# E-Companion

In this Appendix, we present all the mising proofs in the mainbody of the paper. We also prove the result discussed in Remark 7 of Section 3 for a more general definition of clusters.

## EC.1 Proof of Theorem 1

First of all, we define $\widetilde{q}_j := \sum_{i\in\mathcal{N}_j} q_i$ as the probability that a customer views a product from cluster $j$. Then, define the events

$$\begin{aligned}
\mathcal{E}_{N,t} :=& \{\hat{\mathcal{N}}_t = \mathcal{N}_{i_t}\},\\
\mathcal{E}_{B_j,t} :=& \{||\widetilde{\theta}_{j,t} - \theta_j||_2 \le \widetilde{B}_{j,t}\},\\
\mathcal{E}_{V,t} :=& \left\{\lambda_{\min}\left(\sum_{s\in\widetilde{\mathcal{T}}_{j_t,t}} u_s u_s'\right) \ge \frac{\lambda_1 \Delta_0^2 \sqrt{\widetilde{q}_{j_t} t}}{8}\right\},
\end{aligned}$$

where $\lambda_1 = \min(1,\lambda_0)/(1+\overline{p}^2)$ and $\widetilde{\theta}_{j,t}$ is the estimated parameters using data from $\widetilde{\mathcal{T}}_{j,t}$, and

$$\widetilde{B}_{j,t} =: \frac{\sqrt{c(d+2)\log(1+t))}}{\sqrt{\lambda_{\min}(\widetilde{V}_{j,t})}}$$

for some constant $c \ge 20/l_1^2$ and $\widetilde{V}_{j,t} = I + \sum_{s\in\widetilde{\mathcal{T}}_{j,t}} u_s u_s'$. These events hold at least with the following probabilities

$$\begin{aligned}
\mathbb{P}(\mathcal{E}_{N,t}) \ge& 1 - \frac{2n}{t^2} && \text{for } t > \bar{t},\\
\mathbb{P}(\mathcal{E}_{B_j,t}) \ge& 1 - \frac{1}{t} && \text{for any } j \in [m], t \in \mathcal{T},\\
\mathbb{P}(\mathcal{E}_{V,t}) \ge& 1 - \frac{7n}{t} && \text{for } t > 2\bar{t},
\end{aligned}$$

where $\bar{t}$ is defined in (EC.13). The first inequality is from our analysis after Lemma EC.4; the second inequality is from Corollary EC.1; the third inequality is from Lemma EC.5. We further define $\mathcal{E}_{B,t} = \bigcup_{j\in[m]} \mathcal{E}_{B_j,t}$, then it holds with probability at least $1 - m/t$ for any $t \in \mathcal{T}$. Now we define the event $\mathcal{E}_t$ as the union of $\mathcal{E}_{N,t}$, $\mathcal{E}_{B,t}$, and $\mathcal{E}_{V,t}$. This event holds with probability at least $1 - 10n/t$ obviously according to the probability of each event.

We split the regret by considering $t \le 2\bar{t}$ and $t > 2\bar{t}$, i.e.,

$$\sum_{t=1}^{T} \mathbb{E}[r_t(p_t^*) - r_t(p_t)] = \sum_{t\le 2\bar{t}} \mathbb{E}[r_t(p_t^*) - r_t(p_t)] + \sum_{t>2\bar{t}} \mathbb{E}[r_t(p_t^*) - r_t(p_t)].$$

Obviously, the regret of the first summation can be bounded above by $2\overline{p}\bar{t}$. We focus on the second summation. For arbitrary $t > 2\bar{t}$,

$$\begin{aligned}
\mathbb{E}[r_t(p_t^*) - r_t(p_t)] =& \mathbb{E}[(r_t(p_t^*) - r_t(p_t))\mathbf{1}(\mathcal{E}_t)] + \mathbb{E}[(r_t(p_t^*) - r_t(p_t))\mathbf{1}(\bar{\mathcal{E}}_t)] \\
\leq& \mathbb{E}[(p_t^*\mu(\alpha_{i_t}'x_t + \beta_{i_t}p_t^*) - p_t\mu(\alpha_{i_t}'x_t + \beta_{i_t}p_t))\mathbf{1}(\mathcal{E}_t)] + \frac{10\overline{p}n}{t} \\
=& \mathbb{E}[(|2\beta_{i_t}\dot{\mu}(\alpha_{i_t}'x_t + \beta_{i_t}\bar{p}_t) + \beta_{i_t}^2\bar{p}_t\ddot{\mu}(\alpha_{i_t}'x_t + \beta_{i_t}\bar{p}_t)|(p_t^* - p_t)^2)\mathbf{1}(\mathcal{E}_t)] + \frac{10\overline{p}n}{t} \\
\leq& \mathbb{E}[(\widetilde{L}_2(p_t^* - \widetilde{p}_t - \Delta_t)^2)\mathbf{1}(\mathcal{E}_t)] + \frac{10\overline{p}n}{t} \\
\leq& 2\widetilde{L}_2L_0^2\mathbb{E}[||\widetilde{\theta}_{\hat{\mathcal{N}}_t,t-1} - \theta_{i_t}||_2^2\mathbf{1}(\mathcal{E}_t)] + 4\widetilde{L}_2\mathbb{E}[\Delta_t^2\mathbf{1}(\mathcal{E}_t)] + \frac{10\overline{p}n}{t} \\
=& 2\widetilde{L}_2L_0^2\mathbb{E}[||\widetilde{\theta}_{j_t,t-1} - \theta_{j_t}||_2^2\mathbf{1}(\mathcal{E}_t)] + 4\widetilde{L}_2\mathbb{E}[\Delta_t^2\mathbf{1}(\mathcal{E}_t)] + \frac{10\overline{p}n}{t} \\
\leq& 2\widetilde{L}_2L_0^2\mathbb{E}[\widetilde{B}_{j_t,t-1}^2\mathbf{1}(\mathcal{E}_t)] + 4\widetilde{L}_2\mathbb{E}[\Delta_t^2\mathbf{1}(\mathcal{E}_t)] + \frac{10\overline{p}n}{t},
\end{aligned}$$

where the first inequality is from the probability of $\bar{\mathcal{E}}_t$, the second equality is by applying Taylor's theorem (where $\bar{p}_t$ is some price between $p_t^*$ and $p_t$) with Assumption A-1 and Assumption A-2, the second inequality is from Assumption A-2 and $\widetilde{L}_2$ is some constant depending on $L, L_1, L_2, \overline{p}$, and both the last equality and the last inequality are from the definition of $\mathcal{E}_t$ (i.e., events $\mathcal{E}_{N,t}$ and $\mathcal{E}_{B,t}$). Therefore, we have

$$\mathbb{E}[r_t(p_t^*) - r_t(p_t)] \leq 2\widetilde{L}_2L_0^2\mathbb{E}[\widetilde{B}_{j_t,t-1}^2\mathbf{1}(\mathcal{E}_t)] + 4\widetilde{L}_2\mathbb{E}[\Delta_t^2\mathbf{1}(\mathcal{E}_t)] + \frac{10\overline{p}n}{t}. \tag{EC.1}$$

