An Exploration in Mathematical Cartography: Comparative Distortion

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1 Abstract

The purpose of this paper is to find a notion of *comparative distortion* (abbreviated CD in this paper) between two map projections. In historical mathematical cartography, and emphasis has been placed on (1) projecting the globe onto a flat surface and (2) quantifying the necessary distortion that came from it. This paper forms a method to compute some notion of *comparative distortion*, which is a way to quantify how "different" two maps are in terms of distortion. The premise of the methodology is to form a mathematical model that most closely follows human intuition about when graphs are different and how different they are, because "difference" between maps is something inherently subjective.

Procedurally, this is done by dissecting each map into a finite number of points and then quantifying distortion at each of those points. Each map can then be thought of a matrix or vector with a finite number of distortion entries, and two maps can be compared by comparing their vectors.

The method of comparison involves taking the difference of a group of maps' distortion vectors, and then taking the magnitude of that difference vector. This represents some Euclidean notion of "distance" between two maps and some intuitive notion of how "different" they are. It is consistent with qualitative intuition about map differences and it preserves ordinality, so any maps that are closer together than any other maps will be reflected quantitatively. Overall, this method is successful in modeling intuition about comparative distortion.

2 Literature Review

There actually has been a limited amount of economic discussion about map-making and the effects of distortion. In 2020, for example, Abhishek Nagaraj and Scott Stern published a paper called "The Economics of Maps" in , The Journal of Economic Perspectives, published by the American Economic Association. They present an argument regarding the supply and demand for maps themselves, but they argue that maps have been used by economists in every sector of the study including urban economics, economic history, public finance, and political economics. They're used to understand a spatial distribution of resources and form practical/logistical plans of government services. It also helps them understand the different economic decision makers that interact with each other by understanding proximity and separation.

However, little has been studied about the actual maps used and how those could be improved/affected. There have been sparse arguments made, like the one above, about how important maps are to economic decision making, but not about how the maps themselves function.

On the cartographic side, there has also been little discussion about *comparative distortion* between map projections. There has been extensive discussion about absolute distortion from the globe, which is helpful in determining bias from reality. For example, in the book Coordinate Systems and Map Projections, D.H. Maling describes distortion in terms of area, angle, and bearing, which are all from the globe. Many of these metrics have been widely studied and are very available online for most notable map projections. However, these are unhelpful when comparing two maps that have already been projected. This paper will attempt to fill that niche by providing a quantitative way to evaluate *comparative distortion*, which is helpful when using several maps without worrying about bias from a globe.

3 Overview

It is important to define *comparative distortion.* At its core, comparative distortion is an intuitive, subjective, and qualitative evaluation of how different two maps are from each other. For this paper, it's how different the areas on the map are from each other. This paper attempts to summarize that information so it's useful for comparison.

The most prolific mathematical object used in this paper are vectors, because they're a convenient way to organize information. Maps can be turned into vectors, which can be compared to find some difference between them. Then, using a magnitude, which is a convenient way to include all information in a vector, the information can be summarized with one number. Because the vectors would represent those maps, the difference between the vectors would represent some quantitative notion of the difference between the two maps.

This methodology is useful because it's actually consistent with intuition, as it was intended to be. It preserves the fact that two of the same maps have no comparative distortion, because they're not different, and it ensures that two maps that are intuitively more different than another pair of two maps will be reflected by the mathematical analysis (it preserves ordinality).

This methodology also extends to more than just two maps. In fact, one can used a generalized version of the original formula to compare any amount of *map sets*, not just maps. This is done simply by adding up all values from all possible combinations of map comparisons. This also obeys the same intuition as above, because it preserves the fact that a group of the same map would have no comparative distortion and it preserves ordinality.

This analysis could prove useful to policymakers, who could use it to acquire or choose a more consistent group of maps, often used to make geopolitical/geoeconomic decisions, to make more consistent decisions.

4 Assumptions

For this paper, it is important to note that very map can be overlain with some grid involving some coordinate system. Because every map exists in a two-dimensional plane, it can be described with some two-dimensional coordinate system.

For this paper, all rectangular maps with length will be described using a Euclidean coordinate system, with the origin in the bottom left corner. It is not important where the origin is because this analysis is comparative, so as long as they are consistent, the analysis will still be rigorous. One example of a rectangular map is the Mercator projection, overlain with a Euclidean graticule of longitude and latitude. Also, all circular maps will be described with polar coordinates, with the origin in the center of the map.

