AdS$_5$ black hole entropy near the BPS limit

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Abstract: We analyze AdS$_5$ black holes that are nearly supersymmetric. They depart from the BPS limit in two distinct ways: a temperature takes them above extremality and a potential maintains extremality but violates a certain constraint. We study the thermodynamics of these deformations and their interplay in detail. We discuss recent microscopic computations of BPS black hole entropy in $\mathcal{N} = 4$ SYM and generalize the arguments to the nearBPS regime by relaxing constraints imposed by supersymmetry. Our methods recover gravitational results from microscopic theory also for nearBPS black holes.

Contents

1 Introduction

The microscopic understanding of black hole entropy is the linchpin for progress in quantum gravity. However, studies of black hole entropy have been quantitatively successful only in a few systems with a high degree of supersymmetry, e.g. [1, 2]. Such settings offer great control but they appear far removed from the studies of black hole dynamics that is focus of much current research, such as the study of the SYK model, e.g. [3–5] or other fashionable spects of holography e.g. [6–8]. The main motivation for this article is to develop an arena that has the potential to bridge this gap. Our strategy is to leverage results

on BPS ground states established in 4d $\mathcal{N} = 4$ SYM to study the nearBPS properties of these systems as well.

Supersymmetric black holes in AdS$_5$ offer a particularly important setting for the study of holography. The relevant classical geometries have been known for quite some time [9–12]. It is an unfortunate technical complication that these black hole solutions are necessarily somewhat complicated. For example, all supersymmetric AdS$_5$ black holes with regular event horizon must rotate. On the other hand, the entropy of these black holes is relatively simple [13]:

$$S = 2\pi\sqrt{Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - \frac{1}{2}N^2(J_1 + J_2)}\,, \tag{1.1}$$

where Q_I (with $I = 1, 2, 3$) denote the R-charges and J_i (with $i = 1, 2$) the angular momenta within AdS$_5$. The conserved charges of supersymmetric AdS black holes must satisfy not only a conventional BPS mass condition

$$M = \sum_{I=1}^{3} Q_I + \sum_{i=1}^{2} J_i\,, \tag{1.2}$$

but also a certain nonlinear constraint.

$$Q_1 Q_2 Q_3 + \frac{1}{2}N^2 J_1 J_2 = \left(\frac{1}{2}N^2 + Q_1 + Q_2 + Q_3\right)\left(Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - \frac{1}{2}N^2(J_1 + J_2)\right). \tag{1.3}$$

The full significance of the constraint is somewhat mysterious. On the gravity side, some researchers impose it as the condition that closed timelike curves are absent. In subsection 2.2 we show that it follows from the BPS condition (1.2) without any additional assumptions.

It was long thought that supersymmetric indices fail at counting the BPS states underlying the entropy of supersymmetric AdS$_5$ black holes. For example, all possible indices were constructed [14] and their growth was estimated as $\mathcal{O}(1)$. This result was interpreted as due to large cancellations in the partition function that preclude the $\mathcal{O}(N^2)$ growth that is needed to account for the black hole entropy. Further work included [15–17].

There has been progress on this research direction over the last few years. A central insight was the recasting of the entropy (1.1) and the constraint (1.3) in terms of the free energy [18]:

$$\ln Z = -\frac{1}{2}N^2\frac{\tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3}{\tilde{\omega}_1 \tilde{\omega}_2}\,, \tag{1.4}$$

with the potentials satisfying a complex constraint. A straightforward Legendre transform of this expression from the potentials for R-charge Δ_I and the rotational velocities ω_i gives the entropy (1.1). Moreover, the constraint (1.3) on charges follows in the process, from a reality condition on the entropy. Subsequently, the free energy (1.4) and its accompanying constraint were derived from the on-shell action of supergravity [19]. These results (and their generalizations to other dimensions) appear deeply interrelated with the supersymmetric Casimir energy [20, 21] and its relation to anomalies [22, 23].

Over the last few months several microscopic derivations of the BPS entropy of AdS_5 black holes have been presented:

- *Supersymmetric localization* [19]. This ab initio computation is principled: the central point is the deformation of the path integral away from the physical surface while preserving fermion boundary conditions consistent with supersymmetry. However, the argument relies on a somewhat mysterious "generalized supersymmetric Casimir energy" that relates the supersymmetric partition function (which is of order $\mathcal{O}(N^2)$) and the supersymmetric index (which is of order $\mathcal{O}(1)$).

- *Free field construction* [24]. This is the simplest derivation by far, involving nothing but the free field representation of $\mathcal{N} = 4$ SYM at vanishing coupling. However, it is not clear to us that the computation is justified in the regime where it is applied.

- *Superconformal index* [25]. This computation relies on the earlier rewriting of the $\mathcal{N} = 4$ SYM localization on $S^1 \times S^3$ in terms of a complex integral over a circle, with poles inside the disc determined by certain Bethe Ansatz equations [26]. Manipulating the contour and taking advantage of known poles of the integrand [27, 28], the free energy (1.4) is extracted from the supersymmetric index. This derivation is the most rigorous but it is technically more involved and was so far completed only for simplified values of the angular momenta.

A key technical feature of all these derivations is that they invoke complexified potentials Δ_I, ω_i in an essential manner. However, it is not obvious to us that the details of these papers are consistent with one another.[1] Further research prompted by [19, 24, 25] has generalized details of the results in various directions [30–37] but no consensus has emerged yet.

Summary of this article. In this paper we take the substantiation of the BPS free energy (1.4) for granted. Given that basis, we make a significant leap and study the entropy of AdS_5 black holes *away from the supersymmetric limit*. We develop both gravitational and microscopic considerations, and we find several new agreements between these holographically dual descriptions. Our success in this direction develops the emerging microscopics of AdS_5 black holes directly in the regime that is physically relevant.

It is important to recognize that AdS_5 black holes allow *two distinct deformations* away from the BPS limit. Conceptually, the extremal limit $M = M_{\mathrm{ext}}$ indicates the lowest possible mass for given conserved charges. The low lying excitations with energy M in the range $0 < M - M_{\mathrm{ext}} \ll M_{\mathrm{ext}}$ are characterized quantitatively by the low temperature behavior

$$M - M_{\mathrm{ext}} = \frac{1}{2} \left(\frac{C_T}{T} \right) T^2 \,, \tag{1.5}$$

where C_T is the specific heat.[2] The regime where this mass formula applies is also studied in research inspired by the SYK model [3, 4], such as [5, 38–40].

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[1]For some discussions see [19, 29].

[2]The specific heat C_T at low temperature is proportional to the temperature so the combination $\frac{C_T}{T}$ is a constant in the regime we study. In our notation $\frac{C_T}{T}$ is the coefficient in front of $\frac{1}{2}T^2$, evaluated with all charges Q_I, J_i kept fixed. Some might refer to this variable as $C_{Q,J}$ but that is not what we do.

However, in the context of AdS$_5$ black holes, it is equally possible to consider excitations away from the BPS limit that remain on the extremal surface $M = M_{\text{ext}}$ but with the conserved charges Q_I, J_i taking values that violate the constraint (1.3). We find that such excitations have mass

$$M - M_{\text{BPS}} = \frac{1}{2} \left(\frac{C_\varphi}{T} \right) \left(\frac{\varphi}{2\pi\ell_5} \right)^2 , \tag{1.6}$$

where φ is a potential parametrizing departures from the BPS surface that preserve extremality and so have $T = 0$. The coefficient C_φ is the *capacitance* of the black hole.[3] We give the precise relation between the deformations φ within the extremal surface and the generic potentials Δ_I, ω_i in (4.32) with $T = 0$. To complete the set of parameters describing linear response we also introduce a thermoelectric coefficient C_E that quantifies the interplay between the temperature T and the potential φ.

The computation of the response parameters C_T, C_φ, and C_E is conceptually straightforward on the supergravity side, albeit not trivial from a technical point of view. Surprisingly, we find that the heat capacity and the capacitance are identical

$$C_T = C_\varphi. \tag{1.7}$$

Each of these physical quantities are quite nontrivial functions of black hole parameters so this agreement can hardly be an accident. They may be related by the broken supersymmetry but the realization of this mechanism in supergravity is not entirely straightforward and we did not pursue it here. Whatever its origin, this type of relation is novel and interesting in $\mathcal{N} = 4$ SYM.

On the microscopic side any progress may seem implausible because advances in the BPS limit rely heavily on supersymmetry, not to mention that they face some unresolved questions. However, the nonrenormalization due to supersymmetry may well generalize to the linear order that we study. Certainly the low temperature heat capacity is subject to a nAttractor mechanism [40] so $\frac{C_T}{T}$ can be interpreted as a symmetry breaking parameter of scale invariance and $\frac{C_\varphi}{T}$ as its partner under broken extended supersymmetry. It is therefore reasonable to expect that the parameters we compute are protected. Although we have not worked out a detailed argument in the present context we note that the mass terms depending quadratically on potentials were previously derived from BPS considerations in the case of asymptotically flat black holes [41, 42].

In our microscopic computations we proceed pragmatically. The BPS free energy (1.4) is justified only in the strict supersymmetric limit but smoothness is sufficient to find the linear dependence on temperature even away from the BPS limit. Moreover, our gravitational study motivates relaxing the constraint imposed by supersymmetry as well. It is satisfying that the two independent deformations combine nicely, to a complex parameter $\varphi + 2\pi i T$. With these minimal and conservative ingredients we recover gravitational results through a Legendre transform.

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[3]It is admittedly $\frac{C_\varphi}{T}$ that is the capacitance according to the definition (1.5) and standard terminology in electrodynamics. We find this abuse of language an acceptable price for making the symmetry between C_T and C_φ manifest.

This paper is organized as follows. In section 2 we develop the gravitational thermodynamics of nearBPS AdS$_5$ black holes. We carefully distinguish between the near-extremal limit and the extremal nearBPS limit, and we study their interplay. In section 3 we review the partition function of $\mathcal{N} = 4$ SYM in the free field limit, which can be written as a matrix model. We derive the free energy (1.4) from its leading contribution in the large-N and low-temperature limit. In section 4 we study the resulting AdS$_5$ black hole entropy function. We derive some of its implications beyond the regime where it was originally derived, by relaxing constraints on potentials and exploiting general principles such as the first law of thermodynamics. This leads to statistical physics of nearBPS AdS$_5$ black holes that agrees with results from the gravity side. Some open questions and future directions are discussed in section 5.

2 Black hole thermodynamics

In this section we develop the thermodynamics of AdS$_5$ nearBPS black holes. As benchmarks, we first review the general AdS$_5$ thermodynamics and its BPS limit. We then study small but finite deviations from the strict BPS limit by allowing for temperature and for physical charges that violate the constraint. Specifically, in subsection 2.3 we consider the near-extremal regime (the mass exceeding the BPS value but the constraint between charges retained), in subsection 2.4 we study the extremal nearBPS regime (keep vanishing temperature but allow for small violations of the charge constraint), and then in subsection 2.5 we take on the general two-parameter deviations from the BPS surface.

The general AdS$_5$ black hole solution (for any M, Q_I, J_i) is known [12] but has not been analyzed in much detail [43]. In this section we follow most of the literature and focus on diagonal R-charges $Q_1 = Q_2 = Q_3$.

2.1 General AdS$_5$ black holes

We consider the most general charged rotating black holes in five-dimensional minimal gauged supergravity [11]. The solutions are identified by the mass M, the charge Q, and two independent angular momenta J_i (with $i = 1, 2$). These physical quantum numbers are parametrized by four variables (m, q, a, b) as

$$M = \frac{\pi}{4G_5} \frac{m(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b) + 2qabg^2(\Xi_a + \Xi_b)}{\Xi_a^2\Xi_b^2} \,, \tag{2.1}$$

$$Q = \frac{\pi}{4G_5} \frac{q}{\Xi_a\Xi_b} \,, \tag{2.2}$$

$$J_1 = \frac{\pi}{4G_5} \frac{2ma + qb(1 + a^2g^2)}{\Xi_a^2\Xi_b} \,, \tag{2.3}$$

$$J_2 = \frac{\pi}{4G_5} \frac{2mb + qa(1 + b^2g^2)}{\Xi_a\Xi_b^2} \,. \tag{2.4}$$

Here G_5 is Newton's gravitational constant in five dimensions, the coupling of gauged supergravity $g = \ell_5^{-1}$ with ℓ_5 the AdS$_5$ radius, and

$$\Xi_a \equiv 1 - a^2g^2 \,, \quad \Xi_b \equiv 1 - b^2g^2 \,. \tag{2.5}$$

The coordinate r_+ that locates the event horizon is the largest real root of $\Delta_r = 0$, where Δ_r is given by

$$\Delta_r = \frac{(r^2 + a^2)(r^2 + b^2)(1 + g^2 r^2) + q^2 + 2abq}{r^2} - 2m \,. \tag{2.6}$$

We can expresses m in terms of r_+^2 and other variables using this equation. Thus the temperature and entropy are presented conveniently in terms of (r_+, q, a, b). They are

$$T = \frac{r_+^4[1 + g^2(2r_+^2 + a^2 + b^2)] - (ab + q)^2}{2\pi r_+[(r_+^2 + a^2)(r_+^2 + b^2) + abq]} \,, \tag{2.7}$$

$$S = 2\pi \frac{\pi}{4G_5} \frac{(r_+^2 + a^2)(r_+^2 + b^2) + abq}{(1 - a^2 g^2)(1 - b^2 g^2)r_+} \,. \tag{2.8}$$

The electric potential and angular velocities on the horizon are given in the same notation by

$$\Phi = \frac{3qr_+^2}{(r_+^2 + a^2)(r_+^2 + b^2) + abq} \,,$$

$$\Omega_1 = \frac{a(r_+^2 + b^2)(1 + g^2 r_+^2) + bq}{(r_+^2 + a^2)(r_+^2 + b^2) + abq} \,, \tag{2.9}$$

$$\Omega_2 = \frac{b(r_+^2 + a^2)(1 + g^2 r_+^2) + aq}{(r_+^2 + a^2)(r_+^2 + b^2) + abq} \,.$$

2.2 BPS AdS$_5$ black holes

The general black holes introduced in the previous subsection depend on independent physical variables (M, Q, J_1, J_2). Supersymmetry of the theory guarantees that their mass satisfies

$$M - (3Q + gJ_1 + gJ_2)) \geq 0 \,.$$

The BPS black holes saturate this inequality so their mass M is given by the BPS condition

$$M^* = 3Q^* + g(J_1^* + J_2^*) \,. \tag{2.10}$$

We use the starred symbols (M^*, Q^*, J_1^*, J_2^*) instead of (M, Q, J_1, J_2) when we stress that the variables refer to the BPS case.[4]

The general formula for the mass can be written as

$$M - (3Q + gJ_1 + gJ_2) = \frac{\pi}{4G_5} \frac{3 + (a + b)g - abg^2}{(1 - ag)(1 + ag)^2(1 - bg)(1 + bg)^2} [m - q(1 + g(a + b)] \,, \tag{2.11}$$

in terms of the parameters (m, q, a, b). The coefficient in front of the square bracket is always positive, so the physical BPS condition (2.10) is equivalent to the relation

$$q^* = \frac{m^*}{1 + ag + bg} \,. \tag{2.12}$$

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[4]The star must not be confused with complex conjugation.