Summing over $t$, the sum of the last terms above obviously lead to the regret $O(n\log T)$. For the rest, we have

$$\begin{aligned}
\sum_{t>2\bar{t}}\mathbb{E}[\widetilde{B}_{j_t,t-1}^2\mathbf{1}(\mathcal{E}_t)] \leq& \frac{k_2 d\log T}{\Delta_0^2}\sum_{t>2\bar{t}}\mathbb{E}\left[\frac{1}{\sqrt{\widetilde{q}_{j_t}t}}\right] = \frac{k_2 d\log T}{\Delta_0^2}\sum_{t>2\bar{t}}\sum_{j\in[m]}\sqrt{\frac{\widetilde{q}_j}{t}} \\
\leq& \frac{k_2 d\log T}{\Delta_0^2}\sum_{j\in[m]}\sqrt{\widetilde{q}_jT} \leq \frac{k_2 d\log T}{\Delta_0^2}\sqrt{mT}
\end{aligned}$$

for some constant $k_2$, where the first inequality is from $\mathcal{E}_t$ (i.e., $\mathcal{E}_{V,t}$) and the definition of $\widetilde{B}_{j_t,t}^2$, the equality is by conditioning on $j_t = j$ for all $j \in [m]$, and the last inequality is because $\sum_j \widetilde{q}_j = 1$ and apply Cauchy-Schwarz. Hence

$$\sum_{t>2\bar{t}}\mathbb{E}[\widetilde{B}_{j_t,t-1}^2\mathbf{1}(\mathcal{E}_t)] \leq \frac{k_2 d\log T}{\Delta_0^2}\sqrt{mT}. \tag{EC.2}$$

On the other hand, because $\hat{\mathcal{N}}_t = \mathcal{N}_{i_t}$ for all $t > 2\bar{t}$ on $\mathcal{E}_t$,

$$\sum_{t>2\bar{t}}\mathbb{E}[\Delta_t^2\mathbf{1}(\mathcal{E}_t)] \leq \sum_{j\in[m]}\mathbb{E}\left[\sum_{t\in\widetilde{\mathcal{T}}_{j,T}}\frac{\Delta_0^2}{\sqrt{\widetilde{T}_{j,t}}}\right] \leq \Delta_0^2\sum_{j\in[m]}\mathbb{E}\left[\sqrt{\widetilde{T}_{j,T}}\right] \leq \Delta_0^2\sqrt{mT}. \tag{EC.3}$$

Putting (EC.1), (EC.2), and (EC.3) together, we have

$$\sum_{t>2\bar{t}} \mathbb{E}[(r_t(p_t^*) - r_t(p_t))] \leq c_5 d \log(T)\sqrt{mT} + c_5 n \log T$$

for some constant $c_5$, and together with the regret for $t < 2\bar{t}$, we are done with the regret upper bound.

In the rest of this subsection, we prove the lemmas used in the proof of Theorem 1.

LEMMA EC.1. *For each $j \in [m]$ and $t \in \mathcal{T}$, with probability at least $1-\Delta$, $\widetilde{T}_{j,t} \in [\widetilde{q}_j t - \widetilde{D}(t), \widetilde{q}_j t + \widetilde{D}(t)]$ for all $j \in [m]$, $t \in \mathcal{T}$, where $\widetilde{D}(t) = \sqrt{t \log(2/\Delta)}$.*

*Proof.* Obviously $\widetilde{T}_{j,t}$ is a binomial random variable with parameter $t$ and $\widetilde{q}_j$. Then we simply use Hoeffding inequality applied on sequence of i.i.d. Bernoulli random variable and a simple union bound on all $j \in [m]$ and $t \in \mathcal{T}$. □

LEMMA EC.2. *For any $i \in [n]$ and $t \in \mathcal{T}$, let $V_{i,t} = I + \sum_{s \in \mathcal{T}_{i,t}} u_s u_s'$, we have that*

$$||\hat{\theta}_{i,t} - \theta_i||_{V_{i,t}} \leq \frac{2\sqrt{(d+2)\log(1 + T_{i,t}R^2/(d+2)) + 2\log(1/\Delta)} + 2l_1 L}{l_1}$$

*with probability at least $1-\Delta$.*

*Proof.* We first fix some $i \in [n]$, and we drop the index dependency on $i$ for convenience of notation. At round $s$, the gradient of likelihood function $\nabla l_s(\phi)$ is equal to

$$\nabla l_s(\phi) = (\mu(u_s'\phi) - d_s)u_s. \tag{EC.4}$$

And its Hessian is

$$\nabla^2 l_s(\phi) = \dot{\mu}(u_s'\phi) u_s u_s'. \tag{EC.5}$$

Applying Taylor's theorem, we obtain

$$\begin{aligned} 0 &\geq \sum_s l_s(\hat{\theta}_t) - l_s(\theta) \\ &= \sum_s \nabla l_s(\theta)'(\hat{\theta}_t - \theta) + \frac{1}{2}\sum_s \dot{\mu}(u_s'\bar{\theta}_t)(u_s'(\hat{\theta}_t - \theta))^2 + \frac{l_1}{2}||\hat{\theta}_t - \theta||_2^2 - \frac{l_1}{2}||\hat{\theta}_t - \theta||_2^2, \end{aligned} \tag{EC.6}$$

where the first inequality is from the optimality of $\hat{\theta}_t$, and $\bar{\theta}_t$ is a point on line segment between $\hat{\theta}_t$ and $\theta$. Note that by our assumption and boundedness of $u_s$ and $\theta$, we have $\dot{\mu}(u_s'\bar{\theta}_t) \geq l_1$. Therefore, we have

$$\sum_s \dot{\mu}(u_s'\bar{\theta}_t)(u_s'(\hat{\theta}_t - \theta))^2 + l_1||\hat{\theta}_t - \theta||_2^2 \geq l_1||\hat{\theta}_t - \theta||_{V_t}^2, \tag{EC.7}$$

where $V_t = I + \sum_s u_s u_s'$. On the other hand, we have

$$\nabla l_s(\theta_i) = -\epsilon_s u_s, \tag{EC.8}$$

where $\epsilon_s$ is the zero-mean error, which is obviously sub-Gaussian with parameter 1 as it is bounded.

Now combining (EC.6), (EC.7), and (EC.8), we have

$$\frac{l_1}{2}||\hat{\theta}_t - \theta||_{V_t}^2 \le \sum_s \epsilon_s u_s'(\hat{\theta}_t - \theta) + 2l_1 L^2 \le ||\hat{\theta}_t - \theta||_{V_t} ||Z_t||_{V_t^{-1}} + 2l_1 L^2, \tag{EC.9}$$

where $Z_t := \sum_s \epsilon_s u_s$, and the second inequality is from Cauchy-Schwarz and $||\hat{\theta}_t - \theta||_2 \le 2L$. This leads to $||\hat{\theta}_t - \theta||_{V_t} \le \frac{2}{l_1}||Z_t||_{V_t^{-1}} + 2L$.

To bound $||Z_t||_{V_t^{-1}}$, according to Theorem 1 in Abbasi-Yadkori et al. (2011), we have

$$||Z_t||_{V_t^{-1}} \le \sqrt{(d+2)\log(1 + \frac{T_{i,t}R^2}{d+2}) + 2\log(1/\Delta)}$$

with probability at least $1 - \Delta$ and we are done. □

Corollary EC.1. *For any $j \in [m]$ and $t \in \mathcal{T}$, let $\widetilde{V}_{j,t} := I + \sum_{s \in \widetilde{\mathcal{T}}_{j,t}} u_s u_s'$, we have that*

$$||\widetilde{\theta}_{j,t} - \theta_j||_{\widetilde{V}_{j,t}} \le \frac{2\sqrt{(d+2)\log(1 + \widetilde{T}_{j,t}R^2/(d+2)) + 2\log(1/\Delta)} + 2l_1 L}{l_1}$$

*with probability at least $1 - \Delta$.*

Next result is the minimum eigenvalue of the empirical Fisher's information matrix.