It's also important to note that the way to calculate distortion is not specified here. This way, the method can be generalized for area distortion, angular distortion, etc. Note that it's not guaranteed that such a method of calculating distortion even exists, because this is simply a theoretical framework for a comparative analysis. It is not a data-based model, so its base-level assumptions are all debatable but will be taken axiomatically for this paper.

5 Comparing Two Maps

To analyze the distortion of some group of maps comparatively, it will be important to come up with some way to quantify their distortion, and then find some notion of comparison that is consistent with intuition.

Methodologically, we make the claim that the more information one can gather about the distortion of a map, the more accurate the resulting measure will be, so it would be optimal to gather as much information as possible. In context, this would ideally mean gathering information about distortion at *every* singly point on the map. .

To do so, note that each map can be represented at a matrix, with distortion information at each point on the map being an entry. (To calculate "distortion information," use some accepted metric of distortion that can be evaluated at a single point, like the area of Tissot's indicatrix, or some equivalent for angular distortion). Here, "point" is described as the intersection between two lines on a Euclidean grid, or between some radius and angular line on a polar grid. Of course, on these grids, there are infinite points, so theoretically, the matrix would be ∞ x ∞, which is not practically useful. Because of this, it is important in execution to pick a finite number of points on each map. It is not important what that number is, only that the number of points on compared maps are the same.

To pick the number of points on a rectangular map, choose n vertical lines and m horizontal lines. There will be a resulting number of $n * m$ points (intersections). For a circular maps, choose n radial circles and m angular lines.

There will be a resulting number of $n * m$ points. From this form, we will form a vector, which will allow us to perform algebraic operations on the map easily. To form the map vector, enter the distortion values into an (n*m) x 1 vector in some deterministic order. Be sure to form the vector for all compared maps in the same order, starting at the same point. Only compare maps of the same shape to each other. This will be called the map's *distortion vector*

Illustratively, consider the following example for some rectangular map M with distortion values represented in the rectangular matrix below. . Each distortion entry on horizontal line i and vertical line j is denoted d_{ij}. In this case, thew vector was formed by stacking all of the columns of the map's matrix on top of each other from left to right.

$$\begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{n1} & \cdots & d_{nn} \end{bmatrix} = \begin{bmatrix} d_{11} \\ \vdots \\ d_{n1} \\ \vdots \\ \vdots \\ d_{1n} \\ \vdots \\ d_{nn} \end{bmatrix}$$

Now, to calculate comparative distortion (CD), it is necessary to find some measure of separation between the two maps, which are now represented by their distortion vectors. There exists a natural and simple way to describe the separation between two vectors: difference. Simply by subtracting the two vectors, we can obtain a quantitative notion of how "different" their distortions are at each point. This vector will from now be called the *difference vector* between two maps. The difference vector is a useful way to visualize the difference in distortion at many places on the map, but it is not a concise way to understand overall comparative distortion between the two maps. To find this, we will take the magnitude of the difference vector, which gives us one number to describe the difference between the two maps, and includes all information from all points on the two maps.

To calculate the difference between two vectors, take the difference between each of their components. To calculate the magnitude, take the square root of the sum of the squares of all components:

$$\begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} - \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} m_1 - p_1 \\ \vdots \\ m_n - p_n \end{bmatrix}$$

$$\| \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} \| = \sqrt{(m_1)^2 + \ldots + (m_n)^2}$$

For the rest of this paper, define function $d(M, P) = \|\vec{m} - \vec{p}\|$ as the function that computes the *comparative distortion* between the two maps. In a Euclidean sense, this can be thought of the "distance" between the two maps, which could each be represented as vectors pointing to points in some $n * m$ dimensional Euclidean space.

6 Usefulness

Now that we have provided the reasoning for forming a map's distortion matrix, and then its distortion vector, as well as using difference as a measure of comparison, we need to see if these methodological steps actually serve a useful purpose. To do this, we need to consider whether they are consistent with our intuition about what comparative distortion should look like qualitatively, because then we can be confident that the quantitative measure works as as it should. Specifically, there is one question we need to answer:

1. Does this measure preserve ordinality? (For the comparisons between maps, does the measure accurately measure when some maps have more comparative distortion than others?)

We first consider our only edge case: when there is no comparative distortion. This would happen when we have two of the exact same map, because there is no difference between the two maps. Assume that if one map is a linear scale of the other (that is, it's a bigger or smaller version of the same map) that it's scaled up or down so it's identical to the first map. Intuitively, we would consider two maps that are a linear scaling of each other to be *the same map*, so we should mathematically treat them as such. This is considered to be the only edge case because there is no intuitive upper bound on how different two maps can be.

