The F-signature and the Frobenius-Alpha Invariant of Projective Varieties

by

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To my friends and family

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ABSTRACT

This thesis concerns two invariants of projective varieties in positive characteristic, namely, the F-signature and a new invariant called the Frobenius-alpha invariant. For a projective variety X and an ample divisor L on X, both invariants measure asymptotic properties of Frobenius splittings of $|mL|_{m\geq 1}$, i.e., the linear systems defined by multiples of L. In the first part of the thesis, we present joint work with Seungsu Lee, where we prove that for a fixed projective variety X, the F-signature of ample divisors on X extends to a continuous function on the ample cone of X. Moreover, we show that this function has a continuous extension to the non-zero part of the Nef cone of X. In the second part of this thesis, we define and study the Frobenius-alpha invariant in analogy with Tian's alpha-invariant from complex algebraic geometry. In particular, we show that the Frobenius-alpha invariant of Fano varieties is at most $1/2$ and prove upper and lower bounds for the F-signature in terms of the Frobenius-Alpha invariant. Finally, we study the behaviour of the Frobenius-alpha invariant in geometric families, and prove that this invariant is lower semicontinuous in a family of globally F-regular $\mathbb{Q}$-Fano varieties.

CHAPTER I
Introduction

I.1: Motivation

The topic of this thesis is the study of *singularities* of algebraic varieties. A singularity is a point on an algebraic variety V near which V cannot be well-approximated by a vector space (the tangent space). A point that is not a singularity is called a smooth point. This difference between smooth points and singular points leads to deep differences in the behaviour of smooth varieties (ones with no singularities) and singular varieties (ones with one or more singular points).

Our focus will be on singularities in positive characteristics. It turns out that many traditional techniques used in the study of complex singularities are not suitable while working in the positive characteristic setting. Indeed, key results like resolution of singularities and vanishing theorems are either unknown or are known to fail. However, many different techniques, mostly involving the Frobenius map, have been developed that act as a substitute. We now briefly introduce some of these ideas.

The Frobenius Map: Let k be a field of prime characteristic p. Consider an affine algebraic variety V over k, i.e., $V = \operatorname{Spec}(R)$ where R is a finitely generated algebra over k. The *Frobenius map* of R is the natural self-map

$$F : R \to R$$

defined by sending

$$r \mapsto r^p.$$

This is a ring homomorphism thanks to the identity $(r + s)^p = r^p + s^p$ which holds true in every ring of characteristic p, but of course, has no analogue over the real or complex numbers. Note that since F is a ring homomorphism, we get a new R-module structure on the target copy of R defined by restriction of scalars along F. More explicitly, denote the target copy of R by F_*R and for any $r \in R$, let F_*r be the corresponding element in F_*R. Then F_*R is a new R-module where an element $r \in R$ acts on an element F_*s by the rule

$$rF_*s = F_*(r^p s).$$

Similarly, we can define the module F^e_*R for any $e \geq 1$ by restricting scalars along the e^{th}-iterate of F. Then, the key idea in this thesis is that the sequence of R-modules $(F^e_*R)_{e\geq 1}$ encodes the singularity properties of R in various ways. For example, the following theorem of Kunz provides a characterization of regular (the algebraic version of smooth) points:

Theorem I.1.1. *[Kun69] Assume that R is reduced. Then, R is regular if and only if F^e_*R is a flat R-module for some (equivalently, any) $e \geq 1$.*

Therefore, the singularities of R are quantified by the non-flatness of F^e_*R. This idea is made precise via the notion of *strong F-regularity*, a central topic of this thesis:

Definition I.1.2. Let R be a Noetherian local domain of positive characteristic $p > 0$. Then, R is said to be strongly F-regular if for every non-zero element c of R, there exists an $e \gg 0$ such that the map

$$R \to F^e_*R$$

mapping

$$1 \mapsto F^e_*c$$

splits as a map of R-modules.

Strongly F-regular singularities were introduced in the work of Hochster and Huneke in the celebrated tight closure theory [HH89]. Such singularities satisfy many desirable properties; for instance, they are normal, Cohen-Macaulay and have pseudo-rational singularities in the sense of Lipman and Tessier (see [Smi97a]). Though Kunz's theorem implies that regular rings are strongly F-regular, there are many strongly F-regular rings that fail to be regular. Thus, strongly F-regular singularities form a large class of well-behaved, yet non-regular class of singularities and have occupied a very important role in positive characteristic singularity theory.

Singularities are also closely linked to the global geometry of projective varieties. For instance, one way to produce singularities is to take a closed subvariety V of $\mathbb{P}^n$ and consider the *cone over V*. The singularity of the vertex of the cone encodes the global geometry of V. For example, the cone over V is smooth if and only if V is a linear subspace of $\mathbb{P}^n$. Thus, studying the singularities of cones over projective varieties leads to insights into their geometry.

Overview: In this thesis, we will be interested in a singularity invariant called the *F-signature* in the context of cones over projective varieties. The F-signature, that we will introduce in Section I.2, captures asymptotic properties of the Frobenius map in strongly F-regular local rings. In Section I.3, we will describe joint work with Seungsu Lee, that establishes results about how the F-signature of the cone over a projective variety V varies as we change the embedding of V into various projective spaces. The proofs of these results are contained in Chapter III of the thesis. In Section I.4, we introduce a new invariant of the singularity of a cone called the *Frobenius-alpha* invariant. This invariant turns out to be closely related to the F-signature and helps us estimate the F-signature of a projective variety. The detailed results and proofs regarding the Frobenius-alpha invariant are contained in Chapter IV and Chapter V of the thesis.

I.2: F-regularity and the F-signature

In this section, we introduce the two central topics of this thesis: global F-regularity, and the F-signature. Throughout, we fix a perfect field k of characteristic $p > 0$.

I.2.1: Global F-regularity

Globally F-regular varieties were introduced in [Smi00] as the positive characteristic analogs of *log-Fano* type varieties and the global versions of strongly F-regular singularities.

Definition I.2.1. [SS10, Definition 3.2] Let X be a normal variety over k and $\Delta \geq 0$ be an effective $\mathbb{Q}$-divisor. The pair (X, Δ) is said to be *globally F-regular* (resp. locally F-regular) if for any effective Weil divisor D on X, there exists an integer $e \gg 0$, such that, the natural map

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D)$$

splits (resp. splits at every stalk) as a map of $\mathcal{O}_X$-modules. A normal variety X is said to globally (resp. locally) F-regular if the pair $(X, 0)$ is globally (resp. locally) F-regular.

It was proved in [SS10] that globally F-regular varieties are of log Fano type, i.e., if X is globally F-regular, then there is a $\mathbb{Q}$-divisor $\Delta \geq 0$ such that (X, Δ) has klt singularities and $-K_X - \Delta$ is ample. However, not all log Fano type varieites (or even Fano varieties) are globally F-regular; indeed, globally F-regular varieties enjoy additional properties such as satisfying a *Kawamata-Viehweg vanishing* theorem [SS10] that is known to fail in general even for smooth Fano varities in positive characteristics [Tot19]:

Theorem I.2.2 ([SS10], Theorem 6.8)**.** *Let (X, Δ) be a projective, globally F-regular pair over k. Suppose L is a Cartier divisor such that $L - K_X - \Delta$ is big and nef. Then,*

$$H^i(X, \mathcal{O}_X(-L)) = 0 \quad \text{for all } i < \dim(X).$$

Thus, globally F-regular varieties are log Fano varieties with additional favorable properties. They have found various applications, for instance, to the three dimensional minimal model program in positive characteristic [HX15] and in the study of Fano type complex varieties [GOST15]. For other investigations regarding globally F-regular varieties, see [GLP⁺15], [GT19] and [Kaw21].

I.2.2: The F-signature:

Building on the idea from Theorem I.1.1 that we may measure the non-flatness of the modules $F^e_* R$ to quantify singularities of R, we now turn to an asymptotic invariant of these modules. Let R be a local domain essentially of finite type over k.

Definition I.2.3 (F-signature)**.** For any $e \geq 1$, let $a_e(R)$ denote the free rank of $F^e_* R$. In other words, $a_e(R)$ is the maximum integer a such that we have an R-module decomposition

$$F^e_* R \cong R^{\oplus a} \oplus N$$

for some R-module N. Then the *F-signature* of R is defined to be the limit:

$$s(R) := \lim_{e \to \infty} \frac{a_e(R)}{p^{ed}}$$

where d is an integer such that $F_* R$ has generic rank p^d over R (in other words, p^d is the degree of the field extension $\mathrm{Frac}(R) \subset \mathrm{Frac}(R)^{1/p}$).

This invariant was first considered implicitly by Smith and Van den Bergh in [SVdB97], and defined formally by Huneke and Leuschke [HL02]. The fact that the limit in the definition exists was proved only later by Tucker [Tuc12]. Note that since R is local, a finitely generated module over R is flat if and only if it is free. Thus, the F-signature is an asymptotic measurement of the flatness of $F^e_* R$. Indeed, Huneke and Leuschke proved the following asymptotic version of Kunz's theorem (Theorem I.1.1): R is regular if and only $s(R) = 1$.

The F-signature is most interesting in the case of strongly F-regular singularities (Definition I.1.2); the positivity of the F-signature of a local F-finite domain R corresponds exactly to R being strongly F-regular [AL03]. Furthermore, the F-signature has been fruitfully applied to gain insights into the structure of strongly F-regular local rings as seen in the following theorem due to Carvajal-Rojas, Schwede and Tucker that bounds the size of the local étale fundamental group:

Theorem I.2.4. *[CRST18] Let $(R, \mathfrak{m}, k)$ be an F-finite strongly F-regular local ring of characteristic $p > 0$ and dimension ≥ 2. Assume that R is strictly henselian. Then, we have*

$$|\pi_1^{\acute{e}t}(Spec^\circ(R))| \leq \frac{1}{s(R)}$$

where $Spec^\circ(R)$ denotes $Spec(R) \setminus \mathfrak{m}$, and $s(R)$ denotes the F-signature of R. In particular, the size of the above fundamental group is finite.

Similar theorems for complex klt singularities have been established by completely different techniques; see [Xu14], [GKP16], [Bra21] and [XZ21]. Other applications of the F-signature include bounding the torsion subgroup of the divisor class group; see [Mar22], [CR17] and [Pol22].

I.3: Variation on the ample cone

In this section, we summarize the results contained in Chapter III of the thesis. These results are part of joint work with Seungsu Lee. Chapter III is dedicated to studying how the F-signature of a projective variety X varies with the embedding of X into various projective spaces. This is made precise in terms of section rings and the F-signature function on the ample cone. Throughout this section, we fix an algebraically closed field k of characteristic $p > 0$.

Let X be a projective variety over k and $\mathcal{L}$ be an ample line bundle over X. Then, we can construct a graded ring, the *section ring* $S(X, \mathcal{L})$ of X with respect to $\mathcal{L}$, defined by

$$S(X, \mathcal{L}) := \bigoplus_{j \geq 0} H^0(X, \mathcal{L}^j)$$

with multiplication given by the tensor product of global sections. We call $Y = \mathrm{Spec}(S(X, \mathcal{L}))$ the cone over X with respect to $\mathcal{L}$.

It was shown in [Smi00] that a projective variety X is globally F-regular (Definition I.2.1) if and only if the section ring $S(X, \mathcal{L})$ is strongly F-regular for some (equivalently, every) ample line bundle $\mathcal{L}$. Since the F-signature is positive for all strongly F-regular rings, it is natural to ask:

Question I.3.1. Let X be a globally F-regular projective variety. How does the F-signature of the section ring $S(X, \mathcal{L})$ vary with $\mathcal{L}$?

In Chapter III, we consider the following function:

Definition I.3.2. The F**-signature function** s_X is the assignment:

$$\mathcal{L} \mapsto s(S(X, \mathcal{L})) = F\text{-signature of } S(X, \mathcal{L})$$

where $\mathcal{L}$ is any ample line bundle over X.

Note that this function is non-zero for every ample line bundle $\mathcal{L}$ since $S(X, \mathcal{L})$ is strongly F-regular. Building on the work in [VK12] and [CR17], our first theorem allows us to extend the F-signature function to the *rational ample cone* of X as follows:

Theorem I.3.3. *(Theorem III.1.3) Fix a positive dimensional globally F-regular projective variety X over k. Then, the F-signature function s_X (Definition I.3.2) naturally extends to a unique, well-defined, real-valued function*

$$s_X : Amp_{\mathbb{Q}}(X) \to \mathbb{R}$$

on the set of rational classes in the ample cone of X satisfying the identity:

$$s_X(\lambda L) = \frac{1}{\lambda} s_X(L) \quad \textit{for all ample } \mathbb{Q}\textit{-divisors } L \textit{ and all } \lambda \in \mathbb{Q}_{>0}.$$

Recall that the rational ample cone of a projective variety is the set of numerical classes of ample $\mathbb{Q}$-divisors on X. Now that we have a well-defined function on the rational ample cone, the next theorem describes the continuity properties of the F-signature function.

Theorem I.3.4. *Fix any globally F-regular projective variety X over k of positive dimension. Then, the F-signature function of X satisfies the following properties:*

1. *(Theorem III.2.1) The function s_X is continuous on the rational ample cone of X, with respect to the usual topology on the Néron-Severi space.*

2. *(Corollary III.2.3) The function s_X extends continuously to all real classes in the ample cone of X.*

Theorem I.3.3 and Theorem I.3.4 parallel the theory of the *F-signature function* on the ample cone of a globally F-regular variety with the volume function on the big cone. This perspective was first considered in [VK12], where Theorem I.3.4 was proved in the special case when X is a toric variety. Recall that on a projective variety X, to any Cartier divisor D on X, we can associate a non-negative real number called the *volume of* D, measuring the growth of the global sections of multiples of D. A foundational result in the theory of volumes is that the volume of a big divisor D depends only on its numerical equivalence class. Moreover, it extends suitably to all $\mathbb{R}$-divisors and varies continuously as D varies on the Néron-Severi space of X. See [Laz04, Section 2.2] and [LM09] for the details. The study of volumes of divisors has been important in birational geometry; for example, see [Laz04, Section 2.2], [LM09], [Bou02], [ELM$^+$05], [HM06], [Tak06b], and [KÖ6].

The ample cone is an open cone in the Néron-Severi space of a projective variety X, and its closure is represented by the set of nef divisors on X. Hence, it is natural to ask if the F-signature function $\mathcal{s}_X$ from Theorem I.3.4 has a natural extension to the nef cone. We show that this is indeed true:

Theorem I.3.5 (Theorem III.3.1)**.** *Suppose X is a globally F-regular projective variety. Then the F-signature function $\mathcal{s}_X$ extends continuously to all non-zero classes of the Nef cone of X. Moreover, if L is a nef Cartier divisor which is not big, then $\mathcal{s}_X(L) = 0$.*

The proofs of Theorem I.3.3 and Theorem I.3.4 consist of several steps: First, we need to verify that the F-signature function is well-defined on the rational ample cone of X. The main result here is that on globally F-regular projective varieties, numerical equivalence, and $\mathbb{Q}$-linear equivalence coincide (Theorem III.1.4). This is reminiscent of the same fact for log-Fano type varieties over the complex numbers. Next, we analyze the Frobenius splittings of linear systems $|mL|$ for multiples of an ample divisor L. We also develop some techniques to compare the Frobenius splittings of the linear systems $|mL|$ and $|m(L+H)|$, where H is some effective divisor. Along the way, we prove some uniform (with respect to the ample divisors) upper bounds on the F-signature function for a fixed globally F-regular variety X. For more details, we refer to Section III.2. We further utilize these ideas to extend the F-signature function to all non-zero nef divisors in Section III.3. Lastly, in Section III.4 we also prove a local effective upper bound for the F-signature function.

I.4: The Frobenius-Alpha Invariant

The second part of this thesis is dedicated to the study of another invariant of cones over projective varieties, called the α_F-invariant (the *"Frobenius-alpha" invariant*), and its connections to the F-signature (Definition I.2.3). The α_F-invariant is motivated by a connection between singularities in characteristic zero with the F-singularities in positive characterstic that we first recall .

I.4.1: Connections to singularities in characteristic zero

Strongly F-regular singularities (introduced in Definition I.1.2) are the positive characteristic analogs of *Kawamata log terminal (klt)* singularities, an important class of singularities that arise in the minimal model program. Klt singularities are defined using a resolution of singularities over the complex numbers (which is known to exist by the celebrated theorem of Hironaka [Hir64]), and thanks to the Kodaira and Kawamata-Viehweg vanishing theorems, they are normal, Cohen-Macaulay, and rational singularities. Though defined by completely different methods, klt singularities are related to strongly F-regular singularities in a precise way by the following remarkable theorem established in the works of Smith, Hara, Mehta and Srinivas:

Theorem I.4.1 ([Smi97a], [Har98], [MS97])**.** *Let R be a finitely generated $\mathbb{Z}$-algebra and assume that R is a Gorenstein (i.e., the canonical module ω_R is free) domain. Then, the ring $R \otimes_{\mathbb{Z}} \mathbb{C}$ has klt singularities if and only if $R \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ has strongly F-regular singularities for all $p \gg 0$.*

Note that for a Gorenstein domain R, having klt (and similarly, strongly F-regular when R has characteristic $p > 0$) singularities is equivalent to having rational (and respectively, F-rational) singularities. The correspondence between rational and F-rational singularities was established in [Smi97a], [Har98], [MS97]. For the correspondence between more general klt and F-regular singularities, see [HW02] and [Tak06a]. Theorem I.4.1 has a global version that relates *Fano varieties* (varieties with ample anti-canonical bundle) and globally F-regular varieties.

Theorem I.4.2. *[Smi00] Let X be a Fano variety defined over $\mathbb{Z}$. Then, $X_{\mathbb{C}} = X \times_{\mathbb{Z}} Spec(\mathbb{C})$ is a klt Fano variety (i.e., $X_{\mathbb{C}}$ has klt singularities) if and only if $X_p = X \times_{\mathbb{Z}} Spec(\mathbb{Z}/p\mathbb{Z})$ is globally F-regular for all $p \gg 0$.*

I.4.2: The α_F-invariant:

The *α-invariant* of a complex Fano variety X was introduced by Tian in [Tia87] to provide a sufficient criterion for *K-stability* of X, a condition that guarantees the existence of a Kähler-Einstein metric on X. Though initially defined analytically, Demailly later reinterpreted the α-invariant in terms of klt singularities [CtS08] as follows:

Definition I.4.3. Let X be a smooth Fano variety over $\mathbb{C}$. Then,

$$\alpha(X) := \sup \{\, t \geq 0 \,|\, (X, t\Delta) \text{ is klt } \forall \, \text{effective } \mathbb{Q}\text{-divisor } \Delta \sim_{\mathbb{Q}} -K_X \}.$$

Understanding the α-invariant, and K-stability more generally, has led to many fundamental advances in our understanding of complex Fano varieties; see [OS12], [Bir21], [Xu21]. Our focus will be to study a positive characteristic analog of the α-invariant. Fix k to be a perfect field of positive characteristic $p > 0$. Following Theorem I.4.1 and Theorem I.4.2, we may replace the klt singularities with global F-regularity (Definition I.2.1) in Definition I.4.3 to obtain a positive characteristic analogue of Tian's α-invariant, which we call the α_F-invariant as follows:

Definition I.4.4. Let X be a smooth, globally F-regular Fano variety over a perfect field of positive characteristic. Then, we define the α_F-invariant of X as

$$\alpha_F(X) := \sup\{t \geq 0 \,|\, (X, t\Delta) \text{ is globally } F\text{-regular } \forall \text{ effective } \mathbb{Q}\text{-divisors } \Delta \sim_{\mathbb{Q}} -K_X\}.$$

Since we intend for the α_F-invariant to capture global properties of divisors on X, we use global F-regularity instead of just local strong F-regularity (Definition II.3.2). This is justified by noting that simply replacing globally F-regular by klt in characterisitic zero, we obtain the minimum value between the usual α-invariant of X and 1 (see Remark IV.3.11). Thus, at least for Fano varieties with $\alpha(X) \leq 1$, the α_F-invariant is a "Frobenius-analog" of Tian's α-invariant.

Our first theorem proves some surprising properties of the α_F-invariant in contrast to the complex version, and establishes connections to the F-signature of $(X, -K_X)$ (see Definition I.2.3 and Definition I.3.2):

Theorem I.4.5. *Let X be a globally F-regular Fano variety over a perfect field of positive characteristic. Then,*

1. *The α_F-invariant of X (denoted by $\alpha_F(X)$) is at most 1/2 (Theorem IV.3.5).*
2. *Assume X is geometrically connected over the (perfect) base field. We have $\alpha_F(X) = 1/2$ if and only if the F-signature of X (with respect to $-K_X$) equals $\frac{\mathrm{vol}(-K_X)}{2^d(d+1)!}$, where d is the dimension of X (Corollary IV.3.8).*
3. *More generally (and still assuming that X is geometrically connected), the F-signature of X is at most $\frac{\mathrm{vol}(-K_X)}{2^d(d+1)!}$ (Corollary IV.3.8).*
4. *In case X is a toric Fano variety corresponding to a fan $\mathfrak{F}$, then $\alpha_F(X)$ is the same as the complex α-invariant of $X_{\mathbb{C}}(\mathfrak{F})$, the complex toric Fano variety corresponding to $\mathfrak{F}$ (Theorem IV.3.12).*

Part (1) of Theorem I.4.5 is surprising since many complex Fano varieties have α-invariants greater than $1/2$ (and less than 1). Combined with Part (4), this recovers, and provides a positive characteristic proof of the well-known fact that the α-invariant of toric Fano varieties is at most $1/2$ (see [LZ22, Corollary 3.6]).

Note that the F-signature has attracted attention as a candidate for the positive characteristic analog of the *normalized volume* of a Kawamata log-terminal (klt) singularity, extending the established analogy between strongly F-regular and klt singularities; see [LLX20], [Tay19], [MPST19]. The normalized volume has been successfully used in the stability theory of complex klt singularities and the moduli theory of Fano varieties; see the recent survey [Zhu23] for the details.

As in the case of the complex α-invariant, we may consider the α_F-invariant much more generally for arbitrary polarizations of projective varieties. In Section IV.2, we develop the theory of the α_F-invariant in this more general setting. From this perspective, the α_F-invariant is an invariant of a section ring of a projective varieties that shares many properties and relations with the F-signature. In this direction, we prove:

Theorem I.4.6. *Let S denote a section ring of a globally F-regular projective variety (with respect to some ample line bundle) over a perfect field k. Then,*

1. *(Theorem IV.2.8): The number $\alpha_F(S)$ can be calculated as the following limit:*

$$\alpha_F(S) = \lim_{e\to\infty} \frac{m_e(S)}{p^e}$$

   where $m_e(S)$ denotes that maximum integer m such that for each non-zero homogeneous element f of degree m, the map $S \to F^e_* S$ sending 1 to $F^e_* f$ splits.

2. *Since S is strongly F-regular, we have $\alpha_F(S)$ is positive (Theorem IV.2.10).*

3. *Base-change (Corollary IV.2.16): Assume that $S_0 = k$ and K is any perfect field extension of k. Then,*

$$\alpha_F(S) = \alpha_F(S \otimes_k K).$$

Our third set of results concern the semicontinuity properties of the α_F-invariant, analogus to the results of [BL22] about the complex version. In this direction, we prove:

Theorem I.4.7 (Theorem V.2.1)**.** *Let $f : \mathcal{X} \to Y$ be family of globally F-regular Fano varieties such that $-K_{\mathcal{X}|Y}$ is $\mathbb{Q}$-Cartier and f-ample. Assume that Y is regular. Then, the map from $Y \to \mathbb{R}_{\geq 0}$ given by*

$$y \mapsto \alpha_F(\mathcal{X}_{y^\infty})$$

is lower semicontinuous, where $\mathfrak{X}_{y^\infty}$ is the perfectified-fiber over $y \in Y$ (see Notation V.1.1).

We also prove a weaker version of Theorem I.4.7 for any polarized family of globally F-regular varieties. This is analogus to the corresponding result for the F-signature proved in [CRST21] and relies on uniform convergence for the α_F-invariant (Theorem V.1.4) in a family, which may be of independent interest.

Theorem I.4.5 and Theorem I.4.6 are proved in Chapter IV of this thesis. Chapter V is dedicated to the proof of Theorem I.4.7 and related results.

CHAPTER II

Frobenius splittings, F-regularity, and the F-signature of Projective Varieties

In this chapter, we present some background and prove preliminary results about Frobenius splittings, F-regularity and the F-signature in the case of *section rings* of projective varieties. First, we review the construction of section rings of projective varieties in Section II.1. Next, we discuss the F-signature of section rings in Section II.2. Finally, we review the notion of F-regularity and global F-regularity of projective varieties in Section II.3.

Notation II.0.1. Throughout this thesis, all rings are assumed to be Noetherian and commutative with a unit. Unless specified otherwise, k will denote a perfect field of characteristic p. A *variety over* k is an integral (in particular, connected), separated scheme of finite type over k. For a point x on a scheme X, the residue field $\mathcal{O}_{X,x}/\mathfrak{m}_x$ will be denoted by $\kappa(x)$ (where $\mathcal{O}_{X,x}$ is the local ring at x and $\mathfrak{m}_x$ is the maximal ideal of the local ring).

Notation II.0.2 (Divisors and Pairs)**.** A *prime Weil-divisor* on a scheme X is a reduced and irreducible subscheme of X of codimension one. An *integral Weil-divisor* is a formal $\mathbb{Z}$-linear combination of prime Weil-divisors. A $\mathbb{Q}$-*divisor* is a formal $\mathbb{Q}$-linear combination of prime Weil-divisors. By a pair (X, Δ), we mean that X is a Noetherian, normal scheme and Δ is an effective $\mathbb{Q}$-divisor over X.

II.1: Section Rings and Modules

Definition II.1.1. Let A be a Noetherian ring and X be a projective scheme over A. Given an ample invertible sheaf $\mathcal{L}$ on X and $\mathcal{F}$ a coherent sheaf on X, the $\mathbb{N}$-graded ring S defined by

$$S = S(X, \mathcal{L}) := \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^n)$$

is called the *section ring* of X with respect to $\mathcal{L}$. The affine scheme $\operatorname{Spec}(S)$ is called the (affine) *cone over* X with respect to $\mathcal{L}$. The *section module* of $\mathcal{F}$ with respect to $\mathcal{L}$ is a

$\mathbb{Z}$-graded S-module M defined by

$$M = M(X, \mathcal{L}) := \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{F} \otimes \mathcal{L}^n).$$

Similarly, the sheaf corresponding to M on $\operatorname{Spec}(S)$ is called the *cone over* $\mathcal{F}$ with respect to $\mathcal{L}$.

Let S be a Noetherian, $\mathbb{N}$-graded domain, and T denote the set of positive degree homogeneous elements of S. For a finitely generated, torsion-free, $\mathbb{Z}$-graded module M over S, let M' denote the localization $M' = T^{-1}M$. Note that M' is naturally a $\mathbb{Z}$-graded module over $T^{-1}S$. Since M is torsion-free, we can think of M naturally as a subset of M'. In this setting, we define the *saturation* of M to be the $\mathbb{Z}$-graded module

$$M^{\text{sat}} = \{m \in M' \,|\, \mathfrak{p}^n m \in M \quad \text{for some } n > 0\}$$

where $\mathfrak{p}$ is the irrelevant ideal $\mathfrak{p} = \bigoplus_{j>0} S_j$. We say M is saturated if $M = M^{\text{sat}}$.

Lemma II.1.2. *Let A be a Noetherian domain, X be an integral, projective scheme over A and $\mathcal{L}$ an ample invertible sheaf over X.*

1. *The section ring S of X with respect to $\mathcal{L}$ is a finitely generated algebra over A and hence, is Noetherian. If X is normal, and $A = k$, then the section ring is also characterized as the unique normal $\mathbb{N}$-graded ring S such that Proj(S) is isomorphic to X and the corresponding $\mathcal{O}_X(1)$ is isomorphic to $\mathcal{L}$.*

2. *The section module of any torsion-free coherent sheaf over X with respect to $\mathcal{L}$ is finitely generated over S. It is also characterized as the unique saturated, torsion-free, $\mathbb{Z}$-graded S-module M (with respect to the irrelevant ideal $I = \bigoplus_{j>0} S_j$) such that the associated coherent sheaf $\tilde{M}$ on X is isomorphic to $\mathcal{F}$.*

3. *For two torsion-free coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, we have a natural isomorphism:*

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}_S^{gr}(M(\mathcal{F}, \mathcal{L}), M(\mathcal{G}, \mathcal{L}))$$

where $\operatorname{Hom}_S^{gr}(\,,\,)$ denotes the set of grading preserving S-module maps between two graded S-modules.

Proof. See [Sta, Tag 0BXF]. □

In the next lemma, we record some useful principles concerning direct summands of sheaves on a proper variety.

Lemma II.1.3. *On a proper variety X over k, let $\mathcal{L}$, $\mathcal{M}$ be invertible sheaves, and $\mathcal{F}$, $\mathcal{G}$ be coherent sheaves over X. Then,*

1. *If $\mathcal{L}$ is not a direct summand of $\mathcal{F}$ and $\mathcal{G}$, then $\mathcal{L}$ is also not a summand of $\mathcal{F} \oplus \mathcal{G}$.*
2. *If $\mathcal{F} \cong \mathcal{L}^{\oplus n} \oplus \mathcal{G}$ and $\mathcal{L}$ is not a summand of $\mathcal{G}$, then, n is the maximum number of $\mathcal{L}$ summands of $\mathcal{F}$ (in any decomposition).*
3. *Assume $\mathcal{L} \not\cong \mathcal{M}$, and both $\mathcal{L}$ and $\mathcal{M}$ are summands of $\mathcal{F}$, then $\mathcal{L} \oplus \mathcal{M}$ is a summand of $\mathcal{F}$.*

Proof. Note that giving an $\mathcal{O}_X$-summand (i.e., a summand isomorphic to $\mathcal{O}_X$) of a coherent sheaf $\mathcal{F}$ is equivalent to giving a non-zero global section $s \in H^0(X, \mathcal{F})$ and a map $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ such that $\varphi(s) \neq 0$. This is because since $\Gamma(X, \mathcal{O}_X)$ is a field, for any non-zero element $u \in \Gamma(X, \mathcal{O}_X)^\times$, multiplication by u^{-1} is an $\mathcal{O}_X$-module automorphism of $\mathcal{O}_X$. Thus, given s and φ as above with $\varphi(s) \neq 0$, we may assume that $\varphi(s) = 1$ by post-multiplying by $\varphi(s)^{-1}$. Then, φ defines a splitting of the map $\mathcal{O}_X \to \mathcal{F}$ defined by $1 \mapsto s$. Moreover, we may then write $\mathcal{F} = \mathcal{O}_X \oplus \ker(\varphi)$, with the copy of $\mathcal{O}_X$ corresponding to the $\mathcal{O}_X$-submodule of $\mathcal{F}$ generated by s.

1. By twisting by $\mathcal{L}^{-1}$, we may assume that $\mathcal{L} = \mathcal{O}_X$. An $\mathcal{O}_X$-summand of $\mathcal{F} \oplus \mathcal{G}$ is given by a global section

   $$s = (s_1, s_2) \in H^0(X, \mathcal{F} \oplus \mathcal{G}) = H^0(X, \mathcal{F}) \oplus H^0(X, \mathcal{G})$$

   and a map

   $$\varphi = (\varphi_1, \varphi_2) \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \oplus \mathcal{G}, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \oplus \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{O}_X)$$

   such that $\varphi(s) \neq 0$. However, $\varphi(s) = \varphi_1(s_1) + \varphi_2(s_2)$. So, if $\varphi(s) \neq 0$, then $\varphi_i(s_i) \neq 0$ for some $i = 1, 2$, giving an $\mathcal{O}_X$-summand of either $\mathcal{F}$ or $\mathcal{G}$, which is a contradiction.

