

**RESEARCH ARTICLE****Morsifications and mutations**

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**Abstract**

We describe and investigate a connection between the topology of isolated singularities of plane curves and the mutation equivalence, in the sense of cluster algebra theory, of the quivers associated with their morsifications.

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## INTRODUCTION

We present and explore a remarkable connection between two seemingly unrelated subjects: the combinatorics of quiver mutations (which originated in the theory of cluster algebras) and the topology of plane curve singularities. Our constructions build on the elegant approach to the latter subfield of singularity theory that was pioneered in the 1970s by A'Campo [1] and Guseĭn-Zade [38, 39]. Given a real form of an isolated plane curve singularity, one begins by finding its real *morsification*, a real nodal local deformation that has the maximal possible number of real hyperbolic nodes. From the combinatorial topology of the morsification (more precisely, of its *divide*, the set of real points of the deformed curve in the vicinity of the singularity, viewed up to isotopy), one constructs the associated *A'Campo–Guseĭn-Zadediagram* (or  $A\Gamma$ -diagram), a certain tricolored planar graph. The  $A\Gamma$ -diagram can be used to explicitly compute the monodromy and the intersection form in the vanishing homology of the singularity. In fact, more is true: the  $A\Gamma$ -diagram uniquely determines the complex topological type of the underlying singularity; see [11] (for totally real singularities) and [50] (in full generality).

A given complex singularity may have many distinct *real forms*. These are real plane singular curves which, when viewed over the complex numbers, are locally homeomorphic to each other — but over the reals, they are not. Their real morsifications are also different from each other, and so are the associated  $A\Gamma$ -diagrams. How, then, can we tell, looking at two morsifications, whether we are dealing with the same complex singularity or not?

One answer to this question was given by A'Campo [3] in the late 1990s, in terms of a certain link that can be constructed from the divide of a given morsification. In this paper, we propose an alternative answer, which comes from the theory of *cluster algebras* [28, 31], specifically from the combinatorics of *quiver mutations*.

Our approach starts with a small but important change of perspective: we replace  $A\Gamma$ -diagrams by closely related *quivers*, removing the marking of the vertices but introducing orientations of the edges. We show (see Theorem 4.4) that the quiver constructed from a morsification determines the topological type of the singularity. The question then becomes: how can we tell, by looking at two quivers coming from morsifications of two plane curve singularities, whether these singularities are topologically equivalent or not?

The answer comes from the theory of *quiver mutations*. (A quiver can be mutated in different ways, depending on the choice of a vertex. One then applies a mutation to the resulting quiver, and

so on. The set of quivers obtained in this way defines a cluster algebra.) We conjecture that two singularities are topologically equivalent if and only if the quivers associated with their respective morsifications are mutation equivalent to each other, that is, if and only if one quiver can be transformed into another by iterated mutations. Thus, different real forms of the same complex singularity — and different morsifications of these real forms — should give rise to mutation equivalent quivers. Conversely, topologically distinct singularities are expected to produce quivers of different mutation type. Succinctly put, plane curve singularities are classified by the cluster algebras defined by their morsifications.

Our main results establish this relationship between the topology of plane curve singularities and the mutation equivalence of associated quivers modulo some technical assumptions, which we optimistically expect to be redundant. More concretely, we obtain the following results.

First, we establish the direction ‘combinatorial equivalence implies topological equivalence’ in the version where on the combinatorial side, mutation equivalence of quivers associated with morsifications is replaced by the move equivalence of the corresponding *plabic* (that is, planar bicolored) graphs, in the sense of Postnikov [62]. The connection between mutation equivalence and move equivalence is well-known (cf. Proposition 6.9) in the context of morsifications, we expect these notions to be interchangeable (see Conjecture 6.13). Our first main result, Corollary 9.10, demonstrates that move equivalence of plabic graphs constructed from different morsifications implies the topological equivalence of the corresponding singularities.

The key ingredient of the proof of this result is the construction of a link associated with an arbitrary plabic graph. This link is invariant under Postnikov’s local moves (see Corollary 9.5). Furthermore, the links associated with plabic graphs are naturally transverse, and local moves translate into transverse isotopies (see Corollaries 10.13 and 10.23). (For a plabic graph of a divide, one recovers its A’Campo link.)

Our second main result concerns the opposite direction: ‘topological equivalence implies combinatorial equivalence’. In Corollary 14.11, we show that different morsifications of the same singularity produce mutation-equivalent quivers. This result is predicated on the following two assumptions: (i) the existence of sequences of Yang–Baxter moves transforming each of the given divides into a *scannable* form; and (ii) the existence of a *positive isotopy* relating the positive braids associated with the respective scannable divides. We expect the assumptions (i)–(ii) to be redundant (see Conjectures 13.2 and 14.10). These assumptions are satisfied in all examples of real morsifications known to us (cf. Remarks 13.8. and 15.8).

The link between morsifications and quiver mutations revealed in this paper is suggestive of a deep intrinsic relationship between singularities and cluster algebras. To give one example, a quasi-homogeneous singularity  $x^a + y^b = 0$  is described by the same mutation class of quivers as the standard cluster structure on the homogeneous co-ordinate ring of the Grassmannian  $\text{Gr}_{a,a+b}(\mathbb{C})$  (see Remark 16.2). The underlying reasons for these combinatorial coincidences are yet to be uncovered.

Our investigations naturally lead us to a number of questions concerning the topology of plane curve singularities, their real forms, morsifications, divides, braids, quivers, and plabic graphs (see in particular Conjectures 6.17, 13.2, and 14.10 and Problems 7.12, 10.14, 14.3, and 14.8). The recent paper [50] by Leviant and the third author was motivated by this work; cf. in particular Conjecture 1.12 and Theorem 3.4.

The initial impetus for this project came from the desire to understand the reasons behind the common appearance of the *ADE* classification, in its version involving quivers, in two ostensibly unrelated contexts: Arnold’s celebrated classification [6] of simple singularities and the much more recent classification of cluster algebras of finite type [29]. Indeed, we prove — uncondition-

ally — that our main conjecture holds true for simple singularities (respectively, quivers of finite type) (see Theorem 18.3).

## ORGANIZATION OF THE PAPER

This paper has at least two intended audiences: the readers whose primary interests lie either in singularity theory or in the theory of cluster algebras. Bearing this in mind, we aimed to make our presentation as accessible as possible to the mathematicians who might be unfamiliar with one of the two subjects. Additional details from singularity theory can be found in the textbooks [8, Sections 2.1–2.2; 10, Section 4.1] and in the papers [1, 11, 38, 39, 50]. For a thorough exposition of the fundamentals of quiver mutations, and their relations with cluster algebras, the reader is referred to [31].

In Sections 1–3, we review the requisite singularity theory background: isolated singularities of complex and real plane curves, and their morsifications (Section 1); divides and their role in the study of plane curve singularities (Section 2); and the A’Campo–Guseĭn–Zade diagrams (Section 3).

In Section 4, we introduce the quivers of divides, and show that the topology of a complex singularity can be recovered from the quiver of its morsification. In Section 5, we define quiver mutations, and formulate the first version of our main conjecture (Conjecture 5.5), describing the putative correspondence between topological equivalence of singularities and mutation equivalence of associated quivers.

In Sections 6 and 7, we reformulate the problem on both sides of the conjectural correspondence. Section 6 is devoted to the combinatorics of plabic graphs and local moves on them, in a version slightly different from Postnikov’s original treatment [62]. We explain that move equivalent plabic graphs produce mutation equivalent quivers. (The converse implication is false, but can conjecturally be fixed by allowing an additional class of transformations called ‘switches’.) We also explain how divides give rise to plabic graphs, sharing the same quivers, up to mutation equivalence.

In Section 7, we turn to topology. Following A’Campo, we define the links of divides, and review their main properties. We then recast the topological equivalence of singularities in terms of divide links: as shown by A’Campo, two singularities are equivalent if and only if the links of divides coming from their morsifications are isotopic to each other. This enables us to reformulate the main conjecture in the language of divide links and plabic graphs (see Conjectures 7.10 and 7.11).

Sections 8 and 9 are dedicated to the proof of our first main result, Corollary 9.10 (‘move equivalence implies link equivalence’). In Section 8, we introduce and study oriented divides and associated links, in particular showing that local moves on oriented divides translate into smooth isotopies of their links. In Section 9, we define the links of plabic graphs, and show that Postnikov moves result in link isotopy.

Section 10, though optional as far as the main results are concerned, clarifies the relationships between the links considered in this paper and several important classes of links studied in the literature. In particular, we show that all links associated with plabic graphs (or divides) are both quasipositive and transverse; that local moves result in transverse isotopies; and that for divide links, ordinary isotopy and transverse isotopy are equivalent. We note that a related (but different) construction due to Shende, Treumann, Williams, and Zaslow [70] provided a link between cluster algebras and Legendrian (rather than transverse) links (cf. Remark 10.27).

Sections 11–14 are devoted to the reverse direction of the main correspondence (‘link equivalence implies move equivalence’). In Section 11, we discuss scannable divides and associated positive braids. In particular, we recall a beautiful palindromic rule, due to O. Couture and B. Perron [23], for constructing the A’Campo link of a scannable divide as the closure of a certain positive braid.

A distinguished choice of a plabic graph attached to a scannable divide, which we call a *plabic fence*, is introduced in Section 12. These fences are closely related to the corresponding braids. In Section 13, we introduce the notion of *positive braid isotopy*, and relate it to move equivalence of plabic fences (see Theorem 13.10). This establishes the desired implication in the case of scannable divides, modulo the assumption of positive isotopy (cf. condition (ii)).

In Section 14, we review the relevant properties of triangle (or ‘Yang-Baxter’) moves on divides. These moves preserve the associated links, and they can be emulated by local moves on plabic graphs. This enables us to extend the aforementioned result from the scannable case to the generality of arbitrary divides which can be converted into a scannable form via a sequence of triangle moves (cf. assumption (i)). This class conjecturally includes all divides coming from morsifications.

Sections 15 and 16 are devoted to concrete constructions producing scannable real morsifications of plane curve singularities. Here we describe a large class of examples coming from transversal overlays of quasi-homogeneous singularities, and their morsifications built from *Lissajous divides* and *wiring diagrams*.

Section 17 presents an alternative approach to the construction of divide links which employs a particular kind of orientations of plabic graphs. These orientations, which are closely related to Postnikov’s notion of *perfect* orientations, always exist in the scannable case. Such an orientation can be used to describe the A’Campo link of a divide (or plabic graph) by a simple local combinatorial rule. Moreover, these links are preserved under Postnikov moves, allowing for an elegant and elementary development of the theory. Unfortunately, this approach has limited applicability, since there exist divides (including some which come from morsifications) whose associated plabic graphs do not possess an orientation with requisite properties.

Section 18 is devoted to the special case of simple singularities. In this case, everything works precisely as intended: we show that any morsification of a simple singularity gives rise to a quiver of finite type, and conversely, if a morsification produces a quiver of finite type, then the singularity is simple. Moreover, the *ADE* type of a simple singularity matches the cluster type of the corresponding quiver.

## 1 | SINGULARITIES AND MORSIFICATIONS

Throughout this paper, the term *singularity* means a germ  $(C, z) \subset \mathbb{C}^2$  of a reduced analytic curve  $C$  in the complex plane  $\mathbb{C}^2$  at a singular point  $z \in \mathbb{C}^2$ . We can postulate, without loss of generality, that  $z = (0, 0)$ .

We shall always assume that our singularity is *isolated*: there exists a closed ball  $\mathbf{B} = \mathbf{B}_{C,z} \subset \mathbb{C}^2$  centered at  $z$  such that  $z$  is the only singular point of  $C$  in  $\mathbf{B}$ . Moreover, we can assume that any sphere centered at  $z$  and contained in  $\mathbf{B}$  intersects our curve  $C$  transversally; we then call  $\mathbf{B}$  the *Milnor ball* at  $z$ .

The simplest example of a singularity is a *node*, that is, a transversal intersection of two locally smooth branches.

Isolated singularities of plane complex curves can be studied up to different types of equivalence. Here we focus on the *topological* theory, which considers singularities up to homeomorphisms of a neighborhood of an isolated singular point. We note that this point of view is substantially different from treating singularities up to diffeomorphisms (cf. Example 1.1).

**Example 1.1.** All singularities consisting of four smooth branches transversally crossing at a point  $z \in \mathbb{C}^2$  are topologically equivalent to each other. On the other hand, any diffeomorphism of a neighborhood of  $z$  preserves the cross-ratio of the tangent lines (at  $z$ ) to the four branches, so configurations with different cross-ratios are not equivalent to each other in the smooth category.

By a theorem of Weierstrass (see, for example, [37, Theorems I.1.6 and I.1.8] or [15, Section III.8.2]), any locally convergent power series  $f(x, y)$  splits into a product of irreducible factors that are also locally convergent. In the case under consideration (an isolated singularity  $f(x, y) = 0$ ), we can choose  $\mathbf{B}$  so that everything converges there. The factors are determined uniquely up to permutation, and up to multiplication by a unit (that is, by a nonvanishing function  $\mathbf{B} \rightarrow \mathbb{C}$ ). Since we assume that the curve is reduced, the factors are pairwise distinct: no two of them differ by a unit. These factors correspond to the *local branches* of the singularity.

**Definition 1.2.** A singularity  $(C, z)$  is called *quasi-homogeneous* of type  $(a, b)$  (for  $a \geq b \geq 2$ ) if, in suitable local co-ordinates,  $(C, z)$  can be given by an equation of the form

$$f(x, y) = \sum_{\substack{bi+aj=ab \\ i, j \geq 0}} c_{ij} x^i y^j = 0, \tag{1.1}$$

with  $z = (0, 0)$ ; here  $f(x, y)$  must not contain multiple irreducible factors. Any quasi-homogeneous singularity of type  $(a, b)$  is topologically equivalent to the singularity

$$x^a + y^b = 0. \tag{1.2}$$

The number of (complex) branches of the singularity (1.2) is  $\text{gcd}(a, b)$ .

*Remark 1.3.* Suppose a singularity  $(C, z)$  is given (in some local co-ordinates) by an equation of the form

$$\sum_{\substack{bi+aj \geq ab \\ i, j \geq 0}} c_{ij} x^i y^j = 0,$$

with  $z = (0, 0)$ . Assume that the corresponding equation (1.1) defines a quasi-homogeneous singularity of type  $(a, b)$ . Then  $(C, z)$  is topologically equivalent to the quasi-homogeneous singularity (1.2). This follows by combining [9, p. 194, Theorem in Section 12.2] and [72, Theorem 2.1].

**Definition 1.4.** One important topological invariant of a singularity is its *Milnor number* [54, Section 7]. Let  $(C, z)$  be a singularity given by an equation  $f(x, y) = 0$ . Algebraically, the Milnor number  $\mu(C, z)$  is given by

$$\mu(C, z) = \dim_{\mathbb{C}} \left( \mathbb{C}[[x, y]] / \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \right),$$

the dimension of the quotient of the algebra of power series in  $x$  and  $y$  by its Jacobian ideal. To illustrate, the Milnor number of the quasi-homogeneous singularity (1.2) is equal to  $(a - 1)(b - 1)$ .

The Milnor number can also be defined as the maximal number of critical points (in the vicinity of  $z$ ) that a small deformation of  $f$  may have. Informally, the Milnor number measures the complexity of the singular point considered as the critical point of the defining function. See, for example, [37, Section 2.1] for further discussion.

Any deformation of an isolated plane curve singularity that keeps the Milnor number constant yields topological equivalence (see [72]).

**Definition 1.5.** A plane curve singularity  $(C, z)$  is called *real* if  $C \subset \mathbb{C}^2$  is an analytic curve invariant under complex conjugation, and  $z \in C$  its real singular point. Equivalently,  $C$  is given by an equation  $f(x, y) = 0$  where all coefficients in the power series expansion of  $f$  (at  $z$ ) are real.

The simplest example of a real singularity is a real node of a real plane curve. Such a node can be either *hyperbolic* or *elliptic*, that is, analytically equivalent over  $\mathbb{R}$  to  $x^2 - y^2 = 0$  or to  $x^2 + y^2 = 0$ , respectively.

**Definition 1.6.** A (real) singularity  $(C, z)$  is called *totally real* if all its local branches are real. For example, a hyperbolic node is totally real, but an elliptic one is not.

**Theorem 1.7** [11, Theorem 1.1; 39, Theorem 3]. *Every complex plane curve singularity is topologically equivalent to a totally real one.*

A real singularity topologically equivalent to a given complex singularity is called a *real form* of the latter. By Theorem 1.7, any complex plane curve singularity has a real form. There are typically several distinct real forms, up to conjugation-equivariant topological equivalence. For example, a complex node has two essentially different real forms: hyperbolic and elliptic. An irreducible complex singularity (that is, one that has a single branch) has only one real form.

One of our implicit goals is to better understand the relations between different real forms of the same complex singularity.

**Definition 1.8.** A *nodal deformation* of a singularity  $(C, z)$  inside the Milnor ball  $\mathbf{B}$  is an analytic family of curves  $C_t \cap \mathbf{B}$  such that

- the complex parameter  $t$  varies in a (small) disk centered at  $0 \in \mathbb{C}$ ;
- for  $t = 0$ , we recover the original curve:  $C_0 = C$ ;
- each curve  $C_t$  is smooth along  $\partial\mathbf{B}$ , and intersects  $\partial\mathbf{B}$  transversally;
- for any  $t \neq 0$ , the curve  $C_t$  has only ordinary nodes inside  $\mathbf{B}$ ;
- the number of these nodes does not depend on  $t$ .

The maximal number of nodes in a nodal deformation is  $\delta(C, z)$ , the  $\delta$ -invariant of the singularity (see, for example, [54, Section 10]).

**Definition 1.9.** A *real nodal deformation* of a real singularity  $(C, z)$  is obtained by taking a nodal deformation  $(C_t \cap \mathbf{B})$  which is equivariant with respect to complex conjugation, and restricting the parameter  $t$  to a (small) interval  $[0, \tau) \subset \mathbb{R}$ .



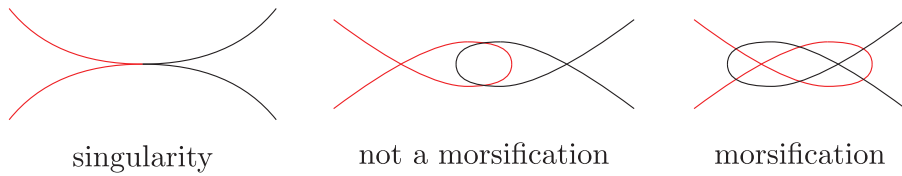


FIGURE 1 A quasi-homogeneous plane curve singularity of type (6,4) (two cusps sharing a tangent) and its two real nodal deformations. The first deformation is not a morsification, as it only has 4 real nodes. The second one has 8 real nodes, and is a morsification. In this example,  $\delta(C, z) = \delta_{\mathbb{R}}(C, z) = 8$

**Definition 1.10.** A real morsification of a real singularity  $(C, z)$  is a real nodal deformation  $C_t = \{f_t(x, y) = 0\}$  as above such that

- all critical points of  $f_t$  are real and Morse (that is, with nondegenerate Hessian);
- all saddle points of  $f_t$  are at the zero level (that is, lie on  $C_t$ ).

Real morsifications of totally real singularities have been successfully used to compute the monodromies and the intersection forms of plane curve singularities [1, 3, 38, 40].

**Proposition 1.11** [50, Lemma 2]. *The number of real hyperbolic nodes in any real nodal deformation of a real singularity  $(C, z)$  is at most*

$$\delta_{\mathbb{R}}(C, z) \stackrel{\text{def}}{=} \delta(C, z) - \text{ImBr}(C, z), \tag{1.3}$$

where  $\delta(C, z)$  is the  $\delta$ -invariant of the singularity, and  $\text{ImBr}(C, z)$  denotes the number of pairs of distinct complex conjugate local branches of  $C$  centered at  $z$ . Moreover, it is equal to  $\delta_{\mathbb{R}}(C, z)$  if and only if this real nodal deformation is a real morsification.

Thus, a real morsification is a real nodal deformation that has  $\delta_{\mathbb{R}}(C, z)$  real hyperbolic nodes, the maximal possible number (cf. Figure 1).

**Conjecture 1.12.** *Any real plane curve singularity possesses a real morsification.*

The totally real case of Conjecture 1.12 was settled long time ago in [1, Theorem 1; 39, Theorem 4]. Much more recently, Conjecture 1.12 was established in [50, Theorem 1] for a wide class of singularities that in particular includes all singularities that can be represented as a union of a totally real singularity with semi-quasi-homogeneous singularities having distinct non-real tangents.

**Example 1.13.** Consider the complex singularity with four smooth branches intersecting transversally at the point  $z = (0, 0)$  (cf. Example 1.1). Its three essentially distinct real forms, and their respective morsifications, are shown in Figure 2.

**Example 1.14.** Two morsifications of (different real forms of) the quasi-homogeneous singularity of type (4,2) (cf. Definition 1.2) are given by  $y^2 + x^4 = tx^2$  (a lemniscate) and  $(x^2 - t)^2 = y^2$  (two parabolas).



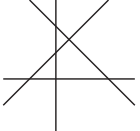
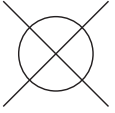
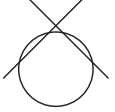
real singularity	morsifications	
$x^3y - xy^3 = 0$ (four real branches)	$xy(x - y + t)(x + y - 2t) = 0$	
$x^4 - y^4 = 0$ (two real branches, two complex conjugate branches)	$(x^2 - y^2)(x^2 + y^2 - t^2) = 0$	
$(x^2 + 4y^2)(4x^2 + y^2) = 0$ (two pairs of complex conjugate branches)	$(x^2 - (y - 1.2t)^2)(x^2 + y^2 - t^2) = 0$	

FIGURE 2 Three real forms of the singularity from Example 1.13, and their morsifications

*Remark 1.15.* The topology of complex singularities of the kind considered in this paper is completely characterized by certain combinatorial invariants defined either in terms of Puiseux expansions (or ‘resolution trees’, with multiplicities) or equivalently in terms of the topology of a certain link (the Burau–Zariski theorem [16, 76], see [15, Chapter 8]); cf. also Proposition 7.5.

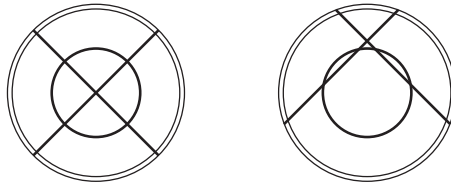
## 2 | DIVIDES

In this section, we recall and discuss the concept of a divide, introduced and extensively studied by A’Campo (see [4, 5, 42] and references therein). There are several versions of this notion in the existing literature; we will use the following one.

**Definition 2.1.** Loosely speaking, a *divide*  $D$  in a closed disk  $\mathbf{D} \subset \mathbb{R}^2$  is the image of a generic relative immersion of a finite set of intervals and circles into  $\mathbf{D}$ . More precisely, the images of immersed intervals and circles, collectively called the *branches* of  $D$ , must satisfy the conditions (D1)–(D6). In particular:

- (D1) the immersed circles do not intersect the boundary  $\partial\mathbf{D}$ ;
- (D2) the immersed intervals have pairwise distinct endpoints which lie on  $\partial\mathbf{D}$ ; moreover these immersed intervals intersect  $\partial\mathbf{D}$  transversally;
- (D3) all intersections and self-intersections of the branches are transversal;
- (D4) no triple (self-)intersections are allowed.

We are only interested in the topology of a divide. That is, we do not distinguish between divides related by a diffeomorphism between their respective ambient disks.



**FIGURE 3** Divides associated with two real morsifications of the real singularity  $x^4 - y^4 = 0$  shown in Figure 2 (second row). The circle  $\partial\mathbf{D}$  is represented by double lines. In each case, the two real branches  $x \pm y = 0$  get deformed into two immersed segments, and the two complex conjugate branches  $x \pm iy = 0$  get deformed into an immersed circle. Each divide has 4 regions and 5 nodes

The connected components of the complement  $\mathbf{D} \setminus D$  which are disjoint from  $\partial\mathbf{D}$  are the *regions* of  $D$ . The closure of the union of all regions and all singular points of  $D$  (its *nodes*) is called the *body* of the divide, denoted  $I(D)$ . We require that

- (D5) the body of the divide is connected, as is the union of its branches;
- (D6) each region is homeomorphic to an open disk.

In what follows, we don't always draw the boundary of the ambient disk  $\mathbf{D}$ .

**Definition 2.2.** Any real morsification  $(C_t)_{t \in [0, \tau]}$  of a real plane curve singularity  $(C, z)$  defines a divide in the following natural way. The sets  $\mathbb{R}C_t$  of real points of the deformed curves  $C_t$ , for  $0 < t < \tau$ , are all isotopic to each other in the ‘Milnor disk’  $\mathbf{D} = \mathbb{R}\mathbf{B} \subset \mathbb{R}^2$  consisting of the real points of the Milnor ball  $\mathbf{B}$ . Each real curve  $\mathbb{R}C_t \cap \mathbf{D}$ , viewed up to isotopy, defines the divide associated with the morsification. Conditions (D1)–(D4) and (D6) of Definition 2.1 are readily checked. Condition (D5) follows from the connectedness of the Dynkin diagram of a singularity [34] and from Guseĭn-Zade’s algorithm [39] that constructs this diagram from a divide (cf. Section 3).

A simple example is given in Figure 3.

Several examples of divides associated with morsifications of (various real forms of) quasi-homogeneous singularities  $x^a + y^b = 0$  are shown in Figure 4.

A few additional examples are given in Figures 5–7.

**Definition 2.3.** The divides arising via the construction of Definition 2.2 are called *algebraic*. Thus, an algebraic divide is a divide that comes from a real morsification of (a real form of) some complex isolated plane curve singularity.

Any divide  $D$  in which some proper subset of branches does not form a divide is not algebraic. In particular, if  $D$  contains two branches which are disjoint, then  $D$  is non-algebraic.

*Remark 2.4.* We are not aware of any (efficiently testable) necessary and sufficient conditions — even conjectural ones — ensuring that a given divide  $D$  represents

- a real morsification of a given real singularity; or
- a real morsification of some real form of a given complex singularity; or
- a real morsification of a real form of some complex singularity (that is,  $D$  is algebraic).









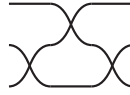

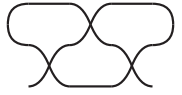
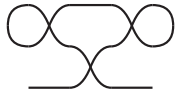



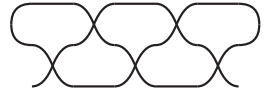

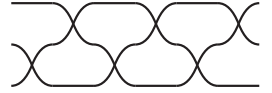
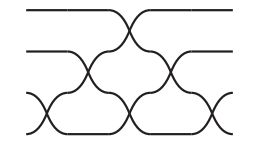
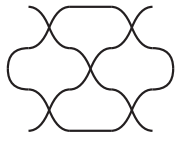
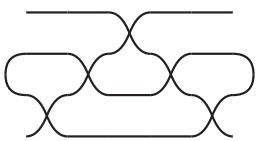
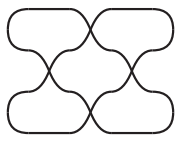
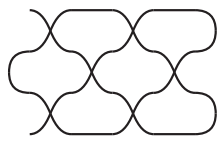
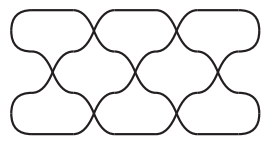
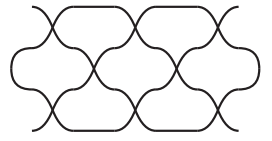
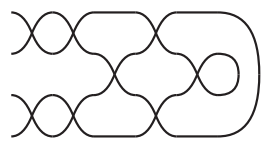

	$a=2$	$a=3$	$a=4$	$a=5$	$a=6$
$b=2$	  $A_1$ node	 $A_2$ cusp	  $A_3$ tacnode	 $A_4$	  $A_5$ order 3 tangency
$b=3$		  $D_4$ 3 lines	  $E_6$	   $E_8$	   $E_8^{(1,1)}$ 3 tangent branches
$b=4$			    $E_7^{(1,1)}$ 4 lines (cf. Fig. 2)		    2 cusps with the same tangent

FIGURE 4 Divides associated with real morsifications of (different real forms of) quasi-homogeneous singularities  $x^a + y^b = 0$ , for  $2 \leq b \leq a \leq 6$  and  $b \leq 4$

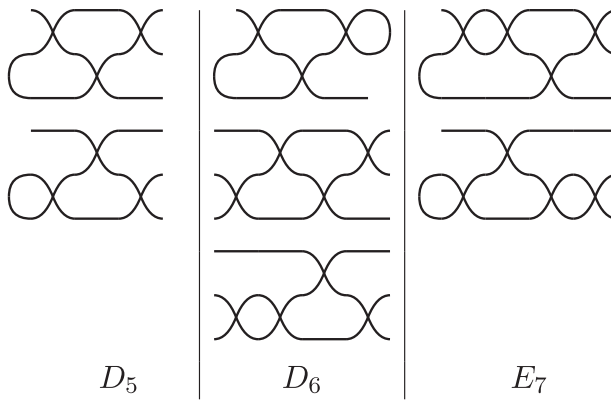


FIGURE 5 Divides associated with singularities of types  $D_5$  (a cusp and a transversal line),  $D_6$  (a tacnode and a transversal line), and  $E_7$  (a cusp and its cuspidal tangent)

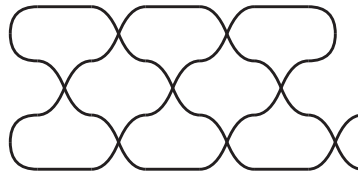


FIGURE 6 A divide associated with the non-quasihomogeneous singularity defined by the Puiseux parameterization  $y = x^{3/2} + x^{7/4}$  (see [23, Figure 31])

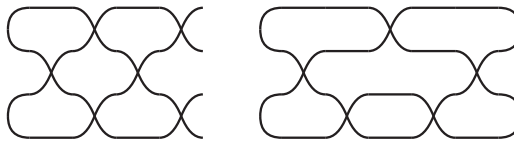


FIGURE 7 Two divides associated with two different real forms of the singularity  $(y^2 + x^3)(x^2 + y^3) = 0$  (two transversal cusps)

While a given real singularity typically has several inequivalent real morsifications, giving rise to distinct divides (cf., for example, Figure 3), some of the basic features of the resulting divide are uniquely determined by the real singularity at hand (see Propositions 2.5 and 2.6). Proofs, further details, and references can be found in [50].