In this case, the two maps, which are the same, would necessarily have the same distortion vectors, because they have the exact same distortion at each point. In this case, for two maps $\vec{m}$ and $\vec{p}$ that are the same, $d(M, P) = \|\vec{m} - \vec{p}\| = d(M, M) = \|\vec{m} - \vec{m}\| = 0$, which says quantitatively that there is zero comparative distortion between these two maps. This is consistent with our intuition about maps with no comparative distortion.

Also, intuitively, this should be the minimal value comparative distortion. It makes sense for two maps to have no comparative distortion, which is when they are the same (there is no difference between them), but it is hard to imagine two maps with *less* difference than that. This indicates that this should be the minimal value our function should take.

And it is. Because our comparative distortion is a magnitude of two vectors, it can only be greater than or equal to 0. 0 is the minimal value it can take. From now on, the mathematical notion of "difference" being subtraction will be refereed to as "separation", so it is not confused with the qualitative word "difference" that refers to a quality of being "not similar."

For the general case, consider two maps with some comparative distortion. Qualitatively, this would mean that when comparing some set of points on two maps, the maps have separate values for distortion. Quantitatively, the more separate those values are, the more comparatively distorted they should be. Specifically, If two maps have *very* separate distortion values at a *few* points or if they have *somewhat* separate values at *many points*, they should have a high comparative distortion. This is the contextual notion of ordinality: two maps are "more separate" from each other than some other two maps if their distortion values have a higher separation at more points, and the function should reflect that.

In fact, the function does reflect that. Consider two sets of maps, M, P and N, Q. Consider the first case where both sets of maps are different at the same number of points i, but have different values of separation at each of those points. Imagine the separation at all other points is 0 for simplicity. Suppose now that M, P are more separate at each of those points than N, Q.

Illustratively, consider $\vec{m}$ and $\vec{p}$ of M and P that are different at i points (and the same at all others). Here it is considered that all i components are consecutive without loss of generality. Then $\vec{m} - \vec{p} =$

$$\begin{bmatrix} m_1 \\ \vdots \\ m_r \\ \vdots \\ m_{r+i} \\ \vdots \\ m_k \end{bmatrix} - \begin{bmatrix} m_1 \\ \vdots \\ p_r \\ \vdots \\ p_{r+i} \\ \vdots \\ m_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ m_r - p_r \\ \vdots \\ m_{r+i} - p_{r+i} \\ \vdots \\ 0 \end{bmatrix}$$

Consider also $\vec{n}$ and $\vec{q}$ of N and Q that are different at i points. Here it is considered that all i components are consecutive without loss of generality. Then $\vec{n} - \vec{q} =$

$$\begin{bmatrix} n_1 \\ \vdots \\ n_r \\ \vdots \\ n_{r+i} \\ \vdots \\ n_k \end{bmatrix} - \begin{bmatrix} n_1 \\ \vdots \\ q_r \\ \vdots \\ q_{r+i} \\ \vdots \\ n_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ n_r - q_r \\ \vdots \\ n_{r+i} - q_{r+i} \\ \vdots \\ 0 \end{bmatrix}$$

with $m_w - p_w > n_w - q_w$, with $r \leq w \leq r + i$.

In that case, i components of $\vec{m} - \vec{p}$ are greater than the same i components of $\vec{n} - \vec{q}$. In that case, $\|\vec{m} - \vec{p}\| > \|\vec{n} - \vec{q}\|$, because those i components contribute more to $\|\vec{m} - \vec{p}\|$ than the same i components contribute to $\|\vec{n} - \vec{q}\|$.

Consider a second case where both sets have the same separation in distortion at each point they're separate, but they're separate at a different number of points. Imagine the difference at all other points is 0. Let M, P be different at i points and N, Q be different at j points, $i > j$.

Illustratively, consider $\vec{m}$ and $\vec{p}$ of M and P that are different at i points. Here it is considered that all i components are consecutive without loss of generality. Then $\vec{m} - \vec{p} =$

$$\begin{bmatrix} m_1 \\ \vdots \\ m_r \\ \vdots \\ m_{r+i} \\ \vdots \\ m_k \end{bmatrix} - \begin{bmatrix} m_1 \\ \vdots \\ p_r \\ \vdots \\ p_{r+i} \\ \vdots \\ m_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ m_r - p_r \\ \vdots \\ m_{r+i} - p_{r+i} \\ \vdots \\ 0 \end{bmatrix}$$

Consider also $\vec{n}$ and $\vec{q}$ of N and Q that are different at j points. Here it is considered that all j components are consecutive without loss of generality. Then $\vec{n} - \vec{q} =$

$$\begin{bmatrix} n_1 \\ \vdots \\ n_r \\ \vdots \\ n_{r+j} \\ \vdots \\ n_k \end{bmatrix} - \begin{bmatrix} n_1 \\ \vdots \\ q_r \\ \vdots \\ q_{r+j} \\ \vdots \\ n_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ n_r - q_r \\ \vdots \\ n_{r+j} - q_{r+j} \\ \vdots \\ 0 \end{bmatrix}$$

with $i > j$.