It turns out that once we saturate the BPS bound we must further impose a constraint between the black hole charges. This constraint takes the form

$$q^* = \frac{1}{g}(a+b)(1+ag)(1+bg)\,, \tag{2.13}$$

when expressed in terms of the parameters (q, a, b). When the BPS conditions (2.12)–(2.13) are imposed, the largest root r_+ of the horizon equation $\Delta_r(r_+) = 0$ where Δ_r is given in (2.6) is a double root that has the simple value:

$$r^* \equiv r_+ = \sqrt{\frac{1}{g}(a+b+abg)}\,. \tag{2.14}$$

It is important to stress that the constraint (2.13) is a *consequence* of the BPS mass formula (2.10) rather than an additional requirement due to an independent physical principle. To see this, we rewrite the horizon equation $\Delta_r(r_+) = 0$ as

$$m - q[1 + g(a+b)] = \frac{1}{2r_+^2}(r_+^2 - r_*^2)^2 + \frac{1}{2}\left[(1 + g(a+b))^2(r_+^2 - r_*^2) - (q - q_*)\right]^2\,, \tag{2.15}$$

without any restrictions on black hole parameters (m, q, a, b). It follows from (2.11) that the left hand side of this equation vanishes when the physical BPS bound is saturated. Since the right hand side is the sum of two complete squares we then establish the constraint (2.13) without further assumptions, along with finding the coordinate position of the horizon (2.14). Thus saturation of the BPS bound implies both the BPS mass formula (2.10) and the constraint.[5]

The two conditions (2.12)–(2.13) that must be satisfied by BPS black holes together yield starred variables (m^*, q^*) that are specified functions of the "rotation" parameters (a, b). It is therefore convenient to pick (a, b) as the two independent coordinates on the BPS surface. This choice gives the physical variables:

$$M^* = \frac{\pi}{4G_5} \frac{\left(3(a+b) - (a^3+b^3)g^2 - ab(a+b)^2g^3\right)}{g(1-ag)^2(1-bg)^2}\,, \tag{2.16}$$

$$Q^* = \frac{\pi}{4G_5} \frac{a+b}{g(1-ag)(1-bg)}\,, \tag{2.17}$$

$$J_1^* = \frac{\pi}{4G_5} \frac{(a+b)(2a+b+abg)}{g(1-ag)^2(1-bg)}\,, \tag{2.18}$$

$$J_2^* = \frac{\pi}{4G_5} \frac{(a+b)(a+2b+abg)}{g(1-ag)(1-bg)^2}\,. \tag{2.19}$$

These expressions satisfy the BPS condition (2.10) for any (a, b), as they must.

For BPS black holes the mass M^* is never an independent parameter and in our setting it is the linear function (2.10) of the other charges. It is a less familiar feature that the

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[5]Specifically, we do not appeal to the absence of closed time-like curves (CTCs). It was shown in [11] that black holes satisfying both the BPS condition (2.12) and the constraint (2.13) do not have CTCs outside the event horizon. On the other hand, supersymmetric "solutions" satisfying $M = \pm 3Q \pm gJ_1 \pm gJ_2$ with positive charges $Q, J_{1,2}$ do have CTCs unless all signs are "+" [44]. Regular black holes correspond to "all +" so CTCs do not play any role and the supersymmetry algebra is sufficient to impose the constraint.

condition (2.13) imposes an additional relation. This is the reason that the three conserved charges $Q^*, J_{1,2}^*$ (2.17)–(2.19) are expressed in terms of just two coordinates (a, b) on the BPS surface. It means these charges are not independent in the BPS limit, they satisfy the *constraint*:

$$Q^{*3} + \frac{\pi}{4G_5} J_1^* J_2^* = \left(\frac{\pi}{4g^2 G_5} + 3Q^* \right) \left(3Q^{*2} - \frac{\pi}{4gG_5}(J_1^* + J_2^*) \right). \tag{2.20}$$

This is the special case of the constraint (1.3) with diagonal R-charges $Q_1 = Q_2 = Q_3$.

The general formulae (2.9) for the electric potential and the angular velocities are nearly trivial in the BPS limit: they give $\Phi^* = 3$ and $\Omega_1^* = \Omega_2^* = g$ for all BPS black holes, independent of the values of (a, b). These values are guaranteed by the first law of thermodynamics

$$T^* dS^* = dM^* - \Phi^* dQ^* - \Omega_1^* dJ_1^* - \Omega_2^* dJ_2^*, \tag{2.21}$$

because in the BPS case $T^* = 0$ and the mass $M = M^*$ is given by (2.10).

As we have noted already, when the BPS conditions (2.12)–(2.13) are imposed, the largest root r_+ of the horizon equation $\Delta_r(r) = 0$ is a double root. This situation corresponds to temperature $T = 0$ and is expected for any extremal black hole. Moreover, with the value for r_+ given in (2.14) and the BPS value for q^* given in (2.12), the general formula (2.8) gives the black hole entropy entirely in terms of the rotation parameters (a, b):

$$S^* = 2\pi \cdot \frac{\pi}{4G_5} \frac{a+b}{g(1-ag)(1-bg)} \sqrt{\frac{1}{g}(a+b+abg)}. \tag{2.22}$$

In our manipulations we will often need the inverse of (2.17)–(2.19) and so express (a, b) in terms of the physical variables Q^* and $J_{1,2}^*$ of the BPS black hole. The resulting formulae are not unique, because the BPS charges are subject to the constraint (2.20). A simple version is

$$a = \frac{2gQ^{*2} - \frac{\pi}{4G_5}J_2^*}{Q^*(2g^2 Q^* + \frac{\pi}{4G_5})}, \tag{2.23}$$

$$b = \frac{2gQ^{*2} - \frac{\pi}{4G_5}J_1^*}{Q^*(2g^2 Q^* + \frac{\pi}{4G_5})}. \tag{2.24}$$

As an example, we can use these equations and the constraint (2.20) to recast the BPS entropy (2.22) as [13]

$$S^* = 2\pi \sqrt{3(\ell_5 Q^*)^2 - \frac{\pi}{4G_5}\ell_5^3 (J_1^* + J_2^*)}. \tag{2.25}$$

In this section we have so far made the effort to retain the dimensionful scales G_5 and $g = \ell_5^{-1}$ in all our equations. This is common in supergravity equations that are written in terms of parametric variables (m, q, a, b), and not terribly inconvenient. However, the practice becomes cumbersome when rewriting formulae in terms of conserved charges M^*, Q^*, and $J_{1,2}^*$. It is better to employ the dimensionless quantity

$$\frac{1}{2}N^2 = \frac{\pi}{4G_5}\ell_5^3, \tag{2.26}$$

where N is the rank of the dual $SU(N)$ gauge group. For example, it is superior to express the BPS entropy (2.25) as

$$S^* = 2\pi\sqrt{3(\ell_5 Q^*)^2 - \frac{1}{2}N^2\left(J_1^* + J_2^*\right)}\,. \qquad (2.27)$$

To compare with microscopic results we additionally record that the supergravity charge Q (with dimension length^{-1}) and the (dimensionless) angular momenta $J_{1,2}$ are normalized such that $Q\ell_5$ and $J_{1,2}$ can be identified with the quantized charges in the microscopic theory. For clarity, we will retain the scale $g = \ell_5^{-1}$ explicitly in the remainder of this section, but these units will be dropped later in the article.

2.3 Near-extremal limit

Conceptually, the simplest way to deform away from the BPS surface is by adding energy to the BPS black hole, while *keeping electric charge and angular momenta fixed*. Such deviations take the black hole away from extremality and lead to nonzero temperature. Since all charges are kept fixed the constraint is satisfied in this region of parameter space. This is the situation we consider in this subsection.

Since we consider charges that are fixed at their BPS values Q^*, $J_{1,2}^*$ we can write the first law of thermodynamics in the near-extremal limit as

$$TdS = dM - \Phi dQ^* - \Omega_1 dJ_1^* - \Omega_2 dJ_2^*\,. \qquad (2.28)$$

Subtracting the corresponding BPS expression (2.21) we have

$$TdS = d(M - M^*) - (\Phi - \Phi^*)dQ^* - (\Omega_1 - \Omega_1^*)dJ_1^* - (\Omega_2 - \Omega_2^*)dJ_2^*\,. \qquad (2.29)$$

Variations *along* the BPS surface have $M = M^*$ identically and correspond to the limit $T \to 0$ so they are described by [45]

$$dS = -(\partial_T\Phi)dQ^* - (\partial_T\Omega_1)dJ_1^* - (\partial_T\Omega_2)dJ_2^*\,. \qquad (2.30)$$

This formula describes the dependence on charges of the BPS entropy (2.27). That is interesting but not our focus in this subsection.

Instead, we keep the charges strictly fixed and consider the heat added to the black hole as we raise the temperature from $T = 0$ [46]. After dividing by the small temperature T, we have

$$dS = \left(\frac{\partial S}{\partial T}\right)_{Q, J_{1,2}} dT = \left.\frac{C_T}{T}\right|_{\text{nearExt}} dT\,, \qquad (2.31)$$

where C_T is the heat capacity that was introduced already in (1.5), along with comments on our notation. The ratio $\frac{C_T}{T}$ is constant in the near-extremal regime so:

$$dM = TdS = \left.\frac{1}{2}\frac{C_T}{T}\right|_{\text{nearExt}} d\left(T^2\right)\,. \qquad (2.32)$$

After integration we have the leading behavior at small temperature

$$M - M^* = \left.\frac{1}{2}\frac{C_T}{T}\right|_{\text{nearExt}} T^2\,, \qquad (2.33)$$

where $M^* = 3Q^* + g(J_1^* + J_2^*)$ is our reference point on the BPS surface.

We can compute C_T explicitly from the definition (2.31) by brute force. We first use the general entropy formula (2.8) to evaluate the dependence of the entropy on black hole parameters through $\frac{\partial S}{\partial(r_+,q,a,b)}$. We similarly compute the entries of the matrix $\frac{\partial(T,Q,J_1,J_2)}{\partial(r_+,q,a,b)}$, from the general formulae for temperature (2.7) and charges (2.2)–(2.4), with the parameter m eliminated in terms of (r_+, q, a, b) using the horizon equation (2.6). Inversion of this matrix (using Mathematica) gives the Jacobian $\frac{\partial(r_+,q,a,b)}{\partial(T,Q,J_1,J_2)}$ and then we can form

$$
\begin{aligned}
\left.\frac{C_T}{T}\right|_{\text{nearExt}} &= \left(\frac{\partial S}{\partial r_+}\right)_{q,a,b}\left(\frac{\partial r_+}{\partial T}\right)_{Q,J_{1,2}} + \left(\frac{\partial S}{\partial q}\right)_{r_+,a,b}\left(\frac{\partial q}{\partial T}\right)_{Q,J_{1,2}} \\
&\quad + \left(\frac{\partial S}{\partial a}\right)_{r_+,q,b}\left(\frac{\partial a}{\partial T}\right)_{Q,J_{1,2}} + \left(\frac{\partial S}{\partial b}\right)_{r_+,q,a}\left(\frac{\partial b}{\partial T}\right)_{Q,J_{1,2}} \\
&= \frac{\pi}{4G_5}\frac{\pi^2(a+b)^2(3+(a+b)g-abg^2)}{g^2(1-ag)(1-bg)\left(1+3(a+b)g+(a^2+3ab+b^2)g^2\right)}\,.
\end{aligned}
\tag{2.34}
$$

In the final step we exploited (2.13) and (2.14) to eliminate q and r_+.

The result for the heat capacity (2.34) can be expressed as a function of BPS physical charges Q^* and $J_{1,2}^*$. We first rewrite (a,b) using (2.23)–(2.24) and then simplify using the constraint (2.20) between charges. The result is not unique, because of the constraint, but we find the manageable expression

$$
\left.\frac{C_T}{T\ell_5}\right|_{\text{nearExt}} = \pi^2 \frac{8(Q^*\ell_5)^3 + \frac{1}{4}N^4(J_1^* + J_2^*)}{3(Q^*\ell_5)^2 - \frac{1}{2}N^2\left(J_1^* + J_2^*\right) + \left(3Q^*\ell_5 + \frac{1}{2}N^2\right)^2}\,.
\tag{2.35}
$$

It is this form of the heat capacity that we can compare with microscopic considerations.

There is an alternative computation that leads to the heat capacity (2.34) with less effort and more insight. It is known as the nAttractor mechanism [40]. The key observation is that for fixed conserved charge the derivative with respect to temperature T and the horizon coordinate r_+ are equivalent. Therefore, it is sufficient to consider the effect caused by the change of r_+ for computation of the heat capacity. Moreover, the departure from the BPS mass $M - M^*$ given in (2.33) is quadratic in the temperature while the entropy is only linear. Therefore, at the linear order, it is sufficient to consider the BPS geometry, there is no need for the general black hole solution. This leads to the economical computation

$$
\left.\frac{C_T}{T}\right|_{\text{nearExt}} = \left(\frac{\partial S}{\partial T}\right)_{Q,J_{1,2}} = \left(\frac{\partial S}{\partial r_+}\right)_{q,a,b}\left(\frac{\partial T}{\partial r_+}\right)_{q,a,b}^{-1}\,.
\tag{2.36}
$$

This expression can be evaluated by hand in a few lines and gives the same result as (2.34).

It is similarly useful to think of the electric potentials Φ and rotational velocities Ω_i as radially dependent "attractor flows" that take their fixed values $\Phi^* = 3$, $\Omega_1^* = \Omega_2^* = g$ at the horizon. The final approach to the horizon is determined in each case by a radial

derivative along the flow. For the electric potential we have

$$
\begin{aligned}
\left(\frac{\partial \Phi}{\partial T}\right)_{Q, J_{1,2}} &= \left(\frac{\partial \Phi}{\partial r_+}\right)_{q,a,b} \left(\frac{\partial T}{\partial r_+}\right)_{q,a,b}^{-1} \\
&= -\frac{3\pi(a+b)\left(1 - abg^2\right)}{g\sqrt{\frac{abg+a+b}{g}}\left(1 + 3(a+b)g + (a^2 + 3ab + b^2)g^2\right)} \\
&= -\frac{\pi^2 N^2 \ell_5}{S^*} \frac{(J_1 + J_2)\left(3Q\ell_5 + \frac{1}{2}N^2\right) - 2\left(\frac{S^*}{2\pi}\right)^2}{\left(\frac{S^*}{2\pi}\right)^2 + \left(3Q\ell_5 + \frac{1}{2}N^2\right)^2} \, .
\end{aligned}
\tag{2.37}
$$

In the final formula we used the BPS entropy S^* given in (2.27) as a preferred combination of charges, in order to avoid an expression that is overly unwieldy.