**Proposition 2.5.** *The branches of a divide associated with a real morsification are obtained by deforming the local branches of the original real singularity. Each real local branch of the singularity deforms into an immersed interval with endpoints on the boundary of the Milnor disk. Each pair of distinct complex conjugate local branches deforms into an immersed circle in the interior of the Milnor disk.*

In particular, among algebraic divides, the ones corresponding to *totally real* singularities are precisely those which contain no closed curves.

**Proposition 2.6.** *Given a real plane curve singularity, the following collections of numbers do not depend on the choice of its morsification (or the associated divide):*

- the numbers of self-intersections of the individual branches of the divide;
- the numbers of intersections of the pairs of branches of the divide;
- the total number of regions in a divide.

*Specifically, the number of regions is equal to  $\mu(C, z) - \delta_{\mathbb{R}}(C, z)$ , where  $\mu(C, z)$  is the Milnor number of the singularity; and the aforementioned intersection numbers are determined by the  $\delta$ -invariants and the intersection and self-intersection numbers of the local branches.*

We note that while the numbers appearing in Proposition 2.6 do not depend on the choice of a morsification (or divide), they do depend on the choice of a real form of a particular complex singularity.

The importance of divides in the context of singularity theory stems from the fact that an algebraic divide completely determines the topological type of the underlying complex singularity. (It also contains some information concerning the real form at hand.) See [50] and references therein, as well as Remark 7.8.

Let  $D$  be a divide in a disk  $\mathbf{D}$ , and  $I(D)$  its body (cf. Definition 2.1). Since we assumed the regions to be homeomorphic to open disks, the body  $I(D)$  has a natural structure of a cell complex:

- the nodes of  $D$  are the 0-cells;
- the components of the set of nonsingular points of  $D$  which are disjoint from  $\partial\mathbf{D}$  are the 1-cells;
- the regions are the 2-cells.

If  $D$  is a hyperbolic node (that is, two embedded segments with a single transverse intersection), then  $I(D)$  is a single point. Otherwise  $I(D)$  is a connected (cf. (D5)) and simply connected 2-dimensional cell complex.

Propositions 1.11 and 2.6 imply the following statement.

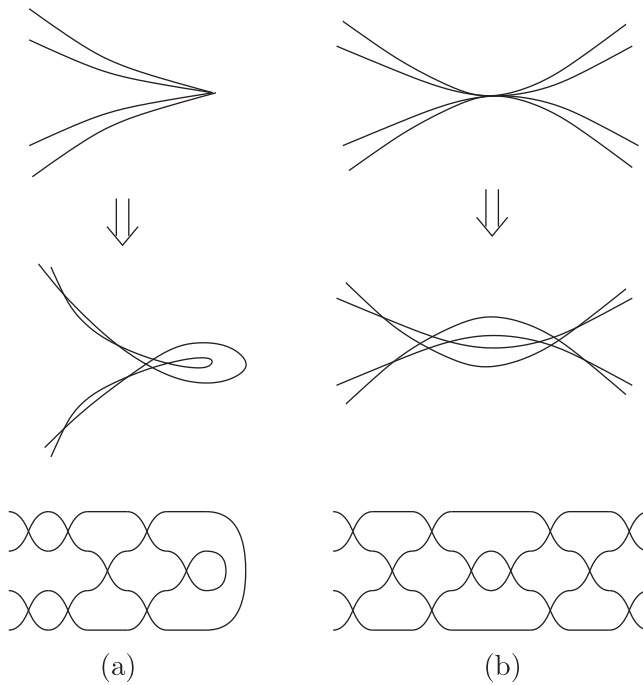
**Proposition 2.7.** *Let  $D$  be an algebraic divide. The sum of the number of 0-cells and the number of 2-cells of the cell complex  $I(D)$  is equal to the Milnor number of the associated singularity. In particular, this number does not depend on the choice of morsification, nor on the choice of the real form of the given complex singularity.*

**Example 2.8.** The three divides in the lower-right corner of Figure 4 correspond to morsifications of the following real forms of the same complex singularity:

- two complex conjugate cusps with the common tangent;
- two real cusps with the common tangent and opposite orientation;
- two co-oriented real cusps with the common tangent (cf. Figure 8a).

In each of the three cases, the combined number of nodes and regions is equal to 15, matching the Milnor number of the singularity.

*Remark 2.9.* For  $D$  an algebraic divide, the cell complex  $I(D)$  is not necessarily *regular*: the closure of a  $d$ -cell does not have to be a closed  $d$ -ball. Even if  $I(D)$  is regular, the intersection of the closures of two  $d$ -cells may be disconnected. Figure 8 (borrowed from [50]) illustrates each of these possibilities, for both  $d = 1$  and  $d = 2$ .



**FIGURE 8** (a) A real morsification of the singularity  $(y^2 + x^3)(y^2 + 2x^3) = 0$  (two co-oriented real cuspidal branches with the common cuspidal tangent) defined by  $(y^2 + x^2(x - \varepsilon_1))(y^2 + 2(x - \varepsilon_2)^2(x - \varepsilon_3)) = 0$ , with  $0 < \varepsilon_2 < \varepsilon_3 \ll \varepsilon_1 \ll 1$ , and the corresponding divide (cf. Figure 4,  $a = 6, b = 4$ , at the bottom). Here we see that the closure of a cell in  $I(D)$  does not have to be simply connected. (b) A real morsification of the real quasi-homogeneous singularity of type  $(8,4)$  given by  $(y^2 - x^4)(y^2 - 2x^4) = 0$  (four real smooth branches quadratically tangent to each other), and its divide. Here we see that the intersection of two  $d$ -cells may be disconnected, for  $d = 1, 2$

### 3 | A’CAMPO–GUSEIN–ZADE DIAGRAMS

In this section, we review the basics of  $A\Gamma$ -diagrams, originally introduced by A’Campo [1] and Gusein-Zade [38]. These diagrams also appeared in the literature under other names: Coxeter–Dynkin diagrams of singularities,  $R$ -diagrams, and so on. See [11] for another overview of this construction, and for additional references.

Two regions of a divide are called *adjacent* if the intersection of their closures contains a 1-cell (which is said to *separate* these two regions).

**Definition 3.1.** Given a divide  $D$  as in Definition 2.1, its A’Campo–Gusein–Zade diagram  $A\Gamma(D)$  ( $A\Gamma$ -*diagram* for short) is a vertex-colored graph constructed as follows:

- place a vertex at each node of  $D$ , and color it black;
- place one vertex into each region of  $D$ ; color these vertices  $\oplus$  or  $\ominus$  so that adjacent regions receive different colors (signs), and non-adjacent regions sharing a node receive the same color;
- for each 1-cell separating two regions, draw an edge connecting the vertices located inside these regions;
- for each region  $R$ , say bounded by  $k$  1-dimensional cells, draw  $k$  edges connecting the nodes on the boundary of  $R$  to the vertex located inside  $R$ ; these edges correspond to the  $k$  distinct

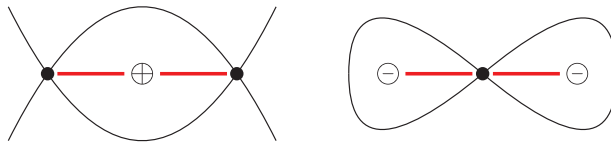


FIGURE 9 Two divides of type  $A_3$ , and their associated  $\text{AF}$ -diagrams

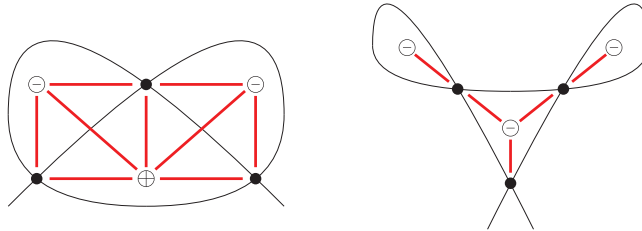


FIGURE 10 Two divides of type  $E_6$ , and their associated  $\text{AF}$ -diagrams

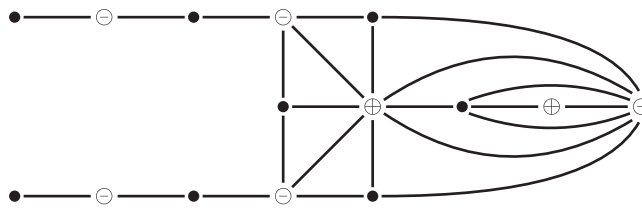


FIGURE 11 The  $\text{AF}$ -diagram for the divide/morsification shown in Figure 8(a)

(up to isotopy) ways to draw a simple curve contained in  $R$  (except for one of the endpoints) connecting the interior vertex to a boundary node.

Figures 9 and 10 show  $\text{AF}$ -diagrams of divides associated with different real morsifications of real singularities of types  $A_3$  and  $E_6$ , respectively.

*Remark 3.2.* The last rule in Definition 3.1 allows for the possibility of double edges in case the closure of  $R$  is not simply connected. For example, this situation arises in the  $\text{AF}$ -diagram associated with the morsification in Figure 8a (see Figure 11).

Definition 3.1 specifies the coloring of the vertices in the  $\text{AF}$ -diagram up to a global change of sign. This coloring is *proper*: every edge in  $\text{AF}(D)$  connects vertices of different color. Thus,  $\text{AF}(D)$  is a *tripartite* graph.

*Remark 3.3.* Any  $\text{AF}$ -diagram is a (vertex-colored) *planar* graph. Although its construction given in Definition 3.1 supplies an embedding of this graph into the real plane, the notion of an  $\text{AF}$ -diagram does *not* include a choice of a planar embedding. Moreover, a given  $\text{AF}$ -diagram can have two non-homeomorphic planar embeddings, and can correspond to two topologically distinct divides (see, for example, [11, Figure 4]). We do not know whether this can happen for algebraic divides.



For an algebraic divide  $D$  coming from a real morsification of a real singularity, the vertices of the  $AG$ -diagram  $AG(D)$  correspond to the critical points of the morsified curve  $C_t = \{f_t(x, y) = 0\}$ . Furthermore, one can choose the coloring so that

- the vertices colored  $\oplus$  are located in the regions where  $f_t > 0$ , and correspond to the local maxima of  $f_t$ ;
- the vertices colored  $\ominus$  are located in the regions where  $f_t < 0$ , and correspond to the local minima of  $f_t$ ;
- the black vertices are located on the curve  $f_t = 0$ , and correspond to the saddle points of  $f_t$ .

By Proposition 2.7, the number of vertices in  $AG(D)$  is equal to the Milnor number of the singularity.

**Theorem 3.4** [50]. *The  $AG$ -diagram of a real morsification of a real isolated plane curve singularity determines the complex topological type of the singularity.*

We emphasize that Theorem 3.4 does not require the knowledge of a specific planar embedding of the  $AG$ -diagram (cf. Remark 3.3).

In the case of totally real singularities, Theorem 3.4 was proved by Balke and Kaenders [11, Theorem 2.5 and Corollary 2.6] under an additional assumption concerning the topology of the intersections of cell closures in  $I(D)$ ; this is related to the discussion in Remark 2.9.

## 4 | QUIVERS

**Definition 4.1.** A *quiver* is a finite directed graph. Oriented cycles of length 1 or 2 are not allowed. In other words, there must be no loops, and all arrows between a given pair of vertices must have the same direction. We do not distinguish between quivers (on the same vertex set) which differ by simultaneous reversal of the direction of all arrows.

We will not need a more general notion of quivers with ‘frozen’ vertices, just the simple setup described in Definition 4.1.

Throughout this paper, we use the standard Dynkin diagram nomenclature, along with Saito’s notation for the extended affine exceptional types (cf. [30, Section 12]), to assign names to some of the quivers appearing in various examples (cf. in particular Figure 4).

**Definition 4.2.** Given a divide  $D$ , its associated *quiver*  $Q(D)$  is constructed from the  $AG$ -diagram  $AG(D)$  as follows:

- first, orient the edges of  $AG(D)$  using the rule  $\bullet \rightarrow \oplus \rightarrow \ominus \rightarrow \bullet$ ;
- then remove the marking of the vertices.

Since we consider quivers up to global reversal of arrows, the choice of signs in the  $AG$ -diagram does not matter.

Examples of quivers associated with divides coming from morsifications can be found in Figure 12 (compare with Figure 10) and Figure 13 (cf. Figure 2).

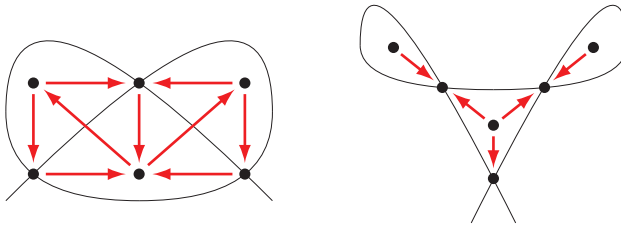


FIGURE 12 Quivers associated with divisors of type  $E_6$

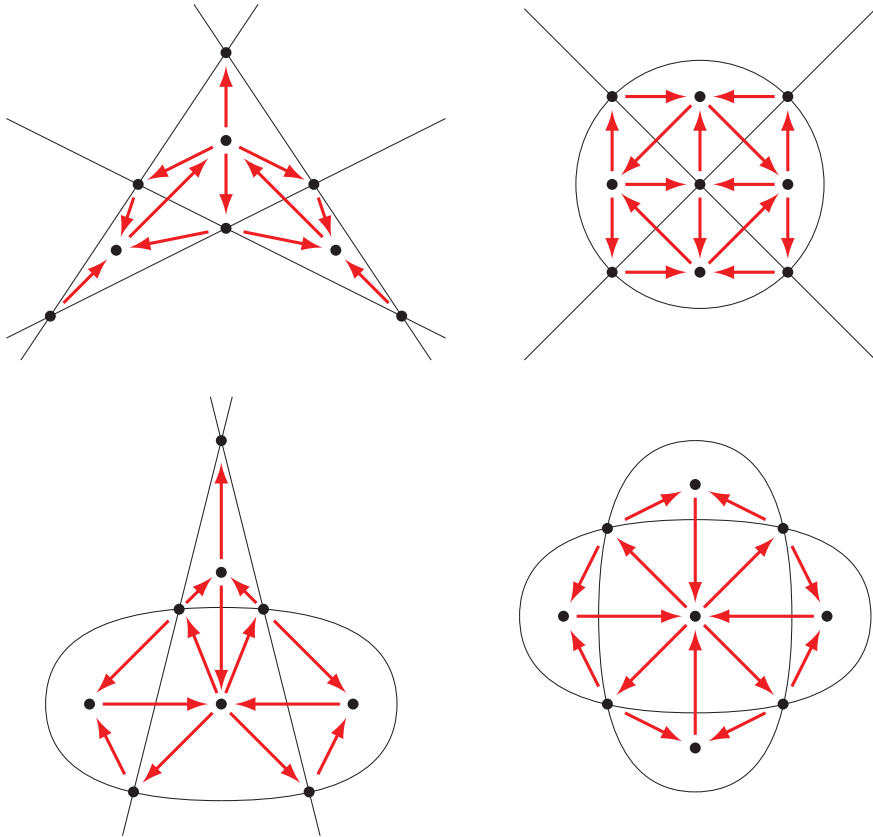


FIGURE 13 The quivers associated with morsifications from Figure 2

*Remark 4.3.* While the quiver  $Q(D)$  is very closely related to the AF-diagram  $\text{AF}(D)$ , some information is lost in the transition from  $\text{AF}(D)$  to  $Q(D)$ . For example, although the AF-diagrams shown in Figure 9 are different from each other, the corresponding quivers are both isomorphic to the quiver  $\bullet \rightarrow \bullet \leftarrow \bullet$ .

In the case of algebraic divisors, Theorem 3.4 can be used to establish the following result.

**Theorem 4.4.** *Let  $Q = Q(D)$  be a quiver constructed from an algebraic divisor  $D$  corresponding to a real morsification of a real isolated plane curve singularity. Then  $Q$  uniquely determines the complex topological type of the singularity.*

To prove this theorem, we will need the following lemma.

**Lemma 4.5.** *Let  $D$  be any divide (not necessarily algebraic). Denote*

$$\begin{aligned} \rho &= \text{number of regions in } D, \\ \nu &= \text{number of nodes in } D, \text{ and} \\ \iota &= \text{number of interval branches in } D. \end{aligned}$$

*Then  $\nu = \rho + \iota - 1$ . In particular,  $\nu \geq \rho - 1$ .*

*Proof.* Let  $K(D) = \overline{D \cup I(D)}$ , the closure of the union of  $D$  and its body  $I(D)$  (see Definition 2.1). Since  $K(D)$  is contractible, its Euler characteristic is equal to 1. Let  $\varepsilon$  denote the number of 1-cells in  $K(D)$ . Then  $2\varepsilon = 4\nu + 2\iota$ , so  $\varepsilon = 2\nu + \iota$ . Hence,

$$1 = \chi(K(D)) = (\nu + 2\iota) - (2\nu + \iota) + \rho = \iota - \nu + \rho,$$

as desired. □

*Proof of Theorem 4.4.* In light of Theorem 3.4, all we need to do is reconstruct the  $\{\bullet, \oplus, \ominus\}$ -coloring of the vertices of the quiver  $Q = Q(D)$ . More precisely, we aim to reconstruct the coloring up to a global color switch  $\oplus \leftrightarrow \ominus$ , which corresponds to changing the sign of the morsified function  $f_t$  (see Section 3) and consequently does not affect the topological type. As we shall demonstrate, this reconstruction can be accomplished in all but a few exceptional cases; treating each of those cases separately, we will show that the quiver  $Q$  determines the topology of the singularity.

The requisite coloring of the vertices of  $Q$  must obey the cyclic orientation rule  $\bullet \rightarrow \oplus \rightarrow \ominus \rightarrow \bullet$ , (or  $\bullet \rightarrow \ominus \rightarrow \oplus \rightarrow \bullet$ , after a global reversal of arrows). This enables us to determine the coloring of the vertices into three colors  $\{1, 2, 3\}$ , without specifying a bijective identification  $\{1, 2, 3\} \leftrightarrow \{\bullet, \oplus, \ominus\}$ . Namely, assign color 1 to some vertex  $v$ , then propagate the coloring away from  $v$  following the cyclic rule  $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ .

At this stage, we would like to determine which of the three colors 1,2,3 is black, that is, corresponds to the nodes of  $D$ ; the other two colors would correspond to  $\oplus$  and  $\ominus$  (the regions of the divide).

Let  $c_1, c_2$ , and  $c_3$  denote the number of vertices in  $Q$  which have color 1, 2, or 3, respectively. Without loss of generality, we may assume that  $c_1 \leq c_2 \leq c_3$ .

*Case 1:*  $c_1 = 0$ . That is, at most two colors are present. One of them must correspond to the nodes. We conclude that all regions of  $D$  must have the same color; every node is adjacent to at most two regions; and the  $AG$ -diagram cannot have cycles. Thus,  $Q$  is an oriented tree. If  $Q$  has a vertex  $v$  of degree  $\geq 3$ , then  $v$  is not a node, and we are done. Otherwise,  $Q$  is a chain (with an alternating orientation), so we are dealing with a singularity of type  $A$  (cf. Figure 4, top row ( $b = 2$ )).

*Case 2:*  $c_1 \geq 2$ . The inequality  $\nu \geq \rho - 1$  (see Lemma 4.5) then implies that the number of nodes  $\nu$  is strictly greater than the number of regions of either color, so we can identify which color is black.

*Case 3:*  $c_1 = 1$ . That is, each color is present, and there is a unique vertex  $v$  of color 1. This case splits into two subcases.

Case 3A:  $c_1 = 1$  and  $c_3 > c_2$ . Then color 3 must be black (otherwise  $\nu \leq c_2 \leq c_3 - 1 \leq \rho - 2$ , contradicting Lemma 4.5), and we are done.

Case 3B:  $c_1 = 1$  and  $c_2 = c_3$ . Let  $Q_v$  denote the induced subquiver of  $Q$  obtained by removing  $v$  together with all incident arrows. Within each connected component  $Q'$  of  $Q_v$ , the vertices connected to  $v$  in  $Q$  form a connected subquiver  $Q'' \subset Q'$  with all degrees  $\leq 2$ . Now, there are two possibilities.

Case 3B(i): Each subquiver  $Q''$  as above is a chain. Then its endpoints must be nodes, and we are done.

Case 3B(ii):  $Q_v$  is connected, and  $Q'' \subset Q' = Q_v$  is a cycle, necessarily with alternating orientation. If there is a vertex  $u$  in  $Q''$  connected to a vertex not belonging to the ‘wheel’  $Q'' \cup \{v\}$ , then  $u$  is a node. Otherwise,  $Q$  is the wheel quiver on  $2m + 1$  vertices, for some  $m \in \mathbb{Z}_{>0}$ ; cf. Figure 13 (lower right), with  $m = 4$ . Although in this case, we cannot uniquely determine which color on the periphery of the wheel is black, the two choices produce isomorphic  $A\Gamma$ -diagrams, so Theorem 3.4 applies. □

Theorem 4.4 implies that any topological invariant of an isolated plane curve singularity is uniquely determined by the quiver of an arbitrary real morsification. While for particular invariants, this may be a very challenging task (cf. also Remark 7.9), there are a couple of cases where the answer is relatively easy. To discuss them, we need to recall the following standard notion.

**Definition 4.6.** Let  $Q$  be a quiver with the vertex set  $V$  of size  $n$ . The skew-symmetric matrix associated with  $Q$  is the  $n \times n$  matrix  $B(Q) = (b_{ij})_{i,j \in V}$  defined by

$$b_{ij} = \begin{cases} \text{number of arrows } i \rightarrow j \text{ in } Q & \text{if } Q \text{ contains such arrows;} \\ -(\text{number of arrows } j \rightarrow i \text{ in } Q) & \text{if } Q \text{ contains such arrows;} \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 4.7.** Given a real morsification of an isolated plane curve singularity  $(C, z)$ , let  $Q$  be its quiver, and let  $B = B(Q)$  the corresponding skew-symmetric matrix. Then

- the Milnor number  $n = \mu(C, z)$  is the size of the matrix  $B$ ;
- the number  $r$  of complex local branches is given by  $r = n - \text{rank}(B) + 1$ ;
- the  $\delta$ -invariant of the singularity is given by  $\delta = n - \frac{1}{2} \text{rank}(B)$ .

*Proof.* The first statement follows from Proposition 2.7.

The  $n \times n$  skew-symmetric matrix  $B(Q)$  is the intersection matrix of 1-cycles in the Milnor fiber  $M$ . The latter is a surface with  $r$  holes, where  $r$  is the number of complex local branches. More precisely: given a pair of 1-cycles  $a, b \in H_1(M)$ , map one of them, say  $b$ , to the relative group  $H_1(M, \partial M)$ ; then apply the (non-degenerate) Poincare pairing  $H_1(M) \times H_1(M, \partial M) \rightarrow \mathbb{Z}$ . The defect in the rank comes from the fact that the homomorphism  $H_1(M) \rightarrow H_1(M, \partial M)$  has  $(r - 1)$ -dimensional kernel, which can be seen from the exact homology sequence

$$H_2(M, \partial M) \rightarrow H_1(\partial M) \rightarrow H_1(M) \rightarrow H_1(M, \partial M).$$

Finally, by the Milnor formula [54],  $n = 2\delta - r + 1$ , and the last claim follows. □

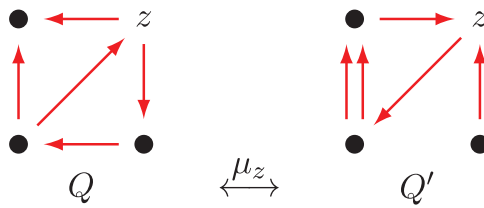


FIGURE 14 Two quivers related by a quiver mutation at the vertex  $z$

### 5 | MAIN CONJECTURE

Although by Theorem 4.4, all information about the topology of a complex plane curve singularity is encoded in the quiver constructed from its real morsification, this does not yield a satisfactory topological classification of such singularities. One reason is that we do not know which quivers (or which divides) can arise from such morsifications — this problem is wide open, and likely hopeless. Another, more practical problem has to do with deciding whether two singularities are isomorphic to each other or not. The same singularity will typically have many real forms; each of them will have many topologically different morsifications, each with its own quiver. What do all these quivers have in common? For example, Figure 13 shows four quivers arising from morsifications of different real forms of the same singularity, namely the quasi-homogeneous singularity of type  $(4,4)$ . What features set these four quivers apart from all other quivers arising from similar constructions?

To rephrase, how can we tell, looking at two quivers associated to morsifications of two isolated plane curve singularities, whether these singularities are topologically the same? In light of Theorem 4.4, this should in principle be possible.

The conjectural answer to the last question (see Conjecture 5.5) comes from the concept of quiver mutation, which we shall now recall. While quiver mutations play a fundamental role in the theory of cluster algebras, we will not rely on any results from this theory, such as, for example, the Laurent Phenomenon [28] or the finite type classification [29]. We refer the interested reader to [31, 74] for further details, examples, and motivation.

**Definition 5.1.** Given a vertex  $z$  in a quiver  $Q$ , the *quiver mutation* at  $z$  is a transformation of  $Q$  into a new quiver  $Q' = \mu_z(Q)$  constructed in three steps.

1. For each path  $x \rightarrow z \rightarrow y$  of length 2 passing through  $z$ , introduce a new edge  $x \rightarrow y$ .
2. Reverse the direction of all edges incident to  $z$ .
3. Remove oriented 2-cycles, one by one (Figure 14).

**Definition 5.2.** Two quivers  $Q$  and  $Q'$  are called *mutation equivalent* if  $Q$  can be transformed into a quiver isomorphic to  $Q'$  by a sequence of mutations. It is easy to see that quiver mutation is involutive (that is,  $\mu_z(\mu_z(Q)) = Q$ ), and consequently mutation equivalence is indeed an equivalence relation.

**Example 5.3.** The two quivers in Figure 12 are mutation equivalent to each other. This is an instance of a general phenomenon discussed in Section 14: divides related via triangle moves have mutation equivalent quivers.

*Remark 5.4.* The problem of deciding whether two given quivers are mutation equivalent or not is notoriously difficult. Furthermore, there is a dearth of known invariants of quiver mutation, even though experimental evidence strongly suggests that many independent invariants must exist.

**Conjecture 5.5** (Main conjecture). *Given two real morsifications of real isolated plane curve singularities, the following are equivalent:*

- *the two singularities have the same complex topological type;*
- *the quivers associated with the two morsifications are mutation equivalent.*

To rephrase, Conjecture 5.5 asserts that isolated plane curve singularities are topologically classified by the mutation classes of associated quivers. Put another way:

- different morsifications of (different real forms of) the same complex plane curve singularity have mutation equivalent quivers;
- morsifications of (real forms of) topologically different complex plane curve singularities have quivers which are not mutation equivalent to each other.


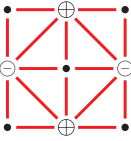
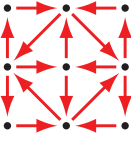
Figure 15 illustrates the essence of Conjecture 5.5, and the relationships between its various ingredients.

**Example 5.6.** Recall that Figure 13 shows the quivers associated with four different morsifications of the same complex singularity, the quasi-homogeneous singularity of type (4,4) (cf. Figure 2). It is not hard to verify (with the help of one of the widely available software tools for quiver mutations) that these four quivers are mutation equivalent to each other, in agreement with Conjecture 5.5. Moreover, it can be shown that any real morsification of a complex singularity that is topologically inequivalent to the one referenced above gives rise to a quiver which is not mutation equivalent to these four quivers — again agreeing with Conjecture 5.5.

**Example 5.7.** For each cell  $(a, b)$  of the table in Figure 4, the quivers associated to the divides shown therein are mutation equivalent to each other. Moreover, the quivers appearing in different cells of the table are not mutation equivalent to each other.

*Remark 5.8.* Conjecture 5.5 only applies to quivers which are already known to have come from morsifications. We note that for a typical singularity, the corresponding mutation equivalence class contains infinitely many pairwise non-isomorphic quivers; among them, the quivers associated with real morsifications of a given singularity form a finite subset. Even if the mutation equivalence class is finite, most quivers appearing in it do not arise from morsifications. For example, the mutation class of a quiver of type  $A_n$  (that is, an arbitrary orientation of a Dynkin diagram of type  $A_n$ ) contains exponentially many (as a function of  $n$ ) pairwise nonisomorphic quivers (see, for example, [71]); among them, at most two come from morsifications of a type  $A_n$  singularity, see Proposition 18.7 or the top ( $b = 2$ ) row of Figure 4.

