C}$ which puts each eigenstate in its domain of definition in correspondence with exactly one classical state in its range will be called the *actualizability-actuality* correspondence.
- The converse of the map $\mathfrak{g}$ will be called the *actualization relation* $\mathfrak{g}^{-1}$ and is postulated for the elements of any subset $\mathcal{B}_\alpha$ to be subject to a probability distribution governed by the Born Rule, and the inverse of $\mathfrak{f}$, a partial function, will be called the *deactualization map* $\mathfrak{f}^{-1}$.
- As in standard quantum mechanics, the value of the variable obtained upon measurement is represented by the eigenvalue obtained from operating on a quantum state with a Hermitian operator, but here it belongs to an element of $\mathcal{C}$, not $\mathcal{H}$.

There is also an auxiliary assumption which is necessary for consistency, namely that the concept of mass is split into two mutually exclusive concepts:

- *Actualizable mass* m, which characterizes physical systems the states of which are elements of $\mathcal{H}$
- *Actual mass* $\mathfrak{f}(m)$, which characterizes physical systems the states of which are elements of $\mathcal{C}$. The notation is merely for convenience, as the actualizability-actuality correspondence characterizes states, not masses.

Actual mass is just what we normally think of as mass, whereas actualizable mass is that concept as an actualizability. Quite obviously, if the state of a system is actualizable, then so is its mass and vice versa. A fuller discussion of these two concepts of mass can be found in [11].
To better understand these relations, it may be helpful to visualize them. A commutative diagram illustrating the composition of relations in equation (1) is shown in figure 4.:

[5]Concretely, the postulates which say that:

1. Quantum states are represented by vectors in Hilbert Space,
2. Measurements are represented by linear Hermitian operators that are functions of the position and/or momentum operator acting on those vectors,
3. The time evolution of quantum states obeys the Schrödinger equation

are all exactly the same as in standard quantum mechanics[10].

$$\begin{tikzcd}
\mathcal{H} & & \mathcal{C} \arrow[ll, "\mathfrak{g}"'] \\
\\
\mathcal{H} \arrow[uu, "\mathfrak{e}"'] \arrow[uurr, "\mathfrak{f}"] & &
\end{tikzcd}$$

Figure 4: The eigenstate map $\mathfrak{e}$ as a composition of the classical-quantum correspondence $\mathfrak{g}$ after the actualizability-actuality correspondence $\mathfrak{f}$

The *physical* relationships we are interested in are represented in terms of the converses of these relations:

$$\mathfrak{f}^{-1} \circ \mathfrak{g}^{-1} = \mathfrak{e}^{-1} \tag{2}$$

which is illustrated by the following commutative diagram in figure 5:

$$\begin{tikzcd}
\mathcal{H} \arrow[rr, "\mathfrak{g}^{-1}"] \arrow[dd, "\mathfrak{e}^{-1}"] & & \mathcal{C} \arrow[ddll, "\mathfrak{f}^{-1}"'] \\
\\
\mathcal{H} & &
\end{tikzcd}$$

Figure 5: The collapse relation $\mathfrak{e}^{-1}$ as a composition of the deactualization map $\mathfrak{f}^{-1}$ after the actualization relation $\mathfrak{g}^{-1}$

Our interest in the latter composition arises from the fact that it permits us to break "quantum collapse" into two distinct steps. A concrete example in terms of the reduction of a spin 1/2 state in the spin-x basis is given below in figure 6.:

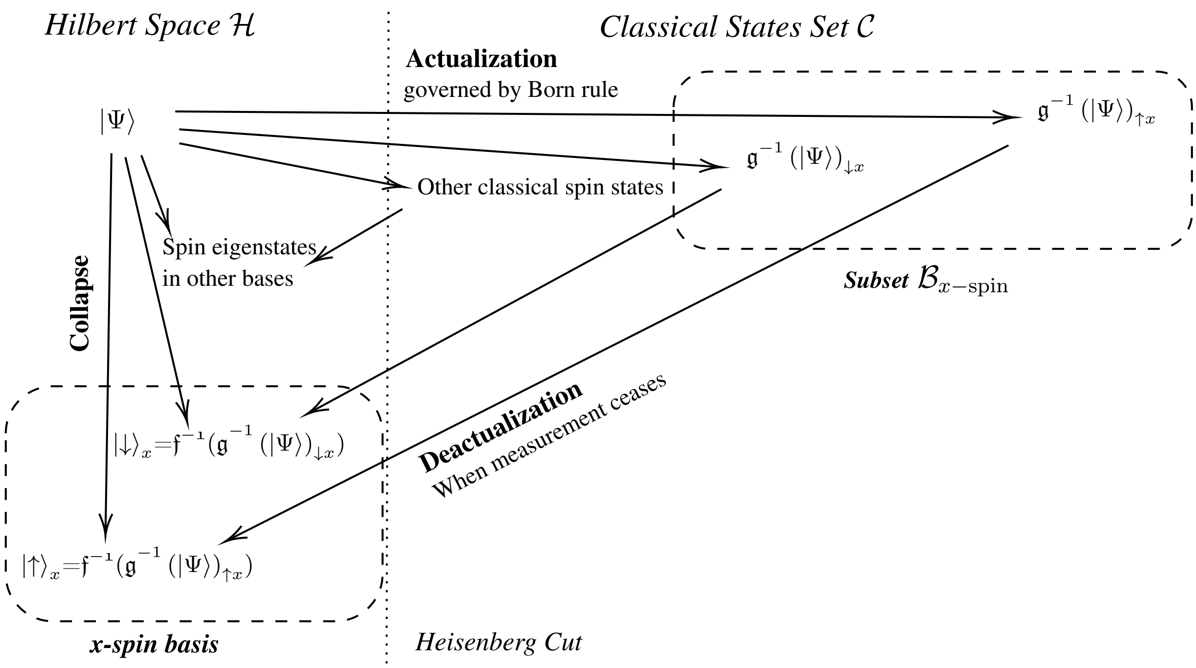

Figure 6: The composition $\mathfrak{e}^{-1} = \mathfrak{f}^{-1} \circ \mathfrak{g}^{-1}$ illustrated by a spin 1/2 state for the x-spin basis.

