RL-.

ILLINOIS INSTITUTE OF TECHNOLOGY

INTERACTION OF ELECTROMAGNETIC WAVES WITH A PLASMA SURROUNDING AN INFINITELY CONDUCTING WEDGE

BY

JAYANTILAL G. SHAH

Submitted in Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy in Electrical Engineering
in the Graduate School of
Illinois Institute of Technology

Approved ____________
Technical Adviser

____
Administrative Adviser

CHICAGO, ILLINOIS
June, 1962

RL-211 = RL-211

ABSTRACT

The problem chosen is "The Interaction of Electromagnetic Waves with a Plasma Surrounding an Infinitely Conducting Wedge". With a simple model of plasma, representing it as a scalar permittivity, present only on one side of the wedge, this problem becomes essentially a three region problem, with Dirichlet and Neumann boundary conditions on the ϕ-planes. The wave equations in the two regions containing non-zero fields are transformed in the ν-plane by means of Kontorowich-Lebedev transform. This method has been found effective in the solution of wave equation with boundary conditions on the ϕ-planes. The boundary conditions arising from the requirement of matching non-zero fields across the surface of contact of one pair of media, give rise to an integral equation, which contains all the information of the problem. The situation in which more than one pair of regions exist in which non-zero fields are required to be matched, has to be described by a system of integral equations. The geometry considered, therefore, is a three region geometry, one of the regions being an infinitely conducting medium, in which the fields are considered zero. The solution of the integral equation has been considered in four different classes, characteristic of the integral equation. The integrability conditions of functions tabulated in Tables II and IV give the ranges of the parameters,

for which a particular class of solution is convergent. Existence condition for the lowest class, gives the locations of the source for which this lowest class of solution holds. Consideration of a three dimensional wave propagation is expected to yield a richer and more practical variation of parameters. Introduction of simple dyadic permittivity does not involve any new technique as long as two dimensional problem is considered. A simple problem on the application of this method reveals it as an extension of method of images. The ranges of the parameters θ_k, σ and σ_1 give all information regarding wave propagation.

TABLE OF CONTENTS

	Page
ACKNOWLEDGEMENT	v
LIST OF TABLES	vi
LIST OF ILLUSTRATIONS	vii
CHAPTER	
I. INTRODUCTION	1
II. SETTING UP THE INTEGRAL EQUATION OF OF THE SIMPLIFIED PROBLEM	6
III. SOLUTION OF THE INTEGRAL EQUATION. . . .	25
IV. INVESTIGATION OF DIFFERENT CLASSES OF SOLUTIONS	47
The Class h(0, 1)	
The Class h(0)	
The Class h_0	
Conditions for Integrability over the μ-Variable	
V. COMPLETION OF AND OBSERVATIONS ON THE SOLUTION	74
VI. INTEGRAL EQUATION WITH PLASMA REPRESENTED BY A SIMPLE DYADIC PERMITTIVITY.	83
Electric Current Line Source.	
Magnetic Current Line Source	
VII. AN APPLICATION OF THE METHOD	90
VIII. CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK.	100
Conclusions	
Suggestions for Further Work	

TABLE OF CONTENTS
(Cont'd.)

APPENDIX		Page
A.	SOME PROPERTIES OF TRANSCENDENTAL FUNCTIONS	113
B.	BEHAVIOUR OF $T(\nu,\mu)$ AS EITHER $\|\mu\|$ OR $\|\nu\|$ BECOMES LARGE	117
C.	EVALUATION OF INTEGRAL I_3	118
D.	RELATIONS AMONG THE PARAMETERS θ_k, θ_{k_1}, σ and σ_1	128

BIBLIOGRAPHY 131

ACKNOWLEDGEMENT

The author wishes to express his gratitude to Professor K. M. Siegel for the invaluable help he has given throughout this research. He suggested the thesis problem and took a personal interest in every stage of its preparation.

To Drs. Ralph E. Kleinman and Surendra Nath Samaddar of the Radiation Laboratory of the University of Michigan, the author is indebted for their very valuable and creative discussions and suggestions.

Special thanks are due Dr. E. R. Whitehead of the Electrical Engineering Department of the Illinois Institute of Technology for acting as co-advisor at a critical time in this program.

The research reported in this thesis was supported in part by Air Force contracts at the Radiation Laboratory.

LIST OF TABLES

Table		Page
I.	Forms of $V_1(\nu,\mu)$ and $f_1(\mu)$ for the Four Classes..........	72
II.	Order of $Z_1(\nu,\mu)$ as Im $\mu \to \infty$ for the Four Classes.......	73
III.	Forms of F_1 and F_2 for the Four Classes.................	77
IV.	Order of $F_1(\eta,\mu)$ as Im $\eta \to \infty$ for the Four Classes.......	77

LIST OF ILLUSTRATIONS

Figure Page

1. The Geometry of the Problem.......................... 6

2. Contour to Help Evaluation of I_9 on the $\eta = \eta_r + j\eta_i$ Plane..... 39

3. Geometry of the Simplified Problem..................... 99

4. Contour to Help Evaluation of Integrals Similar to I_6 and I_7.. 117

CHAPTER I

INTRODUCTION

Problems of electromagnetic wave propagation in a two region geometry have been usually solved by expressing the fields in each of the two regions as a sum of a complete set of orthogonal functions and matching the fields at the boundary of the two regions. The boundary conditions thus give rise to a Dirichlet, Neumann, or Robin boundary value problem. This method becomes more and more difficult as the number of regions increases, especially when different kinds of boundary conditions are encountered; but seems to be very difficult when the set of orthogonal functions for expanding the fields in each of the regions is not known or is incomplete.