*Remark 5.9.* Conjecture 5.5, once established, would imply that any topological invariant  $\alpha = \alpha(C, z)$  of an isolated plane curve singularity  $(C, z)$  is uniquely determined by the mutation equivalence class of the quiver  $Q$  of some (equivalently, any) real morsification of the singularity  $(C, z)$ . Viewing  $\alpha$  as a function of  $Q$ , we conclude that this function must take the same value at all quivers in a given mutation equivalence class which are known to come from a morsification. Put differ-

general concept	illustration
complex singularity $\uparrow$	four smooth branches intersecting transversally at the point $(0, 0) \in \mathbb{C}^2$
real singularity $\uparrow$	$x^4 - y^4 = 0$ (two real branches and two complex conjugate branches)
morsification $\downarrow$	$(x^2 - y^2)(x^2 + y^2 - t^2) = 0$
divide $\downarrow$	
AΓ-diagram $\downarrow$	
quiver $\downarrow$	
mutation class $\downarrow$	$E_7^{(1,1)}$
cluster algebra	Grassmannian $\text{Gr}_{4,8}(\mathbb{C})$

Conjecture 5.5

**FIGURE 15** Unpacking Conjecture 5.5. A complex plane curve singularity has at least one real form. According to Conjecture 1.12, each of these real singularities has a real morsification. A morsification defines a divide. A divide has the associated AΓ-diagram. The AΓ-diagram produces a quiver. The quiver determines a mutation equivalence class (which can in turn be used to define a cluster algebra or category). Conjecture 5.5 asserts that this mutation class and the topology of the original complex singularity uniquely determine each other



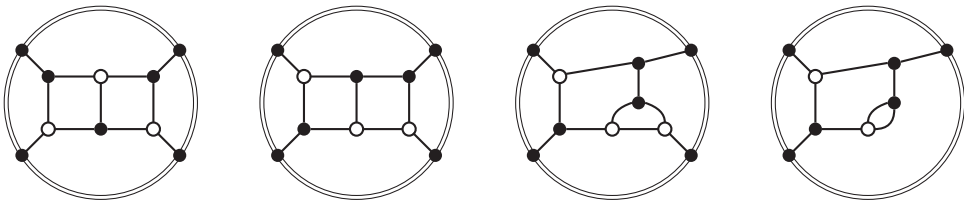


FIGURE 16 Plabic graphs. The first two graphs are related by a square move; the second and the third by a flip move; the third and the fourth by tail removal

ently,  $\alpha(Q)$  should be (a restriction of) a mutation-invariant function of quivers. It would be very interesting to understand the combinatorial meaning of  $\alpha(Q)$  for various well-studied topological invariants  $\alpha$ .

To illustrate, let us recall that Proposition 4.7 provided direct descriptions of three topological invariants of a singularity in terms of a quiver  $Q$  constructed from its real morsification. These three invariants (the Milnor number, the number of complex local branches, and the  $\delta$ -invariant) are all expressed as functions of the number of vertices in  $Q$  (which is obviously a mutation invariant) and the rank of  $B(Q)$ , the skew-symmetric matrix associated with the quiver  $Q$ . It is well-known (see [12, Lemma 3.2]) that the rank of  $B(Q)$  is invariant under mutations of  $Q$ , so the formulae of Proposition 4.7 are mutation invariant, as expected.

## 6 | PLABIC GRAPHS

Plabic graphs were introduced by A. Postnikov [62, Section 12], who used them to describe parameterizations of cells in totally nonnegative Grassmannians. We review the basic notions of this construction below, adapting it for our current purposes. The differences between our setting and Postnikov’s are discussed in Remark 6.6.

**Definition 6.1.** A finite connected planar graph  $P$  properly embedded into a disk  $\mathbf{D}$  (as a 1-dimensional cell complex) is called a *plabic graph* if

- each vertex of  $P$  is colored in one of the two colors, either black or white; the coloring does not have to be proper;
- each vertex of  $P$  lying in the interior of  $\mathbf{D}$  is trivalent (that is, has degree 3);
- each vertex of  $P$  lying on the boundary  $\partial\mathbf{D}$  is univalent (that is, has degree 1);
- each internal face of  $P$  (that is, a face not adjacent to  $\partial\mathbf{D}$ ) is separated from at least one other internal face by an edge whose endpoints have different colors. (This condition does not apply if  $P$  has a single internal face.)

We view plabic graphs up to isotopy, and up to *color reversal*, which switches the color of all vertices in  $P$ . Examples are shown in Figure 16.

There are several types of transformations of plabic graphs which play an important role in this theory. First, there are three types of local moves.

**Definition 6.2.** Local *moves* on plabic graphs are defined as follows (cf. Figure 17):

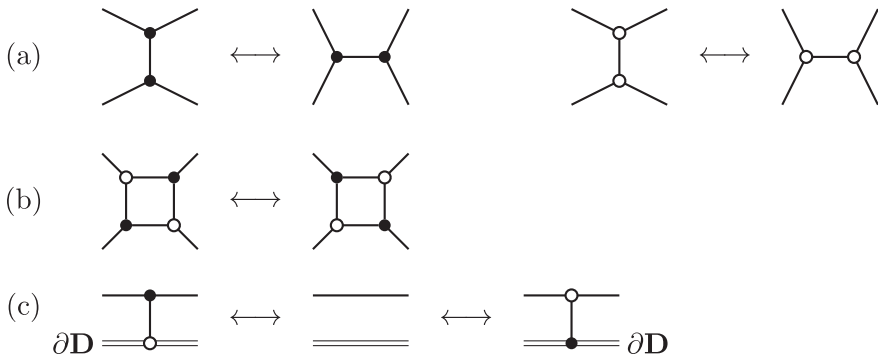


FIGURE 17 Local moves in plabic graphs. (a) The flip move (two versions). (b) The square move. (c) The tail attachment/removal moves. For this last type of move, the colors of the two vertices involved can be arbitrary

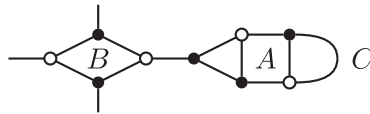


FIGURE 18 A fragment of a plabic graph. The square move is allowed at  $A$ , but not at  $B$ , because face  $C$  is adjacent to two consecutive sides of  $B$

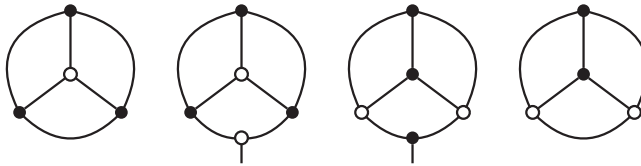
- The *flip* move replaces two adjacent trivalent vertices of the same color with two other vertices of the same color, connected in a different way.
- The *square* move switches the colors on a 4-cycle of vertices of alternating colors, subject to Restriction 6.3. (This restriction is rarely relevant, so a casual reader may skip this technical detail.)
- The *tail removal* move removes an edge  $e$  (a ‘tail’) connected at one end to  $\partial\mathbf{D}$  and at the other end to a trivalent vertex  $v$ . After removing  $e$ , we also remove  $v$ , and merge the two remaining edges which were incident to  $v$ . The reverse move, called *tail attachment*, inserts a vertex  $v$  (of any color) into an edge bordering a region  $R$  adjacent to  $\partial\mathbf{D}$ , and adds a new edge  $e$  connecting  $v$  across  $R$  to a vertex (of any color) lying on  $\partial\mathbf{D}$ .

Two plabic graphs related via a sequence of local moves are called *move equivalent* (see Figure 16).

**Restriction 6.3.** We impose a restriction on the square move of Figure 17(b): among the four faces surrounding the square, the opposite ones are allowed to coincide, but the consecutive ones must be distinct (see Figure 18).

*Remark 6.4.* Using a tail removal followed by a tail attachment, one can change the color of any boundary vertex (or the vertex connected to it). For this reason, when drawing a plabic graph, we sometimes do not show the boundary of the ambient disk, and accordingly do not specify the colors of boundary vertices.

*Remark 6.5.* Applying repeated tail removals, any plabic graph can be transformed into a trivalent one, with no vertices on the boundary  $\partial\mathbf{D}$ . Note however that restricting the setup to trivalent



**FIGURE 19** The trivalent plabic graphs on the left and on the far right are related by a sequence of three moves: a tail attachment, a square move, and a tail removal. These two trivalent graphs are not related via flip and square moves alone, since such moves do not change the number of vertices of each color

plabic graphs would have resulted in a different equivalence relation among them, for the reasons explained in Figure 19.

*Remark 6.6.* As explained by Postnikov [62], Definitions 6.1 and 6.2 naturally extend to arbitrary planar graphs embedded in a disk. We find it easier, for our current purposes, to work in the restricted generality of trivalent-univalent graphs. We also require that for each internal face, there is a black-and-white edge separating it from another internal face. This condition, which propagates under all types of moves, ensures that the moves do not create monogons, nor digons with vertices of the same color.

A more significant difference between our setting and Postnikov's is the introduction of the tail attachment/removal moves, which were not present in [62].

*Remark 6.7.* A plabic graph defines a dual triangulation of the disk  $\mathbf{D}$ , with each triangle colored black or white. A flip move in a plabic graph corresponds to a flip in the dual triangulation (hence the terminology), that is, to removing an interior arc  $\alpha$  and replacing it by another 'diagonal' of the quadrilateral region formed by the two triangles separated by  $\alpha$ . Note that we are only allowed to do this when the triangles are of the same color. Incidentally, this process will never create self-folded triangles (in the terminology of [30]) since those correspond to monogons in a plabic graph.

It is well-known that local moves on plabic graphs are a special case of quiver mutation. To see this, one needs the following definition.

**Definition 6.8.** The quiver  $Q(P)$  associated with a plabic graph  $P$  is constructed as follows. Place a vertex of  $Q(P)$  into each internal face of  $P$ . For each edge  $e$  in  $P$  such that

- the endpoints of  $e$  are of different color, and
- the faces  $F_1, F_2$  on the two sides of  $e$  are internal and distinct,

draw an arrow of  $Q(P)$  across  $e$  connecting the vertices of  $Q(P)$  located inside the faces  $F_1$  and  $F_2$ , and orient this arrow so that the black endpoint of  $e$  appears on its right as one moves in the chosen direction. If this construction produces oriented cycles of length 2, that is, pairs of arrows connecting the same vertices but going in opposite directions, then remove such pairs, one by one (see Figure 20).

We note that the colors of boundary vertices do not affect the quiver.

The following observation is implicit in Postnikov's original work [62].

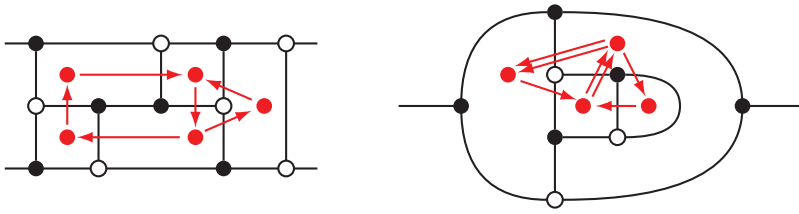


FIGURE 20 Quivers associated with plabic graphs. The double arrows in the right quiver correspond to the instances where a pair of faces of the plabic graph share two disconnected boundary segments

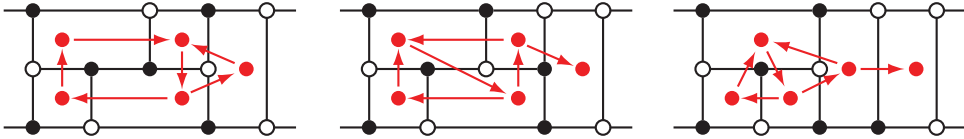


FIGURE 21 The first two plabic graphs are related by a square move; their quivers are obtained from each other by a single mutation. The second and the third plabic graphs are related by a flip move, and have isomorphic quivers

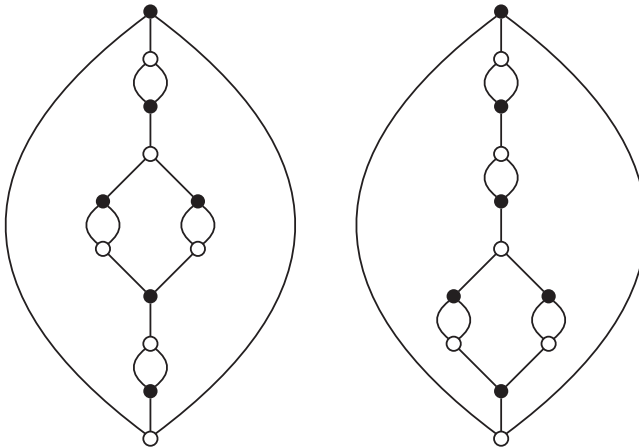


FIGURE 22 Two plabic graphs whose quivers are isomorphic but which are not related to each other by local moves. In fact, the only moves that can be applied to either graph are tail attachments/removals

**Proposition 6.9.** *If two plabic graphs are move equivalent to each other, then their associated quivers are mutation equivalent.*

*Proof.* It is straightforward to check that a square move in a plabic graph translates into a quiver mutation, and that the quiver associated with a plabic graph does not change under a flip move, or a tail attachment/removal (see Figure 21). □

*Remark 6.10.* The converse to Proposition 6.9 is unfortunately false: there exist plabic graphs which are not move equivalent even though their quivers are isomorphic (hence mutation equivalent). An example is shown in Figure 22 (see also Example 9.7/Figure 25).

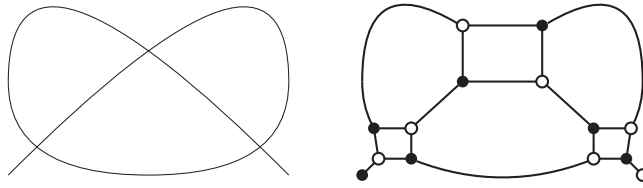


FIGURE 23 A divide coming from a morsification of a type  $E_6$  singularity, and one of the plabic graphs attached to it

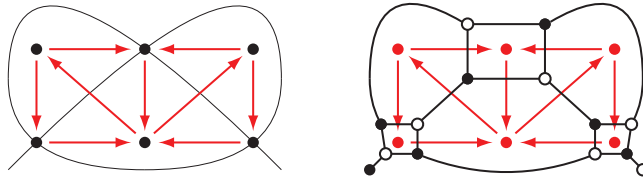
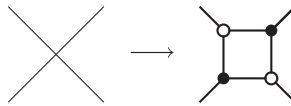


FIGURE 24 Left: the quiver obtained from the AF-diagram (cf. Figure 12). Right: the quiver obtained from the plabic graph (cf. Figure 23)

We next relate plabic graphs to divides.

**Definition 6.11.** The set  $\mathbf{P}(D)$  of *plabic graphs attached to a divide*  $D$  is defined as follows. Replace each node of  $D$  by a ‘roundabout’ involving four trivalent vertices connected into a square, and colored alternately black and white, as shown below:



There are two choices of coloring at each node, related to each other by a square move. We then color the endpoints in  $D \cap \partial \mathbf{D}$  in an arbitrary way. The set  $\mathbf{P}(D)$  consists of the plabic graphs which can be obtained from the divide  $D$  via this procedure. All plabic graphs in  $\mathbf{P}(D)$  are obviously move equivalent to each other.

An example is shown in Figure 23.

Definition 6.11 is justified by the following simple but important observation.

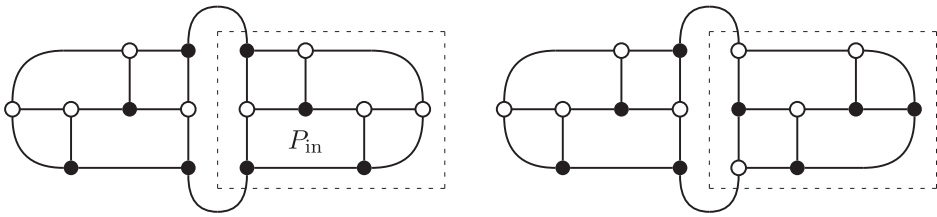
**Proposition 6.12.** For any divide  $D$  and any plabic graph  $P \in \mathbf{P}(D)$  attached to  $D$ , the quivers  $Q(D)$  and  $Q(P)$  are mutation equivalent to each other.

In fact, there is always a choice of  $P \in \mathbf{P}(D)$  such that  $Q(D) = Q(P)$ .

(See Definitions 4.2, 6.8, and 6.11 for the explanations of the notations involved.)

In other words, the quiver of a plabic graph attached to a divide  $D$  is the same (up to mutation equivalence) as the quiver associated with  $D$  (that is, the oriented AF-diagram of  $D$ ; see Figure 24).

Experimental evidence suggests that in the case of plabic graphs attached to algebraic divides, the converse to Proposition 6.9 holds (cf. Remark 6.10).



**FIGURE 25** Plabic graphs related by a switch. The dotted line represents a simple closed curve  $C$ . The portions outside  $C$  are the same. The portions inside  $C$  are related by flipping upside down and changing the colors of all vertices

**Conjecture 6.13.** *Plabic graphs attached to algebraic divides are move-equivalent if and only if the corresponding quivers are mutation equivalent.*

*Remark 6.14.* By Proposition 6.12, it does not matter whether the words ‘the corresponding quivers’ appearing in Conjecture 6.13 are interpreted as ‘the quivers associated with the divides’ or as ‘the quivers associated with the plabic graphs’.

We conclude this section by describing an equivalence relation on (arbitrary) plabic graphs that conjecturally corresponds to mutation equivalence of their quivers. The readers not interested in this digression may proceed directly to the next section.

The key idea is to complement Postnikov’s local moves by certain non-local transformations which do not change the quiver associated with a plabic graph. These transformations are closely related to Whitney’s 2-switching operations which relate different planar embeddings of a given planar graph (see, for example, [55, Section 2.6]). See also Remark 3.3.

**Definition 6.15.** We say that two plabic graphs  $P$  and  $P'$  are related to each other by a *switch* if  $P'$  can be obtained from  $P$  in the following way. Suppose a closed simple curve  $C$  in the interior of the disk  $\mathbf{D}$  intersects (the drawing of)  $P$  exactly twice, at two different edges. Since we consider our plabic graphs up to isotopies of the disk, we may assume, without loss of generality, that  $C$  encloses a rectangle  $R$ , and moreover  $C$  intersects  $P$  at two points located at the top and the bottom sides of  $R$ , respectively, precisely opposite each other.

Let  $P_{in}$  denote the portion of (the drawing of)  $P$  contained inside  $C$ . To obtain  $P'$ , we flip  $P_{in}$  upside down (that is, replace it by its mirror image with respect to the horizontal axis of symmetry of  $R$ ), and reverse the colors of all vertices in  $P_{in}$ . The remaining portion of  $P$  is kept intact.

It is easy to see that applying the same transformation to  $P'$  recovers  $P$ .

Two plabic graphs related to each other via a sequence of local moves (see Definition 6.2) and/or switches are called *move-and-switch equivalent*.

An example is shown in Figure 25.

Proposition 6.9 can be strengthened as follows.

**Proposition 6.16.** *If two plabic graphs are move-and-switch equivalent to each other, then their associated quivers are mutation equivalent.*

*Proof.* Let  $P$  and  $P'$  be two plabic graphs related by a switch, as in Definition 6.15. Let us verify that the corresponding quivers  $Q(P)$  and  $Q(P')$  are isomorphic to each other. Indeed, for every edge

of  $P$  contained entirely inside  $P_{\text{in}}$ , the flipping of  $P_{\text{in}}$  reverses the direction of the corresponding arrow in the quiver; the subsequent reversal of colors restores the original direction. It remains to examine the edges of the plabic graph which cross the boundary of  $P_{\text{in}}$  (denoted by  $C$  in Definition 6.15). A case-by-case inspection shows that the combined contribution of the corresponding arrows remains unchanged under a switch.

The statement now follows by Proposition 6.9. □

It seems reasonable to expect that the converse to Proposition 6.16 holds as well. The following conjecture is inspired by our communications with Michael Shapiro.

**Conjecture 6.17** (M. Shapiro’s conjecture). *Two plabic graphs are move-and-switch equivalent if and only if their associated quivers are mutation equivalent.*

Recall that according to Conjecture 6.13, in the case of algebraic divides the switch transformations are not required.

*Remark 6.18.* We cannot resist stating a closely related version of Shapiro’s conjecture which can be formulated entirely in terms of quivers, without any mention of plabic graphs. This version asserts that if two mutation equivalent quivers  $Q$  and  $Q'$  are both planar (that is, each of them can be drawn on the plane without crossings), then  $Q$  can be transformed into  $Q'$  by a sequence of mutations in which each intermediate quiver is planar.

It is important to note that in the course of these mutations, it may be necessary to alter the topology of a planar embedding of (a portion of) the quiver at hand. To illustrate, the quivers associated with the plabic graphs shown in Figure 25 are isomorphic to each other (so no mutations are necessary) — but their respective planar embeddings naturally associated with these drawings are different.

## 7 | LINKS FROM DIVIDES

As mentioned in Remark 2.4, it is very difficult to distinguish algebraic divides, that is, those associated with real morsifications, from the divides which do not arise in this way. Luckily, this problem can be circumvented using an elegant construction introduced by A’Campo [3], which we recall in Definition 7.1. For surveys of some of the related research, see [42, Sections 1 and 6; 68, Sections 4–5].

The main idea is to extend the equivalence of divides based on the topology of the associated singularity (which can only be defined for algebraic divides) to a more general equivalence relation — defined for all divides — based on the topology of a certain link constructed from a given divide.

**Definition 7.1.** Let  $D$  be a divide in the unit disk  $\mathbf{D} = \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$ . The (A’Campo) link  $L(D)$  of  $D$  is constructed inside the unit 3-sphere

$$\mathbf{S}^3 = \{(x, y, u, v) \in \mathbb{R}^4 \mid x^2 + y^2 + u^2 + v^2 = 1\},$$

as follows. Assume that  $D$  is given by a smooth immersion of a collection of intervals and circles. For each regular (respectively, nodal) point  $(x, y) \in D$  in the interior of  $\mathbf{D}$ , find the two (respectively, four) different points  $(x, y, u, v) \in \mathbf{S}^3$  such that  $(u, v)$  is a tangent vector to  $D$



at  $(x, y)$ . The link  $L(D)$  is defined as the set of all such points  $(x, y, u, v)$ , together with the points  $(x, y, 0, 0)$  for  $(x, y) \in D \cap \partial D$ . We can view  $L(D)$  as a subset of  $\mathbb{C}^2$  via the identification  $(x, y, u, v) \simeq (x + \sqrt{-1}u, y + \sqrt{-1}v)$ .

Two divides are called *link equivalent* if their associated links are isotopic.

*Remark 7.2.* While the original construction in [3] was for divides without closed branches, it can be extended to full generality (cf. [2, 23, 45]).

*Remark 7.3.* All links appearing in this paper are naturally *oriented*. Accordingly, the term ‘link’ will generally mean ‘oriented link’ (with the natural orientation).

We next review the relationship between A’Campo’s construction presented in Definition 7.1 and the classical notion of the link of an isolated singularity.

**Definition 7.4.** The link  $L(C, z)$  associated with an isolated plane curve singularity  $(C, z)$  (as in Section 1) is the intersection of the curve  $C$  with a small sphere centered at  $z$ .

The importance of this construction comes from the following fundamental fact (see [24, 58] for historical background).

**Proposition 7.5.** *The link  $L(C, z)$  completely determines — and is determined by — the local topology of a given singular complex plane curve  $(C, z)$ .*

The crucial property established by N. A’Campo is that the constructions of Definitions 7.1 and 7.4 produce the same link. More precisely:

**Theorem 7.6** (A’Campo). *For an algebraic divide  $D$  arising from a real morsification of an isolated plane curve singularity  $(C, z)$ , the links  $L(D)$  and  $L(C, z)$  are isotopic to each other inside  $\mathbb{S}^3$ .*

Combining Proposition 7.5 with Theorem 7.6, we obtain the following statement.

**Corollary 7.7.** *Algebraic divides are link equivalent if and only if corresponding singularities are topologically equivalent.*

*Remark 7.8.* Proposition 7.5 and Theorem 7.6 imply that for a divide  $D$  coming from a real morsification, the link  $L(D)$  (hence the divide  $D$ ) determines the topological type of the underlying singularity. This does not however imply Theorems 3.4 and/or 4.4 because the same  $A\Gamma$ -diagram/quiver may potentially come from several distinct divides (either coming from morsifications or not, cf. Figure 9).

*Remark 7.9.* By Corollary 7.7, any topological invariant of a plane curve singularity can be in principle recovered from the A’Campo link  $L(D)$  of a divide  $D$  coming from a real morsification. In practice, extracting such invariants from  $L(D)$  can be challenging. For example, the *multiplicity* of a singularity is equal to the *braid index* of its link [51, 73], that is, the smallest number of strands in a braid defining it. However, computing the braid index of a link is, in general, a very difficult problem.

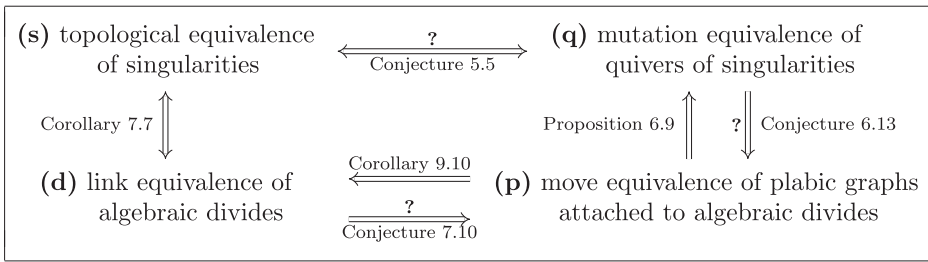


FIGURE 26 Unpacking Conjecture 7.11

We propose the following conjecture.

**Conjecture 7.10.** *Algebraic divides are link equivalent if and only if the plabic graphs attached to them are move equivalent.*

Since all plabic graphs attached to a given divide are move equivalent to each other, the particular choices of attached plabic graphs in Conjecture 7.10 are immaterial.

Conjectures 5.5, 6.13, and 7.10 and Corollary 7.7 are subsumed within the following statement, which is diagrammatically represented in Figure 26.

**Conjecture 7.11** (Main conjecture, expanded). *Let  $D_1$  and  $D_2$  be algebraic divides. Let  $Q_1 = Q(D_1)$  and  $Q_2 = Q(D_2)$  be their quivers. Let  $P_1 \in \mathbf{P}(D_1)$  and  $P_2 \in \mathbf{P}(D_2)$  be plabic graphs attached to  $D_1$  and  $D_2$ . Then the following are equivalent:*

- (s) *the singularities giving rise to the divides  $D_1$  and  $D_2$  are topologically equivalent;*
- (d) *the divides  $D_1$  and  $D_2$  are link equivalent;*
- (q) *the quivers  $Q_1$  and  $Q_2$  are mutation equivalent;*
- (p) *the plabic graphs  $P_1$  and  $P_2$  are move equivalent.*

The key part of Conjecture 7.11 is Conjecture 7.10 (the equivalence  $(\mathbf{d}) \Leftrightarrow (\mathbf{p})$ ); it is arguably more important than the original Conjecture 5.5 (the equivalence  $(\mathbf{s}) \Leftrightarrow (\mathbf{q})$ ). On the singularity theory side, replacing topological equivalence of singularities by the link equivalence of divides makes the issue at hand more tractable computationally, and might allow extensions to non-algebraic divides and their links (cf. Problem 7.12 and Remark 10.26). On the cluster side, replacing mutation equivalence of quivers by the move equivalence of plabic graphs makes even more sense: in light of Remark 5.4, it seems reasonable to restrict the mutation dynamics to a manageable subset of allowed directions.

In the rest of the paper, we focus on Conjecture 7.10 (the equivalence  $(\mathbf{d}) \Leftrightarrow (\mathbf{p})$ ). In Section 9, we prove the implication  $(\mathbf{p}) \Rightarrow (\mathbf{d})$  (see Corollary 9.10). In subsequent sections, we make partial progress towards the converse implication  $(\mathbf{d}) \Rightarrow (\mathbf{p})$ .

It is tempting to extend Conjecture 7.11 to a larger generality.

**Problem 7.12.** Identify a class of divides — as broad as possible — within which the various equivalences in Conjecture 7.11 hold.

**Remark 7.13.** It may well be that  $(\mathbf{d}) \Rightarrow (\mathbf{q})$  for arbitrary divides. It is even possible that  $(\mathbf{d}) \Rightarrow (\mathbf{p})$  for arbitrary plabic graphs, provided one uses transverse equivalence (cf. Problem 10.14).

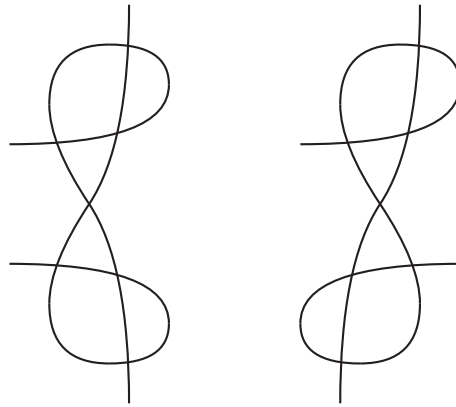


FIGURE 27 Two non-algebraic divides  $D_1$  and  $D_2$ . The quivers  $Q(D_1)$  and  $Q(D_2)$  are isomorphic. On the other hand, the links  $L(D_1)$  and  $L(D_2)$  are not isotopic: the link  $L(D_2)$  has an unknotted component but the link  $L(D_1)$  has not



FIGURE 28 Triangle moves on oriented divides

Remark 7.14. For general plabic graphs, **(q)** does not imply **(p)** (see Remark 6.10). Likewise, for general (non-algebraic) divides, **(q)** does not imply **(d)**; a counterexample, borrowed from [11, Figure 4], is shown in Figure 27.