Lemma EC.3. *Denote $u_t' = (\widetilde{p}_t + \Delta_t, x_t')$. For any $i \in [n]$ and*

$$t > \max\left\{ \left(\frac{8R\log((d+2)T)}{\lambda_1 \Delta_0^2 \min_{i \in [n]} q_i}\right)^2, \left(\frac{\Delta_0^2}{c_0}\right)^2, \frac{2t_0}{\min_{i\in[n]} q_i} \right\},$$

*where $R := 2 + \bar{p}^2$ and $\lambda_1 := 1/((1 + \bar{p}/c_0)^2 + 1)$, we have*

$$\mathbb{P}\left(\lambda_{\min}\Big(\sum_{s \in \mathcal{T}_{i,t}} u_s u_s'\Big) < \frac{\lambda_1 \Delta_0^2 q_i \sqrt{t}}{2}\right) < \frac{1}{t^2}.$$

*Proof.* Define

$$M_s := \mathbb{E}[\mathbf{1}(i_s = i) u_s u_s' | \mathcal{F}_{s-1}],$$

and

$$M := \sum_{s=1}^t \mathbb{E}[\mathbf{1}(i_s = i) u_s u_s' | \mathcal{F}_{s-1}] = \sum_{s=1}^t M_s = \begin{bmatrix} A & B \\ B' & C \end{bmatrix}$$

where

$$A := \sum_{s=1}^{t} \mathbb{E}[\mathbf{1}(i_s = i)(\widetilde{p}_s^2 + \Delta_s^2)|\mathcal{F}_{s-1}]$$
$$B := \sum_{s=1}^{t} \mathbb{E}[\mathbf{1}(i_s = i)\widetilde{p}_s x_s'|\mathcal{F}_{s-1}]$$
$$C := \sum_{s=1}^{t} \mathbb{E}[\mathbf{1}(i_s = i)x_s x_s'|\mathcal{F}_{s-1}]$$

According to Proposition 3 in Walton and Zhang (2020), we have

$$\lambda_{\min}(M) \geq \frac{\lambda_{\min}(C)^2}{(\|B\|_2 + \lambda_{\min}(C))^2 + \lambda_{\min}(C)^2} \min\{\lambda_{\min}(A - BC^{-1}B'), \lambda_{\min}(C)\}.$$

Now let us analyze each term individually. By Assumption A-3, $q_i t \geq \lambda_{\min}(C) \geq c_0 q_i t$. It is also not difficult to get $\|B\|_2 \leq \bar{p} q_i t$. All we let to show is the lower bound of $\lambda_{\min}(A - BC^{-1}B')$, which is summarized in the following claim.

**Claim:** $\lambda_{\min}(A - BC^{-1}B') = A - BC^{-1}B' \geq \sum_{s=1}^{t} \mathbb{E}[\mathbf{1}(i_s = i)\Delta_s^2|\mathcal{F}_{s-1}]$

To prove this claim, let us define

$$\widetilde{u}_s := (\widetilde{p}_s, x_s)\,.$$

That is, $\widetilde{u}_s$ is the same as $u_s$ except without price perturbation. Obviously, $\widetilde{M} := \sum_{s=1}^{t} \mathbb{E}[\mathbf{1}(i_s = i)\widetilde{u}_s\widetilde{u}_s'|\mathcal{F}_{s-1}]$ satisfies $\widetilde{M} \geq 0$. Moreover, by Schur complement, we have $\widetilde{A} - BC^{-1}B' \geq 0$ where $\widetilde{A} = \sum_{s=1}^{t} \mathbb{E}[\mathbf{1}(i_s = i)\widetilde{p}_s^2|\mathcal{F}_{s-1}]$. Since $A - BC^{-1}B' = \widetilde{A} - BC^{-1}B' + \sum_{s=1}^{t} \mathbb{E}[\mathbf{1}(i_s = i)\Delta_s^2|\mathcal{F}_{s-1}]$, we are done with the claim.

Above all, we are able to show that

$$\lambda_{\min}(M) \geq \frac{1}{(1 + \bar{p}/c_0)^2 + 1} \min\{c_0 q_i t, q_i \Delta_0^2 \sqrt{t}\} \geq \frac{\Delta_0^2 q_i \sqrt{t}}{(1 + \bar{p}/c_0)^2 + 1} = \lambda_1 \Delta_0^2 q_i \sqrt{t}$$

where the second inequality is because $t > (\Delta_0^2/c_0)^2$. Since

$$\sum_{s \in \mathcal{T}_{i,t}} u_s u_s' = \sum_{s=1}^{t} \mathbb{1}(i_s = i) u_s u_s',$$

then we have that

$$\begin{aligned}
&\mathbb{P}\left(\lambda_{\min}(\sum_{s \in \mathcal{T}_{i,t}} u_s u_s') < \frac{\lambda_1 \Delta_0^2 q_i \sqrt{t}}{2}\right) \\
=&\mathbb{P}\left(\lambda_{\min}(\sum_{s \in \mathcal{T}_{i,t}} u_s u_s') < \frac{\lambda_1 \Delta_0^2 q_i \sqrt{t}}{2}, \lambda_{\min}\left(\sum_{s=1}^{t} \mathbb{E}[\mathbb{1}(i_s = i) u_s u_s'|\mathcal{F}_{s-1}]\right) \geq \lambda_1 \Delta_0^2 q_i \sqrt{t}\right) \\
\leq&(d+2)e^{-\frac{\lambda_1 \Delta_0^2 q_i \sqrt{t}}{4R}},
\end{aligned}$$

where the last inequality is from Theorem 3.1 in Tropp (2011) with $\zeta = 1/2$.

So for any $i \in [n]$ and

$$t > \left( \frac{8R\log(T(d+2))}{\lambda_1 \Delta_0^2 \min_{i\in[n]} q_i} \right)^2,$$

we have the simple union bound over $i \in [n], t \in \mathcal{T}$, $(d+2)\exp(-\lambda_1 \Delta_0^2 q_i \sqrt{t}/(4R)) < 1/t^2$, and the proof is complete. □

Clearly, if we combine Lemma EC.3 and Lemma EC.2, for any $i \in [n]$, $t > \bar{t}_1$ where

$$\bar{t}_1 = \max\left\{ \left( \frac{8R\log((d+2)T)}{\lambda_1 \Delta_0^2 \min_{i\in[n]} q_i} \right)^2, \left( \frac{\Delta_0^2}{c_0} \right)^2, \frac{2t_0}{\min_{i\in[n]} q_i} \right\}, \tag{EC.10}$$

we have that

$$\begin{aligned} ||\hat{\theta}_{i,t} - \theta_i||_2 &\le \frac{2\sqrt{(d+2)\log(1+tR^2/(d+2)) + 2\log t^2} + 2l_1 L}{l_1\sqrt{\lambda_{\min}(V_{i,t})}} \\ &\le \frac{\sqrt{c(d+2)\log(1+t)}}{\sqrt{\lambda_{\min}(V_{i,t})}} = B_{i,t} \end{aligned} \tag{EC.11}$$

for some constant $c > 20/l_1^2$, and

$$B_{i,t} \le \frac{\sqrt{2c(d+2)\log(1+t)}}{\Delta_0\sqrt{\lambda_1 q_i \sqrt{t}}} \tag{EC.12}$$

with probability at least $1 - 2/t^2$.