For the temperature dependence of the rotational velocity we similarly find

$$
\begin{aligned}
\left(\frac{\partial \Omega_1}{\partial T}\right)_{Q, J_{1,2}} &= \left(\frac{\partial \Omega_1}{\partial r_+}\right)_{q,a,b} \left(\frac{\partial T}{\partial r_+}\right)_{q,a,b}^{-1} \\
&= -\frac{\pi(1 - ag)(a + 2b + (2a+b)bg)}{\sqrt{\frac{abg+a+b}{g}}\left(1 + 3(a+b)g + (a^2 + 3ab + b^2)g^2\right)} \\
&= -\frac{\pi^2 N^2 \ell_5}{S^*} \frac{J_2\left(3Q\ell_5 + \frac{1}{2}N^2\right) - \left(\frac{S^*}{2\pi}\right)^2}{\left(\frac{S^*}{2\pi}\right)^2 + \left(3Q\ell_5 + \frac{1}{2}N^2\right)^2} \, .
\end{aligned}
\tag{2.38}
$$

There is an analogous formula for $\partial_T \Omega_2$ given by exchanging $a \leftrightarrow b$ (or $J_1 \leftrightarrow J_2$).

The temperature dependence of the potentials given in (2.37)–(2.38) is such that the BPS limit of the first law (2.30) is satisfied. It is also interesting that

$$
\left(\frac{\partial \Phi}{\partial T}\right)_{Q, J_{1,2}} = \left(\frac{\partial \Omega_1}{\partial T}\right)_{Q, J_{1,2}} + \left(\frac{\partial \Omega_2}{\partial T}\right)_{Q, J_{1,2}} \, .
\tag{2.39}
$$

We will see shortly that this is a consequence of temperature *respecting* the constraint on charges (2.20).

2.4 Extremal nearBPS limit

In this subsection we consider departures from the BPS surface that preserve extremality $T = 0$. This situation is somewhat unusual. The nearBPS black holes we study remain extremal in the conventional sense of the black hole attaining its minimal possible mass for given charges. However, charges are modified such that the constraint (2.20) is violated. We recall that preserved supersymmetry requires BPS saturation which in turn implies the constraint on charges. We therefore conclude that these black holes break supersymmetry and their the mass must exceed the BPS bound (2.10).

The most important challenge will be to understand the extremal surface in more detail. According to (2.11), the mass M generally exceeds the BPS mass M^* by an amount that

is proportional to (2.15) which we now write as

$$m - (1 + ag + bg)q$$

$$= \frac{g^2 r_+^2 (q - q^*)^2 + \left(\left[(1 + ag + bg)^2 + g^2 r_+^2 \right] (r_+^2 - r^{*2}) - (1 + ag + bg)(q - q^*) \right)^2}{2 r_+^2 \left[(1 + ag + bg)^2 + g^2 r_+^2 \right]} \,. \quad (2.40)$$

We previously applied this expression to show that the BPS mass formula implies the constraint on charges. Presently we consider instead the extremal surface which is characterized by vanishing temperature, not by the BPS condition. Near the BPS surface we can take

$$r_+^2 - r^{*2} \sim q - q^* \sim \epsilon \,, \quad (2.41)$$

small and approximate the temperature T (2.7) as

$$T = \frac{\left[1 + 3(a + b)g + (a^2 + b^2 + 3ab)g^2 \right](r_+^2 - r^{*2}) - (1 + (a + b)g)(q - q^*)}{\pi r^* q^*} \,. \quad (2.42)$$

Thus extremality corresponds to a correlation between the magnitudes of $r_+^2 - r^{*2}$ and $q - q^*$ such that the two terms in the temperature (2.42) cancel at linear order. In this regime the second term in the numerator of (2.40) vanishes, since it is proportional to the temperature, but the first term does not. Thus the BPS condition is preserved at linear order in ϵ, but it is broken at quadratic order. This structure is reminiscent of the nAttractor arguments reviewed in the preceding subsection, but with departures from the BPS surface now due to charges that violate the constraint, rather than nonzero temperature.[6]

We previously determined that the potentials are constants $\Phi = \Phi^* = 3$ and $\Omega_{1,2} = \Omega_{1,2}^* = g$ on the BPS surface. The combination of potentials,

$$\varphi = \Phi - \frac{\Omega_1 + \Omega_2}{g} - 1 \,, \quad (2.43)$$

therefore vanishes there and otherwise gives a physical measure of the "distance" away from the BPS configurations. Departure of the potentials from their BPS values due to a small *temperature* cancels from the expression due to (2.39) so the variable φ measures distance *along* the extremal surface, maintaining vanishing temperature. The general expressions for Φ, $\Omega_{1,2}$ (2.9) give:

$$\varphi = \frac{(3r_+^2 - r^{*2})(q - q^*) - (1 + ag + bg)(r_+^2 - r^{*2})^2}{(r_+^2 + a^2)(r_+^2 + b^2) + abq} \,, \quad (2.44)$$

after some rewriting. The second term in the numerator is negligible near the BPS surface where (2.41) instructs us to take $q - q^*$ and $r_+^2 - r^{*2}$ small and of the same order. We find

$$\varphi = \frac{2(q - q^*)}{q^*} \,, \quad (2.45)$$

---

[6]The formula (2.42) facilitates explicit comparison with Silva's early study of thermodynamics above the BPS limit [45]. It corresponds to $\mu \equiv m - (1 + ag + bg)$ and $q = q^*$. It thus involved temperature alone.

at leading order. Therefore the difference $q - q^*$ is a good measure of departures from the BPS surface that preserve extremality.

To summarize so far, within the extremal surface we can take

$$m = (1 + ag + bg)q \, , \tag{2.46}$$

at the linear order and we can relate the horizon coordinate $r_+^2 - r_*^2$ to $q - q^*$ through the condition that the temperature (2.42) vanishes. This leaves $q - q^*$ as the only variable that is sensitive to the deviation from the BPS surface. It is equivalent to φ through (2.45). The remaining variables (a, b) parametrize the base point on the BPS surface, equivalent to Q, $J_{1,2}$ subject to the constraint (2.20).

After this lengthy discussion of principles, we can return to the general formulae (2.9) for the potentials Φ, $\Omega_{1,2}$. In each expression we take fixed (a, b) and expand around $r^2 = r_*^2$, $q = q^*$. We then eliminate $r_+^2 - r_*^2$ in favor of $q - q^*$ by imposing vanishing temperature and introduce φ through (2.45). The resulting deviations away from the BPS values $\Phi^* = 3$ and $\Omega_{1,2}^* = g$ can be presented at linear order in φ as derivatives of the potentials with respect to φ:

$$\frac{\partial \Phi}{\partial \varphi} = \frac{3(a + b)(2 + (a + b)g)}{2[1 + 3(a + b)g + (a^2 + 3ab + b^2)g^2]} \tag{2.47}$$

$$= 1 - \frac{1}{4}N^2 \frac{J_1 + J_2 + 2(3Q\ell_5 + \frac{1}{2}N^2)}{\left(\frac{S^*}{2\pi}\right)^2 + 2(3Q\ell_5 + \frac{1}{2}N^2)^2} \, , \tag{2.48}$$

and

$$\frac{\partial \Omega_1 \ell_5}{\partial \varphi} = -\frac{(1 - a^2g^2)g}{2[1 + 3(a + b)g + (a^2 + 3ab + b^2)g^2]} \tag{2.49}$$

$$= -\frac{1}{4}N^2 \frac{J_2 + 3Q\ell_5 + \frac{1}{2}N^2}{\left(\frac{S^*}{2\pi}\right)^2 + 2(3Q\ell_5 + \frac{1}{2}N^2)^2} \, . \tag{2.50}$$

The analogous expression for $\partial_\varphi \Omega_2$ is obtained by the substitutions $a \leftrightarrow b$ and $J_1 \leftrightarrow J_2$. As a consistency check, we have

$$\partial_\varphi \Phi - \partial_\varphi (\Omega_1 + \Omega_2) \ell_5 = 1 \tag{2.51}$$

as expected from the definition (2.43).

Having computed the potentials $\Phi - \Phi^*$ and $\Omega_{1,2} - \Omega_{1,2}^*$ at linear order in φ, we turn to the first law of thermodynamics

$$0 = TdS = d(M - M^*) - (\Phi - \Phi^*)dQ - (\Omega_1 - \Omega_1^*)dJ_1 - (\Omega_2 - \Omega_2^*)dJ_2 \, , \tag{2.52}$$

where $T = 0$ because we consider extremal black holes. The charges are given by the general expressions (2.2)–(2.4) with m eliminated in favor of q through (2.46). The differentials dQ, $dJ_{1,2}$ therefore become linear combinations of dq and da, db. We are only interested in the first of these, because the others correspond to motion within the BPS surface. Introducing φ though (2.45) we then find:

$$-(\Phi - \Phi^*)dQ - (\Omega_1 - \Omega_1^*)dJ_1 - (\Omega_2 - \Omega_2^*)dJ_2 = -\frac{C_\varphi}{T} \frac{\varphi \, d\varphi}{(2\pi\ell_5)^2} \, , \tag{2.53}$$

with the temperature independent combination $\frac{C_\varphi}{T}$ given by

$$\frac{C_\varphi}{T} = \frac{\pi}{4G_5} \frac{\pi^2(a+b)^2}{g^2(1-ag)(1-bg)} \frac{3+(a+b)g-abg^2}{1+3(a+b)g+(a^2+b^2+3ab)g^2}\,. \tag{2.54}$$

After integration of the first law (2.52) we have

$$M - M^* = \frac{1}{2}\frac{C_\varphi}{T}\left(\frac{\varphi}{2\pi\ell_5}\right)^2\,, \tag{2.55}$$

to the leading order. The identical result can be derived directly from the general mass formula (2.11). Indeed, the central manipulation is given already in (2.15), with the 2nd term in the numerator absent when the temperature vanishes $T = 0$.

We refer to C_φ as the *capacitance*. This terminology is motivated by standard electromagnetism insofar as φ can be identified with the electric potential. The capacitance quantifies the energy (2.55) required to violate the constraint by an amount measured by the potential φ. Physically, this is quite distinct from the heat capacity C_T, a measure of the energy needed to increase the temperature. It is therefore surprising that

$$C_\varphi = C_T\,,$$

for the black holes we consider.

We have repeatedly invoked the intuition that the potential φ introduced in (2.45) measures the violation of the constraint (2.20) that must be satisfied by all BPS configurations. To make this precise we define the height function:

$$h \equiv \left(3Q\ell_5 + \frac{1}{2}N^2\right)\left(3(Q\ell_5)^2 - \frac{1}{2}N^2(J_1 + J_2)\right) - (Q\ell_5)^3 - \frac{1}{2}N^2 J_1 J_2\,, \tag{2.56}$$

that quantifies the distance from the constraint surface $h = 0$ explicitly. The differential form

$$dh = 3\left(8(Q\ell_5)^2 + N^2 Q\ell_5 - \frac{1}{2}N^2(J_1 + J_2)\right)dQ\ell_5 - \frac{1}{2}N^2\left(J_2 + 3Q\ell_5 + \frac{1}{2}N^2\right)dJ_1 + J_1 \leftrightarrow J_2\,, \tag{2.57}$$

realizes the surfaces $h = $ const as 2D planes that are generated by the three one-forms dQ, dJ_1, dJ_2 subject to the constraint $dh = 0$. In this construction h literally measures the distance along the normal to the 2D constraint surface $h = 0$. The BPS surface can be viewed as the intersection of the constraint surface $h = 0$ with the extremal surface $T = 0$.

Near the BPS surface we can eliminate the parameter m through (2.46) and then (2.2)–(2.4) express the three charges $Q, J_{1,2}$ as functions of the parameters (q, a, b). On general grounds, the differential form dh above becomes a linear combination of dq, da, db after inserting these formulae for the charges. Explicit computation gives

$$dh = \left[8(Q\ell_5)^3 + \frac{1}{4}N^4(J_1 + J_2)\right]\frac{dq}{q^*}\,.$$

The absence of da, db in this formula shows that the curve dh has no components along the BPS surface, as expected. The nontrivial component along dq relates the normalization of h and φ through (2.45). We find

$$h = \frac{1}{2}\left[8(Q\ell_5)^3 + \frac{1}{4}N^4(J_1 + J_2)\right]\varphi \,. \tag{2.58}$$

In the next subsection we will uncover a term in the entropy that is proportional to φ with a positive coefficient. Stability therefore motivates us to focus on the halfline where both height functions are nonnegative

$$\varphi \geq 0 \,. \tag{2.59}$$

2.5 General nearBPS limit

In this subsection we consider the general nearBPS regime where deviations from the BPS surface may have neither fixed charge nor vanishing temperature. Some aspects of this amount to reconsidering the effects discussed in the previous two subsections at the same time. However, their interplay gives important new insights.

The energy of excitations is the thermodynamic quantity that is conceptually most straightforward in the nearBPS limit. A good starting point is the general mass formula (2.11). It depends on the combination of parameters $m - (1 + (a+b)g)q$ that was rewritten in (2.15) without invoking any assumptions black hole variables. Inserting the first order expressions (2.42) for the temperature T and (2.45) for the potential φ we immediately find the excitation energy above the BPS bound:

$$M - M^* = \frac{1}{2}\frac{C_T}{T}\bigg|_{\text{nearExt}}\left[T^2 + \left(\frac{\varphi}{2\pi\ell_5}\right)^2\right] \,, \tag{2.60}$$

at quadratic order. We have highlighted the identity of the heat capacity and the electric capacitance by avoiding reference to the latter altogether. The new fact featured in the general nearBPS limit is the *absence* of any cross-terms $T\varphi$ in the mass formula (2.60). This demonstrates some sort of rotational symmetry that must be present in the regime we explore. The existence of a continuous structure is much stronger than the equality of C_T and C_φ that we stressed in the preceding subsection.