### 8 | ORIENTED DIVIDES AND THEIR LINKS

Our proof of the implication **(p)** $\Rightarrow$ **(d)** in Conjecture 7.11 will rely on a construction that associates a link to an arbitrary plabic graph. But first, we need to discuss a more flexible notion of oriented divides (and their associated links). This notion, due to Gibson and Ishikawa [35, 36], is a variation of Arnold’s description of links associated to plane curves [7].

**Definition 8.1.** An oriented divide  $\vec{D}$  in a disk  $\mathbf{D}$  is an immersion into  $\mathbf{D}$  of a finite set of oriented circles satisfying conditions (D1), (D3), and (D4) of Definition 2.1. (Condition (D2) is unnecessary, since there are no intervals. The connectivity restrictions (D5) and (D6) are not required for our purposes here.)

The link  $L(\vec{D})$  of an oriented divide is defined as in Definition 7.1, except that we only take the vectors  $(x, y, u, v)$  where  $(u, v)$  points in the direction of the orientation of  $\vec{D}$ . As before, we consider oriented divides up to isotopy inside  $\mathbf{D}$ .

**Definition 8.2.** Two oriented divides are called *move equivalent* if they can be related to each other by a sequence of local moves of the following three kinds:

- *triangle moves*, as in Figure 28 (with any orientations);



FIGURE 29 Safe tangency moves on oriented divides

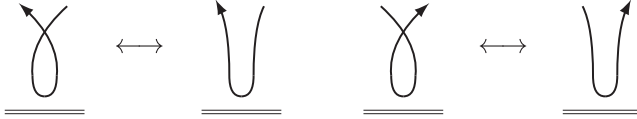


FIGURE 30 (Boundary) U-turn moves on oriented divides. The double horizontal lines at the bottom represent the boundary of the ambient disk  $\mathbf{D}$

- *safe tangency moves* with oppositely oriented strands, as in Figure 29; and/or
- *U-turn moves*, only allowed near the boundary of the disk  $\mathbf{D}$ , as in Figure 30.

*Remark 8.3.* The U-turn move is not an explicit move in the works of Gibson and Ishikawa, but appears implicitly in [35, Proposition 4.2; 36, Lemma 2.5], which assert that adding a loop on an outside arc does not change the isotopy class of the link. In addition to our U-turn move, they allow another move adding an inward-pointing rather than outward-pointing loop. We do not include inward-pointing loops because, on the one hand, it is not needed for the cluster algebra applications, and, on the other hand, it changes the transverse isotopy class of the link (Definition 10.6; see Proposition 10.12).

**Proposition 8.4.** *If oriented divides  $\vec{D}_1$  and  $\vec{D}_2$  are move equivalent, then the links  $L(\vec{D}_1)$  and  $L(\vec{D}_2)$  are smoothly isotopic to each other.*

To clarify: we care about smooth isotopy since we will later work in the category of transverse links.

*Proof.* It suffices to show the existence of a  $C^1$  isotopy. The existence of a  $C^\infty$  isotopy would follow, since everything is compact and there is a polynomial approximation.

For the triangle move, there is a path of immersions of the branches in the plane connecting the two oriented divides, passing through a diagram that has a triple intersection point. We can lift this path of immersions to  $\mathbf{S}^3$  as in Definition 8.1. Since the (oriented) tangents never agree, we get an isotopy of links.

The case of safe tangencies is similar.

It remains to treat the case of a U-turn move, which is a bit trickier. We note that the link of an oriented divide always avoids the equatorial circle in  $\mathbf{S}^3$  given by  $\{(x, y, 0, 0) \mid x^2 + y^2 = 1\}$ . Loosely, a U-turn move corresponds to letting  $L(\vec{D})$  pass through that circle once. To verify this, we use the following explicit construction. Consider the 1-parameter family of oriented curves  $\vec{C}_\varepsilon = \{(x_\varepsilon(t), y_\varepsilon(t))\} \subset \mathbf{D}$  given by

$$x = x_\varepsilon(t) = \varepsilon t + t^3, \tag{8.1}$$

$$y = y_\varepsilon(t) = -(1 - \frac{1}{2}\varepsilon^2)(1 - \frac{1}{2}t^2), \tag{8.2}$$

where  $\varepsilon$  and  $t$  are small real parameters. (To be more precise, we consider  $|\varepsilon| \leq \frac{1}{2}\delta^2$  and  $|t| \leq \delta$ , for a small positive  $\delta$ . Additional tweaking is required to have the two curve segments for  $\varepsilon = \pm\frac{1}{2}\delta^2$  match at the endpoints.) For  $\varepsilon \neq 0$ , the curves  $\vec{C}_\varepsilon$  are segments of an oriented divide in the interior of  $\mathbf{D}$ , differing by a U-turn move near the boundary. For  $\varepsilon = 0$ , we get a curve  $\vec{C}_0$  with a cusp at the boundary point  $(0, -1)$ .

For  $\varepsilon \neq 0$ , the corresponding (segment of the) link  $L_\varepsilon = L(\vec{C}_\varepsilon)$  is given by

$$L_\varepsilon(t) = (x, y, \beta\dot{x}, \beta\dot{y}), \tag{8.3}$$

where

$$\dot{x} = \varepsilon + 3t^2, \tag{8.4}$$

$$\dot{y} = (1 - \frac{1}{2}\varepsilon^2)t, \tag{8.5}$$

$$\beta = \beta_\varepsilon(t) = \left( \frac{1 - x^2 - y^2}{\dot{x}^2 + \dot{y}^2} \right)^{1/2}. \tag{8.6}$$

These formulae can be extended to the case  $\varepsilon = 0$ , with the convention  $\beta_0(0) = 1$ .

Each curve  $L_\varepsilon$  does not intersect itself. We are going to show that the family  $L_\varepsilon$  gives a  $C^1$ -isotopy between the two sides of a U-turn move. More precisely, we will prove that each of the 4 co-ordinates of  $L_\varepsilon(t)$  (cf. (8.3)) is a differentiable function of two variables  $\varepsilon$  and  $t$ , and each of its partial derivatives is continuous in the vicinity of the point  $\varepsilon = t = 0$ . For the first two co-ordinates  $x$  and  $y$ , this statement is obvious (cf. (8.1)–(8.2)). Let us treat the remaining co-ordinates  $\beta\dot{x}$  and  $\beta\dot{y}$ .

Straightforward calculations show that

$$x^2 + y^2 = 1 - \varepsilon^2 - t^2 + \text{poly}_{\geq 3}(\varepsilon, t), \tag{8.7}$$

$$\dot{x}^2 + \dot{y}^2 = \varepsilon^2 + t^2 + \text{poly}_{\geq 3}(\varepsilon, t), \tag{8.8}$$

where the notation  $\text{poly}_{\geq 3}(\varepsilon, t)$  stands for a polynomial in  $\mathbb{Q}[\varepsilon, t]$  in which each monomial has degree  $\geq 3$ . Substituting (8.7)–(8.8) into (8.6), we see that

$$\beta_\varepsilon(t) = \left( \frac{\varepsilon^2 + t^2 + \text{poly}_{\geq 3}(\varepsilon, t)}{\varepsilon^2 + t^2 + \text{poly}_{\geq 3}(\varepsilon, t)} \right)^{1/2} = 1 + O(\sqrt{\varepsilon^2 + t^2}).$$

Since both  $|\varepsilon|$  and  $|t|$  do not exceed  $\sqrt{\varepsilon^2 + t^2}$ , we conclude that

$$\beta_\varepsilon(t) = 1 + O(\sqrt{\varepsilon^2 + t^2}) \xrightarrow{(\varepsilon, t) \rightarrow (0, 0)} 1, \tag{8.9}$$

so  $\beta_\varepsilon(t)$  is a continuous function of  $\varepsilon$  and  $t$ .

We next show that both  $\varepsilon\beta_\varepsilon(t)$  and  $t\beta_\varepsilon(t)$  are in  $C^1$ . In view of (8.4)–(8.5), this will imply that  $\beta\dot{x}$  and  $\beta\dot{y}$  are in  $C^1$ , as desired. The two cases are completely analogous, so let us consider  $\varepsilon\beta_\varepsilon(t)$ . This function is clearly smooth at every point other than  $\varepsilon = t = 0$ , so we only need to examine

the latter point. Equation (8.9) implies that

$$\varepsilon\beta_\varepsilon(t) = \varepsilon + O(\varepsilon^2 + t^2),$$

so  $\varepsilon\beta_\varepsilon(t)$  is differentiable at  $\varepsilon=t=0$ , with partial derivatives  $\frac{\partial(\varepsilon\beta)}{\partial\varepsilon}(0,0) = 1$  and  $\frac{\partial(\varepsilon\beta)}{\partial t}(0,0) = 0$ . Away from this point, these derivatives can be computed using (8.6):

$$\begin{aligned} \frac{\partial}{\partial\varepsilon}(\varepsilon\beta) &= \beta + \frac{\varepsilon}{2\beta} \frac{\text{poly}_{\geq 4}(\varepsilon, t)}{(\varepsilon^2 + t^2 + \text{poly}_{\geq 3}(\varepsilon, t))^2}, \\ \frac{\partial}{\partial t}(\varepsilon\beta) &= \frac{\varepsilon}{2\beta} \frac{\text{poly}_{\geq 4}(\varepsilon, t)}{(\varepsilon^2 + t^2 + \text{poly}_{\geq 3}(\varepsilon, t))^2}. \end{aligned}$$

As  $(\varepsilon, t)$  goes to  $(0,0)$ , these expressions converge to 1 and 0, respectively, establishing the continuity of the derivative. This completes the proof of Proposition 8.4. □

*Remark 8.5.* Passing through a *same-direction tangency*, with the two strands oriented in the same direction, does not produce an isotopy of the associated links, since in this case,  $L(\vec{D})$  crosses through itself.

For an oriented divide  $\vec{D}$ , we denote by  $-\vec{D}$  the same divide with all orientations reversed.

**Lemma 8.6.** *For any oriented divide  $\vec{D}$ , the links  $L(\vec{D})$  and  $L(-\vec{D})$  are isotopic.*

*Proof.* The two links are isotopic through the isotopy of  $\mathbf{S}^3$  given by

$$\phi_t(x, y, u, v) = (x, y, \cos(\pi t)u + \sin(\pi t)v, -\sin(\pi t)u + \cos(\pi t)v). \quad \square$$

*Remark 8.7.* The main difference between Arnold’s theory and the Gibson-Ishikawa theory described above is that we are working in the disk and lifting to  $\mathbf{S}^3$ , rather than working in the plane and lifting to the unit tangent bundle of the plane, which is topologically a solid torus. Every link is the link of an oriented divide, either in the solid torus (Arnold’s theory; see Chmutov, Goryunov, and Murakami [20]) or in  $\mathbf{S}^3$  (Gibson and Ishikawa [36]). There is also a concrete set of moves that relate any two oriented divides whose links are isotopic (Gibson [35]), analogous to the Reidemeister moves. This set of moves is slightly larger than the moves above; see Proposition 10.12 for the explanation.

*Remark 8.8.* Since any link is a link of an oriented divide, one can think of oriented divides as a combinatorial representation of links, on the same level as the traditional link diagrams. It is a less intuitive representation, and some features of links are harder to discern from an oriented divide, compared to link diagrams; for instance, computing the linking number is more involved.

A’Campo’s construction of links of ordinary (that is, unoriented) divides, reproduced in Definition 7.1, is a special case of the Gibson–Ishikawa construction of links of oriented divides described in Definition 8.1, in the following precise sense.

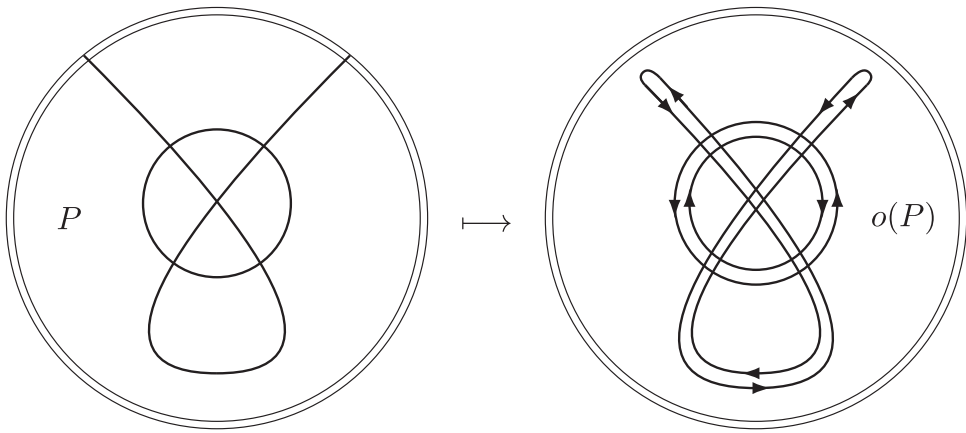


FIGURE 31 A divide  $P$  gives rise to an oriented divide  $o(P)$  via ‘doubling’

**Definition 8.9.** To any divide  $D$ , we can associate an oriented divide  $o(D)$  obtained by the following ‘doubling’ procedure:

- replace each branch of  $D$  with two parallel oriented branches of  $o(D)$ , with opposite orientations, following the ‘rules of the road’ (driving on the right) illustrated in Figure 31;
- near each point where  $D$  hits the boundary  $\partial\mathbf{D}$ , connect the two oriented branches, as shown in Figure 31 on the right (at the top).

**Proposition 8.10.** *The links  $L(D)$  and  $L(o(D))$  are smoothly isotopic to each other.*

*Proof.* Compare Definitions 7.1 and 8.1, and use the isotopy from the proof of Proposition 8.4.  $\square$

*Remark 8.11.* While the construction of the link of an oriented divide is elementary, one may still want to construct a conventional link diagram for  $L(D)$  directly from the combinatorial topology of a divide  $D$ . Several solutions of this problem were suggested by various authors. In particular, Couture and Perron [23] gave an algorithm producing a braid representation for the link  $L(D)$  associated with *any* divide  $D$ . It involves an extension of the basic construction to *signed* divides, wherein each node is labeled by a sign, either  $+$  or  $-$ . (The case when all signs are positive corresponds to the usual notion.) In the special case of ‘scannable’ divides, the Couture–Perron construction simplifies considerably (see Section 11).

Other (related) constructions of braid representations of links of (oriented) divides were given by Chmutov [19], Hirasawa [41], and Gibson–Ishikawa [36]. While those constructions are more direct than the one in [23], and do not involve signs, they are not ‘local’ as they require dragging the strands of the link to the boundary of the disk, and then back. All of these methods involve a non-canonical choice of a preferred ‘Morse direction’ within the ambient disk of the divide.

## 9 | LINKS FROM PLABIC GRAPHS

We next explain how a plabic graph gives rise to an oriented divide, and therefore to a link.



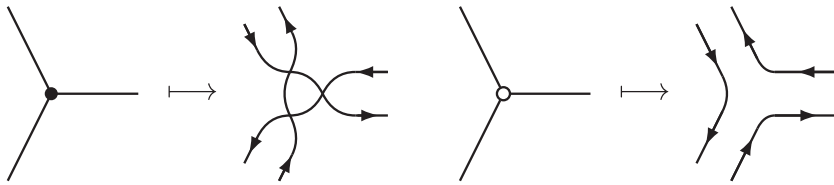


FIGURE 32 Building an oriented divide around internal vertices of a plabic graph



FIGURE 33 Building an oriented divide near the boundary of the ambient disk  $\mathbf{D}$

**Definition 9.1.** The oriented divide  $o(P)$  associated to a plabic graph  $P$  is constructed as follows. Turn each edge in  $P$  into a pair of oppositely oriented strands as in Definition 8.9. (Remember that we are ‘driving on the right’.) At each white trivalent vertex of  $P$ , connect the strands by turning right; at each black vertex connect the strands by turning left (see Figure 32). Note that when we turn left, we introduce transversal crossings in the divide. At the univalent ends of  $P$  lying on  $\partial\mathbf{D}$ , make a U-turn near the boundary by turning either left or right depending on whether the end is white or black, respectively (see Figure 33). (This introduces a crossing if the end is black.)

We then construct a link  $L(P) = L(o(P))$  from the resulting oriented divide  $o(P)$ .

*Remark 9.2.* The construction of the link of a plabic graph given in Definition 9.1 is a special case of the construction given by Kawamura [46] in her theory of *graph divides*, which we will briefly review in Section 10 (cf. in particular Definition 10.2, Remark 10.3, and Figure 37).

Since we view plabic graphs up to a global reversal of colors, we need to check how such a reversal affects the notions introduced in Definition 9.1.

**Proposition 9.3.** *Let  $P$  be a plabic graph, and let  $-P$  denote the plabic graph obtained from  $P$  by reversing the colors of all vertices. Then the oriented divide  $o(-P)$  is move equivalent to  $-o(P)$ . Furthermore, the links  $L(P)$  and  $L(-P)$  are isotopic to each other.*

*Proof.* The oriented divides  $o(P)$  and  $-o(-P)$  differ by isotopy and a safe tangency move for each edge of  $P$  connecting vertices of the same color, as illustrated in Figure 34. The claim then follows by Proposition 8.4 and Lemma 8.6. □

Move equivalence of plabic graphs translates into move equivalence of associated oriented divides.

**Proposition 9.4.** *If plabic graphs  $P_1$  and  $P_2$  are move equivalent, then the associated oriented divides  $o(P_1)$  and  $o(P_2)$  are move equivalent.*

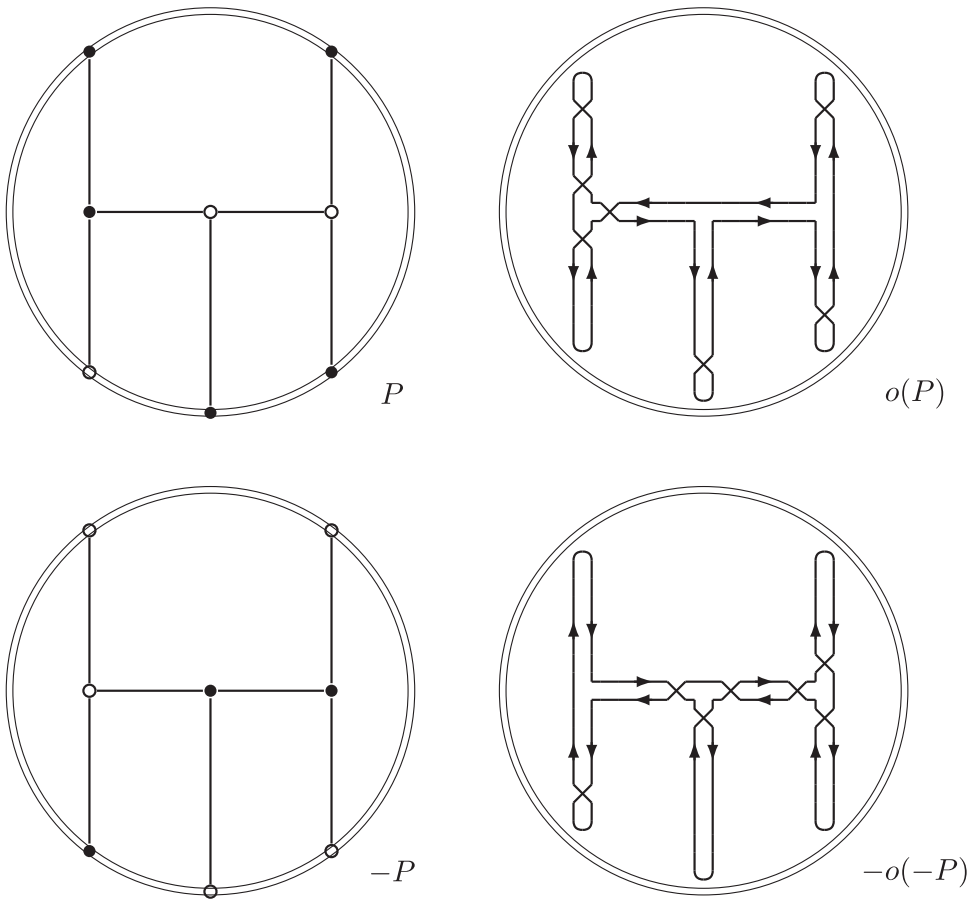


FIGURE 34 Plabic graphs  $P$  and  $-P$ , and oriented divides  $o(P)$  and  $-o(-P)$

*Proof.* The most complicated case is the square move, shown in Figure 35 (from left to right). The transition between the corresponding oriented divides involves a total of four triangle moves and two safe tangency moves.

The other moves on plabic graphs are easier: a flip move between two white vertices changes  $o(P)$  by an isotopy, and a flip move between two black vertices changes  $o(P)$  by two safe tangency moves. A tail attachment/removal changes  $o(P)$  by a safe tangency move (if the internal vertex is black), a U-turn move (if the boundary vertex is black), or both (if both are black).  $\square$

Combining Propositions 8.4 and 9.4, we obtain the following.

**Corollary 9.5.** *If two plabic graphs  $P_1$  and  $P_2$  are move equivalent, then the links  $L(P_1)$  and  $L(P_2)$  are isotopic.*

*Remark 9.6.* Corollary 9.5 provides a powerful tool that can be used to show that a particular pair of plabic graphs are not move equivalent, by verifying that their respective links are not isotopic. (The links can be computed, for example, using Hirasawa’s algorithm [41].) We note that using quiver mutations (that is, a test based on Proposition 6.9) for this purpose is problematic, since

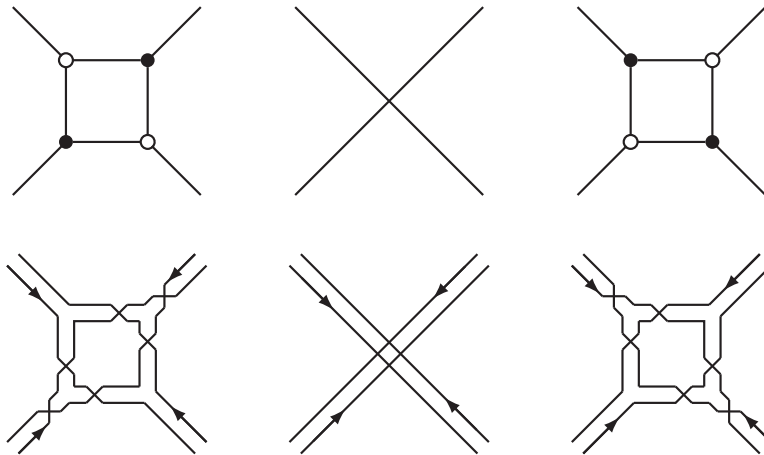


FIGURE 35 The square move and oriented divides

there is no known good algorithm for deciding whether two quivers are mutation equivalent or not (cf. Remark 5.4). Besides, the isotopy class of  $L(P)$  is a finer invariant of a plabic graph  $P$  than the quiver  $Q(P)$  (cf. Example 9.7).

**Example 9.7.** The plabic graphs  $P_1$  and  $P_2$  in Figure 25 have isomorphic quivers. In spite of that,  $P_1$  and  $P_2$  are *not* move equivalent, because the links  $L(P_1)$  and  $L(P_2)$  are *not* isotopic. To be concrete,  $L(P_1)$  (here  $P_1$  is the graph on the left) has an unknotted component, whereas both components of  $L(P_2)$  are trefoils.

To any divide  $D$ , the ‘roundabout’ construction in Definition 6.11 attaches a family of plabic graphs  $P \in \mathbf{P}(D)$ , all of them move equivalent to each other. Using the construction in Definition 9.1, we then obtain a family of oriented divides  $o(P)$ , also move equivalent to each other, by virtue of Proposition 9.4. It is natural to compare the oriented divides  $o(P)$  to the divide  $o(D)$  obtained by the doubling procedure of Definition 8.9.

**Proposition 9.8.** *Let  $D$  be a divide, and  $P \in \mathbf{P}(D)$  a plabic graph attached to  $D$ . Then the oriented divides  $o(D)$  and  $o(P)$  are move equivalent to each other.*

*Proof.* Let us compare the oriented divides  $o(D)$  and  $o(P)$  near a crossing of  $D$ . This is shown in Figure 35 (bottom row):  $o(D)$  is in the middle whereas  $o(P)$  is either on the left or on the right (depending on the choice made when constructing  $P$ , cf. Definition 6.11). As explained in the proof of Proposition 9.4, these oriented divides are move equivalent.

The oriented divides  $o(D)$  and  $o(P)$  may also differ near the points of  $D$  lying on the boundary  $\partial \mathbf{D}$  (depending on the colors chosen for corresponding vertices of  $P$ ). These discrepancies can be straightened out using U-turn moves. □

**Proposition 9.9.** *Let  $D$  be a divide, and  $P \in \mathbf{P}(D)$  be a plabic graph attached to  $D$ . Then the links  $L(D)$  and  $L(P)$  are isotopic to each other.*

*Proof.* We have  $L(D) = L(o(D))$  by Proposition 8.10, and  $L(P) = L(o(P))$  by Definition 9.1. The claim follows by Propositions 8.4 and 9.8. □

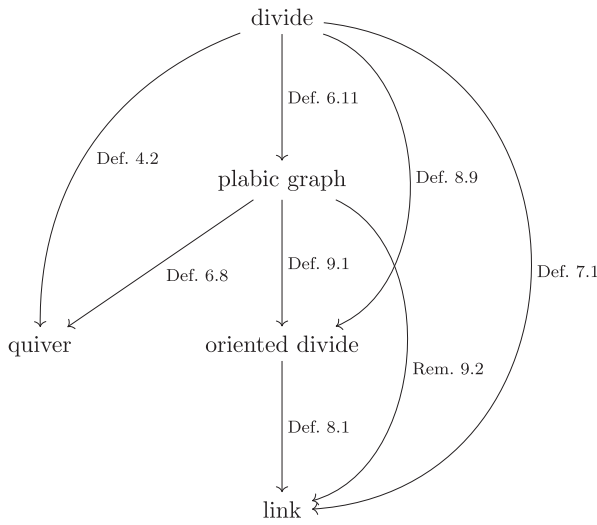


FIGURE 36 Various constructions involving divides, plabic graphs (viewed up to move equivalence), quivers, oriented divides, and links

**Corollary 9.10.** *Let  $P_1$  and  $P_2$  be plabic graphs attached to divides  $D_1$  and  $D_2$ , respectively. If  $P_1$  and  $P_2$  are move equivalent, then  $D_1$  and  $D_2$  are link equivalent.*

*Proof.* This is immediate from Corollary 9.5 and Proposition 9.9. □

Corollary 9.10 establishes the implication **(p)**⇒**(d)** of Conjecture 7.11, even without the assumption of algebraicity.

The diagram in Figure 36 summarizes the correspondences between the various types of objects considered above. This diagram commutes (up to the appropriate equivalences), so that, for example, the link of a divide  $D$  (computed using the A’Campo construction, see Definition 7.1) is isotopic to the link of an oriented divide corresponding to a plabic graph attached to  $D$ .

## 10 | QUASIPOSITIVE AND TRANSVERSE LINKS

In Sections 7–9, we discussed how to construct a link from a divide, a plabic graph, or an oriented divide. It is natural to ask: what kind of links arise via these constructions? It turns out that the links of plabic graphs (and even more generally, of graph divides, see below) are special in two ways. On the one hand, these links are *quasipositive*. On the other hand, they naturally carry extra structure: they are *transverse* links. This additional structure will lead us to formulate some more refined conjectures.

**Definition 10.1.** Let  $\beta$  be a braid. We say that

- $\beta$  is *positive* if it is a product of standard Artin generators  $\sigma_j$ ;
- $\beta$  is *strongly quasipositive* if it is a product of the standard conjugates of the  $\sigma_j$ :

$$\sigma_{i,j} = (\sigma_i \sigma_{i+1} \cdots \sigma_{j-1}) \sigma_j (\sigma_{j-1}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i^{-1});$$

- $\beta$  is *quasipositive* if it is a product of arbitrary conjugates of the  $\sigma_j$ .

Now let  $L$  be an oriented link. We say that

- $L$  is a *positive braid link* if it can be obtained as the closure of a positive braid;
- $L$  is *positive* if it can be represented by a diagram with all crossings positive;
- $L$  is *strongly quasipositive* if it is the closure of a strongly quasipositive braid;
- $L$  is *quasipositive* if it is the closure of a quasipositive braid.

In this listing, each line describes a wider class of links than the previous one. (Rudolph showed [67] that every positive link is strongly quasipositive.)

In addition to braid closures and (more general) link diagrams, links can be constructed using divides (Definition 7.1) or, more generally, plabic graphs (Definition 9.1) or oriented divides (Definition 8.1). Another closely related construction, already mentioned in Remark 9.2, is the following (cf. Kawamura [46]).