In this case again, $\|\vec{m} - \vec{p}\| > \|\vec{n} - \vec{q}\|$, because there are more points contributing to $\|\vec{m} - \vec{p}\|$ than there are contributing to $\|\vec{n} - \vec{q}\|$.

Lastly, consider a case where the two maps are separate at a different number of points, and the values of separation are different as well. In this case it is ambiguous which set has more comparative distortion, but intuitively, we would expect that the map with less different points would only have a higher value if its values of separation are so high that they outweigh the fact that it has less points.

Illustratively, consider $\vec{m}$ and $\vec{p}$ of M and P that are different at i points. Here it is considered that all i components are consecutive without loss of generality. Then $\vec{m} - \vec{p} =$

$$\begin{bmatrix} m_1 \\ \vdots \\ m_r \\ \vdots \\ m_{r+i} \\ \vdots \\ m_k \end{bmatrix} - \begin{bmatrix} m_1 \\ \vdots \\ p_r \\ \vdots \\ p_{r+i} \\ \vdots \\ m_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ m_r - p_r \\ \vdots \\ m_{r+i} - p_{r+i} \\ \vdots \\ 0 \end{bmatrix}$$

Consider also $\vec{n}$ and $\vec{q}$ of N and Q that are different at j points. Here it is considered that all j components are consecutive without loss of generality. Then $\vec{n} - \vec{q} =$

$$\begin{bmatrix} n_1 \\ \vdots \\ n_r \\ \vdots \\ n_{r+j} \\ \vdots \\ n_k \end{bmatrix} - \begin{bmatrix} n_1 \\ \vdots \\ q_r \\ \vdots \\ q_{r+j} \\ \vdots \\ n_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ n_r - q_r \\ \vdots \\ n_{r+j} - q_{r+j} \\ \vdots \\ 0 \end{bmatrix}$$

with $m_w - p_w \neq n_w - q_w$ and $i > j$. In this case it is ambiguous, if the components $n_r - q_r, \ldots, n_{r+j} - q_{r+j}$ are higher, they'll contribute more to $\|\vec{n} - \vec{q}\|$. If that amount is more than than $m_r - p_r, \ldots, m_{r+i} - p_{r+i}$ contribute more to $\|\vec{m} - \vec{p}\|$, then the set N, Q will have more comparative distortion than M, P. This is exactly what we would expect: if the values for those fewer points in N, Q outweigh the many points in M, P, then the comparative distortion of N, Q should have a higher value.

This shows that d preserves ordinality, if a set of maps is expected intuitively to have higher comparative distortion than another set, it does.

7 Comparing Multiple Maps

Now, instead of considering two maps M and P, consider n sets of maps of cardinality s: $\mathcal{M}_1, \ldots, \mathcal{M}_n$, where $\mathcal{M}_i = \{M_{i1}, \ldots, M_{is}\}$, with $M_{i1}, \ldots, M_{is}$ all being maps in the set. Note: the maps are organized in separate sets because its possible that in a real scenario, maps may be grouped in separate collections that are being compared to each other.

it is possible to use the CD function d as described above to not only compare any two maps, but all maps at once from all sets.

To logically extend d, we would intuitively want to compare every map in one set to every map in the other, obtain a comparative distortion value for each of those comparisons, and then somehow combine the comparative distortion values. The simplest way to do that is to use d to compare all possible combinations of maps, and add up all the comparative distortion values. To do this most conveniently, define two functions. Note: The first function is defined to make the second one more clear, but this could be done in one step. The first function is an extension of d, now to deal with two sets of maps instead of just two maps. Let this function D be defined as

$$D(\mathcal{M}_a, \mathcal{M}_b) = \sum_{i=1}^{s} \sum_{j=1}^{s} d(M_{a_i}, M_{b_j})$$

Where M_{d_i} describes the the ith map in set $\mathcal{M}_d$ This is the sum of all comparative distortions from all combinations of maps in sets $\mathcal{M}_a$ and $\mathcal{M}_b$.