2. Again, twisting by $\mathcal{L}^{-1}$, we may reduce to the case when $\mathcal{L} = \mathcal{O}_X$. Suppose that there is another decomposition $\mathcal{F} \cong \mathcal{O}_X^{\oplus(n+m)} \oplus \mathcal{G}'$ for some $m > 0$. Let $\varphi : \mathcal{O}_X^{\oplus(n+m)} \bigoplus \mathcal{G}' \to \mathcal{O}_X^{\oplus n} \bigoplus \mathcal{G}$ be an isomorphism. Now, consider the map $\psi : H^0(X, \mathcal{O}_X^{\oplus(n+m)}) \to H^0(X, \mathcal{O}_X^{\oplus n})$ induced by the inclusion of $\mathcal{O}_X^{\oplus(n+m)}$ into $\mathcal{F}$, the isomorphism φ and the projection onto $\mathcal{O}_X^{\oplus n}$. Since m is positive, there exists a non-zero section $s \in \mathcal{O}_X^{\oplus(n+m)}$ such that $\psi(s) = 0$.

   Now write $\varphi(s, 0) = (0, g)$ for some $g \in H^0(X, \mathcal{G})$. Note that $(s, 0)$ gives an $\mathcal{O}_X$-summand of $\mathcal{F}$. Hence, g must be an $\mathcal{O}_X$-summand of $\mathcal{G}$, which is a contradiction, since $\mathcal{G}$ was assumed to have no $\mathcal{O}_X$-summands.

3. Since $\mathcal{L}$ is a direct summand of $\mathcal{F}$, there is some $\mathcal{G}$ such that $\mathcal{F} \cong \mathcal{L} \oplus \mathcal{G}$. Now, by part (a), if $\mathcal{M}$ is a direct summand of $\mathcal{L} \oplus \mathcal{G}$, then $\mathcal{M}$ is direct summand of either $\mathcal{L}$ or $\mathcal{G}$. However, since $\mathcal{M} \ncong \mathcal{L}$, $\mathcal{M}$ must be a direct summand of $\mathcal{G}$. Hence, $\mathcal{L} \oplus \mathcal{M}$ is a direct summand of $\mathcal{F}$. $\square$

II.2: The F-signature

Let R be any ring of prime characteristic p. Then for any $e \geq 1$, let $F^e : R \to R$ sending $r \mapsto r^{p^e}$ be the e^{th}-iterate of the *Frobenius morphism*. Since R has characteristic p, F^e defines a ring homomorphism, allowing us to define a new R-modules for each $e \geq 1$ obtained via restriction of scalars along F^e. We denote this new R-module by $F^e_* R$ and its elements by $F^e_* r$ (where r is an element of R). Concretely, $F^e_* R$ is the same as R as an abelian group, but the R-module action is given by:

$$r \cdot F^e_* s := F^e_*(r^{p^e} s) \quad \text{for } r \in R \text{ and } F^e_* s \in F^e_* R.$$

Now let $(R, \mathfrak{m})$ denote a normal local domain and X denote the normal scheme $\operatorname{Spec}(R)$. Throughout, we will also assume that R is the localization of a finitely generated k-algebra at a maximal ideal, which also makes it F-finite (i.e., $F^e_* R$ is a finitely generated R-module for any $e \geq 1$). Let Δ be an effective $\mathbb{Q}$-divisor on $X = \operatorname{Spec}(R)$. Then, note that since Δ is effective, for any $e \geq 1$, we have a natural inclusion $R \subset R(\lceil (p^e - 1)\Delta \rceil)$ of reflexive R-modules. Here, $R(\lceil (p^e-1)\Delta \rceil)$ denotes the R-module corresponding to the reflexive sheaf $\mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil)$. Thus, applying $\operatorname{Hom}_R(\_, R)$ to the natural inclusion $F^e_* R \subset F^e_*(R(\lceil (p^e - 1)\Delta \rceil))$, we get

$$\operatorname{Hom}_R\left(F^e_* R(\lceil (p^e - 1)\Delta \rceil), R\right) \subset \operatorname{Hom}_R(F^e_* R, R).$$

Thus, given any element $\varphi \in \operatorname{Hom}_R\left(F^e_* R(\lceil (p^e - 1)\Delta \rceil), R\right)$, it can be naturally viewed as a map from $\varphi : F^e_* R \to R$.

Definition II.2.1 (Splitting Ideals)**.** For any $e \geq 1$, we define the subset $I^\Delta_e \subseteq R$ as

$$I^\Delta_e = \left\{ x \in R \mid \varphi(F^e_* x) \in \mathfrak{m} \text{ for every map } \varphi \in \operatorname{Hom}_R\left(F^e_* R(\lceil (p^e - 1)\Delta \rceil), R\right) \right\}.$$

We observe that I^Δ_e is an ideal of finite colength in R and we call

$$a^\Delta_e = \ell_R(R/I^\Delta_e)$$

the Δ-free rank of $F^e_* R$, where ℓ_R denotes the length as an R-module.

Definition II.2.2. [BST11, Theorem 3.11, Proposition 3.5] Let (R, Δ) be a pair as above, and $a_e^{\Delta}(R)$ denote the Δ-free rank of $F^e_* R$ (Definition II.2.1). Then the F*-signature* of (R, Δ) is defined to be the limit:

$$\mathcal{s}(R, \Delta) := \lim_{e \to \infty} \frac{a_e^{\Delta}}{p^{ed}}$$

where d is the Krull dimension of R. This limit exists by [BST11].

Remark II.2.3. For a local domain R obtained as the localization of a finitely generated k-algebra, the degree of the field extension $\operatorname{Frac}(R) \subset \operatorname{Frac}(R)^{1/p}$ is equal to p^d, where d is the transcendence degree of $\operatorname{Frac}(R)$ over k. Thus, if R was the localization of a finitely generated k-algebra at a maximal ideal, we d is also the Krull dimension of R.

Remark II.2.4. The F-signature of a local ring $(R, \mathfrak{m})$ as above is the F-signature $\mathcal{s}(R, 0)$. By [Tuc12, Proposition 4.5], the definition of the free-rank and the F-signature of R in Definition II.2.2 match the one given in Definition I.2.3.

II.2.1: The F-signature of $\mathbb{N}$-graded rings.

Though the definition of the F-signature is given for a local ring $(R, \mathfrak{m})$, we may also work with $\mathbb{N}$-graded rings $(S, \mathfrak{m}, k)$ i.e. S is $\mathbb{N}$-graded with $S_0 = k$ and $\mathfrak{m} = S_{>0}$. We next relate the local and graded situations.

Definition II.2.5 (Graded free rank)**.** Let $(S, \mathfrak{m}, k)$ be an $\mathbb{N}$-graded ring, finitely generated over k, with $S_0 = k$ and M a finitely generated $\mathbb{Z}$-graded module over S. Then we can decompose M as a graded S-module as:

$$M \cong P \oplus N$$

where P is a graded free S-module (i.e. a direct sum of $S(j)$, the shifted rank 1 free modules, for various $j \in \mathbb{Z}$) and N is a graded module with no graded free summands. Then the rank of P is independent of the chosen decomposition and we define it to be the *graded free rank* of M over S (denoted by $a_{\mathrm{gr}}(M)$).

Lemma II.2.6. *Let $(S, \mathfrak{m}, k)$ and M be as above. Then the free rank of $M_{\mathfrak{m}}$ over the local ring $S_{\mathfrak{m}}$ is the same as the graded free rank of M.*

Proof. See [DSPY22, Proposition 5.7]. □

Now, we describe the F-signature of $\mathbb{N}$-graded rings, relating it to the (local) F-signature *at the vertex*. For similar discussions relating the local and global situations, see [Smi00, Section 3], [Smi97b, Section 4], and [VK12, Section 2.2].

Let S be an $\mathbb{N}$-graded ring. Then F^e_*S is also naturally an $\frac{1}{p^e}\mathbb{N}$-graded S-module by taking

$$(F^e_*S)_{\frac{i}{p^e}} = F^e_*S_i.$$

This gives rise to the $\mathbb{N}$-grading on F^e_*S given by

$$(F^e_*S)_n = \bigoplus_{0\le i\le p^e-1} (F^e_*S)_{\frac{i+np^e}{p^e}}.$$

Thus, F^e_*S decomposes as

$$F^e_*S = \bigoplus_{0\le i\le p^e-1} \bigoplus_{j\ge 0} F^e_*S_{i+jp^e}$$

as an $\mathbb{N}$-graded S-module.

Definition II.2.7 (F-signature of $\mathbb{N}$-graded rings)**.** Let $(S, \mathfrak{m}, k)$ be an $\mathbb{N}$-graded, finitely generated k-algebra, with $S_0 = k$. Then, we define the F-signature of S to be the limit:

$$\lim_{e\to\infty} \frac{a_{e,\mathrm{gr}}(S)}{p^{ed}}$$

where $a_{e,\mathrm{gr}}(S)$ is the graded free rank of F^e_*S and d denotes the Krull dimension of S. We note that by II.2.6, the F-signature of S coincides with the F-signature of $S_{\mathfrak{m}}$, the localization of S at the maximal ideal $\mathfrak{m}$.

II.3: F-regularity:

Definition II.3.1 (Sharp F-splitting)**.** [SS10, Definition 3.1] Let X be a normal variety over k and $\Delta \ge 0$ be an effective $\mathbb{Q}$-divisor. The pair (X, Δ) is said to be *globally sharply F-split* (resp. *locally* sharply F-split) if there exists an integer $e \gg 0$, such that, the natural map

$$\mathcal{O}_X \to F^e_*\mathcal{O}_X(\lceil (p^e-1)\Delta\rceil)$$

splits (resp. splits locally) as a map of $\mathcal{O}_X$-modules. A normal variety X is said to globally F-split if the pair $(X, 0)$ is globally sharply F-split.

Definition II.3.2 (F-regularity)**.** [SS10, Definition 3.1] Let X be a normal variety over k and $\Delta \ge 0$ be an effective $\mathbb{Q}$-divisor. The pair (X, Δ) is said to be *globally F-regular* (resp.

locally strongly F-regular) if for any effective Weil divisor D on X, there exists an integer $e \gg 0$, such that, the natural map

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D)$$

splits (resp. splits locally) as a map of $\mathcal{O}_X$-modules. A normal variety X is said to globally F-regular if the pair $(X, 0)$ is globally F-regular.

Remark II.3.3. When $X = \mathrm{Spec}(R)$ is an affine variety and Δ is an effective $\mathbb{Q}$-divisor, the pair (X, Δ) being globally F-regular (resp. globally sharply F-split) is equivalent to the pair (R, Δ) being locally strongly F-regular (resp. locally sharply F-split) [SS10].

Remark II.3.4. A local ring R is strongly F-regular if and only if its F-signature $\mathcal{s}(R)$ is positive [AL03]. More generally, a pair (R, Δ) (where R is normal, local) is strongly F-regular if and only if the F-signature $\mathcal{s}(R, \Delta)$ (Definition II.2.2) is positive [BST11, Theorem 3.18].

Theorem II.3.5. *[Smi00, Theorem 3.10] Let X be a projective variety over k. Then, X is globally F-regular if and only if the section ring $S(X, L)$ (II.1.1) with respect to some (equivalently, every) ample invertible sheaf L is strongly F-regular.*

Remark II.3.6. (Locally) Strongly F-regular varieties are normal and Cohen-Macaulay. Similarly, globally F-regular varieties enjoy many of nice properties such as:

- As proved in [SS10, Theorem 4.3], they are *log-Fano type*. More precisely, there exists an effective divisor $\Delta \geq 0$ such that the pair (X, Δ) is globally F-regular and $-K_X - \Delta$ is ample.

- A version of the Kawamata-Viehweg vanishing theorem holds on all globally F-regular varieties [SS10, Theorem 6.8].

Theorem II.3.7 ([Smi00], Corollary 4.3)**.** *Let X be a projective, globally F-regular variety over k. Suppose L is a nef invertible sheaf over X. Then,*

$$H^i(X, L) = 0 \quad \textit{for all } i > 0.$$

We need a slight variation of Theorem II.3.7 for $\mathbb{Q}$-ample divisors that we prove here for completeness.

Proposition II.3.8. *Let X be a globally F-split normal variety and L be a $\mathbb{Q}$-ample Weil divisor i.e., L is an integral Weil divisor such that rL is an ample Cartier divisor for some integer $r > 0$. Then,*

$$H^i(X, \mathcal{O}_X(L)) = 0 \quad \textit{for } i > 0.$$

Proof. Let r be an integer such that rL is Cartier. Write $r = p^{e_0}s$ such that s is coprime to p. Pick an $e > 0$ such that s divides $p^e - 1$. Then, since $p^{e_0}(p^{ne}-1)$ is a multiple of r for all $n > 0$, using Serre vanishing theorem, we have
(II.3.1)

$$H^i(X, \mathcal{O}_X(p^{ne+e_0}L)) = H^i(X, \mathcal{O}_X(p^{e_0}L + (p^{ne}-1)p^{e_0}L) = 0 \quad \text{for all } i > 0 \text{ and } n \gg 0.$$

Since the map

$$\mathcal{O}_X \to F_*^{ne+e_0}\mathcal{O}_X$$

is split, twisting by $\mathcal{O}_X(L)$ and reflexifying, we get that

$$\mathcal{O}_X(L) \to F_*^{ne+e_0}\mathcal{O}_X(p^{ne+e_0}L)$$

is split as well. Now the Proposition follows from the vanishing in (II.3.1). □

The next Proposition is a technical result that helps us to restrict $\mathbb{Q}$-Cartier divisors to normal, locally complete intersection subvarieties. This is very close to [PS12, Corollary 3.3], but we will need it in the form stated below.

Proposition II.3.9. *Let $X = Spec(R)$, where $(R, \mathfrak{m})$ is an F-finite, strongly F-regular, local ring and D be an integral Weil-divisor on X such that rD is Cartier for some integer r. Then, for each $m \geq 0$,*

1. *There exists an $e \gg 0$ such that the module $R(mD) = H^0(X, \mathcal{O}_X(mD))$ is isomorphic to an R-module summand of F_*^eR. In particular, $R(mD)$ is a Cohen-Macaulay module over R.*

2. *Suppose that $x_1, \ldots, x_t$ is a regular sequence on R such that the ring $R/(x_1, \ldots, x_t)R$ is normal. Then, the sheaf $\mathcal{O}_X(mD) \otimes_R \mathcal{O}_Y$ is reflexive on $Y = Spec(R/(x_1, \ldots, x_t)R)$. Furthermore, if we assume that the support of D does not contain the subscheme Y, then natural map*

   (II.3.2) $$\mathcal{O}_X(mD) \otimes_R \mathcal{O}_Y \to \mathcal{O}_Y(mD_Y)$$

   is an isomorphism, where D_Y denotes the restriction of D to Y (see the description in the proof below).

Proof. By adding a principal divisor if necessary (which leaves the module $R(mD)$ isomorphic), we assume that D is effective.

1. Since R is strongly F-regular, there exists an $e \gg 0$ such that the map $\mathcal{O}_X \to F^e_*(\mathcal{O}_X(sD))$ splits for each $0 \leq s \leq r$. Once we have such an e, choose $s \leq r$ such that $p^e + s$ is divisible by r. Thus, we may assume that the map (got by twisting by D and reflexifying):
$$\mathcal{O}_X(D) \to F^e_*\big(\mathcal{O}_X((p^e+s)D)\big)$$
is split. Since r divides $p^e + s$, $\mathcal{O}_X((p^e+s)D)$ is isomorphic to $\mathcal{O}_X$ since R is local and rD is Cartier. Therefore, taking global sections, we have that the map
$$R(mD) \to F^e_* R \tag{II.3.3}$$
is split. Note that the e obtained is independent of m. The Cohen-Macaulayness of $R(mD)$ follows because F^e_*R is a Cohen-Macaulay module over R, since R itself is Cohen-Macaulay.

2. Firstly, we may assume $m = 1$ since the discussion holds for an arbitrary Weil divisor and is compatible with addition of Weil-divisors. Now, since Y is a normal, complete intersection subscheme of X, we may "restrict" the rank one reflexive sheaf $\mathcal{F} := \mathcal{O}_X(D)$ on X to a reflexive sheaf $\mathcal{F}_Y$ on Y as follows: Let U be the regular locus of Y. Then there is an open subset $V \subset X_{\mathrm{reg}}$ (where X_{reg} denotes the regular locus of X) such that $V \cap Y = U$. This is possible because Y is a complete intersection in X. Therefore, we may restrict $\mathcal{F}$ to V and then to an invertible sheaf on U, since $\mathcal{F}|_V$ is invertible. Define $\mathcal{F}_Y$ to be
$$\mathcal{F}_Y := i_*(\mathcal{F}|_U)$$
where $i : U \to Y$ is the inclusion. Then, $\mathcal{F}_Y$ is a rank one reflexive sheaf on Y because Y is normal and U contains all the codimension one points of Y. Thus, we can write $\mathcal{F}_Y$ as $\mathcal{O}_Y(D_Y)$ for some Weil-divisor D_Y on Y. Furthermore, if $\mathrm{Supp}(D)$ does not contain Y, then since Y is normal, hence integral, D naturally restricts to a Cartier divisor D_U on U (given by restricting the equation for D) and we may take D_Y to be the closure of D_U. It is also clear from the description of restriction that it commutes with addition of Weil-divisors (since the restriction of Cartier divisors on the regular locus commutes with addition).

   Now, since $\mathcal{F}|_U$ is the restriction of the sheaf $\mathcal{F}$ (i.e., isomorphic to $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_U$), there is a natual map
$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \mathcal{F}_Y$$
which is an isomorphism if and only if $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is reflexive. Therefore, it is suf-

ficient to show that the module $R(D) \otimes_R R/(x_1, \ldots, x_t)R$ satisfies the S_2 condition on $R/(x_1, \ldots, x_t)R$ (since $R/(x_1, \ldots, x_t)R$ is normal). But since $R(mD)$ is Cohen-Macaulay by Part (1) (and clearly full dimensional), and $x_1, \ldots, x_t$ is a regular sequence on R, we get that $R(mD) \otimes R/(x_1, \ldots, x_t)R$ is Cohen-Macaulay as well. This completes the proof of the Proposition. $\square$

CHAPTER III
The F-signature Function on the Ample Cone

In this Chapter, we will study the variation of the F-signature of a fixed projective variety X as we vary the embeddings of X. This is formalized by defining the F-signature function on the (rational) *ample cone* of X in Section III.1. In Section III.2, we prove that the F-signature function is continuous and extends to the real ample cone. We further prove that it extends continuously to the Nef cone of X in Section III.3. Finally, in Section III.4, we prove some effective local upper bounds for the F-signature function. The contents of this Chapter are part of the joint work with Seungsu Lee [LP23].

Notation III.0.1. Throughout this chapter, k will denote an algebraically closed field of positive characteristic p.

III.1: Definition of the F-signature Function

In this section, we will define an F-signature function on the rational ample cone of a globally F-regular projective variety. The *rational ample cone*, consisting of numerical equivalence classes of ample $\mathbb{Q}$-divisors on X will be denoted by $\mathrm{Amp}(X)$. Recall that in the Néron-Severi space $N^1_{\mathbb{R}}(X)$, $\mathrm{Amp}_{\mathbb{Q}}(X)$ is the set of rational points of the open cone $\mathrm{Amp}_{\mathbb{R}}(X)$ (which consists of classes of ample $\mathbb{R}$-divisors on X). Hence, $\mathrm{Amp}_{\mathbb{Q}}(X)$ has a natural topology, inherited from any norm on $\mathrm{N}^1_{\mathbb{R}}(X)$. We refer to [Laz04, Chapter 1] for the details.

Definition III.1.1. Let X be a globally F-regular projective variety over k (Definition II.3.2). Suppose that $\dim(X) > 0$. The F-signature function

$$\mathfrak{s}_X : \mathrm{Amp}_{\mathbb{Q}}(X) \to \mathbb{R}$$

on the rational ample cone of X, is defined as follows:

1. If the class $[L] \in \mathrm{Amp}_{\mathbb{Q}}(X)$ is defined by an integral Cartier divisor L, then we define $\mathfrak{s}_X([L])$ to be the F-signature (Definition II.2.2) of the section ring $S(X, L)$ (Defini-

tion II.1.1) of L:

$$\mathcal{s}_X([L]) := \mathcal{s}(S(X, L)).$$

2. If the class $[L]$ is defined by a rational multiple of an integral Cartier divisor i.e. $L = \frac{a}{b}D$ where D is an integral Cartier divisor on X, then we define:

$$\mathcal{s}_X([L]) := \frac{b}{a}\mathcal{s}_X([D]) = \frac{b}{a}\mathcal{s}(S(X, D)).$$

The rest of this section is devoted to checking that the function $\mathcal{s}$ is indeed well-defined.

Remark III.1.2. If $\dim(X) = 0$, we define the F-signature function as $\mathcal{s}_X(L) = 1$ for any ample divisor on X. Indeed, X is just a point and the only divisor on X is 0.

Theorem III.1.3. *Let X be a globally F-regular projective variety over k with $\dim(X)$ positive. Then, Definition III.1.1 gives a well-defined F-signature function $\mathcal{s}_X$ on the rational ample cone of X, satisfying the identity:*

$$\mathcal{s}_X\left(\frac{a}{b}L\right) = \frac{b}{a}\mathcal{s}_X(L)$$

for any two non-zero natural numbers a and b and any ample $\mathbb{Q}$-divisor L.

Proof. To prove the theorem, we need to check that the function $\mathcal{s}_X$ as defined in Definition III.1.1 is well-defined. There are two issues:

1. The first arising from the choice of a $\mathbb{Q}$-divisor representing a numerical equivalence class (Theorem III.1.4).

2. Having chosen a $\mathbb{Q}$-divisor L representing a numerical class, there is still ambiguity in choosing a representation of L as a rational multiple of an integral Cartier divisor (Theorem III.1.6).

We address the first ambiguity by proving that on a globally F-regular variety, numerical equivalence and linear equivalence are the same conditions. This is an analog of the same result for *log-Fano* varieties over the complex numbers, a well-known consequence of the *Kawamata-Viehweg* vanishing theorem. The following theorem may be well-known to experts, but we do not know a reference.

Theorem III.1.4. *Let X be a projective, globally F-regular variety over k. Suppose $\mathcal{L}$ is a numerically trivial invertible sheaf on X, i.e. $\deg(\mathcal{L}|_C) = 0$ for all curves C on X. Then, $\mathcal{L}$ is isomorphic to the trivial invertible sheaf $\mathcal{O}_X$.*

Proof. First, we note that by [Kle66, Ch. 2, Section 2, Corollary 1], some power $\mathcal{L}^m$ of $\mathcal{L}$ is algebraically equivalent to $\mathcal{O}_X$ i.e. $\mathcal{L}^m$ deforms to $\mathcal{O}_X$. Since the Euler-characteristic (for sheaf-cohomology) is invariant in flat families [Har77, Ch. III, Theorem 9.9], we get that

$$\chi(\mathcal{O}_X) = \chi(\mathcal{L}^m) \tag{III.1.1}$$

for some natural number m. Now, by Theorem II.3.7, since $\mathcal{O}_X$ and $\mathcal{L}^m$ are both nef invertible sheaves, we get that

$$H^i(X, \mathcal{O}_X) = H^i(X, \mathcal{L}^m) = 0 \quad \text{for all } i > 0. \tag{III.1.2}$$

Hence, we get

$$1 = h^0(X, \mathcal{O}_X) = \chi(\mathcal{O}_X) = \chi(\mathcal{L}^m) = h^0(X, \mathcal{L}^m). \tag{III.1.3}$$

Hence, we have shown that $\mathcal{L}^m$ has a non-zero global section. But, since it is also numerically trivial, it must indeed be trivial (since an effective divisor cannot be numerically trivial unless it is the zero divisor). Therefore, $\mathcal{L}^m \cong \mathcal{O}_X$.

Now, by [Kle66, Ch. 1, Section 1], we have that the function $\chi(\mathcal{L}^n)$ is a polynomial function of n (as n varies over all integers). Since $\mathcal{L}^m \cong \mathcal{O}_X$, we must have that $\chi(\mathcal{L}^n) = 1$ for all $n \in \mathbb{Z}$. But, again, since $\mathcal{L}^n$ is nef for all $n \geq 0$, by Theorem II.3.7 we have

$$h^0(X, \mathcal{L}) = \chi(\mathcal{L}) = 1. \tag{III.1.4}$$

Hence, $\mathcal{L} \cong \mathcal{O}_X$ as well because $\mathcal{L}$ is numerically trivial and has a non-zero global section. □

Remark III.1.5. It was proved in [CR17] that torsion divisors (i.e., L such that $nL \sim 0$ for some n) are themselves linearly equivalent to 0. Hence, the last part of the proof above follows from this fact, but we include a proof for the convenience of readers.

Next, we address the second kind of ambiguity in Definition III.1.1. For this, we note the following scaling property for F-signature of section rings under taking Veronese subrings, first observed in [VK12].

Theorem III.1.6. *[VK12, Theorem 2.6.2] Let X be a projective variety over k with* $\dim(X)$ *positive and $\mathcal{L}$ an ample invertible sheaf on X. Let $S(\mathcal{L})$ and $S(\mathcal{L}^n)$ denote the section rings with respect to $\mathcal{L}$ and $\mathcal{L}^n$ respectively, where n is any positive natural number. Then, we* *have the following relation between their F-signatures:*

$$\mathcal{s}(S(\mathcal{L})) = n\,\mathcal{s}(S(\mathcal{L}^n)). \tag{III.1.5}$$

This concludes the proof of Theorem III.1.3. ☐

III.2: Continuity of the F-signature Function

In this section, we prove that the F-signature function (Definition III.1.1) varies continuously on the ample cone. Throughout, we fix a globally F-regular projective variety X over k.

Theorem III.2.1. *The F-signature function is continuous at each rational class in the ample cone of X.*

In fact, much more is true: the F-signature function is *locally Lipschitz* around any real class in the ample cone $\mathrm{Amp}_{\mathbb{R}}(X)$, with respect to any norm chosen on the Néron-Severi space. More precisely, we prove:

Theorem III.2.2. *Fix any norm $\| \, \|$ on the Néron-Severi space $N^1_{\mathbb{R}}(X)$ of a projective globally F-regular variety X. Then for each real class $D \in \mathrm{Amp}_{\mathbb{R}}(X)$, there exist positive real numbers $C(D)$ and $r(D)$ (depending only on D and the norm $\| \, \|$), such that for any two ample $\mathbb{Q}$-divisors L, L' contained in the ball $B_{r(D)}(D) := \{D' \in \mathrm{Amp}_{\mathbb{R}}(X) \mid \|D - D'\| < r(D)\}$, we have*

$$|\mathcal{s}_X(L) - \mathcal{s}_X(L')| \leq C(D)\|L - L'\|. \tag{III.2.1}$$

We will say that the F-signature function $\mathcal{s}_X$ is locally Lipschitz at a real class D with Lipschitz constant $C(D)$ if the inequality (III.2.1) is satisfied for all ample $\mathbb{Q}$-divisors L, L' that are sufficiently close to D.

As an immediate corollary of Theorem III.2.2, we obtain:

Corollary III.2.3. *Let X be a projective variety over k with $\dim(X)$ positive. Then, the F-signature function $\mathcal{s}_X$ extends to a well-defined, continuous, locally Lipschitz function on the real ample cone $\mathrm{Amp}_{\mathbb{R}}(X)$ of X satisfying the identity:*

$$\mathcal{s}_X(\lambda L) = \frac{1}{\lambda}\mathcal{s}_X(L) \quad \text{for all } \lambda \in \mathbb{R}_{>0} \text{ and all } L \in \mathrm{Amp}_{\mathbb{R}}(X).$$

Proof. Let $D \in \mathrm{Amp}_{\mathbb{R}}(X)$ be a real ample class on X. The Lipschitz inequality (III.2.1) implies that for any sequence of ample $\mathbb{Q}$-divisors L_n converging to D, the sequence $\mathcal{s}_X(L_n)$

is Cauchy, hence converges to a unique real number. This gives a well-defined extension of $\mathcal{s}_X$ to the real ample cone $\mathrm{Amp}_{\mathbb{R}}(X)$, that remains locally Lipschitz. Hence, $\mathcal{s}$ is continuous on $\mathrm{Amp}_{\mathbb{R}}(X)$. Finally, the identity $\mathcal{s}_X(\lambda L) = \frac{1}{\lambda}\mathcal{s}_X(L)$ follows by continuity, since it already holds for all rational L and λ. □

III.2.1: Informal sketch of the proof of Theorem III.2.2:

The proof of Theorem III.2.2 consists of several steps. We summarize the ideas in this subsection.

Step 1: First, in Lemma III.2.6, we prove a formula for calculating the F-signature of an ample Cartier divisor L, in terms of *Frobenius splittings* of the linear systems $|mL|$ for $m \gg 0$. This gives us a tool to compare $\mathcal{s}_X(L)$ and $\mathcal{s}_X(L')$ whenever we have a non-zero map $\mathcal{O}_X(mL) \to \mathcal{O}_X(mL')$ for $m \gg 0$ (Lemma III.2.11).

Step 2: Given two ample $\mathbb{Q}$-divisors L and L', we first consider the case when $L' - L$ is big. Since $L' - L$ is big, for $m \gg 0$, we have $|mL' - mL| \neq \emptyset$ allowing us to compare $\mathcal{s}_X(L)$ and $\mathcal{s}_X(L')$ (Lemma III.2.11). Further, we may find a constant α such that $\alpha L - L'$ is big as well. This allows us a reverse comparison between $\mathcal{s}_X(\alpha L)$ and $\mathcal{s}_X(L')$. (Lemma III.2.12).

Step 3: In this step, we estimate the difference in the F-signatures by comparing it to the difference in volumes. Here, we encounter the key difficulty, which is that we don't know the sign of the difference between $\mathcal{s}_X(L')$ and $\mathcal{s}_X(L)$, even if $L' - L$ is effective, which we have already assumed. This is overcome by introducing the difference between $\mathcal{s}_X(L)$ and $\mathcal{s}_X(\alpha L)$ (where α is as in Step 2), along with comparisons to the volume function to estimate the difference between $\mathcal{s}_X(L)$ and $\mathcal{s}_X(L')$. These estimates are the contents of Lemma III.2.13 and Lemma III.2.14.

Step 4: To control the difference in the volumes (from Step 4), we need an additional ingredient: For any $e \geq 1$, we need effective bounds for the degrees m that contribute Frobenius splittings to the e^{th} free-rank for $S(X, L)$ and $S(X, L')$ (Theorem III.2.8).

Step 5: The steps so far give us an inequality of the form

$$|\mathcal{s}_X(L) - \mathcal{s}_X(L')| \leq C(L)\|L - L'\| \tag{III.2.2}$$

for a fixed L and all L' sufficiently close to L and for some constant $C(L)$ depending on L (Lemma III.2.10). One result required here is the (Lipschitz) continuity of the volume function on the ample cone (Lemma III.2.16).

Step 6: Though (III.2.2) proves continuity of $\mathfrak{s}_X$ at a fixed $\mathbb{Q}$-divisor L, it does not prove that $\mathfrak{s}_X$ is locally Lipschitz, since the constant $C(L)$ depends on L. So, in Proposition III.2.17 and Lemma III.2.18, we track the constant $C(L)$ and examine the variation with L. This involves carefully choosing the scalar α from Step 2.

Step 7: As a result, we see that we may pick the constants $C(L)$ such that $C(L) = o(\frac{1}{\|L\|^2})$ as $\|L\| \to \infty$. Now, since $\mathfrak{s}_X(rL) = \frac{1}{r}\mathfrak{s}_X(L)$ by Theorem III.1.3, we see that for a $\mathbb{Q}$-divisor L, we may pick $C(L) = r^2C(rL)$ for any $r \gg 0$. This shows that we may pick uniform Lipschitz constants on compact subsets of the ample cone.

Step 8: Given two ample $\mathbb{Q}$-divisors L and L', we may consider a small (and controlled) perturbation $\lambda L'$ of L' (i.e. $\lambda \approx 1$) so that $\lambda L' - L$ is big (or even ample). Using the transformation rule as in Theorem III.1.3, we may replace L' by $\lambda L'$ and reduce to the case when $L' - L$ is big, concluding the proof.