**Definition 10.2.** A *graph divide* is a connected planar graph  $G = (V, E)$  (multiple edges and loops are allowed) in which each vertex is colored black or white; we moreover fix a proper embedding of  $G$  into the disk  $\mathbf{D}$  (viewed up to isotopy), and additionally assume that each vertex in  $V \cap \partial\mathbf{D}$  is univalent. A *trivalent graph divide* is a graph divide in which all internal vertices (not lying on  $\partial\mathbf{D}$ ) are trivalent. A straightforward extension of Definition 9.1 associates to any graph divide  $G$  the corresponding oriented divide  $o(G)$ , and thus an oriented link  $L(G) = L(o(G))$ .

*Remark 10.3.* Trivalent graph divides are very close to plabic graphs, but are not subject to some technical restrictions. If a general graph divide  $G$  has no internal univalent vertices, then there is a related trivalent graph divide  $G'$  obtained by (a) gluing the pairs of edges which meet at 2-valent vertices and (b) splitting each vertex of degree  $d > 3$  into a tree made of  $d - 2$  trivalent vertices of the same color. It is easy to see that  $L(G)$  is isotopic to  $L(G')$ , so the key restriction that distinguishes plabic graphs from graph divides is the absence of internal univalent vertices. It is unclear how this restriction affects the corresponding class of links.

**Definition 10.4.** We call a link *algebraic* if it can arise as the link  $L(C, z)$  of an isolated plane curve singularity. (Readers beware: some authors ascribe a different meaning to the term ‘algebraic link’.) By Theorem 7.6, algebraic links are precisely the A’Campo links of algebraic divides.

**Definition 10.5.** A link  $L \subset \mathbf{S}^3$  is called  *$\mathbb{C}$ -transverse* if there is an algebraic plane curve  $X \subset \mathbb{C}^2$  such that  $X \cap \mathbf{S}^3$  is isotopic to  $L$ ; here we require that  $X$  is smooth along the sphere  $\mathbf{S}^3$  and intersects it transversally. This is a much larger class than algebraic links, since there may be several singular points of  $X$  inside the ball bounded by  $\mathbf{S}^3$ .

The known relationships between various classes of links mentioned above are shown in Figure 37. In particular,

- any algebraic link is a divide link, by A’Campo’s Theorem 7.6;
- any divide link is the link of a plabic graph, by Proposition 9.9;
- any positive braid link can be represented by a plabic graph (see Section 12);
- quasipositive links are the same as  $\mathbb{C}$ -transverse links, as shown by Boileau and Orevkov [14] and Rudolph [63];

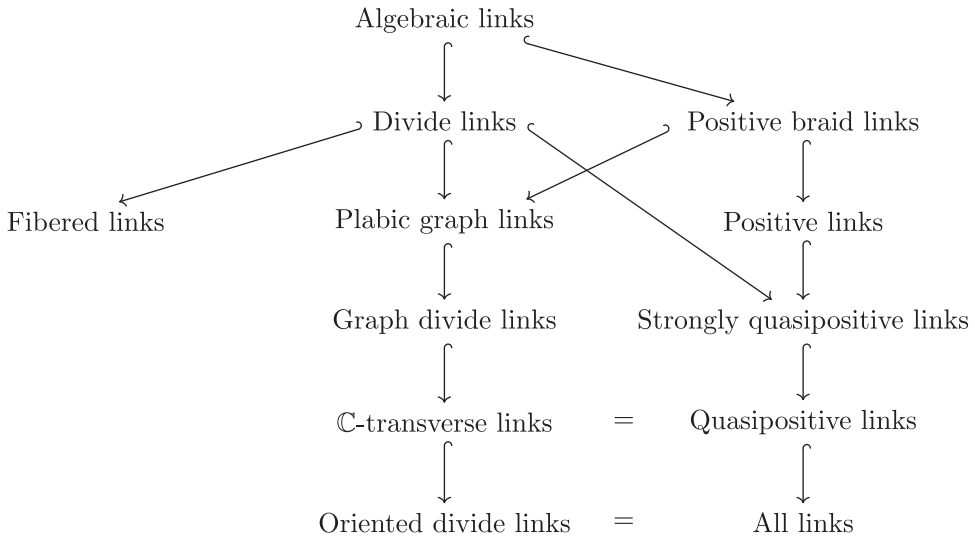


FIGURE 37 The known relationships between the classes of links obtained using different constructions (middle column), or exhibiting various forms of positivity (right column). See the Venn diagram in [47, Figure 13]

- any divide link is strongly quasipositive: Ishikawa [42] showed that divide links are positive Hopf plumbings; Rudolph [66] proved that such plumbings are strongly quasipositive;
- any graph divide link is quasipositive (Kawamura [46]), hence  $\mathbb{C}$ -transverse;
- A’Campo [2] proved that the link of every divide is *fibred* (that is, the complement is a fiber bundle over a circle, with fiber a surface);
- any link is a link of an oriented divide (see Remark 8.7).

In addition to being in a restricted class, the links arising from plabic graphs also have extra structure coming from contact geometry. (For more background on contact geometry, see [25].)

**Definition 10.6.** The *standard contact structure* on  $\mathbf{S}^3 = \{x^2 + y^2 + u^2 + v^2 = 1\}$  is the 2-plane field  $\xi = \ker \omega$ , where  $\omega$  is the 1-form

$$\omega = -u \, dx - v \, dy + x \, du + y \, dv. \tag{10.1}$$

(Up to isotopy, this is the unique contact structure on  $\mathbf{S}^3$  with the additional property of being *tight*.) A link  $L$  embedded in  $\mathbf{S}^3$  is *Legendrian* if its tangent vector  $\dot{L}$  always lies in  $\xi$ , and is *transverse* if  $\dot{L}$  never lies in  $\xi$ . By convention, a transverse link  $L$  is oriented so that  $\langle \dot{L}, \omega \rangle > 0$  everywhere on  $L$ . Two Legendrian/transverse links are *Legendrian/transverse isotopic* if they are isotopic through Legendrian/transverse embeddings of links. There is a natural construction that turns a Legendrian link into its *transverse push-off* (see [25, Section 2.9]).

We will focus on transverse links. One basic invariant of a transverse link is its classical link type, that is, its isotopy type as an ordinary link. For each classical link type, there are many transverse link types. One of the invariants that distinguishes between some of them is the integer-valued self-linking number.

**Definition 10.7.** Let  $T$  be a transverse link. Recall that there is a connected orientable Seifert surface  $\Sigma \subset \mathbf{S}^3$  with  $T = \partial\Sigma$ . Consider  $\xi|_\Sigma$ , the contact plane field restricted to  $\Sigma$ . Since  $\Sigma$  has nonempty boundary,  $\xi|_\Sigma$  is trivial; pick a section of it, a vector field  $v$  defined on  $\Sigma$ . Let  $T'$  be a copy of  $T$ , pushed off in the direction of  $v$ . The *self-linking number*  $sl(T)$  of  $T$  is the linking number of  $T$  and  $T'$ .

*Remark 10.8.* The self-linking number can always be decreased by 2 — while preserving the ordinary isotopy type of the link — by a local operation called *stabilization*. For any link type, there is a maximal realizable value of the self-linking number. For each link type and each value of the self-linking number, there are finitely many possible transverse link types.

The links arising from the various constructions discussed above are naturally transverse, as we will now explain.

Braid closures naturally produce transverse links. Let  $U$  be a standard transversal unknot, say the equator  $y = v = 0$ . Given a braid  $\beta$ , embed the closure of  $\beta$  in a small tubular neighborhood of  $U$  to get a link  $\beta(U)$ . Since the tangent vectors to  $\beta(U)$  are close to the tangents to  $U$ , the link  $\beta(U)$  is also transverse.

**Proposition 10.9.** *Every  $\mathbb{C}$ -transverse link  $L$  (in particular, any algebraic link; see Definition 10.5) has a natural transverse structure.*

*Proof.* Let  $X \subset \mathbb{C}^2$  be the algebraic curve giving rise to  $L$ , as in Definition 10.5. Let  $v \in TL = TX \cap TS^3$  be a tangent vector to  $X$ . Then  $iv$  is also tangent to  $X$ , and by transversality of the intersection is not in  $TS^3$ . Thus,  $v \notin \xi$ . □

We next show that the links of oriented divides are naturally transverse.

If  $\Omega$  is a smooth strictly pseudoconvex (cf., for example, [52, Chapter 3, Definition 1.9]) subset of  $\mathbb{C}^2$ , then  $\partial\Omega$  inherits the structure of a contact manifold, by setting

$$\xi = \{v \in T\partial\Omega \mid iv \in T\partial\Omega\}.$$

Strict pseudoconvexity guarantees that locally  $\xi = \ker \omega$  for a 1-form  $\omega$  such that  $\omega \wedge d\omega \neq 0$ , as required for a contact structure. If  $\Omega$  is the standard unit ball in  $\mathbb{C}^2$ , then this is the contact structure in Definition 10.6. More generally, if  $\Omega$  is any strictly pseudoconvex topological ball, then the contact structure is tight and thus equivalent to the standard one.

**Definition 10.10.** Let  $0 < \lambda \leq 1$ . Following A’Campo [2, Theorem 3], consider the squashed ball  $\mathbf{B}_\lambda = \{(x, y, u, v) \in \mathbb{C}^2 \mid x^2 + y^2 + \lambda^{-2}(u^2 + v^2) < 1\} \subset \mathbb{C}^2$ . As  $\mathbf{B}_\lambda$  is strictly pseudoconvex, there is a natural contact structure on  $\mathbf{S}_\lambda^3 = \partial\mathbf{B}_\lambda$ . (This structure is in fact equivalent to the standard contact structure, by uniqueness of the tight contact structure on  $\mathbf{S}^3$ .)

**Definition 10.11.** For an oriented divide  $\vec{D}$ , let  $T_\lambda(\vec{D})$  denote the lift of  $\vec{D}$  to  $\mathbf{S}_\lambda^3$  obtained via a straightforward extension of Definition 8.1. In particular, for  $\lambda = 1$ , we get  $T_\lambda(\vec{D}) = L(\vec{D})$ . Furthermore, all lifts  $T_\lambda(\vec{D})$  are naturally isotopic to each other, and to  $L(\vec{D})$ .

**Proposition 10.12.** *Let  $\vec{D}$  be an oriented divide. For any sufficiently small  $\lambda$ , the link  $T_\lambda(\vec{D})$  is transverse to the contact structure on  $\mathbf{S}_\lambda^3$ .*

Any move equivalence of oriented divides lifts, for sufficiently small  $\lambda$ , to a transverse isotopy of their associated lifts  $T_\lambda(\vec{D})$ .

*Proof.* Let  $D$  be the unoriented divide underlying  $\vec{D}$ . In the proof of [2, Theorem 3], A’Campo shows that there is a curve  $X \subset \mathbb{C}^2$  such that for  $\lambda$  sufficiently small,  $X \cap \partial\mathbf{B}_\lambda$  is very close to  $T_\lambda(D)$ , which is thus  $\mathbb{C}$ -transverse and hence transverse. The link  $T_\lambda(\vec{D})$  is a union of some of the components of  $T_\lambda(D)$  and is therefore also transverse (but in general not  $\mathbb{C}$ -transverse).

The construction of  $X$  and  $T_\lambda(\vec{D})$  works just as well through triangle moves and safe tangency moves, which therefore give transverse isotopies. To see that a boundary U-turn move is also a transverse isotopy, it suffices to see that the  $C^1$ -isotopy from the proof of Proposition 8.4 is a transverse isotopy. It suffices to check that the curve  $L_\varepsilon$  is transverse to the contact structure on  $\mathbf{S}^3$  at  $(\varepsilon, t) = (0, 0)$ , since by continuity the curve will also be transverse for  $(\varepsilon, t)$  near  $(0, 0)$ . We see that  $L_0(0) = (0, -1, 0, 0)$  and  $\dot{L}_0(0) = (0, 0, 0, 1)$ . Since the 1-form  $\omega$  in Equation (10.1) has a term  $y \, dv$ , the link is transverse at this point as desired. (Similar computations also work if we lift to  $\mathbf{S}_\lambda^3$  instead.) □

For sufficiently small  $\lambda$ , the transverse links  $T_\lambda(\vec{D})$  are transversely isotopic to each other. We thus write simply  $T(\vec{D})$  for their common transverse isotopy class. This isotopy class refines the ordinary isotopy class of the link  $L(\vec{D})$  (cf. Definition 10.11).

We can similarly associate a transverse link to any plabic graph  $P$  by setting  $T(P) = T(o(P))$ ; or to a divide  $D$  by setting  $T(D) = T(o(D))$ .

**Corollary 10.13.** *If two plabic graphs  $P_1$  and  $P_2$  are move equivalent, then  $T(P_1)$  and  $T(P_2)$  are transverse isotopic.*

*Proof.* Immediate from Propositions 9.4 and 10.12. □

It is natural to ask whether the converse is true.

**Problem 10.14.** If two plabic graphs  $P_1$  and  $P_2$  are transverse-equivalent (that is,  $T(P_1)$  is transverse isotopic to  $T(P_2)$ ), does it follow that  $P_1$  and  $P_2$  are move equivalent? (A weaker version: do the quivers  $Q(P_1)$  and  $Q(P_2)$  have to be mutation equivalent?)

**Remark 10.15.** To pose a similar question for general graph divides (cf. Remark 10.3), one would need a proper notion of move equivalence. More precisely, one would need to identify a set of moves (containing the local moves in Definition 6.2) relating any two graph divides whose links are transverse equivalent.

It is not hard to show that any transverse link can be realized as  $T(\vec{D})$  for some oriented divide  $D$ , by adapting the argument in [36]; we omit the details. On the other hand, not every transverse link can be realized as a transverse link of a plabic graph (respectively, a divide, a graph divide). In particular, there is a restriction on the self-linking number.

**Proposition 10.16.** *Let  $P$  be a plabic graph. Then  $sl(T(P)) = -\chi(P)$ , where  $\chi(P)$  denotes the Euler characteristic of  $P$  (viewed as a 1-dimensional simplicial complex).*

The proof of Proposition 10.16 will require some preliminary lemmas.



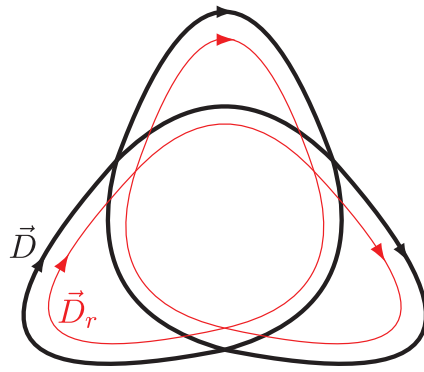


FIGURE 38 An oriented divide  $\vec{D}$  (thick, black) and its push-off  $\vec{D}_r$ , in the direction of the rightward-pointing normal (thin, red)

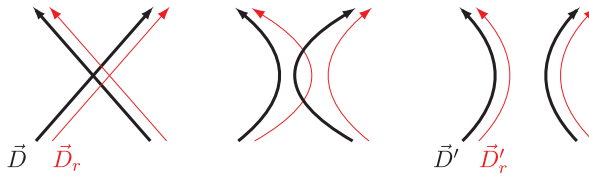


FIGURE 39 An oriented divide with a crossing (left) and its oriented smoothing (right). Both are shown with their rightward push-offs. Shown in the middle is the intermediate step in the proof of Lemma 10.19

**Lemma 10.17.** *Let  $\vec{D}$  be an oriented divide, and let  $\vec{D}_r$  be a small push-off of  $\vec{D}$  in the direction of the right-handed normal of  $\vec{D}$ , as shown in Figure 38. Then  $sl(T(\vec{D}))$  is the linking number between  $L(\vec{D})$  and  $L(\vec{D}_r)$ .*

*Proof.* We work with the model  $T_\lambda(\vec{D}) \subset \mathbf{S}^3_\lambda$ , as in the definition of  $T(\vec{D})$ . The first step is to find a section of the contact plane  $\xi$  on a Seifert surface for  $T_\lambda(\vec{D})$ . In fact, the vector field on  $\mathbf{S}^3_\lambda$  given by

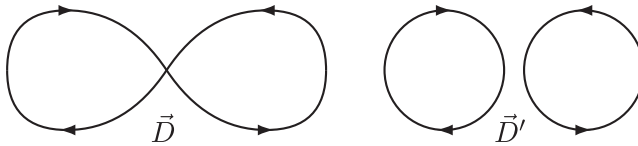
$$\vec{a}(x, y, u, v) = v \frac{\partial}{\partial x} - u \frac{\partial}{\partial y} + \lambda^2 \left( y \frac{\partial}{\partial u} - x \frac{\partial}{\partial v} \right)$$

is a global section of  $\xi$ . Thus, we want to find the linking number of  $T_\lambda(\vec{D})$  with its push-off in the direction of  $\vec{a}$ . For small  $\lambda$ , this push-off is very close to  $T_\lambda(\vec{D}_r)$ . □

We could use Lemma 10.17 to directly prove Proposition 10.16. But, as mentioned in Remark 8.8, it is not very convenient to compute linking numbers from divide presentations of links — so we instead give some reductions.

**Lemma 10.18.** *Let an oriented divide  $\vec{D}$  be a circle inside the disk  $\mathbf{D}$ , with no crossings and an arbitrary orientation. Then  $sl(T(\vec{D})) = -1$ . More generally, if  $\vec{D}$  is a union of  $n$  disjoint non-nested circles, then  $sl(T(\vec{D})) = -n$ .*

*Similarly, if  $\vec{D}$  is an oriented figure-eight curve with one crossing, as shown in Figure 40, then  $sl(T(\vec{D})) = -1$ .*



**FIGURE 40** Smoothing a figure-eight oriented divide into two circles. This translates into smoothing an unknot with self-linking number  $-1$  into two disjoint unknots, each with self-linking number  $-1$

*Proof.* Let  $\vec{D}$  be a circle. With  $\vec{D}_r$  the push-off as in Lemma 10.17, the link  $L(\vec{D} \cup \vec{D}_r)$  is a Hopf link, oriented so that the linking number between the two components is  $-1$ . (Thus,  $T(\vec{D})$  is the standard transverse unknot with self-linking  $-1$ , which is the maximum possible.) If  $\vec{D}$  is a union of  $n$  disjoint non-nested circles, then  $L(\vec{D} \cup \vec{D}_r)$  is a split union of  $n$  Hopf links, each with linking number  $-1$ .

The result for the figure-eight curve follows from Proposition 10.12 by applying a U-turn move to an oriented circle. □

**Lemma 10.19.** *Let  $\vec{D}$  be an oriented divide with a crossing at  $x$ , and let  $\vec{D}'$  be the oriented divide with the crossing at  $x$  smoothed in an orientation-preserving way. Then  $sl(\vec{D}') = sl(\vec{D}) - 1$ .*

*Proof.* By Lemma 10.17, we need to compare the linking numbers  $lk(L(\vec{D}), L(\vec{D}_r))$  and  $lk(L(\vec{D}'), L(\vec{D}'_r))$ . To move from the oriented divide  $\vec{D} \cup \vec{D}_r$  to  $\vec{D}' \cup \vec{D}'_r$ , we perform two operations, illustrated in Figure 39.

First, we smooth a crossing of  $\vec{D}$  with itself, and also a crossing of  $\vec{D}_r$  with itself, both preserving orientations. The effect on the corresponding links is to do two surgeries, one on  $\vec{D}$  and one on  $\vec{D}_r$ . This preserves the linking number, since each surgery is done on the same side of the linking number.

Second, we perform a same-direction tangency move involving both  $\vec{D}$  and  $\vec{D}_r$ . Since  $L(\vec{D})$  crosses once through  $L(\vec{D}_r)$  (cf. Remark 8.5), the linking number changes by  $\pm 1$  (always one or the other). To establish the sign, we consider the case of a figure-eight divide  $\vec{D}$  being smoothed into a divide  $\vec{D}'$  consisting of two circles, as in Figure 40. By Lemma 10.18,  $T(\vec{D})$  is a standard transverse unknot with self-linking number  $-1$  and  $T(\vec{D}')$  is two unlinked unknots with total self-linking number  $-2$ . □

*Proof of Proposition 10.16.* Let  $\vec{D}_0 = \vec{D}(P)$  be the oriented divide associated with the plabic graph  $P$ . The construction of the oriented divide  $D_0$  is explained in Definition 9.1 (cf. especially Figure 32). Each edge  $e$  of  $P$  contributes 0, 1, or 2 crossings to the oriented divide  $\vec{D}_0$ , depending on whether  $e$  connects two white vertices, a black and a white vertex, or two black vertices. To make things more uniform, let  $\vec{D}_1$  be the result of doing a safe tangency move for each edge of  $P$  that connects two white vertices. Now, for each edge  $e$  of  $P$ , the oriented divide  $\vec{D}_1$  has 1 or 2 crossings, depending on whether the endpoints of  $e$  have distinct colors or not. Let  $\vec{D}_2$  be the result of smoothing each crossing of  $\vec{D}_1$  in an orientation-preserving way. Let  $V$ ,  $E$ , and  $K$  denote the number of vertices in  $P$ , edges in  $P$ , and edges in  $P$  connecting vertices of the same color, respectively. Then, by Lemma 10.19,  $sl(T(\vec{D}_2)) = sl(T(\vec{D}_1)) - K - E$ . Since  $\vec{D}_2$  consists of  $V + K$  non-nested circles, we have  $sl(T(\vec{D}_2)) = -V - K$  by Lemma 10.18. Consequently,

$$sl(T(\vec{D}_0)) = sl(T(\vec{D}_1)) = K + E + (-V - K) = E - V = -\chi(P). \quad \square$$

**Corollary 10.20.** *Let  $D$  be a divide with  $a$  nodes and  $b$  interval branches. Let  $n$  denote the number of vertices in the quiver  $Q(D)$ . Then*

$$\text{sl}(T(D)) = 2a - b = n - 1.$$

*Proof.* Let  $G$  be the 1-skeleton of  $D$ ; it is a planar graph with  $a$  vertices of degree 4 and  $2b$  vertices of degree 1. The graph  $G$  has  $v = a + 2b$  vertices and  $e = \frac{1}{2}(4a + 2b) = 2a + b$  edges, so  $\chi(D) = -e + v = -a + b$ . The plabic graph  $P(D)$  has an extra cycle at each node of  $G$ , implying  $\chi(P(D)) = \chi(G) - a = -2a + b$ . By Proposition 10.16, we then have  $\text{sl}(T(D)) = -\chi(P(D)) = 2a - b$ .

Let  $f$  be the number of (bounded) regions of  $D$ . By Euler’s formula,  $v - e + f = 1$ . Also,  $n = a + f$ . Therefore,  $n - 1 = a + f - 1 = a - v + e = 2a - b$ . □

*Remark 10.21.* For any plabic graph  $P$ , the transverse link  $T(P)$  achieves the maximal self-linking number within its topological type, for the following reasons. First, there is a natural smooth surface  $\Sigma(P)$  embedded in the 4-dimensional ball  $\mathbf{B}^4$  such that  $\partial\Sigma(P) = T(P)$  and  $\Sigma(P)$  contains  $P$  as a spine, so that  $\chi(\Sigma(P)) = \chi(P)$ . This is explained by Shende, Treumann, Williams, and Zaslow in [70, Theorem 4.9] in a slightly different setting; we do not repeat the details. Second, the slice-Bennequin inequality [65] says that, for any transverse link  $T$  and smooth surface  $\Sigma \subset \mathbf{B}^4$  with  $\partial\Sigma = T$ , we have  $\text{sl}(T) \leq -\chi(\Sigma)$ . Combining these two facts with Proposition 10.16 implies that the self-linking number is maximal.

Given that the link of a (graph) divide is naturally transverse, one might think that Conjecture 7.11 and Problem 7.12 should be revised, with link equivalence replaced by a more refined notion based on transverse equivalence. This turns out to be unnecessary (see Corollary 10.23).

**Theorem 10.22** (Etnyre and van Horn-Morris [26, Corollary 5.3]). *Let  $L$  be a strongly quasipositive fibered link, with (quasipositive Seifert) fiber surface  $\Sigma$ . Then  $L$  has a unique transverse representative with self-linking number equal to  $-\chi(\Sigma)$ . (This is the maximal possible value of the self-linking number.)*

**Corollary 10.23.** *For any divides  $D_1$  and  $D_2$ , the following are equivalent:*

- (d)  $D_1$  and  $D_2$  are link equivalent (that is, the links  $L(D_1)$  and  $L(D_2)$  are isotopic);
- (t) the transverse links  $T(D_1)$  and  $T(D_2)$  are transverse isotopic.

*Proof.* Clearly (t)⇒(d). Let us prove the converse. The link  $L(D)$  of any divide  $D$  is fibered and strongly quasipositive (see Figure 37). Moreover,  $T(D)$  has the maximal self-linking number; to see this, either use Remark 10.21 or compare Corollary 10.20 with the Euler characteristic of the fiber surface for  $L(D)$ , as computed in [2, Remark 1]. By Theorem 10.22,  $L(D)$  has a unique transverse representative in the maximal self-linking number, so (d)⇒(t). □

Corollary 10.23 implies the transverse version of A’Campo’s Theorem 7.6.

**Corollary 10.24.** *The classical ( $\mathbb{C}$ -transverse, cf. Proposition 10.9) link of an isolated plane curve singularity is transverse isotopic to the transverse link  $T(D)$  associated to any algebraic divide  $D$  coming from a real morsification of this singularity.*

*Remark 10.25.* We can further extend Conjecture 7.11 by adding the statement **(t)** (transverse equivalence of links), which by Corollary 10.23 is equivalent to **(d)**, even without the assumption of algebraicity.

*Remark 10.26.* For a general plabic graph  $P$  (not necessarily coming from a divide), there is no reason to expect transverse simplicity; that is, link equivalence is unlikely to imply transverse equivalence.

*Remark 10.27.* The appearance of transverse (rather than Legendrian) links in the study of plane curve singularities might be puzzling to a reader familiar with Arnold's classical construction (cf. Remark 8.7) which associates to a plane curve a Legendrian link in the solid torus [7, 70]. However, this is too much to hope for in the context of links in the 3-sphere (as opposed to the solid torus). In particular, the boundary U-turn move does not appear to extend to a Legendrian isotopy in any natural way. As an operation on braid closure, it preserves the transverse type of the link but not any natural Legendrian type (cf. Remark 13.9).

## 11 | SCANNABLE DIVIDES

In Corollary 9.10, we established the implication **(p)** $\Rightarrow$ **(d)** of Conjecture 7.11. Our strategy for tackling the converse implication **(d)** $\Rightarrow$ **(p)** is closely aligned with the approach used by O. Couture and Perron [22, 23]. The main idea is to transform a given divide, via local moves preserving the associated link, into a 'scannable' divide (in their terminology, an 'ordered Morse divide'). The links of divides in this class are then described by a simple combinatorial rule.

In this section, we review the basic results of [23] needed for our purposes.

**Definition 11.1.** A *scannable divide* is a divide  $D$  drawn inside a rectangle of the form  $[a_0, a] \times [b_0, b] \subset \mathbb{R}^2$  so that the following conditions hold, for some  $a_0 < a_1 < a_2 < a$ . For every point  $(x_0, y_0)$  on  $D$  such that the tangent line to a local branch of  $D$  at  $(x_0, y_0)$  is vertical (that is, is given by the equation  $x = x_0$ ), we require that

- $(x_0, y_0)$  is a smooth point of  $D$  (that is, not a node);
- either  $x_0 = a_1$  or  $x_0 = a_2$ ;
- if  $x_0 = a_1$ , then the local branch of  $D$  lies to the right of the tangent;
- if  $x_0 = a_2$ , then the local branch of  $D$  lies to the left of the tangent.

In other words, we can parameterize each branch of  $D$  so that, as we move along it, the  $x$ -coordinate makes all of its U-turns at locations of the form  $(a_1, y)$  (approaching them from the right) or  $(a_2, y)$  (approaching from the left).

Every vertical line  $x = x_0$  with  $a_1 < x_0 < a_2$  intersects a scannable divide as above in the same number of points, the number of *strands* in the divide. We say that a scannable divide is of *minimal index* if this number is equal to the braid index of the corresponding link. (Recall that for algebraic divides, the braid index is equal to the multiplicity of the corresponding singularity, see Remark 7.9.)

When considering a scannable divide  $D$ , we ordinarily fix a particular representation of  $D$  of the form described above, up to isotopy within the class of divides with the given num-

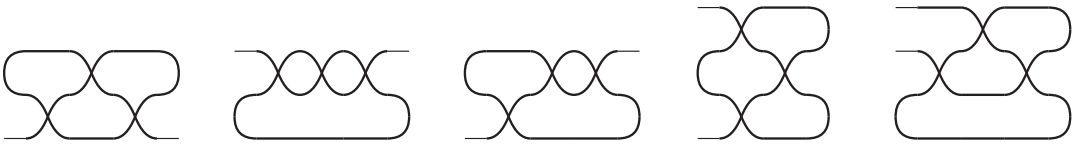


FIGURE 41 Isotopic scannable divides

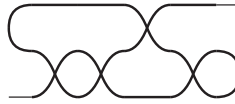


FIGURE 42 For this scannable divide  $D$ , we have  $\text{left}(D) = \sigma_2$ ,  $\text{right}(D) = \sigma_1$ ,  $\text{bulk}(D) = \sigma_1\sigma_1\sigma_2\sigma_1$ ,  $\text{kclub}(D) = \sigma_1\sigma_2\sigma_1\sigma_1$ , and  $\beta(D) = \sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1$

ber of strands inside a fixed ambient rectangle. Such a representation is by no means unique (cf. Remark 11.2).