The problem of diffraction by a wedge, essentially a two region problem, has been tackled by earlier authors as a Dirichlet boundary value problem by considering the wedge to be made of an infinitely conducting material. Senior [15]* has shown that the solution to the problem of diffraction by a semi-infinite metallic sheet changes materially, if the finite conductivity of the metal is taken into account, evidently due to a change in the

---

*The numbered references in brackets pertain to the Bibliography.

impedance at the surface. The preponderate effect of the impedance at two boundaries of a transmission line is very well known in transmission line theory. Senior [16] has also solved the problem of diffraction by an imperfectly conducting wedge, by representing the surface impedance as a boundary condition; and has shown that his solution coincides with that of Sommerfeld for an infinitely conducting wedge. Felsen [17] has considered a wedge and a cone with a linearly varying surface impedance. The solutions of Sommerfeld and Senior use the methods of functions of complex variable and are expressed as integrals over a complex parameter. Felsen's solutions are obtained in terms of Green's functions using orthogonal sets of functions. The author was much interested in reading the article of Kontorowich and Lebedev [1], who have shown that the transform which they introduced in their article enables the solution of the problem of diffraction by a half plane to be represented in either of the two forms - a Sommerfeld integral or a sum of orthogonal functions.

Baker and Copson [18] have introduced a solution to the diffraction by a black half-plane as a sum of two parts, one representing the effect according to the laws of geometrical optics and the other being a correction term which takes account of the diffraction. The diffraction term consists of 'cut' Hankel functions, which lack terms involving negative powers of the argument. These 'cut' Hankel functions cannot be formed from normal orthogonal functions. Both the parts of the solution have discontinuities across the geometrical shadow; the total solution is, however, continuous. Whipple [19] has observed similar phenomena in his work on diffraction by a wedge.

The problem is set up as a three region boundary value problem in Chapter II, by taking a simplified model of plasma representing it as a scalar permittivity. This leads to the singular integral equation (2. 85). The integral equation (2. 85) holds for any suitable three region geometry in the shape of wedges and an electric line source excitation (2. 1), although it has been set up from the consideration of one of the regions consisting of plasma, describable by means of a scalar permittivity (2. 3). All the media must be isotropic and linear, capable of being described by means of scalar parameters. The non-linearity of the plasma region has been considered negligible due to low intensity of waves. In the geometry, there must be present at least one of the regions, the impedance at the surfaces of which must be known. This region is taken as an infinitely conducting region, in order to simplify the integral equation. The other two regions may have any suitable parameters described by (7. 1) and (7. 2). The excitation source is located in the denser medium, so that the waves originate in the denser medium and propagate into the rarer medium. If there are more than one pair of regions, at the surface of contact of which non-zero fields are required to be matched, the situation has to be described in terms of a system of singular integral equations, rather than a single one. The geometry has been further chosen extending to infinity in both the ρ and Z directions, with no variation of either the parameters or the excitation in the Z-direction, in order to reduce the problem to a two dimensional one. The only approximate solution of the problem of a three region boundary value problem in the shape of wedges that the author has seen, is that by Karp and Sollfrey [20] of New York University. Besides their solution being a good approximation for the case in which the difference in the dielectric constants of the two media in contact is very small, the geometry which they have considered has further restrictions of ϕ_0 being equal to $\frac{\pi}{2}$, and of the source of waves being in the rarer medium.

The solution of the integral equation (2.35) has been obtained in class h(1) in Chapter III, by reducing it to an equivalent Fredholm equation of the second type. The remaining three classes are discussed in Chapter IV. Chapter V aims at summarizing and making appropriate observations on the investigations of Chapters III and IV and thus help completion of the solution to the problem set up in Chapter II. Chapter VI takes into consideration the changes introduced in the integral equation (2.35) due to simple anisotropy of medium 2. A simple problem has been worked out in Chapter VII to indicate the physical significance of this method, from the points of view of images and wave propagation. The ranges of the parameters θ_k, σ and σ_1 for the four classes of solutions give all the information about the wave propagation. Conclusions of Chapter VIII show the effectiveness of the method used in the analysis. Suggestions for further work are also given in the same chapter.

CHAPTER II

SETTING UP THE INTEGRAL EQUATION OF THE SIMPLIFIED PROBLEM

The problem chosen is to study the interaction of electromagnetic waves with a plasma surrounding an infinitely conducting wedge.

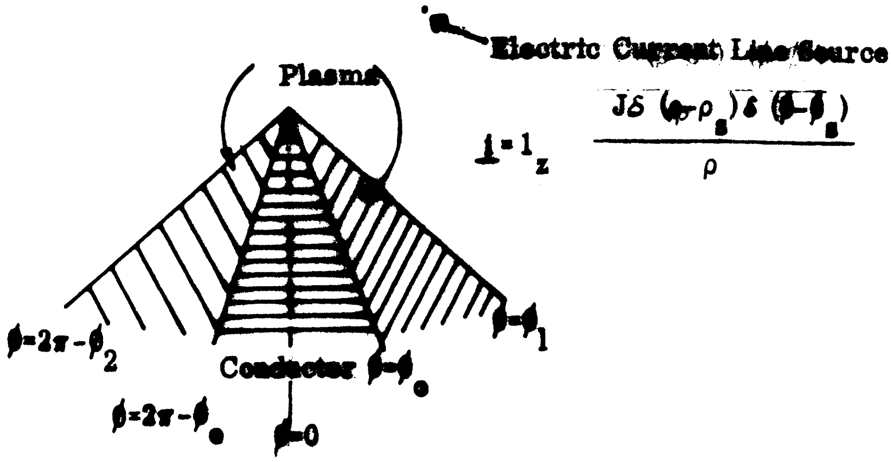

Figure 1. The geometry of the problem.

The wedge is taken having an angle $2\phi_e$, surrounded by a uniform plasma as shown. The system is assumed to extend to infinity in both the z and ρ directions. The rest of the medium is vacuum.

The axes with cylindrical coordinates are chosen such that $\phi = 0$ bisects the angle of the wedge, and the positive z-axis comes out at right angles to the plane of paper, and forms the edge of the wedge.