III.2.2: Proof of Theorem III.2.2

The rest of this section is dedicated to a detailed proof of Theorem III.2.2. Note that if X is 0-dimensional, then the only ample divisor on X is $\mathcal{O}_X$ and the Theorem is trivially true. Hence, we assume for the rest of this section that $\dim X$ is positive.

Notation III.2.4. For any Cartier divisor D, we use the notation $H^0(D)$ to denote the space of global sections $H^0(X, \mathcal{O}_X(D))$.

Definition III.2.5. For any Cartier divisor D on X, define the k-vector subspace $I_e(D)$ of $H^0(D)$ as follows:

$$I_e(D) := \{f \in H^0(D) \mid \varphi(F^e_* f) = 0 \text{ for all } \varphi \in \operatorname{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D), \mathcal{O}_X)\}.$$

That is, $I_e(D)$ is the set of global sections f of $\mathcal{O}_X(D)$ such the map $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$ sending $1 \mapsto F^e_* f$ does not split. A section $f \in H^0(D)$ that is not contained in $I_e(D)$ along with a map $\varphi : F^e_*\mathcal{O}_X(D) \to \mathcal{O}_X$ sending $F^e_* f$ to 1 is called an e^{th}-*Frobenius splitting* of the linear system $|D|$.

Lemma III.2.6. *Let L be an ample Cartier divisor and S denote the section ring of X with respect to L. Then, for any $e \geq 1$, if $a_e(L)$ denotes the free-rank of $F^e_* S$ as an S-module, then $a_e(L)$ is computed by the following formula:*

$$a_e(L) = \sum_{m=0}^{\infty} \dim_k \frac{H^0(mL)}{I_e(mL)}. \tag{III.2.3}$$

Hence, the F-signature of L can be computed as

$$\mathcal{s}_X(L) = \lim_{e\to\infty} \frac{\sum_{m=0}^{\infty} \dim_k \frac{H^0(mL)}{I_e(mL)}}{p^{e(\dim(X)+1)}}$$

Proof. Let $\mathcal{L}$ denote the invertible sheaf $\mathcal{O}_X(L)$. We note that the S-module $F_*^e S$ naturally decomposes as an $\mathbb{N}$-graded module as (see the discussion preceding Definition II.2.7):

$$F_*^e S = \bigoplus_{n=0}^{p^e-1} M_{e,n},$$

where $M_{e,n} := \bigoplus_{i\geq 0} H^0(X, \mathcal{L}^i \otimes F_*^e \mathcal{L}^n)$ is naturally an $\mathbb{N}$-graded S-module. Note also that since $H^0(X, \mathcal{L}^{n+ip^e}) = 0$ for $i < 0$ and $0 \leq n \leq p^e - 1$, the module $M_{e,n}$ is the section module of the sheaf $F_*^e \mathcal{L}^n$ with respect to $\mathcal{L}$. We recall that by Lemma II.2.6, $a_e(L)$ can be calculated as the graded free-rank of $F_*^e S$ i.e.

$$a_e(L) = \max\{r \mid F_*^e S \cong \bigoplus_{t=1}^{r} S(-j_t) \bigoplus N \text{ as graded } S\text{-modules}$$
$$\text{for some } j_t \in \mathbb{Z} \text{ and some graded } S\text{-module } N\}.$$

Since $F_*^e S$ is $\mathbb{N}$-graded, we note that each integer j_t occurring in any decomposition of $F_*^e S$ as above is non-negative. Sheaf theoretically, we have an equivalent description (see [Smi00, Theorem 3.10] and the proof):

$$a_e(L) = \max\{r \mid \widetilde{F_*^e S} \cong \bigoplus_{0\leq n\leq p^e-1} F_*^e \mathcal{L}^n \cong \bigoplus_{t=1}^{r} \mathcal{L}^{-j_t} \bigoplus \mathcal{N} \tag{III.2.4}$$
$$\text{as } \mathcal{O}_X\text{-modules for some } j_t \in \mathbb{N} \text{ and some sheaf } \mathcal{N}\}$$

Now, for any $0 \leq n \leq p^e - 1$, and $j \geq 0$, the maximum number of $\mathcal{L}^{-j}$ summands of $F_*^e \mathcal{L}^n$ is the same as the maximum number of $\mathcal{O}_X$-summands of $F_*^e \mathcal{L}^{n+jp^e}$. Writing $F_*^e \mathcal{L}^n \cong \mathcal{O}_X^{\oplus n} \oplus \mathcal{G}$ such that $\mathcal{G}$ does not have any $\mathcal{O}_X$-summands, we see that the set $I_e(\mathcal{L}^n)$ can be identified with the set $H^0(X, \mathcal{G})$. Hence, the maximum number of $\mathcal{O}_X$-summands of any $F_*^e \mathcal{L}^m$ is exactly given by the dimension of $H^0(mL)/I_e(mL)$ (see Lemma II.1.3, part (b)). Using Lemma II.1.3 again, running over all $0 \leq n \leq p^e - 1$ and $j \geq 0$, we get the desired formula (IV.3.7) for $a_e(L)$. □

Remark III.2.7. Since the free-rank of $F_*^e S$ is bounded by its generic rank (which is exactly $p^{e(\dim(X)+1)}$), the sum in equation (IV.3.7) is indeed finite. Next, we will find uniform bounds for the number of terms in this sum.

Theorem III.2.8. *Let X be a globally F-regular projective variety over k. Fix a norm $\| \|$ on the Néron-Severi space $N^1_{\mathbb{R}}(X)$. There exists a constant $C_1 := C_1(X)$ (depending only on X, and the norm $\| \|$) such that, whenever L and H are any two effective Cartier divisors on X, we have:*

1. $$I_e(mL) = H^0(mL) \text{ for } m > \frac{C_1}{\|L\|}p^e, \quad \text{and,}$$

2. *For all $n > \frac{2\|H\|}{\|L\|}$,*
$$I_e(m(nL+H)) = H^0(m(nL+H)) \text{ for all } m > \frac{C_1 p^e}{n\|L\|}.$$

Proof. Since X is normal, we can consider the canonical (Weil) divisor on X (denoted by K_X), by extending the canonical divisor on the non-singular locus of X. Choosing an ample divisor A such that $A + K_X$ is effective, we may write $A \sim -K_X + E$ for some effective (Weil) divisor E. Let $[A]$ denote the class of A in the ample cone of X.

Let L be any effective Cartier divisor on X. By applying duality for the Frobenius map, we have,
$$\mathscr{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(mL), \mathcal{O}_X) \cong F^e_*\mathcal{O}_X(-(p^e-1)K_X - mL).$$
See [SS10, Section 4.1] for a detailed discussion regarding duality for the Frobenius map. Hence, we have,

$$\mathrm{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(mL), \mathcal{O}_X) \cong F^e_* H^0(X, \mathcal{O}_X(-(p^e-1)K_X - mL)). \tag{III.2.5}$$

This shows that to prove that $H^0(mL) = I_e(mL)$ for any given m, it suffices to show the right hand side in Equation (III.2.5) is zero.

Claim: There exists a positive constant C_1' (depending only on X, the choice of A and the norm $\| \|$), such that for any effective divisor D with $\|D\| > C_1'$, we have $-K_X - D$ is not an effective divisor, i.e., $-K_X - D$ is not $\mathbb{R}$-linearly equivalent to any effective divisor.

Proof of the claim. Recall that the pseudoeffective cone (denoted by $\overline{\mathrm{Eff}}(X)$) is a closed strongly convex cone (i.e. there is no non-zero class $\nu \in \overline{\mathrm{Eff}}(X)$ such that $-\nu \in \overline{\mathrm{Eff}}(X)$), and contains the class of every effective divisor on X [Laz04, Definition 2.2.25]. Thus, the set
$$\kappa := \overline{\mathrm{Eff}}(X) \bigcap ([A] - \overline{\mathrm{Eff}}(X))$$

is a compact subset of $\overline{\mathrm{Eff}}(X)$. Since the norm function achieves a maximum on κ, we may choose C_1' to be bigger than the norm of any class in κ:

$$C_1' > \max\{\|\xi\| \,|\, \xi \in \kappa\}.$$

Note that C_1' depends only on the choice of A and the norm $\|\ \|$.

Since the class of every effective Cartier divisor on X is contained in the pseudoeffective cone of X, if D is an effective divisor with $\|D\| > C_1'$, then D can not belong to κ by the definition of C_1'. Hence, we see that $A - D$ is not effective. Since $A = -K_X + E$ for an effective divisor E, this means that $-K_X - D$ is not effective. This proves the claim. □

Continuation of the proof of Theorem III.2.8: For any effective Cartier divisor L, if $m > \frac{C_1' p^e}{\|L\|}$, we have $\|\frac{m}{p^e-1}L\| > C_1'$, hence, applying the claim above, we conclude that $-K_X - \frac{m}{p^e-1}L$ is not effective. Therefore, the divisor

$$-(p^e-1)K_X - mL$$

is not effective. By (III.2.5), this gives us $H^0(mL) = I_e(mL)$ as required. This proves part (a).

For part (b), we use part (a) of the Theorem by replacing L by $nL + H$, which gives us that $H^0(m(nL+H)) = I_e(m(nL+H))$ for $m > \frac{C_1' p^e}{\|nL+H\|}$. Since by assumption $\|H\| \leq \frac{1}{2}\|nL\|$, we have, $\|nL + H\| \geq \|nL\| - \|H\| \geq \frac{1}{2}\|nL\|$. Therefore,

$$\frac{2C_1' p^e}{n\|L\|} \geq \frac{C_1' p^e}{\|nL + H\|},$$

using which we see that $C_1 = 2C_1'$ works for both parts (a) and (b). This completes the proof of Theorem III.2.8. □

Remark III.2.9. For a more effective, but less uniform version of Theorem III.2.8, see Lemma III.4.2.

Next, we prove Theorem III.2.2 in the special case when the divisor L is fixed and the difference $L' - L$ is big.

Lemma III.2.10 (Key Lemma)**.** *Let L be an integral ample divisor on X. Then, there exists a constant $C(L)$ (depending only on L and the norm $\|\ \|$) such that for any other ample $\mathbb{Q}$-divisor L' sufficiently close to L, and for which $L' - L$ is big, we have:*

$$|s_X(L) - s_X(L')| \leq C(L)\|L - L'\|.$$

Proof of Lemma III.2.10. Throughout the proof, we fix the following set-up: Fixing the ample, integral divisor L on X, we pick an arbitrary ample $\mathbb{Q}$-divisor L' such that $L' - L$ is big. Then, we may write $L' = L + \frac{1}{n}H$, for some $n \gg 0$ and an effective and big Cartier divisor H.

Lemma III.2.11. *For effective divisors D_1 and D_2, consider the natural inclusion:*

$$\phi : F^e_* \mathcal{O}_X(D_1) \subset F^e_* \mathcal{O}_X(D_1 + D_2).$$

Then,

$$\phi(I_e(D_1)) \subset I_e(D_1 + D_2). \tag{III.2.6}$$

Equivalently, viewing $H^0(X, \mathcal{O}_X(D_1))$ as a subset of $H^0(X, \mathcal{O}_X(D_1 + D_2))$ through the map ϕ, we have:

$$\phi(I_e(D_1)) \subset I_e(D_1 + D_2) \cap H^0(D_1) = \{x \in H^0(D_1) | \phi(x) \in I_e(D_1 + D_2)\}. \tag{III.2.7}$$

Proof. This follows from the definitions once we observe that for every map φ in $\mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X(D_1 + D_2), \mathcal{O}_X)$, we get a map $\tilde{\varphi}$ in $\mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X(D_1), \mathcal{O}_X)$ by pre-composing with the map ϕ.

$$\begin{tikzcd}
F^e_* \mathcal{O}_X(D_1) \arrow[r, "\phi"] \arrow[dr, "\tilde{\varphi}"'] & F^e_* \mathcal{O}_X(D_1 + D_2) \arrow[d, "\varphi"] \\
& \mathcal{O}_X
\end{tikzcd}$$

□

Lemma III.2.12. *With L an ample Cartier divisor and H an effective big divisor on X, and any natural number n, suppose that we have a natural number $b := b(n)$, such that $nL - bH$ is big. Consequently, by [Laz04, Corollary 2.2.10], there is a $C_2 \gg 0$ such that for all $m \geq C_2$, we have*

$$H^0(m(nL - bH)) \neq 0.$$

Then, for $m \geq C_2$ and all $e \geq 1$, there is a factorization of inclusions:

$$\begin{tikzcd}
F^e_* \mathcal{O}_X(mnbL) \arrow[r, "\cdot F^e_* mbH"] \arrow[dr, "\cdot F^e_* c"'] & F^e_* \mathcal{O}_X(mb(nL + H)) \arrow[d, "\cdot F^e_* d"] \\
& F^e_* \mathcal{O}_X(mn(b+1)L)
\end{tikzcd}$$

given by a choice of a section $d \in H^0(X, \mathcal{O}_X(mnL - mbH))$.

Proof. Given a section $d \in H^0(X, \mathcal{O}_X(mnL - mbH))$, let D_1 be the corresponding effective divisor. Then, we have $D_2 = mbH + D_1 \sim mbH + mnL - mbH = mnL$. Then, we get inclusions

$$\begin{tikzcd}
\mathcal{O}_X(mnbL) \arrow[r] \arrow[dr] & \mathcal{O}_X(mb(nL+H)) \arrow[d] \\
& \mathcal{O}_X(mnbL + D_2)
\end{tikzcd}$$

since D_1 was effective. We get the required factorization by applying F^e_* to the above diagram and taking c to be the section corresponding to the divisor D_2. $\square$

Lemma III.2.13. *Let $d = \dim X$ and let $C_1 = C_1(X)$ be the constant as obtained in Theorem III.2.8. Fix an ample Cartier divisor L and an effective Cartier divisor H on X. Let n and $b := b(n)$ be positive integers such that $n > \frac{2\|H\|}{\|L\|}$ and that $nL - bH$ is big. Then, we have the following inequality:*

$$\text{(III.2.8)} \quad \begin{aligned} |s_X(L) - s_X(L + \frac{1}{n}H)| \leq \quad & \frac{C_1^{d+1}}{\|L\|^{d+1}(d+1)!}\Bigg(2\operatorname{vol}(L)\,\frac{\left((b+1)^d - b^d\right)}{b^d} \\ & + \left(\operatorname{vol}(L + \frac{1}{n}H) - \operatorname{vol}(L)\right)\Bigg) + 2\,\frac{s_X(L)}{b+1} \end{aligned}$$

Proof. First, fixing n and b, there is a $C_2 \gg 0$ such that $H^0(m(nL - bH)) \neq 0$ for all $m \geq C_2$. Using Lemma III.2.6 and Theorem III.2.8, we have the following formulas for the F-signatures:

$$\text{(III.2.9)} \qquad s_X(nbL) = \lim_{e\to\infty} \frac{1}{p^{e(d+1)}} \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mnbL)}{I_e(mnbL)}$$

and similarly,

$$\text{(III.2.10)} \qquad s_X(b(nL+H)) = \lim_{e\to\infty} \frac{1}{p^{e(d+1)}} \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mb(nL+H))}{I_e(mb(nL+H))}.$$

Note that even though the formula from Lemma III.2.6 requires us to begin the sums (III.2.9) and (III.2.10) at $m = 0$, we may begin the sums at C_2 since changing finitely many terms does not alter the limit.

According to formulas (III.2.9) and (III.2.10), to compare $\jmath_X(nbL)$ with $\jmath_X(b(nL+H))$, we need to understand the difference

$$\dim_k \frac{H^0(mbnL)}{I_e(mbnL)} - \dim_k \frac{H^0(mb(nL+H))}{I_e(mb(nL+H))}.$$

We have an inclusion

$$\frac{H^0(mbnL)}{H^0(mbnL)\cap I_e(mb(nL+H))} \hookrightarrow \frac{H^0(mb(nL+H))}{I_e(mb(nL+H))} \tag{III.2.11}$$

coming from the inclusion of $H^0(mbnL) \hookrightarrow H^0(mb(nL+H))$.

Let $J_e(mbnL) = H^0(mbnL)\cap I_e(mb(nL+H))$. Then using (III.2.11), we have:

$$\dim_k \frac{H^0(mb(nL+H))}{I_e(mb(nL+H))} = \dim_k \frac{H^0(mbnL)}{J_e(mbnL)} + \dim_k \frac{H^0(mb(nL+H))}{H^0(mnbL)+I_e(mb(nL+H))}. \tag{III.2.12}$$

Then, using (III.2.12) and the triangle inequality, we get

$$\begin{aligned}
&\left|\sum_{m=C_2}^{\frac{C_1p^e}{\|L\|nb}} \dim_k \frac{H^0(mbnL)}{I_e(mbnL)} - \sum_{m=C_2}^{\frac{C_1p^e}{\|L\|nb}} \dim_k \frac{H^0(mb(nL+H))}{I_e(mb(nL+H))}\right| \\
&\left|\sum_{m=C_2}^{\frac{C_1p^e}{\|L\|nb}} \dim_k \frac{H^0(mbnL)}{I_e(mbnL)} - \sum_{m=C_2}^{\frac{C_1p^e}{\|L\|nb}} \left(\dim_k \frac{H^0(mbnL)}{J_e(mbnL)} + \dim_k \frac{H^0(mb(nL+H))}{H^0(mnbL)+I_e(mb(nL+H))}\right)\right| \\
\leq &\left|\sum_{m=C_2}^{\frac{C_1p^e}{\|L\|nb}} \dim_k \frac{H^0(mbnL)}{I_e(mbnL)} - \sum_{m=C_2}^{\frac{C_1p^e}{\|L\|nb}} \dim_k \frac{H^0(mbnL)}{J_e(mbnL)}\right| + \sum_{m=C_2}^{\frac{C_1p^e}{\|L\|nb}} \dim_k \frac{H^0(mb(nL+H))}{H^0(mnbL)} \\
\leq &\left|\sum_{m=C_2}^{\frac{C_1p^e}{\|L\|nb}} \dim_k \frac{H^0(mbnL)}{I_e(mbnL)} - \sum_{m=C_2}^{\frac{C_1p^e}{\|L\|nb}} \dim_k \frac{H^0(m(b+1)nL)}{I_e(m(b+1)nL)}\right| \\
+ &\left|\sum_{m=C_2}^{\frac{C_1p^e}{\|L\|nb}} \dim_k \frac{H^0(m(b+1)nL)}{I_e(m(b+1)nL)} - \sum_{m=C_2}^{\frac{C_1p^e}{\|L\|nb}} \dim_k \frac{H^0(mbnL)}{J_e(mbnL)}\right| + \sum_{m=C_2}^{\frac{C_1p^e}{\|L\|nb}} \dim_k \frac{H^0(mb(nL+H))}{H^0(mnbL)}
\end{aligned} \tag{III.2.13}$$

where in the last inequality, we use the triangle inequality again after adding and subtracting the term $\sum_{m=C_2}^{\frac{C_1p^e}{\|L\|nb}} \dim_k \frac{H^0(m(b+1)nL)}{I_e(m(b+1)nL)}$.

To proceed, we need to understand the difference between the spaces $\frac{H^0(m(b+1)nL)}{I_e(m(b+1)nL)}$ and $\frac{H^0(mbnL)}{J_e(mbnL)}$. To this end, we prove the following:

Lemma III.2.14. *Suppose, as in Lemma III.2.12, b is such that $nL - bH$ is big and C_2 is such that for all $m \geq C_2$, we have $H^0(m(nL - bH)) \neq 0$. Then, for $m \geq C_2$ and all $e \geq 1$, choosing a non-zero global section $d \in H^0(mnL - mbH)$ and setting $c = d \otimes h^{mb}$, where h is the section of $\mathcal{O}_X(H)$ that corresponds to the rational function 1, we have the inclusions*

$$I_e(mnbL) \subset J_e(mnbL) \subset \{x \in H^0(mnbL) \mid cx \in I_e(mn(b+1)L)\}. \tag{III.2.14}$$

Moreover, we have the following inequality (with C_1 being the constant from Theorem III.2.8):

(III.2.15)
$$\begin{aligned}
&\left| \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mn(b+1)L)}{I_e(mn(b+1)L)} - \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mnbL)}{J_e(mnbL)} \right| \\
&\leq \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} 2 \dim_k \frac{H^0(mn(b+1)L)}{cH^0(mnbL)} + \left| \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mnbL)}{I_e(mnbL)} - \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mn(b+1)L)}{I_e(mn(b+1)L)} \right|
\end{aligned}$$

Before proving Lemma III.2.14, we note that putting (III.2.15) together with (III.2.13), we obtain:

$$\begin{aligned}
&\left| \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mbnL)}{I_e(mbnL)} - \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mb(nL+H))}{I_e(mb(nL+H))} \right| \\
&\leq \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} 2 \dim_k \frac{H^0(mn(b+1)L)}{H^0(mnbL)} + \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mb(nL+H))}{H^0(mnbL)} \\
&+2 \left| \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mnbL)}{I_e(mnbL)} - \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mn(b+1)L)}{I_e(mn(b+1)L)} \right|.
\end{aligned}$$

Thus, we get

$$
\begin{aligned}
&|\jmath_X(nbL) - \jmath_X(b(nL+H))| = \\
&\lim_{e\to\infty} \frac{1}{p^{e(d+1)}} \left| \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mbnL)}{I_e(mbnL)} - \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mb(nL+H))}{I_e(mb(nL+H))} \right| \\
&\leq \lim_{e\to\infty} \frac{1}{p^{e(d+1)}} \left(\sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} 2\dim_k \frac{H^0(mn(b+1)L)}{H^0(mnbL)} + \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mb(nL+H))}{H^0(mnbL)} \right. \\
&\left. + 2 \left| \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mnbL)}{I_e(mnbL)} - \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mn(b+1)L)}{I_e(mn(b+1)L)} \right| \right) \\
&\leq \lim_{e\to\infty} \frac{C_1^{d+1} p^{e(d+1)}}{\|L\|^{d+1} n^{d+1} b^{d+1} p^{e(d+1)} (d+1)!} \left(2n^d \operatorname{vol}(L) \left((b+1)^d - b^d\right) \right. \\
&\left. + b^d n^d \left(\operatorname{vol}(L + \frac{1}{n}H) - \operatorname{vol}(L)\right) \right) + 2\, |\jmath_X(nbL) - \jmath_X(n(b+1)L)| \\
&= \frac{C_1^{d+1} n^d b^d}{\|L\|^{d+1} n^{d+1} b^{d+1} (d+1)!} \left(2\operatorname{vol}(L) \big(\frac{(b+1)^d - b^d}{b^d}\big) + \left(\operatorname{vol}(L + \frac{1}{n}H) - \operatorname{vol}(L)\right) \right) \\
&+ |\jmath_X(nbL) - \jmath_X(n(b+1)L)|
\end{aligned}
$$

Finally, using the scaling property for $\jmath_X$ (Theorem III.1.3), we get:

$$
\begin{aligned}
&\left| \jmath_X(L) - \jmath_X(L + \frac{1}{n}H) \right| \\
&= nb\, |\jmath_X(nbL) - \jmath_X(b(nL+H))| \\
&\leq \frac{C_1^{d+1}}{\|L\|^{d+1} (d+1)!} \left(2\operatorname{vol}(L) \frac{\left((b+1)^d - b^d\right)}{b^d} + \left(\operatorname{vol}(L + \frac{1}{n}H) - \operatorname{vol}(L)\right) \right) \\
&+ 2\, nb \left| \frac{\jmath_X(L)}{nb} - \frac{\jmath_X(L)}{n(b+1)} \right| \\
&= \frac{C_1^{d+1}}{\|L\|^{d+1} (d+1)!} \left(2\operatorname{vol}(L) \frac{\left((b+1)^d - b^d\right)}{b^d} + \left(\operatorname{vol}(L + \frac{1}{n}H) - \operatorname{vol}(L)\right) \right) + 2\, \frac{\jmath_X(L)}{b+1}
\end{aligned}
$$

This completes the proof of Lemma III.2.13, pending the proof of Lemma III.2.14, which we prove next. □

Notation III.2.15. Recall that L and H are fixed integral Cartier divisors, with L ample and H effective and $n \geq 1$ is any natural number. For any natural number $k \in \mathbb{N}$, we define:

$$I_e(k) := I_e(kL),$$

$$J_e(kn) := H^0(knL) \cap I_e(k(nL+H)),$$

where we view $H^0(knL)$ as a subspace of $H^0(k(nL+H))$ via the inclusion map $\mathcal{O}_X(nkL) \subset \mathcal{O}_X(knL+kH)$.

Proof of Lemma III.2.14. The first inclusion in (III.2.14) follows from Lemma III.2.11 by taking $D_1 = mnbL$ and $D_2 = mnbH$. The second inclusion follows from Lemma III.2.12 and the second part of Lemma III.2.11, by taking $D_1 = mb(nL+H)$ and D_2 to be the effective divisor corresponding to $d \in H^0(mnL - mbH)$. Hence, we get

$$
\begin{aligned}
&\left| \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mn(b+1)L)}{I_e(mn(b+1))} - \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mnbL)}{J_e(mnb)} \right| \\
&= \left| \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mn(b+1)L)}{cH^0(mnbL)} - \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{I_e(mn(b+1))}{cJ_e(mnb)} \right| \quad \text{(rearranging terms)} \\
&\leq \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mn(b+1)L)}{cH^0(mnbL)} + \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{I_e(mn(b+1))}{cJ_e(mnb)} \quad \text{(triangle inequality)} \\
&\leq \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mn(b+1)L)}{cH^0(mnbL)} \\
&\quad + \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{I_e(mn(b+1))}{cI_e(mnb)} \quad \text{(since } cI_e(mnb) \subset cJ_e(mnb) \text{ by (III.2.14))} \\
&= \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mn(b+1)L)}{cH^0(mnbL)} + \left| \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mn(b+1)L)}{cI_e(mnb)} - \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mn(b+1)L)}{I_e(mn(b+1))} \right| \\
&= \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mn(b+1)L)}{cH^0(mnbL)} \\
&\quad + \left| \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \left(\dim_k \frac{H^0(mn(b+1)L)}{cH^0(mnbL)} + \dim_k \frac{H^0(mnbL)}{I_e(mnb)} - \dim_k \frac{H^0(mn(b+1)L)}{I_e(mn(b+1))} \right) \right|
\end{aligned}
$$

$$\leq \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} 2\dim_k \frac{H^0(mn(b+1)L)}{cH^0(mnbL)} + \left| \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mnbL)}{I_e(mnb)} - \sum_{m=C_2}^{\frac{C_1 p^e}{\|L\| nb}} \dim_k \frac{H^0(mn(b+1)L)}{I_e(mn(b+1))} \right|$$

where in the second-last step, we rearrange the terms of the sum, and in the last step use the triangle inequality again. This completes the proof of the lemma. $\square$

To complete the proof of Lemma III.2.10, we need the following lemma about the Lipschitz continuity of the volume function. We record a quick proof for ample classes that works for any algebraically closed field, and in any characteristic:

Lemma III.2.16. *[Laz04, Theorem 2.2.44] Let X be a projective variety of dimension d over k. Fix a norm $\| \ \|$ on the real Néron-Severi space. Then, there exists a positive constant $C > 0$ such that for any two real* ample *classes ξ and ξ', we have:*

$$|\mathrm{vol}(\xi) - \mathrm{vol}(\xi')| \leq C \max(\|\xi\|, \|\xi'\|)^{d-1} \|\xi - \xi'\|.$$

Proof. Since the volume function coincides with the intersection form on the real Nef cone, it is given by a polynomial P of degree d once we choose a basis for $N^1_{\mathbb{R}}(X)$. Hence, there exists a constant C (depending only on X), such that

$$\|P'(x_1, \ldots, x_\rho)\| \leq C \|(x_1, \ldots, x_\rho)\|^{d-1}$$

for any vector $(x_1, \ldots, x_\rho) \in \mathrm{Nef}_{\mathbb{R}}(X)$. With this observation, the Lemma follows from an application of the mean-value theorem.

$\square$

***Completion of the proof of Lemma III.2.10*:** Recall that L is a fixed ample divisor on X (in particular, L is big). Suppose L' is an ample $\mathbb{Q}$-divisor such that $L' - L$ is big. Further assume that $\|L' - L\| < \frac{\|L\|}{2}$. Then, we may write $L' = L + \frac{1}{n} H$ for a suitable effective Cartier divisor H and some natural number $n \geq 1$.

We would like to apply Lemma III.2.13 to this choice of L, H and n. For this, we need to choose a natural number b such that $nL - bH$ is big. We note that we may choose b in the following way: Since L is big, by openness of the big cone of X, there exists a constant $C_4 > 0$ (depending only on L) such that any $\mathbb{Q}$-divisor D satisfying $\|L - D\| \leq C_4$ is also big. Since we need $L - \frac{b}{n} H$ to be big, it is sufficient that $\|\frac{b}{n} H\| \leq C_4$. So we may choose $b(n) = \lfloor \frac{nC_4}{\|H\|} \rfloor$ so that $b(n) \to \infty$ as $n \to \infty$.

Now, applying Lemma III.2.13 to this choice of n and b, we get:

$$
\begin{aligned}
|\jmath_X(L) - \jmath_X(L + \frac{1}{n}H)| \leq \quad & \frac{C_1^{d+1}}{\|L\|^{d+1}(d+1)!} \Bigg(2\operatorname{vol}(L)\, \frac{\left((b+1)^d - b^d\right)}{b^d} \\
& + \left(\operatorname{vol}(L + \frac{1}{n}H) - \operatorname{vol}(L)\right) \Bigg) + 2\,\frac{\jmath_X(L)}{b+1}
\end{aligned} \tag{III.2.16}
$$

Further, we have

$$\frac{(b+1)^d - b^d}{b^d} \leq \frac{2^d}{b}$$

and by Lemma III.2.16, there is a positive constant C_3, depending only on X and the norm $\|\ \|$, such that for any two ample classes $\xi_1, \xi_2 \in N^1_{\mathbb{Q}}(X)$,

$$|\operatorname{vol}(\xi_1) - \operatorname{vol}(\xi_2)| \leq C_3 \left(\max(\|\xi_1\|, \|\xi_2\|) \right)^{d-1} \|\xi_1 - \xi_2\|.$$

Putting these together, along with (III.2.16), and using that $\|\frac{1}{n}H\| = \|L - L'\|$, we get

$$
\begin{aligned}
|\jmath_X(L) - \jmath_X(L')| \leq \quad & \frac{C_1^{d+1}}{(d+1)!} \Bigg(2\operatorname{vol}(L)\, \frac{2^d}{b} \\
& + C_3\, \|L'\|^{d-1} \|L' - L\| \Bigg) + \frac{2}{b} \jmath_X(L)
\end{aligned} \tag{III.2.17}
$$

Next, using the fact b was chosen to be $b(n) = \lfloor \frac{nC_4}{\|H\|} \rfloor$, we have $b \geq \frac{nC_4}{2\|H\|}$, using which we get

$$
\begin{aligned}
|\jmath_X(L) - \jmath_X(L')| \leq \quad & \frac{C_1^{d+1}}{\|L^{d+1}\|(d+1)!} \Bigg(2\operatorname{vol}(L)\, \frac{2^{d+1}}{C_4} \|L - L'\| \\
& + C_3\, \|L'\|^{d-1} \|L - L'\| \Bigg) + \frac{4}{C_4} \jmath_X(L) \|L - L'\|
\end{aligned} \tag{III.2.18}
$$

Lastly, since $\|L - L'\| < \frac{\|L\|}{2}$ we have $\|L'\| < 2\|L\|$. Hence, we have

$$
\begin{aligned}
|\jmath_X(L) - \jmath_X(L')| \leq \quad & \frac{C_1^{d+1}}{\|L\|^{d+1}(d+1)!} \Bigg(2\operatorname{vol}(L)\, \frac{2^{d+1}}{C_4} \|L - L'\| \\
& + 2^{d-1} C_3\, \|L\|^{d-1} \|L - L'\| \Bigg) + \frac{4}{C_4} \jmath_X(L) \|L - L'\|
\end{aligned} \tag{III.2.19}
$$

Hence, we see that for any ample, integral divisor L, we have

$$|\jmath_X(L) - \jmath_X(L')| \leq C(L)\|L - L'\|$$

for all ample $\mathbb{Q}$-divisors L' such that $L' - L$ is big and $\|L - L'\| < \frac{\|L\|}{2}$ where $C(L)$ is given by

$$C(L) = \frac{C_1^{d+1}}{\|L\|^{d+1}(d+1)!}\Big(\mathrm{vol}(L)\frac{2^{d+1}}{C_4} + 2^{d-1}C_3\|L\|^{d-1}\Big) + \frac{4}{C_4}\jmath_X(L).$$

This completes the proof of the Key Lemma III.2.10. $\square$

The proof of the Key Lemma III.2.10 actually shows a stronger and more explicit statement that will be useful to us. We record it in the following Proposition.