When drawing a scannable divide, we only show its part lying within the rectangle  $[a_1, a_2] \times [b_0, b]$ , the rest of it being redundant.

To illustrate, all divides shown in Figure 4 are manifestly scannable, with the exception of the divide at the bottom of the last column, which is not scannable.

*Remark 11.2.* Isotopic scannable divides may have rather different combinatorial types, and may even have a different number of strands (see Figure 41).

**Definition 11.3.** Let  $D$  be a scannable divide with  $k$  strands. The braid  $\beta(D)$  associated with  $D$  is the (positive) braid in the  $k$ -strand braid group defined as follows. Let  $\sigma_i$  denote the positive Artin generator that switches the  $i$ th and  $(i+1)$ st strands. (The strands are numbered bottom up.) Let  $\text{left}(D)$  (respectively,  $\text{right}(D)$ ) be the product of the (commuting) generators  $\sigma_i$  corresponding to the pairs of adjacent strands  $(i, i+1)$  of the divide  $D$  which connect to each other at its left (respectively, right) end, at the points with a vertical tangent. Let  $\text{bulk}(D)$  be the product of Artin generators  $\sigma_i$ , multiplied left to right, corresponding to the pairs of adjacent strands of the divide which get switched as we scan it left to right. Let  $\text{kclub}(D)$  denote the product of the same generators, multiplied right to left. We then set

$$\beta(D) = \text{left}(D)\text{bulk}(D)\text{right}(D)\text{kclub}(D).$$

See Figure 42 for an example. Additional examples appear further in the text.

We can now state the key result by Couture and Perron.

**Theorem 11.4** [23, Proposition 2.3]. *The link  $L(D)$  of a scannable divide  $D$  is isotopic to the closure of the positive braid  $\beta(D)$  given by Definition 11.3.*

Theorem 11.4 does not require the divide  $D$  to be algebraic.

**Example 11.5.** Let  $D_1$  and  $D_2$  be the scannable divides shown in Figure 7. Denote

$$\Delta = \sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2 = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3 = \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1.$$

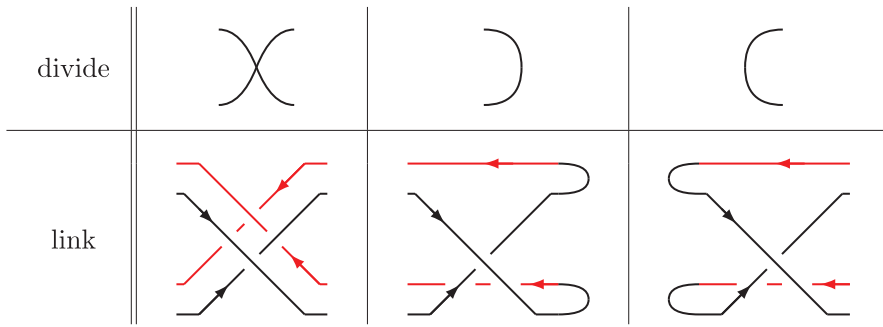


FIGURE 43 Transforming a scannable divide into a link diagram

Then

$$\beta(D_1) = \sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2 = \Delta\sigma_1\sigma_3\Delta = \Delta^2\sigma_1\sigma_3,$$

$$\beta(D_2) = \sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_1\sigma_2 = \sigma_2^{-1}\Delta^2\sigma_1\sigma_3\sigma_2,$$

which shows that the braids  $\beta(D_1)$  and  $\beta(D_2)$  are conjugate to each other. This was to be expected, since the divides  $D_1$  and  $D_2$  come from morsifications of two different real forms of the same complex singularity.

The description of the link  $L(D)$  associated with a scannable divide  $D$  given in Theorem 11.4 and Definition 11.3 can be recast in the language of conventional link diagrams, as follows. Replace each strand of  $D$  by a pair of parallel strands. Transform each crossing in  $D$ , and each coupling of its strands that occurs at either of the two ends of  $D$ , using the recipe shown in Figure 43. Finally, cap each of the remaining ‘loose ends’ of  $D$  by connecting the corresponding two strands of the link to each other. The resulting (oriented) link is isotopic to  $L(D)$ . (This claim is merely a restatement of Theorem 11.4.) An example is shown in Figure 44.

We note that our convention for drawing braids and links is different from [23], as we are using right-handed twists as positive Artin generators.

## 12 | PLABIC FENCES

For a scannable divide  $D$ , there is a natural choice of a plabic graph attached to  $D$  which we call a ‘plabic fence’ (borrowing the term *fence* from Rudolph [64]). The combinatorics of plabic fences is closely connected to positive braids.

A quick note on pictorial conventions: from now on, to simplify the drawing process, we will not always picture the white vertices of a plabic graph as hollow circles. We will continue to depict black vertices as filled circles; all the remaining points in a drawing where three lines (representing edges of the graph) come together, as well as all the remaining endpoints, will be understood to represent the white vertices.

**Definition 12.1.** Fix an integer  $k \geq 2$ . Let  $w$  be an arbitrary word in the alphabet

$$\mathbf{A}_k = \{\sigma_1, \dots, \sigma_{k-1}\} \cup \{\tau_1, \dots, \tau_{k-1}\}. \tag{12.1}$$

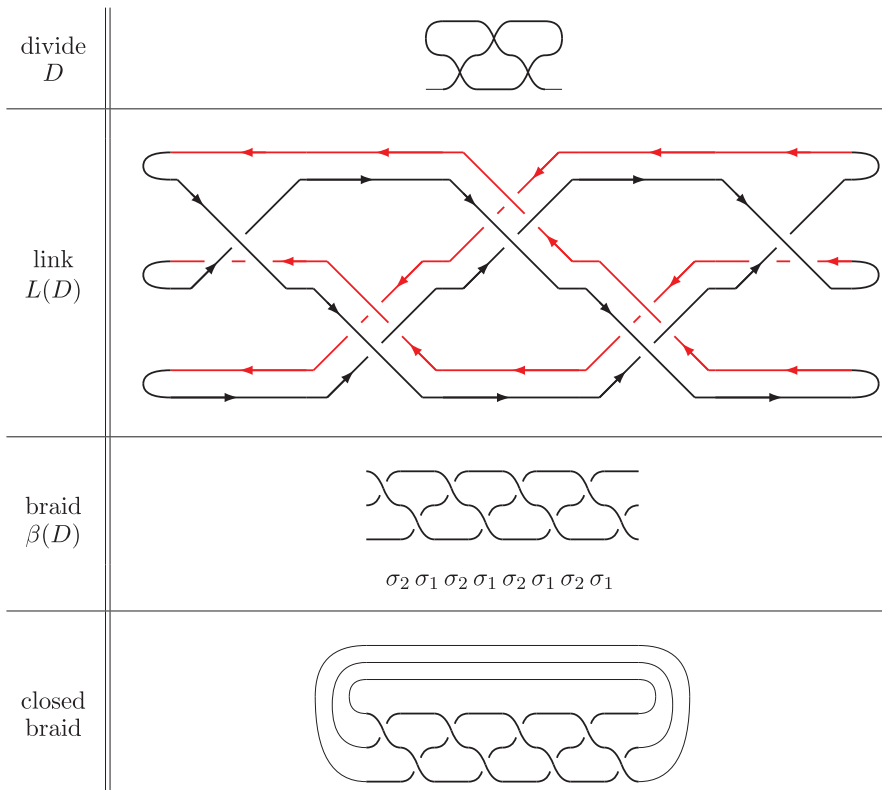


FIGURE 44 Scannable divide  $D$  coming from a morsification of a singularity of type  $E_6$ ; the associated link  $L(D)$  (a (3,4)-torus knot); and the corresponding positive braid  $\beta(D)$ . The closure of the braid recovers the link  $L(D)$

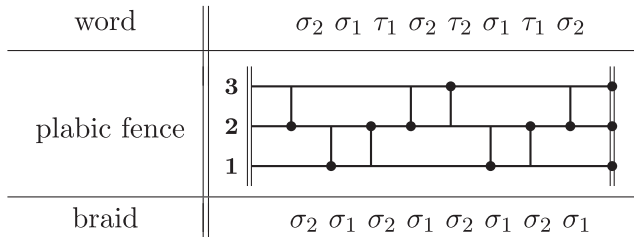




FIGURE 45 A word  $w$ , the associated plabic fence  $\Phi$ , and the braid  $\beta(w) = \beta(\Phi)$

The plabic graph  $\Phi = \Phi(w)$ , called the *plabic fence* associated with  $w$ , is constructed as follows. Begin by stacking  $k$  parallel horizontal line segments (‘strands’) on top of each other, and numbering them  $1, \dots, k$ , bottom to top. Reading the entries of  $w$  left to right, place vertical connectors between pairs of adjacent strands of the fence, representing each entry  $\sigma_i$  as  $\begin{smallmatrix} \perp \\ \downarrow \end{smallmatrix}$ , and each entry  $\tau_i$  as  $\begin{smallmatrix} \perp \\ \uparrow \end{smallmatrix}$ , each time connecting strands numbered  $i$  and  $i + 1$  (see Figure 45).

Conversely, any plabic fence  $\Phi$  as above determines the associated word  $w = w(\Phi)$  in the alphabet  $\mathbf{A}_k$ . To be a bit more precise, since  $\Phi$  is defined up to isotopy, the corresponding word  $w(\Phi)$  is defined up to transpositions of the form  $\sigma_i \sigma_j \leftrightarrow \sigma_j \sigma_i$ , or  $\sigma_i \tau_j \leftrightarrow \tau_j \sigma_i$ , or  $\tau_i \tau_j \leftrightarrow \tau_j \tau_i$ , for  $|i - j| \geq 2$ .

**Definition 12.2.** Let  $D$  be a scannable divide with  $k$  strands. We define the associated plabic fence  $\Phi(D)$  as follows:

- (i) stack  $k$  parallel horizontal line segments (strands) on top of each other;
- (ii) for each node in  $D$ , connect the corresponding strands in  $\Phi(D)$  by a pair of vertical edges, and color their four endpoints as follows: ;
- (iii) for each instance of two adjacent strands in  $D$  connecting to each other at an end of  $D$ , insert a connector  between the corresponding strands in  $\Phi(D)$ ;
- (iv) color the left endpoints of  $\Phi(D)$  white, and color the right endpoints black.

To illustrate, the scannable divide in Figure 44 gives rise to the plabic fence shown in Figure 45.

*Remark 12.3.* For a scannable divide  $D$ , the plabic fence  $\Phi(D)$  is a plabic graph attached to  $D$ , in the sense of Definition 6.11. In particular, the quiver  $Q(\Phi(D))$  associated with the plabic fence  $\Phi(D)$  coincides with the quiver  $Q(D)$  defined by  $D$ .

**Definition 12.4.** Let  $\Phi$  be a plabic fence on  $k$  strands, and let  $w$  be the corresponding word in the alphabet  $\mathbf{A}_k$  (see (12.1)). We define the positive braid  $\beta(\Phi) = \beta(w)$  in the  $k$ -strand braid group  $\mathbb{B}_k$  as follows. Let  $\bar{w}$  be the word obtained from  $w$  by first recording all the entries of the form  $\sigma_i$ , left to right, then all the entries of the form  $\tau_i$ , right to left, then replacing each  $\tau_i$  by  $\sigma_i$ . The braid  $\beta(\Phi)$  is then obtained from  $\bar{w}$  by interpreting each  $\sigma_i$  as an Artin generator of  $\mathbb{B}_k$ .


To illustrate, if  $w = \sigma_1\tau_2\sigma_3\tau_4$ , then  $\beta(w) = \sigma_1\sigma_3\sigma_4\sigma_2$ . Also, see Figure 45.

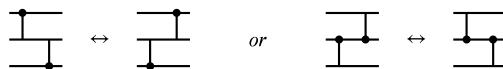
The following statement is easily confirmed by direct inspection.

**Proposition 12.5.** *Let  $D$  be a scannable divide. Then  $\beta(\Phi(D)) = \beta(D)$ . That is, the braid constructed from the plabic fence associated with  $D$  (see Definitions 12.2 and 12.4) coincides with the braid  $\beta(D)$  described in Definition 11.3.*

**Lemma 12.6.** *Let  $w$  be a word in the alphabet  $\mathbf{A}_k$ . Let  $\bar{w}$  be the corresponding braid word from Definition 12.4. Then the plabic fences  $\Phi(w)$  and  $\Phi(\bar{w})$  are move equivalent.*

*Proof.* The braid word  $\bar{w}$  can be obtained from  $w$  by repeatedly applying transformations of the form  $\dots \tau_j \sigma_i \dots \rightsquigarrow \dots \sigma_i \tau_j \dots$  (pushing the letters  $\tau$  to the right of the letters  $\sigma$ ) and/or  $\dots \tau_i \rightsquigarrow \dots \sigma_i$  (replacing  $\tau_i$  by  $\sigma_i$  at the end of the word). Each of these transformations can be viewed as an instance of move equivalence:

- a switch  $\tau_i \sigma_i \leftrightarrow \sigma_i \tau_i$  corresponds to a square move: 
- a switch  $\tau_{i\pm 1} \sigma_i \leftrightarrow \sigma_i \tau_{i\pm 1}$  corresponds to a flip move:



- a switch  $\tau_j \sigma_i \rightsquigarrow \sigma_i \tau_j$  for  $|i - j| \geq 2$  translates into an isotopy of the plabic fence;



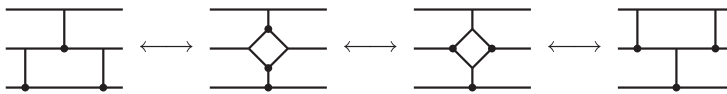
- replacing  $\tau_i$  by  $\sigma_i$  at the end of a word is emulated by a tail removal followed by a tail attachment:

$$\begin{array}{c} \text{I} \\ \text{I} \end{array} \leftrightarrow \begin{array}{c} \text{I} \\ \text{I} \end{array} \leftrightarrow \begin{array}{c} \text{I} \\ \text{I} \end{array} \tag{12.2}$$

□

**Proposition 12.7.** *Let  $\Phi_1$  and  $\Phi_2$  be plabic fences on  $k$  strands. If the braids  $\beta(\Phi_1), \beta(\Phi_2) \in \mathbb{B}_k$  are equal to each other, then  $\Phi_1$  and  $\Phi_2$  are move equivalent.*

*Proof.* By Lemma 12.6, it is enough to treat the case of plabic fences whose associated words only involve the functions  $\sigma$  but not the functions  $\tau$ . In this case, we need to check that each relation in Artin’s presentation of  $\mathbb{B}_k$  translates into an instance of move equivalence for the corresponding plabic fences. Indeed, the switches  $\sigma_j\sigma_i \leftrightarrow \sigma_i\sigma_j$  for  $|i - j| \geq 2$  translate into isotopies of the plabic graph, whereas the braid relations  $\sigma_i\sigma_{i+1}\sigma_i \leftrightarrow \sigma_{i+1}\sigma_i\sigma_{i+1}$  are emulated by flip and square moves, as follows:



□

### 13 | POSITIVE BRAID ISOTOPY

As explained in Section 11, (the diagram of) the link of a scannable divide is described by a simple and explicit combinatorial recipe. Therefore, it is natural to try to establish our main conjectures in the case of scannable divides. We start by noting that for two scannable divides  $D$  and  $D'$ , the following are equivalent:

- $D_1$  and  $D_2$  are link equivalent;
- the closures of the braids  $\beta(D_1)$  and  $\beta(D_2)$  are isotopic;
- the braids  $\beta(D_1)$  and  $\beta(D_2)$  can be obtained from each other by a sequence of Markov moves combined with braid conjugation.

The first two statements are equivalent by Theorem 11.4; the last two statements are equivalent by Markov’s theorem.

**Definition 13.1.** Two positive braids  $\beta_1$  and  $\beta_2$ , or more precisely positive braid words defining them, are called *positive-isotopic* to each other if they are related by a sequence of the following transformations:

- (i) isotopy among positive braids (that is, applying Artin’s braid relations);
- (ii) cyclic shifts (that is, moving the last entry in a braid word to the front);
- (iii) positive Markov moves and their inverses.

(A positive Markov move adds a strand at the top of a  $k$ -strand braid, and inserts the Artin generator  $\sigma_k$  into the braid word, at a single arbitrarily chosen location.) If  $\beta_1$  and  $\beta_2$  can be related to each other using transformations (i)–(ii) only, then we say that they are *positive-isotopic inside the solid torus*.

By definition, positive braid isotopy (respectively, positive isotopy inside the solid torus) corresponds to a particular subclass of isotopies of *closed* positive braids inside  $\mathbb{R}^3$  (respectively, inside the solid torus).

If two positive braids are positive-isotopic inside the solid torus, then they have the same number of strands, and moreover are conjugate to each other. In general, conjugate positive braids do not have to be positive-isotopic inside the solid torus.

If one drops the adjective ‘positive’, the situation simplifies: two braids with the same number of strands are conjugate if and only if they are isotopic in the solid torus (see [56] or [44, Theorem 2.1]).

The positive braids  $\beta(D_1)$  and  $\beta(D_2)$  in Example 11.5 are positive-isotopic to each other inside the solid torus. Another set of examples is provided by Figure 41: all braids associated to the divides shown therein are positive-isotopic to each other. Some of these braids have different number of strands, and consequently are not positive-isotopic inside the solid torus.

**Conjecture 13.2.** *Let  $D_1$  and  $D_2$  be link equivalent scannable algebraic divides. Then the braids  $\beta(D_1)$  and  $\beta(D_2)$  are positive-isotopic.*

**Conjecture 13.3.** *Let  $D_1$  and  $D_2$  be link equivalent scannable algebraic divides of (the same) minimal index. Then the braids  $\beta(D_1)$  and  $\beta(D_2)$  are positive-isotopic inside the solid torus.*

Conjectures 13.2 and 13.3 — especially the latter one — are motivated by a couple of observations due to Stepan Orevkov (see Lemmas 13.4 and 13.5). We thank Stepan for allowing us to cite them here.

The first observation is based on Garside’s solution of the conjugacy problem in the braid group.

**Lemma 13.4** [27, Section 3.2]. *Suppose that  $\beta$  is a positive braid whose closure contains the positive half-twist  $\Delta$ , that is,  $\beta$  is positive isotopic inside a solid torus to a braid of the form  $\gamma\Delta$ , with  $\gamma$  positive. Then any braid conjugate to  $\beta$  is positive isotopic to  $\beta$  inside a solid torus.*

The second observation concerns a certain positive braid of minimal index associated with a given isolated singularity  $f(x, y) = 0$ , as follows. Instead of a Milnor ball, consider a (small) bi-disk

$$\mathbf{D}^2 = \{|x| \leq \varepsilon, |y| \leq \varepsilon\} \subset \mathbb{C}^2.$$

Assume, without loss of generality, that inside  $\mathbf{D}^2$  our curve stays close to the plane  $y = 0$ , so that its intersection with the boundary  $\partial(\mathbf{D}^2)$  is contained in the solid torus  $\mathbf{V} = \{|x| = \varepsilon, |y| \leq \varepsilon\} \subset \mathbb{C}^2$ . Let  $L_f$  denote this intersection:

$$L_f = \{f(x, y) = 0, |x| = \varepsilon, |y| \leq \varepsilon\}. \tag{13.1}$$

By construction,  $L_f$  is a link inside the solid torus  $\mathbf{V}$ . Furthermore,  $L_f$  is the closure of a minimal-index positive braid  $\beta_f$  obtained by cutting  $L_f$  by the plane  $\{x = x_o\}$ , for some  $x_o$  with  $|x_o| = \varepsilon$ . Note that all intersections of the complex line  $x = x_o$  with our complex curve are positive, and moreover no intersection point escapes the bi-disc as the line varies in the vertical pencil. The braid  $\beta_f$  depends on the choice of  $x_o$ , but its closure obviously does not.

**Lemma 13.5** [59; 73, p. 699, Proposition II]. *The braid  $\beta_f$  contains the positive full twist  $\Delta^2$ .*

*Proof.* It is not hard to see that  $\Delta^{-2}\beta_f$  is the (positive) braid corresponding to the blown-up singularity  $f(x, xy) = 0$ .  $\square$

*Remark 13.6.* The fact that  $\beta_f$  is a minimal-index braid also follows from [33, Corollary 2.4]: a positive braid on  $n$  strands containing  $\Delta^2$  has braid index  $n$ .

**Proposition 13.7.** *Conjecture 13.3 holds for link equivalent scannable algebraic divides  $D_1$  and  $D_2$  whose associated link  $L = L(D_1) = L(D_2)$  is isotopic inside the solid torus to the link  $L_f$  of the corresponding singularity (cf. (13.1)).*

Note that Theorem 7.6 only asserts that  $L$  and  $L_f$  are isotopic inside  $\mathbf{S}^3$ .

*Proof.* The fact that  $L$  is isotopic to  $L_f$  inside the solid torus means that the braids  $\beta(D)$  and  $\beta(D')$  are conjugate to  $\beta_f$ , and consequently to each other. Moreover, Lemma 13.5 implies that both braids contain  $\Delta^2$ , so by Lemma 13.4 they are positive isotopic to each other inside the solid torus.  $\square$

*Remark 13.8.* The isotopy condition in Proposition 13.7 is satisfied for all divides of minimal index coming from real morsifications constructed in [1] (see [1, Theorem 1]) and/or [50, Section 2] (see [50, Theorem 2]). Consequently, whenever such constructions are used to produce different real morsifications of (different real forms of) the same complex singularity, the positive braids associated with the corresponding scannable divides are positive isotopic to each other inside the solid torus. In particular, this statement holds for all examples of scannable algebraic divides of minimal index discussed in this paper.

*Remark 13.9.* As shown by Orevkov–V. Shevchishin [60] and Wrinkle [75], two braids (positive or not) are related by braid isotopy, cyclic shifts, and positive Markov moves if and only if their closures are transverse isotopic (using a natural transverse structure on the closure of a braid). By Theorem 10.22, transverse isotopy is the same as isotopy for many of the links we consider, including closures of positive braids like those from scannable divides, and also for (algebraic) links of singularities. Thus, the obstacle to a full proof of Conjecture 13.2 is the difference between positive versus plain isotopy of positive braids inside the solid torus. (Using Proposition 13.7 to establish positive isotopy requires an additional topological condition.)

We next prove the implication **(d)** $\Rightarrow$ **(p)** of Conjecture 7.11 in the case of scannable (not necessarily algebraic) divides, under the assumption of positive isotopy. This assumption is potentially redundant (cf. Conjecture 13.2 and Remark 13.8).

**Theorem 13.10.** *Let  $D_1$  and  $D_2$  be scannable divides whose respective braids  $\beta(D_1)$  and  $\beta(D_2)$  are positive-isotopic. (In particular,  $D_1$  and  $D_2$  are link equivalent.) Then the plabic fences  $\Phi(D_1)$  and  $\Phi(D_2)$  are move equivalent.*

In view of Proposition 12.5, Theorem 13.10 follows from Proposition 13.11.

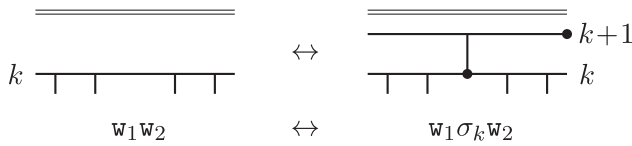


FIGURE 46 Positive Markov move via tail attachments

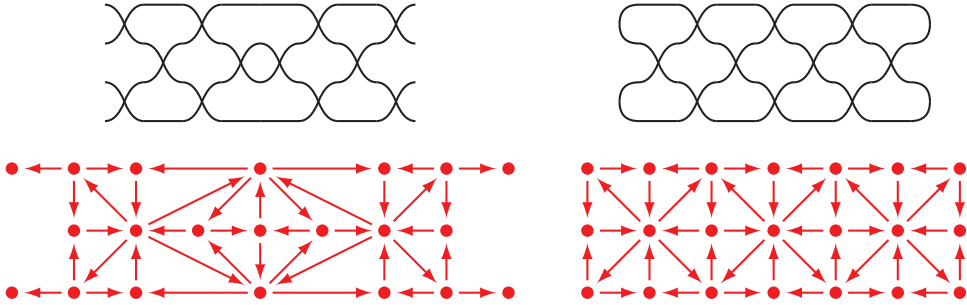


FIGURE 47 Scannable divides associated with two different morsifications of the quasi-homogeneous singularity  $x^8 + y^4 = 0$ , and the corresponding quivers

**Proposition 13.11.** *Let  $\Phi_1$  and  $\Phi_2$  be plabic fences (possibly with a different number of strands) whose associated braids  $\beta(\Phi_1)$  and  $\beta(\Phi_2)$  are positive-isotopic to each other. Then  $\Phi_1$  and  $\Phi_2$  are move equivalent.*

*Proof.* We need to show that each of the transformations (i)–(iii) in Definition 13.1 can be interpreted as an instance of move equivalence. Transformations (i) are covered by Proposition 12.7. A cyclic shift of the form  $w\sigma_i \leftrightarrow \sigma_i w$  can be executed by first replacing  $\sigma_i$  by  $\tau_i$  at the end of the word (cf. (12.2)), then moving  $\tau_i$  all the way to the left (cf. the proof of Lemma 12.6), then replacing  $\tau_i$  by  $\sigma_i$  at the beginning of the word. Finally, a positive Markov move of the form  $w_1 w_2 \leftrightarrow w_1 \sigma_k w_2$ , with  $w_1, w_2 \in \mathbb{B}_k$  and  $w_1 \sigma_k w_2 \in \mathbb{B}_{k+1}$  (respectively, the reverse of it) is emulated by two tail attachments (respectively, tail removals) (see Figure 46). □

**Corollary 13.12.** *Scannable divides whose associated braids are positive-isotopic have mutation equivalent quivers.*

*Proof.* By Propositions 6.9 and 6.12, move equivalence of the plabic fences  $\Phi(D_1)$  and  $\Phi(D_2)$  implies mutation equivalence of the quivers  $Q(D_1)$  and  $Q(D_2)$ . □

**Example 13.13.** Consider the two divides shown in Figure 47. Their associated positive braids are both equal to  $\Delta^4$ , where  $\Delta = \sigma_1 \sigma_3 \sigma_2 \sigma_1 \sigma_3 \sigma_2 = \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_1 \sigma_3$ . (The link equivalence of these two divides is explained by the fact that they arise from morsifications of different real forms of the quasi-homogeneous singularity  $x^8 + y^4 = 0$ , cf. Figure 8(b).) By Corollary 13.12, the quivers associated to these divides must be mutation equivalent to each other. This can be also verified directly using any of the widely available software packages for quiver mutations.

We conclude this section by a discussion of one simple instance of Conjecture 13.2.

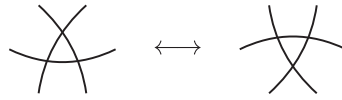


FIGURE 48 A triangle move

*Remark 13.14.* The Klein four-group  $\mathbf{K}$  naturally acts on (isomorphism classes of) scannable divides. For  $D$  a scannable divide rendered as in Definition 11.1, the images of  $D$  under the action of  $\mathbf{K}$  are:

- $D$  itself;
- the reflection of  $D$  with respect to a vertical line, denoted  $D^{\leftrightarrow}$ ;
- the reflection of  $D$  with respect to a horizontal line, denoted  $D^{\updownarrow}$ ;
- the result of rotating  $D$  by a  $180^\circ$  turn, denoted  $D^{\curvearrowright}$ .

Each of the scannable divides  $D^{\leftrightarrow}, D^{\updownarrow}, D^{\curvearrowright}$  is link equivalent to  $D$ . If  $D$  is algebraic, then so are  $D^{\leftrightarrow}, D^{\updownarrow}$ , and  $D^{\curvearrowright}$ . According to Conjecture 13.2, the four positive braids  $\beta(D), \beta(D^{\leftrightarrow}), \beta(D^{\updownarrow})$  and  $\beta(D^{\curvearrowright})$  must be positive-isotopic to each other. It is easy to see that  $\beta(D^{\leftrightarrow})$  and  $\beta(D)$  are related via cyclic shifts, so these braids are positive-isotopic. The challenge is to prove that  $\beta(D)$  is positive-isotopic to  $\beta(D^{\updownarrow})$  and  $\beta(D^{\curvearrowright})$ , in the case of algebraic divides (and possibly beyond). It would suffice to establish this claim for  $\beta(D^{\updownarrow})$ . While  $\beta(D^{\updownarrow})$  is conjugate to  $\beta(D)$  by the half-twist  $\Delta$ , this in itself does not guarantee the existence of positive isotopy. On the other hand, the latter property holds whenever  $\beta(D)$  contains  $\Delta$ , by Lemma 13.4. In view of Lemma 13.5 and Remark 13.8, this condition is satisfied for the scannable algebraic divides of minimal index arising from all common constructions of real morsifications.

## 14 | TRIANGLE MOVES

**Definition 14.1.** A *triangle move* (or a ‘Yang-Baxter move’) is a local deformation of a divide that can be applied for any triangular region, that is, a region whose boundary consists of three 1-cells and three nodes. A triangle move pushes a branch containing one of these three 1-cells across the opposing node (see Figure 48).

**Definition 14.2.** Two divides are called  $\mathbb{X}$ -*equivalent* (‘triangle-move equivalent’) if they are related via a sequence of triangle moves and isotopies. An example is shown in Figure 49.