The next lemma states that when estimation errors are bounded, under certain conditions we have $\hat{\mathcal{N}}_t = \mathcal{N}_{i_t}$.

LEMMA EC.4. *Suppose for all $i \in [n]$ it holds that $||\hat{\theta}_{i,t-1} - \theta_i||_2 \le B_{i,t-1}$ and $B_{i,t-1} < \gamma/4$. Then*

$$\hat{\mathcal{N}}_t = \mathcal{N}_{i_t}.$$

*Proof.* First of all, for $i_1, i_2 \in [n]$, if they belong to different clusters and $B_{i_1,t-1} + B_{i_2,t-1} < \gamma/2$, we must have $||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2 > B_{i_1,t-1} + B_{i_2,t-1}$ because

$$\begin{aligned} \gamma \le& ||\theta_{i_1} - \theta_{i_2}||_2 \le ||\theta_{i_1} - \hat{\theta}_{i_1,t-1}||_2 + ||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2 + ||\hat{\theta}_{i_2,t-1} - \theta_{i_2}||_2 \\ \le& B_{i_1,t-1} + ||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2 + B_{i_2,t-1} < \gamma/2 + ||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2, \end{aligned}$$

which implies that $||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2 > \gamma/2 > B_{i_1,t-1} + B_{i_2,t-1}$.

On the other hand, if $||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2 > B_{i_1,t-1} + B_{i_2,t-1}$, we must have $i_1, i_2$ belongs to different clusters because

$$\begin{aligned} B_{i_1,t-1} + B_{i_2,t-1} <& ||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2 \le ||\theta_{i_1} - \hat{\theta}_{i_1,t-1}||_2 + ||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2 + ||\hat{\theta}_{i_2,t-1} - \theta_{i_2}||_2 \\ \le& B_{i_1,t-1} + ||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2 + B_{i_2,t-1}, \end{aligned}$$

which implies $||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2 > 0$, i.e., they belong to different clusters.

Therefore, if $i \in \hat{\mathcal{N}}_t$, i.e., $||\hat{\theta}_{i_t,t-1} - \hat{\theta}_{i,t-1}|| \leq B_{i_t,t-1} + B_{i,t-1}$, we must have that $i \in \mathcal{N}_{i_t}$ as well or $B_{i_t,t-1} + B_{i,t-1} \geq \gamma/2$ (which is impossible by our assumption that $B_{i,t-1} < \gamma/4$).

On the other hand, if $i \in \mathcal{N}_{i_t}$, then we must have $||\hat{\theta}_{i_t,t-1} - \hat{\theta}_{i,t-1}|| \leq B_{i_t,t-1} + B_{i,t-1}$, which implies that $i \in \hat{\mathcal{N}}_t$ as well.

Above all, we have shown that $\hat{\mathcal{N}}_{i_t} = \mathcal{N}_{i_t}$. □

Note that given (EC.11) and (EC.12), we have that $B_{i,t-1} < \gamma/4$ for all $i$ if

$$t > 1 + \frac{k_1((d+2)\log(1+T))^2}{\gamma^4 \lambda_1^2 \Delta_0^4 \min_{i\in[n]} q_i^2}$$

for some constant $k_1$. Therefore, for each $t > \bar{t}$ where

$$\bar{t} = \max\left\{4\bar{t}_1, 1 + \frac{k_1((d+2)\log(1+T))^2}{\gamma^4 \lambda_1^2 \Delta_0^4 \min_{i\in[n]} q_i^2}\right\}, \tag{EC.13}$$

and $\bar{t}_1$ is defined in (EC.10), $\hat{\mathcal{N}}_t = \mathcal{N}_{i_t}$ with probability at least $1 - 2n/t^2$.

The next lemma shows that the clustered estimation will be quite accurate when most of the $\hat{\mathcal{N}}_t$ is actually equal to $\mathcal{N}_{i_t}$.

Lemma EC.5. *For any $t$ such that $t > 2\bar{t}$, we have*

$$\mathbb{P}\left(\lambda_{\min}\left(\sum_{s\in\widetilde{\mathcal{T}}_{j_t,t}} u_s u_s'\right) < \frac{\lambda_1 \Delta_0^2 \sqrt{\widetilde{q}_{j_t} t}}{8}\right) < \frac{7n}{t},$$

*where $\bar{t}$ is defined in (EC.13).*

*Proof.* The proof is analogous to Lemma EC.3. Let $\mathcal{E}_{N,t}$ be the event such that $\hat{\mathcal{N}}_t = \mathcal{N}_{i_t}$, and $\widetilde{\mathcal{E}}_{j,t}$ be the event such that $\widetilde{T}_{j,t} \leq 3\widetilde{q}_j t/2$. From our previous analysis, we know that given $t > \bar{t}$, $\mathcal{E}_{N,t}$ holds with probability at least $1 - 2n/t^2$. Also, according to Lemma EC.1, event $\widetilde{\mathcal{E}}_{j,t}$ holds with probability at least $1 - 1/t^2$ given $t \geq 8\log(2T)/\min_{j\in[m]} \widetilde{q}_j^2$ (which is satisfied by taking $t > \bar{t}$).

On event $\widetilde{\mathcal{E}}_{j,t}$ and $\mathcal{E}_{N,s}$ for all $s \in [t/2, t]$ (which holds with probability at least $1 - 6n/t$), we have

$$\lambda_{\min}\left(\sum_{s=1}^{t} \mathbb{E}[\mathbb{1}(j_s = j) u_s u_s' | \mathcal{F}_{s-1}]\right) \geq \lambda_{\min}\left(\sum_{s=t/2}^{t} \mathbb{E}[\mathbb{1}(j_s = j) u_s u_s' | \mathcal{F}_{s-1}]\right) \geq \sum_{s=t/2}^{t} \lambda_1 \Delta_0^2 \widetilde{q}_j (\widetilde{T}_{j,s})^{-1/2}$$
$$\geq \lambda_1 \Delta_0^2 \frac{\sqrt{\widetilde{q}_j t}}{4}.$$

by definition of $\widetilde{q}_j$ following a similar procedure as in Lemma EC.3.