Then Define set function δ:

$$\delta(\mathcal{M}_1, \ldots, \mathcal{M}_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} D(\mathcal{M}_i, \mathcal{M}_j)$$

which sums the comparative distortion between all maps in all sets $\mathcal{M}_1, \ldots, \mathcal{M}_n$.

For δ to satisfy a useful measure of comparative distortion between two sets of maps, it has to answer question 1. The justification of this questions are quick and a generalization of the two-map version.

For the edge case of question 1, consider a case where some map sets $\mathcal{M}_1, \ldots, \mathcal{M}_n$ have no distortion. This would be a case when each map in all sets are exactly the same. In this case, $D(\mathcal{M}_i, \mathcal{M}_j)$ for all $i, j = 0$, and therefore $\delta(\mathcal{M}_1, \ldots, \mathcal{M}_n) = 0$, which is a consistent reflection of two sets with no distortion.

To address the general case, notice that δ does preserve ordinality. It is intuitive that a map comparison $\delta(\mathcal{M}_1, \ldots, \mathcal{M}_n)$ is more distorted than some other map comparison $\delta(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ if the maps being compared in $\delta(\mathcal{M}_1, \ldots, \mathcal{M}_n)$ are more distorted from each other *on average* than maps in $\delta(\mathcal{P}_1, \ldots, \mathcal{P}_n)$. (It is unintuitive to distinguish between two groups of different sizes, so this case will be omitted) Specifically, this would mean $\frac{\sum_{j=1}^{n} \sum_{i=1}^{n} D(\mathcal{M}_i, \mathcal{M}_j)}{n^2}$, the arithmetic mean of distortions between all maps in sets $\mathcal{M}_1, \ldots, \mathcal{M}_n$ is greater than $\frac{\sum_{j=1}^{n} \sum_{i=1}^{n} D(\mathcal{P}_i, \mathcal{P}_j)}{n^2}$, the mean distortion between all maps in sets $\mathrm{P}_1, \ldots, \mathcal{P}_n$

Notice that

$$\begin{aligned}\frac{\sum_{j=1}^{n} \sum_{i=1}^{n} D(\mathcal{M}_i, \mathcal{M}_j)}{n^2} > \frac{\sum_{j=1}^{n} \sum_{i=1}^{n} D(\mathcal{P}_i, \mathcal{P}_j)}{n^2} &\Rightarrow \sum_{j=1}^{n} \sum_{i=1}^{n} D(\mathcal{M}_i, \mathcal{M}_j) > \sum_{j=1}^{n} \sum_{i=1}^{n} D(\mathcal{P}_i, \mathcal{P}_j) \\ &\Rightarrow \delta(\mathcal{M}_1, \ldots, \mathcal{M}_n) > \delta(\mathcal{P}_1, \ldots, \mathcal{P}_n)\end{aligned}$$

This shows that δ preserves ordinality, because any group of maps that is more distorted than any other group is reflected by the δ function intuitively. δ is a useful function for comparing distortion between two sets of maps.

8 Discussion and Implications

This paper presents a theoretical argument for a model of comparative map distortion. It aims to quantify some intuitive, qualitative notion of how "different" or "separate" groups of maps are from each other. It does this by organizing each map's distortion information into a vector, and then finding the difference as a useful quantitative way to evaluate the difference between the two maps.

The difference information is then summarized by a magnitude, which preserves intuitive notions of ordinality when comparing any sets of maps, including when it is generalized to some arbitrary finite number of maps being compared. Maps are an extremely useful tool in international politics and economics. They help policy makers visualize geographic relations between nations, which could help them strategically position their resources or facilities. They may use a map to place a strategic military base next to a hostile neighbor or position their resource farming facilities away from borders as protection.

It also helps them analyze geographic relations between other relevant countries. Maps could possibly help them predict the strategic placement that other leaders make, which could influence a military campaign, diplomatic act, or plans for their own strategies.

Because maps are such a useful tool in international strategy, it is important that the maps used are consistent with each other. Massive inconsistencies in map type or quality could lead to contradictory assumptions about where borders are, how big some countries are, or where facility placements actually are. The methodology in this paper can be used to evaluate whether a group of maps used by policymakers is "consistent" - similar to each other, or "inconsistent" - all different from each other. Not only could it be used to see consistency of maps in a group, it can produce a number to show how different the maps are, which could help policymakers evaluate relatively how consistent the maps they're using are with each other. It could allow policymakers to choose similar maps, which would help them make better decisions, because they wouldn't be receiving contradictory evidence from different maps.

9 References

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