The plasma is taken uniform and homogeneous so that the parameters representing the plasma do not vary with the coordinates. No D. C. applied magnetic field is assumed so that the representation of the plasma can be made with a scalar permittivity and not a dyadic one. The intensity of electromagnetic waves is taken low so that the non-linear terms $\underline{U} \cdot \nabla\underline{U}$ and $\underline{U} \times \underline{B}$ in the equations of motion of the particles constituting the plasma can be neglected.

The wedge is assumed to be made of a perfectly conducting material so that the fields inside the wedge are zero $(-\phi_e < \phi < \phi_e)$.

An electric current line source extending to infinity and having no variations in the z-direction is assumed.

$$\underline{i} = 1_z \frac{J\delta(\rho - \rho_s)\,\delta(\phi - \phi_s)}{\rho} \tag{2.1}$$

where J is the strength of the electric current, and

$$2\pi - \phi_2 > \phi_s > \phi_1 \tag{2.2}$$

so that the exciting source is in the region of vacuum.

The fields in the vacuum region will be designated by a subscript 1, $(2\pi - \phi_2 \geq \phi \geq \phi_1)$; those in the plasma region at the right of the wedge by a subscript 2, $(\phi_1 \geq \phi \geq \phi_e)$; and those in the plasma region at the left of the wedge by a subscript 3, $(2\pi - \phi_e \geq \phi \geq 2\pi - \phi_2)$. In the initial analysis, plasma will be assumed to be present only on the right of the conducting wedge $(\phi_2 = \phi_0)$ to avoid the possibility of simultaneous integral equations instead of a simple one.

The simple model of plasma can be represented as a medium having a scalar permittivity ϵ_p given by,

$$\epsilon_p = \epsilon_o \left\{ 1 - \frac{\omega_p^2}{\omega^2 (1 - \frac{j\nu_c}{\omega})} \right\} \tag{2.3}$$

where

$$\omega_p^2 = \frac{N e^2}{\epsilon_o m} \tag{2.4}$$

gives the plasma resonance frequency, ω is the frequency of the exciting source, N is the volume density of electrons, e is the electronic charge, m is the electronic mass, ϵ_o is the permittivity of vacuum, and ν_c is the effective collision frequency.

Since the medium and the source have no variation in the z-direction, the problem reduces to a two dimensional one, and

$$\partial_z = 0 \tag{2.5}$$

If ψ is taken to represent the z-component of the electric field, the problem can be formulated in terms of partial differential equations to be satisfied by the function ψ in the two regions, and boundary conditions as follows:

$$\partial_\rho^2 \psi_1 + \frac{1}{\rho} \partial_\rho \psi_1 + \frac{1}{\rho^2} \partial_\phi^2 \psi_1 + k_1^2 \psi_1 = j\omega\mu_o \frac{J\delta(\rho - \rho_o)\delta(\phi - \phi_o)}{\rho} \tag{2.6}$$

$$\partial_\rho^2 \psi_2 + \frac{1}{\rho} \partial_\rho \psi_2 + \frac{1}{\rho^2} \partial_\phi^2 \psi_2 + k_2^2 \psi_2 = 0. \tag{2.7}$$

$$k_1^2 = \omega^2 \epsilon_o \mu_o \tag{2.8}$$

$$k_2^2 = \omega^2 \epsilon_p \mu_o \tag{2.9}$$

in which a time dependence of $e^{j\omega t}$ is assumed.

The boundary conditions on ψ_i $(i = 1, 2)$ are,

$$\psi_1 = 0 \qquad (\phi = 2\pi - \phi_o) \tag{2.10}$$

$$\psi_2 = 0 \qquad (\phi = \phi_o) \tag{2.11}$$

$$\psi_1 = \psi_2 \qquad (\phi = \phi_1) \tag{2.12}$$

$$\partial_\phi \psi_1 = \partial_\phi \psi_2 \ (\phi = \phi_1) \tag{2.13}$$

Both ψ_1 and ψ_2 are required to go to zero at $\rho = 0$, because of the presence of the conductor at $\rho = 0$.

Multiplying (2.6) and (2.7) through by ρ^2 reduces them to a more suitable form to which the Kontorowich Lebedev transform [1] can be applied.

$$\rho^2 \partial_\rho^2 \psi_1 + \rho\, \partial_\rho \psi_1 + \partial_\phi^2 \psi_1 + k_1^2 \rho^2 \psi_1 = j\omega\mu_o \rho J\ \delta(\rho - \rho_s)\ \delta(\phi - \phi_s). \tag{2.14}$$

$$\rho^2 \partial_\rho^2 \psi_2 + \rho\, \partial_\rho \psi_2 + \partial_\phi^2 \psi_2 + k_2^2 \rho^2 \psi_2 = 0. \tag{2.15}$$

Multiplying (2.14) through by $\dfrac{H_\nu^{(2)}(k_1\rho)}{\rho}$ and integrating from 0 to ∞ gives,

$$\int_0^\infty \left[\rho^2 \partial_\rho^2 \psi_1 + \rho \partial_\rho \psi_1 + \partial_\phi^2 \psi_1 + k_1^2 \rho^2 \psi_1\right] \frac{H_\nu^{(2)}(k_1\rho)}{\rho}\, d\rho =$$

$$\int_0^\infty j\omega\mu_o\, \rho\, J\, \delta(\rho - \rho_s)\ \delta(\phi - \phi_s)\ \frac{H_\nu^{(2)}(k_1\rho)}{\rho}\, d\rho \tag{2.16}$$