Proposition III.2.17. *For any ample, integral divisor L, we have*

$$|\jmath_X(L) - \jmath_X(L')| \leq C(L)\|L - L'\|$$

for all ample $\mathbb{Q}$-divisors L' such that $L' - L$ is big and $\|L - L'\| < \frac{\|L\|}{2}$, where $C(L)$ may be chosen to be of the form

$$C(L) = \frac{C_1^{d+1}}{\|L\|^{d+1}(d+1)!}\Big(\mathrm{vol}(L)\frac{2^{d+1}}{C_4} + 2^{d-1}C_3\|L\|^{d-1}\Big) + \frac{4}{C_4}\jmath_X(L).$$

Here, $C_1 := C_1(X)$ is the constant (depending only on X) obtained in Theorem III.2.8, C_3 depends only on X, and $C_4 := C_4(L)$ is any constant (depending on L) with the property that the closed ball $B = \{D \in N^1_{\mathbb{Q}}(X) \mid \|D - L\| \leq C_4\}$ is contained in the big cone of X.

Next, we examine how the constant $C(L)$ in Proposition III.2.17 varies with L.

Lemma III.2.18. *Let X be projective variety and $\mathcal{C}$ be a closed cone contained in the big cone of X. Then, there exists a constant $\tilde{C}_4$ (depending only on $\mathcal{C}$) such that for any non-zero class $D \in \mathcal{C}$, the closed ball*

$$B(D) = \{\xi \in \mathrm{N}^1_{\mathbb{R}}(X) \mid \|\xi - D\| < C_4\,\|D\|\}$$

is contained in $\mathrm{Big}(X)$.

Proof. Consider the set

$$\kappa := \{D' \in \mathcal{C} \mid \|D'\| = 1\}.$$

Since $\mathcal{C}$ is a closed cone, κ is a compact subset of $\mathcal{C}$. Moreover, since $\mathcal{C}$ is contained in the big cone of X and because $\mathrm{Big}(X)$ is an open subset of $\mathrm{N}^1_{\mathbb{R}}(X)$, there exists a positive real number $\tilde{C}_4 > 0$ such that the ball $B_{\tilde{C}_4}(D) = \{\xi \mid \|D - \xi\| \le \tilde{C}_4\}$ is contained in the big cone for all $D \in \kappa$. Now the lemma follows by considering $\frac{1}{\|D\|}D \in \kappa$ whenever D is a non-zero class in $\mathcal{C}$. $\square$

Lemma III.2.19. *Given any two norms $\| \ \|_1$ and $\| \ \|_2$ on the vector space $\mathbb{R}^N$, we have positive constants μ_1 and μ_2 such that for any vector $\nu \in \mathbb{R}^N$,*

$$\mu_1\|\nu\|_1 \le \|\nu\|_2 \le \mu_2\|\nu\|_1.$$

Proof. See [Fol99, Section 5.1, Ex. 6]. $\square$

Lemma III.2.20. *Let $e_1, \ldots, e_\rho$ be a basis for the Néron-Severi space of X, where each e_i corresponds to a big divisor. Let $\mathcal{C}$ denote the closed cone generated by the e_i's and $\| \ \|$ denote the sup-norm with respect to the basis $\{e_i\}$. For any L in $\mathcal{C}$, let $\lambda_i(L)$ denote the i^{th}-coordinate of L with respect to the basis $\{e_i\}$. Suppose we have two positive numbers $0 < A_1 < A_2$ and a compact subset κ of $\mathcal{C}$ defined by*

$$\kappa = \{\xi \in \mathcal{C} \mid A_1 \le \|\xi\| \le A_2\}.$$

In this situation, for every D in the interior of κ, there exists a positive real number $r(D)$ such that the following three conditions are satisfied:

1. *$r(D) < \frac{A_1}{2}$.*
2. *The closed ball*
$$B_{r(D)} := \{D' \mid \|D' - D\| \le r(D)\}$$
is contained in the interior of κ.
3. *For any two $\mathbb{Q}$-divisors L and L' in $B_{r(D)}$, setting $\lambda = \max_i\{\frac{\lambda_i(L)}{\lambda_i(L')}\}$, we have*
$$A_1 < \lambda\|L'\| < A_2$$
and
$$\|\lambda L' - L\| < \frac{A_1}{2}.$$

Proof. First, pick any positive number $r < \frac{A_1}{4}$ such that B_r, the closed ball of radius r around D is contained in κ (this is possible since D is contained in the interior of κ). Now, there exists a positive number ε such that for any L in $B_{r/2}$, both $(1-\varepsilon)L$ and $(1+\varepsilon)L$ are contained in B_r. Finally pick $0 < r(D) < r/2$ so small that for each i, we have

$$\left|1 - \frac{\lambda_i(L)}{\lambda_i(L')}\right| < \varepsilon$$

for all L, L' in $B_{r(D)}$. This is possible due to the local uniform continuity of the function $\lambda_i(L)$ as L varies. By construction, for any $L, L' \in B_{r(D)}$, we have

$$1 - \varepsilon < \lambda = \max_i \left\{ \frac{\lambda_i(L)}{\lambda_i(L')} \right\} < 1 + \varepsilon.$$

This ensures that $\lambda L'$ is in B_r and since $r < \frac{A_1}{4}$, also that

$$\|\lambda L' - L\| \leq \|\lambda L' - D\| + \|D - L\| < \frac{3r}{2} < \frac{A_1}{2}.$$

$\square$

Finally, we can now prove Theorem III.2.2.

Completion of the proof of Theorem III.2.2: Fix a real class D in the ample cone $\mathrm{Amp}_{\mathbb{R}}(X)$. Then, to prove that s_X is locally Lipschitz around D, by Lemma III.2.19, we may a pick a suitable norm depending on D. Since the ample cone $\mathrm{Amp}_{\mathbb{R}}(X)$ is an open subset of $N^1_{\mathbb{R}}(X)$, given D in $\mathrm{Amp}_{\mathbb{R}}(X)$, we may pick a basis $e_1, \ldots, e_\rho$ for $N^1_{\mathbb{R}}(X)$ such that each e_i is the class of an ample invertible sheaf and such that D in contained in the interior of the cone generated by the e_i's (equivalently, $D = \sum a_i e_i$ with each $a_i > 0$). Let $\mathcal{C} = \{a_i e_i \,|\, a_i \geq 0\}$ denote the closed cone generated by the e_i's and $\| \, \|$ denote the sup-norm with respect to the basis $\{e_i\}$.

Pick two positive real numbers A_1 and A_2 such that $0 < A_1 < \|D\| < A_2$. Let $\kappa = \{D' \in \mathcal{C} \,|\, A_1 \leq \|D'\| \leq A_2\}$. We will first consider the case of any two $\mathbb{Q}$-divisors L and L' in κ such that $L' - L$ is big and $\|L' - L\| < \frac{\|L\|}{2}$. Choose an integer $r \gg 0$ such that rL is integral. Then, we may apply Proposition III.2.17 to rL and rL', to get

$$|\mathcal{s}_X(rL) - \mathcal{s}_X(rL')| \leq C(rL)\|rL - rL'\| \tag{III.2.20}$$

where

$$C(rL) = \frac{C_1^{d+1}}{\|rL\|^{d+1}(d+1)!}\Big(\mathrm{vol}(rL)\frac{2^{d+1}}{C_4(rL)} + 2^{d-1}C_3\|rL\|^{d-1}\Big) + \frac{4}{C_4(rL)}\mathcal{s}_X(rL). \tag{III.2.21}$$

Now, applying Lemma III.2.18 to the cone $\mathcal{C}$, we may pick $C_4(rL)$ with the property that

$$C_4(rL) \geq \tilde{C}_4 \, \|rL\|$$

for some constant $\tilde{C}_4$ (depending only on the basis $\{e_i\}$) and for all r and all $L \in \mathcal{C}$. Using this in (III.2.21), we get

(III.2.22)
$$\begin{aligned} C(rL) &\leq \frac{C_1^{d+1}}{\|rL\|^{d+1}(d+1)!}\Big(\mathrm{vol}(rL)\frac{2^{d+1}}{\tilde{C}_4\|rL\|} + 2^{d-1}C_3\|rL\|^{d-1}\Big) + \frac{4}{\tilde{C}_4\|rL\|}\jmath_X(rL) \\ &= \frac{C_1^{d+1}}{(d+1)!\|L\|^{d+1}r^{d+1}}\Big(r^d\mathrm{vol}(L)\frac{2^{d+1}}{r\tilde{C}_4\|L\|} + 2^{d-1}C_3r^{d-1}\|L\|^{d-1}\Big) + \frac{4}{r^2\tilde{C}_4\|L\|}\jmath_X(L) \\ &= \frac{1}{r^2}\Bigg(\frac{C_1^{d+1}}{(d+1)!\|L\|^{d+1}}\Big(\mathrm{vol}(L)\frac{2^{d+1}}{\tilde{C}_4\|L\|} + 2^{d-1}C_3\|L\|^{d-1}\Big) + \frac{4}{\tilde{C}_4\|L\|}\jmath_X(L)\Bigg) \end{aligned}$$

Now, using the fact that $\mathrm{vol}(L)$ is a continuous function of L [LM09], we may find a constant A_3 (depending only on the compact set κ) such that $\mathrm{vol}(L) \leq A_3$ for all L in κ. Using this together with the bounds $A_1 \leq \|L\| \leq A_2$ in (III.2.22), we get

$$C(rL) \leq \frac{1}{r^2}\Bigg(\frac{C_1^{d+1}}{(d+1)!\|A_1\|^{d+1}}\Big(A_3\frac{2^{d+1}}{\tilde{C}_4A_1} + 2^{d-1}C_3A_2^{d-1}\Big) + \frac{4}{\tilde{C}_4A_1}\Bigg).$$

So setting

$$C'(D) = \Bigg(\frac{C_1^{d+1}}{(d+1)!\|A_1\|^{d+1}}\Big(A_3\frac{2^{d+1}}{\tilde{C}_4A_1} + 2^{d-1}C_3A_2^{d-1}\Big) + \frac{4}{\tilde{C}_4A_1}\Bigg),$$

and using it in (III.2.20), we have

$$|\jmath_X(rL) - \jmath_X(rL')| \leq \frac{1}{r^2}C'(D)\|rL - rL'\|.$$

Using the scaling property of $\jmath_X$ for $\mathbb{Q}$-divisors (Theorem III.1.3), this in turn implies,

(III.2.23)
$$|\jmath_X(L) - \jmath_X(L')| \leq C'(D)\|L - L'\|$$

for any two $\mathbb{Q}$-divisors L, L' in κ such that $L' - L$ is big and $\|L - L'\| \leq \frac{\|L\|}{2}$. Note that $C'(D)$ only depends on the set κ and hence only on D.

To complete the proof of Theorem III.2.2, we need to remove the assumption that $L' - L$

is big from inequality (III.2.23). For any L in $\mathcal{C}$, let $\lambda_i(L)$ denote the i^{th}-coordinate of L with respect to the basis $\{e_i\}$ (for $1 \leq i \leq \rho$). Now, since D is contained in the interior of κ, by Lemma III.2.20 there exists a positive $r(D)$ satisfying the following three conditions:

1. $r(D) < \frac{A_1}{2}$.
2. The closed ball
$$B_{r(D)} := \{D' \mid \|D' - D\| \leq r(D)\}$$
is contained in the interior of κ.
3. For any two $\mathbb{Q}$-divisors L and L' in $B_{r(D)}$, setting $\lambda = \max_i\{\frac{\lambda_i(L)}{\lambda_i(L')}\}$, we have
$$A_1 < \lambda\|L'\| < A_2$$
and
$$\|\lambda L' - L\| < \frac{A_1}{2}.$$

Fix such an $r(D)$. For any two $\mathbb{Q}$-divisors L and L' in $B_{r(D)}$ such that L' is not a multiple of L, setting $\lambda = \max_i\{\frac{\lambda_i(L)}{\lambda_i(L')}\}$, then $\lambda L' - L$ is ample (hence, big). Indeed, recall that e_i's are an ample basis for $N^1_(X)$ and the j-th coordinate of $\lambda L' - L$ is

$$\lambda\lambda_j(L') - \lambda_j(L) = \lambda_j(L')\left(\lambda - \frac{\lambda_j(L)}{\lambda_j(L')}\right) \geq 0$$

The right hand side is non-negative since λ is the maximum of $\lambda_i(L)/\lambda_i(L')$. Now, if $\lambda = \lambda_j(L)/\lambda_j(L')$ for all j, then L' is a multiple of L. Therefore, if L' is not a multiple of L, one of the coefficients of $\lambda L' - L$ is strictly positive, which implies $\lambda L' - L$ is ample.

Furthermore, $\lambda L' \in \kappa$ and $\|\lambda L' - L\| < \frac{\|L\|}{2}$ (these are ensured by condition (c) on $r(D)$). Hence, using (III.2.23) we have

$$|\jmath_X(\lambda L') - \jmath_X(L)| \leq C'(D)\|\lambda L' - L\|$$

for any two ample $\mathbb{Q}$-divisors L and L' contained in $B_{r(D)}$.

Pick a positive constant A_4 (depending only on D and $r(D)$) such that we $\lambda_i(L) \geq A_4$ for any L in $B_{r(D)}$ and all i. This is possible because κ, hence the closed ball $B_{r(D)}$ is contained in the interior of the cone $\mathcal{C}$. Since for some i, we have $\lambda = \frac{\lambda_i(L)}{\lambda_i(L')}$, we have

$$|\lambda - 1| \leq \frac{|\lambda_i - \lambda_i'|}{\lambda_i'} \leq \frac{\|L - L'\|}{A_4}.$$

Similarly, we have

$$\left|\frac{1}{\lambda}-1\right| \leq \frac{\|L-L'\|}{A_4}.$$

To conclude the argument, we note that

$$\begin{aligned}
|s_X(L)-s_X(L')| \leq & |s_X(L)-s_X(\lambda L')| + |s_X(\lambda L')-s_X(L')| \\
& \leq C'(D)\|L-\lambda L'\| + |\frac{1}{\lambda}-1| s_X(L') \\
& \leq C'(D)\|L-L'\| + C'(D)|1-\lambda|\|L'\| + |\frac{1}{\lambda}-1| s_X(L') \\
& \leq C'(D)\|L-L'\| + C'(D)\frac{A_2}{A_4}\|L-L'\| + \frac{1}{A_4}\|L-L'\|.
\end{aligned}$$

Lastly, if L' were a multiple of L, then only the last term in the above inequality suffices. Thus, we see that for our choice of $r(D)$, choosing $C(D) = C'(D) + C'(D)\frac{A_2}{A_4} + \frac{1}{A_4}$ works for the inequlaity (III.2.1), hence proving Theorem III.2.2. □

III.3: Extending the F-signature Function to the Nef Cone.

In this section, we will prove that the F-signature function, originally defined in Section 3 only on the ample cone (Definition II.2.2) extends continuously to the non-zero classes in the nef cone.

Theorem III.3.1. *Suppose that X is a globally F-regular projective variety of dimension d. Then the F-signature function s_X extends continuously to all non-zero classes of the Nef cone $\mathrm{Nef}_{\mathbb{R}}(X)$. Moreover, if D is a nef Cartier divisor which is not big, then $s_X(D) = 0$.*

We prove Theorem III.3.1 in two parts, depending on whether or not L is big. First, we have the following comparison of the F-signature function with the volume function:

Lemma III.3.2. *Let X be a globally F-regular projective variety of dimension d. Fix a norm $\|\ \|$ on the Néron-Severi space of X. Let C_1 be a constant such that for any non-zero effective divisor L, we have (see Definition III.2.5 for the notation),*

$$I_e(mL) = H^0(mL) \text{ for all } m > \frac{C_1}{\|L\|}p^e.$$

The existence of such a constant is guaranteed by Theorem III.2.8. Then, for any ample Cartier divisor D on X, we have

$$s_X(D) \leq \frac{C_1^{d+1}\mathrm{vol}(D)}{\|D\|^{d+1}(d+1)!}. \qquad \text{(III.3.1)}$$

Note that the right-hand side has the same order of decay as the F-signature function, decaying in the order of $1/\|D\|$ as the norm of divisor $\|D\| \to \infty$.

Proof. Using Lemma III.2.6 to calculate the F-signature $\mathcal{s}_X(D)$, we have

$$\begin{aligned}
\mathcal{s}_X(D) &= \lim_{e\to\infty} \frac{1}{p^{e(d+1)}} \sum_{m=0}^{\infty} \dim_k \frac{H^0(mD)}{I_e(mD)} \\
&= \lim_{e\to\infty} \frac{1}{p^{e(d+1)}} \sum_{m=0}^{\frac{C_1 p^e}{\|D\|}} \dim_k \frac{H^0(mD)}{I_e(mD)} \\
&\le \lim_{e\to\infty} \frac{1}{p^{e(d+1)}} \sum_{m=0}^{\frac{C_1 p^e}{\|D\|}} \dim_k H^0(mD) \\
&\le \lim_{e\to\infty} \frac{1}{p^{e(d+1)}} \frac{\mathrm{vol}(D)}{(d+1)!} \left(\frac{C_1 p^e}{\|D\|}\right)^{d+1} \qquad \text{(using the Hilbert-polynomial of } D) \\
&= \frac{C_1^{d+1}\mathrm{vol}(D)}{\|D\|^{d+1}(d+1)!}
\end{aligned}$$

□

Proof of Theorem III.3.1. First suppose that D is a non-zero nef divisor that is not big. Then, for any sequence $\{L_t\}_t$ of ample $\mathbb{Q}$-divisors approaching D, choose a positive integer r_t for each $t \ge 1$ such that $r_t L_t$ is integral Cartier. Then, we see that using Lemma III.3.2,

$$\mathcal{s}_X(L_t) = r_t\, \mathcal{s}_X(r_t L_t) \le \frac{C_1^{d+1}\mathrm{vol}(r_t L_t)}{\|r_t L_t\|^{d+1}(d+1)!} = \frac{C_1^{d+1}\mathrm{vol}(L_t)}{\|L_t\|^{d+1}(d+1)!}.$$

Since $\|D\| \neq 0$, we have that $\|L_t\|$ approaches a non-zero number (namely, $\|D\|$) and $\mathrm{vol}(L_t)$ approaches 0 as $t \to \infty$ (since D is not big), this shows that $\mathcal{s}_X(L_t) \to 0$ as $t \to \infty$. As the sequence $\{L_t\}$ chosen was arbitrary, this shows that the F-signature function $\mathcal{s}_X$ extends continuously by zero to all non-zero nef divisors L that are not big.

Now suppose that D is a big and nef divisor. Following the proof of Theorem III.2.2, to prove that $\mathcal{s}_X$ is locally Lipschitz for ample divisors around D, by Lemma III.2.19, we may a pick a suitable norm depending on D. Since the big cone $\mathrm{Big}(X)$ is an open subset of $N^1_{\mathbb{R}}(X)$, given D in $\mathrm{Big}(X)$, we may pick a basis $e_1, \ldots, e_\rho$ for $N^1_{\mathbb{R}}(X)$ such that each e_i is the class of a big invertible sheaf and such that D in contained in the interior of the cone generated by the e_i's (equivalently, $D = \sum a_i e_i$ with each $a_i > 0$). Let $\mathcal{C} = \{a_i e_i \,|\, a_i \ge 0\}$ denote the closed cone generated by the e_i's and $\| \,\|$ denote the sup-norm with respect to the basis $\{e_i\}$. Then, arguing verbatim as in the final step of the proof of Theorem III.2.2 and applying the argument to all ample $\mathbb{Q}$-divisors L, L' contained in $\mathcal{C}$, we get positive numbers $r(D)$ and $C(D)$ such that

$$|\mathfrak{s}_X(L) - \mathfrak{s}_X(L')| \leq C(D)\|L - L'\|$$

for any two ample $\mathbb{Q}$-divisors L and L' contained in a ball of radius $r(D)$ around D. This proves that $\mathfrak{s}_X$ is uniformly continuous in a neighbourhood of D, which gives us a unique continuous extension of $\mathfrak{s}_X$ to D. $\square$

The F-signature function of the blow-up of $\mathbb{P}^2$ at a point provides an instructive example of the behavior of the function on the boundary. For a formula for general Hirzebruch surfaces, see [HS17].

Example III.3.3. Let $X = \mathrm{Bl}_x(\mathbb{P}^2)$ be the blow-up of $\mathbb{P}^2$ at $x = [0 : 0 : 1]$. Let H denote the pull-back of a line in $\mathbb{P}^2$ passing through x and E be the exceptional divisor for the blow-up. Then H and E form a basis for the Néron-Severi space and the nef cone of X is given by the divisors $aH - bE$ such that $0 \leq b \leq a$. For $L = aH - bE$, we can compute the F-signature of L using the formula described in [VK12], and it is given by

$$\mathfrak{s}_X(L) = \begin{cases} \frac{a-b}{ab}, & \text{if } b \leq a \leq \frac{3}{2}b \\ \frac{2b-a}{2a(a-b)} + \frac{(3b-a)(2a-3b)}{6b(a-b)^2} + \frac{(2a-3b)^2}{2a(a-b)^2}, & \text{if } \frac{3}{2}b \leq a \leq 2b \\ \frac{1}{a} - \frac{b^3+(a-2b)^3}{6ab(a-b)^2} & \text{if } 2b \leq a \leq 3b \\ \frac{1}{a} - \frac{b^2+(a-2b)^2+(a-3b)(a-2b)+(a-3b)^2}{6a(a-b)^2} & \text{if } 3b \leq a \end{cases}$$

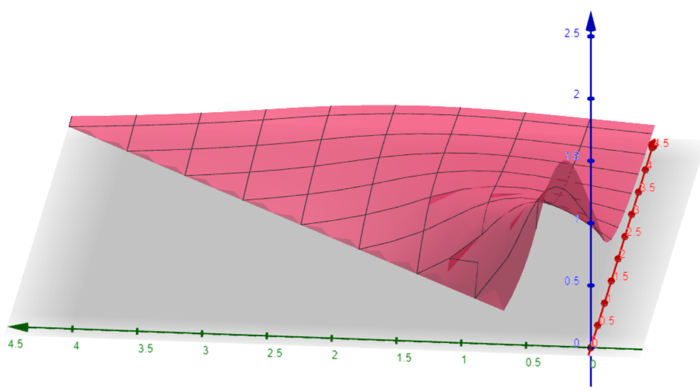

Figure III.1: The F-signature function of the blow up of $\mathbb{P}^2$ at a point.

Note that along the line $a = b$, which corresponds to a nef but not big boundary face, the F-signature extends to the zero function (as proved in Theorem III.3.1). On the other hand, along $b = 0$, which is the big and nef boundary face, letting $b \to 0$ yields $s_X(L) = \frac{1}{a} - \frac{a^2+a^2+a^2}{6a^2} = \frac{1}{2a}$. It turns out that this corresponds to the F-signature of the cone over the pair $(\mathbb{P}^2, \mathfrak{m}_x)$ with respect to the divisor aL on $\mathbb{P}^2$ (see [BST11, Theorem 4.20] for the definition of the F-signature of pairs).

Remark III.3.4. Theorem III.3.1 gives us a unique extension of the F-signature function to the non-zero classes in the nef cone of X. Further, we also know that for nef divisors that are not big, the extension is 0. Thus, it is natural to ask what the extension to a big and nef divisor is. In forthcoming work, we explore this question and provide some answers in terms of F-signature of pairs, as indicated by Example III.3.3.

In particular, we can ask if the extension of the F-signature function to all big and nef divisors is positive. Motivated by this (and Lemma III.3.2), we raise the following question on lower bounds for the F-signature function:

Question III.3.5. Let X be a globally F-regular projective variety and $\| \ \|$ be a fixed norm on $N^1(X)$. Then, does there exist a constant $C > 0$ (depending only on X) such that, we have

$$s_X(L) \geq \frac{C\mathrm{vol}(L)}{\|L\|^{d+1}}$$

for all ample $\mathbb{Q}$-divisors L?

III.4: Local Upper Bounds for the F-signature Function

In this section, we prove effective local upper bounds for the F-signature function (Definition II.2.2).

Theorem III.4.1. *Let X be a globally F-regular projective variety. Let $d = \dim X$ be positive. Fix a basis $e_1, \ldots, e_\rho$ for the Néron-Severi space $N^1_{\mathbb{R}}(X)$ such that each e_i corresponds to the class of an ample and globally generated invertible sheaf. Let $\mathcal{C}$ denote the simplicial cone generated by the e_i's, that is, $\mathcal{C} = \{\sum a_i e_i \mid a_i \in \mathbb{R}_{\geq 0}\}$. Let $\| \ \|$ denote the sup-norm on $N^1_{\mathbb{R}}(X)$ with respect to the e_i's. Then, for any non-zero class L in $\mathcal{C}$, we have*

$$s_X(L) \leq \frac{(d^2+2d)^{d+1}\mathrm{vol}(L)}{\lfloor \|L\| \rfloor^{d+1}(d+1)!}. \tag{III.4.1}$$

Lemma III.4.2. *Suppose L is a globally generated ample divisor and H any nef divisor on X. Then, for all $e \geq 1$, we have:*

1.

$$I_e(mL) = H^0(mL) \text{ for } m > (d^2+d)p^e,$$

2.

$$I_e(m(nL+H)) = H^0(m(nL+H)) \textit{ for all } m > \frac{(d^2+2d)p^e}{n}.$$

Proof. Let S be the section ring of X with respect to L. And for any $j \geq 0$, let M^j be the S-module $\bigoplus_{t\geq 0} \mathcal{O}_X(jH+tL)$.

1. First, we claim that S is generated as a graded ring by homogeneous elements of degree at most d. This follows from Mumford's Theorem [Laz04, Theorem 1.8.5], if we show that the trivial bundle $\mathcal{O}_X$ is d-regular with respect to L. Since X is globally F-regular and L is ample, by Theorem II.3.7, we have that

$$H^i(X, \mathcal{O}_X((d-i)L)) = 0 \quad \text{for all } i > 0.$$

This implies that $\mathcal{O}_X$ is d-regular with respect to L and hence that S is generated by elements of degree at most d.

Since the section ring S is generated by elements of degree $\leq d$, the homogeneous maximal ideal $\mathfrak{m} = S_{>0}$ is generated in degrees $\leq d$. By [HS06, Proposition 8.3.8], there exist elements $x_0, \dots, x_d$ (not necessarily homogeneous), such that all terms of each x_i have degree at most d, and the integral closure $\overline{(x_0, \dots, x_d)}$ is equal to the maximal ideal $\mathfrak{m}$. Now, by using the Briançon-Skoda theorem in the strongly F-regular ring S [HH90, Theorem 5.4], we have

$$\mathfrak{m}^{(d+1)p^e} = \overline{(x_0^{p^e}, \dots, x_d^{p^e})^{d+1}} \subseteq (x_0^{p^e}, \dots, x_d^{p^e}).$$

Therefore, if $m \geq d(d+1)p^e$, for any element $x \in S_m = H^0(mL)$, by the pigeon-hole principle, we have $x \in \mathfrak{m}^{(d+1)p^e}$, and consequently, $x \in (x_0^{p^e}, \dots, x_d^{p^e})$. Hence, the map $\mathcal{O}_X \to F^e_* \mathcal{O}_X(mL)$ sending $1 \mapsto F^e_* x$ cannot split.

2. Similarly as in part (a), we claim that for any $j \geq 0$, M^j is generated over S by elements of degree at most d. For this, again by Mumford's theorem, it is enough to show that $\mathcal{O}_X(jH)$ is d-regular with respect to L. Since H is nef, by Theorem II.3.7 we again have:

$$H^i(X, \mathcal{O}_X(jH + (d-i)L)) = 0 \quad \text{for all } i > 0.$$

Suppose $f \in H^0(m(nL+H) \setminus I_e(m(nL+H))$ and $m > \frac{(d^2+2d)p^e}{n}$, then we may write $f = \sum r_i f_i$ for $r_i \in S$ and $f_i \in M^m$ with degree of f_i at most d. Then the degree of each r_i is at least $(d^2+d)p^e$. Now, since by assumption, the map $\mathcal{O}_X \to F^e_* \mathcal{O}_X(m(nL+H))$ sending 1 to $F^e_* f$ splits, we must have that for some i, $r_i \in H^0(kL) \setminus I_e(kL)$ for a suitable $k > (d^2 + d)p^e$, contradicting part (a) of the lemma. Hence, we must have $I_e(m(nL + H)) = H^0(m(nL + H))$ for all $m > \frac{(d^2+2d)p^e}{n}$. This completes the proof of the lemma. $\square$

Lemma III.4.3. *Fix a basis $e_1, \ldots, e_\rho$ of $N^1_{\mathbb{R}}(X)$ such that each e_i corresponds to the class of an ample and globally generated invertible sheaf. Let $\mathcal{C}$ denote the simplicial cone generated by the e_i's, that is, $\mathcal{C} = \{\sum a_i e_i \,|\, a_i \in \mathbb{R}_{\geq 0}\}$. Let $\| \, \|$ denote the sup-norm on $N^1_{\mathbb{R}}(X)$ with respect to the e_i's. Then, for any invertible sheaf $\mathcal{L}$ such that its class L in the Néron-Severi space satisfies $L \in \mathcal{C}$ and $\|L\| \geq d$ (where d is the dimension of X), we have*

(a) $\mathcal{L}$ is ample and globally generated.

(b) Further,

$$I_e(mL) = H^0(mL) \text{ for all } m > \frac{(d^2 + 2d)p^e}{\lfloor \|L\| \rfloor}.$$

Proof. 1. Ampleness of $\mathcal{L}$ follows from the assumption that L lies in $\mathcal{C}$ and L is non-zero since $\|L\| \neq 0$. It remains to show global generation of $\mathcal{L}$. For this, we note that since $\|L\| \geq d$, there is some i such that we may decompose the divisor L as $L = dL_i + H$ where H is some nef Cartier divisor and L_i is a Cartier divisor corresponding to the class e_i. This follows from the assumption that e_i's are integral, ample and globally generated and the fact that the sup-norm is achieved by some coordinate of L. Hence, applying Theorem II.3.7, we have

$$H^p(X, \mathcal{O}_X(L - p\,L_i)) = 0 \quad \text{for all } p > 0.$$

Therefore, $\mathcal{L}$ is 0-regular with respect to the globally generated ample divisor L_i. Hence, $\mathcal{L}$ is globally generated itself.

2. Since $\|L\| \geq d$, for some $0 \leq i \leq \rho$, we may write $L = \lfloor \|L\| \rfloor e_i + H$ for some integral and nef class H. Now, applying Part (b) of Lemma III.4.2, we get

$$I_e(mL) = H^0(mL) \text{ for all } m > \frac{(d^2 + 2d)p^e}{\lfloor \|L\| \rfloor}.$$

$\square$

Proof of Theorem III.4.1: By Theorem III.2.2, the F-signature function is continuous, hence we may prove Theorem III.4.1 only when L is an ample $\mathbb{Q}$-divisor. Further, since both sides of (III.4.1) scale inverse-linearly, we may assume that L is a Cartier divisor and $\|L\| \geq d$.