We next consider the additional entropy due to simultaneously allowing small temperature and violation of the constraint. Starting from the general expression (2.8) for the entropy, we apply the procedure explained around (2.46). Thus we first eliminate the parameter m using (2.46) and expand to linear order in $q - q^*$ and $r_+^2 - r_*^2$. We then take appropriate linear combinations so those two variables are eliminated in favor of the temperature T (2.42) and the potential φ (2.45). These steps give an entropy of the excitations taking the form

$$S - S^* = \frac{C_T}{T}T + \frac{C_E}{T}\frac{\varphi}{2\pi} \,, \tag{2.61}$$

where the heat capacity C_T agrees with the expression (2.34) found previously by considering temperature on its own and

$$\frac{C_E}{T} = \frac{2\pi(Q\ell_5)^2}{S^*}\frac{\pi^2}{g}\frac{(1+2(a+b)g+abg^2)(3+(a+b)g-abg^2)}{1+3(a+b)g+g^2(a^2+b^2+3ab)}$$
$$= \frac{2\pi}{S^*}\left(\frac{C_T}{T}\right)\left(3Q+\frac{1}{2}N^2\right).\tag{2.62}$$

The value of C_E is subject to a subtle ambiguity. The expression (2.61) gives the entropy $S-S^*$ that is *in excess* of the BPS entropy S^*. However, the BPS entropy is not a proper function of charges, it is only defined modulo the constraint (2.20). This caveat is inconsequential on the BPS surface where the constraint is satisfied identically. In contrast, the constraint is proportional to φ so, for the additional entropy $S-S^*$ the ambiguity can shift the coefficient of C_E arbitrarily, potentially rendering this quantity unphysical.

This issue must be addressed consistently in computations. For example, we variously express the BPS entropy S^* as a function of parameters (a,b) (2.22) or as a function of the charges Q, $J_{1,2}$ (2.27). The differential dS^* computed from the former only gives terms proportional to da and db but when it is evaluated from the latter we get terms of the form dQ, $dJ_{1,2}$ that, because charges depend on all of the parameters (q,a,b), also yield the differential dq. The two forms of the BPS entropy therefore give different coefficients in front of the term $d(q-q^*)=2q^*d\varphi$ even though they agree if we impose the BPS relation $q=q^*$ before computing the differentials.

We "gauge fix" the ambiguity by insisting that the BPS entropy takes the canonical form (2.27) in terms of Q, $J_{1,2}$, rather than something that is equivalent to this formula upon imposing the constraint. Our result for C_E (2.62) is predicated on this convention.

The final aspects of the general nearBPS limit that we consider are the potential terms in the first law of thermodynamics:

$$TdS = d(M-M^*) - (\Phi-\Phi^*)dQ - (\Omega_1-\Omega_1^*)dJ_1 - (\Omega_2-\Omega_2^*)dJ_2.\tag{2.63}$$

We can compute the potentials using the procedure described around (2.46), as in the previous examples. However, for the potentials the results for the general nearBPS limit can equally be inferred from the near extremal ($T\neq 0$ and $\varphi=0$) and nearBPS extremal ($T=0$ and $\varphi\neq 0$) special cases that we studied in the last two subsections. For example, combining (2.38) and (2.50) we find the correct result

$$(\Omega_1-\Omega_1^*)\ell_5 = \frac{1}{4}N^2 \frac{\left[J_2+(3Q\ell_5+\frac{1}{2}N^2)\right]\varphi - \frac{(2\pi)^2}{S^*}\left[J_2\left(3Q\ell_5+\frac{1}{2}N^2\right)-\left(\frac{S^*}{2\pi}\right)^2\right]T}{\left(\frac{S^*}{2\pi}\right)^2+\left(3Q\ell_5+\frac{1}{2}N^2\right)^2},\tag{2.64}$$

for the angular velocity in the general nearBPS limit. The analogous equation for $\Omega_2-\Omega_2^*$ follows by taking $J_1\leftrightarrow J_2$. The one for $\Phi-\Phi^*$ can be computed similarly from (2.37) and (2.48) or by invoking the sum rule

$$(T\partial_T+\varphi\partial_\varphi)\left[\Phi-(\Omega_1+\Omega_2)\ell_5\right]=\varphi,\tag{2.65}$$

that consolidates (2.39) and (2.51) for derivatives with respect to T and φ, respectively.

The explicit formula (2.64) for $\Omega_1 - \Omega_1^*$ and its analogues for other nearBPS potentials are somewhat lengthy and not very illuminating independently. However, they become more instructive when considered together, as a vector in the space of charges. The first law of thermodynamics (2.63) motivates construction of the differential form

$$TdS^* + (\Phi - \Phi^*)dQ + (\Omega_1 - \Omega_1^*)dJ_1 + (\Omega_2 - \Omega_2^*)dJ_2$$
$$= \left(\frac{2\pi}{S^*} \left(3Q + \frac{1}{2}N^2 \right) T + \frac{\varphi}{2\pi} \right) \frac{2\pi dh}{2 \left[\left(\frac{S^*}{2\pi} \right)^2 + (3Q + \frac{1}{2}N^2)^2 \right]}, \qquad (2.66)$$

where dh is the one-form (2.57) generated by the height function. Thus the relative strength of the three potentials is precisely the same as the one appearing in the height function h, except for the components taken into account by dS^* that are within the BPS surface and only relevant for BPS physics.

Introducing φ using (2.58) the potential terms become

$$TdS^* + (\Phi - \Phi^*)dQ + (\Omega_1 - \Omega_1^*)dJ_1 + (\Omega_2 - \Omega_2^*)dJ_2 = \left[-\left(\frac{C_E}{T} \right) T + \left(\frac{C_T}{T} \right) \varphi \right] d\varphi, \quad (2.67)$$

where C_T and C_E agree with the functions of charges defined in (2.54) and (2.62). This is precisely what is needed to satisfy the first law of thermodynamics. Importantly, the values for C_E match only when using the canonical form (2.27) of the BPS entropy S^* when evaluating TdS^* in (2.67). This illustrates the necessity of treating the ambiguity discussed below (2.62) consistently.

3 The BPS partition function

In this section, we review the recent progress on the microscopic origin of the AdS$_5$ black hole entropy. We focus on the free field approach applied to $\mathcal{N} = 4$ SYM and closely follow [14, 24].

In this and subsequent sections we adopt microscopic units with $\ell_5 = g^{-1} = 1$ and eliminate all references to Newton's constant G_5 in favor of the rank of the gauge group N through (2.26).

3.1 The partition function

Following recent work, we seek to compute the physical partition function

$$Z(\beta, \Delta_I, \omega_i) = \text{Tr} \left[e^{-\beta E} e^{\Delta_I Q_I + \omega_i J_i} \right]. \qquad (3.1)$$

The electric charges and the angular momenta are denoted by Q_I and J_i and the corresponding chemical potentials are Δ_I and ω_i. Sums over repeated indices I and i are implied. We stress that we do not consider the supersymmetric index since no grading $(-1)^F$ has been inserted.

The fermionic symmetries $\mathcal{Q}$ and $\mathcal{S}$ transform as spinors under both the SU(4)$_R$ symmetry and the SO(4) little group. We denote the quantum numbers of the supercharge $\mathcal{Q} \equiv \mathcal{Q}_{-t_1,-t_2}^{s_1,s_2,s_3}$ with respect to these groups $\frac{1}{2}s_I$ and $-\frac{1}{2}t_i$ where $s_I = \pm 1$ and $t_i = \pm 1$.

The signs of the spinorial indices are explicitly flipped for the conformal supercharges $\mathcal{S} \equiv \mathcal{S}_{t_1,t_2}^{-s_1,-s_2,-s_3}$ which have opposite chirality so we can take the products $\prod s_I \prod_i t_i = +1$ in both cases.

We write the superalgebra in component form as

$$\{\mathcal{Q}, \mathcal{S}\} = E - \sum_{I=1}^{3} s_I Q_I - \sum_{i=1}^{2} t_i J_i\,. \tag{3.2}$$

The anticommutator $\{\mathcal{Q}, \mathcal{S}\} = 0$ when acting on $\frac{1}{16}$-BPS states so we have the BPS condition

$$E = \sum_I Q_I + \sum_i J_i\,, \tag{3.3}$$

when we restrict to the sector where all $s_I, t_i = +1$ without loss of generality. We can adapt the physical partition function (3.1) to this sector as

$$Z(\beta, \Delta_I, \omega_i) = \mathrm{Tr}\left[e^{-\beta(E - \sum_I Q_I - \sum_i J_i)} e^{\widetilde{\Delta}_I Q_I + \widetilde{\omega}_i J_i} \right]\,, \tag{3.4}$$

where we introduced

$$\widetilde{\Delta}_I \equiv \Delta_I - \beta\,, \quad \widetilde{\omega}_i \equiv \omega_i - \beta\,. \tag{3.5}$$

Then only the states satisfying the BPS condition (3.3) contribute to the partition function in the limit $\beta \to \infty$ with fixed $\widetilde{\Delta}_I$ and $\widetilde{\omega}_i$.

The quantum numbers Q_I, J_i are half-integer valued for all states in the theory so the corresponding chemical potentials satisfy the periodicity conditions

$$\Delta_I \equiv \Delta_I + 4\pi i\,, \quad \omega_i \equiv \omega_i + 4\pi i\,. \tag{3.6}$$

However, the operator $\mathcal{O} = e^{\tilde{\Delta} \cdot Q + \tilde{\omega} \cdot J}$ that is inserted in the partition function (3.4) generally does not anticommute with the supercharge

$$\mathcal{Q}\mathcal{O} = e^{-\frac{1}{2} s \cdot \tilde{\Delta} + \frac{1}{2} t \cdot \tilde{\omega}} \mathcal{O}\mathcal{Q}\,. \tag{3.7}$$

Anticommutation imposes the additional condition:

$$s \cdot \tilde{\Delta} - t \cdot \tilde{\omega} = 2\pi i \pmod{4\pi i}\,. \tag{3.8}$$

Therefore supersymmetry demands that the potentials satisfy

$$\tilde{\Delta}_1 + \tilde{\Delta}_2 + \tilde{\Delta}_3 - \tilde{\omega}_1 - \tilde{\omega}_2 = 2\pi i \pmod{4\pi i}\,, \tag{3.9}$$

for projection to the BPS sector we focus on.

The complex condition (3.9) on the potentials is essential. It is closely related to the supersymmetric index, since insertion of $(-)^F$ in the partition function is equivalent to the shift $\omega_i \to \omega_i + 2\pi i$ for either $i = 1$ or $i = 2$. However, following recent literature we will maintain reference to the partition function rather than the supersymmetric index and we will consider complex potentials. In this terminology the partition function counts protected states (and so is independent of β) on the surface defined by the complex supersymmetry condition (3.9).

3.2 Single particle enumeration

We now turn to the central problem of computing the partition function (3.1) of $\mathcal{N} = 4$ SYM on $S^1 \times S^3$. We impose anti-periodic boundary conditions for fermions along S^1, as usual for any partition function but in contradistinction to the supersymmetric index. We work perturbatively at weak coupling and express the result as a matrix model following [14, 47, 48].

The first step is to enumerate "single-particle states", in the terminology of the bulk AdS$_5$ theory. In the quantum field theory description these are the individual operators that generate the operator algebra. They can be realized as elementary fields, possibly with derivatives since those do not change the particle number. It is useful to decompose the field content of $\mathcal{N} = 4$ SYM under an $\mathcal{N} = 1$ subalgebra and represent is matter as an $\mathcal{N} = 1$ vector multiplet, three $\mathcal{N} = 1$ chiral multiplets, and three $\mathcal{N} = 1$ anti-chiral multiplets. In components, a $\mathcal{N} = 1$ vector multiplet contains a gauge boson and a real spinor, while a $\mathcal{N} = 1$ chiral multiplet contains a complex scalar and a real chiral spinor.

We first consider a chiral multiplet with the spectrum:

Fields	$(E, J_1, J_2; Q_1, Q_2, Q_3)$
$X = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2)$	$(1, 0, 0; 1, 0, 0)$
$Y = \frac{1}{\sqrt{2}}(\phi^3 + i\phi^4)$	$(1, 0, 0; 0, 1, 0)$
$Z = \frac{1}{\sqrt{2}}(\phi^5 + i\phi^6)$	$(1, 0, 0; 0, 0, 1)$
$\bar{\psi}_{\dot{\alpha}, a}$	$(\frac{3}{2}, 0, \pm\frac{1}{2}; -\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2})$
	$(\frac{3}{2}, 0, \pm\frac{1}{2}; +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2})$
	$(\frac{3}{2}, 0, \pm\frac{1}{2}; +\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2})$

The corresponding partition functions become

$$f_B^c(\beta, \Delta_I, \omega_i) = \sum_{I=1}^{3} e^{\Delta_I} \frac{e^{-\beta}(1 - e^{-2\beta})}{(1 - e^{-\beta+\omega_1})(1 - e^{-\beta+\omega_2})(1 - e^{-\beta-\omega_1})(1 - e^{-\beta-\omega_2})}, \quad (3.10)$$

$$f_F^c(\beta, \Delta_I, \omega_i) = \sum_{I=1}^{3} e^{\Delta_I} \frac{e^{-\frac{3}{2}\beta-\Delta}\left((e^{\omega_+} + e^{-\omega_+}) - e^{-\beta}(e^{\omega_-} + e^{-\omega_-})\right)}{(1 - e^{-\beta+\omega_1})(1 - e^{-\beta+\omega_2})(1 - e^{-\beta-\omega_1})(1 - e^{-\beta-\omega_2})}, \quad (3.11)$$

where

$$\Delta \equiv \frac{\Delta_1 + \Delta_2 + \Delta_3}{2}, \quad \omega_\pm \equiv \frac{\omega_1 \pm \omega_2}{2}. \quad (3.12)$$

The common denominators of (3.10)–(3.11) incorporate the quantum numbers of the four components in the gradient ∇_μ, each resummed as a geometric series to take into account any number of derivatives. The positive terms in the numerators encode the data from the table, while the negative ones remove operators that satisfy their equations of motion.