Integrating by parts,

$$\int_0^\infty \rho\, \partial_\rho^2 \psi_1\, H_\nu^{(2)}(k_1\rho)\, d\rho = \rho\, H_\nu^{(2)}(k_1\rho)\, \partial_\rho \psi_1 \Big|_0^\infty -$$

$$- \int_0^\infty \partial_\rho \psi_1 \left[H_\nu^{(2)}(k_1\rho) + \rho\, \partial_\rho H_\nu^{(2)}(k_1\rho) \right] d\rho \tag{2.17}$$

$$- \int_0^\infty \partial_\rho \psi_1 \left[H_\nu^{(2)}(k_1\rho) + \rho\, \partial_\rho H_\nu^{(2)}(k_1\rho) \right] d\rho =$$

$$= - \psi_1 \left[H_\nu^{(2)}(k_1\rho) + \rho\, \partial_\rho H_\nu^{(2)}(k_1\rho) \right] \Big|_0^\infty +$$

$$+ \int_0^\infty \psi_1 \left[\partial_\rho H_\nu^{(2)}(k_1\rho) + \partial_\rho\, \rho\, \partial_\rho H_\nu^{(2)}(k_1\rho) \right] d\rho \tag{2.18}$$

Substituting (2.18) into (2.17) gives,

$$\int_0^\infty \rho\, \partial_\rho^2 \psi_1\, H_\nu^{(2)}(k_1\rho)\, d\rho =$$

$$= \left[\rho\, H_\nu^{(2)}(k_1\rho)\, \partial_\rho \psi_1 - \rho\, \psi_1\, \partial_\rho H_\nu^{(2)}(k_1\rho) - \psi_1 H_\nu^{(2)}(k_1\rho) \right] \Big|_0^\infty +$$

$$+ \int_0^\infty \psi_1 \left[\partial_\rho H_\nu^{(2)}(k_1\rho) + \partial_\rho\, \rho\, \partial_\rho H_\nu^{(2)}(k_1\rho) \right] d\rho. \tag{2.19}$$

Integrating by parts,

$$\int_0^\infty \partial_\rho \psi_1 H_\nu^{(2)}(k_1\rho)\, d\rho = \psi_1 H_\nu^{(2)}(k_1\rho) \Big|_0^\infty - \int_0^\infty \psi_1\, \partial_\rho H_\nu^{(2)}(k_1\rho)\, d\rho. \tag{2.20}$$

$$\int_0^\infty \partial^2_\phi \psi_1 \frac{H_\nu^{(2)}(k_1\rho)}{\rho} d\rho = \partial^2_\phi \int_0^\infty \psi_1 \frac{H_\nu^{(2)}(k_1\rho)}{\rho} d\rho. \tag{2.21}$$

Define the transform of $\psi_1 (\rho, \phi)$ as

$$\overline{\psi_1} (\nu, \phi) \equiv \int_0^\infty \psi_1 (\rho, \phi) \frac{H_\nu^{(2)}(k_1\rho)}{\rho} d\rho \tag{2.22}$$

Substituting (2.22) into (2.21) gives,

$$\int_0^\infty \partial^2_\phi \psi_1 \frac{H_\nu^{(2)}(k_1\rho)}{\rho} d\rho = \partial^2_\phi \overline{\psi_1} \tag{2.23}$$

Substituting (2.19), (2.20) and (2.23) into the L.H.S. of (2.16) gives,

$$\int_0^\infty \left[\rho^2 \partial^2_\rho \psi_1 + \rho \partial_\rho \psi_1 + \partial^2_\phi \psi_1 + k_1^2 \rho^2 \psi_1\right] \frac{H_\nu^{(2)}(k_1\rho)}{\rho} d\rho =$$

$$= \left[\rho H_\nu^{(2)}(k_1\rho)\, \partial_\rho \psi_1 - \rho \psi_1\, \partial_\rho H_\nu^{(2)}(k_1\rho)\right] \Bigg|_0^\infty +$$

$$+ \int_0^\infty \psi_1 \left[\partial_\rho \rho\, \partial_\rho H_\nu^{(2)}(k_1\rho) + k_1^2 \rho H_\nu^{(2)}(k_1\rho)\right] d\rho + \partial^2_\phi \overline{\psi}_1 \tag{2.24}$$

In the brackets on the R.H.S. of (2.24),

$$H_\nu^{(2)}(k_1\rho) \xrightarrow[\rho \to \infty]{} \frac{e^{-jk_1\rho}}{\sqrt{\rho}} \tag{2.25}$$

$$\partial_\rho H_\nu^{(2)}(k_1\rho) \xrightarrow[\rho \to \infty]{} - \frac{e^{-jk_1\rho}}{\rho^{3/2}} \left[jk_1\rho + \frac{1}{2}\right] \tag{2.26}$$

Thus,

$$\left[\rho\, H_\nu^{(2)}(k_1\rho)\, \partial_\rho \psi_1 - \rho\, \psi_1\, \partial_\rho H_\nu^{(2)}(k_1\rho)\right] \xrightarrow[\rho \to \infty]{}$$

$$\frac{e^{-jk_1\rho}}{\sqrt{\rho}} \left[\rho \left\{\partial_\rho \psi_1 + j\, k_1 \psi_1 + \frac{\psi_1}{2\rho}\right\}\right] \longrightarrow 0 \qquad (2.27)$$

if it is assumed that ψ_1 satisfies the Sommerfeld's radiation condition. In a similar way the brackets may be assumed to tend to zero as $\rho \to 0$ for a suitable behavior of the function ψ_1.