Then, applying Part (b) of Lemma III.4.3, the Theorem follows from Lemma III.3.2 by using $\frac{d^2+2d}{\lfloor \|L\| \rfloor}$ instead of $\frac{C_1}{\|L\|}$. $\square$

CHAPTER IV
The Frobenius-Alpha Invariant

In analogy with Tian's α-invariant in complex geometry, we define the "Frobenius-alpha" invariant (denoted by α_F) for any pair (X, L) where X is a globally F-regular projective variety (Definition II.3.2) and L is an ample Cartier divisor on X (Section IV.2). In Section IV.3, we will specialize to the case of Fano varieties polarized by their anti-canonical divisors. Finally, in Section IV.4, we discuss some examples of the α_F-invariant and highlight interesting behaviour in the process of reduction modulo p of a complex Fano variety. We begin with some preliminaries on the geometric aspects of cones over projective varieties and Frobenius splittings on the cones. These results allow us to apply global geometric techniques to the study of the α_F-invariant and the F-signature of section rings. Throughout this chapter, unless specified otherwise, k will denote a perfect field of characteristic $p > 0$.

IV.1: Frobenius Splittings on Cones

IV.1.1: Affine and Projective cones

Given an integral, projective scheme X over a noetherian domain A, and an ample invertible sheaf $\mathcal{L}$ over X, let S be the corresponding section ring. Assume that $S_0 = A$. Let $\overline{X}$ be the projective A-scheme defined as

$$\overline{X} = \operatorname{Proj}(S[z])$$

where $S[z]$ is the $\mathbb{N}$-graded ring obtained by adjoining a new variable z to S in degree 1. Then, $\overline{X}$ is called the *projective cone over* X with respect to $\mathcal{L}$. Denoting by $\mathfrak{m}$ the homogeneous irrelevant ideal $\bigoplus_{j>0} S_j$ of S, we have a map of graded A-algebras

$$S[z] \to (S/\mathfrak{m})[z] \cong A[z]$$

that induces the "zero-section" map

$$\sigma : \operatorname{Spec}(A) \to \overline{X}$$

over A. We call this map the "zero-section of the cone $\overline{X}$". We also have natural maps of graded rings

$$S \subset S[z] \to S[z]/(z) \cong S$$

which, via the construction in [Har77, II, Exercise 2.14 (b)] induce maps

$$i_\infty : X \hookrightarrow \overline{X}$$

called the "section at infinity" and

$$\pi : \overline{X} \setminus \sigma(\mathrm{Spec}(A)) \to X$$

where σ is the zero-section described above. It follows from the Proj construction that π is an $\mathbb{A}^1$-bundle over X. The affine cone $Y = \mathrm{Spec}(S)$ is isomorphic to $\overline{X} \setminus i_\infty(X)$, and the zero section σ actually maps into Y. Thus, π restricts to a map $Y \setminus \sigma(\mathrm{Spec}(A)) \to X$ that is a $\mathrm{Spec}(A[t, t^{-1}])$-bundle over X. See [HS04, Section 2] and [Kol13, Section 3.1] for details.

IV.1.2: Cones over $\mathbb{Q}$-divisors

We follow the description of the cone over a $\mathbb{Q}$-divisor as in [SS10, Section 5], where it is explained for a projective variety over a field. Essentially the same description holds in the following more general relative setting: Let A be a normal domain of finite type over k, and X be an integral, normal, projective scheme over A. Assume that X is flat over A and of positive relative dimension. Fix an ample invertible sheaf $\mathcal{L}$ over X and S be the corresponding section ring. Assume further that $S_0 = A$. Note that this guarantees that the codimension of the zero section σ is at least two in $Y = \mathrm{Spec}(S)$. Thus, it follows that S is normal as well. In this situation, given any integral Weil-divisor $D = \sum a_i D_i$ (for distinct prime Weil divisors D_i) on X, we can construct the corresponding Weil divisor $\tilde{D}$ on Y, the "cone over D", in three equivalent ways:

1. Let $\tilde{D}_i$ be the prime Weil divisor on Y corresponding to the height one prime $\mathfrak{p}_i \subset S$ corresponding to D_i. Then $\tilde{D} = \sum a_i \tilde{D}_i$.

2. Let $\mathcal{O}_X(D)$ be the reflexive sheaf on X corresponding to D. Then, $\tilde{D}$ is the divisor corresponding to the reflexive S-module defined by

$$M := M(D, \mathcal{L}) = \bigoplus_{j \in \mathbb{Z}} H^0(X, \mathcal{O}_X(D) \otimes \mathcal{L}^j).$$

   Note that the fact that M is reflexive can be seen by applying Part (3) of Lemma II.1.2

to each twist of the graded module M.

3. Let $\pi : Y \setminus \sigma(\mathrm{Spec}(A)) \to X$ be the $A[t, t^{-1}]$-bundle map defined in the previous paragraph. Then, we may define $\tilde{D}$ to be the pull back of D to Y. More precisely, near the generic point of a component of D, if D is given by an equation f, then is $\tilde{D}$ is defined by $\pi^* f$. We then take closures to obtain a Weil-divisor on Y. This defines a unique divisor on Y since π is flat and $\mathrm{codim}_Y(\sigma(\mathrm{Spec}(A)))$ is at least 2.

Given a principal divisor D on X defined by the rational function f, the cone over D can be seen to be the principal divisor defined by f again. The construction of the cone clearly preserves addition of divisors. This implies that cone construction extends to $\mathbb{Q}$-divisors and preserves the linear equivalence of Weil-divisors. Furthermore, by taking closures, this construction also extends to the projective cone $\overline{X}$ described in the previous paragraph. Finally, using the third description of the cone, we see that the cone over the canonical (Weil-)divisor K_X is the canonical divisor of K_Y (recall that Y is also normal).

IV.1.3: F-signature of cones over projective varieties.

In this subsection, we collect some results about the F-signature of cones over projective varieties (and pairs) using global splittings on X, extending the discussion from the previous subsection. Note that some of the results stated here are proved in later chapters, but they are collected here since they are used repeatedly.

We begin with a useful lemma that relates global Frobenius splitting of a divisor to splitting "on the cone". This is a slight generalization to the relative setting of [Smi00, Theorem 3.10], where it is proved over a field.

Lemma IV.1.1. *Let A be a regular ring of finite type over k and X be an integral, normal projective scheme over A (with $H^0(X, \mathcal{O}_X) = A$). Assume that X is flat over A and of positive relative dimension. Fix an ample invertible sheaf $\mathcal{L}$ and S be the corresponding section ring. Fix an effective Weil divisor D over X and $\tilde{D}$ be the cone over D with respect to $\mathcal{L}$. Then, for any $e \geq 1$, the natural map*

$$\mathcal{O}_X \to F^e_*(\mathcal{O}_X(D))$$

splits as a map of $\mathcal{O}_X$-modules if and only if the map on the cones

$$S \to F^e_*(S(\tilde{D}))$$

splits as a map of S-modules.

Proof. Using the second description of the cone over a $\mathbb{Q}$-divisor in Section IV.1.2, and Part (3) of Lemma II.1.2, the proof is exactly the same as that in [SS10, Proposition 5.3]. $\square$

Returning to working over a perfect field k, we next recall a formula to compute the F-signature of the section ring of a projective variety proved in Chapter III. Fix a normal projective variety X over k and Δ be an effective $\mathbb{Q}$-divisor over X.

Definition IV.1.2. For any Weil-divisor D on X and $e \geq 1$, define the k-vector subspace $I_e^{\Delta}(D)$ of $H^0(X, D)$ as follows:

$$I_e^{\Delta}(D) := \{f \in H^0(X, D) \mid \varphi(F_*^e f) = 0 \text{ for all } \varphi \in \operatorname{Hom}_{\mathcal{O}_X}(F_*^e \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D), \mathcal{O}_X)\}.$$

Remark IV.1.3. Note that the subspace $I_e^{\Delta}(D)$ only depends on the sheaf $\mathcal{O}_X(D)$ and not on the specific divisor D in its linear equivalence class.

Remark IV.1.4. Let L be an ample divisor on X. Then, it follows from Lemma IV.1.1 that $I_e^{\Delta}(mL)$ is the degree m component of the Δ-splitting ideal of the section ring of S with respect to L (Definition II.2.1).

Lemma IV.1.5. *(Lemma III.2.6) Let L be an ample Cartier divisor on X and S denote the section ring of X with respect to L. Let Δ_S denote the cone over Δ with respect to L (Section IV.1.2). Then, for any $e \geq 1$, if $a_e^{\Delta}(L)$ denotes the Δ_S-free-rank of $F_*^e S$ (Definition II.2.1), then $a_e^{\Delta}(L)$ is computed by the following formula:*

$$a_e^{\Delta}(L) = \frac{1}{[k' : k]} \sum_{m=0}^{\infty} \dim_k \frac{H^0(X, mL)}{I_e^{\Delta}(mL)} \tag{IV.1.1}$$

where k' denotes the field $H^0(X, \mathcal{O}_X)$. Hence, the F-signature of (X, Δ) with respect to L can be computed as

$$\mathfrak{s}_{(X,\Delta)}(L) := \mathfrak{s}(S(X, L), \Delta_S) = \frac{1}{[k' : k]} \lim_{e \to \infty} \frac{\sum_{m=0}^{\infty} \dim_k \frac{H^0(X, mL)}{I_e^{\Delta}(mL)}}{p^{e(\dim(X)+1)}}$$

Proof. Using Lemma IV.1.1, we have that

$$I_e^{\Delta_S} = \bigoplus_{m \geq 0} I_e^{\Delta}(mL).$$

See Definition II.2.1 for the definition of $I_e^{\Delta_S}$. Therefore, we have

$$\ell_S(S/I_e^{\Delta_S}) = \sum_{m\geq 0} \dim_{k'} \frac{H^0(mL)}{I_e^{\Delta}(mL)} = \frac{1}{[k':k]} \sum_{m=0}^{\infty} \dim_k \frac{H^0(X,mL)}{I_e^{\Delta}(mL)}$$

where ℓ_S denotes the length as an S-module. This completes the proof of the lemma. □

IV.1.4: Duality and the trace map

It will be convenient to think of the subspaces I_e^{Δ} (Definition IV.1.2) using a pairing arising out of duality for the Frobenius map. Let (X,Δ) be a projective pair and D be any Weil divisor on X. We continue to work over any perfect field k of characteristic $p > 0$. But in this subsection, we assume that $H^0(X, \mathcal{O}_X) = k$.

Recall that by applying duality to the Frobenius map, we get an isomorphism of reflexive sheaves:

$$\text{(IV.1.2)} \quad \mathscr{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(\lceil (p^e-1)\Delta\rceil + D), \mathcal{O}_X) \cong F^e_*\mathcal{O}_X(-(p^e-1)K_X - \lceil (p^e-1)\Delta\rceil - D).$$

See [SS10, Section 4.1] for a detailed discussion regarding duality for the Frobenius map. Furthermore, when $D = 0$, this gives an isomorphism

$$\text{(IV.1.3)} \qquad \mathscr{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(\lceil (p^e-1)\Delta\rceil, \mathcal{O}_X) \cong F^e_*\mathcal{O}_X(-(p^e-1)K_X - \lceil (p^e-1)\Delta\rceil).$$

Composing this isomorphism (over the global sections) with the evaluation at F^e_*1 map, we obtain the trace map:

$$\text{(IV.1.4)} \qquad \mathrm{Tr}^e_{\Delta} : H^0\Big(X, F^e_*\big(\mathcal{O}_X((1-p^e)K_X - \lceil (p^e-1)\Delta\rceil)\big)\Big) \to k = H^0(X, \mathcal{O}_X).$$

Lemma IV.1.6. *The kernel of the trace map Tr^e_{Δ} in Equation (IV.1.4) is exactly the subspace $F^e_* I^{\Delta}_e((1-p^e)K_X - \lceil (p^e-1)\Delta\rceil)$ (Definition IV.1.2).*

Proof. A section $f \in H^0(X, \mathcal{O}_X((1-p^e)K_X - \lceil (p^e-1)\Delta\rceil))$ is contained in the corresponding I^{Δ}_e-subspace if and only if for every

$$\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(F^e_*(\mathcal{O}_X((1-p^e)K_X)), \mathcal{O}_X) \cong H^0(X, F^e_*\mathcal{O}_X) \cong F^e_*k,$$

we have $\varphi(F^e_*f) = 0$. But, $\mathrm{Hom}_{\mathcal{O}_X}(F^e_*(\mathcal{O}_X((1-p^e)K_X)), \mathcal{O}_X)$ is a one dimensional k-vector space, and is generated by the trace map Tr^e (where $\Delta = 0$). Now, the map Tr^e_{Δ} is just the restriction of the trace map Tr^e to the subspace $H^0\Big(X, F^e_*\big(\mathcal{O}_X((1-p^e)K_X - \lceil (p^e-1)\Delta\rceil)\big)\Big)$.

Thus, the lemma follows. $\square$

Lemma IV.1.7. *Let (X,Δ) be a normal projective pair, and D be any Weil divisor on X. Then, denoting $D_1 = (1-p^e)K_X - \lceil (p^e-1)\Delta \rceil - D$ and $D_2 = (1-p^e)K_X - \lceil (p^e-1)\Delta \rceil$ for any $e \geq 1$, we have a non-degenerate pairing*

$$\frac{H^0(D)}{I_e^\Delta(D)} \times \frac{H^0(D_1)}{I_e^\Delta(D_1)} \to \frac{H^0(D_2)}{I_e^\Delta(D_2)}$$

obtained from multiplication (and reflexifying) global sections. In particular,

$$\dim_k \frac{H^0(D)}{I_e^\Delta(D)} = \dim_k \frac{H^0(D_1)}{I_e^\Delta(D_1)}.$$

Proof. Using Equation (IV.1.2), the natural multiplication map

$$H^0(D) \times H^0(D_1) \to H^0(D_2)$$

can be identified with the evaluation map

$$H^0\Big(F^e_*\big(\mathcal{O}_X(D)\big)\Big) \times \operatorname{Hom}_{\mathcal{O}_X}\Big(F^e_*\big(\mathcal{O}_X(\lceil (p^e-1)\Delta \rceil + D)\big), \mathcal{O}_X\Big) \to \operatorname{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(\lceil (p^e-1)\Delta \rceil), \mathcal{O}_X)$$

where we identify $\operatorname{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(\lceil (p^e-1)\Delta \rceil + D), \mathcal{O}_X)$ and $\operatorname{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(\lceil (p^e-1)\Delta \rceil), \mathcal{O}_X)$ as subspaces of $\operatorname{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(D), \mathcal{O}_X)$ and $\operatorname{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X)$ respectively, both via the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(\lceil (p^e-1)\Delta \rceil)$. Therefore, a section $f \in H^0(D)$ is contained in $I_e^\Delta(D)$ if and only if for all sections $g \in H^0(D_1)$, the multiplication $f\,g$ is contained in $I_e^\Delta(D_2)$. By symmetry, a section $g \in H^0(D_1)$ is contained in $I_e^\Delta(D_1)$ if and only if for all sections $f \in H^0(D)$, $g\,f \in I_e^\Delta(D_2)$. This proves there is a well defined, and non-degenerate pairing as needed.

Finally, we note that since by Lemma IV.1.6, $I_e^\Delta(D_2)$ is the kernel of the trace map (Equation (IV.1.4)), the vector space $H^0(D_2)/I_e^\Delta(D_2)$ is either one-dimensional over k, or equal to 0. In either case, the equality of dimensions follows. $\square$

The next Proposition allows us to perturb by any divisor while computing the F-signature of a section ring.

Proposition IV.1.8. *Let X be a normal projective varitey over k and L an ample divisor on X. Assume $H^0(X,\mathcal{O}_X) = k$. Fix a Weil divisor D on X. Then, there exists a constant*

$C > 0$ (depending only on D and L) such that

$$\left|\dim_k \frac{H^0(mL)}{I_e(mL)} - \dim_k \frac{H^0(mL+D)}{I_e(mL+D)}\right| \leq Cp^{e(\dim(X)-1)}$$

for all $m > 0$ and $e > 0$.

Proof. First we prove the case when D is effective: For any $m \geq 1$, by using the natural map $\mathcal{O}_X(mL) \to \mathcal{O}_X(mL+D)$, we will view $H^0(mL)$ as a subspace of $H^0(mL+D)$. Let $J_e(mL)$ denote the subspace $H^0(mL) \cap I_e(mL+D)$. By Lemma III.2.11, we see that $I_e(mL) \subset J_e(mL)$. Moreover, by Equation (III.2.12) in the proof of Lemma III.2.13, we have

$$\dim_k \frac{H^0(mL+D)}{I_e(mL+D)} = \dim_k \frac{H^0(mL)}{J_e(mL)} + \dim_k \frac{H^0(mL+D)}{H^0(mL)+I_e(mL+D)}.$$

Using this and the triangle inequality, we obtain that

$$\text{(IV.1.5)} \qquad \left|\dim_k \frac{H^0(mL)}{I_e(mL)} - \dim_k \frac{H^0(mL+D)}{I_e(mL+D)}\right| \leq \dim_k \frac{H^0(mL+D)}{H^0(mL)} + \dim_k \frac{J_e(mL)}{I_e(mL)}$$

for all $m, e > 0$. Next, to compute the second term in the above inequality, fix an $e > 0$ and set $\Delta_e = \frac{1}{p^e-1}D$. Then, we observe that the subspace $J_e(mL)$ is exactly the same as $I_e^{\Delta_e}(mL)$ (Definition IV.1.2). Moreover, we also similarly have

$$\text{(IV.1.6)} \quad I_e^{\Delta_e}((1-p^e)K_X - mL - D) = H^0((1-p^e)K_X - mL - D) \cap I_e((1-p^e)K_X - mL).$$

Thus, by Lemma IV.1.7, we have

$$\dim_k \frac{H^0(mL)}{I_e(mL)} = \dim_k \frac{H^0((1-p^e)K_X - mL)}{I_e((1-p^e)K_X - mL)}$$

and similarly,

$$\dim_k \frac{H^0(mL)}{J_e(mL)} = \dim_k \frac{H^0((1-p^e)K_X - mL - D)}{I_e^{\Delta_e}((1-p^e)K_X - mL - D)}.$$

By Equation (IV.1.6), we see the natural map from $H^0((1-p^e)K_X - mL - D)$ to $H^0((1-p^e)K_X - mL)$ restricts to an *injective* map

$$\frac{H^0((1-p^e)K_X - mL - D)}{I_e^{\Delta_e}((1-p^e)K_X - mL - D)} \hookrightarrow \frac{H^0((1-p^e)K_X - mL)}{I_e((1-p^e)K_X - mL)}.$$

By considering the cokernel of this map, we get that

(IV.1.7)

$$\dim_k \frac{J_e(mL)}{I_e(mL)} = \dim_k \frac{H^0(mL)}{I_e(mL)} - \dim_k \frac{H^0(mL)}{J_e(mL)} \leq \dim_k \frac{H^0((1-p^e)K_X - mL)}{H^0((1-p^e)K_X - mL - D)}.$$

Pick an $M \gg 0$ such that mL admits a global section that doesn't vanish along D for all $m \geq M$ (this is possible since L is ample). Using the standard exact sequences to restrict to D, we see that

$$\dim_k \frac{H^0((1-p^e)K_X - mL)}{H^0((1-p^e)K_X - mL - D)} \leq \frac{\mathrm{vol}(-K_X|_D)}{(d-1)!} p^{e(d-1)} + o(p^{e(d-2)}) \tag{IV.1.8}$$

and

$$\dim_k \frac{H^0(mL + D)}{H^0(mL)} \leq \frac{\mathrm{vol}(L|_D)}{(d-1)!} m^{d-1} + o(m^{d-2}). \tag{IV.1.9}$$

Finally, by Theorem III.2.8, we may pick a constant $C_2 > 0$ such that $H^0(mL) = I_e(mL)$ and $H^0(mL + D) = I_e(mL + D)$ for $m > C_2 p^e$. Therefore, to prove the Proposition, it is enough to consider the case when $m \leq C_2 p^e$. In this case, the Proposition now follows by putting together inequalities in IV.1.5, IV.1.7, IV.1.8 and IV.1.9. This completes the proof of the Proposition when D is effective.

More generally, we first pick an $r \gg 0$ such that rL and $D + rL$ are both effective. Then, for any $e \geq 1$ and $m > r$, we have

$$\begin{aligned}\left|\dim_k \frac{H^0(mL)}{I_e(mL)} - \dim_k \frac{H^0(mL+D)}{I_e(mL+D)}\right| &\leq \left|\dim_k \frac{H^0(mL)}{I_e(mL)} - \dim_k \frac{H^0((m-r)L)}{I_e((m-r)L)}\right| \\ &\quad + \left|\dim_k \frac{H^0((m-r)L)}{I_e((m-r)L)} - \dim_k \frac{H^0(mL+D)}{I_e(mL+D)}\right|.\end{aligned}$$

Now, we may apply the previous case of the Proposition (since both $D + rL$ and rL are effective) to each of the two terms in the above inequality. Since r was independent of e, this completes the proof of the Proposition . $\square$

IV.2: The α_F-invariant of Section Rings.

Throughout this section, by a section ring S, we mean that S is the section ring $S(X, L)$ of some projective variety X ($\cong \mathrm{Proj}(S)$) with respect to some ample line bundle L ($\cong \mathcal{O}_X(1)$) on X (see Definition II.1.1 for the definitions).

IV.2.1: Definitions

Definition IV.2.1. Let $(R, \mathfrak{m})$ be an F-finite normal local ring. Assume that R is strongly F-regular. Then, we define the *F-pure threshold* of an effective $\mathbb{Q}$-divisor $D \geq 0$ to be:

$$\mathrm{fpt}_{\mathfrak{m}}(R, D) := \sup\{\, \lambda \,|\, (R, \lambda D) \text{ is sharply } F\text{-split}\,\}.$$

Moreover, by [SS10, Lemma 4.9], it follows that the F-pure threshold of (R, D) is equivalently characterized as the supremum of the set $\{\, \lambda \,|\, (R, \lambda D) \text{ is strongly } F\text{-regular}\}$.

If D is the principal divisor corresponding to a function $f \in R$, we write f instead of D in the notation for the F-pure threshold.

Definition IV.2.2. Let $(S, \mathfrak{m})$ be a strongly F-regular section ring of a projective variety X. Then, we define

$$\alpha_F(S) = \inf\{\, \mathrm{fpt}_{\mathfrak{m}}(S, f)\ \deg(f) \,|\, 0 \neq f \in S \text{ homogeneous element}\,\}.$$

If S is the section ring of a projective variety X with respect to an ample divisor L, we may also use $\alpha_F(X, L)$ to denote $\alpha_F(S)$.

We have the following equivalent ways of characterizing the α_F-invariant of a section ring.

Lemma IV.2.3. *Let X be a globally F-regular projective variety and L be an ample Cartier divisor on X. Let $S = S(X, L)$ be the section ring of X with respect to L. Then, $\alpha_F(S)$ from Definition IV.2.2 is equal to the supremum of any of the following sets:*

1. *The set of $\lambda \geq 0$ such that the pair $(S, \frac{\lambda}{n} D)$ is sharply F-split for every $n \in \mathbb{N}$ and every effective divisor $D \sim nL$.*

2. *The set of $\lambda \geq 0$ such that the pair $(S, \frac{\lambda}{n} D)$ is strongly F-regular for every $n \in \mathbb{N}$ and every effective divisor $D \sim nL$.*

3. *The set of $\lambda \geq 0$ such that the pair $(X, \frac{\lambda}{n} D)$ is globally sharply F-split for every $n \in \mathbb{N}$ and every effective divisor $D \sim nL$.*

4. *The set of $\lambda \geq 0$ such that the pair $(X, \frac{\lambda}{n} D)$ is globally F-regular for every $n \in \mathbb{N}$ and every effective divisor $D \sim nL$.*

5. *The set of $\lambda \geq 0$ such that the pair $(X, \lambda D)$ is globally F-regular for every effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} L$.*

Proof. Statements (1) and (2) follow immediately from the definition of the α_F-invariant and the definition of the F-pure threshold (Definition IV.2.1). Statements (3) and (4) follow from (1) and (2) by using Lemma IV.1.1. Part (5) is just a reformulation of (4) since every effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} L$ is of the form $\frac{1}{n}nD$ for some effective Cartier divisor $nD \sim nL$. $\square$

Next, we explain a more precise connection between the α_F-invariant and Frobenius splittings in S.

Proposition IV.2.4. *Let S be a strongly F-regular section ring and $\alpha'(S)$ denote the supremum of the following set:*

$$\mathscr{A}(S) := \{\lambda \in \mathbb{R}_{\geq 0} \mid \text{ for any integers } e \geq 1 \text{ and } m \leq \lambda(p^e - 1), \text{ we have } I_e(m) = 0\}.$$

Then, $\alpha'(S) = \alpha_F(S)$. Moreover, $\alpha_F(S)$ belongs to the set $\mathscr{A}(S)$.

Remark IV.2.5. Note that in the above Proposition, it is unclear if the set $\mathscr{A}(S)$ contains any non-zero element. This is equivalent to the positivity of $\alpha_F(S)$ and will be addressed in Theorem IV.2.10.

The proof of the Proposition IV.2.4 is based on the following lemma which is a slight generalization of a result of Hernández:

Lemma IV.2.6. *Let $(S, \mathfrak{m})$ be an F-finite, F-regular local ring. Fix an effective Weil-divisor D on $X = Spec(S)$. Then, for any fixed $e_0 > 0$, let ψ_{e_0} denote the natural map*

$$\psi_{e_0} : \mathcal{O}_X \to F^{e_0}_*(\mathcal{O}_X(D)).$$

Then, the following are equivalent:

1. *The map ψ_{e_0} splits as a map of $\mathcal{O}_X$ modules.*
2. *The pair $(X, \frac{1}{p^{e_0}-1}D)$ is sharply F-split (Definition II.3.1).*
3. *The F-pure threshold of (X, D) is at least $\frac{1}{p^{e_0}-1}$ (Definition IV.2.1).*

Proof. It follows immediately from the definitions that (1) implies (2), and (2) implies (3). Hence, it remains to show that (3) implies (1).

Following [Her12, Thoerem 4.9], if $\operatorname{fpt}_{\mathfrak{m}}(X, D) \geq \frac{1}{p^{e_0}-1}$, we must must have that the pair $(X, \frac{1}{p^{e_0}}D)$ is sharply F-split. Thus, there is an $e > 0$ such that the natural map

$$\psi_e(D) : \mathcal{O}_X \to F^e_*(\mathcal{O}_X(\lceil \frac{p^e - 1}{p^{e_0}} \rceil D) \tag{IV.2.1}$$

splits. Since the same holds for $\psi_{ne}(D)$ for any natural number $n \geq 1$ (see [Sch08, Proposition 3.3] for the proof), we get the map:

$$\psi_{ee_0}(D) : \mathcal{O}_X \to F_*^{ee_0}(\mathcal{O}_X(\lceil \frac{p^{ee_0}-1}{p^{e_0}} \rceil D)) = F_*^{ee_0}(\mathcal{O}_X(p^{(e-1)e_0}D))$$

splits. Note that ψ_{e_0} as defined in Equation (IV.2.1) matches with the map considered in the statement of the Lemma.

Let $U \subset X$ denote the regular locus of X. Since ϕ_{e_0} is a map between reflexive sheaves, to show that it splits, it sufficient to show that its restriction of U splits. Over U, we may construct the map ψ_{ee_0} as follows: First consider the map

$$\phi_{(e-1)e_0} : \mathcal{O}_U(D) \to F_*^{(e-1)e_0}\mathcal{O}_U(p^{(e-1)e_0}D)$$

obtained by twisting the $(e-1)e_0^{\text{th}}$-iterate of the Frobenius map by the invertible sheaf $\mathcal{O}_U(D)$. If f denotes the local equation of D, then $\phi_{(e-1)e_0}$ is defined by sending

$$f \mapsto F_*^{(e-1)e_0} f^{p^{(e-1)e_0}}.$$

Then, after restricting to U, we have that

$$\psi_{ee_0} = F_*^{e_0}\phi_{(e-1)e_0} \circ (\psi_{e_0}|_U)$$

where the right hand side is the composition

$$F_*^{e_0}\phi_{(e-1)e_0} \circ (\psi_{e_0}|_U) : \mathcal{O}_U \to F_*^{e_0}\mathcal{O}_U(D) \to F_*^{ee_0}(\mathcal{O}_X(p^{(e-1)e_0}D)).$$

Therefore, if ψ_{ee_0} splits then so does ψ_{e_0}. This proves that part (3) implies part (1), completing the proof of the lemma. $\square$

Proof of Proposition IV.2.4. Set $\alpha = \alpha_F(S)$, and $\alpha' = \alpha'(S)$ (which is defined in the satement of the Proposition). First we will prove that $\alpha \leq \alpha'$. This is clear if $\alpha = 0$, so we assume that α is positive. For any non-zero element $f \in S_m$, by definition of α, we must have $\operatorname{fpt}_{\mathfrak{m}}(S, f) \geq \frac{\alpha}{m}$. So, if $m \leq \alpha(p^e - 1)$, we have

$$\operatorname{fpt}_{\mathfrak{m}}(S, f) \geq \frac{\alpha}{m} \geq \frac{\alpha}{\alpha(p^e-1)} = \frac{1}{p^e-1}.$$

Thus, by Lemma IV.2.6, the map $R \to F_*^e R$ sending 1 to $F_*^e f$ splits. Since f was an arbitrary non-zero element of degree m, this shows that $I_e(m) = 0$ whenever $m \leq p^e - 1$. Therefore,

α belongs to the set $\mathscr{A}(S)$, which proves that $\alpha \geq \alpha'$.

Next we prove that $\alpha \geq \alpha'$. Fix any $\lambda < \alpha'$. Then, whenever $m \leq \lambda(p^e - 1)$ and $f \in S_m$ is any non-zero element, we know that f is not contained in $I_e(m)$. By Lemma IV.2.6, we have $\mathrm{fpt}_{\mathfrak{m}}(S, f) \geq \frac{1}{p^e-1}$. In other words, if e is the smallest integer such that $m \leq \lambda(p^e - 1)$, (equivalently, $e = \lceil \log_p(\frac{m}{\lambda}) \rceil$), then $\mathrm{fpt}_{\mathfrak{m}}(S, f) \geq \frac{1}{p^e-1}$. Now combining this with the fact that $\mathrm{fpt}_{\mathfrak{m}}(S, f^a) = \frac{\mathrm{fpt}_{\mathfrak{m}}(S,f)}{a}$ for any integer a to get:

$$\mathrm{fpt}_{\mathfrak{m}}(S, f) \geq \sup_{a \geq 1} \frac{a}{p^{\lceil \log_p(\frac{am}{\lambda}) \rceil} - 1} \geq \frac{\lambda}{m}. \tag{IV.2.2}$$

To see the right inequality, we make the following observations: Fixing m and λ and for any a, write

$$\lceil \log_p(a) + \log_p(\frac{m}{\lambda}) \rceil = \log_p(a) + \log_p(\frac{m}{\lambda}) + \varepsilon(a)$$

for some non-negative real number $\varepsilon(a)$. Then, we have

$$\inf_{a \geq 1} \frac{p^{\log_p(a) + \log_p(\frac{m}{\lambda}) + \varepsilon(a)} - 1}{a} = \inf_{a \geq 1} \frac{m}{\lambda} p^{\varepsilon(a)} - \frac{1}{a} \leq \inf_{a \geq 1} \frac{m}{\lambda} p^{\varepsilon(a)}$$

for each $a \geq 1$. So it is sufficient to show that

$$\inf_{a \geq 1} p^{\varepsilon(a)} = 1.$$

This is true because given any real number γ, the infimum of the set $\{\lceil \gamma + \log_p(a) \rceil - \gamma - \log_p(a) \,|\, a \in \mathbb{N}\}$ is zero. This proves the inequality in (IV.2.2) .