The anti-chiral multiplet can be obtained from the chiral multiplet by

$$J_1 \leftrightarrow J_2, \quad Q_I \to -Q_I. \quad (3.13)$$

For bosons this just changes the overall factor giving the R-charge, but for fermions the Lorentz indices of the fields and the equation of motion exchange their chirality $\omega_+ \leftrightarrow \omega_-$, in addition to the R-charges changing sign:

$$f_B^a(\beta, \Delta_I, \omega_i) = \sum_{I=1}^{3} e^{-\Delta_I} \frac{e^{-\beta}(1 - e^{-2\beta})}{(1 - e^{-\beta + \omega_1})(1 - e^{-\beta + \omega_2})(1 - e^{-\beta - \omega_1})(1 - e^{-\beta - \omega_2})} , \quad (3.14)$$

$$f_F^a(\beta, \Delta_I, \omega_i) = \sum_{I=1}^{3} e^{-\Delta_I} \frac{e^{-\frac{3}{2}\beta + \Delta}\left((e^{\omega_-} + e^{-\omega_-}) - e^{-\beta}(e^{\omega_+} + e^{-\omega_+})\right)}{(1 - e^{-\beta + \omega_1})(1 - e^{-\beta + \omega_2})(1 - e^{-\beta - \omega_1})(1 - e^{-\beta - \omega_2})} . \quad (3.15)$$

The vector multiplet has the spectrum:

Fields	$(E, J_1, J_2; Q_1, Q_2, Q_3)$
$A_\mu\,(\mu = 1, \cdots, 4)$	$(1, \pm 1, \pm 1; 0, 0, 0)$
ψ_α^a	$(\frac{3}{2}, \pm\frac{1}{2}, 0; +\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2})$
$\bar{\psi}_{\dot\alpha, a}$	$(\frac{3}{2}, 0, \pm\frac{1}{2}; -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$

with the single particle partition functions:

$$f_B^v(\beta, \Delta_I, \omega_i) = \frac{e^{-\beta}(e^{\omega_1} + e^{\omega_2} + e^{-\omega_1} + e^{-\omega_2}) - 1 - e^{-2\beta}}{(1 - e^{-\beta + \omega_1})(1 - e^{-\beta + \omega_2})(1 - e^{-\beta - \omega_1})(1 - e^{-\beta - \omega_2})}(1 - e^{-2\beta}) + 1 , \quad (3.16)$$

$$f_F^v(\beta, \Delta_I, \omega_i) = \frac{e^{-\frac{3}{2}\beta}(e^\Delta - e^{-\Delta}e^{-\beta})(e^{\omega_+} + e^{-\omega_+}) + e^{-\frac{3}{2}\beta}(e^{-\Delta} - e^\Delta e^{-\beta})(e^{\omega_-} + e^{-\omega_-})}{(1 - e^{-\beta + \omega_1})(1 - e^{-\beta + \omega_2})(1 - e^{-\beta - \omega_1})(1 - e^{-\beta - \omega_2})} . \quad (3.17)$$

The terms in the boson numerator indicate the vector field A_μ (that has the same quantum numbers as ∇_μ in the denominator), with subtractions for the neutral scalar gauge function Λ and Lorentz condition $\nabla_\mu A^\mu = 0$. The overall factor $(1 - e^{-2\beta})$ imposes the Klein-Gordon equation on all operators and the additional $+1$ corrects for the fact we erroneously counted Λ with no derivative acting on it as a pure gauge degree of freedom. The gaugini follow from the fermions in chiral and anti-chiral multiplets, upon omission of the overall $\sum_{I=1}^{3} e^{-\Delta_I}$ and subsequent reversal of the correlation between chirality and R-charge.

In order to focus on states that satisfy the BPS condition (3.3) we rewrite the physical partition function as (3.4) by introducing the shifted "tilde" potentials (3.5). The low temperature limit $\beta \to \infty$ taken with these variables fixed is exceptionally simple, because only a few single particle operators contribute. Considering bosons and fermions separately, we have

$$f_B = f_B^v + f_B^c + f_B^a = \frac{\sum_I e^{-\tilde{\Delta}_I} + e^{-\tilde{\omega}_1 - \tilde{\omega}_2}}{(1 - e^{-\tilde{\omega}_1})(1 - e^{-\tilde{\omega}_2})} ,$$

$$f_F = f_F^v + f_F^c + f_F^a = \frac{\sum_I e^{\tilde{\Delta}_I - \tilde{\Delta}} e^{-\tilde{\omega}_+} + e^{-\tilde{\Delta} + \tilde{\omega}_+}}{(1 - e^{-\tilde{\omega}_1})(1 - e^{-\tilde{\omega}_2})} - e^{-\tilde{\Delta} + \tilde{\omega}_+} . \quad (3.18)$$

The analogous expressions for general β are much more complicated. However, if we impose the constraint (3.9) by taking $e^{-\tilde{\Delta} + \tilde{\omega}_+} = -1$, the dependence on β simplifies and in fact the total single particle partition function

$$f_B + f_F = 1 - \frac{\prod_I (1 - e^{-\tilde{\Delta}_I})}{(1 - e^{-\tilde{\omega}_1})(1 - e^{-\tilde{\omega}_2})} , \quad (3.19)$$

becomes *independent* of β. Equivalently, the supersymmetric index $f_B - f_F$ is temperature independent on the constraint surface $e^{-\tilde{\Delta} + \tilde{\omega}_+} = +1$. In fact, it is identical to (3.19). These results verify by explicit computations the expectations from the general arguments familiar from the Witten index. The demonstate that the partition function computed at vanishing temperature $\beta = \infty$ applies at any temperature when the constraint is satisfied.

3.3 Multiparticle enumeration

All of the "letters" realized by single particle operators can be multiplied together to form "words". Starting from any of the operators enumerated by the single particle partition function $f(\beta, \Delta_I, \omega_i)$ we can form n-particle products counted by $f(n\beta, n\Delta_I, n\omega_i)$. An overall trace must be imposed to ensure gauge invariance but then n-fold cyclicity of the trace must be taken account so, after summing over any number of operators, the single trace partition function becomes

$$Z_{\text{ST}} = \sum_{k=1}^{\infty} \frac{f(n\beta, n\Delta_I, n\omega_i)}{n}. \tag{3.20}$$

Furthermore, having determined all single trace operators, taking multi-trace operators into account exponentiates the counting. The full partition function becomes

$$Z_{\text{MP}}(\beta, \Delta_I, \omega_i) = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \left[f_B(n\beta, n\Delta_I, n\omega_i) + (-1)^{n+1} f_F(n\beta, n\Delta_I, n\omega_i)\right]\right]. \tag{3.21}$$

This result is purely combinatorial so the statistics of fermions only enter through the exclusion principle. This accounts for the prefactor $(-1)^{n+1}$ in front of the fermionic terms.

The brief justification of the multiparticle partition function (3.21) presented in this subsection has been cavalier about the combinatorics. The relative ordering of operators within traces is important for the detailed enumeration and somewhat elaborate combinatorics (Polya theory) must be invoked. However, for large N the correct result is in fact (3.21) and the heuristic arguments given in this subsection serve to motivate the key features of the formula.

3.4 The $\mathcal{N} = 4$ SYM perturbative matrix model

One additional feature must taken into account. All of the quantum fields transform in the adjoint of $\text{U}(N)$ and we must incorporate their gauge indices. We (somewhat prematurely) imposed a singlet condition on the operators already in the preceding subsection. Incorporation of the full gauge structure gives the unitary matrix model [14, 47, 48]

$$Z(\beta, \Delta_I, \omega_i) = \int [dU] \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} f_n(n\beta, n\Delta_I, n\omega_i) \chi_{\text{Adj}}(U^n)\right\}, \tag{3.22}$$

where U denotes a $\text{U}(N)$ matrix, χ_{Adj} the character in the adjoint representation, and

$$f_n(n\beta, n\Delta_I, n\omega_i) = f_B(n\beta, n\Delta_I, n\omega_i) + (-1)^{n+1} f_F(n\beta, n\Delta_I, n\omega_i). \tag{3.23}$$

For completeness, we recall that in the case of weakly coupled $\mathcal{N} = 4$ SYM the single particle partition functions are

$$
\begin{aligned}
f_B &= f_B^v + f_B^c + f_B^a \,, \\
f_F &= f_F^v + f_F^c + f_F^a \,,
\end{aligned} \tag{3.24}
$$

where the constituent functions were given in (3.10), (3.14), (3.16) and (3.11), (3.15), (3.17), respectively. Their explicit expressions are not illuminating in general but for $\beta = \infty$ they greatly simplify and reduce to (3.18).

The standard strategy for integrating over all unitary matrices is to represent them in terms of their eigenvalues $e^{i\alpha_a}$ (so those of the adjoint representation become $e^{i\alpha_{ab}}$ where $\alpha_{ab} \equiv \alpha_a - \alpha_b$) and change variables to an integral over eigenvalues

$$
Z(\beta, \Delta_I, \omega_i) = \frac{1}{N!} \oint \prod_{a=1}^{N} \frac{d\alpha_a}{2\pi} \prod_{a<b} \left(2 \sin \frac{\alpha_{ab}}{2} \right)^2 \exp \left[\sum_{a,b=1}^{N} \sum_{n=1}^{\infty} \frac{1}{n} f_n(n\beta,\, n\Delta_I,\, n\omega_i)\, e^{in\alpha_{ab}} \right]. \tag{3.25}
$$

The factor involving $\sin \frac{\alpha_{ab}}{2}$ is the van der Monde determinant that arises as a Jacobian due to the change of variables.

We stress that the BPS constraint (3.9) has not yet been imposed on the matrix model so the chemical potentials Δ_I and ω_i are still independent parameters and the partition function (3.25) includes contributions from nonBPS states.

3.5 The BPS limit

The analysis of matrix models is a highly developed science [49, 50]. The established intuition is that the eigenvalues of the matrices experience a universal repulsion because of the van der Monde determinant that may be balanced by attraction due to a model-dependent potential, which in the current context is closely related to the single particle distribution f. The repulsion favors a uniform eigenvalue distribution that corresponds to a confined phase with free energy of order $\mathcal{O}(1)$ while attraction can prompt localization that gives rise to a deconfined phase where free energy increases to order $\mathcal{O}(N^2)$ [51].

Early studies of the matrix model for $\mathcal{N} = 4$ SYM failed to identify a deconfined phase appropriate for the description of macroscopic black holes [14], but recent research makes claims to the contrary [30]. We will refrain from a nuanced discussion of the evidence one way or another and simply assume a deconfined phase where the phases $e^{in\alpha_{ab}}$ do not give cancellations at the leading order. Then the matrix model (3.25) yields the partition function

$$
\ln Z(\beta,\, \Delta_I,\, \omega_i) = N^2 \sum_{n=1}^{\infty} \frac{1}{n} f_n(n\beta,\, n\Delta_I,\, n\omega_i) \,.
$$

The BPS limit requires vanishing temperature and we must also impose the BPS constraint (3.9). We already computed the single particle partition function f in this limit with the result (3.19). The fermion phase factor $(-)^{n+1}$ in (3.23) conspires with signs from

the constraint $e^{-\tilde{\Delta}+\tilde{\omega}_+} = -1$ such that the multiparticle generalization becomes

$$f_n(n\beta,\, n\Delta_I,\, n\omega_i) = 1 - \frac{\prod_I(1 - e^{-n\tilde{\Delta}_I})}{(1 - e^{-n\tilde{\omega}_1})(1 - e^{-n\tilde{\omega}_2})}\,. \tag{3.26}$$

This is exactly the result reported in [24]. There the analysis focussed on the high temperature limit $\beta \to 0$ while we have discussed the low temperature regime $\beta \to \infty$. The agreement of the results is due to the temperature independence along the BPS surface.

We further restrict the discussion to the Cardy limit $|\tilde{\omega}_i| \ll 1$. In this situation it has been argued that all significant contributions to the sum over n are from sufficiently small n that $n|\tilde{\omega}_i| \ll 1$. Therefore

$$\begin{aligned}
\ln Z(\beta,\, \Delta_I,\, \omega_i) &= \frac{N^2}{\tilde{\omega}_1 \tilde{\omega}_2} \sum_{n=1}^{\infty} \frac{1}{n^3} \prod_I (1 - e^{-n\tilde{\Delta}_I}) \\
&= \frac{N^2}{\tilde{\omega}_1 \tilde{\omega}_2} \sum_{s_1 s_2 s_3 = +1} \left[\mathrm{Li}_3\left(-e^{\frac{s_I \tilde{\Delta}_I}{2}}\right) - \mathrm{Li}_3\left(-e^{-\frac{s_I \tilde{\Delta}_I}{2}}\right) \right].
\end{aligned} \tag{3.27}$$

The identity

$$\mathrm{Li}_3(-e^x) - \mathrm{Li}_3(-e^{-x}) = -\frac{x^3}{6} - \frac{\pi^2 x}{6} \quad \text{for } -\pi < \mathrm{Im}(x) < \pi\,, \tag{3.28}$$

finally gives the free energy

$$\begin{aligned}
\ln Z(\beta,\, \Delta_I,\, \omega_i) &= -\frac{N^2}{6\,\tilde{\omega}_1 \tilde{\omega}_2} \sum_{s_1 s_2 s_3 = +1} \left[\frac{1}{8}\left(s_I \tilde{\Delta}_I\right)^3 + \frac{1}{2}\pi^2 \left(s_I \tilde{\Delta}_I\right) \right] \\
&= -\frac{1}{2} N^2 \frac{\tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3}{\tilde{\omega}_1 \tilde{\omega}_2}\,.
\end{aligned} \tag{3.29}$$

3.6 Discussion

The partition function (3.29) with the complex constraint on potentials (3.9) undoubtedly describes $\mathcal{N} = 4$ SYM in the regime relevant for comparison with AdS_5 BPS black holes. It was first inferred from black hole thermodynamics [18] and recently derived from the Euclidean path integral and supersymmetric localization [19], from free field analysis [24], and from computation of the superconformal index using a sum over Bethe vacua [25]. All these works pursue similar ideas but some aspects of the computations and their interrelation remains unclear, at least to the authors of this article.

The simple computations we presented in this section following [24] give an interesting free field representation that appears to capture some aspects of physics in the strongly coupled region. The situation is akin to a model that represents a CFT_2 with central charge c as a free model such as c free bosons. Such a toy model of CFT_2 generally misses many detailed features of the theory, but it captures some aspects of the CFT_2 robustly (such as the Casimir energy and the entropy) and so it serves as a useful benchmark.

In other words, we are not committed to the free field derivation, but the upshot one way or another is that the partition function for the BPS limit is

$$\ln Z = -\frac{1}{2} N^2 \frac{\tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{\Delta}_3}{\tilde{\omega}_1 \tilde{\omega}_2}\,. \tag{3.30}$$

Our goal in the subsequent section is to leverage this result, whichever way it came about, to account also for nonBPS physics.

4 Black hole statistical physics

In this section we discuss thermodynamics of black holes starting from the BPS result for the microscopic free energy (3.30). Our emphasis is on thermodynamic variables for nearBPS black holes.

4.1 Studying nearBPS using BPS data: introduction

The microscopic considerations in section 3 studied the partition function (3.4) which we reproduce here for convenience:[7]

$$Z(\beta, \Delta_I, \omega_i) = \text{Tr} \left[e^{-\beta(E - E^*)} e^{\tilde{\Delta}_I Q_I + \tilde{\omega}_i J_i} \right] . \tag{4.1}$$

The notation strongly suggests that this partition function depends on all the variables $\beta, \Delta_I, \omega_i$ but in fact the computations in section 3 were restricted to the BPS limit, a surface of real codimension two. The restriction on the domain of Z can be specified by the constraints $\beta = \infty$ and $\sum_I \tilde{\Delta}_I - \sum_i \tilde{\omega}_i = 2\pi i$.