The brackets inside the integral on the R. H. S. of (2.24) is simplified by using,

$$\partial_\rho\, \rho\, \partial_\rho H_\nu^{(2)}(k_1\rho) + k_1^2\, \rho\, H_\nu^{(2)}(k_1\rho) = \frac{\nu^2}{\rho}\, H_\nu^{(2)}(k_1\rho) \qquad (2.28)$$

Using the definition of transform of ψ_1 (2.22) and (2.27) and (2.28) simplifies the R. H. S. of (2.24) to,

$$\int_0^\infty \left[\rho^2 \partial_\rho^2 \psi_1 + \rho\, \partial_\rho \psi_1 + \partial_\phi^2 \psi_1 + k_1^2 \rho^2 \psi_1\right] \frac{H_\nu^{(2)}(k_1\rho)}{\rho}\, d\rho =$$

$$= \partial_\phi^2 \bar{\psi}_1 + \nu^2 \bar{\psi}_1 \qquad (2.29)$$

The integral on the R. H. S. of (2.16) evaluates to,

$$\int_0^\infty j\omega\mu_0\, \rho\, J\, \delta(\rho - \rho_s)\, \delta(\phi - \phi_s)\, \frac{H_\nu^{(2)}(k_1\rho)}{\rho}\, d\rho = D\, \delta(\phi - \phi_s) \qquad (2.30)$$

where

$$D \equiv j\omega\mu_0 \, J \, H_\nu^{(2)}(k_1\rho_s) \tag{2.31}$$

Substituting (2.29) and (2.30) into (2.16) gives the differential equation for $\bar{\psi}_1$

$$\frac{d^2\bar{\psi}_1}{d\phi^2} + \nu^2 \bar{\psi}_1 = D\,\delta(\phi - \phi_s) \tag{2.32}$$

For solution of the inhomogeneous equation (2.32), the procedure indicated by Collin [2] is followed. Two independent solutions,

$$\bar{\psi}_{11} = \cos \nu\phi \tag{2.33}$$

and

$$\bar{\psi}_{12} = \sin\nu\phi \tag{2.34}$$

of the homogeneous equation obtained from (2.32) by putting the R. H. S. equal to zero are chosen. The Wronskian of these two functions is equal to,

$$W = \bar{\psi}_{11}\bar{\psi}_{12}' - \bar{\psi}_{12}\bar{\psi}_{11}' = \nu \tag{2.35}$$

The particular integral of the inhomogeneous differential equation is then given by,

$$\bar{\psi}_{1p} = \bar{\psi}_{12}\int^{\phi} \frac{\bar{\psi}_{11} f}{W}\, d\phi - \bar{\psi}_{11}\int^{\phi} \frac{\bar{\psi}_{12} f}{W}\, d\phi \tag{2.36}$$

where f is the forcing function,

$$f = D\,\delta(\phi - \phi_s). \tag{2.37}$$

Substituting (2.33) through (2.35) and (2.37) into (2.36) and choosing appropriate lower limits gives for the particular integral,

$$\bar{\psi}_{1p} = \frac{D}{\nu}\left[\sin\nu\phi\cos\nu\phi_s - \cos\nu\phi\sin\nu\phi_s\right] \quad (\phi > \phi_s) \tag{2.38}$$

and

$$\bar{\psi}_{1p} = -\frac{D}{\nu}\left[\sin\nu\phi\cos\nu\phi_s - \cos\nu\phi\sin\nu\phi_s\right] \quad (\phi < \phi_s) \tag{2.39}$$

The complete solution of (2.32) is given by,

$$\bar{\psi}_1 = A_1\cos\nu\phi + B_1\sin\nu\phi \pm \frac{D}{\nu}\sin\nu\;(\phi - \phi_s) \tag{2.40}$$

the positive sign being used for $\phi > \phi_s$ and the negative for $\phi < \phi_s$.

Using the boundary condition (2.10) into (2.40),

$$A_1\cos\nu(2\pi - \phi_0) + B_1\sin\nu(2\pi - \phi_0) + \frac{D}{\nu}\sin\nu(2\pi - \phi_0 - \phi_s) = 0 \tag{2.41}$$

$$B_1 = -A_1\frac{\cos\nu(2\pi - \phi_0)}{\sin\nu(2\pi - \phi_0)} - \frac{D}{\nu}\frac{\sin\nu(2\pi - \phi_0 - \phi_s)}{\sin\nu(2\pi - \phi_0)} \tag{2.42}$$

Substituting (2.42) into (2.40), gives,

$$\bar{\psi}_1(\nu,\phi) = A_1\frac{\sin\nu(2\pi - \phi_0 - \phi)}{\sin\nu(2\pi - \phi_0)} - \frac{D}{\nu}\frac{\sin\nu\phi\sin\nu(2\pi - \phi_0 - \phi_s)}{\sin\nu(2\pi - \phi_0)}$$

$$\pm\frac{D}{\nu}\sin\nu(\phi - \phi_s). \tag{2.43}$$

Taking the inverse transform [1] of $\bar{\psi}_1(\nu,\phi)$, gives,

$$\psi_1(\rho,\phi) = -\frac{1}{2}\int_{-j\infty}^{j\infty} \mu\, \overline{\psi}_1(\mu,\phi)\, J_\mu(k_1\rho)\, d\mu \tag{2.44}$$

or

$$\psi_1(\rho,\phi) = \frac{1}{4j}\int_{-j\infty}^{j\infty} \mu\, \overline{\psi}_1(\mu,\phi)\, e^{-j\mu\pi} \sin\mu\pi\, H_\mu^{(2)}(k_1\rho)\, d\mu \tag{2.45}$$

The two forms of inversion are written down, since it will help to visualize the poles of the integrand better, and thus the behavior of the function. The two forms also lead to different forms of representation of the function. In (2.43) if the function $A_1(\nu)$ is known, the inversion (2.44) or (2.45) will give the field function ψ_1 in vacuum. Use of the boundary conditions (2.12) and (2.13), leads to an integral equation for a similar function $A_2(\nu)$ appearing in the expression for the field function $\overline{\psi}_2$ in the plasma region. It is the purpose of this chapter to set up this integral equation. The following quantities required in (2.12) and (2.13) are calculated below.

$$\overline{\psi}_1(\nu,\phi)\Big|_{\phi=\phi_1} = \frac{A_1 \sin\nu\phi_v}{\sin\nu(2\pi-\phi_0)} - \frac{D}{\nu}\,\frac{\sin\nu\phi_1 \sin\nu(2\pi-\phi_0-\phi_s)}{\sin\nu(2\pi-\phi_0)} +$$

$$+ \frac{D}{\nu}\sin\nu(\phi_s-\phi_1) \tag{2.46}$$

where

$$\phi_v = 2\pi - \phi_0 - \phi_1 \tag{2.47}$$

is the angular sector occupied by vacuum region.