Since f was an arbitrary non-zero homogeneous element of degree m, it follows from Equation (IV.2.2) that $\alpha \geq \lambda$. Since λ was an arbitrary number smaller than α', we must have $\alpha \geq \alpha'$ as well. This completes the proof that $\alpha = \alpha'$. $\square$

IV.2.2: Finite-degree approximations

Now we will define finite-degree approximations to the α_F-invariant. This establishes a limit formula for the α_F-invariant that is analogous to the F-signature (see Definition II.2.2 and the classical definition in [Tuc12]).

Definition IV.2.7. Let S be an $\mathbb{N}$-graded section ring over k. For each integer $e \geq 1$, we define

$$m_e(S) := \max\{m \geq 0 \,|\, I_e(m) = 0\}$$

and define

$$\alpha_e(S) := \frac{m_e(S)}{p^e}.$$

Theorem IV.2.8. *Let S be a strongly F-regular $\mathbb{N}$-graded section ring over k. Then, we have*

$$\lim_{e\to\infty} \alpha_e(S) = \alpha_F(S).$$

In particular, the limit exists. See Definition IV.2.2 for the definition of $\alpha_F(S)$.

Lemma IV.2.9. *Let S be a strongly F-regular $\mathbb{N}$-graded section ring over k. For any $e \geq 1$, we have*

$$\alpha_e(S) + \frac{1}{p^e} \geq \alpha_{e+1}(S) + \frac{1}{p^{e+1}}.$$

Proof. First note that since S is strongly F-regular, S is a normal domain. For any $e \geq 1$, let $0 \neq f$ be an element of $I_e(m_e+1)$. Then, we have that f^p is a non-zero element of $I_{e+1}(p(m_e+1))$ (see [Tuc12, Lemma 4.4]). This proves that

$$p(m_e+1) \geq m_{e+1} + 1.$$

Dividing both sides by p^{e+1}, we obtain the required inequality. □

Proof of Theorem IV.2.8. The sequence $\{\alpha_e + \frac{1}{p^e}\}_{e\geq 1}$ is decreasing, by Lemma IV.2.9. Since it is a decreasing sequence of non-negative real numbers, the sequence converges to its infimum. Moreover, since the sequence $\frac{1}{p^e}$ converges to zero, the sequence $\{\alpha_e\}_{e\geq 1}$ also converges and

$$\lim_{e\to\infty} \alpha_e = \inf_{e\geq 1}\{\alpha_e + \frac{1}{p^e}\}.$$

It remains to show that the limit is equal to $\alpha_F(S)$. Using the definition of α_e, we have that

$$\alpha_F(S) \leq \frac{p^e}{p^e-1}(\alpha_e + \frac{1}{p^e}) = \frac{m_e+1}{p^e-1}$$

for each $e \geq 1$. This is because we know that $\alpha_F(S)$ belongs to the set $\mathscr{A}(S)$ from Proposition IV.2.4. Taking a limit over e, we obtain

$$\alpha_F(S) \leq \lim_{e\to\infty} \alpha_e.$$

For the reverse inequality, setting $\alpha := \lim \alpha_e$, we note that

$$\alpha(p^e-1) \leq (\alpha_e + \frac{1}{p^e})(p^e-1) < p^e\alpha_e + 1.$$

By the definition of $m_e = p^e \alpha_e$, the subspace $I_e(m)$ is equal to zero for each $m \leq \alpha(p^e - 1) \leq m_e$. Thus, α belongs to the set $\mathscr{A}(S)$ defined in Proposition IV.2.4. Since $\alpha_F(S)$ is the supremum of $\mathscr{A}(S)$, we get that $\alpha_F(S) \geq \alpha$. This completes the proof of Theorem IV.2.8. $\square$

IV.2.3: Positivity and comparison to the F-signature.

Next we will show that the α_F-invariant is positive by comparing it to the F-signature (Definition II.2.2). Recall that for a section ring S of any globally F-regular projective variety, there exists a positive constant C such that for any $e > 0$ and any $m \leq C\, p^e$, we have $I_e(m) = S_m$. This follows from Theorem III.2.8.

Theorem IV.2.10. *The α_F-invariant of a strongly F-regular section ring is positive. Moreover, setting $\alpha = \alpha_F(S)$ and fixing a constant C as discussed above (so that for any $e > 0$ and any $m \leq C\, p^e$, we have $I_e(m) = S_m$), we have the following comparisons:*

$$\frac{e(S)\, \alpha^{\dim(S)}}{\dim(S)!} \leq \mathcal{s}(S) \leq \frac{e(S)}{\dim(S)!}\big(C^{\dim(S)} - (C - \alpha)^{\dim(S)}\big), \tag{IV.2.3}$$

where $e(S)$ denotes the Hilbert-Samuel multiplicity of S.

Lemma IV.2.11. *Given a non-zero homogeneous element f in S of degree n, let $\lambda > n\, fpt_{\mathfrak{m}}(S, f)$ be a real number. Then,*

$$\mathcal{s}(S) \leq \frac{e(S)}{\dim(S)!}\big(C^{\dim(S)} - (C - \lambda)^{\dim(S)}\big). \tag{IV.2.4}$$

Proof. Since we have assumed that $\lambda > n\, \mathrm{fpt}_{\mathfrak{m}}(S, f)$, there exist integers $a, e_0 > 0$ such that

$$\mathrm{fpt}_{\mathfrak{m}}(S, f) < \frac{a}{p^{e_0} - 1} < \frac{\lambda}{d}.$$

Replacing f by f^a, by Lemma IV.2.6 we may assume that the map $S \to F^{e_0}_* S$ defined by $1 \to F^{e_0}_* f$ does not split (since $\mathrm{fpt}_{\mathfrak{m}}(S, f^a) < \frac{1}{p^{e_0} - 1}$). We may also assume that $n \leq \lambda\, (p^{e_0} - 1)$. Now, since f belongs to the ideal I_{e_0}, we have $S_{m-n} \cdot f \subset I_{e_0}(m)$ for any $m \geq d$ yielding the inequality

$$\dim_k \frac{S_m}{I_{e_0}(m)} \leq \dim_k S_m - \dim_k S_{m-n}. \tag{IV.2.5}$$

for all $m \geq n$. Further, setting $v_r = \frac{p^{re_0} - 1}{p^{e_0} - 1}$ for any integer r, we have that $\mathrm{fpt}_{\mathfrak{m}}(S, f^{v_r}) < \frac{1}{p^{re_0} - 1}$, and so f^{v_r} belongs to $I_{re_0}(nv_r)$. Therefore, we similarly have

$$\text{(IV.2.6)} \qquad \dim_k \frac{S_m}{I_{ne_0}(m)} \leq \dim_k S_m - \dim_k S_{m-nv_r}$$

for all $m \geq nv_r$. Then, using the Lemma IV.1.5 we may compute the F-signature $s(S)$ as follows:

$$\begin{aligned} s(S) &= \frac{1}{[k':k]} \lim_{r\to\infty} \frac{\sum_{m=0}^{m=Cp^{re_0}} \dim_k \frac{S_m}{I_{re_0}(m)}}{p^{re_0 \dim(S)}} \\ &\leq \frac{1}{[k':k]} \left(\lim_{r\to\infty} \frac{\sum_{m=0}^{m=Cp^{re_0}} \dim_k S_m}{p^{re_0 \dim(S)}} - \frac{\sum_{m=dv_r}^{m=Cp^{re_0}} \dim_k S_{m-nv_r}}{p^{re_0 \dim(S)}} \right), \end{aligned}$$

where k' is the field S_0. Here we have used Equation (IV.2.6) and the defining property of the constant C. Finally, calculating the dimensions in the above inequality using the formula

$$\dim_k S_m = [k':k] \frac{e(S)}{(\dim S - 1)!} m^{\dim S - 1} + o(m^{\dim S - 2}),$$

we obtain

$$s(S) \leq \frac{e(S)}{\dim(S)!} \big(C^{\dim(S)} - (C - \frac{n}{p^{e_0} - 1})^{\dim(S)} \big).$$

The proof of the lemma is now complete by using the fact that $\lambda \geq \frac{n}{p^{e_0}-1}$.

$\square$

Proof of Theorem IV.2.10. We note that if $\alpha_F(S) = 0$, then the rightmost inequality of Equation (IV.2.3) implies that the F-signature $s(S)$ is zero. But this is a contradiction since S was assumed to be strongly F-regular (see [AL03, Theorem 0.2]). So the positivity of $\alpha_F(S)$ follows from Equation (IV.2.3), which we will now prove.

The rightmost inequality follows from Lemma IV.2.11 by taking a limit as $\lambda \to \alpha_F(S)$, since the Lemma applies to each λ such that $\lambda > \mathrm{fpt}_{\mathfrak{m}}(S, f) \deg(f)$ for some non-zero f.

To prove the leftmost inequality, we use Lemma IV.1.5 again to compute the F-signature of S and we observe that for any $e \geq 1$,

$$\text{(IV.2.7)} \qquad s(S) = \lim_{e\to\infty} \frac{\sum_{m=0}^{m=Cp^e} \dim_k \frac{S_m}{I_e(m)}}{p^{e \dim(S)}} \geq \lim_{e\to\infty} \frac{\sum_{m=0}^{m=m_e} \dim_k S_m}{p^{e \dim(S)}}.$$

Recall that for any $e \geq 1$, m_e is the largest m such that $I_e(m) = 0$, which justifies Equation (IV.2.7). But the right hand side is equal to

$$\lim_{e\to\infty} \frac{e(S)}{\dim(S)!}\Big(\frac{m_e}{p^e}\Big)^{\dim(S)}.$$

We conclude the proof by using Theorem IV.2.8, which says that $\lim_{e\to\infty} \frac{m_e}{p^e} = \alpha_F(S)$. This concludes the proof of Equation (IV.2.3) and thus of Theorem IV.2.10. $\square$

Remark IV.2.12. The positivity of the α_F-invariant proved in Theorem IV.2.10 can also be deduced from the main theorem of [Sat18].

IV.2.4: Behaviour under certain ring extensions.

In this subsection, we record some useful results on the behaviour of the α_F-invariant under suitably nice extensions of section rings.

Proposition IV.2.13. *Let S and S' be two $\mathbb{N}$-graded section rings (of possibly different varieties) and $S \subset S'$ be an inclusion such that for a fixed integer $n \geq 1$ and any other m, all degree m elements of S are mapped to degree nm elements of S'. Further, assume that the inclusion $S \subset S'$ splits as a map of S-modules. Then, we have*

$$\alpha_F(S) \geq \frac{\alpha_F(S')}{n}. \tag{IV.2.8}$$

Moreover, equality holds in (IV.2.8) if S is the n^{th}-Veronese subring of S'.

Proof. The first part follows immediately from Theorem IV.2.8 and the fact that a homogeneous element f of S splits from $F^e_* S$ if it splits from $F^e_* S'$. In other words, we have

$$I_e(S_m) \subset I_e(S'_{mn})$$

for any e and m. For the statement about Veronese subrings, the key observation is that since S' is a section ring (of (X, L), say), then

$$I_e(S'_{mn}) = I_e(S_m) = I_e(X, L^{mn}).$$

Thus, the equality again follows from Theorem IV.2.8. $\square$

Remark IV.2.14. The F-signature of section rings also transforms in a similar manner to the α_F-invariant above. Indeed, see Theorem III.1.6 and its generalization in [CR17, Theorem 4.8]. A simple proof of this transformation rule for section rings can also be obtained using Proposition IV.1.8.

Proposition IV.2.15. *Let $(S, \mathfrak{m}) \subset (S', \mathfrak{n})$ be a degree preserving map of $\mathbb{N}$-graded, strongly F-regular section rings (of possibly different varieties and over possibly different perfect fields). Assume that both S and S' are generated in degree one , S' is flat over S, and that the ring $S'/\mathfrak{m}S'$ is regular. Then, we have*

$$\alpha_F(S') = \alpha_F(S).$$

Proof. First note that since S and S' are generated in degree 1, the α_F-invariant must be at most 1 for both of them. Next, for any $e \geq 1$ and $m \leq p^e - 1$, we may apply Claim 3.4 in [CRST21, Proof of Theorem 3.1] to get that

$$I_e(S, m)\, S' = I_e(S', m).$$

This implies that $m_e(S) = m_e(S')$ for any $e \geq 1$. Now, the Proposition follows immediately by using Theorem IV.2.8. $\square$

Recall that k was assumed to be a perfect field of characteristic $p > 0$.

Corollary IV.2.16. *Let S be a strongly F-regular section ring over k and K be an arbitrary perfect field extension of k. Then, the base-change $S \otimes_k K$ is isomorphic to a product of strongly F-regular section rings S^i over finite extensions of K and for each i, we have*

$$\alpha_F(S) = \alpha_F(S^i).$$

Proof. Firstly, we may assume that S is generated in degree one by using Proposition IV.2.13. Set $S' = S \otimes_k K$. Since k is perfect and $S/\mathfrak{m}$ is a finite separable extension of k, we see that

$$S'/\mathfrak{m}S' \cong S/\mathfrak{m} \otimes_k K \cong \prod_i L_i$$

is a finite product of perfect fields L_i. Thus, if S was the section ring of X, then $S' \cong \prod_i S^i$ where S^i is the section ring of $X \times_{S/\mathfrak{m}} L_i$. Note that S^i is isomorphic to $S \otimes_{S/\mathfrak{m}} L_i$, and hence each S^i is strongly F-regular by [CRST21, Theorem 3.6]. The corollary now follows from Proposition IV.2.15 by applying it to each inclusion $S \subset S^i$. $\square$

Remark IV.2.17. In the setting of Corollary IV.2.16, the same results also holds for the F-signature of S instead of the α_F-invariant by using [Yao06, Theorem 5.6] in place of Proposition IV.2.15. This allows to reduce computing both the F-signature and the α_F-invariant of a section ring to the geometrically connected case, i.e., when $S_0 = k$.

IV.3: The α_F-invariant of Globally F-regular Fano Varieties.

In this section, we specialize the study of the α_F-invariant to the case of globally F-regular Fano varieties (and when the ample divisor is a multiple of $-K_X$). We begin by defining what we mean by a $\mathbb{Q}$-*Fano variety* in positive characteristic. Recall that k denotes a perfect field of characteristic $p > 0$.

IV.3.1: $\mathbb{Q}$-Fano varieties and the main theorems.

Definition IV.3.1. A $\mathbb{Q}$-*Fano variety* X is a projective variety over k such that

1. X is *locally* strongly F-regular (Definition II.3.2).
2. K_X is a $\mathbb{Q}$-Cartier divisor.
3. $-K_X$ is ample.

Note that since X has only strongly F-regular singularities, X is automatically normal and Cohen-Macaulay. In particular, we may define the canonical Weil-divisor K_X by extending a canonical divisor from the smooth locus. In fact, $\omega_X = \mathcal{O}_X(K_X)$ is a dualizing sheaf over X. In particular, in case $H^0(X, \mathcal{O}_X) = k$, we have

$$H^d(X, \omega_X) \cong k \tag{IV.3.1}$$

where d is the dimension of X. Moreover, the second and third conditions in Definition IV.3.1 guarantee that there is a positive integer r such that rK_X is Cartier and $-rK_X$ is ample. The smallest such r is called the index of K_X.

Definition IV.3.2. Let X be a globally F-regular $\mathbb{Q}$-Fano variety over k and r be a positive integer divisible by the index of K_X. Let

$$S := S(X, -rK_X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(-mrK_X))$$

denote the section ring of X with respect to $-rK_X$. Then, the α_F-*invariant* of X is defined to be

$$\alpha_F(X) := r\, \alpha_F(S)$$

where $\alpha_F(S)$ denotes the α_F-invariant of the strongly F-regular ring S, as defined in Defintion IV.2.2.

By taking the affine cone over a $\mathbb{Q}$-Fano variety, we also define the global F-signature of the a $\mathbb{Q}$-Fano variety.

Definition IV.3.3 (F-signature of a Fano variety)**.** Let X be a $\mathbb{Q}$-Fano variety over k and r denote a positive integer divisible by the index of K_X. Let

$$S := S(X, -rK_X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(-mrK_X))$$

denote the section ring of X with respect to $-rK_X$. Then, the *F-signature* of X is defined to be

$$\mathcal{s}(X) := r\, \mathcal{s}(S)$$

where $\mathcal{s}(S)$ denotes the F-signature of S, as defined in Defintion II.2.2.

Remark IV.3.4. Though the definitions of the α_F-invariant and the F-signature involve making a choice of a multiple of the index of K_X, both invariants are well-defined thanks to Proposition IV.2.13 (for the α_F-invariant) and [VK12, Theorem 2.6.2] (for the F-signature).

Theorem IV.3.5. *Let X be a globally F-regular $\mathbb{Q}$-Fano variety of positive dimension. Then, $\alpha_F(X)$ is at most $1/2$.*

Proof of Theorem IV.3.5: Let d denote the dimension of X, and $r \gg 0$ be an integer divisible by the index of K_X and such that $H^0(X, \mathcal{O}_X(-mK_X)) \neq 0$ for all $m \geq r$. First, we claim that there is an integer $n > 0$ such that

$$\dim_k H^0(X, \mathcal{O}_X(-mK_X)) < \dim_k H^0(X, \mathcal{O}_X(-(m+n)K_X))$$

for all $m \gg 0$. This is clear if $-K_X$ is a Cartier divisor (and we may take $n = 1$ in this case), since $\dim_k H^0(X, \mathcal{O}_X(-mK_X))$ is a polynomial in m of degree $d > 0$ and a positive leading term (because $-K_X$ is ample). More generally, by the asymptotic Riemann-Roch formula ([Laz04, Example 1.2.19]), for each $0 \leq i \leq r-1$, there exists polynomials P_i of degree d such that for all $m \gg 0$

$$\dim_k H^0(X, \mathcal{O}_X(-(i+mr)K_X)) = P_i(m).$$

Moreover, setting $V = (-rK_X)^d$, each P_i has the form

$$P_i(m) = \frac{V}{d!} m^d + Q_i(m)$$

for polynomials Q_i of degree at most $d-1$. In this situation, the existence of an integer n as required is guaranteed by Lemma IV.3.6 stated and proved below.

Assume, for the sake of contradiction, that $\alpha_F(X) > \frac{1}{2} + \varepsilon$ for some small $\varepsilon > 0$. Then, note that by Proposition IV.2.4, for all $e \gg 0$, we have $I_e(-mrK_X) = 0$ for all $m \leq \frac{p^e-1}{2r} + \frac{\varepsilon}{r}(p^e-1)$.

Now, for $e \gg 0$, we can find an integer m satisfying the following properties:

- $mr < \frac{p^e-1}{2}$,
- $p^e - 1 - mr + 2r < \frac{p^e-1}{2} + \varepsilon(p^e-1)$, and,
- $n \leq p^e - 1 - 2mr$.

This is equivalent to finding an integer m such that

$$\frac{p^e-1}{2r} + 2 - \frac{\varepsilon}{r}(p^e-1) \leq m \leq \frac{p^e-1}{2r} - \frac{n}{2r},$$

which is possible since n is fixed and $\frac{\varepsilon}{2}(p^e-1) \to \infty$ as $e \to \infty$. The third condition on m guarantees that

$$\dim_k H^0(X, -mrK_X) < \dim_k H^0(X, -(p^e-1-mr)K_X). \tag{IV.3.2}$$

The second condition guarantees that there exists a non-zero effective Weil divisor $E \geq 0$ that induces an injective map

$$\mathcal{O}_X(-(p^e-1-mr)K_X) \hookrightarrow \mathcal{O}_X(-m'rK_X)$$

for some $m < m' \leq \frac{p^e-1}{2r} + \frac{\varepsilon}{r}(p^e-1)$. Therefore, we know that $I_e(-mrK_X) = I_e(-m'rK_X) = 0$ as noted above. By Lemma III.2.11, this implies that $I_e(-(p^e-1-mr)K_X) = 0$ as well. Finally, note that Lemma IV.1.7 applied to $D = -mrK_X$ tells us that

$$\dim_k \frac{H^0(X, \mathcal{O}_X(-mrK_X))}{I_e(-mrK_X)} = \dim_k \frac{H^0(X, \mathcal{O}_X(-(p^e-1-mr)K_X))}{I_e(-(p^e-1-mr)K_X)}$$

But since both the I_e-subspaces in the above equation are zero, this is in contradiction to Equation (IV.3.2). This proves that $\alpha_F(X)$ is at most 1/2. □

The following lemma was used in the proof of Theorem IV.3.5.

Lemma IV.3.6. *For each $0 \leq i < r$, let P_i be a polynomial with real coefficients. Moreover, assume that all the P_i's have the same positive degree and the same positive leading term.*

In other words, for each $0 \leq i \leq r-1$, there exists a real polynomial Q_i of degree at most $d-1$ such that

$$P_i(m) = a_d m^d + Q_i(m)$$

for some real number $a_d > 0$ (independent of i). Then, there is an integer $n \gg 0$ such that for all integers $m \gg 0$, and all pairs $0 \leq i, j \leq r-1$, we have

$$P_i(m) \leq P_j(m+n).$$

Proof. Since each Q_i is a real polynomial of degree at most $d-1$, we may find positive constants C_1, C_2 such that

$$-C_2 m^{d-1} \leq Q_i(m) \leq C_1 m^{d-1}$$

for each $m \geq 0$ and each $0 \leq i \leq r-1$. Now, it is sufficient to find a constant n such that

$$a_d m^d + C_1 m^{d-1} \leq a_d (m+n)^d - C_2 (m+n)^{d-1} \tag{IV.3.3}$$

for all $m \gg 0$. For this, expanding $(m+n)^d$ using the binomial theorem, Equation (IV.3.3) is equivalent to

$$C_1 m^{d-1} \leq a_d (dnm^{d-1} + \cdots + dmn^{d-1} + n^d) - C_2 (m+n)^{d-1}.$$

First, we note that we may choose $n \gg 0$ such that $\frac{a_d dn}{2}$ is larger than both C_1 and C_2. This implies that $\frac{a_d dn}{2} m^{d-1} > C_1 m^{d-1}$ for all $m \geq 0$. Furthermore, having chosen the $n \gg 0$ as before, we have $\frac{a_d dn}{2} m^{d-1} \geq C_2 (m+n)^{d-1}$ for all $m \gg 0$. This proves the lemma. $\square$

For the rest of this section, we assume that $H^0(X, \mathcal{O}_X) = k$, i.e., that X is geometrically connected. However, see Remark IV.2.17 for ways to extend the results to more general cases. In the case of Fano varieties, we have a stronger version of comparison of the α_F-invariant to the F-signature than the formula in Theorem IV.2.10:

Theorem IV.3.7. *Let X be a d-dimensional globally F-regular $\mathbb{Q}$-Fano variety over k. Assume that $H^0(X, \mathcal{O}_X) = k$ and that d is positive. Set $\alpha = \alpha_F(X)$. Then, we have the following inequalities relating the F-signature and the α_F-invariant:*

$$\frac{2\,\alpha^{d+1}\,\mathrm{vol}(X)}{(d+1)!} \leq \mathcal{s}(X) \leq \frac{2\left((\frac{1}{2})^{d+1} - (\frac{1}{2} - \alpha)^{d+1}\right)\mathrm{vol}(X)}{(d+1)!} \tag{IV.3.4}$$

Here, $\mathrm{vol}(X)$ *denotes the volume of the* $\mathbb{Q}$*-Cartier divisor* $-K_X$*.*

Corollary IV.3.8. *Let* X *be a globally* F*-regular* $\mathbb{Q}$*-Fano variety of dimension* $d > 0$*. Then,*

1. *We have*
$$\mathcal{s}(X) \leq \frac{\mathrm{vol}(X)}{2^d(d+1)!}.$$

2. *Moreover,* $\alpha_F(X)$ *is equal to* $1/2$ *if and only if the value of the* F*-signature* $\mathcal{s}(X)$ *is equal to*
$$\frac{\mathrm{vol}(X)}{2^d(d+1)!}.$$

For $\mathbb{Q}$-Fano varieties, the F-signature has a more refined formula than Lemma IV.1.5, which we prove next.

Proposition IV.3.9. *Let* X *be a globally* F*-regular* $\mathbb{Q}$*-Fano variety over* k*. Assume* $H^0(X, \mathcal{O}_X) = k$ *and that* X *is positive dimensional. Let* r *be an integer such that* rK_X *is Cartier. Then, the* F*-signature of* X *can be computed as*

$$\mathcal{s}(X) = \lim_{e\to\infty} \frac{2r \sum_{m=0}^{\lfloor \frac{p^e-1}{2r} \rfloor} \dim_k \frac{H^0(-mrK_X)}{I_e(-mrK_X)}}{p^{e(\dim(X)+1)}}.$$

Lemma IV.3.10. *Let* X *be a globally* F*-regular* $\mathbb{Q}$*-Fano variety. Then, for any* $r > 0$*, we have* $H^0(-mrK_X) = I_e(-mrK_X)$ *whenever* $m > \frac{p^e-1}{r}$*.*

Proof. We will prove this lemma in two different ways, since both ideas may be useful in other situations.

Proof 1: Let $m > \frac{p^e-1}{r}$. By definition of the subspace I_e (Definition IV.1.2), it is sufficient to show that there are no non-zero maps $\phi : F^e_*\mathcal{O}_X(-mrK_X) \to \mathcal{O}_X$. We have

$$\mathrm{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(-mrK_X, \mathcal{O}_X) \cong H^0(X, (1-p^e+mr)K_X).$$

By assumption, we have $1-p^e+mr > 0$. Since $-rK_X$ is ample, this means that $H^0(X, (1-p^e+mr)K_X) = 0$. Hence, there are no non-zero maps $\phi : F^e_*\mathcal{O}_X(-mrK_X) \to \mathcal{O}_X$, which proves the lemma.

Proof 2: Let $m > \frac{p^e-1}{r}$ and suppose that there is a non-zero global section $f \in H^0(-mrK_X)$ that is not in $I_e(-mrK_X)$. Then, by definition of I_e, we have a map

$$\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(-mrK_X), \mathcal{O}_X)$$

such that $\phi(F^e_* f) = 1$. Thus, we have the splitting

$$\mathcal{O}_X \hookrightarrow F^e_* \mathcal{O}_X(-mrK_X) \twoheadrightarrow \mathcal{O}_X \tag{IV.3.5}$$

where the first map is got by sending $1 \to F^e_* f$ and the second map is ϕ. Twisting equation IV.3.5 by $\mathcal{O}_X(K_X)$ and reflexifying, we obtain a splitting:

$$\mathcal{O}_X(K_X) \hookrightarrow F^e_* \mathcal{O}_X(-(mr - p^e)K_X) \twoheadrightarrow \mathcal{O}_X(K_X). \tag{IV.3.6}$$

By assumption, $mr - p^e$ is non-negative. Hence, $H^d(X, F^e_* \mathcal{O}_X(-(mr - p^e)K_X)) = 0$ by Proposition II.3.8, in turn implying that $H^d(X, \mathcal{O}_X(K_X)) = 0$ (using the splitting in equation IV.3.6). This is a contradiction, since $\mathcal{O}_X(K_X)$ is the canonical sheaf of X. This completes the proof of Lemma IV.3.10. $\square$

Proof of Proposition IV.3.9. Let S denote the section ring of X with respect to $-rK_X$. Fix an $e > 0$ and let a_e denote the free rank of $F^e_* S$ as an S-module (Definition II.2.1). Recall that by Lemma IV.1.5, we have

$$a_e = \sum_{m=0}^{\infty} \dim_k \frac{H^0(-mrK_X)}{I_e(-mrK_X)} \tag{IV.3.7}$$

so that

$$\mathcal{s}(X) = r \lim_{e \to \infty} \frac{a_e}{p^{e(\dim(X)+1)}}.$$

Then, Lemma IV.3.10 shows that the terms of the sum in Equation (IV.3.7) are zero for $m > \frac{p^e-1}{r}$. Furthermore, using Lemma IV.1.7, we have

$$\sum_{m=\lceil \frac{p^e-1}{2r} \rceil}^{\lfloor \frac{p^e-1}{r} \rfloor} \dim_k \frac{H^0(-mrK_X)}{I_e(-mrK_X)} = \sum_{m=\lceil \frac{p^e-1}{2r} \rceil}^{\lfloor \frac{p^e-1}{r} \rfloor} \dim_k \frac{H^0(-(p^e-1-mr)K_X)}{I_e(-(p^e-1-mr)K_X)}.$$

Let a be an integer between 0 and r such that $p^e - 1 \equiv a \mod r$. Hence, we have

$$\sum_{m=\lceil \frac{p^e-1}{2r} \rceil}^{\lfloor \frac{p^e-1}{r} \rfloor} \dim_k \frac{H^0(-(p^e-1-mr)K_X)}{I_e(-(p^e-1-mr)K_X)} = \sum_{m=0}^{\lfloor \frac{p^e-1}{2r} \rfloor} \dim_k \frac{H^0(-(a+mr)K_X)}{I_e(-(a+mr)K_X)}.$$

Thus, we have

$$a_e = \sum_{m=0}^{\lfloor \frac{p^e-1}{2r} \rfloor} \dim_k \frac{H^0(-mrK_X)}{I_e(-mrK_X)} + \sum_{m=0}^{\lfloor \frac{p^e-1}{2r} \rfloor} \dim_k \frac{H^0(-(a+mr)K_X)}{I_e(-(a+mr)K_X)}.$$

Moreover, using Proposition IV.1.8, we see that there is a constant $C > 0$ such that

$$\left| \frac{H^0(-(a+mr)K_X)}{I_e(-(a+mr)K_X)} - \frac{H^0(-mrK_X)}{I_e(-mrK_X)} \right| < Cp^{e(\dim(X)-1)}.$$

Thus, we have that

$$\left| a_e - 2\sum_{m=0}^{\lfloor \frac{p^e-1}{2r} \rfloor} \dim_k \frac{H^0(-mrK_X)}{I_e(-mrK_X)} \right| < Cp^{e\dim(X)}.$$

The proof is now complete since the right hand side limits to zero when divided by $p^{e(\dim(X)+1)}$ and as $e \to \infty$. $\square$

Proof of Theorem IV.3.7. The proof of this Theorem is exactly the same proof as the proof of Equation (IV.2.3) in Theorem IV.2.10 (see the proof of Lemma IV.2.11), once we replace the formula from Lemma IV.1.5 with the formula from Proposition IV.3.9 to compute the F-signature of $S = S(X, -rK_X)$. $\square$

Proof of Corollary IV.3.8. Part (1) follows immediately from the right-hand inequality in Theorem IV.3.7, since we know that $\alpha_F(X) \leq \frac{1}{2}$ by Theorem IV.3.5. We also see that if $\alpha_F(X) < \frac{1}{2}$, we must have

$$\mathfrak{s}(X) < \frac{\mathrm{vol}(X)}{2^d(d+1)!}.$$

Thus, Part (2) also follows from Equation (IV.3.4) once we note that when $\alpha_F(X) = \frac{1}{2}$, both sides of the inequality in Equation (IV.3.4) are equal to $\frac{\mathrm{vol}(X)}{2^d(d+1)!}$. $\square$

Remark IV.3.11. Let X be a $\mathbb{Q}$-Fano variety over $\mathbb{C}$. This means that X is a normal variety, $-K_X$ is $\mathbb{Q}$-Cartier and ample and X has only klt singularities. Let r be such that $-rK_X$ is Cartier, and $S = S(X, -rK_X)$. Then, for any effective $\mathbb{Q}$-divisor Δ on X with $\Delta \sim_{\mathbb{Q}} -rK_X$, we have

$$\mathrm{lct}_{\mathfrak{m}}(S, \Delta_S) = \min\{\, \mathrm{lct}(X, \Delta),\, \frac{1}{r}\,\}$$

where lct denotes the log canonical threshold and Δ_S denotes the cone over Δ. This follows from [Kol13, Lemma 3.1]. Thus, if we let

$$\tilde{\alpha}(X) = r\ \inf\{\operatorname{lct}_{\mathfrak{m}}(S, \Delta_S) \mid \Delta \geq 0 \text{ is a } \mathbb{Q}\text{-divisor on } X \text{ such that } \Delta \sim_{\mathbb{Q}} -K_X\},$$

then, we have that

$$\tilde{\alpha}(X) = \min\{\alpha(X), 1\}.$$

Therefore, for any $\mathbb{Q}$-Fano variety with $\alpha(X) \leq 1$, the α_F-invariant from Definition IV.3.2 is a "Frobenius analog" of the complex α-invariant.