Therefore, we have for any $t > 2\bar{t}$,

$$\begin{aligned}
&\mathbb{P}\left(\lambda_{\min}\left(\sum_{s\in\widetilde{\mathcal{T}}_{j_t,t}} u_s u_s'\right) < \frac{\lambda_1\Delta_0^2\sqrt{\widetilde{q}_{j_t}}t}{8}\right)\\
=&\sum_{j\in[m]}\mathbb{P}\left(\lambda_{\min}\left(\sum_{s\in\widetilde{\mathcal{T}}_{j_t,t}} u_s u_s'\right) < \frac{\lambda_1\Delta_0^2\sqrt{\widetilde{q}_{j_t}}t}{8}\middle| j_t = j\right)\mathbb{P}(j_t = j)\\
=&\sum_{j\in[m]}\mathbb{P}\left(\lambda_{\min}\left(\sum_{s\in\widetilde{\mathcal{T}}_{j,t}} u_s u_s'\right) < \frac{\lambda_1\Delta_0^2\sqrt{\widetilde{q}_j}t}{8}\right)\widetilde{q}_j.
\end{aligned}$$

For each $j\in[m]$, we have

$$\begin{aligned}
&\mathbb{P}\left(\lambda_{\min}\left(\sum_{s\in\widetilde{\mathcal{T}}_{j,t}} u_s u_s'\right) < \frac{\lambda_1\Delta_0^2\sqrt{\widetilde{q}_j}t}{8}\right)\\
\le&\mathbb{P}\left(\lambda_{\min}\left(\sum_{s\in\widetilde{\mathcal{T}}_{j,t}} u_s u_s'\right) < \frac{\lambda_1\Delta_0^2\sqrt{\widetilde{q}_j}t}{8}, \bigcup_{s\in[t/2,t]}(\mathcal{E}_{N,t}\cup\widetilde{\mathcal{E}}_{j,t})\right) + \frac{6n}{t}\\
=&\mathbb{P}\left(\lambda_{\min}\left(\sum_{s\in\widetilde{\mathcal{T}}_{j,t}} u_s u_s'\right) < \frac{\lambda_1\Delta_0^2\sqrt{\widetilde{q}_j}t}{8}, \lambda_{\min}\left(\sum_{s\in\widetilde{\mathcal{T}}_{j,t}}\mathbb{E}[u_s u_s'|\mathcal{F}_{s-1}]\right) \ge \frac{\lambda_1\Delta_0^2\sqrt{\widetilde{q}_j}t}{4}, \bigcup_{s\in[t/2,t]}(\mathcal{E}_{N,t}\cup\widetilde{\mathcal{E}}_{j,t})\right)\\
&+\frac{6n}{t} \le \frac{7n}{t},
\end{aligned}$$

where the first inequality is from the probability of the complement of $\bigcup_{s\in[t/2,t]}(\mathcal{E}_{N,t}\cup\widetilde{\mathcal{E}}_{j,t})$, and the last inequality is by Theorem 3.1 in Tropp (2011), and we take

$$t > \left(\frac{8R\log(2(d+2)T)}{\lambda_1\Delta_0^2\min_{j\in[m]}\sqrt{\widetilde{q}_j}}\right)^2.$$

Since $\bar{t} > \left(8R\log(2(d+2)T)/(\lambda_1\Delta_0^2\min_{j\in[m]}\sqrt{\widetilde{q}_j})\right)^2$ by definition, we complete the proof. □

## EC.2 Different $\theta_i$ for the Same Cluster

In this section we present the technical lemmas in proving the regret of the modified CSMP when parameters $\theta_i$ within the same cluster are different. Note that we assume $||\theta_{i_1} - \theta_{i_2}||_2 \le \gamma_0$ for any $i_1, i_2$ in any cluster $\mathcal{N}_j$.

The first result is a corollary of Lemma EC.4.

Corollary EC.2. *Suppose for all $i\in[n]$ it holds that $||\hat{\theta}_{i,t-1} - \theta_i||_2 \le B_{i,t-1}$ and $B_{i,t-1}\in(\gamma_0/2, \gamma/6)$. Then (with $\gamma > 3\gamma_0$) we have that $\hat{\mathcal{N}}_t = \mathcal{N}_{i_t}$. Moreover, if we only have $B_{i,t-1} < \gamma/6$, we have $\hat{\mathcal{N}}_t \subset \mathcal{N}_{i_t}$.*

*Proof.* For the first part of the corollary, the proof is almost identical to Lemma EC.4. First of all, for $i_1, i_2 \in [n]$, if they belong to different clusters and $B_{i_1,t-1} + B_{i_2,t-1} < \gamma/3$, we must have $||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2 > 2B_{i_1,t-1} + 2B_{i_2,t-1}$ because

$$\begin{aligned}\gamma \leq &||\theta_{i_1} - \theta_{i_2}||_2 \leq ||\theta_{i_1} - \hat{\theta}_{i_1,t-1}||_2 + ||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2 + ||\hat{\theta}_{i_2,t-1} - \theta_{i_2}||_2 \\ \leq &B_{i_1,t-1} + ||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2 + B_{i_2,t-1} < \gamma/3 + ||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2,\end{aligned}$$

which implies that $||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2 > 2\gamma/3 > 2B_{i_1,t-1} + 2B_{i_2,t-1}$.

On the other hand, if $||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2 > 2B_{i_1,t-1} + 2B_{i_2,t-1}$, we must have $i_1, i_2$ belongs to different clusters because

$$\begin{aligned}2B_{i_1,t-1} + 2B_{i_2,t-1} < &||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2 \leq ||\theta_{i_1} - \hat{\theta}_{i_1,t-1}||_2 + ||\theta_{i_1,t-1} - \theta_{i_2,t-1}||_2 + ||\hat{\theta}_{i_2,t-1} - \theta_{i_2}||_2 \\ \leq &B_{i_1,t-1} + ||\theta_{i_1,t-1} - \theta_{i_2,t-1}||_2 + B_{i_2,t-1}\end{aligned}$$

which implies $||\theta_{i_1,t-1} - \theta_{i_2,t-1}||_2 > B_{i_1,t-1} + B_{i_2,t-1} \geq \gamma_0$ (where the second inequality is because $B_{i,t-1} \geq \gamma_0/2$), i.e., they belong to different clusters.

Therefore, if $i \in \hat{\mathcal{N}}_t$, i.e., $||\hat{\theta}_{i_t,t-1} - \hat{\theta}_{i,t-1}|| \leq 2B_{i_t,t-1} + 2B_{i,t-1}$, we must have that $i \in \mathcal{N}_{i_t}$ as well or $B_{i_t,t-1} + B_{i,t-1} \geq \gamma/3$ (which is impossible by our assumption that $B_{i,t-1} < \gamma/6$).

On the other hand, if $i \in \mathcal{N}_{i_t}$, then we must have $||\hat{\theta}_{i_t,t-1} - \hat{\theta}_{i,t-1}|| \leq 2B_{i_t,t-1} + 2B_{i,t-1}$, which implies that $i \in \hat{\mathcal{N}}_t$ as well. Summarizing, we have shown that $\hat{\mathcal{N}}_{i_t} = \mathcal{N}_{i_t}$.