The left side of figure 6. represents standard quantum collapse, taking us from the original state to

one of its eigenfunctions in the measurement basis $\alpha = x - \text{spin}$. In the Heisenberg interpretation, this is the result of the sequence of two steps: first, the state is "actualized" by being related to the classical counterparts of its eigenstates in the measurement basis, with a probability distribution on elements of each classical basis set $\mathcal{B}_\alpha$ given by the Born Rule. Then, the classical counterpart in $\mathcal{C}$ is mapped to its corresponding eigenstate in $\mathcal{H}$ to account for the fact that shortly after the measurement, the quantum state once again undergoes time evolution.
There is a 1-to-1 correspondence between $\mathcal{B}_\alpha$ and the measurement basis α, though notice that there are an infinite number of other subsets which correspond to other measurement bases, one for each possible direction along which a spin measurement could be performed on the original quantum state.
The modification of the quantum formalism in the Heisenberg interpretation has a number of interesting consequences. For instance, the Cartesian product $\mathcal{H} \times \mathcal{C}$ naturally induces a "cut" between the quantum and the classical world, which is here with arguable justification named after Heisenberg[6]. Further, the interpretation leads to the resolution of certain *other* longstanding conceptual difficulties in the standard formalism (e.g. it resolves the problem of incompatible time evolutions having the same domain of applicability [13], it provides an explanation for the identicality or absolute identify of quantum particle states[14][15][16]. etc.), but these will be discussed in detail elsewhere [9]. Our concern here is how the *UC-A* correspondence manifests itself in quantum mechanics with these modifications in place.

7 Quantum Entanglement and Pro-actuality

Since $\mathfrak{f}$ puts eigenstates in particular bases into correspondence with their classical counterparts, and these counterparts are actualities, it is natural to consider it to be the quantum analog of the map f in the enriched axiomatization of probability. Further, analogous to situations involving singleton sample spaces, when performing a measurement on an eigenstate in the measurement basis it becomes a *UC-A* correspondence, since the outcome of the measurement, if performed, is certain. The eigenstate in that case is therefore a conditionally actualizable pro-actuality!
It is, however, only a *temporary* or *time-dependent* pro-actuality, since under Schrödinger evolution it will soon evolve into a different state. Are there states under the Heisenberg interpretation which can be considered to be *permanent* or *time-independent* conditionally actualizable pro-actualities? Indeed there are!
Consider, for instance, a singlet state of unbounded spin 1/2 particles A and B, and suppose a spin-measurement is performed on A. Under the standard formalism, we have to say that a measurement on A renders the outcome of a measurement on B in the same basis certain. But there seems to be something curious: the state cannot evolve because that would violate the quantum correlation, yet it also cannot be said to have been 'measured' until a measurement on B is actually performed. One might be tempted to claim that the measurement "collapses" the singlet state into a product state. Say, if A is measured to be spin up along the measurement direction, we might want to write

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \longrightarrow |\uparrow\rangle_A |\downarrow\rangle_B \tag{3}$$

but this can't be quite right: not only does the representation in terms of a product state fail to account for the fact that one particle has been measured while the other has not, but *it behaves differently than an ordinary product state*: in an ordinary product state, both particles will undergo time evolution, whereas in the partially measured singlet state only the measured particle will. The other one will stay "stuck" in its state until measured. So, the difference is not merely "philosophical" but has observable consequences!
It appears, therefore, that standard quantum mechanics does not possess the conceptual and mathematical resources to exactly represent the situation. Under the Heisenberg interpretation, it is perfectly clear what is happening: a measurement on A partially actualizes the singlet state, and due to constraints imposed by the properties of the actualized part in conjunction with the appropriate conservation law (angular momentum in the case of spin), the unactualized part, i.e.

[6] Usually the Heisenberg cut is interpreted epistemically[12], but in this interpretation it is ontic. The argument is that even though Heisenberg often seemed to refer to his cut in epistemic terms, it is reasonable to interpret his distinction between "possibilities or potentialities" and "things or facts" ontologically, which implies an ontic cut.

B, remains 'locked in' as a *permanent conditionally actualizable pro-actuality* (in that measurement basis) until such time that it is measured. Mathematically, this can be represented as follows:

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \longrightarrow \mathfrak{g}^{-1}_{\uparrow A}(|\psi\rangle) + |\downarrow\rangle^p_B \tag{4}$$

Here, $\mathfrak{g}^{-1}_{\uparrow A}(|\psi\rangle)$ is that image in $\mathcal{C}$ of $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\uparrow\downarrow\rangle)$ under this relation which is the classical counterpart to the spin-up state of A in that measurement basis. By the compositions (1) and (2)

$$\mathfrak{g}^{-1}_{\uparrow A}(|\psi\rangle) = \mathfrak{f}(|\uparrow\rangle_A) \tag{5}$$

and the only other available image of A's state in that basis is the spin down state as an actuality, $\mathfrak{f}(|\downarrow\rangle_A)$. Of course, the classical states can only represent the state of A while under measurement, as after cessation of measurement it deactualizes and subsequently undergoes time evolution. On the other hand, B has become a permanent pro-actuality, and this is denoted by the superscript on its state. Notice that in standard quantum mechanics, adding such a superscript would be meaningless because that formalism has no concept of pro-actuality, and cannot even accommodate it.
So, the image set for $|\psi\rangle$ under $\mathfrak{g}^{-1}$ but expressed in terms of $\mathfrak{f}$ for *this* particular measurement basis is

$$\mathcal{B}_\alpha = \{\mathfrak{f}(|\uparrow\rangle_A), \mathfrak{f}(|\downarrow\rangle_A), \mathfrak{f}(|\uparrow\rangle_B), \mathfrak{f}(|\downarrow\rangle_B)\} \tag{6}$$