The goal of this section is to apply the microscopic description discussed in section 3 also beyond the BPS regime and so allow deformations of both these constraints by small amounts. Our strategy is to make mild smoothness assumptions on the partition function, which are then validated by comparison with our results from gravity. To clarify how it is possible to learn anything interesting from such minimal assumptions, it is instructive to consider a simple example.

As a start, we must address an issue of conventions. In our gravitational computations, the thermodynamic variables satisfy the first law (2.63). Comparison with the microscopic partition function (3.4) indicates a relative factor of β in the potentials, leading to the identifications $\beta(\Phi_I - \Phi_I^*) = \tilde{\Delta}_I$ and $\beta(\Omega_i - \Omega_i^*) = \tilde{\omega}_i$.[8] The shift of the gravitational potentials by their BPS values Φ_I^*, Ω_i^* corresponds precisely to the definition (3.5) of microscopic potentials with a tilde from those without tilde.

The meticulous tracking of conventions gives an immediate payoff in the extremal limit $T \to 0$. The limit is taken such that variables with tilde are kept fixed. Therefore, the potentials that appear in the microscopic partition function are identified with the *thermal derivative* of the potentials employed in the gravity description [13, 45]:

$$\begin{aligned} \text{Re } \tilde{\Delta}_I &= \partial_T \Phi_I , \\ \text{Re } \tilde{\omega}_i &= \partial_T \Omega_i . \end{aligned} \tag{4.2}$$

---

[7]The potentials (Δ_I, ω_i) in the microscopic theory can be identified with (Φ_I, Ω_i) in supergravity. Using different symbols let us stress the distinct provenance of various results and allows for easier comparison with some key references.

[8]The gravitational computations in section 2 were restricted to the diagonal case where the three potentials are equal but this limitation does not apply to formulae in this section. When we "compare" with section 2 we only literally compare for diagonal charges. The generic formula we derive in this section constitute microscopic *predictions* for the gravitational side.

The application of the microscopic BPS partition function (4.1) also in the nearBPS regime gives values for the left hand side that should coincide with the gravitational results on the right hand side. We will verify this expectation in subsection 4.3.

The identifications (4.2) also illustrate the strategy for going beyond BPS. These equations were found by taking the extremal limit $T \to 0$ but supersymmetry was not invoked. This provenance suggests their validity also when the constraint is violated. We will confirm this expectation below. This successful comparison is a simple example of leveraging BPS results to study the nearBPS regime.

Recall that the supersymmetry condition (3.9) is complex and by continuity potentials remain complex in the entire nearBPS region. In our identifications (4.2) we identified the *real* part of the potentials with the corresponding physical field in spacetime. This is justified because the physical charges are real and so their conjugate potentials are real, according to the partition function (4.1). We will later find more data about the nearBPS by exploiting both the real and imaginary parts of the potential, as well as their interplay.

BPS configurations have energy $E = E^*$ and so the saddlepoint approximation to the partition function (4.1) gives the entropy

$$S = \ln Z - \widetilde{\Delta}_I Q_I - \widetilde{\omega}_i J_i - \Lambda \left(\sum_I \widetilde{\Delta}_I - \sum_i \widetilde{\omega}_i - 2\pi i \right). \qquad (4.3)$$

The constraint on potentials (3.9) was imposed by introducing a Lagrange multiplier Λ. The expression (4.3) is referred to as the *entropy function*. Unlike the usual entropy it is a function of potentials but, after they are extremized over, it gives the black hole entropy as function of charges. In the present context there is little more to the terminology than a basic change of thermodynamic ensemble through Legendre transform (but there are impressive generalizations [52]). In the next subsection we will extremize the entropy function (4.3) for BPS black holes explicitly. Later, in subsection 4.4, we will generalize the entire entropy function to the nearBPS regime using minimal assumptions.

4.2 Entropy extremization for BPS black holes

In this subsection we review the computation of BPS black hole entropy. We start from the entropy function S (4.3) derived from the BPS partition function (3.30). Our analysis mainly follow [19].

Extremization of the entropy function (4.3) with $\ln Z$ given by (3.30) gives

$$\frac{\partial S}{\partial \Delta_I} = -\frac{1}{2} N^2 \frac{1}{\widetilde{\Delta}_I} \frac{\widetilde{\Delta}_1 \widetilde{\Delta}_2 \widetilde{\Delta}_3}{\widetilde{\omega}_1 \widetilde{\omega}_2} - (Q_I + \Lambda) = 0, \qquad (4.4)$$

$$\frac{\partial S}{\partial \omega_i} = \frac{1}{2} N^2 \frac{1}{\widetilde{\omega}_i} \frac{\widetilde{\Delta}_1 \widetilde{\Delta}_2 \widetilde{\Delta}_3}{\widetilde{\omega}_1 \widetilde{\omega}_2} - (J_i - \Lambda) = 0, \qquad (4.5)$$

$$\frac{\partial S}{\partial \Lambda} = \sum_I \widetilde{\Delta}_I - \sum_i \widetilde{\omega}_i - 2\pi i = 0. \qquad (4.6)$$

The last equation imposes the constraint (3.9) on $\widetilde{\Delta}_I$ and $\widetilde{\omega}_i$.

We obtain the BPS entropy by simplifying the entropy function (4.3) using the extremization conditions (4.4), (4.5):

$$S^* = 2\pi i \Lambda \,. \tag{4.7}$$

It is therefore essential to find the Lagrange multiplier Λ. To do so, we combine (4.4) and (4.5) to find

$$\prod_I (Q_I + \Lambda) + \frac{1}{2} N^2 \prod_i (J_i - \Lambda) = 0 \,. \tag{4.8}$$

This is a cubic equation that yields the Lagrange multiplier Λ, and so the BPS entropy S^*, as a function of the black hole charges Q_I and J_i. The explicit form of the cubic equation is

$$\Lambda^3 + A\Lambda^2 + B\Lambda + C = 0 \,, \tag{4.9}$$

where

$$\begin{aligned}
A &= Q_1 + Q_2 + Q_3 + \frac{1}{2} N^2 \,, \\
B &= Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - \frac{1}{2} N^2 (J_1 + J_2) \,, \\
C &= Q_1 Q_2 Q_3 + \frac{1}{2} N^2 J_1 J_2 \,.
\end{aligned} \tag{4.10}$$

Imposing reality of the physical entropy (4.7) demands a purely imaginary Λ. Since the charges Q_I and J_i are all real, the purely imaginary roots of (4.9) appear in pairs and the cubic equation must factorize as

$$(\Lambda^2 + B)(\Lambda + A) = 0 \,. \tag{4.11}$$

Thus coefficients in the cubic satisfy $C - AB = 0$ or

$$\left(Q_1 Q_2 Q_3 + \frac{1}{2} N^2 J_1 J_2 \right) - \left(Q_1 + Q_2 + Q_3 + \frac{1}{2} N^2 \right) \left(Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - \frac{1}{2} N^2 (J_1 + J_2) \right) = 0 \,. \tag{4.12}$$

This is the constraint on black hole charges (2.20) that must be satisfied on the BPS surface, as a consequence of the BPS formula for the mass. It generalizes the constraint (2.20) previously found from gravity, with perfect agreement when the three charges are identical. When the constraint is satisfied, the root for Λ with negative imaginary part gives the BPS entropy

$$S^* = 2\pi i \Lambda = 2\pi \sqrt{Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - \frac{1}{2} N^2 (J_1 + J_2)} \,. \tag{4.13}$$

This formula similarly generalizes the result from gravity (2.27) with agreement when the three charges are identical.

We also need the potentials $\widetilde{\Delta}_I$ and $\widetilde{\omega}_i$ at the extremum of the entropy function. We consider $\Lambda = \frac{S^*}{2\pi i}$ the known function of the charges given through (4.13). The extremization conditions (4.4), (4.5) give the ratios

$$\frac{\widetilde{\Delta}_I}{\widetilde{\omega}_i} = -\frac{J_i - \Lambda}{Q_I + \Lambda} \,, \tag{4.14}$$

for any $I = 1, 2, 3$ and $i = 1, 2$. Comparing (4.5) with $i = 1$ and $i = 2$ we also find the ratio

$$\frac{\widetilde{\omega}_1}{\widetilde{\omega}_2} = \frac{J_2 - \Lambda}{J_1 - \Lambda}. \tag{4.15}$$

The constraint (3.9), i.e. the equation of motion for Λ (4.6), now gives

$$\frac{\widetilde{\omega}_1 + \widetilde{\omega}_2 + 2\pi i}{\widetilde{\omega}_i} = \frac{\widetilde{\Delta}_1 + \widetilde{\Delta}_2 + \widetilde{\Delta}_3}{\widetilde{\omega}_i} = -(J_i - \Lambda) \left(\frac{1}{Q_1 + \Lambda} + \frac{1}{Q_2 + \Lambda} + \frac{1}{Q_3 + \Lambda} \right), \quad (4.16)$$

where we have used the ratios $\widetilde{\Delta}_I/\widetilde{\omega}_i$ given in (4.14). The ratio between the $\widetilde{\omega}_i$'s (4.15) let us reorganize as

$$\frac{2\pi i}{\widetilde{\omega}_i} = -(J_i - \Lambda) \left(\frac{1}{J_1 - \Lambda} + \frac{1}{J_2 - \Lambda} + \frac{1}{Q_1 + \Lambda} + \frac{1}{Q_2 + \Lambda} + \frac{1}{Q_3 + \Lambda} \right), \tag{4.17}$$

$$\frac{2\pi i}{\widetilde{\Delta}_I} = (Q_I + \Lambda) \left(\frac{1}{J_1 - \Lambda} + \frac{1}{J_2 - \Lambda} + \frac{1}{Q_1 + \Lambda} + \frac{1}{Q_2 + \Lambda} + \frac{1}{Q_3 + \Lambda} \right). \tag{4.18}$$

The second line was found by invoking the ratio (4.14). The inverses of these equations give

$$\frac{\widetilde{\omega}_i}{2\pi i} = \frac{1}{2} N^2 \frac{\prod_k (J_k - \Lambda)}{J_i - \Lambda} \frac{1}{2\Lambda(\Lambda + Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2)}, \tag{4.19}$$

$$\frac{\widetilde{\Delta}_I}{2\pi i} = \frac{\prod_K (Q_K + \Lambda)}{Q_I + \Lambda} \frac{1}{2\Lambda(\Lambda + Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2)}. \tag{4.20}$$

We used (4.8) to simplify the algebra. These are the explicit results for the potentials on the BPS surface expressed in terms of charges. We recall that $\Lambda = \frac{S^*}{2\pi i}$ is the known function of the charges given through (4.13).

4.3 NearBPS microscopics

We now want to leverage the microscopic results derived in the BPS limit to study nearBPS black holes as well.

The real part of the potentials (4.19) are

$$\text{Re } \widetilde{\omega}_1 = -\frac{\pi^2 N^2}{S^*} \frac{J_2(Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2) - \left(\frac{S^*}{2\pi}\right)^2}{\left(\frac{S^*}{2\pi}\right)^2 + (Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2)^2}, \tag{4.21}$$

with an analogous expression for $\text{Re } \widetilde{\omega}_2$, and the real part of the potentials (4.20) similarly are

$$\text{Re } \widetilde{\Delta}_1 = -\frac{2\pi^2}{S^*} \frac{(Q_2 Q_3 - \left(\frac{S^*}{2\pi}\right)^2)(Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2) + \left(\frac{S^*}{2\pi}\right)^2 (Q_2 + Q_3)}{\left(\frac{S^*}{2\pi}\right)^2 + (Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2)^2}, \tag{4.22}$$

with analogous equations for $\text{Re } \widetilde{\Delta}_2$ and $\text{Re } \widetilde{\Delta}_3$. These formulae for the potentials $\text{Re } \widetilde{\omega}_i$, $\text{Re } \widetilde{\Delta}_I$ in the microscopic theory all agree precisely with the corresponding gravitational formulae for $\partial_T \Phi_I$ and $\partial_T \Omega_i$ given in (2.37)–(2.38). This confirms the identifications (4.2).

It is significant that these comparisons *all* agree. The differentials $(d\mathrm{Re}\,\widetilde{\omega}_i, d\mathrm{Re}\,\widetilde{\Delta}_I)$ form a vector in the five dimensional space that is generated (locally) by the direct sum of the tangent space to the BPS surface and its normal, and the latter violates the constraint. It is also noteworthy that the comparisons agree *precisely*. Formulae in the BPS limit are only defined modulo the constraint on charges but the agreements found here apply with no need for the constraint. These facts suggest that the microscopic description goes beyond BPS.

We can make these observations quantitative by forming the differential

$$dS^* + \mathrm{Re}\,\widetilde{\Delta}_I\, dQ_I + \mathrm{Re}\,\widetilde{\omega}_i\, dJ_i = \frac{2\pi^2}{S^*} \frac{(Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2)\, dh}{\left(\frac{S^*}{2\pi}\right)^2 + \left(Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2\right)}\,, \qquad (4.23)$$

where the height function h defined through

$$h = \left(Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2\right)\left(Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - \frac{1}{2}N^2(J_1 + J_2)\right) - \left(Q_1 Q_2 Q_3 + \frac{1}{2}N^2 J_1 J_2\right), \qquad (4.24)$$

is a measure of the violation of the constraint (4.12) on black hole charges. The addition of the differential dS^* on the left hand side of (4.23) removes the terms that are attributable to BPS physics. We can interpret the remainder as a formula for the entropy in excess of the BPS entropy

$$S - S^* = \frac{2\pi^2 h}{S^*} \frac{Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2}{\left(\frac{S^*}{2\pi}\right)^2 + \left(Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2\right)}\,, \qquad (4.25)$$

due to the violation of the constraint $h = 0$. This formula is consistent with the gravitational result (2.66). Our definition (4.24) of the height function is such that $h \geq 0$ corresponds to positive entropy.