For the second part, suppose this is not true. That is, there is some $i \in \hat{\mathcal{N}}_t$ with $i \notin \mathcal{N}_{i_t}$, which implies $||\theta_{i_t} - \theta_i|| \geq \gamma$ and $||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2 \leq 2B_{i_t,t-1} + 2B_{i,t-1}$. Note that

$$\begin{aligned}\gamma \leq ||\theta_{i_t} - \theta_i||_2 \leq ||\theta_{i_t} - \hat{\theta}_{i_t,t-1}||_2 + ||\hat{\theta}_{i_t,t-1} - \hat{\theta}_{i,t-1}||_2 + ||\hat{\theta}_{i,t-1} - \theta_i||_2 \\ \leq B_{i_t,t-1} + ||\hat{\theta}_{i_t,t-1} - \hat{\theta}_{i,t-1}||_2 + B_{i,t-1} < \gamma/3 + ||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2;\end{aligned}$$

Thus we have $||\hat{\theta}_{i_1,t-1} - \hat{\theta}_{i_2,t-1}||_2 > 2\gamma/3$ and we have $B_{i_t,t-1} + B_{i,t-1} > \gamma/3$, contradicting with $B_{i,t-1} < \gamma/6$ for all $i$. □

Suppose in some time period $t$, product $i_t$ is in some neighborhood $\hat{\mathcal{N}}_t$ which satisfies $||\theta_{i_1} - \theta_{i_2}||_2 \leq \widetilde{\gamma}_0$ with some constant $\widetilde{\gamma}_0$ for any $i_1, i_2 \in \hat{\mathcal{N}}_t$. Let $\widetilde{\theta}_{i_t,t}$ denote the estimated parameter by clustering all data in neighborhood $\hat{\mathcal{N}}_t$. The next lemma measures the confidence bound of $\widetilde{\theta}_{i_t,t}$ compared with any true parameter $\widetilde{\theta}_i \in \hat{\mathcal{N}}_t$.

Lemma EC.6. *When $T_{i,t} \geq q_i t/2$ for all $i \in [N]$, we have for any $i \in \hat{\mathcal{N}}_t$,*

$$||\widetilde{\theta}_{i_t,t} - \theta_i||_2 \leq \frac{2\sqrt{(d+2)\log\left(1 + \frac{tR^2}{d+2}\right) + 4\log t}}{l_1\sqrt{\lambda_{\min}(V_{\hat{\mathcal{N}}_t})}} + \frac{L_1 R^2 \widetilde{\gamma}_0 \widetilde{q}_{\hat{\mathcal{N}}_t} t}{l_1 \lambda_{\min}(V_{\hat{\mathcal{N}}_t})} + \frac{2L}{\sqrt{\lambda_{\min}(V_{\hat{\mathcal{N}}_t})}}$$

*with probability at least $1 - O(1/t^2)$, where $V_{\hat{\mathcal{N}}_t} = I + \sum_{t \in \mathcal{T}_{\hat{\mathcal{N}}_t}} u_s u_s'$ and $\widetilde{q}_{\hat{\mathcal{N}}_t} = \sum_{i \in \hat{\mathcal{N}}_t} q_i$.*

*Proof.* The proof is quite similar to Lemma EC.2. We drop the index $i_t$ for convenience. Note that for an arbitrary parameter $\phi \in \Theta$, since $\widetilde{\theta}_t$ is the MLE, we have

$$\begin{aligned} 0 \geq \sum_s l_s(\widetilde{\theta}_t) - \sum_s l_s(\phi) = \sum_s \nabla l_s(\phi)'(\widetilde{\theta}_t - \phi) + \frac{1}{2}\sum_s \dot{\mu}(u_s'\bar{\phi}_t)(u_s'(\widetilde{\theta}_t - \phi))^2 \\ + \frac{l_1}{2}||\widetilde{\theta}_t - \phi||_2^2 - \frac{l_1}{2}||\widetilde{\theta}_t - \phi||_2^2 \geq \sum_s \nabla l_s(\phi)'(\widetilde{\theta}_t - \phi) + \frac{l_1}{2}||\widetilde{\theta}_t - \phi||^2_{V_{\hat{\mathcal{N}}_t}} - 2l_1L^2, \end{aligned} \tag{EC.14}$$

where the first inequality is from the optimality of $\widetilde{\theta}_t$, and $\bar{\phi}_t$ is a point on line segment between $\widetilde{\theta}_t$ and $\phi$.

Now we consider $\nabla l_s(\phi)$. By Taylor's theorem, $\nabla l_s(\phi) = \nabla l_s(\theta_s) + \nabla^2 l_s(\check{\theta}_s)'(\phi - \theta_s)$, where $\theta_s$ is the true parameter at time $s$, and $\check{\theta}_s$ is a point between $\phi$ and $\theta_s$. As a result,

$$\nabla l_s(\phi) = -\epsilon_s u_s + \dot{\mu}(u_s'\check{\theta}_s)u_s u_s'(\phi - \theta_s). \tag{EC.15}$$

Since $\phi \in \Theta$ is an arbitrary vector, we can let $\phi = \theta_i$ for any $i \in \mathcal{N}_j$. Combining (EC.14) and (EC.15), we have that with probability at least $1 - 1/t^2$.

$$\begin{aligned} \frac{l_1}{2}||\widetilde{\theta}_t - \theta_i||^2_{V_{\hat{\mathcal{N}}_t}} &\leq \sum_s \epsilon_s u_s'(\widetilde{\theta}_t - \theta_i) - \sum_s \dot{\mu}(u_s'\check{\theta}_s)(\theta_i - \theta_s)'u_s u_s'(\widetilde{\theta}_t - \phi) + 2l_1L^2 \\ &\leq ||\sum_s \epsilon_s u_s||_{V^{-1}_{\hat{\mathcal{N}}_t}} ||\widetilde{\theta}_t - \theta_i||_{V_{\hat{\mathcal{N}}_t}} + \sum_s ||\dot{\mu}(u_s'\check{\theta}_s)u_s u_s'(\theta_i - \theta_s)||_{V^{-1}_{\hat{\mathcal{N}}_t}} ||\widetilde{\theta}_t - \theta_i||_{V_{\hat{\mathcal{N}}_t}} + 2l_1L^2 \\ &\leq \sqrt{(d+2)\log\left(1 + \frac{tR^2}{d+2}\right) + 4\log t}\,||\widetilde{\theta}_t - \theta_i||_{V_{\hat{\mathcal{N}}_t}} \\ &\quad + \frac{\sum_s ||\dot{\mu}(u_s'\check{\theta}_s)u_s u_s'(\theta_i - \theta_s)||_2 ||\widetilde{\theta}_t - \theta_i||_{V_{\hat{\mathcal{N}}_t}}}{\sqrt{\lambda_{\min}(V_{\hat{\mathcal{N}}_t})}} + 2l_1L^2 \\ &\leq \sqrt{(d+2)\log\left(1 + \frac{tR^2}{d+2}\right) + 4\log t}\,||\widetilde{\theta}_t - \theta_i||_{V_{\hat{\mathcal{N}}_t}} + \frac{L_1R^2\widetilde{\gamma}_0\widetilde{q}_{\hat{\mathcal{N}}_t}t||\widetilde{\theta}_t - \theta_i||_{V_{\hat{\mathcal{N}}_t}}}{2\sqrt{\lambda_{\min}(V_{\hat{\mathcal{N}}_t})}} + 2l_1L^2, \end{aligned}$$

where the second inequality is from Theorem 1 in Abbasi-Yadkori et al. (2011) and the last inequality is because $T_{i,t} \geq q_i t/2$. By some simple algebra, above inequality implies that

$$||\widetilde{\theta}_t - \theta_i||_{V_{\hat{\mathcal{N}}_t}} \leq \frac{2\sqrt{(d+2)\log\left(1 + \frac{tR^2}{d+2}\right) + 4\log t}}{l_1} + \frac{L_1R^2\widetilde{\gamma}_0\widetilde{q}_{\hat{\mathcal{N}}_t}t}{l_1\sqrt{\lambda_{\min}(V_{\hat{\mathcal{N}}_t})}} + 2L.$$

This inequality further implies that

$$||\widetilde{\theta}_t - \theta_i||_2 \leq \frac{2\sqrt{(d+2)\log\left(1 + \frac{tR^2}{d+2}\right) + 4\log t}}{l_1\sqrt{\lambda_{\min}(V_{\hat{\mathcal{N}}_t})}} + \frac{L_1R^2\widetilde{\gamma}_0\widetilde{q}_{\hat{\mathcal{N}}_t}t}{l_1\lambda_{\min}(V_{\hat{\mathcal{N}}_t})} + \frac{2L}{\sqrt{\lambda_{\min}(V_{\hat{\mathcal{N}}_t})}}$$

and we are done. □

The previous lemma shows that the estimation error is basically dependent on the value of $\lambda_{\min}(V_{\hat{\mathcal{N}}_t})$, which is described by the following lemma.

