Appendices

Appendix A: Proofs of Propositions

A.1. Proof of Proposition 1

Consider the case where items are ordered at the start of the selling horizon, and online demands are fulfilled over T fulfillment periods. Assume that $C_{T+1}(\mathbf{x^{T+1}}, \tilde{\mathbf{D}}^{T+1}) = 0$ without loss of generality. Thus, from (2),(3),(4), $C_T(\mathbf{x^T}, \tilde{\mathbf{D}}^T)$ is the optimal value of a linear program which is jointly convex in $(\mathbf{x^T}, \tilde{\mathbf{D}}^T)$. This leads to the base case result that $C_T(\mathbf{x^T}, \tilde{\mathbf{D}}^T)$ is convex in x^T given any $\tilde{\mathbf{D}}^T$. By backward induction, we need to show that $C_t(\mathbf{x^t}, \tilde{\mathbf{D}}^t)$ is convex in $\mathbf{x^t}$ for any given $\tilde{\mathbf{D}}^t$, with the assumption that $C_{t+1}(\mathbf{x^{t+1}}, \tilde{\mathbf{D}}^{t+1})$ is convex in $\mathbf{x^{t+1}}$ given any $\tilde{\mathbf{D}}^{t+1}$. The cost-to-go function can be represented by $C_t(\mathbf{x^t}, \tilde{\mathbf{D}}^t) = \min_{\mathbf{z^t}, \mathbf{Z^t} \in \Delta} \mathcal{G}(\mathbf{x^t}, \tilde{\mathbf{D}}^t, \mathbf{z^t}, \mathbf{Z^t})$, where

$$\mathcal{G}(\mathbf{x^t}, \tilde{\mathbf{D}}^t, \mathbf{z^t}, \mathbf{Z^t}) = \Big[P(\mathbf{x^t}, \tilde{\mathbf{D}}^t, \mathbf{z^t}, \mathbf{Z^t}) + \mathbb{E}C_{t+1}(x_i^t - z_i^t - \sum_{j=1}^{N} Z_{ij}^t, \tilde{\mathbf{D}}^{t+1})\Big] \tag{28}$$

Consider any $\mu \geq 0$, and $\mathbf{x_1^t}, \mathbf{x_2^t} \geq 0$. Let $(\mathbf{z_i^t}, \mathbf{Z_i^t}) = \arg\min_{\mathbf{z^t}, \mathbf{Z^t} \in \Delta} \mathcal{G}(\mathbf{x_i^t}, \tilde{\mathbf{D}}^t, \mathbf{z^t}, \mathbf{Z^t})$. Note that P is a linear function in its variables (Equation 3), and $\mathbb{E}C_{t+1}(\mathbf{x^{t+1}}, \tilde{\mathbf{D}}^{t+1})$ is convex in $\mathbf{x^{t+1}}$, as expectation preserves convexity. Let $\bar{\mathbf{x}}^\mathbf{t} = \mu\mathbf{x_1^t} + (1-\mu)\mathbf{x_2^t}$, $\bar{\mathbf{z}}^\mathbf{t} = \mu\mathbf{z_1^t} + (1-\mu)\mathbf{z_2^t}$ and $\bar{\mathbf{Z}}^\mathbf{t} = \mu\mathbf{Z_1^t} + (1-\mu)\mathbf{Z_2^t}$. We have:

$$\begin{aligned} C_t(\bar{\mathbf{x}}^\mathbf{t}, \tilde{\mathbf{D}}^t) &= \min_{\mathbf{z^t}, \mathbf{Z^t} \in \Delta} \Big[P(\bar{\mathbf{x}}^\mathbf{t}, \tilde{\mathbf{D}}^t, \mathbf{z^t}, \mathbf{Z^t}) + \mathbb{E}C_{t+1}(\bar{x}_i^t - z_i^t - \sum_{j=1}^{N} Z_{ij}^t, \tilde{\mathbf{D}}^{t+1})\Big] \\ &\leq P(\bar{\mathbf{x}}^\mathbf{t}, \tilde{\mathbf{D}}^t, \bar{\mathbf{z}}^\mathbf{t}, \bar{\mathbf{Z}}^\mathbf{t}) + \mathbb{E}C_{t+1}(\bar{x}_i^t - \bar{z}_i^t - \sum_{j=1}^{N} \bar{Z}_{ij}^t, \tilde{\mathbf{D}}^{t+1}) \\ &\leq \mu P(\mathbf{x_1^t}, \tilde{\mathbf{D}}^t, \mathbf{z_1^t}, \mathbf{Z_1^t}) + (1-\mu) P(\mathbf{x_2^t}, \tilde{\mathbf{D}}^t, \mathbf{z_2^t}, \mathbf{Z_2^t}) + \mathbb{E}C_{t+1}(\bar{x}_i^t - \bar{z}_i^t - \sum_{j=1}^{N} \bar{Z}_{ij}^t, \tilde{\mathbf{D}}^{t+1}) \end{aligned} \tag{29}$$

The first inequality follows from the feasibility of $\bar{\mathbf{z}}^\mathbf{t}, \bar{\mathbf{Z}}^\mathbf{t}$ in Δ, as $(\mathbf{z_1^t}, \mathbf{Z_1^t})$ and $\mathbf{z_2^t}, \mathbf{Z_2^t})$ are feasible in Δ. The second inequality follows from the convexity of P. As $\mathbb{E}C_{t+1}(\mathbf{x^{t+1}}, \tilde{\mathbf{D}}^{t+1})$ is convex in $\mathbf{x^{t+1}}$, we have:

$$\begin{aligned} \mathbb{E}C_{t+1}\big(\bar{x}_i^t - \bar{z}_i^t - \sum_{j=1}^{N} \bar{Z}_{ij}^t, \tilde{\mathbf{D}}^{t+1}\big) &= \mathbb{E}C_{t+1}\left[\mu\left(x_{1,i}^t - z_{1,i}^t - \sum_{j=1}^{N} Z_{1,ij}^t\right) + (1-\mu)\left(x_{2,i}^t - z_{2,i}^t - \sum_{j=1}^{N} Z_{2,i}^t\right), \tilde{\mathbf{D}}^{t+1}\right] \\ &\leq \mu\mathbb{E}C_{t+1}\left[x_{1,i}^t - z_{1,i}^t - \sum_{j=1}^{N} Z_{1,ij}^t, \tilde{\mathbf{D}}^{t+1}\right] + (1-\mu)\mathbb{E}C_{t+1}\left[x_{2,i}^t - z_{2,i}^t - \sum_{j=1}^{N} Z_{2,i}^t, \tilde{\mathbf{D}}^{t+1}\right] \end{aligned} \tag{30}$$