It is also well worth the effort to extract the imaginary parts of the complex potentials (4.19)–(4.20) found in the microscopic computation. Representative components are

$$\frac{\mathrm{Im}\,\widetilde{\omega}_1}{2\pi} = -\frac{1}{4}N^2 \frac{J_2 + Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2}{\left(\frac{S^*}{2\pi}\right)^2 + \left(Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2\right)^2}\,, \qquad (4.26)$$

$$\frac{\mathrm{Im}\,\widetilde{\Delta}_1}{2\pi} = \frac{(Q_2 + Q_3)(2Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2) - \frac{1}{2}N^2(J_1 + J_2)}{2\left[\left(\frac{S^*}{2\pi}\right)^2 + \left(Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2\right)^2\right]}\,, \qquad (4.27)$$

and the complete set of potentials follow by appropriate permutation of indices. It is instructive to collect the entire vector of imaginary potentials as a one-form

$$\frac{\mathrm{Im}\,\widetilde{\Delta}_I\, dQ_I + \mathrm{Im}\,\widetilde{\omega}_i\, dJ_i}{2\pi} = -\frac{dh}{2\left[\left(\frac{S^*}{2\pi}\right)^2 + \left(Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2\right)^2\right]}\,. \qquad (4.28)$$

The entropy function formalism for the BPS black holes in AdS$_5$ [18] employs complex potentials and identify their real part as the physical potential, but their imaginary part is enigmatic. It is therefore satisfying that the imaginary potentials computed in the

microscopic description of BPS black holes vanish on the constraint surface $h = 0$ but increase proportionally to violations of the constraint.

Combining the real and imaginary parts of the potentials (4.23), (4.28), we can write the first law of thermodynamics as a complex-valued expression:

$$
\begin{aligned}
& T dS^* + (\widetilde{\Delta}_I - \widetilde{\Delta}_I^*)dQ_I + (\widetilde{\omega}_i - \widetilde{\omega}_i^*)dJ_i \\
& = \left(\frac{2\pi}{S^*} \left(Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2 \right) T - i \right) \frac{\pi\, dh}{\left(\frac{S^*}{2\pi} \right)^2 + \left(Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2 \right)^2} \, .
\end{aligned}
\tag{4.29}
$$

It is not a coincidence that this formula is a close analogue of (2.66), the consolidated formula for all the potentials we computed from gravity in the nearBPS region. In the following subsection we relate them quantitatively.

4.4 Extremizing free energy near BPS

In this subsection, we generalize the BPS entropy extremization reviewed in subsection 4.2 to an extremization principle that describes the nearBPS region.

4.4.1 The free energy function

As we stressed repeatedly in section 3, all microscopic computations of the BPS arrive at the partition function (3.30) subject to the subsidiary condition (3.9) and the black hole entropy is subsequently extracted therefrom through the extremization procedure reviewed in subsection 4.2. The variables of the BPS partition function are potentials that must be interpreted as thermal derivatives of physical potentials. Our discussion in subsection 4.1 arrived at this identification in the course of comparing microscopic conventions with those of gravity, but it applies equally as a general relation between the extremal and near-extremal partition functions. Thus the adaptation of the BPS partition function (3.30) to a notation appropriate for the nearBPS regime is

$$
\ln Z(\beta, \Delta_I, \omega_i) = -\frac{N^2}{2T} \frac{(\Delta_1 - \Delta_1^*)(\Delta_2 - \Delta_2^*)(\Delta_3 - \Delta_3^*)}{(\omega_1 - \omega_1^*)(\omega_2 - \omega_2^*)} \, .
\tag{4.30}
$$

It is significant to note that there are *no* tilde's on the variables in this formula: Δ_I and ω_i refer to the full potentials rather than their thermal derivatives. The BPS reference values Δ_I^* and ω_i^* all equal 1 numerically, because the BPS mass is the sum of the conjugate conserved charges with coefficient 1 (in units where $\ell_5 = 1$). We maintain the more elaborate notation for conceptual clarity. The original BPS partition function (3.30) is recovered in the extremal limit $T \to 0$, as it should be.

The microscopic partition function (4.1) was introduced generally, without restricting to the BPS (or even the nearBPS) regime. In the saddle point approximation it gives

$$
\ln Z = S - \beta(M - M^*) + \beta(\Delta_I - \Delta_I^*)Q_I + \beta(\omega_i - \omega_i^*)J_i \, .
\tag{4.31}
$$

It is tempting to identify $\ln Z$ in this formula with the BPS partition function (4.30) but that is incorrect even in the strict BPS limit where it is crucial that we require the potentials to satisfy the complex constraint (3.9). Moreover, in the BPS limit the real part of the

potentials satisfy $\sum_I(\Delta_I - \Delta_I^*) - \sum_i(\omega_i - \omega_i^*) = 0$ by definition but this equation must be relaxed in the nearBPS region. We impose the constraint

$$\sum_I(\Delta_I - \Delta_I^*) - \sum_i(\omega_i - \omega_i^*) = \varphi + 2\pi i T. \qquad (4.32)$$

The imaginary part of this equation reformulates the BPS condition (3.9) in a manner that is meaningful also for small but nonvanishing temperature. The real part allows potentials to depart from their BPS values and parametrize this violation by φ, in conformity with the convention in gravity (2.43).

We now combine the general saddle point approximation (4.31) with the microscopic computation of the free energy (4.30) and subject the result to the complex constraint (4.32). This gives the free energy in the nearBPS regime

$$\begin{aligned}
\mathcal{F} &\equiv (M - M^*) - TS \\
&= \frac{1}{2}N^2 \frac{(\Delta_1 - \Delta_1^*)(\Delta_2 - \Delta_2^*)(\Delta_3 - \Delta_3^*)}{(\omega_1 - \omega_1^*)(\omega_2 - \omega_2^*)} + (\Delta_I - \Delta_I^*)Q_I + (\omega_i - \omega_i^*)J_i + \\
&\quad + \Lambda\left(\sum_I(\Delta_I - \Delta_I^*) - \sum_i(\omega_i - \omega_i^*) - \varphi - 2\pi i T\right).
\end{aligned} \qquad (4.33)$$

We have not attempted to argue that there can be no additional contributions to the free energy in the nearBPS regime. On the contrary, we conservatively claim that it *at least* includes the ingredients incorporated in (4.33).

An *effective* theory of nearBPS black holes can be found be extremizing the free energy (4.33) over the potentials Δ_I, ω_i, and the Lagrange multiplier Λ. It will depend on the remaining potentials φ, T, the dynamical fields in the effective description. As usual, the effective theory will also feature dimensionful parameters that are unspecified *a priori* but computable from the UV completion in principle. In the present context a complete microscopic theory relates the effective parameters (C_T, C_φ, C_E) to conserved charges.

4.4.2 Extremization of free energy

The free energy (4.33) differs from the entropy function (4.3) only by an overall factor $-T$ and the addition of a simple term $-\Lambda\varphi$. Therefore the extremization is nearly unchanged from the BPS computation in subsection 4.2. The equations of motion (4.4)–(4.5) are entirely unchanged so the ratios (4.14)–(4.15) remain, and so the steps needed for finding the potentials explicitly are exactly the same. The key modification is the equation of motion for Λ, ie. the constraint (4.32). For the potentials the constraint enters in (4.16) where its role is to provide an overall normalization for the potentials that are otherwise determined by relations between their ratios. The new (complex) normalization modifies the potentials from (4.19)–(4.20) to

$$\frac{\omega_i - \omega_i^*}{\varphi + 2\pi i T} = \frac{1}{2}N^2 \frac{\prod_k(J_k - \Lambda)}{J_i - \Lambda} \frac{1}{2\Lambda\left(\Lambda + Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2\right)}, \qquad (4.34)$$

$$\frac{\Delta_I - \Delta_I^*}{\varphi + 2\pi i T} = \frac{\prod_K(Q_K + \Lambda)}{Q_I + \Lambda} \frac{1}{2\Lambda\left(\Lambda + Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2\right)}. \qquad (4.35)$$

Consider ω_1 for definiteness. For small φ the constraint on the charges is only violated mildly so we assume that Λ is unchanged to leading order, and then the right hand of the first equation is $\frac{1}{2\pi i}$ (Re $\widetilde{\omega}_1 + i$Im $\widetilde{\omega}_1$) where Re $\widetilde{\omega}_1$ and Im $\widetilde{\omega}_1$ are the BPS values for these variables given in (4.21), (4.26). The physical potential is the real part so the leading dependence on the constraint violation φ is due to the *imaginary* part of $\widetilde{\omega}_1$:

$$\text{Re}(\omega_1 - \omega_1^*)\big|_{\varphi \text{ dependence}} = \varphi \, \frac{\text{Im } \widetilde{\omega}_1}{2\pi} = -\frac{1}{4} N^2 \frac{J_2 + Q_1 + Q_2 + Q_3 + \frac{1}{2} N^2}{\left(\frac{S^*}{2\pi}\right)^2 + \left(Q_1 + Q_2 + Q_3 + \frac{1}{2} N^2\right)^2} \varphi. \quad (4.36)$$

This agrees exactly with the gravitational computation of the change of the rotational velocity Ω_1 due to small violations of the constraint (2.50). The dependences of all other potentials on φ similarly agree with gravitational results.

At this point we have showed that the extremization principle based on the free energy (4.33) reproduces the dependence of all physical potentials on temperature and on violations of the constraint parametrized by the field deformation φ. We now turn to the mass and entropy in the nearBPS regime.

As we mentioned earlier in this subsection, the equations of motion are entirely unchanged from the BPS case (4.4)–(4.5), except for the equation of motion for the Lagrange multiplier Λ, ie. the constraint (4.32). Therefore the cubic equation (4.8) for Λ still holds. However, since the potentials Δ_I, ω_i do not satisfy the BPS constraint (3.9) in the nearBPS theory, the charges Q_I, J_i may also violate their constraint (4.12).

We already anticipated this situation in the preceding subsection when introducing a height function h (4.24) parametrizing the violation of the constraint (4.12). In the schematic notation introduced in (4.9)–(4.10) the height function $h \equiv C - AB$ deforms the cubic equation to

$$\Lambda^3 + A\Lambda^2 + B\Lambda + C = (\Lambda^2 + B)(\Lambda + A) + h = 0. \quad (4.37)$$

Modifying charges and Λ away from $h = 0$ at first order in perturbation theory

$$(2\Lambda\delta\Lambda + \delta B)(\Lambda + A) + h = 0,$$

and recalling that Λ is purely imaginary at leading order we find

$$\text{Im } \delta\Lambda = \frac{Ah}{2\text{Im}\Lambda(B + A^2)} + \frac{\delta B}{2\text{Im}\Lambda}. \quad (4.38)$$

The second term

$$\frac{\delta B}{2\text{Im}\Lambda} = -\frac{\delta S_*}{2\pi}, \quad (4.39)$$

takes into account the change of the BPS entropy due to changes of the conserved charges. Therefore the entropy in excess of the BPS entropy due to the violation of the constraint becomes

$$\delta(S - S^*) = -2\pi\text{Im } \delta\Lambda = \frac{2\pi^2 h}{S^*} \frac{Q_1 + Q_2 + Q_3 + \frac{1}{2} N^2}{\left(\frac{S^*}{2\pi}\right)^2 + (Q_1 + Q_2 + Q_3 + \frac{1}{2} N^2)^2}. \quad (4.40)$$

This formula agrees with the expression (4.25) inferred from applying the thermodynamic potentials derived in the BPS case also off the BPS surface. Moreover, the same expression was found also from gravity considerations (2.66). The considerations here are based on a minimal generalization of the *microscopic* theory.

4.4.3 Parameters in effective field theory

We have expressed possible violations of the constraint in two distinct ways. We defined the height function h (4.24) that measures departures from the constraint $h = 0$ on charges, and the effective potential φ introduced through the deformed constraint (4.32) that is required to take the value $\varphi = 0$ by the conditions for supersymmetry (3.9). The "charge" h and the "potential" φ are proportional near the BPS surface we computed their constant of proportionality (2.58) on the gravity side. It is interesting to understand their relation more generally.

The height function h (4.24) is fairly elaborate so the expression for small departures $\delta h = \partial_{Q_I} h \delta Q_I + \partial_{J_i} h \delta J_i$ from the constraint $h = 0$ due to general variations δQ_I, $\delta \omega_i$ of the charges is lengthy and not illuminating. However, variations that are *proportional* to the charges themselves $\delta Q_I = \lambda Q_I$, $\delta J_i = \lambda J_i$ yield a manageable formula

$$\delta h = \Big[\big(Q_1^2(Q_2 + Q_3) + \text{cyclic} \big) + 2Q_1Q_2Q_3 + \frac{1}{4}N^4(J_1 + J_2) \Big]\lambda, \qquad (4.41)$$

after simplifications using the constraint $h = 0$. However, the transformation of the effective potential is canonical (with dimension 2 relative to the charge):

$$\delta\varphi = 2\lambda\varphi. \qquad (4.42)$$

Comparison of the two preceding equations gives

$$h = \frac{1}{2}\Big[\big(Q_1^2(Q_2 + Q_3) + \text{cyclic} \big) + 2Q_1Q_2Q_3 + \frac{1}{4}N^4(J_1 + J_2) \Big]\varphi. \qquad (4.43)$$

This generalizes the constant of proportionality (2.58) computed in the gravity to the case of three distinct charges.

We can now consolidate our results by presenting the first law of thermodynamics in a coherent manner. We collected most of them already in the complex form of the first law (4.29). In the course of this subsection we have rederived each of the terms in this equation from free energy extremization. Additionally, the imaginary part of the potentials acquired a more satisfying interpretation through its relation with the deformation parameter φ given in (4.36) and its analogues for other potentials. The conversion (4.43) between h with φ finally let us rewrite (4.29) as

$$TdS^* + (\Delta_I - \Delta_I^*)dQ_I + (\omega_i - \omega_i^*)dJ_i = \frac{C_T}{T}\left[\frac{2\pi}{S^*}\left(Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2 \right)T + \frac{\varphi}{2\pi} \right]\frac{d\varphi}{2\pi}, \qquad (4.44)$$

where C_T is the heat capacity that is linear in temperature with proportionality constant

$$\left.\frac{C_T}{T}\right|_{\text{nearExt}} = \pi^2 \frac{Q_1^2(Q_2 + Q_3) + Q_2^2(Q_3 + Q_1) + Q_3^2(Q_1 + Q_2) + 2Q_1Q_2Q_3 + \frac{1}{4}N^4(J_1 + J_2)}{Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \frac{1}{2}N^2(J_1 + J_2) + (Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2)^2}.$$

We previously computed this expression in gravity (2.35). In some parts of this paper we introduced the effective field theory parameters C_φ and C_E in addition to C_T. They all have different physical significance but numerically $C_\varphi = C_T$ and

$$\left(\frac{C_E}{T} \right) = \frac{2\pi}{S^*}\left(\frac{C_T}{T} \right)\left(Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2 \right), \qquad (4.45)$$

so in the presentation of these final results we opt for writing the formulae more explicitly.

The first law of thermodynamics

$$TdS = d(M - M^*) - (\Delta_I - \Delta_I^*)dQ_I - (\omega_i - \omega_i^*)dJ_i \,, \qquad (4.46)$$

now gives the energy and the entropy of the excitations above the BPS ground state as

$$M - M^* = \frac{1}{2}\frac{C_T}{T}\left(\frac{\varphi}{2\pi}\right)^2 \,, \qquad (4.47)$$

$$S - S^* = \frac{2\pi}{S^*}\frac{C_T}{T}\left(Q_1 + Q_2 + Q_3 + \frac{1}{2}N^2\right)\frac{\varphi}{2\pi} \,. \qquad (4.48)$$

These expressions agree with results earlier in the paper, including computations in gravity (2.55), (2.61).