whereas the co-domain $\mathcal{C}$ contains the actual counterparts of spin states in all possible measurement bases (i.e. along all possible directions). If we measure the spins of each particle along a different direction than the other, then the image set changes accordingly, with a corresponding change in the probability distribution on its elements in accordance with the Born Rule. Note again that because $\mathcal{C}$ is not a Hilbert space, it does not allow us to express classical states in terms of a combination of other classical states in a different basis.
Some quantum states are not a superposition of all eigenstates in a particular measurement basis. For example, measuring the state $|\uparrow\rangle_y$ will never yield $\mathfrak{f}(|\downarrow\rangle_y)$, but for simplicity I will adopt the convention that its image set under g^{-1} is still $\mathcal{B}_{y-\mathrm{spin}} = \{\mathfrak{f}(|\uparrow\rangle_y), \mathfrak{f}(|\downarrow\rangle_y)\}$ with the probability distribution such that $P(\mathfrak{f}(|\downarrow\rangle_y)|\,|\uparrow\rangle_y) = 0$. Similarly, if we wish to compute the probability for the above partially actualized state, we can only consider the unactualized part, and so

$$\langle\downarrow|\downarrow\rangle^p_B \longrightarrow P(\mathfrak{f}(|\downarrow\rangle_B)|\,|\downarrow\rangle^p_B) = 1 \tag{7}$$

the fact that this equation holds for any amount of time up until the state is measured indicates in physical terms that the state of B is a permanent pro-actuality in that measurement basis.
Let us now consider a situation in which A and B are spacelike separated. In that case, the order of measurements becomes dependent on the observer's frame. Taking the same outcomes as before, suppose the spins of both are measured along the same direction, and suppose that the two spacelike separated particles are considered in a frame in which the spin of B is measured first. Then, the transformation to a partially actualized state can be written as

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \longrightarrow |\uparrow\rangle^p_A + \mathfrak{f}(|\downarrow\rangle_B) \tag{8}$$

It is sometimes claimed that spacelike separated quantum correlations are evidence for *non-locality*, i.e. a violation of the *principle of locality*[17][18]. Stated in generality, the principle says that no physical influence of any kind can travel from one physical system to another without passing through all intermediate points in space between the two. Because special relativity imposes an upper speed limit on such influences, any such influence beyond that speed limit may also be considered to violate the principle.
Do the quantum correlations violate the principle of locality? One can already see that such purported violations are at least potentially problematic by considering that what enforces the correlations in the first place is the relevant conservation law, that by Noether's theorem the conservation law corresponds to a symmetry of spacetime, and that that spacetime symmetry is just what ensures the validity of special relativity with respect to that conservation law, but in the end whether the violations occur or not will depend on the particular interpretation at hand.
Let us consider the issue then within the context of the Heisenberg Interpretation. Discussions of

locality have so far never made a serious distinction between actualities and unactualized possibilities (as far as I know), and so it is safe to suppose that the locality principle is meant to apply to actualities. This is supported by the fact that special relativity (outside the context of quantum field theory) is unambiguously a theory the objects of which are actualities.
So, under the Heisenberg Interpretation, assuming that the domain of applicability of the locality principle only extends over actualities in spacetime and their associated properties (e.g. charges which set up fields), quantum mechanics is plainly local. What a special relativistic observer in the first frame observes is the emergence in spacetime of A with classical state $\mathfrak{f}(|\uparrow\rangle)_A$ followed by the emergence of B with classical state $\mathfrak{f}(|\downarrow\rangle_B)$, and what the one in the second frame observes is the emergence of B followed by the emergence of A. In either frame, no physical influence of any kind is transmitted or could have been transmitted from one to the other because the supposed "recipient" did not even exist until it was "measured"!
Discussions of Bell's theorem[22] are sometimes framed in terms of a trade-off between *locality* and *realism*, the idea that systems represented by quantum states have at least some pre-existing physical properties prior to being measured[20]. To the extent that the actualizability concept denotes a complete negation of "realism" applied to physical systems, the Heisenberg interpretation gives up realism in favor of locality full stop. Notice that this could not have been the case if the Heisenberg interpretation had used the potentiality concept to describe quantum states, as that concept still maintains a measure of "realism", given that it posits a system to exist in some form before the triggering event.
Within this interpretation, the spacelike separated correlations are to be understood in terms of *correlated actualizations*: they are enforced at the level of actualizabilities, as are other similar phenomena, such as the abrupt vanishing of the wave function even over spacelike distances upon a measurement[7]. So any question regarding how they are enforced has to be answered in terms of whatever it is that the classical states emerge from. Such a discussion is outside the scope of this paper, but will be addressed in detail in future work.

8 Born Rule and the UC-A Correspondence

It has been argued that quantum mechanics is a kind of generalized probability[21]. The perspective behind the arguments given in this paper is similar in some respects, since it utilizes comparisons and analogies between probability and quantum mechanics. But if so, then there has to be a clearly identifiable analog of the function g of the enriched axiomatization in the Heisenberg Interpretation, since this interpretation is meant to incorporate the $UC - A$ correspondence.
The most obvious candidate for such an analog is the actualization relation $\mathfrak{g}^{-1}$, *if* we take it to be governed by the Born rule, by which I mean that the Born rule assigns the probability for the actualization of each possible classical system out of the measurement interaction. But is the governing by the Born rule of the actualization relation really necessary in order for it to embody the $UC - A$ correspondence?
To answer this, we have to identify the domain and co-domain of the $UC - A$ correspondence in the Heisenberg Interpretation. According to the Born rule, probability enters the theory by means of the absolute square of inner products, but because in the Heisenberg interpretation the rule is associated with the actualization (as opposed to the collapse) relation, the interpretation of these inner products is slightly different:

$$|\langle\phi|\psi\rangle|^2 \longrightarrow P(\mathfrak{f}(|\phi\rangle)) \text{ given the state } |\psi\rangle \tag{9}$$

as we also saw in equation (7). The change vis-a-vis standard quantum mechanics $|\phi\rangle \longrightarrow \mathfrak{f}(|\phi\rangle)$ does not affect the distribution of probabilities but rather the range of objects to which the distribution applies. The superset including all such ranges is just $\mathcal{C}$, and this identifies the "actuality" part of the $UC - A$ correspondence, which is to say, its co-domain.
Now we have to identify its domain. We start by attempting to identify the quantum analogs of the elements of the sample space under this interpretation, since in probability, Ω is the domain of of the $UC - A$ correspondence. If we consider the inner product $|\langle\phi|\psi\rangle|^2$ for the case $|\phi\rangle = |\psi\rangle$, so that now we have $\langle\psi|\psi\rangle$, then this expresses a situation involving a pro-actuality.