Thus, from Equation 28, we have:

$$\begin{aligned} C_t(\bar{\mathbf{x}}^\mathbf{t}, \tilde{\mathbf{D}}^t) &\leq \mu\mathcal{G}(\mathbf{x_1^t}, \tilde{\mathbf{D}}^t, \mathbf{z_1^t}, \mathbf{Z_1^t}) + (1-\mu)\mathcal{G}(\mathbf{x_2^t}, \tilde{\mathbf{D}}^t, \mathbf{z_2^t}, \mathbf{Z_2^t}) \\ &= \mu C_t(\mathbf{x_1^t}, \tilde{\mathbf{D}}^t) + (1-\mu) C_t(\mathbf{x_2^t}, \tilde{\mathbf{D}}^t) \end{aligned} \tag{31}$$

The equality follows from the definitions of $(\mathbf{z_1^t}, \mathbf{Z_1^t})$ and $(\mathbf{z_2^t}, \mathbf{Z_2^t})$. □

A.2. Proof of Lemma 1

By recursion on x_i^t, we have: $x_i^T - z_i^T - \sum_j Z_{ij}^T = y_i - \sum_{t=1}^{T} z_i^t - \sum_{t=1}^{T}\sum_j Z_{ij}^t$. Thus, we have the following coefficients for the decision variables in the objective:

$$\begin{aligned} &z_i^t : -p_s - h \quad , && \forall i, \forall t \leq T \\ &Z_{ii}^t : s - p_o - h \quad , && \forall i, \forall t \leq T \\ &Z_{ij}^t : s_{ij} - p_o - h && \forall i, j \neq i, \forall t \leq T \end{aligned}$$

Note that based on the assumptions in Equation 1, we have: $-p_s - h > s - p_o - h \geq s_{ij} - p_o - h$. Then, by greedy allocation for each i, we will have $\sum_{t=1}^{T} z_i^t = \min(y_1, \sum_{t=1}^{T} D_{is}^t$, followed by $\sum_{t=1}^{T} Z_{ii}^t = \min\left(\left(y_i - \sum_{t=1}^{T} D_{is}^t\right)^+, \sum_{t=1}^{T} D_{io}^t\right)$. Finally, $\sum_{t=1}^{T}\sum_{i,j} Z_{ij}^t = \min\left(\sum_{i=1}^{N}\left(y_i - \sum_{t=1}^{T} D_{is}^t\right)^+, \sum_{i=1}^{N}\sum_{t=1}^{T} D_{io}^t\right)$. □

A.3. Proof of Proposition 2

First we eliminate x_i^t variables using $x_i^t = y_i - \sum_{t'=1}^{t-1} z_i^{t'} - \sum_{t'=1}^{t-1} Z_{ij}^{t'}$. Thus, (6) is equivalent to:

$$\begin{aligned}
\underline{C}(\mathbf{y}, \tilde{\mathbf{D}}) = \min_{\mathbf{z}^t, \mathbf{Z}^t} \quad & \sum_{t=1}^{T}\left[\sum_{i=1}^{N} p_s(D_{is}^t - z_i^t) + \sum_{j=1}^{N} p_o\left(D_{jo}^t - \sum_{i=1}^{N} Z_{ij}^t\right)\right. \\
& \left. + \sum_{i=1}^{N} sZ_{ii}^t + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} s_{ij} Z_{ij}^t\right] + \sum_{i=1}^{N} h\left(y_i - \sum_{t=1}^{T} z_i^t - \sum_{t=1}^{T}\sum_{j=1}^{N} Z_{ij}^t\right) \\
\text{s.t.} \quad & \sum_{t'=1}^{t} z_i^{t'} + \sum_{t'=1}^{t}\sum_{j=1}^{N} Z_{ij}^{t'} \leq y_i, \quad \forall i \in [N], \forall t \in [T], \\
& z_i^t \leq D_{is}^t, \quad \forall i \in [N], \forall t \in [T], \\
& \sum_{i=1}^{N} Z_{ij}^t \leq D_{jo}^t, \quad \forall j \in [N], \forall t \in [T], \\
& \mathbf{z^t}, \mathbf{Z^t} \geq 0, \quad \forall t \in [T]
\end{aligned} \tag{32}$$

First, note that the first constraint can be replaced by $\sum_{t'=1}^{T} z_i^{t'} + \sum_{t'=1}^{T}\sum_{j=1}^{N} Z_{ij}^{t'} \leq y_i, \quad \forall i \in [N]$, since $\mathbf{z^t}, \mathbf{Z^t} \geq 0$. Since the objective in (6) contains the decision variables z_i^t, Z_{ij}^t only occurring in the sum over T (i.e. as $\sum_{t=1}^{T} z_i^t$ and $\sum_{t=1}^{T} Z_{ij}^t$), we can replace the second and third constraints by $\sum_{t=1}^{T} z_i^t \leq \sum_{t=1}^{T} D_{is}^t, \quad \forall i \in [N]$ and $\sum_{t=1}^{T}\sum_{i=1}^{N} Z_{ij}^t \leq \sum_{t=1}^{T} D_{jo}^t, \quad \forall j \in [N]$ respectively. Note that this replacement relaxes the problem, but we show that the objective solution does not change in value. Consider the second constraint involving z_i^t variables. Any feasible solution to the relaxed problem can be modified to be feasible in the original problem without altering the objective, as the objective only contains terms of the form $\sum_{t=1}^{T} z_i^t$. The proof is by contradiction, as if the solution cannot be modified to be feasible in the original problem, then it cannot be feasible in the relaxed problem. Similar arguments can be made for the third constraint involving Z_{ij}^t variables. Thus, an equivalent formulation of (6) is:

$$\begin{aligned}
\underline{C}(\mathbf{y}, \tilde{\mathbf{D}}) = \min_{\mathbf{z}^t, \mathbf{Z}^t} \quad & \sum_{t=1}^{T}\left[\sum_{i=1}^{N} p_s(D_{is}^t - z_i^t) + \sum_{j=1}^{N} p_o\left(D_{jo}^t - \sum_{i=1}^{N} Z_{ij}^t\right)\right. \\
& \left. + \sum_{i=1}^{N} sZ_{ii}^t + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} s_{ij} Z_{ij}^t\right] + \sum_{i=1}^{N} h\left(y_i - \sum_{t=1}^{T} z_i^t - \sum_{t=1}^{T}\sum_{j=1}^{N} Z_{ij}^t\right) \\
\text{s.t.} \quad & \sum_{t=1}^{T} z_i^t + \sum_{t=1}^{T}\sum_{j=1}^{N} Z_{ij}^t \leq y_i, \quad \forall i \in [N], \\
& \sum_{t=1}^{T} z_i^t \leq \sum_{t=1}^{T} D_{is}^t, \quad \forall i \in [N], \\
& \sum_{t=1}^{T}\sum_{i=1}^{N} Z_{ij}^t \leq \sum_{t=1}^{T} D_{jo}^t, \quad \forall j \in [N], \\
& \mathbf{z^t}, \mathbf{Z^t} \geq 0, \quad \forall t \in [T]
\end{aligned} \tag{33}$$