IV.3.2: The α_F-invariant of toric Fano varieties.

Let k denote an algebraically closed field of prime characteristic $p > 0$. Fix a lattice $N \cong \mathbb{Z}^d$ and let M be the dual lattice (where d is some positive integer).

Theorem IV.3.12. *Let X_p be a $\mathbb{Q}$-Fano toric variety over k defined by a fan $\mathcal{F}$ in N. Let $X_{\mathbb{C}}$ be the corresponding complex toric variety (which is also automatically $\mathbb{Q}$-Fano). Then, we have*

$$\alpha_F(X_p) = \alpha(X_{\mathbb{C}}).$$

Proof. Let $v_1, \dots, v_n$ denote the primitive generators for the one dimensional cones in $\mathcal{F}$ and write $-K_X = \sum_i b_i v_i$ for rational numbers b_i.

First we choose an $r > 0$ such that $rb_i \in \mathbb{Z}$ for each i and the section ring $S(X, -rK_X)$ is generated in degree one. Let $P \subset M$ denote the polytope associated to $-rK_X$, and defined by:

$$P = \{u \in M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle u, v_i\rangle \geq -b_i \text{ for all } 1 \leq i \leq n\}.$$

Since we are assuming that $S(X, -rK_X)$ is generated in degree 1, the vertices of P are lattice points of M. For any $u \in P \cap M$, let D_u be the corresponding effective divisor in the linear system $|-rK_X|$. By [BJ20, Corollary 7.16], we have that

$$\alpha(X_{\mathbb{C}}) = \min_{u \in P \cap M} r \operatorname{lct}(X_{\mathbb{C}}, D_u) \tag{IV.3.8}$$

where $\operatorname{lct}(X_{\mathbb{C}}, -)$ denotes the log canonical threshold of a divisor on $X_{\mathbb{C}}$. Note that since the vertices of P are lattice points, just the vertices are sufficient to compute $\alpha(X_{\mathbb{C}})$.

Let $\tilde{P}$ denote the polytope $P \times \{1\} \subset M \times \mathbb{Z}$. Then, the section ring $S(X, -rK_X)$ is the semigroup ring associated to the cone over $\tilde{P}$ in $(M \times \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Note that S is $\mathbb{Q}$-Gorenstein. Therefore, by [Bli04, Theorem 3], we see that for any $\tilde{u} \in \tilde{P} \cap (M \times \mathbb{Z})$, we have

$$\operatorname{fpt}_{\mathfrak{m}}(S(X_p, -K_{X_p}), D_{\tilde{u}}) = \operatorname{lct}_{\mathfrak{m}}(S(X_{\mathbb{C}}, -K_{X_{\mathbb{C}}}), D_{\tilde{u}}). \tag{IV.3.9}$$

Next, note that since X is a normal toric variety, it is automatically globally F-regular. Now we prove that the α_F-invariant of X_p can also be computed by only considering the torus invariant divisors. To see this, let $S = S(X_p, -rK_{X_p})$ and let $f \in S$ be a non-zero homogeneous element. Then, following the discussion in [BJ20, Section 7.4] and [Eis95, Theorem 15.17], there exists an integral weight vector $\mu = (\mu_1, \ldots, \mu_{d+1})$ with $\mu_i \in \mathbb{Z}_{>0}$ such that $\mathrm{in}_{>_\mu}(f) = \mathrm{in}_>(f)$. Here $>_\mu$ denotes the weight monomial order with respect to μ and $>$ denotes the graded lexicographic monomial order on S. Then, we have a flat degeneration of f to its initial term. In other words, if $f = \sum_u \beta_u \chi^u$ for monomials $\chi^u \in S$, then setting $w = \max\{\langle \mu, u\rangle \mid \beta_u \neq 0\}$, the element

$$\tilde{f} = t^w \sum_u \beta_u t^{-\langle \mu, u\rangle} \chi^u \in S[t]$$

satisfies the following properties:

- Viewing $S[t]$ as a $k[t]$-algebra, the ring $S[t]/(\tilde{f})$ is a flat $k[t]$-module.
- The image of $\tilde{f}$ modulo t is equal to $\mathrm{in}_>(f)$, the initial term of f with respect to the graded lex monomial order on S.
- For any point $0 \neq \lambda \in k$, the image f_λ of $\tilde{f}$ in $S[t]/(t-\lambda)$ satisfies

$$\mathrm{fpt}_{\mathfrak{m}}(S, f_\lambda) = \mathrm{fpt}_{\mathfrak{m}}(S, f).$$

With this construction in place, we conclude the proof of the theorem with the following lemma:

Lemma IV.3.13. *For any non-zero homogeneous element f of S, we have*

$$\mathit{fpt}_{\mathfrak{m}}(S, f) \geq \mathit{fpt}_{\mathfrak{m}}(S, \mathit{in}_>(f)).$$

Assuming this lemma for a moment, we see that

$$\alpha_F(X_p) = \inf_{\tilde{u} \in \tilde{P} \cap (M \times \mathbb{Z})} \mathrm{fpt}_{\mathfrak{m}}(S, D_{\tilde{u}}).$$

Furthermore, by Equation (IV.3.9), we have

$$\alpha_F(X_p) = \inf_{\tilde{u} \in \tilde{P} \cap (M \times \mathbb{Z})} \mathrm{lct}_{\mathfrak{m}}(S(X_{\mathbb{C}}, -rK_{X_{\mathbb{C}}}), D_{\tilde{u}}). \tag{IV.3.10}$$

Since by Theorem IV.3.5 we have $\alpha_F(X_p) \leq 1/2$, we must have $\mathrm{lct}(S(X_{\mathbb{C}}, -rK_{X_{\mathbb{C}}}), D_{\tilde{u}}) < \frac{1}{nr}$

for some $\tilde{u} = (u, n) \in \tilde{P} \cap (M \times \mathbb{Z})$. Note that D_u corresponds to a torus-invariant divisor on $X_{\mathbb{C}}$ linearly equivalent to $-nrK_{X_{\mathbb{C}}}$. Therefore, by Remark IV.3.11, we have

$$\operatorname{lct}(X_{\mathbb{C}}, D_u) = \operatorname{lct}_{\mathfrak{m}}(S(X_{\mathbb{C}}, -rK_{X_{\mathbb{C}}}), D_{\tilde{u}})$$

for any $\tilde{u} = (u, n)$ such that $\operatorname{lct}(S(X_{\mathbb{C}}, -rK_{X_{\mathbb{C}}}), D_{\tilde{u}}) < \frac{1}{nr}$. Putting this together with Equation (IV.3.10) and Equation (IV.3.8), we get that

$$\alpha_F(X_p) = \alpha(X_{\mathbb{C}})$$

as required. □

Finally, it remains to prove Lemma IV.3.13.

Proof of Lemma IV.3.13. By Lemma IV.2.6, it is sufficient to show that for all rational numbers of the form $\frac{a}{p^e-1}$ such that

$$\frac{a}{p^e - 1} < \operatorname{fpt}_{\mathfrak{m}}(S, \operatorname{in}_{>}(f)),$$

the map $S \to F^e_* S$ sending 1 to $F^e_* f^a$ splits. Equivalently, for all such $\frac{a}{p^e-1}$, it suffices to show that $f^a \notin I_e(S)$. Since the pair $(S, \operatorname{in}_{>}(f)^{\frac{a}{p^e-1}})$ is strongly F-regular, in particular it is sharply F-split. By Lemma IV.2.6 again, we know that $\operatorname{in}_{>}(f^a) = (\operatorname{in}_{>}(f))^a \notin I_e(S)$. Now, since S is a toric ring, $I_e(S)$ is a monomial ideal of S. Therefore, if $\operatorname{in}_{>}(f^a) \notin I_e(S)$, we also have $f^a \notin I_e(S)$ as required. □

Remark IV.3.14. Combining Theorem IV.3.12 with Theorem IV.3.5, we recover the well-known fact that the α-invariant of a toric Fano variety is at most 1/2 (see [LZ22, Corollary 3.6]).

IV.4: Examples

In this section, we compute some examples of the α_F-invariant for non-toric varieties and highlight some interesting features.

IV.4.1: Quadric hypersurfaces

Fix any algebraically closed field k of characteristic $p \neq 2$ and let $Q_d \subset \mathbb{P}^{d+1}$ be the d-dimensional smooth quadric hypersurface over k. Note that by the adjunction formula, $-K_{Q_d} = dH$ where H denotes a hyperplane section.

Example IV.4.1. Then, $\alpha_F(Q_d) = \frac{1}{d}$. Equivalently, if S denotes the section ring

$$S := S(Q_d, \mathcal{O}_X(1) \cong k[x_0, \dots, x_{d+1}]/(x_0^2 + \dots x_{d+1}^2),$$

then $\alpha_F(S) = 1$. This follows from a description of the structure of the sheaves $F_*^e(\mathcal{O}_{Q_d}(m))$ proved in [Lan08] and [Ach12]. More precisely, for any $e \geq 1$ and $0 \leq m \leq p^e - 1$, [Ach12, Theorem 2] tells us that $F_*^e(\mathcal{O}_{Q_d}(m))$ is a direct sum of $\mathcal{O}_{Q_d}(-t)$ and $\mathcal{S}(-t)$ for $t \geq 0$, where $\mathcal{S}$ is an ACM bundle that sits in an exact sequence of the form

$$0 \to \mathcal{O}_{\mathbb{P}^{d+1}}(-2)^{\oplus a} \to \mathcal{O}_{\mathbb{P}^{d+1}}(-1)^{\oplus b} \to i_*\mathcal{S} \to 0$$

for suitable positive integers a and b. Here $i : Q_d \hookrightarrow \mathbb{P}^{d+1}$ is the inclusion. See [Ach12, Section 1.3] for the details. Since $H^1(\mathbb{P}^{d+1}, \mathcal{O}_{\mathbb{P}^{d+1}}(-2)) = 0$, we deduce from the exact sequence above that $\mathcal{S}(-t)$ has no global sections for any $t \geq 0$. Therefore, all global sections of $F_*^e(\mathcal{O}_{Q_d}(m))$ appear in the trivial summands. In other words, $I_e(S(m)) = 0$ for any $e \geq 1$ and any $m \leq p^e - 1$. Moreover, since $S_1 \neq 0$, we know that $m_e = p^e - 1$. Therefore, by Theorem IV.2.8, we have

$$\alpha_F(S) = \lim_{e \to \infty} \frac{p^e - 1}{p^e} = 1.$$

Remark IV.4.2. This example shows that the α_F-invariant does not characterize regularity of section rings, since the α_F-invariant of a polynomial ring is also equal to 1.

Remark IV.4.3. Another interesting feature of this example is that the α_F-invariant of smooth quadrics is independent of the characteristic p (for $p \neq 2$). This is far from true in general (as seen in the next example). Furthermore, for any $d > 2$, the F-signature of Q_d is known to depend on p in a rather complicated way (see [Tri23]).

IV.4.2: Comparison to the complex α-invariant

Let $k = \mathbb{F}_p$ for some prime number $p \geq 5$ and $X_p \subset \mathbb{P}^3$ be the diagonal cubic surface defined by $x^3 + y^3 + z^3 + w^3 = 0$ over k.

Example IV.4.4. For each $p \geq 5$, we have $\alpha_F(X_p) < \frac{1}{2}$. However,

$$\lim_{p \to \infty} \alpha_F(X_p) = \frac{1}{2}.$$

To see this, we recall the following result proved by Shideler (see [Shi, Example 4.2.2 and Section 5.1]), building on the techniques of Han and Monsky: Let $\mathcal{s}_p$ denote the F-signature of X_p, equivalently, of the ring $\mathbb{F}_p[x, y, z, w]/(x^3 + y^3 + z^3 + w^3)$. Then for any $p \geq 5$, we have $\mathcal{s}_p < \frac{1}{8}$. Moreover,

$$\lim_{p \to \infty} \mathcal{s}_p = \frac{1}{8}.$$

Using this, our claims about the α_F-invariant of X_p follow from Theorem IV.3.7 and Corollary IV.3.8, once we observe that

$$\frac{\mathrm{vol}(-K_{X_p})}{2^2\,3!} = \frac{1}{8}.$$

Remark IV.4.5. The complex α-invariant of the cubic surface defined by $x^3 + y^3 + z^3 + w^3$ is equal to $2/3$ (see [Che08, Theorem 1.7]). Note that by [HY03], we know that for a fixed divisor D on a variety X

$$\lim_{p \to \infty} \mathrm{fpt}(X_p, D_p) = \mathrm{lct}(X, D)$$

where X_p and D_p denote the reduction to characteristic p of X and D respectively. Example IV.4.4 points to limitations of approximating the log canonical threshold by F-pure threshold for an unbounded family of divisors on X.

CHAPTER V
Semicontinuity Properties of the Frobenius-Alpha Invariant

In this chapter, we will examine the behaviour of the α_F-invariant in geometric families. First we prove a weak semicontinuity result for a family of globally F-regular varieties polarized by an arbitrary family of ample divisors (Section V.1). This is analogous to the case of the F-signature which was proved in [CRST21]. Next, we prove that the α_F-invariant is lower semicontinuous in a family of globally F-regular Fano varieties (Section V.2). The analogus statement for the complex α-invariant is proved in [BL22]. Throughout this chapter, k will denote an *algebraically closed* field of characteristic $p > 0$.

V.1: Weak Semicontinuity of the α_F-invariant.

First, we will prove a result for a family of arbitrary globally F-regular varieties.

Notation V.1.1. Recall that the perfection of a field K (of positive characteristic) is the union

$$K^\infty := \bigcup_{e=1}^{\infty} K^{1/p^e}$$

of all the p^e-th roots of elements of K. For a map $f : \mathfrak{X} \to Y$ of varieties over k, and a point $y \in Y$ (not necessarily closed), we denote the *perfectified fiber* over y as

$$X_{y^\infty} := \mathfrak{X} \times_Y \operatorname{Spec}(\kappa(y)^\infty).$$

Similarly, if $Y = \operatorname{Spec}(A)$ and S is a finitely generated A-algebra, then for any $y \in Y$, we denote the perfectified fiber $S \otimes_A \kappa(y)^\infty$ by S_{y^∞}.

Notation V.1.2. By a *family of globally F-regular varieties*, we mean that

1. We have a flat and projective morphism $f : \mathfrak{X} \to Y$ where $\mathfrak{X}$ is normal and Y is smooth over k.

2. We assume additionally that f has connected fibers (i.e., $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_Y$).

3. For each point $y \in Y$ (not necessarily closed), the fiber X_{y^∞} (Notation V.1.1) is globally F-regular (Definition II.3.2).

Recall that as in Definition IV.2.2, we can consider the α_F-invariant of a pair (X, L) where X is a globally F-regular projective variety (over a perfect field) and L is an ample line bundle over X.

Theorem V.1.3. *Let $f : (\mathcal{X}, \mathcal{L}) \to Y$ be a flat family of globally F-regular varieties, where $\mathcal{L}$ is an ample line bundle over $\mathcal{X}$. Let K denote the fraction field of Y and α_{gen} denote the α_F-invariant of $(X_{K^\infty}, \mathcal{L}|_{X_{K^\infty}})$, the perfectified generic fiber. Then, for each real number $0 < \alpha < \alpha_{gen}$, there exists a dense open subset $U_\alpha \subset Y$ such that*

$$\alpha_F(X_{y^\infty}, \mathcal{L}|_{X_{y^\infty}}) > \alpha \quad \textit{for every point } y \textit{ in } U_\alpha.$$

Recall that in Theorem IV.2.8, we defined the sequence $(\alpha_e)_{e\geq 1}$ that converges to the α_F-invariant. To prove Theorem V.1.3, we need to understand the rate of convergence in Theorem IV.2.8. For this we will use "degree-lowering operators" as below.

Theorem V.1.4. *Let X be a projective globally F-regular variety over k and L be an ample invertible sheaf over X. Suppose we have positive integers N and e such that the sheaf*

$$\mathscr{H}om_{\mathcal{O}_X}\big((F^e_*(L^m), L^N \big)$$

is generically globally generated for each $0 \leq m \leq p^e - 1$. Then, if S denotes the section ring $S(X, L)$, we have

$$|\alpha_F(S) - \alpha_e(S)| \leq \frac{N}{p^e - 1}.$$

Proof. Set $d = \dim(X)$. First, we claim that for each $0 \leq m \leq p^e - 1$, we can find an injective map of $\mathcal{O}_X$-modules

$$\iota_{e,m} : F^e_*(L^m) \hookrightarrow (L^N)^{\oplus p^{ed}}. \tag{V.1.1}$$

To see this, let η denote the generic point of X and consider the following restriction map to the generic stalk:

$$\operatorname{Hom}_{\mathcal{O}_X}(F^e_*(L^m), L^N) \to \mathscr{H}om_{\mathcal{O}_X}(F^e_*(L^m), L^N)_\eta \cong \operatorname{Hom}_{\mathcal{O}_{X,\eta}}(F^e_*(L^m_\eta), L^N_\eta). \tag{V.1.2}$$

The assumption that $\mathscr{H}om_{\mathcal{O}_X}\big((F^e_*(L^m), L^N \big)$ is generically globally generated means that the image of the map in Equation (V.1.2) generates $\mathrm{Hom}_{\mathcal{O}_{X,\eta}}(F^e_*(L^m_\eta), L^N_\eta)$ as an $\mathcal{O}_{X,\eta}$-module. Recall that $\mathcal{O}_{X,\eta}$ is just the fraction field of X. Since $F^e_*(L^m_\eta)$ is a free $\mathcal{O}_{X,\eta}$-vector space of rank p^{ed}, we can choose p^{ed} maps $\phi_1, \ldots, \phi_{p^{ed}}$ in $\mathrm{Hom}_{\mathcal{O}_X}(F^e_*(L^m), L^N)$ such that their images under the map in Equation (V.1.2) forms a basis of $\mathrm{Hom}_{\mathcal{O}_{X,\eta}}(F^e_*(L^m_\eta), L^N_\eta)$ over $\mathcal{O}_{X,\eta}$. Thus, defining $\iota_{e,m}$ to be the product map

$$\iota_{e,m} := \phi_1 \times \cdots \times \phi_{p^{ed}} : F^e_*(L^m) \to (L^N)^{\oplus p^{ed}},$$

we see that $\iota_{e,m}$ is generically an isomorphism by construction. Furthermore, since $F^e_*(L^m)$ is a torsion-free sheaf of rank p^{ed} over $\mathcal{O}_X$, $\iota_{e,m}$ is injective since it is generically an isomorphism. This completes the proof of the claim that maps as in Equation (V.1.1) exist.

Now, fix a map $\iota_{e,m}$ for each $0 \leq m \leq p^e - 1$ as in Equation (V.1.1). Then, by taking section modules with respect to L (Definition II.1.1), we get a corresponding map of graded S-modules

$$\iota_{e,m} : (F^e_* S)_{m \bmod p^e} \hookrightarrow S(N)^{\oplus p^{ed}}$$

and let

$$\iota_e : F^e_* S \hookrightarrow S(N)^{\oplus p^{e(d+1)}}$$

denote the direct sum of the $\iota_{e,m}$'s. Here, for any $m \in \mathbb{Z}$, $(F^e_* S)_{m \bmod p^e}$ denotes the S-module

$$(F^e_* S)_{m \bmod p^e} := \bigoplus_{j \in \mathbb{Z}} F^e_*(S_{m+jp^e}),$$

and we naturally have an $\mathbb{N}$-graded S-module decomposition

$$F^e_* S \cong \bigoplus_{0 \leq m \leq p^e - 1} (F^e_* S)_{m \bmod p^e}.$$

Note that ι_e is injective because the $\iota_{e,m}$'s were injective. Furthermore, the key property of ι_e that we will use is the following: for each non-zero homogeneous element f of degree m in S, $\iota_e(F^e_*(f))$ is a non-zero homogeneous element of degree

$$\lfloor \frac{m}{p^e} \rfloor + N. \tag{V.1.3}$$

Thus, we may use this map as a "degree-lowering operator".

From Theorem IV.2.8, recall that $\alpha_e := \alpha_e(S) = \frac{m_e}{p^e}$ where m_e is defined to be the number $\max\{m \,|\, I_e(m) = 0\}$. The condition that $I_e(m) = 0$ means that for all non-zero elements

$f \in S_m$, there exists a splitting $F^e_* S \to S$ that sends $F^e_* f$ to 1.

Claim: For any $\ell \geq 1$, let $e' = \ell e$, and m be an integer such that $m \leq (\alpha_e - \sum_{t=1}^{\ell-1} \frac{N}{p^{te}})p^{\ell e}$. Then $I_{e'}(m) = 0$.

We will prove the claim by induction on ℓ. If $\ell = 1$, note that the sum in the claim is empty, and hence we have $m \leq m_e = \alpha_e p^e$. In this case, $I_e(m) = 0$ by the definition of m_e.

Now let $\ell > 1$ and f be any non-zero element of S_m. We need to show that $F^e_* f$ splits from $F^e_* S$. To see this, note that by Equation (V.1.3), $\iota_e(F^e_*(f))$ is a non-zero element of degree at most

$$\lfloor \frac{1}{p^e} (\alpha_e - \sum_{t=1}^{\ell-1} \frac{N}{p^{te}}) \, p^{\ell e} \rfloor + N = m_e \, p^{(\ell-2)e} - N \, p^{(\ell-2)e} - \cdots - N + N = (\alpha_e - \sum_{t=1}^{\ell-2} \frac{N}{p^{te}}) \, p^{(\ell-1)e}.$$

Thus, the inductive hypothesis applies to $\iota_e(F^e_*(f))$, implying that $\iota_e(F^e_*(f))$ is not contained in $I_{e'-e}$. Let $\varphi : F^{e'-e}_* S \to S$ denote a splitting of $\iota_e(F^e_*(f))$. Then, we see that (forgetting the degrees) $\varphi \circ F^{e'-e}_* \iota_e : F^{e'}_* S \to S$ defines a splitting of $F^{e'}_*(f)$, as required. This completes the proof of the claim.

To complete the proof of the Theorem, we note that the claim above implies that for any $\ell \geq 1$,

$$\alpha_{\ell e} = \frac{m_{\ell e}}{p^{\ell e}} \geq \alpha_e - \sum_{t=1}^{\ell-1} \frac{N}{p^{te}}.$$

Letting $\ell \to \infty$ and using Theorem IV.2.8, we get

$$\alpha_e(S) - \alpha_F(S) \leq \frac{N}{p^e - 1}. \tag{V.1.4}$$

Lastly, note that by Lemma IV.2.9, we already have

$$\alpha_F(S) - \alpha_e(S) \leq \frac{1}{p^e},$$

which, together with Equation (V.1.4) completes the proof of the theorem. ☐

Lemma V.1.5. *Let X be a globally F-regular variety of dimension d and L be an ample and globally generated invertible sheaf. Suppose $e_0 > 0$ is such that $(1 - p^e)K_X$ is linearly equivalent to an effective Weil divisor for all $e \geq e_0$. Then, for all $e \geq e_0$, and each $0 \leq m \leq p^e - 1$, the sheaf*

$$\mathscr{H}om_{\mathcal{O}_X}(F^e_*(L^m), L^{d+1})$$

is generically globally generated.

Proof. First, by an application of Castelnuovo-Mumford regularity, we prove the following statement:

Claim: For all $e \geq 1$ and all $0 \leq n \leq p^e - 1$, the sheaf $F^e_*(L^{n+dp^e})$ is 0-regular with respect to L, and hence globally generated.

To see this, we check that

$$H^i(X, F^e_*(L^{n+dp^e}) \otimes L^{-i}) = H^i(X, L^{n+(d-i)p^e}) = 0 \quad \text{for all } i > 0.$$

Here, we have used the projection formula to see that $F^e_*(L^{n+dp^e}) \otimes L^{-i} \cong F^e_*(L^{n+(d-i)p^e})$ and the cohomology vanishing follows from (Theorem II.3.7), since $L^{n+(d-i)p^e}$ is nef for all $i \leq d$. Thus, $F^e_*(L^{n+dp^e})$ is 0-regular with respect to L, and hence is globally generated (see [Laz04, Theorem 1.8.5] for the details regarding Castelnuovo-Mumford regularity).

Next, for any $e \geq e_0$ and $n \geq 0$, write $(1-p^e)K_X \sim D_e$ where D_e is an effective Weil-divisor. Note that it is always possible to find such an e_0 thanks to [SS10, Theorem 4.3]. Let $\phi_e(L^n)$ denote the map obtained by twisting the defining map for D_e by L^n and pushing forward under F^e:

$$\phi_e(L^n) : F^e_*(L^n) \hookrightarrow F^e_*(\mathcal{O}_X(D_e) \otimes L^n).$$

Note that for any point $x \notin \operatorname{Supp}(D_e)$, ϕ_e restricts to an isomorphism in an open neighbourhood around x. For any sheaf $\mathcal{F}$, let $\mathcal{F}_x$ denote the stalk of $\mathcal{F}$ at x.

Applying duality for the Frobenius map (Equation (IV.1.2)), we have (for any $m \in \mathbb{Z}$):

$$\mathscr{H}om_{\mathcal{O}_X}(F^e_*(L^m), L^{d+1}) \cong F^e_*(\mathcal{O}_X(D_e) \otimes L^{dp^e+p^e-m}).$$

Therefore, for any $e \geq e_0$ and $0 \leq m \leq p^e - 1$, set $n = dp^e + p^e - m$ and consider the diagram

$$\begin{tikzcd}
H^0(X, F^e_*(L^n)) \arrow[r, hook, "H^0(\phi_e(L^n))"] \arrow[d] & H^0(X, F^e_*(\mathcal{O}_X(D_e) \otimes L^n)) \arrow[d] \\
F^e_*(L^n)_x \arrow[r, hook, "\phi_e(L^n)_x", "\sim"'] & F^e_*(\mathcal{O}_X(D_e) \otimes L^n)_x
\end{tikzcd}$$

where x is any point not contained in $\operatorname{Supp}(D_e)$. Since the horizontal arrows are injective and the bottom horizontal arrow is an isomorphism, any set of global sections generating $F^e_*(L^n)_x$, viewed as global sections of $F^e_*(\mathcal{O}_X(D_e) \otimes L^n)$, will also generate $F^e_*(\mathcal{O}_X(D_e) \otimes L^n)_x$. Since by the claim above $F^e_*(L^n)$ is globally generated, we have that $\mathscr{H}om_{\mathcal{O}_X}(F^e_*(L^m), L^{d+1}) \cong F^e_*(\mathcal{O}_X(D_e) \otimes L^n)$ is globally generated at any $x \notin \operatorname{Supp}(D_e)$ (and hence generically globally generated). This completes the proof of the lemma. $\square$

Lemma V.1.6. *Let $f : \mathcal{X} \to Y$ be a flat family of globally F-regular varieties where Y is regular. Let $\mathcal{D}$ be an integral Weil-divisor such that $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(r\mathcal{D})$ is Cartier. We also assume* *that $\mathcal{L}$ is ample.*

1. *Then, for any integer $m \geq 0$, the sheaf $f_*(\mathcal{L}^m)$ is locally free on Y and for any point $y \in Y$ (not necessarily closed), the natural map*

$$f_*(\mathcal{L}^m) \otimes_{\mathcal{O}_Y} \kappa(y) \to H^0(\mathcal{X}_y, \mathcal{L}^m|_{\mathcal{X}_y})$$

is an isomorphism. Moreover, for any $y \in Y$, there exists an affine open neighbourood $Spec(B) = U \subset Y$ of y such that for y is defined by a regular sequence $b_1, \dots b_t$ and for any $m \geq 0$, the natural map

$$H^0(f^{-1}(U), \mathcal{L}^m) \otimes_B B/\mathfrak{p}B \to H^0(\mathcal{X}_{B/\mathfrak{p}B}, \mathcal{L}^m|_{\mathcal{X}_{B/\mathfrak{p}B}})$$

is an isomorphism where $\mathfrak{p}$ is the ideal $(b_1, \dots, b_t)$.

2. *Suppose, in addition that $\mathcal{X}$ is a locally strongly F-regular variety. Then, for any $m \geq 0$, setting $\mathcal{F} := \mathcal{O}_{\mathcal{X}}(m\mathcal{D})$, we have*

   (a) $\mathcal{F}$ is flat over Y.

   (b) For any $y \in Y$, the restriction $\mathcal{F}_y := \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}_y}$ is reflexive.

   (c) $f_*\mathcal{F}$ is locally free on Y and for any $y \in Y$, the natural map

$$f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \kappa(y) \to H^0(\mathcal{X}_y, \mathcal{F}_y)$$

   is an isomorphism.

   (d) Assume that $Supp(\mathcal{D})$ does not contain any fiber of f. Then,

$$f_*(\mathcal{O}_{\mathcal{X}}(m\mathcal{D})) \otimes_{\mathcal{O}_Y} \kappa(y) \to H^0(\mathcal{X}_y, \mathcal{O}_{\mathcal{X}_y}(m\mathcal{D}|_{\mathcal{X}_y}))$$

   is an isomorphism as well. Here, we restrict the Weil-divisor $m\mathcal{D}$ to Y as explained in Proposition II.3.9.

Proof. Note that since $\mathcal{L}^m$ is an invertible sheaf on $\mathcal{X}$, it is flat over Y for any m.

1. Since $\mathcal{O}_{\mathcal{X}}(m\mathcal{L})$ is flat over Y, the claim follows from Grauert's Theorem [Har77, Chapter III, Corollary 12.9] once we note that the function

$$y \mapsto \dim_{\kappa(y)} H^0(X_y, \mathcal{L}_y)$$

is constant on Y. But we have

$$\dim_{\kappa(y)} H^0(X_y, \mathcal{L}_y^m) = \chi(\mathcal{L}_y^m) = \chi(\mathcal{L}_\eta^m) = H^0(X_\eta, \mathcal{L}_\eta^m)$$

where η denotes the generic point of Y. Here, we are using the higher cohomology vanishing (Theorem II.3.7) for nef invertible sheaves on the fibers (which are globally F-regular varieties by assumption), and the fact that the Euler-characteristic is constant for all fibers of a flat map [Har77, Chapter III, Theorem 9.9]. Since the restriction of $\mathcal{L}^m$ to each fiber of f has vanishing higher cohomology, Grauert's Theorem also implies that $R^i f_*(\mathcal{L}^m)$ is zero for any $m \geq 0$. Fix a $y \in Y$ and choose an open neighbourhood $\mathrm{Spec}(B) = U \subset Y$ of y such that $\mathfrak{p}$ is generated by a regular sequence $(b_1, \ldots, b_t)$ and $B/\mathfrak{p}B$ is regular (where $\mathfrak{p}$ is the prime ideal of B corresponding to y). Then, note that the following exact sequence of sheaves on $f^{-1}(U)$:

$$0 \longrightarrow \mathcal{L}^m \xrightarrow{\cdot b_1} \mathcal{L}^m \longrightarrow \mathcal{L}^m \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_{\mathcal{X}_{B/x_1B}} \longrightarrow 0$$

remains exact after applying $H^0(f^{-1}(U), -)$ since $H^1(f^{-1}(U), \mathcal{L}^m) = 0$. This tells us that

$$H^0(f^{-1}(U), \mathcal{L}^m) \otimes_B B/b_1B \cong H^0(\mathcal{X}_{B/x_1B}, \mathcal{L}^m|_{\mathcal{X}_{B/x_1B}}).$$

Now, since B/x_1B is also regular, we may proceed inductively to complete the proof of Part (1).

2. Fix any $y \in Y$ and let A be the local ring at $y \in Y$ and R be the local ring of any point $x \in X$ mapping to y.