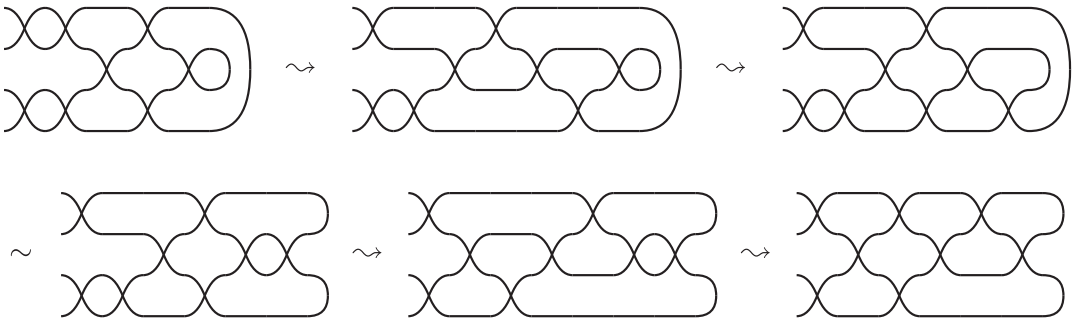
*Problem 14.3.* Can an algebraic divide be  $\mathbb{X}$ -equivalent to a non-algebraic one?

The following result, in different guises, has been a part of the ‘cluster folklore’ for at least a decade; we do not claim any originality for it (see, for example, [48, Section 2.3]).

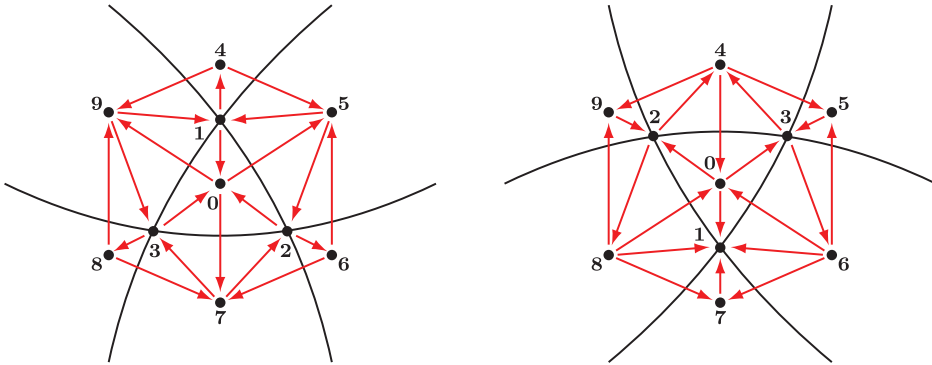
**Proposition 14.4.**  $\mathbb{X}$ -equivalent divides have mutation equivalent quivers.

We sketch two (implicitly related) proofs of Proposition 14.4.

*Proof.* Let us examine what happens to the quiver in the vicinity of a triangle move. The case where all neighboring connected components of the complement of a divide are bounded is shown in Figure 50. There are many other cases (cf., for example, Figure 12). □



**FIGURE 49** A sequence of triangle moves (denoted  $\rightsquigarrow$ ) and isotopies (denoted  $\sim$ ). All these divides are  $\mathbb{A}$ -equivalent to each other. The first two divides are not scannable. While the shown drawing of the third divide is not scannable, this divide is isotopic to the first divide in the bottom row, which is scannable (as are all divides in this row)



**FIGURE 50** Quivers of two divides related by a triangle move. Only the arrows connecting pairs of vertices labeled 0, 1, 2, ..., 9 are shown. The two quivers are related via the composition of 5 mutations  $\mu_0 \circ \mu_1 \circ \mu_2 \circ \mu_3 \circ \mu_4$

An alternative proof of Proposition 14.4 uses the machinery of plabic graphs. In view of Propositions 6.9 and 6.12, it suffices to establish the following claim.

**Proposition 14.5.** *Let  $D_1$  and  $D_2$  be two  $\mathbb{A}$ -equivalent divides. Then any plabic graphs  $P_1 \in \mathbf{P}(D_1)$  and  $P_2 \in \mathbf{P}(D_2)$  are move equivalent.*

*Proof.* For  $D_1$  and  $D_2$  related by a triangle move, Figure 51 shows a sequence of local moves relating a plabic graph  $P_1 \in \mathbf{P}(D_1)$  to a plabic graph  $P_2 \in \mathbf{P}(D_2)$ . □

*Remark 14.6.* The above argument can be recast in the language of plabic fences and associated words (cf. Section 12). Look at the fragments of plabic graphs shown in Figure 51 in the upper-left and lower-right corners. These fragments can be drawn as plabic fences on 3 strands (see Figure 52). Their associated words are  $\sigma_1\tau_1\sigma_2\tau_2\sigma_1\tau_1$  and  $\sigma_2\tau_2\sigma_1\tau_1\sigma_2\tau_2$ , respectively. These are related to each other via switches of the form  $\tau_j\sigma_i \leftrightarrow \sigma_i\tau_j$  combined with the braid relations  $\sigma_1\sigma_2\sigma_1 \leftrightarrow \sigma_2\sigma_1\sigma_2$  and  $\tau_1\tau_2\tau_1 \leftrightarrow \tau_2\tau_1\tau_2$ .

Proposition 14.5 and Corollary 9.5 imply the following result.

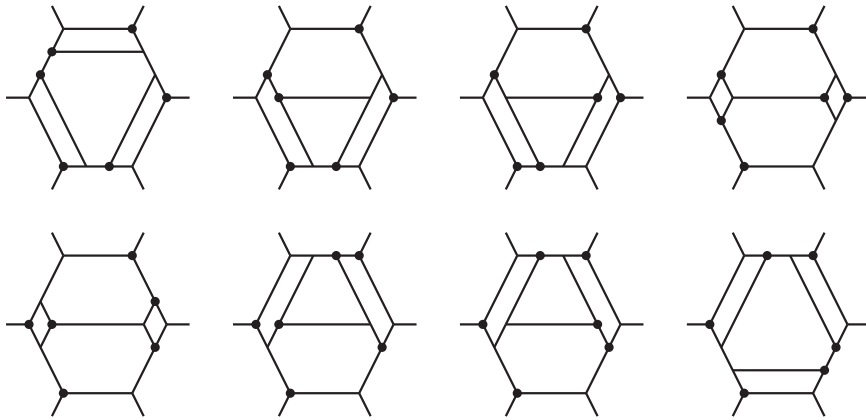


FIGURE 51 Viewing a triangle move as a sequence of local moves

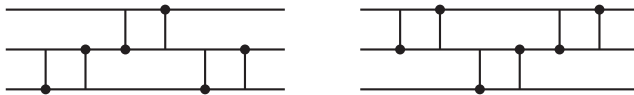


FIGURE 52 Interpreting a triangle move in the language of fences

**Proposition 14.7** [23, Lemma 1.3].  *$\mathbb{A}$ -equivalent divides are link equivalent.*

*Problem 14.8.* Let  $\mathcal{D}$  denote the set of all (algebraic) divides coming from morsifications of the same real singularity. Are any two divides in this set  $\mathbb{A}$ -equivalent? If  $D'$  is a divide  $\mathbb{A}$ -equivalent to  $D \in \mathcal{D}$ , does it follow that  $D' \in \mathcal{D}$ ? (cf. Problem 14.3).

By Propositions 14.4, 14.5, and 14.7, triangle moves preserve:

- (d) the isotopy class of the A'Campo link of a divide;
- (q) the mutation class of the associated quiver; and
- (p) the move equivalence class of the associated plabic graphs.

This means that once a connection between the statements **(d)**, **(q)**, and **(p)** appearing in Conjecture 7.11 has been established for a particular class of divides, it can be immediately extended to all divides  $\mathbb{A}$ -equivalent to a divide in this class. With this in mind, we make the following definition.

**Definition 14.9.** A divide is *malleable* if it is  $\mathbb{A}$ -equivalent to a scannable divide.

To illustrate, all divides in Figure 49 are malleable.

**Conjecture 14.10.** *Every algebraic divide is malleable.*

Theorem 14.11 establishes the implication **(d)** $\Rightarrow$ **(p)** of Conjecture 7.11 under the assumptions of positive isotopy and malleability. Each of these assumptions is potentially redundant (cf. Conjectures 13.2 and 14.10, respectively).

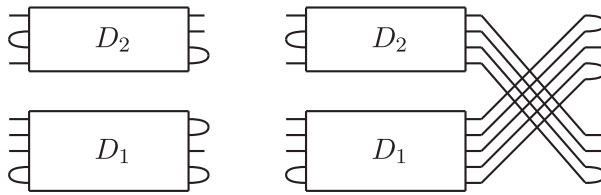


FIGURE 53 Two scannable divides, and their transversal overlay

**Theorem 14.11.** *Let  $D_1$  and  $D_2$  be (link equivalent) malleable divides which are  $\mathbb{X}$ -equivalent to scannable divides whose respective braids are positive-isotopic. Then the associated plabic graphs  $P_1 \in \mathbf{P}(D_1)$  and  $P_2 \in \mathbf{P}(D_2)$  are move equivalent. Consequently, the quivers  $Q(D_1)$  and  $Q(D_2)$  are mutation equivalent.*

*Proof.* Follows from Theorem 13.10 and Propositions 14.5 and 14.7. □

**Example 14.12.** Consider the three divides  $D_1, D_2, D_3$  in the lower-right corner of Figure 4, representing the morsifications of three different real forms of the quasi-homogeneous singularity  $x^6 + y^4 = 0$ . The first two divides are scannable, with the same associated braid  $\beta(D_1) = \beta(D_2) = \Delta^3$ , where  $\Delta = (\sigma_1\sigma_3\sigma_2)^2 = (\sigma_2\sigma_1\sigma_3)^2$  is the positive half-twist. The divide  $D_3$  is not scannable but malleable: it is  $\mathbb{X}$ -equivalent to the scannable divide  $D$  shown in Figure 49 at the right end of the bottom row. The braid associated with  $D$  is given by

$$\beta(D) = \sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2 \cdot \sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3 \cdot \sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3 = \Delta^3 = \beta(D_1) = \beta(D_2).$$

This means (see Proposition 12.7) that the plabic graphs associated with  $D_1, D_2,$  and  $D$  (and hence  $D_3$ ) are pairwise move equivalent, as asserted by Theorem 14.11. Furthermore the quivers  $Q(D_1), Q(D_2), Q(D_3)$  are mutation equivalent to each other.

## 15 | TRANSVERSAL OVERLAYS

In this section, we discuss operations which combine scannable divides to produce new divides, also scannable. We show that by changing the order in which these operations are applied, one can obtain different scannable divides which are link equivalent to each other, with positive-isotopic braids. By Corollary 13.12, it follows that the corresponding quivers are mutation equivalent.

**Definition 15.1.** Let  $D_1$  and  $D_2$  be two scannable divides, with  $k_1$  and  $k_2$  strands respectively, oriented on the co-ordinate plane as in Section 11. Let us place  $D_2$  above  $D_1$ , so that their respective ambient rectangles have the same width. Stretch their strands horizontally near the right ends, then bend these extensions up (for  $D_1$ ) and down (for  $D_2$ ), as shown in Figure 53, thereby creating  $k_1k_2$  new nodes in the form of a grid. The resulting divide  $D_1 \# D_2$ , which is scannable by construction, is called the *transversal overlay* of  $D_1$  and  $D_2$ .

We note that this operation depends on a choice of ‘scanning directions’ for the input divides. If two divides  $D_1$  and  $D'_1$  are isotopic to each other but have different scanning directions (in particular, they might have a different number of strands, cf. Figure 41), then the overlays  $D_1 \# D_2$  and  $D'_1 \# D_2$  do not have to be isotopic.



*Remark 15.2.* The construction of Definition 15.1 allows for a multitude of modifications, obtained as follows. Stretch both divides  $D_1$  and  $D_2$  very wide, then rotate one of them and overlay on top of the other, making sure that each strand of  $D_1$  transversally intersects each strand of  $D_2$  exactly once. All such overlays are  $\mathbb{A}^1$ -equivalent to each other, and consequently have equivalent (that is, isotopic or move/mutation equivalent) links, braids, quivers, and plabic graphs.

*Remark 15.3.* Transversal overlays have a natural interpretation in the context of plane curve singularities. To explain this, we shall make some simplifying assumptions which can in principle be relaxed. Let  $(C, z)$  and  $(C', z)$  be two complex isolated plane curve singularities with a common singular point  $z$ . Suppose  $C$  and  $C'$  are in general position at  $z$  with respect to each other, that is,  $C$  and  $C'$  have no common tangents. Then the topological type of the overlay  $(C \cup C', z)$  is canonically defined.

Let  $(C, z)$  be a real singularity, and  $L$  a line transversal to it. Suppose that  $(C_t)_{0 \leq t \leq \zeta}$  is a real morsification of  $(C, z)$  such that each curve  $\mathbb{R}C_t$ , for  $0 < t \leq \zeta$ , is scannable (in an appropriate neighborhood of  $z$ ) by a pencil of lines parallel to  $L$ . Each of these lines intersects  $\mathbb{R}C_t$  in the same number of points equal to the multiplicity of  $(C, z)$ . The direction of  $L$  corresponds to the vertical direction in Definition 11.1.

Now let  $(C, z)$  and  $(C', z)$  be real singularities, say of multiplicities  $k$  and  $k'$ , respectively. Suppose that each of them has a scannable morsification as above. Then the same is true for the (generic) transversal overlay  $(C \cup C', z)$ . To obtain a scannable morsification for  $(C \cup C', z)$ , let us shrink each of the two input morsifications along their respective transversal directions, then overlay them at an angle, so that they intersect in  $kk'$  points. The divide corresponding to the resulting morsification is obtained from the two input divides via the procedure outlined in Remark 15.2.

**Definition 15.4.** We define the equivalence relation  $\overset{+}{\sim}$  on scannable divides as follows. Let  $D$  and  $D'$  be two scannable divides with the same number of strands. The notation  $D \overset{+}{\sim} D'$  means that the associated braids  $\beta(D)$  and  $\beta(D')$  are positive-isotopic to each other inside the solid torus (see Definition 13.1).

The operation of transversal overlay of scannable divides descends to the level of equivalence classes with respect to the equivalence relation  $\overset{+}{\sim}$ .

**Lemma 15.5.** *Let  $D_1, D_2, D'_1, D'_2$  be scannable divides. If  $D_1 \overset{+}{\sim} D'_1$  and  $D_2 \overset{+}{\sim} D'_2$ , then  $D_1 \# D_2 \overset{+}{\sim} D'_1 \# D'_2$ .*

*Proof.* Direct inspection shows that the positive braid  $\beta(D_1 \# D_2)$  associated with the transversal overlay of two scannable divides  $D_1$  and  $D_2$  is obtained by ‘linking’ the braids  $\beta(D_1)$  and  $\beta(D_2)$  as shown in Figure 54. The closure of  $\beta(D_1 \# D_2)$  is thus obtained by placing the closures of  $\beta(D_1)$  and  $\beta(D_2)$  in the vicinities of the two components of a Hopf link. The claim follows.  $\square$

**Lemma 15.6.** *Transversal overlay of scannable divides is associative and commutative modulo the equivalence  $\overset{+}{\sim}$ .*

*Proof.* It is easy to see that transversal overlay is associative modulo  $\mathbb{A}^1$ -equivalence. More precisely, if  $D_1, D_2, D_3$  are scannable divides, then the divides  $(D_1 \# D_2) \# D_3$  and  $D_1 \# (D_2 \# D_3)$  are  $\mathbb{A}^1$ -equivalent

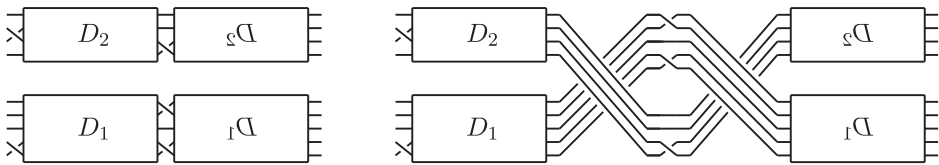


FIGURE 54 Positive braids associated with scannable divides  $D_1$  and  $D_2$  (on the left) and with their transversal overlay (on the right)

to each other. Moreover, the corresponding braids are positive-isotopic inside the solid torus (cf. Remark 14.6).

To prove commutativity, we need to show that the positive braid  $\beta(D_1 \# D_2)$  shown in Figure 54 and the analogous braid  $\beta(D_2 \# D_1)$  are positive-isotopic to each other inside the solid torus. To see that, pull the strands of  $\beta(D_1 \# D_2)$  coming from  $\beta(D_1)$  (respectively, from  $\beta(D_2)$ ) along the corresponding components of the Hopf link, so that the fragments marked  $D_1$  and  $\iota\mathbb{A}$  in Figure 54 get repositioned above the fragments marked  $D_2$  and  $\varepsilon\mathbb{A}$ , matching — up to a cyclic shift — the braid  $\beta(D_2 \# D_1)$ . □

*Remark 15.7.* Let  $D_1, \dots, D_m$  be scannable divides. In view of Remark 15.2 and Lemmas 15.5 and 15.6, we can construct different versions of the transversal overlay of these  $m$  divides, as follows. Take a generic planar arrangement of  $m$  straight lines. Pick an arbitrary bijection between these lines and the given divides. Stretch each divide  $D_i$  along its scanning direction (horizontally, in our usual rendering), then place  $D_i$  near the corresponding line. For each  $i$ , there are, generally speaking, four ways to do this, related by the action of the Klein four-group. We assume that all these four versions yield the same result modulo the equivalence  $\overset{+}{\sim}$ . This assumption is satisfied for all scannable divides arising from commonly used constructions (see Remark 13.14).

Different choices of a line arrangement, of an assignment of the divides  $D_1, \dots, D_m$  to the lines in the arrangement, and of a placement of each divide  $D_i$  along the corresponding line will produce different overlays of  $D_1, \dots, D_m$ . All of them will be  $\overset{+}{\sim}$ -equivalent to each other.

*Remark 15.8.* Iterated transversal overlays, as described in Remark 15.7, can be used to construct numerous examples in support of our main conjectures. As inputs, we take two  $m$ -tuples of scannable divides  $D_1, \dots, D_m$  and  $D'_1, \dots, D'_m$  such that, for  $i = 1, \dots, m$ ,

- the divides  $D_i$  and  $D'_i$  come from scannable morsifications of (potentially different real forms of) the same complex singularity, as in Remark 15.3;
- the divides  $D_i$  and  $D'_i$  are  $\overset{+}{\sim}$ -equivalent.

(For example, one can take  $D_i = D'_i$ , or let  $D_i$  and  $D'_i$  be related by the action of a Klein group element (cf. Remark 13.14). In any case, the number of strands in both  $D_i$  and  $D'_i$  should be equal to the multiplicity of the singularity.) For each of these two  $m$ -tuples of divides, we then construct some version of their transversal overlay (see Remark 15.7). This will produce a pair of divides  $D$  and  $D'$  such that

- $D$  and  $D'$  come from morsifications of (real forms of) the same complex singularity (see Remark 15.3); consequently,  $D$  and  $D'$  are link equivalent;
- $D$  and  $D'$  are  $\overset{+}{\sim}$ -equivalent, by Remark 15.2 and Lemmas 15.5 and 15.6; hence the quivers  $Q(D)$  and  $Q(D')$  are mutation equivalent, by Corollary 13.12.

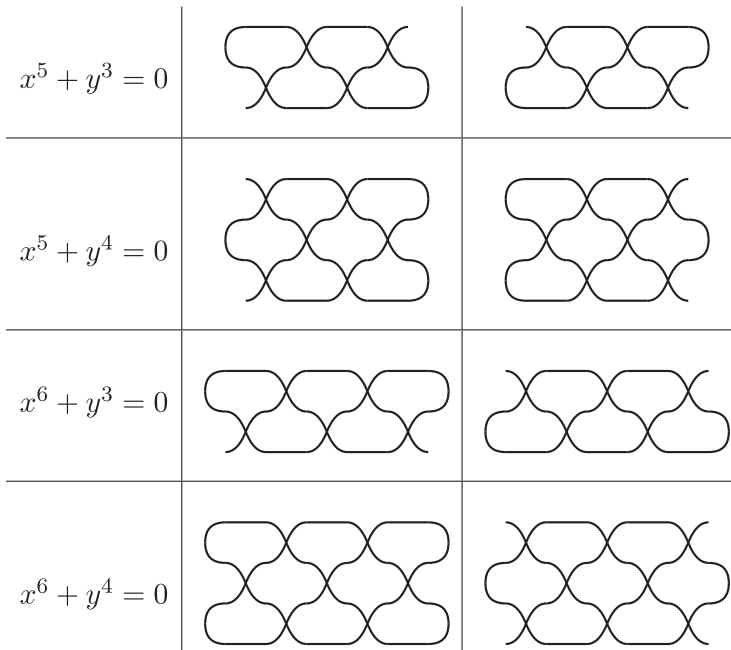


FIGURE 55 Lissajous divides (cf. Figure 4)

Summing up, morsifications of the same complex singularity obtained via different versions of transversal overlays, as described above, give rise to mutation equivalent quivers, thereby providing evidence in support of Conjecture 5.5.

A number of concrete examples are given in Section 16.

## 16 | LISSAJOUS DIVIDES AND WIRING DIAGRAMS

In this section, we illustrate the construction of Section 15 using a class of divides coming from quasi-homogeneous singularities. These examples show that mutation equivalence of quivers obtained via different iterated transversal overlays (as in Remark 15.8) may be far from obvious from a combinatorial standpoint.

**Definition 16.1.** A *Lissajous divide* of type  $(a, b)$  (here  $a \geq b \geq 2$ ) is one of the two scannable divides on  $b$  strands constructed as follows. Begin by drawing  $b$  horizontal and  $a$  vertical line segments in the form of a  $(b - 1) \times (a - 1)$  grid. Pick one of the two proper black-and-white (‘checkerboard’) colorings of the  $(b - 1)(a - 1)$  squares of the grid. Finally, replace each black square by a crossing  $\times$  (see Figure 55).

It is straightforward to check that the two Lissajous divides of type  $(a, b)$  are  $\overset{+}{\sim}$ -equivalent to each other.

It is well-known, and not hard to see, that a Lissajous divide of type  $(a, b)$  represents a morsification of (an appropriate real form of) the quasi-homogeneous singularity  $x^a + y^b = 0$  (cf. Definition 1.2). The conditions of Remark 15.8 are readily checked, providing a large class of examples

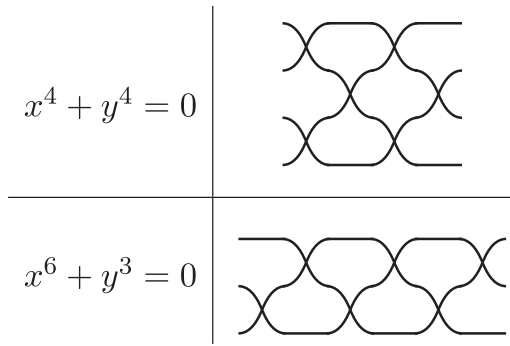


FIGURE 56 Alternating wiring diagrams of types (4,4), and (6,3) (cf. Figure 4)

of pairs of divides (namely, overlays of Lissajous divides) whose associated quivers are — provably — mutation equivalent.

This phenomenon is already combinatorially nontrivial for transversal overlays of singularities of types  $A_1$  and  $A_2$  (that is, nodes and/or cusps). Let  $D_1, \dots, D_m$  and  $D'_1, \dots, D'_m$  be such that for each  $i = 1, \dots, m$ , we have one of the following:

- each of  $D_i$  and  $D'_i$  is either an ellipse  $\circ$  or a pair of lines  $\times$ ; or
- each of  $D_i$  and  $D'_i$  is a nodal cubic  $\alpha$ .

When we construct transversal overlays for each of the two input  $m$ -tuples, we have the freedom of rotating each morsified cusp  $\alpha$ , by  $180^\circ$ , and more importantly, the freedom of choosing the cyclic ordering of the ingredient divides; this ordering determines the placement of their stretched versions near the  $m$  lines of a planar line arrangement. All the resulting divides will have mutation equivalent quivers.

*Remark 16.2.* The readers familiar with cluster algebras will recognize the quiver associated to a Lissajous divide of type  $(a, b)$  as the quiver defining the standard cluster structure on the corresponding Plücker ring, the homogeneous co-ordinate ring of the Grassmannian  $\text{Gr}_{a,a+b}(\mathbb{C})$  (see [69]). This suggests the existence of an intrinsic connection between the quasi-homogeneous complex singularity  $x^a + y^b = 0$  and the standard cluster structure on the affine cone over  $\text{Gr}_{a,a+b}(\mathbb{C})$ . One algebraic interpretation of this connection (via additive categorification) was proposed in [43]. It would be very interesting to find an explanation of this connection that directly relates Plücker rings, viewed as cluster algebras, to quasi-homogeneous singularities (viewed up to topological equivalence).

**Definition 16.3.** An *alternating wiring diagram* of type  $(a, b)$  (here  $a, b \geq 2$ ) is a scannable divide constructed via the following modification of Definition 16.1 Draw  $b$  horizontal and  $a + 1$  vertical line segments forming a grid of size  $(b - 1) \times a$ . Pick a checkerboard coloring of the squares of the grid. Replace each black square by a crossing  $\times$ . Finally, remove all the remaining vertical segments (see Figure 56).

In the special case  $a = b$ , we get a wiring diagram with  $b$  branches (‘pseudolines’) each pair of which intersect each other exactly once. This wiring diagram represents a morsification of the real singularity  $\prod_{i=1}^b (y - r_i x) = 0$ , with distinct real slopes  $r_i$ . More generally, when  $b$  divides  $a$ , an

alternating wiring diagram of type  $(a, b)$  represents a quasi-homogeneous singularity of the same type. (For arbitrary  $a$  and  $b$ , this can be false.) To be more precise, one can show that an alternating wiring diagram of type  $(bc, b)$  represents a morsification of a quasi-homogeneous singularity of type  $(bc, b)$ , specifically its real form involving  $b$  smooth branches each pair of which have a tangency of order  $c$ . (For  $c = 1$ , each pair is transversal.) It is also straightforward to check that this wiring diagram is  $\overset{+}{\sim}$ -equivalent to a Lissajous divide of type  $(bc, b)$ . We can consequently use this wiring diagram as a replacement for a Lissajous divide of type  $(bc, b)$  in any transversal overlay.

## 17 | ORIENTED PLABIC GRAPHS AND THEIR LINKS

In this section, we present an alternative approach to the theory of divide links. Unlike the Gibson–Ishikawa/Kawamura constructions (cf. Definition 9.1), this approach utilizes conventional link diagrams, so it is more explicit and combinatorial. Unfortunately, diagrammatic constructions based on uniform local rules appear to only exist for scannable divides (respectively, the corresponding class of plabic graphs), hence the limitations of the method. On the other hand, many morsifications arising via commonly used constructions are scannable (cf. Sections 15–16), so the technique described in this section has a fairly wide applicability.

Recall that in Theorem 13.10, we showed, modulo some technical assumptions, that link equivalence of scannable divides implies move equivalence of associated plabic graphs (fences). In this section, we provide a combinatorial proof of the converse implication, again modulo some (conjecturally redundant) assumptions. The end result (cf. Proposition 17.15) is thus substantially weaker than Corollary 9.5, which required neither those assumptions nor scannability. Nevertheless, we opted to include this section in the paper since the machinery developed herein is much simpler (though less powerful) than the one employed in Section 9; it also suggests connections with Postnikov’s theory of perfect orientations of plabic graphs (cf. Remark 17.2).

**Definition 17.1.** Let  $P$  be a plabic graph. An orientation of the edges of  $P$  is called *admissible* if it satisfies the following requirements:

- (B) at each trivalent black vertex, two edges are incoming, and one outgoing; at each univalent black vertex, one edge is incoming;
- (W) at each trivalent white vertex, two edges are outgoing, and one is incoming; at each univalent white vertex, one edge is outgoing;
- (F) the edges at the boundary of each internal face of  $P$  form a directed graph (an orientation of a cycle) with exactly one source and one sink.

*Remark 17.2.* In Postnikov’s original treatment [62], the orientations satisfying conditions (B) and (W) in Definition 17.1 (for the interior trivalent vertices) were called *perfect*. For our purposes however, ‘perfect’ is not enough, as we do need condition (F), or some variant thereof. (One acceptable way to relax condition (F) is to only forbid two types of orientations of the boundaries of internal faces, namely (a) oriented cycles and (b) orientations with two sources and two sinks.)

**Lemma 17.3.** *An admissible orientation of a plabic graph is acyclic.*

*Proof.* Suppose not. Let  $C$  be an oriented cycle in an admissible orientation. We may assume that  $C$  is simple (that is, it does not visit the same vertex more than once) and moreover  $C$  does

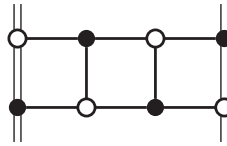


FIGURE 57 A balanced plabic graph that does not have an admissible orientation. (It does have perfect orientations, but they do not satisfy condition (F).)

not enclose another oriented cycle. If  $C$  encloses a single face, then we are in contradiction with condition (F). Otherwise  $C$  contains a vertex  $v$  incident to an edge  $e$  located inside  $C$ . Assume that  $e$  is oriented away from  $v$ , as the other case is similar. Starting with  $e$ , keep moving in the direction of the orientation. When arriving at a black vertex, make the unique choice of the outgoing edge; at a white vertex, choose any of the two outgoing edges. Eventually, this walk will either hit itself or hit  $C$ , thereby creating an oriented cycle enclosed by  $C$ , a contradiction.  $\square$

While an admissible orientation does not have to exist, it is always unique.