Applying the transformations completes the proof:

$$D_{is} \leftarrow \sum_{t=1}^{T} D_{is}^{t}, \quad D_{io} \leftarrow \sum_{t=1}^{T} D_{io}^{t}$$
$$z_i \leftarrow \sum_{t=1}^{T} z_i^t, \quad Z_{ij} \leftarrow \sum_{t=1}^{T} Z_{ij}^t$$

□

A.4. Proof of Proposition 3

Proof: Consider the linear program representation $\tilde{C}(\mathbf{y}, \tilde{\mathbf{D}})$, where z_i represents the amount of inventory at R_i used to fulfill its in-store demand, and Z_{ij} represents the amount of inventory of R_i used to fulfill online demand from region j.

$$\begin{aligned} \tilde{C}(\mathbf{y}, \tilde{\mathbf{D}}) = \min_{z_i, Z_{ij}} \sum_i h \quad & (y_i - z_i - \sum_j Z_{ij}) + \sum_i p_s (D_{is} - z_i) \\ & + \sum_i p_o (D_{io} - \sum_j Z_{ji}) + \sum_i s Z_{ii} + \sum_i \sum_{j \neq i} s_{ij} Z_{ij} \\ \text{subject to } z_i + \sum_j Z_{ij} & \leq y_i, \quad \forall i \\ z_i & \leq D_{is}, \quad \forall i \\ \sum_j Z_{ji} & \leq D_{io}, \quad \forall i \\ z_i, Z_{ij} & \geq 0, \quad \forall i, j \end{aligned} \tag{34}$$

Note that $C^{IIP}(\mathbf{y}) = \mathbb{E}(\tilde{C}(\mathbf{y}, \tilde{\mathbf{D}}))$. The structure of C^{IIP} as an expectation of a linear program draws direct comparison with the value function in newsvendor networks (van Mieghem and Rudi 2002). Similar to proposition 2 in Harrison and van Mieghem (1999), the gradient of the function $\tilde{C}(\mathbf{y}, \tilde{\mathbf{D}})$ with respect to $\mathbf{y} = (y_1, y_2)$ can be written as:

$$\nabla_{\mathbf{y}} \tilde{C}(\mathbf{y}, \tilde{\mathbf{D}}) = (h, h)^T - \lambda(\mathbf{y}, \tilde{\mathbf{D}}) \tag{35}$$

where $\lambda(\mathbf{y}, \tilde{\mathbf{D}})$ is the dual-price vector corresponding to the constraints with y_1 and y_2 in (34). For a given $\mathbf{y}$, the 4-dimensional demand space $(D_{1s}, D_{1o}, D_{2s}, D_{2o})$ can be divided into domains $(\Omega_k(\mathbf{y}))_{k=1}^{20}$ such that in each domain, the optimal values of the decision variables z_i, z_{ii} and z_{ij} are linear in y_i, and hence the dual-price vector $\lambda(\mathbf{y}, \tilde{\mathbf{D}})$ is constant (refer to Appendix B for a discussion). The first-order conditions are:

$$0 = \nabla_{\mathbf{y}} C^{IIP}(\mathbf{y}) = \nabla_{\mathbf{y}} \mathbb{E}\left(\tilde{C}\left(\mathbf{y}, \tilde{\mathbf{D}}\right)\right) \tag{36}$$

We can interchange the gradient and expectation on the right hand side of Equation 36 (see Harrison and van Mieghem (1999) for a proof), and thus Equation 36 becomes

$$\begin{aligned} 0 = \nabla_{\mathbf{y}} C^{IIP}(\mathbf{y}) = \mathbb{E}_{\tilde{D}} \nabla_{\mathbf{y}} \tilde{C}\left(\mathbf{y}, \tilde{\mathbf{D}}\right) &= (h, h)^T - \mathbb{E}_{\tilde{D}} \lambda\left(\mathbf{y}, \tilde{\mathbf{D}}\right) \\ &= (h, h)^T - \sum_k \lambda^k \mathbb{P}\left(\Omega_k\left(\mathbf{y}\right)\right) \end{aligned} \tag{37}$$

where λ^k is the constant $\lambda(\mathbf{y}, \tilde{\mathbf{D}})$ for $\tilde{\mathbf{D}} \in \Omega_k(\mathbf{y})$. □

A.5. Proof of Proposition 4

Based on the approximation used to formulate C^{LB}, the difference in costs between C^{IIP} and C^{LB} is:

$$\begin{aligned}
C^{IIP}(\mathbf{y}) - C^{LB}(\mathbf{y}) &= (h+p_o-s_{12})\,\mathbb{E}\Big[\Big(\sum_i D_{io} - \sum_i (y_i - D_{is})^+\Big)^+ + \sum_i (D_{is}-y_i)^+ - \Big(D - \sum_i y_i\Big)^+\Big] \\
&\geq (h+p_o-s_{12})\,\mathbb{E}\Big[\Big(\sum_i D_{io} - \sum_i (y_i - D_{is})^+ + \sum_i (D_{is}-y_i)^+\Big)^+ - \Big(D - \sum_i y_i\Big)^+\Big] \\
&= 0
\end{aligned}$$

The first inequality follows from : $a^+ + b^+ \geq (a+b)^+$, and further simplification uses $x^+ - (-x)^+ = x$. □

The proof follows for any number of stores, as long as the cross-shipping cost is a constant and $s_{12} < h + p_o$.