[7] under the standard formalism, one would have to add the qualifier:..."except in the small region where the measurement takes place.", but not in the Heisenberg Interpretation: The wave function "really" vanishes because it is replaced by the position state of an actual system while under measurement, and the delta function has to be interpreted as representing the quantum state only after it has "reappeared" upon deactualization.

Recall that in the case of probability, one can build up the domain of the $UC - A$ correspondence by collecting all the elements of the domain of f, each of which in a singleton sample space is also a pro-actuality. This suggests that one should also form a collection of pro-actualities in the quantum case, but then one immediately runs into a problem since

$$\{\langle\psi|\psi\rangle : \langle\psi| \in \overline{\mathcal{H}}, |\psi\rangle \in \mathcal{H}\} = \{1\} \tag{10}$$

where $\overline{H}$ is the dual Hilbert space. All the unactualized certainties associated with different quantum states collapse into a single element! Since the proper quantum analog of the map g should really have *this* set as its domain, it seems that the actualization relation does not really embody the $UC - A$ correspondence.

However, there is a way to resolve this: the difficulty here is caused by the fact that we cannot distinguish between the unactualized certainties associated with the absolute square of different quantum states. But if we labeled each by an index, then we could! The most natural index label is $|\psi\rangle$ itself, so we can consider the following re-interpretation:

$$|\psi\rangle = \langle\psi|\psi\rangle\, |\psi\rangle \tag{11}$$

Although the equation is mathematically trivial, it represents a radical conceptual departure from standard quantum mechanics. According to this re-interpretation, $|\psi\rangle$ (and, of course, also $\langle\psi|$) is merely a stand-in for $\langle\psi|\psi\rangle$. Putting it more sharply, $|\psi\rangle$ is an index, albeit an "index with structure", and the Hilbert space is essentially a "structured index set" in order to keep track of the distinct elements that *would* have been in the domain of the $UC - A$ correspondence, but which cannot be represented distinctly in a direct manner[8].

The purpose of the label is to represent the inner concept of the unactualized certainty, its *intension*, by means of an extension which has a scope of applicability in the Hilbert space but also fulfills the necessary and sufficient conditions which allow us to specify *that* unactualized certainty and distinguish it from all others.

Under this re-interpretation, the Hilbert space functions essentially as a "substitute" for the domain of the $UC - A$ correspondence: its basis elements $|\psi_i\rangle$ for any measurement basis α are really just labels for $\langle\psi_i|\psi_i\rangle$, so under this substitution of domains the actualization relation can be considered to be the quantum analog of the map g after all, *provided* it is governed by the Born Rule.

9 Measurement Contexts and Unamendable Probability

The argument of the previous section reveals that under the Heisenberg interpretation, the concept of a quantum state is profoundly different than the way it is usually thought of today. Under the standard view, we tend to think of the quantum state as being directly connected to a physical system. For example, given an electron state, we might visualize an electron, even when the state is not being measured.

Under the Heisenberg interpretation, on the other hand, the quantum state is merely directly connected to an unactualized certainty; it is an index for an unactualized certainty, necessary because all unactualized certainties are *mathematically* identical. Moreoever, unactualized certainties are not physical systems, they have no physical properties and presuming so is a category error.

Rather, the differences captured by the index are in terms of the conditions necessary so that *if* a "measurement" is carried out under those conditions, one will measure *with certainty* the property of a physical system the state of which is the classical counterpart to one of the eigenstates in the measurement basis of *that* index state. For instance, when we talk about the position of an electron that is not being measured, under the Heisenberg interpretation, what we really mean is not the position of an electron-*since there is no electron!*-but the conditions necessary so that if a measurement is performed under those conditions, one will *with certainty* obtain an electron position (as opposed to, say, some other property of an electron, or the position or other property of some other particle, or nothing)[9]. It is useful to give such conditions a distinct name:

[8] Of course, one can define an unstructured set $\{|\psi\rangle : |\psi\rangle \in \mathcal{H}\}$ as the domain of the *UC-A* correspondence under the reinterpretation of equation (11), but such a set does not add anything over and above the Hilbert space, and so by Occam's razor it is cut. Alternatively, one could collect the unactualized certainties in a *multiset*, but a different application of Occam's razor compels one not to use mathematical structures rarely used in physics unless there is an appreciable benefit to doing so.

[9] assuming ideal conditions, of course.

- A *Measurement context* M_α is a set of experimental or natural conditions or configurations such that if a quantum measurement occurs in connection with a given quantum state under those conditions, an observable which represents the property of a system with a classical state which is the image of one of the eigenstates of that quantum state in the measurement basis α under $\mathfrak{f}$ will be found with certainty.

Any change in the measurement context can effect a change in how an unactualized certainty connects to a given set of classical states. In terms of quantum states, since they label something which has no inherent physical properties, it opens the possibility for contriving measurement contexts which reveal a lack of pre-existing or context-independent properties. In particular, even if the change does not directly affect an entity described by the quantum state, it might still change the observables associated with its image as, for instance, a measurement on one part of an entangled system can change other parts which were left unmeasured. This, of course, points directly to *quantum contextuality*[22][23], and the entangled system discussed in section 7. already gave a concerete example of that.