Our microscopic discussion in this section has not at all touched on the conventional heat capacity, ie. the term in the entropy that is linear in temperature (and correlated with a mass term that is quadratic in the temperature). What is needed to get this term is an equation of motion where the height parameter h in the cubic equation (4.37) is traded for φ through (4.43) and subsequently complexified to $\varphi + 2\pi iT$. The last step is very natural in that much of the theory apparently depends holomorphically on a complex symmetry breaking parameter. However, we are not yet able to present a principled argument based on microscopic theory.

5 Summary and outlook

In this paper we have discussed the thermodynamics and the statistical physics of AdS$_5$ black holes, addressing both the BPS and the nearBPS configurations. In the nearBPS region, we made an important distinction between the near-extremal ($T \neq 0$, $\varphi = 0$), the extremal nearBPS ($T = 0$, $\varphi \neq 0$), and the general nearBPS ($T \neq 0$, $\varphi \neq 0$) black holes. The unfamiliar potential φ parametrizes the possible violation of a constraint on charges that must be imposed in the strict BPS limit.

In the gravitational theory we studied all thermodynamic potentials in great detail, especially their interrelation through the first law of thermodynamics, in the entire BPS and nearBPS domain. In the holographically dual theory, we reviewed an elementary version of the microscopic description, based on the free field representation of $\mathcal{N} = 4$ SYM. It yields the semiclassical partition function found in [18] that is common to all the recent proposals for a microscopic theory of BPS black holes in AdS$_5$ [19, 24, 25].

We found that, with minor additional assumptions, the same semiclassical partition function also describes aspects of nearBPS black holes. This approach to the theory without supersymmetry is particularly successful for extremal nonBPS black holes. It is the main basis for our claim of a microscopic description of the nearBPS black holes. Our generalization of the entropy function to a free energy function captures the effective field theory of the nearBPS region succinctly and in a manner that relates directly to the microscopic theory.

The recent progress towards a statistical description of BPS black holes [19, 24, 25] is not yet fully satisfactory. It is not even clear that the reports are consistent with one another. Our study gives general support for these advances. For example, we find that the BPS limit is robust in that it can be approached from any direction.

However, there are still many open questions, particularly on the microscopic side. The free energy function could surely be improved. The more important open problem is why the microscopic description of nearBPS black holes is even possible. It appears that some non-renormalization due to supersymmetry persists also away from the BPS limit. We suspect this follows from softly broken superconformal symmetry but we have not developed a principled argument.

Furthermore, we expect that the additional microscopic degrees of freedom in the nearBPS region can equally be modeled by a simple gas of free particles, much like the free model of the BPS limit that we review. However, it might well be necessary to invoke other sectors of $\mathcal{N} = 4$ SYM that are BPS, but preserving different supersymmetries than the ground state. This is the structure of successful microscopic models for near-extremal black holes with flat asymptotic space, such as the D1-D5 system [53].

There are also problems in gravitational physics that we leave for the future. We found that the potential for constraint violations φ exhibits entropic preference for $\varphi \geq 0$, reminiscent of the behavior of conventional temperature T. It would be interesting to develop the geometric underpinnings of φ. Additionally, the near horizon AdS_2 expected for any near-extremal black hole must have an analogue for extremal nearBPS configurations and the full nearBPS region of parameter space promises an interesting interplay between the low temperature limit and a mildly violated constraint.

We look forward to pursue these and related directions in future research.

Acknowledgments

References

[1] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett.* **B 379** (1996) 99 [hep-th/9601029] [INSPIRE].

[2] F. Benini, K. Hristov and A. Zaffaroni, *Black hole microstates in AdS$_4$ from supersymmetric localization*, *JHEP* **05** (2016) 054 [arXiv:1511.04085] [INSPIRE].

[3] S. Sachdev and J. Ye, *Gapless spin fluid ground state in a random, quantum Heisenberg magnet*, *Phys. Rev. Lett.* **70** (1993) 3339 [cond-mat/9212030] [INSPIRE].

[4] A. Kitaev, *A simple model of quantum holography*, http://online.kitp.ucsb.edu/online/entangled15/.

[5] J. Maldacena, D. Stanford and Z. Yang, *Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space*, *PTEP* **2016** (2016) 12C104 [arXiv:1606.01857] [INSPIRE].

[6] P. Gao, D.L. Jafferis and A.C. Wall, *Traversable Wormholes via a Double Trace Deformation*, *JHEP* **12** (2017) 151 [arXiv:1608.05687] [INSPIRE].

[7] M. Rangamani and T. Takayanagi, *Holographic Entanglement Entropy*, *Lect. Notes Phys.* **931** (2017) 1 [arXiv:1609.01287] [INSPIRE].

[8] D. Harlow, *TASI Lectures on the Emergence of Bulk Physics in AdS/CFT*, PoS(TASI2017)002 (2018) [arXiv:1802.01040] [INSPIRE].

[9] J.B. Gutowski and H.S. Reall, *Supersymmetric AdS$_5$ black holes*, *JHEP* **02** (2004) 006 [hep-th/0401042] [INSPIRE].

[10] J.B. Gutowski and H.S. Reall, *General supersymmetric AdS$_5$ black holes*, *JHEP* **04** (2004) 048 [hep-th/0401129] [INSPIRE].

[11] Z.W. Chong, M. Cvetič, H. Lü and C.N. Pope, *General non-extremal rotating black holes in minimal five-dimensional gauged supergravity*, *Phys. Rev. Lett.* **95** (2005) 161301 [hep-th/0506029] [INSPIRE].

[12] S.-Q. Wu, *General Nonextremal Rotating Charged AdS Black Holes in Five-dimensional U(1)^3 Gauged Supergravity: A Simple Construction Method*, *Phys. Lett.* **B 707** (2012) 286 [arXiv:1108.4159] [INSPIRE].

[13] S. Kim and K.-M. Lee, *1/16-BPS Black Holes and Giant Gravitons in the AdS$_5 \times S^5$ Space*, *JHEP* **12** (2006) 077 [hep-th/0607085] [INSPIRE].

[14] J. Kinney, J.M. Maldacena, S. Minwalla and S. Raju, *An Index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209 [hep-th/0510251] [INSPIRE].

[15] M. Berkooz, D. Reichmann and J. Simon, *A Fermi Surface Model for Large Supersymmetric AdS$_5$ Black Holes*, *JHEP* **01** (2007) 048 [hep-th/0604023] [INSPIRE].

[16] L. Grant, P.A. Grassi, S. Kim and S. Minwalla, *Comments on 1/16 BPS Quantum States and Classical Configurations*, *JHEP* **05** (2008) 049 [arXiv:0803.4183] [INSPIRE].

[17] C.-M. Chang and X. Yin, *1/16 BPS states in $\mathcal{N} = 4$ super-Yang-Mills theory*, *Phys. Rev.* **D 88** (2013) 106005 [arXiv:1305.6314] [INSPIRE].

[18] S.M. Hosseini, K. Hristov and A. Zaffaroni, *An extremization principle for the entropy of rotating BPS black holes in AdS$_5$*, *JHEP* **07** (2017) 106 [arXiv:1705.05383] [INSPIRE].

[19] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *Microscopic origin of the Bekenstein-Hawking entropy of supersymmetric AdS$_5$ black holes*, *JHEP* **10** (2019) 062 [arXiv:1810.11442] [INSPIRE].

[20] B. Assel, D. Cassani and D. Martelli, *Localization on Hopf surfaces*, *JHEP* **08** (2014) 123 [arXiv:1405.5144] [INSPIRE].

[21] B. Assel, D. Cassani, L. Di Pietro, Z. Komargodski, J. Lorenzen and D. Martelli, *The Casimir Energy in Curved Space and its Supersymmetric Counterpart*, *JHEP* **07** (2015) 043 [arXiv:1503.05537] [INSPIRE].

[22] N. Bobev, M. Bullimore and H.-C. Kim, *Supersymmetric Casimir Energy and the Anomaly Polynomial*, *JHEP* **09** (2015) 142 [arXiv:1507.08553] [INSPIRE].

[23] F. Brünner, D. Regalado and V.P. Spiridonov, *Supersymmetric Casimir energy and* SL(3, ℤ) *transformations*, *JHEP* **07** (2017) 041 [arXiv:1611.03831] [INSPIRE].

[24] S. Choi, J. Kim, S. Kim and J. Nahmgoong, *Large AdS black holes from QFT*, arXiv:1810.12067 [INSPIRE].

[25] F. Benini and P. Milan, *Black holes in 4d* $\mathcal{N} = 4$ *Super-Yang-Mills*, *Phys. Rev.* **X 10** (2020) 021037 [arXiv:1812.09613] [INSPIRE].

[26] F. Benini and P. Milan, *A Bethe Ansatz type formula for the superconformal index*, arXiv:1811.04107 [INSPIRE].

[27] S.M. Hosseini, A. Nedelin and A. Zaffaroni, *The Cardy limit of the topologically twisted index and black strings in* AdS_5, *JHEP* **04** (2017) 014 [arXiv:1611.09374] [INSPIRE].

[28] J. Hong and J.T. Liu, *The topologically twisted index of* $\mathcal{N} = 4$ *super-Yang-Mills on* $T^2 \times S^2$ *and the elliptic genus*, *JHEP* **07** (2018) 018 [arXiv:1804.04592] [INSPIRE].

[29] A. Zaffaroni, *Lectures on AdS Black Holes, Holography and Localization*, 2019, arXiv:1902.07176 [INSPIRE].

[30] S. Choi, J. Kim, S. Kim and J. Nahmgoong, *Comments on deconfinement in AdS/CFT*, arXiv:1811.08646 [INSPIRE].

[31] M. Honda, *Quantum Black Hole Entropy from 4d Supersymmetric Cardy formula*, *Phys. Rev.* **D 100** (2019) 026008 [arXiv:1901.08091] [INSPIRE].

[32] A. Arabi Ardehali, *Cardy-like asymptotics of the 4d* $\mathcal{N} = 4$ *index and* AdS_5 *blackholes*, *JHEP* **06** (2019) 134 [arXiv:1902.06619] [INSPIRE].

[33] S. Choi and S. Kim, *Large* AdS_6 *black holes from* CFT_5, arXiv:1904.01164 [INSPIRE].

[34] J. Kim, S. Kim and J. Song, *A 4d N = 1 Cardy Formula*, arXiv:1904.03455 [INSPIRE].

[35] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *The asymptotic growth of states of the 4d* $\mathcal{N} = 1$ *superconformal index*, *JHEP* **08** (2019) 120 [arXiv:1904.05865] [INSPIRE].

[36] A. Amariti, I. Garozzo and G. Lo Monaco, *Entropy function from toric geometry*, arXiv:1904.10009 [INSPIRE].

[37] J. Nian and L.A. Pando Zayas, *Microscopic entropy of rotating electrically charged* AdS_4 *black holes from field theory localization*, *JHEP* **03** (2020) 081 [arXiv:1909.07943] [INSPIRE].

[38] A. Almheiri and J. Polchinski, *Models of* AdS_2 *backreaction and holography*, *JHEP* **11** (2015) 014 [arXiv:1402.6334] [INSPIRE].

[39] A. Almheiri and B. Kang, *Conformal Symmetry Breaking and Thermodynamics of Near-Extremal Black Holes*, *JHEP* **10** (2016) 052 [arXiv:1606.04108] [INSPIRE].

[40] F. Larsen, *A nAttractor mechanism for* $nAdS_2/nCFT_1$ *holography*, *JHEP* **04** (2019) 055 [arXiv:1806.06330] [INSPIRE].

[41] F. Larsen and E.J. Martinec, U(1) *charges and moduli in the D1-D5 system*, *JHEP* **06** (1999) 019 [hep-th/9905064] [INSPIRE].

[42] F. Larsen and E.J. Martinec, *Currents and moduli in the* (4, 0) *theory*, *JHEP* **11** (1999) 002 [hep-th/9909088] [INSPIRE].

[43] T. Birkandan and M. Cvetič, *Wave Equation for the Wu Black Hole*, *JHEP* **09** (2014) 121 [`arXiv:1407.8173`] [ɪɴSPIRE].

[44] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *Rotating black holes in gauged supergravities: Thermodynamics, supersymmetric limits, topological solitons and time machines*, `hep-th/0504080` [ɪɴSPIRE].

[45] P.J. Silva, *Thermodynamics at the BPS bound for Black Holes in AdS*, *JHEP* **10** (2006) 022 [`hep-th/0607056`] [ɪɴSPIRE].

[46] J. Preskill, P. Schwarz, A.D. Shapere, S. Trivedi and F. Wilczek, *Limitations on the statistical description of black holes*, *Mod. Phys. Lett.* **A 6** (1991) 2353 [ɪɴSPIRE].

[47] B. Sundborg, *The Hagedorn transition, deconfinement and $N = 4$ SYM theory*, *Nucl. Phys.* **B 573** (2000) 349 [`hep-th/9908001`] [ɪɴSPIRE].

[48] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk, *The Hagedorn-deconfinement phase transition in weakly coupled large N gauge theories*, *Adv. Theor. Math. Phys.* **8** (2004) 603 [`hep-th/0310285`] [ɪɴSPIRE].

[49] P.H. Ginsparg and G.W. Moore, *Lectures on* 2-*D gravity and* 2-*D string theory*, in *Proceedings, Theoretical Advanced Study Institute (TASI 92): From Black Holes and Strings to Particles*, Boulder, U.S.A., 1–26 June 1992, pp. 277–469 (1993) [`hep-th/9304011`] [ɪɴSPIRE].

[50] M. Mariño, *Lectures on localization and matrix models in supersymmetric Chern-Simons-matter theories*, *J. Phys.* **A 44** (2011) 463001 [`arXiv:1104.0783`] [ɪɴSPIRE].

[51] D.J. Gross and E. Witten, *Possible Third Order Phase Transition in the Large N Lattice Gauge Theory*, *Phys. Rev.* **D 21** (1980) 446 [ɪɴSPIRE].

[52] A. Sen, *Black Hole Entropy Function, Attractors and Precision Counting of Microstates*, *Gen. Rel. Grav.* **40** (2008) 2249 [`arXiv:0708.1270`] [ɪɴSPIRE].

[53] C.G. Callan and J.M. Maldacena, *D-brane approach to black hole quantum mechanics*, *Nucl. Phys.* **B 472** (1996) 591 [`hep-th/9602043`] [ɪɴSPIRE].