A.6. Proof of Proposition 5

A similar result is proved in Dong and Rudi (2004, Lemma 1), who consider the case of traditional transshipment. Substituting $\mathbf{y^{DIP}}$ into the first order condition for C^{LB} in Equation 19, we have:

$$\begin{aligned}
(h+p_o-s_{12})F_D\left(\sum_j y_j^{DIP}\right) + (s_{12}-s)F_{D_i}(y_i^{DIP}) + (p_s - p_o + s)F_{D_{is}}((y_i^{DIP}) - p_s \\
= (h+p_o-s_{12})\left(\Phi\left(z^{DIP}\sum_i \sigma_i/\sigma\right) - \Phi\left(z^{DIP}\right)\right)
\end{aligned}$$

where Φ is the CDF of the standard normal distribution. The equality follows from the fact that $\mathbf{y^{DIP}}$ satisfies Equation 12, and the normality of demands, as we can write $y_i^{DIP} = \mu_i + z^{DIP}\sigma_i$, where $D_i \sim \mathcal{N}(\mu_i, \sigma_i)$, and $D \sim \mathcal{N}(\mu, \sigma)$. As $\sum_i \sigma_i/\sigma \geq 1$, it follow that the gradient of C^{LB} at $\mathbf{y^{DIP}}$ is $\geq 0 (\leq 0)$ whenever $z^{DIP} \leq (\geq) \mu_i$. Also, writing $\sigma = \sqrt{\sum_i \sigma_i^2 + \sum_j 2\rho_l\sigma_i\sigma_j}$, where ρ_l is the correlation coefficient between locations, y^{DIP} is optimal to C^{LB} and C^{IIP} when $\rho_l = 1$. □

A.7. Proof of Proposition 6

The proof follows from Govindarajan et al. (2020), by noting that the nested structure provides a closed-form expression for the total shipping cost, as opposed to a linear program, by summing the shipping costs in each level. The key difference from Govindarajan et al. (2020) is that the available inventory levels at location i is $(y_i - D_{is})^+)$, rather than just y_i, which gives rise to nested piecewise linear terms in the cost function.

In level 0, the shipping cost is $\sum_{i\in[N]} s\cdot\min\Big(D_{io}, (y_i - D_{is})^+\Big) = s\cdot\mathbf{e}^\top\mathbf{D_o} - \sum_{i\in[N]} s\cdot\Big(D_{io} - (y_i - D_{is})^+\Big)$. For any level $\ell \geq 1$, the number of fulfilled units of demand from regions in set $\mathcal{I}_k^{(\ell)}$ at level ℓ is

$$\underbrace{\sum_{m\in\mathcal{K}_k^{(\ell)}}\left(\sum_{i\in\mathcal{I}_m^{(\ell-1)}} D_{io} - \sum_{i\in\mathcal{I}_m^{(\ell-1)}} (y_i - D_{is})^+\right)^+}_{\text{unmet demand in } \mathcal{I}_k^{\ell} \text{ after level } \ell-1} - \underbrace{\left(\sum_{i\in\mathcal{I}_k^{(\ell)}} D_{io} - \sum_{i\in\mathcal{I}_k^{(\ell)}} (y_i - D_{is})^+\right)^+}_{\text{unmet demand in } \mathcal{I}_k^{(\ell)} \text{ after level } \ell}, \tag{38}$$

where $\mathcal{K}_k^{(\ell)}$ is the set of level $\ell-1$ children of set $\mathcal{I}_k^{(\ell)}$. Note that the per-unit cost of this fulfillment is $s_{\ell,k}$. The total cost is thus given by:

$$\begin{aligned}
C^{IIP}(\mathbf{y}) = \mathbb{E}\Bigg[& h\cdot(\mathbf{e}^\top\mathbf{y} - \mathbf{e}^\top\mathbf{D})^+ + p_s\cdot\mathbf{e}^\top(\mathbf{D}_s - \mathbf{y})^+ + p_o\cdot\left(\mathbf{e}^\top\mathbf{D}_o - \mathbf{e}^\top(\mathbf{y}-\mathbf{D}_s)^+\right)^+ \\
&+ s\cdot\mathbf{e}^\top\mathbf{D}_o + \sum_{\ell=0}^{L-2}\sum_{k\in[n_\ell]}(s_{\ell+1,m^{(\ell+1)}(k)} - s_{\ell,k})\cdot\left(\sum_{i\in\mathcal{I}_k^{(\ell)}} D_{io} - \sum_{i\in\mathcal{I}_k^{(\ell)}} (y_i - D_{is})^+\right)^+\Bigg]
\end{aligned}$$

where $m^{(\ell+1)}(k) \in [n_{\ell+1}]$ is the level $\ell+1$ parent of $k \in [n_\ell]$. The proof is completed using the definition of η_ℓ as given in the Proposition statement. ■

A.8. Proof of Proposition 7

Proof of (1): The proof is similar to that of Proposition 4 and is hence omitted.

Proof of (2): C_2^{LB} is convex in the inventory levels, and its first order conditions can be solved to yield a heuristic solution y^{IIPH} characterized by the first order conditions:

$$(h+p_o-s)F_D\left(\sum_{j\in\mathcal{S}} y_j^{IIPH}\right)+(p_s-p_o+s)F_{D_{is}}(y_i^{IIPH})=p_s, \quad \forall i\in\mathcal{S} \tag{39}$$

Rewriting the above equation, we have:

$$y_i^{IIPH}=F_{D_{is}}^{-1}\left(\frac{p_s-(h+p_o-s)\cdot F_{D_\mathcal{S}}\left(\sum_{j\in\mathcal{S}} y_j^{IIPH}\right)}{p_s-p_o+s}\right)$$

Let $m=p_s-(h+p_o-s)\cdot F_{D_\mathcal{S}}\left(\sum_{j\in\mathcal{S}} y_j^{IIPH}\right)$. Thus, we have:

$$y_i^{IIPH}=F_{D_{is}}^{-1}\left(\frac{m}{p_s-p_o+s}\right) \tag{40}$$

Substituting the above equation into the definition of m, we have:

$$\sum_{j\in\mathcal{S}} F_{D_{is}}^{-1}\left(\frac{m}{p_s-p_o+s}\right)=F_D^{-1}\left(\frac{p_s-m}{h+p_o-s}\right) \tag{41}$$

The left hand side is increasing in m, whereas the right hand side is decreasing in m. Note that $p_s-(h+p_o-s)\le m\le p_s-(p_o-s)$. Due to the monotonicity of the left and right hand sides and their extreme values in this range, there must be a unique value of m that satisfies this equation, thus yielding a unique solution from (40). □

Proof of (3): Since we can solve for a unique solution for m in (41) which yields a unique solution $\mathbf{y^{IIPH}}$ from (40), it directly follows that stores stocks at the same critical fractile of their in-store demand. □

Proof of (4): Consider a square of unit area in which N stores are uniformly distributed. Let the square be divided into $\sqrt{N}$ identical cells, such that each cell contains $\sqrt{N}$ stores. The dimensions of each cell are thus $\frac{1}{N^{\frac{1}{4}}}\times\frac{1}{N^{\frac{1}{4}}}$. The superscript l for a demand variable (e.g. D_{is}^l) denotes that the demand belongs to a store in cell l.