Lemma EC.7. *The value of* $\lambda_{\min}(V_{\hat{\mathcal{N}}_t})$ *satisfies the following.*

*(a) If* $t \geq \Omega(\bar{t})$ *and* $t \leq k_3/(\max_i q_i^2\gamma_0^4)$ *for some constant* $k_3$ *(so that* $k_3/(\max_i q_i^2\gamma_0^4) \geq \Omega(\bar{t})$ *without loss of generality),* $\lambda_{\min}(V_{\hat{\mathcal{N}}_t}) \geq \lambda_1\Delta_0^2\sqrt{\widetilde{q}_{j_t}t}/8$ *with probability at least* $1-O(n/t)$*.*

*(b) If* $t \geq k_3/(\max_i q_i^2\gamma_0^4)$*,* $\lambda_{\min}(V_{\hat{\mathcal{N}}_t}) \geq \lambda_1\Delta_0^2\sqrt{\widetilde{q}_{\hat{\mathcal{N}}_t}/\widetilde{q}_{j_t}}\sqrt{\widetilde{q}_{\hat{\mathcal{N}}_t}t}/8$ *with probability at least* $1-O(n/t)$*.*

*Proof.* For part (a), it follows from the same procedure as Lemma EC.5. The reason we want $t \leq O(1/(\max_i q_i^2\gamma_0^4))$ is to guarantee $B_{i,s} > \gamma_0/2$ for $s \leq t$ so that we will have $\hat{\mathcal{N}}_s = \mathcal{N}_{i_s}$ by Corollary EC.2.

For part (b), with probability at least $1-O(n/t)$, we have $\hat{\mathcal{N}}_s \subset \mathcal{N}_{i_s}$ for any $s \geq k_3/(\max_i q_i^2\gamma_0^4)$ according to Corollary EC.2. Thus $\widetilde{T}_{\hat{\mathcal{N}}_s,s} \in [T_{i_s,s}, \widetilde{T}_{j_s,s}]$ and following the proof of Lemma EC.5, with probability at least $1-O(n/t)$, $\lambda_{\min}(V_{\hat{\mathcal{N}}_t}) \geq \lambda_1\Delta_0^2\sqrt{\widetilde{q}_{\hat{\mathcal{N}}_t}/\widetilde{q}_{j_t}}\sqrt{\widetilde{q}_{\hat{\mathcal{N}}_t}t}/8$. □

The implication of the previous lemmas is the following. When $t \geq \Omega(\bar{t})$ and $t \leq k_3/\max_i q_i^2\gamma_0^4$, we have most of the time $\hat{\mathcal{N}}_t = \mathcal{N}_{i_t}$, and thus everything basically resembles the main setting of this paper. However, as $t$ keeps growing, we start to have only $\hat{\mathcal{N}}_t \subset \mathcal{N}_{i_t}$ according to Corollary EC.2. That is, the $n$ products are no longer clustered into the $m$ clusters (with high probability) as we want. Therefore, the regret after $t \geq k_3/\max_i q_i^2\gamma_0^4$ has to be analyzed more carefully. Now we provide the proof (sketch) of the theorem of regret of modified algorithm.

Theorem EC.1. *The expected regret of the modified algorithm of CSMP is*

$$R(T) = O\left(\frac{d^2\log^2(dT)}{\min_{i\in[n]} q_i^2} + d\log T\sqrt{\widetilde{m}(T)T} + \Gamma(T)\right)$$

*where* $\widetilde{m}(T)$ *and* $\Gamma(T)$ *are functions of* $T$*. In particular, when* $T \leq k_3/\max_i q_i^2\gamma_0^4$*, we have* $\widetilde{m}(T) = m, \Gamma(T) = \min\{\gamma_0^2\sum_j \widetilde{q}_j^2T^2, T\}$*; as* $T \to \infty$*,* $\widetilde{m} \to n$*,* $\Gamma(T) \to \bar{\Gamma}$ *where* $\bar{\Gamma}$ *is a constant depending on the minimum gap between* $\theta_{i_1}, \theta_{i_2}$ *within any of the same neighborhood.*

*Sketch of the Proof.* Note that in this proof, we will calculate everything on all nice events (e.g., $||\hat{\theta}_{i,t} - \theta_i||_2 \leq B_{i,t-1}$ for all $i \in \mathcal{N}$) as in the proof of Theorem 1 which hold with high probability, as the regret on their complement can be controlled to at most $O(n)$. Now let the results in Lemma EC.6 hold. If $T \leq k_3/\max_i q_i^2\gamma_0^4$, Corollary EC.2 shows that $\hat{\mathcal{N}}_t = \mathcal{N}_{i_t}$ for all $t \geq \Omega(\bar{t})$. Thus, combining part (a) of Lemma EC.7 and Lemma EC.6,

$$||\theta_{i_t} - \widetilde{\theta}_{j_t,t}||_2 \leq O(\sqrt{d\log T}/(\widetilde{q}_{j_t}t)^{1/4} + \min\{\gamma_0\sqrt{\widetilde{q}_{j_t}t}, 1\}), \tag{EC.16}$$

where the part $\min\{\gamma_0\sqrt{\widetilde{q}_{j_t}t}, 1\}$ is because any estimated parameter is bounded. Thus, to bound the regret when $T \leq k_3/\max_i q_i^2\gamma_0^4$, the proof is almost identical to Theorem 1 so we neglect most part of the proof. We want to bound $r_t(p_t^*) - r_t(p_t) = O(r_t(p_t^*) - r_t(p_t') + \Delta_t^2)$. Note that $\Delta_t^2 = O\left(\widetilde{T}_{\hat{\mathcal{N}}_t,t}^{-1/2}\right)$, thus for the part of regret $\sum_t O\left(\widetilde{T}_{\hat{\mathcal{N}}_t,t}^{-1/2}\right)$, it is bounded as in Theorem 1.

To bound $r_t(p_t^*) - r_t(p_t')$, note that we have $r_t(p_t^*) - r_t(p_t') \le O\left(||\theta_{i_t} - \widetilde{\theta}_{j_t,t}||_2^2\right)$. Thus combining (EC.16) and sum over $t$, we have this part of the regret is at most $O(d\log T\sqrt{mT} + \min\{\gamma_0^2 \sum_j \widetilde{q}_j^2 T^2, T\})$.