In standard quantum mechanics, contextuality is connected to the Born rule through Gleason's theorem[24]. Under the Heisenberg interpretation, the connection is conceptual and transparent: if the Born rule is interpreted to define $\mathcal{H}$ as the (substitute) domain of the $UC-A$ correspondence between unactualized certainties labeled by quantum states and that of (sets of) classical states, then quantum states *must* be contextual: they do not represent physical systems with pre-existing or non-contextual properties but label unactualized certainties which attain their representation in terms of quantum states by means of measurement contexts. More formally, let

$$\mathcal{S} = \{\mathcal{B}_\alpha : \mathcal{B}_\alpha \subsetneq \mathcal{C}\} \tag{12}$$

be the set of all classical basis sets $\mathcal{B}_\alpha$ such that each $\mathcal{B}_\alpha$ contains the image of every eigenstate of a quantum state in a particular measurement basis α under $\mathfrak{g}^{-1}$.

Let

$$\mathfrak{H} = \bigsqcup_\alpha \mathcal{H}_\alpha \tag{13}$$

be the disjoint union of copies of a given $\mathcal{H}$, each expressed in a different measurement basis α than all others, over all possible measurement bases. The idea here is to treat any given quantum state when it is expressed in a different measurement basis as though it were a different object. So, for instance, whereas $\sum c_i |\psi\rangle_i$ might be the same state in $\mathcal{H}$ as $\sum c_j |\phi\rangle_j$ but expressed in a different basis, they correspond to different elements in $\mathfrak{H}$. If we consider each disjoint $(\mathcal{H}_\alpha, \alpha)$ to be an element of a set $\mathbb{H}$, then equation (13) can also be rewritten as

$$\mathfrak{H} = \bigsqcup \mathbb{H} \tag{14}$$

Finally, define the *measurement context set* $\mathcal{M}$ as

$$\mathcal{M} = \{M_\alpha : \forall(\mathcal{H}_\alpha, \alpha) \in \mathbb{H}, \varphi((\mathcal{H}_\alpha, \alpha)) = \eta(M_\alpha)\} \tag{15}$$

where the one-to-one correspondence $\varphi : \mathbb{H} \longrightarrow \mathcal{S}$ will be called the *fate map*, and the one-to-one correspondence $\eta : \mathcal{M} \longrightarrow \mathcal{S}$ will be called the *emergence map.*

The interpretation of the fate map is that it represents how a given unactualized certainty connects a measurement context to a set of classical states: given a quantum state in measurement basis α, a measurement in that basis is *certain* to yield an observable associated with an element of its image $\mathcal{B}_\alpha \in \mathcal{S}$ under φ (its "fate"), which is also the image of a particular measurement context $M_\alpha \in \mathcal{M}$ under η.

Quantum states, when represented in a particular measurement basis, will be called *indirect representations* of unactualized certainties in Hilbert space, rather than mere index labels. The reason is that we can always write

$$|\psi\rangle = \langle\psi|\psi\rangle |\psi\rangle = \langle\psi|\psi\rangle \sum c_i |\psi_i\rangle = \sum c_i \langle\psi_i|\psi_i\rangle |\psi_i\rangle \tag{16}$$

which tells us how much the unactualized certainty labeled by each eigenstate contributes to the *representation* of the unactualized certainty (the last term) in Hilbert space.

In contrast, a *direct representation*

$$\langle\psi|\psi\rangle = \sum |c_i|^2 \langle\psi_i|\psi_i\rangle \tag{17}$$

tells us how much the unactualized certainty labeled by each eigenstate contibutes to the unactualized certainty *itself*. Whereas equation (16) is a vector equation, equation (17) is scalar, which seems appropriate, given that an unactualized certainty is a scalar quantity with unit magnitude. The interpretation of the emergence map is that it connects a measurement context M_α directly to the emergence of a classical state, such that if a measurement occurs in the presence of an unactualized certainty under that measurement context, the classical state is *certain* to be an element of $\mathcal{B}_\alpha$. $\mathcal{M}$ is defined such that $\mathcal{B}_\alpha$ is precisely that quantum state's image in $\mathcal{S}$ under φ which is expressed in a corresponding measurement basis α.
It should be noted that the measurement context is an idealization insofar that many distinct experimental or natural conditions are compatible with connecting a given unactualizable certainty to actual states in accordance with these maps. We ignore such differences here, but in situations where they might become important, they can always be accommodate by representing them as distinct measurement contexts which are members of a common subset $\mathcal{U} \subsetneq \mathcal{M}$, such that the elements of $\mathcal{U}$ under η are all mapped to the same subset $\mathcal{B}_\alpha$ of the classical states set $\mathcal{C}$. If such cases are admitted, then the emergence map is of course no longer bijective but merely surjective.
So, a measurement context M_α, when connected to an unactualized certainty, permits it to be indirectly represented as a quantum state in a particular measurement basis α. The representation of measurement contexts by measurement bases can be formalized through what will be called the *context representation map* $\kappa : \mathcal{M} \longrightarrow \mathbb{H}$. This is a one-to-one correspondence, except when there are subsets $\mathcal{U} \subsetneq \mathcal{M}$ with more than one member mapped to the same element of $\mathbb{H}$, in which case κ is a surjection, as was also mentioned in the case of the mapping η. Because there is a one-to-one correspondence between the elements of $\mathbb{H}$ and those of $\mathcal{S}$ under φ, any subsets $\mathcal{U}$, if they exist, are identical sets in the domains of both η and κ.
The context representation map formally represents a given measurement context in terms of a particular measurement basis in Hilbert space. The emergence map then becomes the composition of the fate map after the context representation map

$$\eta = \varphi \circ \kappa \tag{18}$$

This can be visualized by what will be called an *unactualized certainty diagram* because it visualizes how κ permits distinct representations of unactualized uncertainties either explicitly in terms of quantum states via ϕ or implicitly via η:

$$\begin{tikzcd} & \mathbb{H} \arrow[dr, "\varphi"] & \\ \mathcal{M} \arrow[ur, "\kappa"] \arrow[rr, "\eta"] & & \mathcal{S} \end{tikzcd}$$

Figure 7: The emergence map η as a composition of the fate map φ after the context representation map κ. Its interpretation is that κ connects an *explicit* representation of unactualized certainties in terms of quantum states via φ to their *implicit* representation via η. For this reason, it will be called an *unactualized certainty diagram*.