Since the solution $\mathbf{y^{IIPH}}$ yields identical quantities at each location when the demands and costs are identical across locations, we simplify notation for the sake of this proof by replacing $C(\mathbf{y})$ by $C(y)$, where y is the inventory level at each location as specified by the solution $\mathbf{y}$. Let $C^{LB'}$ be the cost function obtained from C^{IIP} by lowering all cross-shipping costs to the within-region shipping cost s. Let C^{IIP_c} and $C^{LB'_c}$ be the functions obtained by restricting C^{IIP} and $C^{LB'}$ respectively, so that cross-shipments can only be made between two stores belonging to the same cell. Clearly, $C^{IIP}(y)\le C^{IIP_c}(y)$ and $C^{LB'}(y)\le C^{LB'_c}(y)$ for any $y\ge 0$. Let $g(y,N)$ denote the cost incurred by N stores starting with inventory y each, without the option of cross-shipping:

$$g(y,N)=\sum_{i=1}^{N}\left[h\,(y-D_i)^+ +p_s\,(D_{is}-y)^+ +p_o\left(D_{io}-(y-D_{is})^+\right)^+ +s\min\left(D_{io},(y-D_{is})^+\right)\right]$$

Note that $g(y,N)$ represents the sum of costs incurred by individual stores, and hence, $\mathbb{E}g(y,N) = \mathbb{E}\sum_{l=1}^{\sqrt{N}} g(y,\sqrt{N}) = \sqrt{N}g(y,\sqrt{N})$. Let $CS_{ij}(y,N)$ denote the cross-shipped quantity between stores i and j, when there are N stores with order-up-to quantity y each (CS^l_{ij} when defined within a cell). Note that both the functions g and CS_{ij} also depend on the demand vector, but the dependency is ignored for notational convenience. As the cells are identical in terms of demands and costs, we have:

$$\begin{aligned}
C^{IIP_c}(y^{IIPH}) = \quad & \mathbb{E}\left(\sum_{l=1}^{\sqrt{N}}\left[g(y^{IIPH},\sqrt{N}) + \sum_{i=1}^{\sqrt{N}}\sum_{j=1,j\neq i}^{\sqrt{N}}(s^l_{ij}-h-p_o)CS^l_{ij}(y^{IIPH},\sqrt{N})\right]\right)\\
= \quad & \mathbb{E}g(y^{IIPH},N) + \mathbb{E}\left(\sum_{l=1}^{\sqrt{N}}\left(\sum_{i=1}^{\sqrt{N}}\sum_{j=1,j\neq i}^{\sqrt{N}}(s^l_{ij}-h-p_o)CS^l_{ij}(y^{IIPH},\sqrt{N})\right)\right)\\
C^{LB'}(y^{IIPH}) = \quad & C^{LB'_c}(y^{IIPH})\\
& +(s-h-p_o)\mathbb{E}\left[\sum_{l=1}^{\sqrt{N}}\left(\sum_{i=1}^{\sqrt{N}}D^l_{io}-\left(y^{IIPH}-D^l_{is}\right)^+\right)^+ - \left(\sum_{i=1}^{N}D_{io}-\left(y^{IIPH}-D_{is}\right)^+\right)^+\right]\\
= \quad & \mathbb{E}g(y^{IIPH},N) + \mathbb{E}\left(\sum_{l=1}^{\sqrt{N}}\left(\sum_{i=1}^{\sqrt{N}}\sum_{j=1,j\neq i}^{\sqrt{N}}(s-h-p_o)CS^l_{ij}(y^{IIPH},\sqrt{N})\right)\right)\\
& +(s-h-p_o)\left[\sqrt{N}\mathbb{E}\left(\sum_{i=1}^{\sqrt{N}}D^l_{io}-\left(y^{IIPH}-D^l_{is}\right)^+\right)^+ - \mathbb{E}\left(\sum_{i=1}^{N}D_{io}-\left(y^{IIPH}-D_{is}\right)^+\right)^+\right]
\end{aligned}$$

The expression for $C^{LB'}$ is written as the sum of $C^{LB'_c}$ which restricts cross-shipping to within each cell, and the cost of the additional cross-shipped units with this restriction removed. We know that $C_2^{LB}(y^{IIPH}) \le C^{LB'}(y^{IIPH}) \le C^{IIP}(y^{IIPH}) \le C^{IIP_c}(y^{IIPH})$. We first show that $\frac{C^{IIP_c}(y^{IIPH})}{C^{LB'}(y^{IIPH})} \to 1$ as $N\to\infty$. We have:

$$\begin{aligned}
\frac{C^{IIP_c}(y^{IIPH})}{C^{LB'}(y^{IIPH})} - 1 = & \frac{\mathbb{E}\left(\sum_{l=1}^{\sqrt{N}}\left(\sum_{i=1}^{\sqrt{N}}\sum_{j=1,j\neq i}^{\sqrt{N}}(s^l_{ij}-s)CS^l_{ij}(y^{IIPH},\sqrt{N})\right)\right)}{C^{LB'}(y^{IIPH})}\\
& + \frac{(h+p_o-s)\left[\sqrt{N}\mathbb{E}\left(\sum_{i=1}^{\sqrt{N}}D^l_{io}-(y^{IIPH}-D^l_{is})^+\right)^+ - \mathbb{E}\left(\sum_{i=1}^{N}D_{io}-(y^{IIPH}-D_{is})^+\right)^+\right]}{C^{LB'}(y^{IIPH})}
\end{aligned}$$

We have $s^l_{ij}-s = f(d^l_{ij}) \le f\left(\frac{\sqrt{2}}{N^{\frac{1}{4}}}\right)$, as the maximum distance within a cell is $\frac{\sqrt{2}}{N^{\frac{1}{4}}}$. Thus, using $C^{LB'}(y^{IIPH}) \ge \mathbb{E}\left(\sum_{l=1}^{\sqrt{N}}\left(\sum_{i=1}^{\sqrt{N}}\sum_{j=1,j\neq i}^{\sqrt{N}}(s)CS^l_{ij}(y^{IIPH},\sqrt{N})\right)\right)$ for the first term, and $C^{LB'}(y^{IIPH}) \ge s\mu_o N$ for the second term, we have

$$\frac{C^{IIP_c}(y^{IIPH})}{C^{LB'}(y^{IIPH})} - 1 \le \frac{f\left(\frac{\sqrt{2}}{N^{\frac{1}{4}}}\right)}{s} + \left(\frac{h+p_o-s}{s\mu_o\sqrt{N}}\right)\mathbb{E}\left(\sum_{i=1}^{\sqrt{N}}D_{io}-\left(y^{IIPH}-D_{is}\right)^+\right)^+ \tag{42}$$

The first term on the right hand side vanishes to zero as $N\to\infty$, as $f(d)\to 0$ as $d\to 0$. To simplify the second term, we need the following lemmas.