$$\partial_\phi \bar{\psi}_1 = - A_1 \frac{\nu \cos \nu(2\pi - \phi_e - \phi)}{\sin \nu(2\pi - \phi_e)} - \frac{D \cos \nu\phi \sin \nu(2\pi - \phi_e - \phi_s)}{\sin \nu(2\pi - \phi_e)} \pm$$

$$\pm D \cos \nu(\phi - \phi_s) \tag{2.48}$$

$$\partial_\phi \bar{\psi}_1 \Big|_{\phi = \phi_1} = - \frac{A_1 \nu \cos \nu \phi_v}{\sin \nu (2\pi - \phi_e)} - \frac{D \cos \nu \phi_1 \sin \nu (2\pi - \phi_e - \phi_s)}{\sin \nu (2\pi - \phi_0)} -$$

$$- D \cos \nu (\phi_s - \phi_1) \tag{2.49}$$

The function ψ_2 in region 2, obeys the equation,

$$\rho^2 \partial_\rho^2 \psi_2 + \rho \partial_\rho \psi_2 + \partial_\phi^2 \psi_2 + k_2^2 \rho^2 \psi_2 = 0. \tag{2.50}$$

Multiplying (2.50) through by $\frac{H_\nu^{(2)}(k_2\rho)}{\rho}$ and integrating from 0 to ∞, and following similar arguments giving (2.16) through (2.29), gives the equation for $\bar{\psi}_2$,

$$\frac{d^2 \bar{\psi}_2}{d\phi^2} + \nu^2 \bar{\psi}_2 = 0 \tag{2.51}$$

where

$$\bar{\psi}_2 (\nu, \phi) \equiv \int_0^\infty \frac{\psi_2(\rho, \phi) H_\nu^{(2)}(k_2 \rho)}{\rho} \, d\rho \tag{2.52}$$

General solution of (2.51) is,

$$\overline{\psi}_2 = A_2 \cos \nu \phi + B_2 \sin \nu \phi . \tag{2.53}$$

The boundary condition on $\overline{\psi}_2$ from (2.11) is,

$$\overline{\psi}_2 = 0 \quad (\phi = \phi_o) \tag{2.54}$$

$$A_2 \cos \nu \phi_o + B_2 \sin \nu \phi_o = 0 \tag{2.55}$$

$$B_2 = - A_2 \frac{\cos \nu \phi_o}{\sin \nu \phi_o} \tag{2.56}$$

$$\overline{\psi}_2 (\nu , \phi) = - \frac{A_2 \sin \nu (\phi - \phi_o)}{\sin \nu \phi_o} \tag{2.57}$$

Taking the inverse transform of $\overline{\psi}_2$ gives,

$$\psi_2 (\rho, \phi) = - \frac{1}{2} \int_{-j\infty}^{j\infty} \mu \, \overline{\psi}_2 (\mu , \phi) \, J_\mu (k_2 \rho) \, d\mu \tag{2.58}$$

$$\psi_2(\rho, \phi) = \frac{1}{4j} \int_{-j\infty}^{j\infty} \mu \, \overline{\psi}_2 (\mu , \phi) \, e^{-j\mu\pi} \sin \mu\pi \, H_\mu^{(2)}(k_2\rho) \, d\mu \tag{2.59}$$

Now in order to make use of the boundary conditions (2.12) and (2.13) which relate the functions ψ_1 and ψ_2 and their normal derivatives at $\phi = \phi_1$, it is necessary to obtain relations between the transforms $\overline{\psi}_1$ and $\overline{\psi}_2$ with respect to the same wave number.

From the boundary condition,

$$\psi_1(\rho,\phi)\Big|_{\phi=\phi_1} = \psi_2(\rho,\phi)\Big|_{\phi=\phi_1} \tag{2.12}$$

Multiplying both sides by $\dfrac{H_\nu^{(2)}(k_1\rho)}{\rho}$ and integrating from 0 to ∞, gives,

$$\int_0^\infty \psi_1(\rho,\phi)\,\frac{H_\nu^{(2)}(k_1\rho)}{\rho}\,d\rho\,\Big|_{\phi=\phi_1} = \int_0^\infty \psi_2(\rho,\phi)\,\frac{H_\nu^{(2)}(k_1\rho)}{\rho}\,d\rho\,\Big|_{\phi=\phi_1}. \tag{2.60}$$

The L.H.S. of (2.60) is $\bar{\psi}_1(\nu,\phi)$ as defined by (2.22), and substituting for $\psi_2(\rho,\phi)$ in the R.H.S. of (2.60) from (2.58) gives,

$$\bar{\psi}_1(\nu,\phi)\Big|_{\phi=\phi_1} = -\frac{1}{2}\int_0^\infty \frac{H_\nu^{(2)}(k_1\rho)}{\rho}\,d\rho \int_{-j\infty}^{j\infty} \mu\,\bar{\psi}_2(\mu,\phi)\,J_\mu(k_2\rho)\,d\mu\,\Big|_{\phi=\phi_1} \tag{2.61}$$

Interchanging the order of integration over ρ and μ,

$$\bar{\psi}_1(\nu,\phi)\Big|_{\phi=\phi_1} = -\frac{1}{2}\int_{-j\infty}^{j\infty} \mu\,\bar{\psi}_2(\mu,\phi)\,d\mu \int_0^\infty \frac{J_\mu(k_2\rho)\,H_\nu^{(2)}(k_1\rho)}{\rho}\,d\rho\,\Big|_{\phi=\phi_1} \tag{2.62}$$

$$\bar{\psi}_1(\nu,\phi)\Big|_{\phi=\phi_1} = -\frac{1}{2}\int_{-j\infty}^{j\infty} \mu\,\bar{\psi}_2(\mu,\phi)\,d\mu\,I_1\,\Big|_{\phi=\phi_1} \tag{2.63}$$

where

$$I_1 = \int_0^{\infty} \frac{J_{\mu}(k_2\rho)\, H_{\nu}^{(2)}(k_1\rho)}{\rho}\, d\rho \,. \tag{2.64}$$