Notice that because an unactualized certainty is not material or actual, it can only be explicitly represented in terms of the domain of of the $UC - A$ correspondence, which in the case of quantum mechanics is Hilbert space. In contrast, unactualized certainties are merely implicit in the map connecting $\mathcal{M}$ to $\mathcal{S}$, in the sense that they have to be there in order for a classical state to emerge out of a measurement context upon a measurement, but nothing associated with η or its domain and co-domain represents these unactualized certainties.
The practical advantage of going through $\mathcal{H}$ instead of directly from measurement contexts to sets of classical states is that it permits us to decompose the unactualized certainty connecting $\mathcal{M}$ to $\mathcal{S}$ into individual probabilities $|c_i|^2$ associated with each element of $\mathcal{B}_\alpha$. This reflects the probability distribution on the actualization relation governed by the Born Rule.
The vector space structure of $\mathcal{H}$, absent in $\mathcal{M}$, indicates that the probabilities connected to an unactualized uncertainty given a particular measurement context are related to probabilities connected to it given other measurement contexts in concrete ways which give rise to just that kind of structure. This structure is absent in $\mathcal{M}$ as its elements are connected exclusively to actualities (e.g. properties of classical "environment plus measurement apparatus").

Why do unactualized uncertainties, which are all mathematically identical to each other, give rise to distinct quantum states with different associated probabilities in particular measurement contexts? For two reasons: first, each measurement can be thought of as a preparation of a new state (in the Heisenberg Interpretation, after deactualization), which is to say, a new orientation of the indirect representation of the unactualized certainty in Hilbert space relative to the measurement basis, and second, because Schrödinger evolution itself also effects this.

In terms of an imperfect classical analogy, one could imagine a hypothetical coin which has unit probability of yielding heads on one particular type of surface, but only one half on a different kind, whereas another coin has unit probability of yielding heads on the second surface but only one-half on the first. If we now imagine that in each case there was no coin until a coin "flip" occurred, then this illustrates how unactualized certainties themselves do not have any "orientations", but that their indirect representation in terms of quantum states in particular measurement bases do.

Finally, the above construction can also help with the conceptualization of *why* quantum mechanics under the Heisenberg interpretation is (ontologically) *unamendably probabilistic*.

In order to have a deterministic physical theory, two ingredients are needed:

1. A set of deterministic laws, *complete* in the sense that they apply to all those properties of all physical systems in the theory which under those laws dispose systems to behave deterministically.

2. A set of initial conditions, *complete* in the sense that they specify definite values for all properties to which those laws apply

Classical physics has both ingredients: the laws are deterministic and cover all properties of all physical systems which are needed to dispose the systems to behave deterministically, namely (for those specific laws) positions and velocities (or momenta). Further, one can, at least in principle, specify the positions and velocities/momenta of all systems in the theory so that the set of initial conditions can also be considered complete. Thus, classical physics is deterministic.

Under the Heisenberg Interpretation, the second condition is impossible to satisfy in quantum mechanics: since there is no physical system prior to a measurement, there are no properties of a system to specify. The relevant properties of the measurement context can be specified in principle, but they are insufficient for a complete determination of a classical state upon a measurement unless the unactualized certainty happens to be a pro-actuality in that measurement context (in standard parlance or in terms of indirect representation: the quantum system is in an eigenstate in the measurement basis). The absence of the system prior to a measurement is therefore akin to a "hole" in the required complete list of initial conditions and breaks deterministic causal chains connecting times before a measurement to times after.

To illustrate this by contrast, suppose there was an interpretation which, like the Heisenberg Interpretation, distinguished between classical and quantum states, but in which physical systems existed before measurements and had definite values for all relevant properties, even if unknown to us. That would turn it into a hidden variable theory and ensure that any deterministic causal chains continue under a measurement. The unactualized certainty diagram in fig. 7 would have have to modified to include those values for the relevant properties, as in figure 8.:

$$\begin{tikzcd}
 & \mathbb{H} \arrow[r, "\varphi"] & \mathcal{S} \\
\mathcal{M} \arrow[ur, "\kappa"] \arrow[r, "\pi_1"] & \mathcal{P} \arrow[u, "\pi_2"] \arrow[r, "\pi_3"] & \mathcal{B}_\alpha
\end{tikzcd}$$

Figure 8: The unactualized certainty diagram of a hypothetical "hidden variable variant" of the Heisenberg interpretation. Here, whereas φ merely specifies with certainty a set $\mathcal{B}_\alpha$, π_3 specifies with certainty one of its elements because $\mathcal{P}$ supplies the specifications needed (i.e. hidden variables) to continue any deterministic causal chain from before to after a measurement.

In figure 8, the values of relevant properties needed for the specification of the initial conditions of *the system* are supplied by the set $\mathcal{P}$, while those of the *measurement context* are, as before, specified by M_α. That makes it possible to define three new maps: π_1 maps measurement contexts to definite values for properties, so that both together can provide a complete list of specifications for precisely specifying a classical state, π_2 maps those values necessary to at least specify the characteristics of the quantum state in the Hilbert space under a given basis, though these specifications may not be sufficient for the precise specification of a classical state unless the quantum state is an eigenstate in the measurement basis, and π_3 is the map which precisely specifies a classical state. In short, what the diagram in figure 8. shows is that whereas the best quantum mechanics can do is to specify with certainty a *set of classical states*, the hidden variables characterizing the pre-existing system in conjunction with the measurement context can specify with certainty a single classical state, thereby continuing the deterministic causal chain from before the measurement to after.

Notice that if eigenstates in a measurement basis in this theory are conceived as unactualized possibilities, they cannot be actualizabilities because the system exists before the measurement. We see, then, that the actualizability concept not only enforces the locality, antirealism and contextuality of the Heisenberg interpretation, but also supplies a reason for the unamendable probability of the quantum formalism.