   (a) Since R is strongly F-regular by assumption, we may apply Part (1) of Proposition II.3.9 to conclude that $R(m\mathcal{D})$ is isomorphic to a summand of $F_*^e R$. Since A is regular, we have $F_*^e A$ is flat over A and by assumption $F_*^e R$ is flat over $F_*^e A$. Thus, we see that $F_*^e R$ is flat over A and consequently, $R(m\mathcal{D})$ is flat over A.

   (b) Fix a regular sequence $b_1, \ldots, b_t$ on A generating the maximal ideal (this is possible because A is regular). Because R is flat over A, $b_1, \ldots, b_t$ is also a regular sequence on R. Now, it is sufficient to show that $R(m\mathcal{D}) \otimes_R R'$ is reflexive over $R' = R/(b_1, \ldots, b_t)R$. But this is guaranteed by Part (2) of Proposition II.3.9 once we note that by [Mat89, Theorem 23.9], since all fibers of f are normal, R' is itself normal.

   (c) Using the vanishing theorem for $\mathbb{Q}$-Cartier ample divisors proved in Proposition II.3.8 instead, we can repeat the argument from the first part using Grauert's Theorem, since $\mathcal{F}$ is flat over Y by part (a).

(d) This is immediate by combining Part (c) with the last part of Proposition II.3.9. □

Proof of Theorem V.1.3. We divide the proof into several steps, since some of the steps will be used again in the next subsection.

Step 1: By Proposition IV.2.13, we may replace $\mathcal{L}$ by a multiple of $\mathcal{L}$ if necessary and assume that $\mathcal{L}$ is globally generated on $\mathfrak{X}$. In that case, the restriction of $\mathcal{L}$ to each fiber of f is also automatically globally generated.

Step 2: Putting together Lemma V.1.5 and Theorem V.1.4, we get that for each $y \in Y$, and each $e \geq 1$,

$$|\alpha_e(S_{y^\infty}) - \alpha_F(S_{y^\infty})| \leq \frac{d+1}{p^e}. \tag{V.1.5}$$

Here, S_{y^∞} denotes the section ring of the perfect fiber of f over y with respect to the restriction of $\mathcal{L}$, and d denotes the dimension of every fiber of f (which is well-defined because f is flat).

Now, given any $\alpha < \alpha_{\text{gen}}$, let $\varepsilon = \alpha_{\text{gen}} - \alpha$. Choose an $e \gg 0$ such that $\frac{d+1}{p^e} < \varepsilon/2$. Then, we have

$$\alpha_{\text{gen}} - \varepsilon/2 < \alpha_e(S_{K^\infty}). \tag{V.1.6}$$

We claim that there exists a dense open set $U_\alpha \subset Y$ such that

$$\alpha_e(S_{y^\infty}) \geq \alpha_e(S_{K^\infty})$$

for each $y \in U_\alpha$. Assuming the claim, Equation (V.1.5), and Equation (V.1.6) together imply that

$$\alpha_F(S_{y^\infty}) > \alpha_e(S_{y^\infty}) - \varepsilon/2 \geq \alpha_e(S_{K^\infty}) - \varepsilon/2 > \alpha_{\text{gen}} - \varepsilon = \alpha$$

for every $y \in U_\alpha$ as required.

Step 3: We now proceed to prove the claim used above, i.e., that there exists a dense open set $U_\alpha \subset Y$ such that $\alpha_e(S_{y^\infty}) \geq \alpha_e(S_{K^\infty})$ for every $y \in U_\alpha$. Working locally, we may assume that $Y = \mathrm{Spec}(A)$ and let S_A denote the section ring $S(\mathcal{X}, \mathcal{L})$. Set $m_e = m_e(S_{K^\infty})$. By Lemma V.1.6, $H^0(\mathcal{X}, \mathcal{L}^m)$ if a locally free A-module for any $m \geq 0$. So by shrinking Y around any point if necessary, we may assume that $H^0(\mathcal{X}, \mathcal{L}^m)$ is a free A-module with a basis $\mathscr{B}_m$ for each $0 \leq m \leq m_e$. By Lemma V.1.6 again, $\mathscr{B}_m$ restricts to a basis of $H^0(X_y, \mathcal{L}_y)$ for any $y \in Y$. Let $\mathscr{B} = \sqcup_{m=0}^{m_e} \mathscr{B}_m$. Note that $m_e \leq p^e - 1$ since for any non-zero section $f \in H^0(X_{K^\infty}, \mathcal{L}_{K^\infty})$, which exists because $\mathcal{L}$ is assumed to be globally generated, we have $0 \neq f^{p^e} \in I_e(p^e)$.

Step 4: By definition of m_e, for any $m \leq m_e$ and any non-zero $f \in H^0(X_{K^\infty}, \mathcal{L}^m_{K^\infty})$, the map $S_{K^\infty} \to F^e_* S_{K^\infty}$ sending 1 to $F^e_* f$ splits. Thus, using Lemma II.1.3 repeatedly on the set $\mathscr{B}_m$, we can construct a surjective map $\psi_m : F^e_*(\mathcal{L}^m_{K^\infty}) \to \mathcal{O}_{X_{K^\infty}}^{\oplus \mathscr{B}_m}$ such that if $f \in \mathscr{B}_m$ is a basis element, then $\psi_m(F^e_* f) = \mathbf{1}_f$, where $\mathbf{1}_f$ is the standard basis element corresponding to f of $\mathcal{O}_{X_{K^\infty}}^{\oplus \mathscr{B}_m}$. Taking induced map on the section modules and putting together the maps ψ_m for all $0 \leq m \leq m_e$, along with the zero map for $m_e < m \leq p^e - 1$, we get a surjective S_{K^∞}-module map

$$\psi_{K^\infty} : F^e_* S_{K^\infty} \to S_{K^\infty}^{\oplus \mathscr{B}} \tag{V.1.7}$$

which satisfies the following property: if $0 \neq f \in H^0(X_{K^\infty}, \mathcal{L}^m_{K^\infty})$ and $m \leq m_e$, then $\psi(F^e_* f)$ is a basis element of $S_{K^\infty}^{\oplus \mathscr{B}}$. In other words, ψ_{K^∞} simultaneously splits all the non-zero sections of degree at most m_e on X_{K^∞}. By [CRST21, Lemma 4.8], there is an integer $d_e > 0$, a non-zero element $g \in A$ such that if $B = A[g^{-1}]$, we have a map

$$\psi_{B^{1/p^{e+d_e}}} : S^{1/p^e}_{B^{1/p^{e+d_e}}} \to S^{\oplus \mathscr{B}}_{B^{1/p^{e+d_e}}}$$

which satisfies $\psi_{B^{1/p^{e+d_e}}} \otimes_{B^{1/p^{e+d_e}}} K^\infty = \psi_{K^\infty}$. Here, $S^{1/p^e}_{B^{1/p^{e+d_e}}}$ denotes the e^{th}-relative Frobenius over the base change $S_{B^{1/p^{e+d_e}}} := S_A \otimes_A B^{1/p^{e+d_e}}$. Note also that after base-changing to K^∞, we are identifying the relative and absolute Frobenius over K^∞. Since every $f \in \mathscr{B}$ is mapped to a basis element of $S^{\oplus \mathscr{B}}_{B^{1/p^{e+d_e}}}$ after tensoring to K^∞, we may assume that the same is true for $\psi_{B^{1/p^{e+d_e}}}$ after inverting another element of A if necessary. Finally, for any $y \in \mathrm{Spec}(B)$, base changing to $\kappa(y)^\infty$, we see that

$$\psi_{\kappa(y)^\infty} = \left(\psi_{B^{1/p^{e+d_e}}} \otimes_{B^{1/p^{e+d_e}}} \kappa(y)^{1/p^{e+d_e}}\right) \otimes_{\kappa(y)^{1/p^{e+d_e}}} \kappa(y)^\infty.$$

simultaneously splits each non-zero element $0 \neq f \in H^0(X_{y^\infty}, \mathcal{L}^m_{y^\infty})$. Note that we are again identifying the absolute and relative Frobenius over the perfect field $\kappa(y)^\infty$. This shows that $\alpha_e(S_{y^\infty}) \geq \alpha_e(S_{K^\infty})$. This completes the proof of Theorem V.1.3. $\square$

V.2: Semicontinuity for a Family of $\mathbb{Q}$-Fano Varieties

In this subsection, we will prove that the α_F-invariant is lower semicontinuous in a family of globally F-regular $\mathbb{Q}$-Fano varieties:

Theorem V.2.1. *Let $f : \mathcal{X} \to Y$ be flat family of globally F-regular $\mathbb{Q}$-Fano varieties such that $-K_{\mathcal{X}|Y}$ is $\mathbb{Q}$-Cartier and f-ample. Assume that Y is regular. Then, the map from $Y \to \mathbb{R}_{\geq 0}$ given by*

$$y \mapsto \alpha_F(\mathcal{X}_{y^\infty})$$

is lower semicontinuous, where $\mathcal{X}_{y^\infty}$ is the perfectified-fiber over $y \in Y$.

See Definition IV.2.2 for the definition of the α_F-invariant of a $\mathbb{Q}$-Fano variety, and Notation V.1.2 for the meaning of a family of globally F-regular varieties.

Remark V.2.2. For related semicontinuity results for the F-signature and the Hilbert-Kunz multiplicity, see [Pol18], [PT18], [Smi16] and [Smi20]. Similarly, for the corresponding lower semicontinuity result for the complex α-invariant, see [BL22].

Idea of the proof: Roughly, the proof of Theorem V.2.1 involves combining Theorem V.1.3 with the inversion of adjunction for strong F-regularity as proved in [PSZ18]. The main technical difficulty arises when p divides the index of $K_{\mathcal{X}|Y}$. In this situation, we use a standard perturbation trick similar to [Pat14, Lemma 3.15] and [HX15, Lemma 2.13]. But to do this, we need to be able to restrict $\mathbb{Q}$-Cartier, ample Weil-divisors in a family to the fibers of the family. So we begin by observing that this can indeed be done in our situation.

Setup and Notation: Let A be a regular k-algebra of finite type. Given a flat family $f : \mathcal{X} \to Y = \mathrm{Spec}(A)$ of globally F-regular Fano varieties such that $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}|Y})$ is Cartier and ample for some integer $r > 0$, we can form the section ring

$$S_A := S(\mathcal{X}, \mathcal{L}).$$

Then, since $\mathcal{X}$ is normal and $\mathcal{L}$ is ample, S_A is a normal, finitely generated , $\mathbb{N}$-graded algebra over A. For any A-algebra B, let S_B denote the section ring $S(\mathcal{X}_B, \mathcal{L}|_{\mathcal{X}_B})$, where $\mathcal{X}_B$ denotes the base change $\mathcal{X} \times_A \mathrm{Spec}(B)$.

Lemma V.2.3. *With notation as above, the construction of S_A satisfies the following properties:*

1. *S_A is flat over A and for any prime ideal $\mathfrak{p} \subset A$ we have that*

$$S_A \otimes_A \kappa(\mathfrak{p}) \cong S_{\kappa(\mathfrak{p})}.$$

2. *For each prime $\mathfrak{p} \in Spec(A)$, there is an affine open neighbourhood $Spec(B) = U \subset Y$ containing $\mathfrak{p}$ such that the restriction $K_{\mathcal{X}}$ to $Spec(B/\mathfrak{p}B)$ (as explained in Proposition II.3.9) is linearly equivalent to $K_{\mathcal{X}_{B/\mathfrak{p}B}}$. In particular, for any $\mathfrak{p} \in Spec(A)$, $-rK_{\mathcal{X}_{\mathfrak{p}}}$ is Cartier. Moreover, $\mathcal{X}$ and S_A are both $\mathbb{Q}$-Gorenstein.*

3. *For each $\mathfrak{p} \in Spec(A)$, there is an affine open neighbourhood $Spec(B) = U \subset Y$ containing $\mathfrak{p}$ and a regular sequence $b_1, \ldots, b_t$ on B generating $\mathfrak{p}$ such that S_B, and $S_{B/(b_1,\ldots,b_i)B}$ is strongly F-regular for each $1 \leq i \leq t$. In particular, $\mathcal{X}$ and S_A are both globally F-regular.*

4. *For any Weil-divisor $\mathcal{D}$ on $\mathcal{X}$ such that $r\mathcal{D}$ is Cartier and ample for some $r > 0$, the section module*
$$M_A(\mathcal{D}) = \bigoplus_{m \geq 0} H^0(\mathcal{X}, \mathcal{O}_X(\mathcal{D}) \otimes \mathcal{L}^m)$$
is flat over A, and compatible with base change to fibers. In other words, for any $\mathfrak{p} \in Spec(A)$, the natural map

$$M_A(\mathcal{D}) \otimes \kappa(\mathfrak{p}) \to M_{\kappa(\mathfrak{p})}(\mathcal{O}_{\mathcal{X}}(\mathcal{D})|_{\mathcal{X}_{\mathfrak{p}}})$$

is an isomorphism. Moreover, if $Supp(\mathcal{D})$ does not contain any fiber of f then the natural map
$$M_A(\mathcal{D}) \otimes \kappa(\mathfrak{p}) \to M_{\kappa(\mathfrak{p})}(\mathcal{D}|_{\mathfrak{p}})$$
is an isomorphism as well.

Proof. Since all parts of the lemma can be proved locally on Y, we may shrink Y if necessary to assume that ω_Y is a free A-module.

1. This is immediate from Part (1) of Lemma V.1.6.

2. By inverting an element $g \in A \setminus \mathfrak{p}$, and setting $B = A[g^{-1}]$, we may assume that $\mathfrak{p}$ is generated by a regular sequence $b_1, \ldots, b_t$ on B (this is possible since $A_{\mathfrak{p}}$ is regular). Fix any $1 \leq i \leq t$ and let $B_i = B/(b_1, \ldots, b_i)B$. By [Mat89, Theorem 23.9], since all fibers of f are normal, we in particular know that each $\mathcal{X}_{B_i}$ (and hence, S_{B_i}) is normal. By shrinking U further if necessary, using Part (1) of Lemma V.1.6, we may also assume $S_{B_i} \cong S_B \otimes_B B_i$.

Now we will show that $K_{\mathfrak{X}}$ restricted to $\mathfrak{X}_{B_i}$ is linearly equivalent to the divisor $K_{\mathfrak{X}_{B_i}}$. Let $V' \subset \mathfrak{X}_{B_i}$ denote the smooth locus of $\mathfrak{X}_{B_i}$, and $V \subset \mathfrak{X}_{\text{sm}}$ be an open set such that $V \cap \mathfrak{X}_{B_i} = V'$. This is possible because $\mathfrak{X}_{B_i}$ is a complete intersection in $\mathfrak{X}_B$. Then, applying the adjunction formula for the complete intersection $V' \subset V$, we have $K_{V'} \sim K_V|_{V'}$. Since $V' \subset \mathfrak{X}_{B_i}$ is the smooth locus and $\mathfrak{X}_{B_i}$ is normal, it contains all the codimension one points of $\mathfrak{X}_{B_i}$. Thus, taking closures we get that

$$-rK_{\mathfrak{X}}|_{\mathfrak{X}_{B_i}} = \overline{-rK_{\mathfrak{X}}|_{V'}} = \overline{-rK_V|_{V'}} \sim -rK_{\mathfrak{X}_{B_i}}.$$

This proves that $\mathcal{L}$ restricted to $\mathfrak{X}_{B_i}$ is linearly equivalent to $-rK_{\mathfrak{X}_{B_i}}$. In particular $-rK_{\mathfrak{X}_{B_i}}$ is Cartier. Furthermore, it follows from the discussion in [SS10, Section 5.2]) that the canonical divisor $K_{S_{B_i}}$ is the cone over the canonical divisor $K_{\mathfrak{X}_{B_i}}$ (as Weil-divisor). Thus, we get that $\mathcal{O}_{S_{B_i}}(-rK_{S_{B_i}}) \cong \mathcal{O}_{S_{B_i}}(1)$ as a graded module over S_{B_i}. Also note that for any maximal ideal $\mathfrak{m} \subset A$, the fiber $S_{\kappa(\mathfrak{m})}$ is strongly F-regular. In particular, all fibers over closed points of $\operatorname{Spec}(A)$ are Cohen-Macaulay, we see that S_A is Cohen-Macaulay as well. Therefore, S_A (and similarly S_{B_i}) is $\mathbb{Q}$-Gorenstein.

3. From part (2), we may assume that $\mathfrak{p}$ is generated by a regular sequence $a_1, \dots a_t$ on A and $-rK_{\mathfrak{X}_{A_i}}$ is Cartier for each $1 \le i \le t$, where $A_i = A/(a_1, \dots, a_i)A$. Next, note that by assumption (and Part (1)), we have $S_{\kappa(\mathfrak{p})}$ is strongly F-regular. So, there exists a non-zero element $g \in A \setminus \mathfrak{p}$ such that S_{B_t} is strongly F-regular where $B_t = A_t[g^{-1}]$. Here, we are using the fact that the non-strongly F-regular locus of S_{B_t} is closed, compatible with localization and homogeneous with respect to the $\mathbb{N}$-grading on S_{B_t}. Let $B_{t-1} = A_{t-1}[g^{-1}]$. Then, since $-K_{S_{B_{t-1}}}$ is $\mathbb{Q}$-Cartier and a_t is a non-zero divisor on B_{t-1}, by [Das15, Theorem A], we conclude that $S_{B_{t-1}}$ is strongly F-regular in a neighbourhood of $\mathbb{V}(b_t)$. Since the locus of points where $S_{B_{t-1}}$ is not strongly F-regular is defined by a homogeneous ideal (and is disjoint from $\mathbb{V}(a_t)$), we may pick a non-zero element $h \in A[g^{-1}] \setminus (a_1, \dots, a_t)$ such that if we set $B'_{t-1} = B_{t-1}[h^{-1}]$, $S_{B'_{t-1}}$ is strongly F-regular. Replacing $\mathfrak{p}$ with $(a_1, \dots, a_{t-1})$, we may proceed inductively to get a localization B of A (at finitely many elements), and an open neighbourhood $U = \operatorname{Spec}(B) \subset \operatorname{Spec}(A)$ of $\mathfrak{p}$ such that S_B and $S_{B/(a_1,\dots,a_i)B}$ is strongly F-regular for each $1 \le i \le t$, as required.

   Applying this to maximal ideals in A, we see that S_A is strongly F-regular. Hence, $\mathfrak{X}$ is globally F-regular.

4. Using Part (3) above, $\mathfrak{X}$ is in particular, locally strongly F-regular. Now the claim is an immediate consequence of Part (2) of Lemma V.1.6. $\square$

Lemma V.2.4. *Let $f : \mathcal{X} \to Y = Spec(A)$ be as above and assume A is regular. Suppose $\mathcal{D}$ is a $\mathbb{Q}$-divisor on $\mathcal{X}$ satisfying the following two properties:*

1. *there is some $e > 0$ for which $(p^e - 1)\mathcal{D}$ is an integral Weil divisor linearly equivalent to $(1 - p^e)K_{\mathcal{X}|Y}$ as Weil divisors, and*

2. *Supp$(\mathcal{D})$ does not contain any fiber of f.*

Then, if $y_0 \in Y$ is a point such that the pair $(\mathcal{X}_{y_0^\infty}, \lambda\,\mathcal{D}_{y_0^\infty})$ is globally F-regular for some $\lambda \in \mathbb{Q}_{\geq 0}$, then there is an open neighbourhood $U \subset Y$ of y_0 such that $(\mathcal{X}_{y^\infty}, \lambda\mathcal{D}_{y^\infty})$ is also globally F-regular for all $y \in U$.

Proof. The proof is divided into several steps, but the strategy is to apply [PSZ18, Corollary 4.19] carefully. See Section IV.1.2 for a detailed discussion of the process of taking cones over divisors in family.

Step 1: By shrinking Y to a neighbourhood of y_0 if necessary, we may also assume that ω_Y is isomorphic to A. Fix an $r > 0$ such that $-rK_{\mathcal{X}}$ is an ample Cartier divisor, and set $\mathcal{L} := -rK_{\mathcal{X}}$ and $S_A = S(\mathcal{X}, \mathcal{L})$ be the corresponding section ring and Spec(S_A) is the cone over $\mathcal{X}$. By Lemma V.1.6, taking the cone over $\mathcal{X}$ commutes with base change to fibers of f. For any integral Weil divisor $\mathcal{E}$, let $M_A(\mathcal{E}) = \bigoplus_{m\geq 0} H^0(\mathcal{X}, \mathcal{O}_X(\mathcal{E}) \otimes \mathcal{L}^m)$ denote the corresponding section module over S_A.

Step 2: For any e sufficiently divisible, let $\mathcal{D}_e = (p^e - 1)\mathcal{D}$, which we may assume is an integral Weil-divisor. Since $-K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier, so is $\mathcal{D}_e$. Therefore, by Part (4) of Lemma V.2.3, the section module $M_A(\mathcal{D}_e)$ is compatible with base change to fibers. Thus, we may consider the cone over $\mathcal{D}_e$ as a Weil-divisor on Spec(S_A) defined by the reflexive sheaf $M_A(\mathcal{D}_e)$. The compatibility with base changing to fibers guarantees that taking the cone over $\mathcal{D}_e$ commutes with restricting $\mathcal{D}_e$ to the fibers of f.

Step 3: Since the pair $(\mathcal{X}_{\kappa(y_0)^\infty}, \mu\,\mathcal{D}_{\kappa(y_0)^\infty})$ is globally F-regular for $\mu = \lambda$, it remains so for all $\mu < \lambda + \varepsilon$ for some small $\varepsilon > 0$. Now we set $\mu := \frac{\ell_1}{\ell_2}$ to be a rational number such that $\lambda \leq \mu < \lambda + \varepsilon$ for positive integers ℓ_1 and ℓ_2 with the following properties:

1. $\ell_2 - \ell_1$ is divisible by r (the Cartier index of $K_{\mathcal{X}}$).

2. ℓ_2 is not divisible by p.

Furthermore, choose $e_0 > 0$ such that ℓ_2 divides $p^{e_0} - 1$ and $(p^{e_0} - 1)\mathcal{D}$ and $-(p^{e_0} - 1)K_{\mathcal{X}}$ are both integral Weil-divisors. Such an e_0 exists by our assumptions on $\mathcal{D}$. Lastly, set $\Delta = \mu\,\mathcal{D}$. With this notation, we note that the following properties are satisfied:

- $\Delta = \frac{\ell_1}{(p^{e_0}-1)\ell_2}\,(p^{e_0} - 1)\mathcal{D}$, where $(p^{e_0} - 1)\mathcal{D}$ is a Weil-divisor and p does not divide $(p^{e_0} - 1)\ell_2$.
- $(p^{e_0} - 1)^2\,(K_{\mathcal{X}} + \Delta)$ is linearly equivalent to

  $$(p^{e_0} - 1)^2\,(K_{\mathcal{X}} + \Delta) \sim (p^{e_0} - 1)^2(1 - \frac{\ell_1}{\ell_2})\,K_{\mathcal{X}} = \frac{(p^{e_0} - 1)^2(\ell_2 - \ell_1)}{\ell_2}\,K_{\mathcal{X}}. \tag{V.2.1}$$

  And since r divides $\ell_2 - \ell_1$, we get that $(p^{e_0} - 1)^2(K_{\mathcal{X}} + \Delta)$ is an integral Cartier divisor. Also note that $(p^{e_0} - 1)^2$ clearly divides $(p^{e_0} - 1)\ell_2$ and is not divisible by p.
- The fibers of f are geometrically normal since the perfect fibers are globally F-regular. They are also geometrically connected by assumption. Thus, our assumption that $\mathrm{Supp}(\mathcal{D})$ does not contain any fibers of f guarantees that $\mathrm{Supp}(\mathcal{D})$ does not contain any generic point of any geometric fiber of f.

Step 4: In this context, we may use [PSZ18, Corollary 4.19] applied to the projective cone of f with respect to $\mathcal{L}$ to conclude the proof (see Section IV.1.1 for details about the projective cone construction). More precisely, consider the map

$$\overline{f} : \overline{\mathcal{X}} := \mathrm{Proj}(S_A[z]) \to \mathrm{Spec}(A)$$

where z is just another variable adjoined to S_A in degree 1. Note that $-rK_{\overline{\mathcal{X}}}$ is Cartier on $\overline{\mathcal{X}}$ (since this is true at the zero-section by construction, and away from the zero section we know that $\overline{\mathcal{X}}$ is an $\mathbb{A}^1$-bundle over $\mathcal{X}$.

The construction of the projective cone with respect to $\mathcal{L}$ is compatible with base change to fibers. In other words, for any $y \in Y$, the fiber $\overline{f}_y : \overline{\mathcal{X}}_y \to \mathrm{Spec}(\kappa(y))$ is the map

$$\overline{f}_y : \overline{\mathcal{X}}_y = \mathrm{Proj}(S_{\kappa(y)}[z]) \to \mathrm{Spec}(\kappa(y)).$$

Let $\overline{\Delta}$ denote the $\mathbb{Q}$-divisor obtained as the closure in $\overline{\mathcal{X}}$ of the cone over Δ. For any $r > 0$ such that $r\Delta$ is integral, the section module corresponding to $r\overline{\Delta}$ is $M_A(r\Delta)[z]$ by construction. Thus, by Step 2,the construction of the projective cone over Δ is compatible with restricting to fibers. Thus, by Equation (V.2.1) and the fact that away from the zero-section, $\overline{\mathcal{X}}$ is an $\mathbb{A}^1$-bundle over $\mathcal{X}$, we have that $K_{\overline{\mathcal{X}}}+\overline{\Delta}$ is $\mathbb{Q}$-Cartier with index not divisible by p.

Step 5: Finally, we observe that for any $y \in Y$, the local strong F-regularity of the cone

$$(\overline{\mathcal{X}}_{\kappa(y)^\infty}, \overline{\Delta}_{\kappa(y)^\infty})$$

is equivalent to the global F-regularity of $(\mathcal{X}_{y^\infty}, \Delta_{y^\infty})$, since both correspond to the strong F-regularity of the pair (S_A, Δ) . With these observations in place, [PSZ18, Corollary 4.19] gives us an open neighbourhood $U \subset Y$ of y_0 such that $(\mathcal{X}_{y^\infty}, \Delta_{y^\infty})$ is globally F-regular for all $y \in U$. Since, $\Delta = \mu\mathcal{D}$ and $\lambda \leq \mu$, the same is true for $(\mathcal{X}_{y^\infty}, \lambda\mathcal{D}_{y^\infty})$ as required. □

Proof of Theorem V.2.1. Recall that to prove that the given map is lower semicontinuous, we need to show that given any point $y_0 \in Y$ and $\alpha > 0$ such that $\alpha_F(\mathcal{X}_{y_0^\infty}) > \alpha$, there exists an open neighbourhood $U(y_0, \alpha) \subset Y$ such that $\alpha_F(X_{y^\infty}) > \alpha$ for all $y \in U(y_0, \alpha)$. The idea of the proof is similar to the proof of Theorem V.1.3, but we need a slight variation since we need an open neighbourhood of y_0 instead of just any open subset of Y.

Firstly, by shrinking Y to a neighbourhood of y_0, we assume $Y = \mathrm{Spec}(A)$ is affine and ω_Y is a free A-module. Next, using Proposition IV.2.13, it is sufficient to prove the lower semicontinuity of the function

$$y \mapsto \alpha_F(S(\mathcal{X}_{y^\infty}, -rK_{\mathcal{X}_{y^\infty}}))$$

for any $r \gg 0$. So we pick an $r \gg 0$ such that $-rK_{\mathcal{X}}$ is a globally generated ample divisor on $\mathcal{X}$. In particular, $-rK_{\mathcal{X}}$ is Cartier. Therefore, Part (2) of Lemma V.2.3, we have that $-rK_{\mathcal{X}_{y^\infty}}$ is a globally generated ample Cartier divisor on $\mathcal{X}_{y^\infty}$ for any $y \in Y$. Additionally, fix an integer $t > 0$ such that $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-mK_{\mathcal{X}})) \neq 0$ for all $m \geq t$.

Let d be the relative dimension of f, let α_0 denote $\alpha_F(S_{y_0^\infty})$, and $\varepsilon := \alpha_0 - \alpha$. Recall that for any A-algebra B, S_B denotes the section ring $S(\mathcal{X}_B, \mathcal{L}|_{\mathcal{X}_B})$, where $\mathcal{X}_B$ denotes the base change $\mathcal{X} \times_A \mathrm{Spec}(B)$ and $\mathcal{L} = \mathcal{O}_X(-rK_{\mathcal{X}})$. By the argument in Step 2 of the proof of Theorem V.1.3 (replacing the generic point with y_0), it is sufficient to show that there exists an $e \gg 0$ such that $\frac{d+1}{p^e} < \varepsilon/2$ and an open neighbourhood $U(y_0, \alpha)$ of y_0 such that

$$\alpha_e(S_{y^\infty}) \geq \alpha_0 - \frac{\varepsilon}{2}$$

for all $y \in U(y_0, \alpha)$. To prove this, choose an $e \gg 0$ such that $\frac{d+1}{p^e} < \varepsilon/2$, and

$$\alpha_0 \, (p^e - 1) > p^e \, (\alpha_0 - \varepsilon/2) + t. \tag{V.2.2}$$

Let n be an integer such that $r(p^e(\alpha_0 - \varepsilon/2) + t) < n < r \, \alpha_0 \, (p^e - 1)$ be an integer that is not divisible by p. By Part 2c of Lemma V.1.6, we may assume (by shrinking Y if necessary) that $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-mK_{\mathcal{X}}))$ is a free A-module for each $0 \leq m \leq n$ with a basis $\mathscr{B}_m$. Let $\mathcal{D}$ be any element of $\mathscr{B}_n$ and $\frac{n}{p^e-1} < \lambda < r \, \alpha_0$ be any rational number. Since $\mathcal{D}$ is a basis element using Part 2c of Lemma V.1.6 again, we see that $\mathcal{D}$ does not contain any fibers of f, since $\mathcal{D}$ restricts to a non-zero global section on each fiber. Then, we apply Lemma V.2.4, to the $\mathbb{Q}$-divisor $\frac{1}{n}\mathcal{D}$ to get an open neighbourhood $U = U(y_0, \alpha)$ such that for each $y \in U$, the pair $(\mathcal{X}_{y^\infty}, \frac{\lambda}{n}\mathcal{D}|_{\mathcal{X}_{y^\infty}})$ is globally F-regular. This is because since $\lambda < r\alpha_0$, the pair $(\mathcal{X}_{y_0^\infty}, \frac{\lambda}{n}\mathcal{D}|_{y_0^\infty})$ is globally F-regular by the definition of α_0. Since $\frac{\lambda}{n} > \frac{1}{p^e-1}$ by construction, Lemma IV.2.6 tells us that for each $y \in Y$, and each $\mathcal{D}$ in $\mathscr{B}_n$, the map

$$S_{y^\infty} \to F^e_*(S_{y^\infty}(\mathcal{D}_{y^\infty})) \tag{V.2.3}$$

splits. Furthermore, we may pick an integer $m \leq n$ satisfying: r divides m, $n - m \geq t$, and $m > rp^e \, (\alpha_0 - \varepsilon/2)$. This is possible by our choice of n. Since $n - m \geq t$, we can pick a non-zero $\mathcal{E} \in \mathscr{B}_{n-m}$ that restricts to a non-zero Weil-divisor on each fiber (by base-change). Thus, for any element $\mathcal{D}$ in $\mathscr{B}_m$, since the corresponding map for $(\mathcal{D} + \mathcal{E})_{y^\infty}$ splits for each $y \in U$ by Equation (V.2.3), the corresponding map for $\mathcal{D}_{y^\infty}$ also splits for each $y \in U$. Finally, we apply Lemma II.1.3 repeatedly to the basis $\mathscr{B}_m$ to conclude that $I_e(m) = 0$ for S_{y^∞} for each $y \in U$ and thus

$$\alpha_e(S_{y^\infty}) \geq \frac{m}{rp^e} > \alpha_0 - \varepsilon/2$$

for all $y \in U$ as required. This completes the proof of Theorem V.2.1. □

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