I_1 can be broken up into two integrals,

$$I_1 = \int_0^{\infty} \frac{J_{\mu}(k_2\rho)\, J_{\nu}(k_1\rho)}{\rho}\, d\rho - j \int_0^{\infty} \frac{J_{\mu}(k_2\rho)\, Y_{\nu}(k_1\rho)}{\rho}\, d\rho = I_2 - j I_3 \tag{2.65}$$

where

$$I_2 = \int_0^{\infty} \frac{J_{\mu}(k_2\rho)\, J_{\nu}(k_1\rho)}{\rho}\, d\rho \tag{2.66}$$

and

$$I_3 = \int_0^{\infty} \frac{J_{\mu}(k_2\rho)\, Y_{\nu}(k_1\rho)}{\rho}\, d\rho \,. \tag{2.67}$$

The value of I_2 is taken from Watson, [3],

$$I_2 = \frac{k_2^{\mu}\, \Gamma\left(\frac{\mu+\nu}{2}\right)}{2k_1^{\mu}\, \Gamma\left(1 - \frac{\mu-\nu}{2}\right)\Gamma(\mu+1)}\; {}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}; \mu+1; \frac{k_2^2}{k_1^2}\right)$$

$$\left[0 < k_2 < k_1 ;\ \mathrm{Re}(\mu + \nu + 1) > -1\right] \tag{2.68}$$

In order to evaluate I_3, use is made of another formula in Watson, [4] which reads,

$$\int_0^\infty J_\mu(k_2\rho)\left[\cos\frac{1}{2}(\mu-\nu)\pi\, J_\nu(k_1\rho)+\sin\frac{1}{2}(\mu-\nu)\pi\cdot Y_\nu(k_1\rho)\right]\frac{\rho\, d\rho}{\rho^2-r^2}=$$

$$=\frac{1}{2}\pi\, j\, e^{\frac{1}{2}(\nu-\mu)\pi j}\, J_\mu(k_2 r)\, H_\nu^{(1)}(k_1 r)=I_3^1$$

$$\left[k_1\geq k_2>0;\ \ \mathrm{Re}\,(\mu)>\left|\mathrm{Re}(\nu)\right|-2\right] \tag{2.69}$$

To compare I_3^1 with the integrals I_2 and I_3 required in this development, limit of I_3^1 is taken as $r\to 0$.

$$\lim_{r\to 0} I_3^1=\cos\frac{1}{2}(\mu-\nu)\pi\cdot I_2+\sin\frac{1}{2}(\mu-\nu)\pi\cdot I_3=$$

$$=\lim_{r\to 0}\frac{1}{2}\pi\, j\, e^{\frac{1}{2}(\nu-\mu)\pi j}\, J_\mu(k_2 r)\, H_\nu^{(1)}(k_1 r). \tag{2.70}$$

Now,

$$\lim_{r\to 0} J_\mu(k_2 r)\, H_\nu^{(1)}(k_1 r)=$$

$$=\lim_{r\to 0}\left\{\frac{\left(\frac{k_2 r}{2}\right)^\mu}{\Gamma(\mu+1)}\right\}\left\{\frac{\left(\frac{k_1 r}{2}\right)^\nu}{\Gamma(\nu+1)}(1+j\cot\nu\pi)-\frac{j\csc\nu\pi}{\left(\frac{k_1 r}{2}\right)^\nu\Gamma(1-\nu)}\right\}$$

$$=0\qquad\left[\mathrm{Re}\,\mu>\left|\mathrm{Re}\,\nu\right|\right] \tag{2.71}$$

Substituting (2.71) in R.H.S. of (2.70) gives,

$$I_3 = -\cot \frac{1}{2}(\mu - \nu)\pi \cdot I_2 \tag{2.72}$$

Substituting (2.72) into (2.65) gives,

$$I_1 = I_2 \left[1 + j\cos \frac{1}{2}(\mu - \nu)\pi\right] = \frac{je^{-\frac{1}{2}(\mu-\nu)\pi j}}{\sin \frac{1}{2}(\mu - \nu)\pi} \cdot I_2 \tag{2.73}$$

Substituting the value of I_2 from (2.68) into (2.73) gives,

$$I_1 = \frac{je^{-\frac{1}{2}(\mu-\nu)\pi j}}{2\sin \frac{1}{2}(\mu - \nu)\pi} \frac{k_2^{\mu}}{k_1^{\mu}} \frac{\Gamma(\frac{\mu+\nu}{2})}{\Gamma(1 - \frac{\mu-\nu}{2})\ \Gamma(\mu+1)} \times$$

$$\times\ {}_2F_1(\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}; \mu+1; \frac{k_2^2}{k_1^2}) \quad \left[\mathrm{Re}\,\mu > \left|\mathrm{Re}\,\nu\right|\right] \tag{2.74}$$

Substituting the values of $\overline{\psi}_1(\nu, \phi)\Big|_{\phi = \phi_1}$ and $\overline{\psi}_2(\nu, \phi)\Big|_{\phi = \phi_1}$ from (2.46) and (2.57), into (2.63) gives,

$$\frac{A_1 \sin \nu \phi_v}{\sin \nu (2\pi - \phi_o)} - \frac{D \sin \nu \phi_1 \sin \nu (2\pi - \phi_o - \phi_s)}{\nu \sin \nu (2\pi - \phi_e)} + \frac{D}{\nu} \sin \nu (\phi_s - \phi_1) =$$

$$= \frac{1}{2} \int_{-j\infty}^{j\infty} \frac{A_2(\mu)\mu \sin \mu \phi_p}{\sin \mu \phi_e} I_1 \, d\mu \, . \tag{2.75}$$

where

$$\phi_p = \phi_1 - \phi_o \tag{2.76}$$

is the angular sector occupied by the plasma region on the right of the conducting wedge.