10 The Concept of an Unactualized Possibility

The ideas discussed above can be applied in such a way that they arguably lead to a deep insight into the concept of an unactualized possibility. To demonstrate this, we will construct a simplification of the structures and relations of the previous section, and show that a formally identical set of structures and relations between them can be constructed for classical probability, which is meant to buttress the idea that behind each unactualized possibility there lies an unactualized certainty. First, let us define the unstructured set

$$\Omega_\alpha(|\Psi\rangle) = \{|\psi_i\rangle \in \mathcal{H}_\alpha : |\psi_i\rangle \text{ is an eigenstate of } |\Psi\rangle \text{ in the basis } \alpha\} \tag{19}$$

Then we can define a one-to-many relation $\kappa_\alpha : \{M_\alpha\} \longrightarrow \Omega_\alpha(|\Psi\rangle)$, where $\{M_\alpha\}$ is the singleton the element of which specifies the experimental or natural conditions which are required to express $|\Psi\rangle$ in basis α, and κ_α relates it to every element of $\Omega_\alpha(|\Psi\rangle)$. We can then define another one-to-many relation $\eta_\alpha : \{M_\alpha\} \longrightarrow \mathcal{B}_\alpha$, which relates the set of these conditions to the emergence of each one of a set of classical states $\mathcal{B}_\alpha$. We also define the one-to-one correspondence $\varphi_\alpha : \Omega_\alpha(|\Psi\rangle) \longrightarrow \mathcal{B}_\alpha$ which ensures that we can compose these relations as follows

$$\eta_\alpha = \varphi_\alpha \circ \kappa_\alpha \tag{20}$$

Finally, let us define the function $\xi_\alpha : \Omega_\alpha(|\Psi\rangle) \longrightarrow \mathcal{B}_\alpha$ such that its inverse ξ_α^{-1} is a fiber on each element of $\mathcal{B}_\alpha$. Notice that if we apply the Born rule, we can define a probability measure on $\Omega_\alpha(|\Psi\rangle)$ which in a frequentist conception is the result of a hypothetically infinite sequence of identical measurements on $|\Psi\rangle$ to obtain a probability for each $\mathfrak{f}(|\psi_i\rangle) \in \mathcal{B}_\alpha$.

This construction is not very useful for quantum mechanics because it tells us nothing about probabilities associated with eigenstates of $|\Psi\rangle$ in other bases. It is useful, however, for illuminating the relationship between quantum mechanics and classical probability because we can recover a formally identical set of structures and relations which include those of the enriched axiomatization of probability as an integral component by making the following substitutions:

$$\begin{aligned}
\Omega_\alpha(|\Psi\rangle) &\longrightarrow \Omega \\
\mathcal{B}_\alpha &\longrightarrow \Gamma \\
M_\alpha &\longrightarrow \mathcal{E} \\
\varphi_\alpha &\longrightarrow f \\
\xi_\alpha &\longrightarrow g \\
\kappa_\alpha &\longrightarrow k \\
\eta_\alpha &\longrightarrow e
\end{aligned} \tag{21}$$

which can be visualized by the following diagram in figure 9. This diagram commutes under f but not under g (unless Ω and Γ are singletons), and so g is represented without an arrowhead.

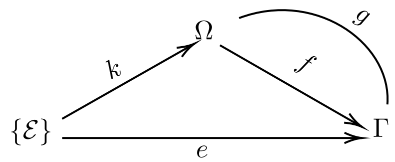

Figure 9: A diagram which shows how the event context, sample space, and outcome set are related to each other. Under the reverse substitutions of (21) identical structures can represent the relationship between Measurement context, a complete set of eigenstates of a quantum state in a given measurement basis, and the corresponding classical basis set.

where Ω and Γ are the sample space and outcome set, respectively, and f and g are our old friends the possibility-actuality correspondence and the $UC - A$ correspondence, respectively.
The *event context* $\mathcal{E}$ is a set of conditions or configurations such that if a certain "triggering event" occurs under those conditions, its outcome is *certain* to be an element of Γ. The relation $k : \{\mathcal{E}\} \longrightarrow \Omega$ relates $\mathcal{E}$ to each element of Ω. Its interpretation is that it represents how the set of conditions imposes a constraint on what is possible upon the "triggering event", so that only the elements of Ω can count as a possibilities, i.e. the images of $\mathcal{E}$ under k. Thus, we can write

$$P(\Omega|\mathcal{E}) \equiv P(\Omega) \tag{22}$$

which in a very natural way makes explicit that every probability is in reality conditioned upon "something" and that notations which express it as though it were unconditional are at best merely shorthand.
The relation $e : \{\mathcal{E}\} \longrightarrow \Gamma$ relates $\mathcal{E}$ to each element of Γ. Its interpretation is that it directly connects the event context to outcomes, rendering the unactualized possibilities implicit. Since f is a bijection, the diagram in figure 9. commutes under f.
This diagram treats probability as though it were situated in a truly indeterministic world, and for many purposes it is perfectly adequate to treat it that way. However, within a classical context, we know that such indeterminism is merely epistemic, and if we wanted to represent probability in way that takes this into account, we can instead construct the following diagram:

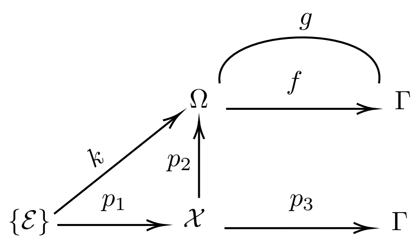

Figure 10: A variant of the diagram in figure 9. which takes into account that in a classical context, probability is situated in a deterministic world.

where $\mathcal{X}$ represents the set of all the specifications necessary so that a definite outcome can be determined. As in figure 8., it permits the definition of three new relations: p_1 relates the event context to this set of specifications, p_2 represents the mapping of any specifications which influence the probability distribution on Ω, and p_3 maps the complete set of specifications to the definite outcome γ_d. Even though Γ is the co-domain of f and p_1, they obviously do not commute (unless Γ is a singleton). This can already be seen by the fact that whereas Γ is also the range of f, the range of p_3 is $\{\gamma_d\} \subseteq \Gamma$, where, again, $\gamma_d \in \Gamma$ is *the* definite outcome which is "hidden" as such without $\mathcal{X}$.
The point of these constructions is to show that via substitutions as in equation (21), one can demonstrate isomorphisms between structures and relations involving representations of quantum states in a given measurement basis and quite natural structures and relationships involving elements of the sample space. This, in turn, strongly suggests that applying the same re-interpretation as in equation (11) to the elements of Ω is not just a "mathematical trick" or a reflection of an epistemic attitude, but (for ontic probability) something which has basis in reality.