Lemma 2. *If $h < p_o - s$, then $y^{IIPH} > \mu$ where $\mu = \mu_s + \mu_o$, and if additionally $h < (p_s - p_o + s)F_s(\mu)$,*

$$y^{IIPH} \to F_s^{-1}\left(\frac{p_s-p_o+s-h}{p_s-p_o+s}\right) \in (0,\infty), \text{ as } N\to\infty \tag{43}$$

Proof: Lemma 1 is proved from the optimality equations of C^{LBN} (Equation 25) for identical stores:

$$(h+p_o-s)\mathbb{P}\left(\sum_{i=1}^{N} D_i \le N y^{IIPH}\right) + (p_s-p_o+s)F_{D_{1s}}(y^{IIPH}) = p_s$$

From the above equation, when $h < p_o - s$, we have $p_s < 2\,(p_o-s)\,\mathbb{P}\left(\sum_{i=1}^{N} D_i \le N y^{IIPH}\right) + (p_s - p_o + s)$. This simplifies to yield $y^{IIPH} > \mu$. Now, by applying the central limit theorem as $N \to \infty$ and $y^{IIPH} > \mu$, $\mathbb{P}\Big(\sum_{i=1}^{N} D_i/N \le y^{IIPH}\Big) \to 1$, and the result follows. Note that the asymptotic solution should also satisfy $y^{IIPH} > \mu$, which translates to the condition $h < (p_s - p_o + s)F_s(\mu)$. □

LEMMA 3. *When $h < \min(p_o - s,\ p_s - p_o + s)$, and the demands are bounded above as $D_{is} \le M_s$ and $D_{io} \le M_o$ for all i,*

$$\mathbb{P}\left(\sum_{i=1}^{\sqrt{N}} D_{io} > \sum_{i=1}^{\sqrt{N}} \left(y^{IIPH} - D_{is}\right)^+\right) \le \exp\left\{\frac{-2\sqrt{N}(y^{IIPH}-\mu)^2}{M_o + M_s}\right\} \tag{44}$$

Proof:

$$\mathbb{P}\left(\sum_{i=1}^{\sqrt{N}} D_{io} > \sum_{i=1}^{\sqrt{N}} \left(y^{IIPH} - D_{is}\right)^+\right) = \mathbb{P}\left(\sum_{i=1}^{\sqrt{N}} \left(D_i - \left(D_{is} - y^{IIPH}\right)^+\right) > \sqrt{N} y^{IIPH}\right) \le \mathbb{P}\left(\sum_{i=1}^{\sqrt{N}} D_i > \sqrt{N} y^{IIPH}\right)$$
$$\le \exp\left\{\frac{-2\sqrt{N}(y^{IIPH}-\mu)^2}{M_o + M_s}\right\} \to 0, \ \textit{as}\ N \to \infty$$

The final inequality follows from the Hoeffding bound for tail probabilities Hoeffding (1963), as $y^{IIPH} > \mu$ and demands are bounded, and the limit exists as y^{IIPH} approaches a finite positive quantity as $N \to \infty$ by Lemma 1. The expectation in the second term of Equation 42 can be bounded as follows:

$$\begin{aligned}
&\mathbb{E}\left(\sum_{i=1}^{\sqrt{N}} \left(D_{io} - \left(y^{IIPH} - D_{is}\right)^+\right)\right)^+ \\
&= \mathbb{E}\left[\left(\sum_{i=1}^{\sqrt{N}} \left(D_{io} - \left(y^{IIPH} - D_{is}\right)^+\right)\right)^+ \middle| \sum_{i=1}^{\sqrt{N}} D_{io} > \sum_{i=1}^{\sqrt{N}} \left(y^{IIPH} - D_{is}\right)^+\right] \mathbb{P}\left(\sum_{i=1}^{\sqrt{N}} D_{io} > \sum_{i=1}^{\sqrt{N}} \left(y^{IIPH} - D_{is}\right)^+\right) \\
&\le \mathbb{E}\left[\sum_{i=1}^{\sqrt{N}} D_{io} \middle| \sum_{i=1}^{\sqrt{N}} D_{io} > \sum_{i=1}^{\sqrt{N}} \left(y^{IIPH} - D_{is}\right)^+\right] \mathbb{P}\left(\sum_{i=1}^{\sqrt{N}} D_{io} > \sum_{i=1}^{\sqrt{N}} \left(y^{IIPH} - D_{is}\right)^+\right) \\
&\le M_o\sqrt{N} \exp\left\{\frac{-2\sqrt{N}(y^{IIPH}-\mu)^2}{M_o + M_s}\right\}
\end{aligned}$$

The last inequality follows from Lemma 2 and the boundedness of the demands as $D_{is} \le M_s$, and $D_{io} \le M_o$ for all i with $0 < M_s, M_o < \infty$. □

Thus, we have:

$$\begin{aligned}
\frac{C^{IIP_c}(y^{IIPH})}{C^{LB'}(y^{IIPH})} &\le 1 + \frac{f\left(\frac{\sqrt{2}}{N^{\frac{1}{4}}}\right)}{s} + \left(\frac{h+p_o-s}{s\mu_o}\right)\left(M_o\sqrt{N}\exp\left\{\frac{-2\sqrt{N}(y^{IIPH}-\mu)^2}{M_o+M_s}\right\}\right) \\
&\to 1, \ \textit{as}\ N \to \infty
\end{aligned} \tag{45}$$

The next step is to show the C_2^{LB} is off by a constant factor from the $C^{LB'}$. From the proof of Proposition 4, the difference simplifies to:

$$C^{LB'}(y^{IIPH}) - C_2^{LB}(y^{IIPH})$$
$$= (h+p_o-s)\,\mathbb{E}\left[\left(\sum_{i=1}^{N} D_{io} - \left(y^{IIPH} - D_{is}\right)^+\right)^+ + \sum_{i=1}^{N}\left(D_{is} - y^{IIPH}\right)^+ - \left(D - \sum_{i=1}^{N} y^{IIPH}\right)^+\right]$$

where $D = \sum_{i=1}^{N} D_{is} + D_{io}$.