**Proposition 17.4.** *A plabic graph has at most one admissible orientation.*

(This statement relies on the convention that we do color boundary vertices.)

*Proof.* Let  $O_1$  and  $O_2$  be distinct admissible orientations of the same plabic graph  $P$ . At each univalent boundary vertex, the two orientations coincide by conditions (B) and (W). At a trivalent vertex  $v$  in the interior of  $P$ , they either coincide for all three edges, or else they coincide at one edge, and are opposite at the remaining two edges  $e_1$  and  $e_2$ ; moreover  $e_1$  and  $e_2$  form an oriented two-edge path in both  $O_1$  and  $O_2$ . It follows that the edges whose orientations in  $O_1$  and  $O_2$  differ from each other form a collection of disjoint oriented cycles (with different orientations in  $O_1$  and in  $O_2$ ). This contradicts Lemma 17.3.  $\square$

A plabic graph is *balanced* if it contains equally many black and white vertices.

To illustrate, any plabic fence (cf. Definition 12.1) is balanced, assuming its boundary vertices located at the left (respectively, right) end are colored white (respectively, black), as in Figure 45.

**Lemma 17.5.** *Any plabic graph possessing an admissible orientation is balanced.*

*Proof.* In an admissible orientation, at each white (respectively, black) vertex, whether internal or located on the boundary, the number of outgoing edges minus the number of incoming edges is equal to 1 (respectively,  $-1$ ). Summing over all the half-edges, we obtain the claim.  $\square$

*Remark 17.6.* The converse to Lemma 17.5 is false: a balanced plabic graph does not have to allow an admissible orientation. A counterexample is shown in Figure 57.

*Remark 17.7.* The property of being balanced is preserved by both flip and square moves (cf. Figure 17a,b). For it to be preserved by a tail attachment/removal move, one needs to require that the two vertices involved in the move have opposite colors, as in Figure 17(c). Similarly, a switch transformation (see Definition 6.15) preserves balanceness provided the portion of the

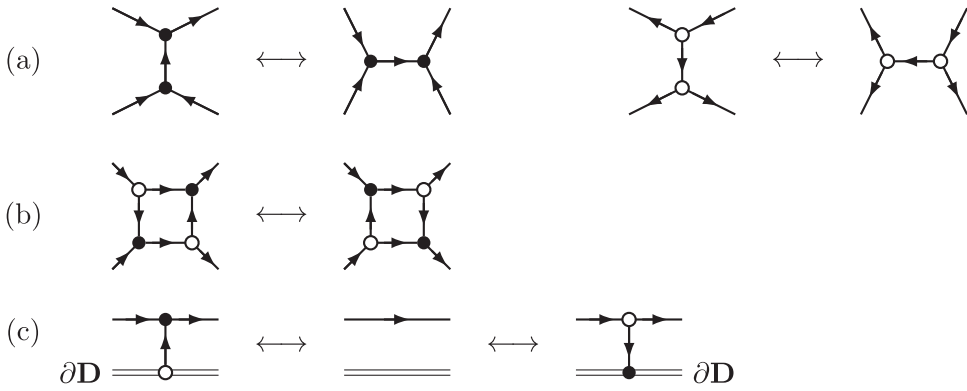


FIGURE 58 Transporting admissible orientations via moves in plabic graphs

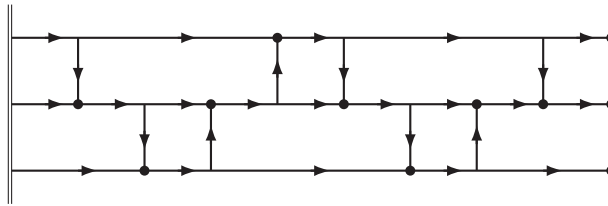


FIGURE 59 An admissible orientation of a plabic fence

plabic graph being flipped over (cf. Figure 25 inside the dotted line) contains an equal number of black and white vertices.

**Definition 17.8.** Two balanced plabic graphs are *move equivalent through balanced plabic graphs* if they are related to each other by a sequence of local moves which only involve balanced plabic graphs. As noted in Remark 17.7, Postnikov’s flip and square moves preserve the property of being balanced.

**Proposition 17.9.** *If two balanced plabic graphs are move equivalent through balanced plabic graphs, and one of them has a (necessarily unique) admissible orientation, then so does the other.*

*Proof.* All we need to do is check that for each type of local move transforming a balanced plabic graph  $P_1$  into another balanced plabic graph  $P_2$  (see Figure 17), we can transport an admissible orientation of  $P_1$  into an admissible orientation of  $P_2$ . This is demonstrated in Figure 58. It is straightforward to verify that each of these local transformations preserves the conditions (B), (W), and (F) of Definition 17.1. We note that Figure 58 shows all possible edge orientations (up to isotopy, rotation, and/or reflection) which are consistent with these conditions.  $\square$

**Proposition 17.10.** *Any plabic fence has an admissible orientation.*

*Proof.* Orient all horizontal edges of a plabic fence left to right, and orient each vertical edge from the white vertex to the black one (see Figure 59).  $\square$

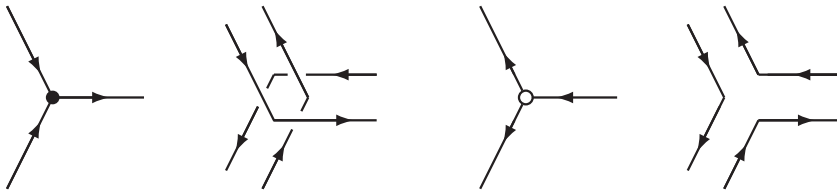


FIGURE 60 Building a link diagram around a black, respectively white, vertex of a plabic graph carrying an admissible orientation (cf. Figure 32)

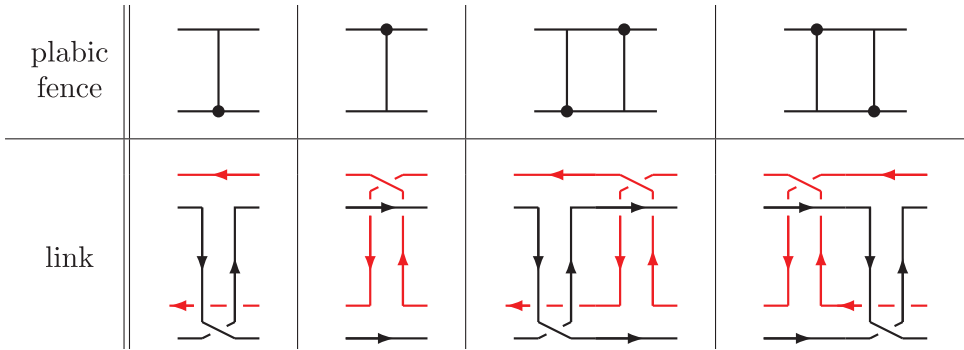


FIGURE 61 Transforming a plabic fence  $\Phi$  into a link diagram for  $L^\circ(\Phi)$

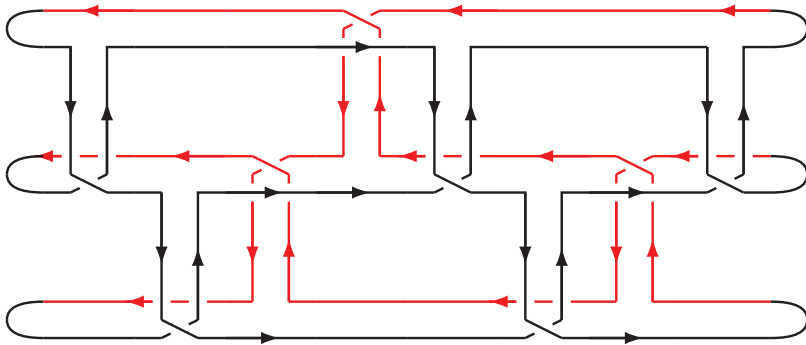


FIGURE 62 The link  $L^\circ(\Phi)$  for the plabic fence  $\Phi$  in Figure 59 (cf. Figure 44)

**Definition 17.11.** Let  $P$  be a (necessarily balanced) plabic graph allowing a (necessarily unique) admissible orientation. The (oriented) *link*  $L^\circ(P)$  associated with  $P$  is defined in terms of a link diagram constructed as follows. Replace each edge of  $P$  by a pair of parallel strands, oriented according to the ‘drive on the right side’ rule. Connect these strands at each internal trivalent vertex of  $P$  according to the recipe shown in Figure 60. Finally, at each univalent boundary vertex of  $P$ , connect the two strands to each other.

Definition 17.11 is illustrated, for the special case of plabic fences, in Figures 61 and 62. For plabic fences associated to scannable divides, we recover the A’Campo links:



**Proposition 17.12.** *Let  $D$  be a scannable divide, and  $\Phi = \Phi(D)$  the corresponding plabic fence (cf. Definition 12.2). Then the links  $L(D) = L(\Phi)$  and  $L^\circ(\Phi)$  are isotopic to each other.*

*Proof.* Compare the constructions of the links  $L(D)$  and  $L^\circ(\Phi)$  given in Theorem 11.4 (due to Couture and Perron) and Definition 17.11, respectively. □

*Remark 17.13.* More generally, for any plabic graph  $P$  supporting an admissible orientation, the link  $L^\circ(P)$  constructed in Definition 17.11 is isotopic to the Kawamura link  $L(P)$  from Definition 9.1. Since we will not rely on this statement, we shall only sketch an argument justifying it.

The construction of Definition 17.11 naturally produces a link in the solid torus  $UT\mathbf{D}$ , the unit tangent bundle of the disk  $\mathbf{D}$ . (Collapsing the boundary of  $UT\mathbf{D}$  yields a link in  $S^3$ .) Links in the solid torus are a bit difficult to visualize, especially since rotating the tangent vector of an oriented divide  $\vec{D}$  moves a short distance in the disk while the resulting link makes a full revolution around the solid torus. Now suppose that  $\vec{v}$  is a nonzero vector field on  $\mathbf{D}$  such that  $\vec{D}$  is never tangent to  $\vec{v}$ , with the same orientation. Then  $\vec{v}$  gives a section of  $UT\mathbf{D}$ , a meridional disk  $M_{\vec{v}}$  cutting across the solid torus. The complement  $UT\mathbf{D} \setminus M_{\vec{v}}$  is homeomorphic to  $\mathbb{R}^3$ , and it is much easier to draw pictures there. To concretely draw  $L(\vec{D})$  in this space, given a point  $(x, w) \in L(\vec{D})$ , two of the three co-ordinates of its image in  $\mathbb{R}^3 \cong UT\mathbf{D}$  are the co-ordinates of  $x$  in  $\mathbf{D}$ , and the third co-ordinate (which in our pictures would be perpendicular to the page) measures the clockwise rotation from  $\vec{v}(x)$  to  $w$ . Thus, to convert a crossing in  $\vec{D}$  into a crossing in the traditional link diagram for  $L(\vec{D})$ , we designate as the bottom strand the one whose tangent you reach first when rotating clockwise from  $\vec{v}(x)$ . (Hirasawa’s technique for visualizing divides [41] can be viewed as the special case of this construction, with  $\vec{v}$  the constant vector field pointing to the left, and with additional elaboration for cases when the link passes through the forbidden set  $M_{\vec{v}}$ .)

From an admissible orientation of a plabic graph  $P$ , we can construct a non-vanishing vector field  $\vec{w}$ , arranged to be parallel to each oriented edge (with the same orientation) except near the vertices, and turning a minimal amount at each vertex. Condition (F) ensures that  $\vec{w}$  extends continuously across the faces. (To extend a non-zero vector field across a disk, it suffices to check that the index on the boundary is 0, which is exactly what Condition (F) ensures.) Now let  $\vec{v}$  be the clockwise  $90^\circ$  rotation of  $\vec{w}$ . By construction, away from univalent vertices,  $\vec{v}$  is never parallel with the same orientation to the oriented divide  $o(P)$  from Definition 9.1.

We need to see that the link diagram in Figure 60 represents  $L(o(P))$ . Since  $o(P)$  is not tangent to  $\vec{v}$  away from the boundary, we can use the rule above to establish which strand is on top at the crossings of  $o(P)$  near black vertices. The result differs by a Reidemeister III move from Figure 60. Finally, near the univalent vertices lying on  $\partial\mathbf{D}$ , the conventions from Figure 33 produce an oriented divide  $o(P)$  that does pass through the forbidden set  $M_{\vec{v}}$ . We can fix this by applying a boundary U-turn move to  $o(P)$  near each boundary univalent vertex. This introduces extra crossings near the univalent vertices; these crossings can be removed with Reidemeister I moves.

In light of Remark 17.13, it comes as no surprise that local moves on oriented plabic graphs preserve the associated links (cf. Proposition 9.5).

**Proposition 17.14.** *Let  $P_1$  and  $P_2$  be balanced plabic graphs which are move equivalent through balanced plabic graphs. Suppose that one of them (hence the other, see Proposition 17.9) has an admissible orientation. Then the links  $L^\circ(P_1)$  and  $L^\circ(P_2)$  are isotopic to each other.*

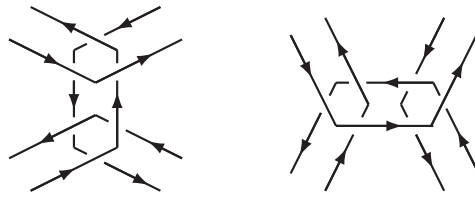


FIGURE 63 Link diagrams for two oriented plabic graphs related via a flip move involving two black vertices. The two links are isotopic to each other

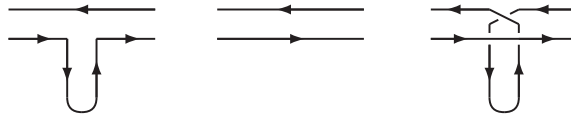


FIGURE 64 Transforming the link diagram under a tail removal/attachment

*Proof.* We need to check that each type of local move shown in Figure 58 preserves the isotopy type of the link  $L^\circ(P)$  associated with a plabic graph  $P$  carrying an admissible orientation. For a flip move involving two white vertices, the statement is clear (no strands cross each other). The case of a flip move involving two black vertices is shown in Figure 63. For the square move, examine the last two columns of Figure 61 and verify that the isotopy type of the link does not change. Finally, the case of tail attachment/removal is treated in Figure 64.  $\square$

**Proposition 17.15.** *Let  $D_1$  and  $D_2$  be scannable divides whose respective plabic fences  $\Phi(D_1)$  and  $\Phi(D_2)$  are move equivalent through balanced plabic graphs. Then  $D_1$  and  $D_2$  are link equivalent.*

*Proof.* By Proposition 17.10, the plabic fences  $\Phi(D_1)$  and  $\Phi(D_2)$  have admissible orientations. Since they are move equivalent through balanced plabic graphs, their links are isotopic, by Proposition 17.14. Moreover, by Proposition 17.12 these links are isotopic to the respective A’Campo links  $L(D_1)$  and  $L(D_2)$ , and we are done.  $\square$

*Remark 17.16.* The following example illustrates the difficulties involved in extending the arguments presented in this section to arbitrary (not necessarily scannable) algebraic divides. Let  $D$  be the non-scannable divide shown in Figure 49 on the upper left (or in Figure 8(a)). It is not hard to verify that any plabic graph  $P \in \mathbf{P}(D)$  (see Definition 6.11) does not have an admissible orientation. (To see this, examine the portion of  $P$  corresponding to the vicinity of the rightmost node in  $D$ .) Consequently  $P$  cannot be obtained from a plabic graph attached to a scannable algebraic divide of the same type (see, for example, Figure 49 on the lower right) via local moves through balanced plabic graphs. This means that some of the moves in Figure 49, when translated into the language of plabic graphs, must pass through non-balanced graphs. (Note that the plabic graph for any divide can be chosen to be balanced.)

## 18 | SIMPLE SINGULARITIES

In this section, we verify that the main construction of this paper provides a direct link between two ADE classifications: (i) Arnold’s classification of simple singularities and (ii) the classification

of skew-symmetric cluster algebras of finite type. We begin by reviewing these two classifications, starting with the latter.

Any quiver  $Q$  gives rise to a *cluster algebra*  $\mathcal{A}(Q)$  (with trivial coefficients and a skew-symmetric exchange matrix). The cluster algebra  $\mathcal{A}(Q)$  only depends on the mutation equivalence class of  $Q$ . It is generated inside a field of rational functions in several variables by a distinguished set of generators called *cluster variables* (see, for example, [31, Chapter 3]). The cluster algebra  $\mathcal{A}(Q)$  (or the quiver  $Q$ ) is said to have *finite type* if this generating set is finite. While cluster algebras as such do not play a role in our arguments, we would like to recall the following classification result.

**Theorem 18.1** [29]. *A connected quiver  $Q$  has finite type if and only if  $Q$  is mutation equivalent to an orientation of a simply laced Dynkin diagram.*

Orientations of two Dynkin diagrams are mutation equivalent if and only if the (unoriented) diagrams are isomorphic. Thus, each connected quiver of finite type has a well-defined *type* in the standard *ADE* nomenclature of the simply laced Dynkin diagrams. These types are  $A_n$  ( $n \geq 1$ ),  $D_n$  ( $n \geq 4$ ), and  $E_n$  ( $6 \leq n \leq 8$ ).

A singularity is called *simple* if any two curves in its topological equivalence class are locally diffeomorphic. Thus, simple singularities are those for which the topological classification coincides with the contact analytic one, that is, the classification up to local diffeomorphism in the source and multiplication by a unit of the local ring.

A classical result by Arnold asserts that simple singularities are classified by the simply laced Dynkin diagrams. In the language of Section 3, Arnold's classification can be stated as follows.

**Theorem 18.2** [6]. *A plane curve singularity is simple if and only if it has a real morsification whose unlabeled  $A\Gamma$ -diagram is a simply laced Dynkin diagram.*

Furthermore, every *ADE* type comes up as a type of a plane curve singularity (cf. Figures 4 and 5). See [37, Section I.2.4] for a detailed treatment.

A quiver corresponding to an arbitrary real morsification of a simple singularity does not have to be an orientation of an *ADE* Dynkin diagram, in the traditional Lie-theoretic sense of the term; see, for example, Figure 12 on the left. What, then, distinguishes these quivers among those arising from real morsifications of general singularities? The answer is suggested by Conjecture 5.5.

**Theorem 18.3.** *Let  $(C, z)$  be a plane curve singularity, as in Section 1. Let  $D$  be a divide associated with its real morsification, and  $Q = Q(D)$  the corresponding quiver. Then the following are equivalent:*

- $(C, z)$  is a simple singularity;
- $Q$  is a quiver of finite type.

*Furthermore, if these statements hold, then the type of the singularity  $(C, z)$  matches the type of the quiver  $Q$ .*

To rephrase, Theorem 18.3 (proved below in this section) asserts that a plane curve singularity is simple if and only if a quiver arising from some (equivalently, any) real morsification of this singularity gives rise to a cluster algebra of finite type. Moreover, the type of this cluster algebra matches the type of the singularity.

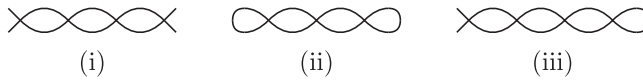


FIGURE 65 Divides arising from real morsifications of type  $A_n$  singularities. For  $n$  odd, there are two isotopy classes, shown in (i)–(ii), for  $n = 7$ . For  $n$  even, there is one isotopy class, shown in (iii), for  $n = 8$



FIGURE 66 Divides arising from real morsifications of type  $D_n$  singularities. Each of them is obtained as a transversal overlay of a real morsification of a type  $A_{n-3}$  singularity and a single smooth branch (shown as a vertical line)

**Example 18.4.** A quasi-homogeneous singularity of type  $(a, b)$ ,  $a \geq b \geq 2$ , is simple if and only if  $b = 2$  or else  $b = 3$  and  $a \leq 5$ . These are precisely the cases in which the  $(a - 1) \times (b - 1)$  grid quiver defines a cluster algebra of finite type (cf. Remark 16.2).

*Remark 18.5.* Most quivers of finite type do not arise from any morsification (nor from any divide, for that matter). The complete list of the quivers of type  $ADE$  which do arise in this way can be obtained from Proposition 18.7.

Theorem 18.3 implies that Conjecture 5.5 holds for simple singularities.

**Corollary 18.6.** *Given two real morsifications of isolated plane curve singularities, one of which is known to be simple, the following are equivalent:*

- the two singularities have the same complex topological type;
- the quivers associated with the two morsifications are mutation equivalent.

Our proof of Theorem 18.3 will rely on the classification of morsifications of  $ADE$  singularities (see Callahan [17, Section 4] and Chislenko [18, Section 4]).

**Proposition 18.7** [17, 18]. *Divides arising from real morsifications of simple singularities are classified, up to a homeomorphism of an ambient disk, as follows:*

- for each type  $A_n$  ( $n$  odd), there are two possible divides (see Figure 65, (i)–(ii));
- for each type  $A_n$  ( $n$  even), there is one possible divide (see Figure 65, (iii));
- for each type  $D_n$ , there are  $\lfloor \frac{n}{2} \rfloor$  possible divides (see Figure 66);
- for type  $E_6$ , there are two possible divides (see Figure 12);
- for type  $E_7$ , there are two possible divides (see Figure 5);
- for type  $E_8$ , there are three possible divides (see Figure 4).

We will also need the following lemma, listing several properties of the quivers which come from divides.

**Lemma 18.8.** *Let  $D$  be a divide, and  $Q = Q(D)$  the associated quiver. Then,*

- (1)  $Q$  does not contain an improperly oriented 3-cycle. That is, if  $Q$  contains arrows  $a \rightarrow b$  and  $b \rightarrow c$ , then it does not contain an arrow  $a \rightarrow c$ ;

- (2) if  $Q$  contains arrows  $a \rightarrow b$  and  $b \rightarrow c$ , then it contains a 3-cycle that includes at least one of these arrows;
- (3) let  $\Delta$  be an oriented 3-cycle in  $Q$ . Then  $Q$  contains another oriented 3-cycle which is different from  $\Delta$  but shares an arrow with  $\Delta$ ;
- (4) if  $Q$  contains arrows  $a \rightarrow b \rightarrow c \rightarrow d \rightarrow e$ , then it contains a 3-cycle that includes two consecutive arrows among these four;
- (5) if  $v$  is a vertex in  $Q$  with exactly two incoming arrows  $a \rightarrow v$  and  $c \rightarrow v$ , and exactly two outgoing arrows  $v \rightarrow b$  and  $v \rightarrow d$ , with  $a, b, c, d$  distinct, then  $Q$  contains arrows  $a \leftarrow b \rightarrow c \leftarrow d \rightarrow a$ ;
- (6) if a vertex  $b$  is incident in  $Q$  to exactly three arrows  $a \rightarrow b, b \rightarrow c, d \rightarrow b$ , and  $c$  is incident in  $Q$  to exactly three arrows  $c \rightarrow a, b \rightarrow c, c \rightarrow d$ , then  $Q$  is the 4-vertex quiver with vertices  $a, b, c, d$  and the five arrows listed above.

*Proof.*

- (1) This statement is immediate from the orientation rule of Definition 4.2.
- (2) Note that by the construction of  $Q(D)$ , each arrow  $\oplus \rightarrow \ominus$  is contained in a 3-cycle, and each path  $\ominus \rightarrow \bullet \rightarrow \oplus$  is contained in a 3-cycle.
- (3) If  $Q$  contains an arrow parallel to one of the arrows in  $\Delta$  (that is, if two vertices of  $\Delta$  are connected in  $Q$  by two or more arrows), then the claim is obvious. Otherwise,  $\Delta$  contains an arrow  $a \rightarrow b$  corresponding to two adjacent (inner) regions of  $D$  separated by a curve segment connecting two distinct nodes  $u$  and  $v$ . This yields two oriented 3-cycles  $a \rightarrow b \rightarrow u \rightarrow a$  and  $a \rightarrow b \rightarrow v \rightarrow a$ , one of which is  $\Delta$ .
- (4) The oriented path  $a \rightarrow b \rightarrow c \rightarrow d \rightarrow e$  must include three consecutive vertices labeled  $\ominus \rightarrow \bullet \rightarrow \oplus$ . This two-arrow path has to be contained in a 3-cycle.
- (5) If  $v$  is labeled  $\bullet$  (that is, comes from a node in  $D$ ), then  $a, b, c, d$  correspond to the four incident regions, and the claim follows by the construction of  $Q(D)$ . If  $v$  is labeled  $\oplus$  or  $\ominus$ , then it comes from a region of  $D$  bounded by two 1-cells (that is, a digon), with the vertices  $a, b, c, d$  corresponding to the two vertices of the digon and the two adjacent regions; the claim follows.
- (6) If  $b$  is labeled  $\bullet$  (that is, comes from a node), then none of  $a, c, d$  is labeled  $\bullet$ . Hence,  $c$  comes from a region adjacent to a single node (that is, a monogon), and consequently  $c$  cannot be incident to three arrows. One similarly shows that  $c$  cannot be labeled  $\bullet$ . Hence, both  $a$  and  $d$  are labeled  $\bullet$ . Furthermore  $b$  and  $c$  come from adjacent digons, each having two vertices  $a$  and  $d$ . Moreover, these digons are not adjacent to any other regions of  $D$ . The statement now follows from the connectedness of  $Q(D)$ . □

*Proof of Theorem 18.3.* If  $(C, z)$  is a simple singularity, then  $D$  is one of the divides listed in Proposition 18.7. It is straightforward to check that for each of these divides, the quiver  $Q(D)$  is mutation equivalent to an orientation of the Dynkin diagram of the corresponding type.

It remains to show that if  $Q(D)$  is a quiver of finite type, then  $(C, z)$  is a simple singularity. If  $n \leq 8$ , then the claim follows from the well-known fact (see, for example, [9, Preamble to Part II]) that all singularities with the Milnor number  $n \leq 8$  are simple. Thus, we may assume that  $Q(D)$  is a quiver of type  $A_n$  or  $D_n$ , with  $n \geq 9$ ; that is,  $D$  is mutation equivalent to an orientation of the corresponding Dynkin diagram. We next show that this assumption implies that  $D$  is one of the divides catalogued in Proposition 18.7 (cf. Figures 65 and 66), and therefore the underlying singularity is simple, of the corresponding type ( $A_n$  or  $D_n$ ) (cf. Theorem 3.4). We will treat the types  $A_n$  and  $D_n$  separately, assuming the reader’s familiarity with the basic combinatorics of triangulations giving rise to quivers of these types (see, for example, [32, Chapter 5]).

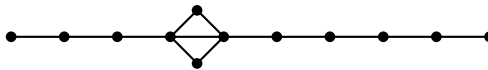
**Type  $A_n$ .** Each quiver  $Q$  of type  $A_n$  describes the signed adjacencies of diagonals in a triangulation  $T$  of a convex  $(n + 3)$ -gon. If  $T$  contains three diagonals forming a triangle, then  $Q$  contains an oriented 3-cycle whose sides are not contained in any other 3-cycles. By Lemma 18.8(3), such a quiver cannot come from a divide. If  $T$  does not include a triple of diagonals forming a triangle, then  $Q$  is an orientation of the Dynkin diagram of type  $A_n$ . If such a quiver  $Q$  includes two co-oriented arrows  $a \rightarrow b \rightarrow c$ , then  $Q$  cannot come from a divide, by Lemma 18.8(2). We conclude that  $Q$  has an alternating orientation, that is, every vertex is either a source or a sink. The only divides producing such quivers are the ones appearing in Figure 65.

**Type  $D_n$ .** Each quiver  $Q$  of type  $D_n$  comes from a (tagged) triangulation  $T$  of an  $n$ -gon with a single puncture  $p$ . Removing the arcs of  $T$  incident to  $p$ , we obtain a partial triangulation  $T'$ , with  $p$  lying inside a punctured  $k$ -gon  $\mathbf{P}$ , for some  $k \geq 2$ .

Suppose  $k \geq 4$ . Then  $Q$  contains a chordless oriented  $k$ -cycle, and therefore cannot come from a divide, by Lemma 18.8(4).

Suppose  $k = 3$ . If exactly one side of the triangle  $\mathbf{P}$  is a diagonal of the  $n$ -gon, then Lemma 18.8(6) applies, and  $Q$  is a quiver of type  $D_4$  which comes from a divide obtained by crossing a circle with a line. If two or three sides of  $\mathbf{P}$  are diagonals, then we run into a contradiction with Lemma 18.8(5).

Suppose  $k = 2$ . Then  $p$  is incident to two arcs in  $T$ . If these two arcs connect  $p$  to two distinct vertices of the digon  $\mathbf{P}$  (with the same tagging), then  $Q$  contains a chordless oriented 4-cycle, contradicting Lemma 18.8(4). Thus, let us assume that the two aforementioned arcs connect  $p$  to the same vertex (and are tagged differently). Arguing as in the type  $A_n$  case above, we conclude that the partial triangulation  $T'$  cannot include three diagonals forming a triangle. It follows that the associated quiver  $Q$  is an orientation of a graph of the form



(with both triangles properly oriented). Moreover, each of the chains on either side of this graph must have an alternating orientation, by Lemma 18.8(2) (cf. the argument in type  $A_n$ ). Finally, none of the degree 4 vertices in  $Q$  can have two incoming and two outgoing arrows, by Lemma 18.8(5). These considerations completely fix the orientation of all the arrows in the quiver  $Q$ , up to a global reversal. We conclude that this quiver must come from one of the divides shown in Figure 66. □

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