Consider, for example, the toss of a coin. Analogously to equation (11), we can rewrite the elements of the sample space as

$$\Omega = \{1 \cdot H, 1 \cdot T\} \tag{23}$$

so that H and T are re-interpreted as mere index labels for the unactualized certainties associated with a particular set of conditions, analogous to the measurement context in the quantum case.
Just as happened in equation (10), we cannot represent these unactualized certainties directly in a distinct manner because they all are mathematically identical, hence the need for labels.
Since the set in equation (23) contains two unactualized certainties, we have to qualify them as *unnormalized*. Under normalization, *and not due to any intrinsic properties but solely because of the availability of an alternative*, the unactualized certainties become "reduced" to unactualized possibilities which fall short of being certain.
This allows us to conceptualize an unactualized possibility as an unactualized certainty associated with a set of conditions such that if a "triggering event" happens under those conditions, the outcome which labels the unactualized certainty obtains with certainty. The availability of alternatives under normalization reduces the certainty to a mere possibility, and probability quantifies how much each unactualized certainty contributes relative to all available alternatives.
The general applicability of this argument to any probabilistic sample space validates that the elements of a sample space are not actually outcomes, but unnormalized unactualized certainties, and that, similar to the quantum case, outcomes merely label their *intension* so that we can tell them apart.
Adopting the notational convention

$$1.\gamma \longrightarrow 1_\gamma \tag{24}$$

where $\gamma \in \Gamma$, we can for example visualize the die throw example under the enriched axiomatization as depicted in figure 3. with this re-interpretation as in figure 11.:

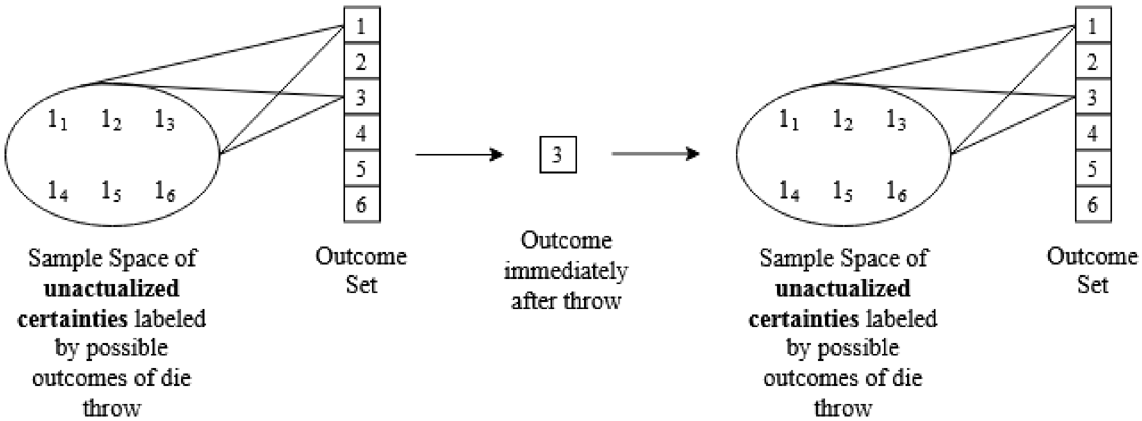

Figure 11: The representation of the die throw example under the enriched axiomatization has been suggestively modified to emphasize that the elements of the sample space are not outcomes, but *unnormalized unactualized certainties* labeled by outcomes or actualities to extensionally reflect their *intension*.

From the point of view of this discussion, the primary difference between the unactualized possibilities of probability and those of quantum mechanics is that the former are potentialities and the latter actualizabilities. But in either case, an unactualized possibility is "really" an unactualized certainty, demoted to a mere possibility among others through the availability of alternatives by means of the context of its actualization and subsequent normalization. And it is just this unactualized certainty which stands in opposition to the actualized certainty of an actuality. That is perhaps the deepest level of what the UC-A correspondence apprehends.

11 Conclusion

This paper investigated the unactualized certainty-actuality correspondence, a correspondence which arises naturally when unactualized certainties are formally distinguished from actualities, as is the case in the axiomatic enrichment of probability and the Heisenberg interpretation of quantum mechanics.
The main results can be summarized as follows:

1. The $UC - A$ correspondence is embodied by the map g in the axiomatic enrichment of probability, and by the actualization relation $\mathfrak{g}^{-1}$ in the Heisenberg Interpretation.

2. A situation in which there is only one unactualized possibility available involves an unactualized certainty deserving of its own name- pro-actuality- as it highlights situations in which the standard axiomatization of probability conflates it with an actuality, and elucidates the conceptualization of unmeasured parts of partially measured entangled quantum states under the Heisenberg Interpretation.

3. Unactualized possibilities are potentialities in probability and actualizabilities in quantum mechanics under the Heisenberg interpretation. The actualizability concept enforces the locality, anti-realism and contextuality of that interpretation and provides a reason for the unamendable probability in the quantum formalism.

4. The objects of probability and of quantum mechanics are unactualized certainties. Outcomes and quantum states, respectively, are *labels* for unactualized certainties which connect event contexts to actual outcomes in probability and measurement contexts to classical states in quantum mechanics under the Heisenberg interpretation as follows: if a triggering event or measurement occurs given an event context or a measurement context, respectively, then an outcome or classical state which obtains will *with certainty* belong to Γ or $\mathcal{B}_\alpha$, respectively.

5. Both in the axiomatic enrichment and in the Heisenberg Interpretation, unactualized possibilities are fundamentally unactualized certainties which merely through the availability of alternatives by means of the actualization context and subsequent normalization become demoted to less-than-certain possibilities. The opposite of an impossibility is not a possibility, but a certainty; alternative certainties make it into a possibility.

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