Similar to what was done to bound the second term in Equation 42, we can show that whenever the conditions in Lemma 2 are satisfied, $\mathbb{E}\left(\sum_{i=1}^{N} D_{io} - (y^{IIPH} - D_{is})^+\right)^+ \leq M_o N \exp\left\{\frac{-2N(y^{IIPH}-\mu)^2}{M_o+M_s}\right\}$. Thus, we have:

$$C^{LB'}(y^{IIPH}) - C_2^{LB}(y^{IIPH}) \leq (h+p_o-s)\left[M_o N \exp\left\{\frac{-2N(y^{IIPH}-\mu)^2}{M_o+M_s}\right\} + \sum_{i=1}^{N}\left(D_{is} - y^{IIPH}\right)^+\right]$$

Using $C_2^{LB}(y^{IIPH}) \geq s\mu_o N$ and $C_2^{LB}(y^{IIPH}) \geq (p_s - p_o + s)\sum_{i=1}^{N}(D_{is} - y^{IIPH})^+$, we have:

$$\frac{C^{LB'}(y^{IIPH})}{C^{IIPH}(y^{IIPH})} - 1 \leq \left(\frac{h+p_o-s}{s\mu_o}\right)\left(M_o \exp\left\{\frac{-2N(y^{IIPH}-\mu)^2}{M_o+M_s}\right\}\right) + \left(\frac{h+p_o-s}{p_s-p_o+s}\right) \tag{46}$$

Thus, from Equations 45 and 46, as $N \to \infty$, we have

$$\frac{C^{IIP_c}(y^{IIPH})}{C_2^{LB}(y^{IIPH})} \leq 1 + \frac{h+p_o-s}{p_s-p_o+s}$$
$$\Rightarrow \frac{C^{IIP}(\mathbf{y^{IIPH}})}{C^{IIP}(\mathbf{y^{IIP}})} \leq \frac{h+p_s}{p_s-p_o+s}$$

The final step follows from $C^{IIP_c}(y^{IIPH}) \geq C^{IIP}(y^{IIPH})$, and $C_2^{LB}(\mathbf{y^{IIPH}}) \leq C^{IIP}(\mathbf{y^{IIP}})$. □

The result may hold subject to some generalizations, such as the unit square can be replaced with any finite area, and non-identical cells as long as the number of stores in each cell grows to infinity as $N \to \infty$. The resulting cases may call for a more complicated proof, and is outside the scope of this study.

Appendix B: Demand Regions for the IIP Solution

We illustrate the identification of demand regions in which the dual vector λ is constant (as discussed in Section 3.1.3) and the calculation of the corresponding probabilities. For any given (y_1, y_2), the demand space $(D_{1s}, D_{1o}, D_{2s}, D_{2o})$ can be divided into a number of independent regions. Based on the values taken by the variables in the optimal solution in (34), Table 5 shows the different cases that are possible given y_1 and y_2. From these cases, the independent demand regions are listed in Table 6 along with the constant dual prices in those regions. The underlined cases are redundant, and can be discarded while calculating the probability for each region. The dual prices λ_1, λ_2 are the shadow prices of the constraints which contain y_1 and y_2 respectively, namely the first set of constraints $z_i + \sum_{j=1}^{2} z_{ij} \leq y_i, \forall i$ in the linear program in (34), and can be obtain in a standard fashion from linear programming theory. For example, for the demand regions with the case D1, that is, $y_1 \geq D_1 + D_{2o}$, irrespective of the value of y_2, there will be inventory left over at retail store 1 at the end of the period. Thus the constraint $z_1 + \sum_{j=1}^{2} z_{1j} \leq y_1$ will not bind, and hence $\lambda_1 = 0$.

Table 5 Table showing the various demand cases based on the values of y_1, y_2

	A	B	C	D
1	$y_1 < D_{1s}$	$D_{1s} \leq y_1 < D_1$	$D_1 \leq y_1 < D_1 + D_{2o}$	$y_1 \geq D_1 + D_{2o}$
2	$y_2 < D_{2s}$	$D_{2s} \leq y_2 < D_2$	$D_2 \leq y_2 < D_2 + D_{1o}$	$y_2 \geq D_2 + D_{1o}$
3	$y_1 + y_2 < D_1 + D_2$	$y_1 + y_2 \geq D_1 + D_2$		

Table 6 Table showing the various demand regions and the corresponding constant dual-prices. (underlined notation indicates redundant cases)

Region	Case	λ_1	λ_2	Region	Case	λ_1	λ_2
Ω_1	A1,A2,A3	$h+p_s$	$h+p_s$	Ω_{11}	C1,A2,A3	$h+p_o-s_{12}$	$h+p_s$
Ω_2	A1,B2,A3	$h+p_s$	$h+p_o-s$	Ω_{12}	C1,B2,A3	$h+p_o-s_{12}$	$h+p_o-s$
Ω_3	A1,C2,A3	$h+p_s$	$h+p_o-s_{12}$	Ω_{13}	C1,B2,B3	0	$s_{12}-s$
Ω_4	A1,D2,A3	$h+p_s$	0	Ω_{14}	C1,C2,B3	0	0
Ω_5	A1,D2,B3	$h+p_s$	0	Ω_{15}	C1,D2,B3	0	0
Ω_6	B1,A2,A3	$h+p_o-s$	$h+p_s$	Ω_{16}	D1,A2,A3	0	$h+p_s$
Ω_7	B1,B2,A3	$h+p_o-s$	$h+p_o-s$	Ω_{17}	D1,A2,B3	0	$h+p_s$
Ω_8	B1,C2,A3	$h+p_o-s$	$h+p_o-s_{12}$	Ω_{18}	D1,B2,B3	0	$s_{12}-s$
Ω_9	B1,C2,B3	$s_{12}-s$	0	Ω_{19}	D1,C2,B3	0	0
Ω_{10}	B1,D2,B3	$s_{12}-s$	0	Ω_{20}	D1,D2,B3	0	0

The probability for each region is calculated as follows, when demands follow normal distributions. The region is expressed as an inequality of the form $R_k\tilde{D} <= S_kY$, where $\tilde{D} = [D_{1s}, D_{1o}, D_{2s}, D_{2o}]^\intercal$ and $Y = [y_1, y_2]^\intercal$. For example, $\Omega_3 = (A1, C2) = \{y_1 < D_{1s}, D_2 \leq y_2 < D_2 + D_{1o}\}$. This can be expressed as:

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} D_{1s} \\ D_{1o} \\ D_{2s} \\ D_{2o} \end{bmatrix} \leq \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$R_k\tilde{D}$ is multivariate normal with mean $R_k\mu$ and covariance matrix $R_k\Sigma\Sigma^\intercal R_k^\intercal$, where μ and Σ are the mean and covariance matrices of $\tilde{D}$. The probability of region k reduces to evaluating the cumulative distribution function of $A_k\tilde{D}$ at B_kY. For general demand distributions, numerical methods have to be employed.