If $T > k_3/\max_i q_i^2\gamma_0^4$, for any $t \ge k_3/\max_i q_i^2\gamma_0^4$, since Corollary EC.2 shows that $\hat{\mathcal{N}}_t \subset \mathcal{N}_{i_t}$. Thus at any time $t$, we can take any subset of all estimated neighborhood of all $i$ (i.e., $\{\hat{\mathcal{N}}_{i,t} : i \in [n]\}$) whose union is equal to $\mathcal{N}$. Without loss of generality, let $\widetilde{m}_t$ denote the number of such neighborhoods (denoted by $\{\hat{\mathcal{N}}_{[k],t} : k \in [\widetilde{m}_t]\}$) and $\widetilde{\gamma}_{0,t}$ denote the maximum distance between any two parameters within any $\hat{\mathcal{N}}_{[k],t}$ (for instance, when all $\hat{\mathcal{N}}_t = \mathcal{N}_{i_t}$, we have $\widetilde{m}_t = m, \widetilde{\gamma}_{0,t} = \gamma_0$). Obviously, we have $\widetilde{m}_t \in [m, n]$ and $\widetilde{\gamma}_{0,t} \le \gamma_0$. Then we basically follow the same procedure as earlier, and the regret in each time $t \ge k_3/\max_i q_i^2\gamma_0^4$ is at most

$$\begin{aligned}
&O\left(\mathbb{E}\left[d\log T\sum_{k=1}^{\widetilde{m}_t}\sqrt{\widetilde{q}_{j_{[k]}}/\widetilde{q}_{\hat{\mathcal{N}}_{[k],t}}}\sqrt{\widetilde{q}_{\hat{\mathcal{N}}_{[k],t}}/t} + \min\left\{\widetilde{\gamma}_{0,t}^2\sum_{k=1}^{\widetilde{m}_t}\widetilde{q}_{\hat{\mathcal{N}}_{[k],t}}^2 t, 1\right\} + \sum_{k=1}^{\widetilde{m}_t}\sqrt{\widetilde{q}_{\hat{\mathcal{N}}_{[k],t}}/t}\right]\right) \\
=&O\left(\mathbb{E}\left[d\log T\sum_{k=1}^{\widetilde{m}_t}\sqrt{\widetilde{q}_{j_{[k]}}/t} + \min\left\{\widetilde{\gamma}_{0,t}^2\sum_{k=1}^{\widetilde{m}_t}\widetilde{q}_{\hat{\mathcal{N}}_{[k],t}}^2 t, 1\right\} + \sqrt{\widetilde{m}_t/t}\right]\right)
\end{aligned} \tag{EC.17}$$

where $j_{[k]}$ denote the index of the true neighborhood that $\hat{\mathcal{N}}_{[k],t} \subset \mathcal{N}_{j_{[k]}}$, and the expectation is taken over the realization of all neighborhoods $\{\hat{\mathcal{N}}_{i,t} : i \in [n]\}$. Thus, we can choose the $\widetilde{m}_t$ neighborhood so that $\sum_{k=1}^{\widetilde{m}_t}\sqrt{\widetilde{q}_{j_{[k]}}}$ is minimized, and denote $\widetilde{m}$ as a number so that $\sqrt{\widetilde{m}} \ge \mathbb{E}\left[\sum_{k=1}^{\widetilde{m}_t}\sqrt{\widetilde{q}_{j_{[k]}}}\right]$ and $\widetilde{m} \ge \mathbb{E}[\widetilde{m}_t]$ for all $t$. Thus (EC.17) is bounded above as

$$O\left(d\log T\sqrt{\widetilde{m}/t} + \mathbb{E}\left[\min\left\{\widetilde{\gamma}_{0,t}^2\sum_{k=1}^{\widetilde{m}_t}\widetilde{q}_{\hat{\mathcal{N}}_{[k],t}}^2 t, 1\right\}\right] + \sqrt{\widetilde{m}/t}\right),$$

and we are done with the expression of regret by summing over all $t$. Note that as $T \to \infty$, it is obvious that each $\hat{\mathcal{N}}_{i,t}$ becomes $\{i\}$ itself when $B_{i,t-1}$ is sufficiently small compared with the minimum gap between any two parameters within the same neighborhood. Thus $\widetilde{\gamma}_{0,t}$ becomes 0 (implying $\Gamma(T)$ is bounded) and $\widetilde{m} \to n$ as $\widetilde{m}_t \to n$. □

## EC.3 Price Dependency of the Real Dataset

In this section, we conduct a data analysis of the real dataset in Section 4.2. The purpose is to show that the demand of each product mainly depends on its own price. For illustration of the effect of price, we exclude the features and adopt a simple logistic regression model for each product, which is defined as, if we make the dependency on index $i$ implicit,

$$\mathbb{E}[d_t(p)] = \frac{1}{1 + e^{-\alpha - \beta p}}$$

where $p$ is the price of interest, which can be the product's own price or any other product's price.

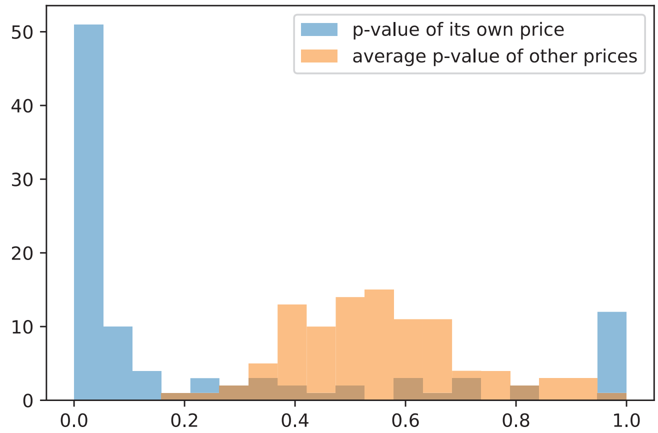


**Figure EC.1** P-value of own price versus other prices.

To test the significance of each price, we evaluate the p-value of the hypothesis test with

$$H_0 : \beta = 0 \quad \text{VS} \quad H_a : \beta \neq 0.$$

First, for each product $i$, we calculate the p-value of its own price and the average p-value of other prices, and results are summarized in Figure EC.1, which is a histogram of the two p-values of all products. From this histogram, we can clearly see that most products have significantly lower p-value of its own price than other prices, showing that the demand is mainly dependent on its own price.

Of course, this experiment mainly shows that overall other products' prices do not have significant impact on each product's demand, but we still do not know how specifically price of product $i$ affects demand of product $i'$. Next, we will investigate one by one of each product's price on other products. For instance, fix product $i$, we calculate the p-value of price $p_{i,t}$ on the demand of any other product $i' \neq i$, and then count how many products $i' \neq i$ that price of product $i$ has significant (i.e., p-value<0.05) impact on.

Table EC.1 summarize this result. On average, the price of each product only significantly affects the demand of 9.44 other products, compared with the fact that number of products, whose demand is significantly affected by its own price, is equal to 51. Note that it is not surprising some products' demands are not significantly affected by their prices because of the data scarcity due to low sales and popularity. For the purpose of simulation in Section 4.2, we will still fit the data of these 100 low-sale products as it is for illustrative purposes.

| | Mean | Standard Deviation | Maximum | Minimum |
|---|---|---|---|---|
| Number of significant p-value | 9.44 | 4.49 | 22 | 2 |

**Table EC.1** Number of significant p-value on demand of other products.