Appendix C: Heuristic based on Constant Shipping Costs for a Network of Omnichannel Stores and OFCs

We obtain the heuristic solution y^{IIPH} for multiple locations with $\mathcal{S}_o \neq \emptyset$ by calculating order quantities for the OFCs separately, and using them in Equation 39 to compute order quantities for the omnichannel stores. The order-up-to quantities for OFCs are calculated from the pooled total order quantity for OFCs, which is determined using the newsvendor quantity for the combined online demand $D_{\mathcal{S}_o} = \sum_{i \in \mathcal{S}_o} D_{io}$.

$$\sum_{j \in \mathcal{S}_o} y_j^{IIPH} = F^{-1}_{D_{\mathcal{S}_o}} \left(\frac{p_o - s}{h + p_o - s} \right) \tag{47}$$

The actual underage cost for online demands at the OFCs would be less than $p_o - s$ and would depend on inventory information of stores, as stores can fulfill these online orders with available inventory. The calculation of inventory levels at stores and OFCs are dependent on each other, but since we are forced to estimate the inventory at OFCs separately, we inflate the underage cost to $p_o - s$ which yields a higher overall inventory level at the OFCs. This is a limitation that arises out of our heuristic approximation, but it allows us to extend the heuristic to the case where OFCs have a different shipping cost (s_o) compared to the stores (s), as the inventory calculation for the OFCs is done separately.

To calculate the individual order quantities at the OFCs, y_i^{IIPH}, $i \in \mathcal{S}_o$, we use the method of obtaining order-up-to quantities for multiple products with capacity constraints, as described in Chopra and Meindl (2007, p. 367). Each unit from $\sum_{j \in \mathcal{S}_o} y_j^{IIPH}$ is allocated incrementally to the OFCs based on the individual expected marginal costs. Once the order-up-to quantities for the OFCs are obtained, they are used in Equation 48 to determine order-up-to levels for other omnichannel stores.

$$(h + p_o - s) F_{D_{\mathcal{S}}} \left(\sum_{j \in \mathcal{S}} y_j^{IIPH} \right) + (p_s - p_o + s) F_{D_{is}} \left(y_i^{IIPH} \right) = p_s, \quad \forall i \in \mathcal{S}_{so} \tag{48}$$

Calculating this heuristic solution $\mathbf{y^{IIPH}}$ is also computationally fast, as Proposition 7(3) still applies to Equation 48. The cost of the heuristic solution is given by $C^{IIPH} = C^{IIP}(\mathbf{y^{IIPH}})$. We capture the effect of virtual pooling among the facilities in this heuristic, and the systematic approach is shown in Algorithm 2.

Algorithm 2 Procedure to calculate the heuristic solution $\mathbf{y^{IIPH}}$

1: For physical stores in set $\mathcal{S}_s$, set $y_i^{IIPH} = F_{is}^{-1}\left(\frac{p_s}{h+p_s}\right), \forall i \in \mathcal{S}_s$.
2: **for** $i \in \mathcal{S}_o$ (OFCs) **do**
3: Calculate total order quantity: $y^{TOT} = F_{D_{\mathcal{S}_o}}^{-1}\left(\frac{p_o - s}{h + p_o - s}\right)$, where $D_{\mathcal{S}_o} = \sum_{i \in \mathcal{S}_o} D_{io}$.
4: Set $y_i^{IIPH} = 0, \forall i \in \mathcal{S}_o$, and $rem = \lfloor y^{TOT} \rfloor$.
5: Calculate marginal cost $MC_i\left(y_i^{IIPH}\right) = -\left(p_o - s\right)\left(1 - F_{D_{io}}(y_i^{IIPH})\right) + hF_{D_{io}}\left(y_i^{IIPH}\right)$
6: Choose $i^* = \min_{i \in \mathcal{S}_o} MC_i(y_i^{IIPH})$. Set $y_{i^*}^{IIPH} \leftarrow y_{i^*}^{IIPH} + 1$
7: Set $rem \leftarrow rem - 1$. If $rem > 0$, go to Step 3.
8: **for** $i \in \mathcal{S}_{so}$ **do**
9: Calculate order quantities implicitly from the optimality equations: $(h + p_o - s) F_{D_{\mathcal{S}}} \left(\sum_{j \in \mathcal{S}} y_j^{IIPH} \right) + (p_s - p_o + s) F_{D_{is}} \left(y_i^{IIPH} \right) = p_s$, $\forall i \in \mathcal{S}_{so}$.

Appendix D: Additional Details for Numerical Analyses

All numerical analyses were done on a desktop computer (i7-3770 CPU @3.7GHz, 16GB RAM). The total market is assumed to be the top 300 most populous cities in mainland US. We take the sum of the mean in-store and online demands in each region to be a fixed proportion of the cities' populations. This represents the average market size of the region, and the mean in-store and online total demands over the horizon are calculated as $1 - \alpha$ and α proportions respectively of this mean market size in each region. The demands for the OFCs are calculated based on the population not covered by omnichannel stores. This online demand is allocated to each OFC based on the optimal throughput rates estimated by Chicago Consulting (2016).

D.1. Simulation Procedure

A brief overview of the simulation is listed below:

1. The parameters for demands in each fulfillment period are calculated based on demands over the horizon estimated from population data. The starting inventory level vectors $\mathbf{y^{DIP}}$ and $\mathbf{y^{IIPH}}$ are calculated using the demand information based on Equation 12 and Algorithm 1 respectively.
2. We generate a sample of size 10^4, where each sample is a realization of demands over the entire selling horizon, although fulfillment decisions in each fulfillment period are made without knowing future demands. For each sample, we iterate over steps 3-7, and take the sample averages as approximations for expectations.
3. The fulfillment thresholds for the TF policy are calculated based on Equation 27. For the MF policy, these thresholds are set to zero.
4. For $t = 1, \ldots, T$, iterate over steps 5-6. The starting inventory levels are set based on the inventory policy followed (IIPH or DIP).
5. Implement Algorithm 2 based on the fulfillment policy followed (MF or TF) and the corresponding thresholds calculated in Step 3.
6. At the end of each fulfillment period, penalty and shipping costs are calculated. The ending inventory at a location becomes the starting inventory for the next fulfillment period.
7. The total cost is the sum of the costs in each fulfillment period over the selling horizon.