In the boundary condition,

$$\partial_\phi \psi_1(\rho,\phi)\Big|_{\phi=\phi_1} = \partial_\phi \psi_2(\rho,\phi)\Big|_{\phi=\phi_1} \tag{2.13}$$

the expression (2.58) for $\psi_2(\rho,\phi)$ is substituted on the R.H.S. giving,

$$\partial_\phi \psi(\rho,\phi)\Big|_{\phi=\phi_1} = \partial_\phi \left[-\frac{1}{2}\int_{-j\infty}^{j\infty} \mu\, \bar{\psi}_2(\mu,\phi) J_\mu(k_2\rho)\, d\mu\right]\Bigg|_{\phi=\phi_1} \tag{2.77}$$

Multiplying both sides of (2.77) by $\dfrac{H_\nu^{(2)}(k_1\rho)}{\rho}$ and integrating from 0 to ∞ gives,

$$\partial_\phi \left[\int_0^\infty \frac{\psi_1(\rho,\phi) H_\nu^{(2)}(k_1\rho)}{\rho}\, d\rho\right]\Bigg|_{\phi=\phi_1} =$$

$$= \partial_\phi \left[-\frac{1}{2}\int_0^\infty \frac{H_\nu^{(2)}(k_1\rho)}{\rho}\, d\rho \int_{-j\infty}^{j\infty} \mu\, \bar{\psi}_2(\mu,\phi) J_\mu(k_2\rho)\, d\mu\right]\Bigg|_{\phi=\phi_1} \tag{2.78}$$

Substituting (2.22) on the L.H.S. and interchanging the order of integration over ρ and μ in the R.H.S. of (2.78) gives,

$$\partial_\phi \overline{\psi}_1(\nu, \phi) \Big|_{\phi = \phi_1} =$$

$$= -\frac{1}{2} \int_{-j\infty}^{j\infty} \mu \, \partial_\phi \overline{\psi}_2(\mu, \phi) \, d\mu \int_0^{\infty} \frac{J_\mu(k_2\rho) \, H_\nu^{(2)}(k_1\rho)}{\rho} \, d\rho \Big|_{\phi = \phi_1} \tag{2.79}$$

From (2.57),

$$\partial_\phi \overline{\psi}_2(\nu, \phi) \Big|_{\phi = \phi_1} = -\frac{A_2(\nu)\nu \cos \nu \phi_p}{\sin \nu \phi_o} \; . \tag{2.80}$$

Substituting (2.49) and (2.80) and (2.64) into (2.79) gives,

$$-\frac{A_1 \nu \cos \nu \phi_v}{\sin \nu (2\pi - \phi_o)} - \frac{D \cos \nu \phi_1 \sin \nu (2\pi - \phi_o - \phi_s)}{\sin \nu (2\pi - \phi_o)} - D \cos \nu (\phi_s - \phi_1) =$$

$$= \frac{1}{2} \int_{-j\infty}^{j\infty} \frac{\mu^2 A_2(\mu) \cos \mu \phi_p}{\sin \mu \phi_o} \, I_1 \, d\mu \; . \tag{2.81}$$

From (2.78),

$$A_1 = \frac{D \sin \nu \phi_1 \sin \nu (2\pi - \phi_o - \phi_s)}{\nu \sin \nu \phi_v} - \frac{D \sin \nu (\phi_s - \phi_1) \sin \nu (2\pi - \phi_o)}{\nu \sin \nu \phi_v} +$$

$$+ \frac{1}{2} \int_{-j\infty}^{j\infty} \frac{A_2(\mu)\mu \sin \mu \phi_p \cdot \sin \nu (2\pi - \phi_o)}{\sin \mu \phi_o \sin \nu \phi_v} \, I_1 \, d\mu . \tag{2.82}$$

$$\frac{A_1 \nu \cos\nu\phi_v}{\sin\nu(2\pi - \phi_0)} = \frac{D\sin\nu\phi_1 \sin\nu(2\pi - \phi_0 - \phi_s)\cot\nu\phi_v}{\sin\nu(2\pi - \phi_0)} -$$

$$- D\sin\nu(\phi_s - \phi_1)\cot\nu\phi_v + \frac{1}{2}\int_{-j\infty}^{j\infty} \frac{A_2(\mu)\mu\ \nu\sin\mu\phi_p\cot\nu\phi_v}{\sin\mu\phi_0} I_1\, d\mu \qquad (2.83)$$

Equating the values of $\dfrac{A_1 \nu\cos\nu\phi_v}{\sin\nu(2\pi - \phi_0)}$ obtained from (2.81) and (2.83) gives the integral equation for $A_2(\mu)$

$$\int_{-j\infty}^{j\infty} \frac{A_2(\mu)}{2\sin\mu\phi_0} I_1 \left[\mu^2\cos\mu\phi_p + \mu\nu\sin\mu\phi_p\cot\nu\phi_v\right] d\mu$$

$$= D\sin\nu(\phi_s - \phi_1)\cot\nu\phi_v - D\cos\nu(\phi_s - \phi_1)$$

$$- \frac{D\sin\nu(2\pi - \phi_0 - \phi_s)}{\sin\nu(2\pi - \phi_0)}\left[\cos\nu\phi_1 + \sin\nu\phi_1\cot\nu\phi_v\right] \qquad (2.84)$$

This equation can be further simplified to,

$$\int_{-j\infty}^{j\infty} \frac{A_2(\mu)}{2\sin\mu\phi_0} I_1 \left[\mu^2\cos\mu\phi_p\sin\nu\phi_v + \mu\nu\sin\mu\phi_p\cos\nu\phi_v\right] d\mu$$

$$= -2D\sin\nu(2\pi - \phi_0 - \phi_s)\,. \qquad (2.85)$$

CHAPTER III

SOLUTION OF THE INTEGRAL EQUATION

In this chapter, the integral equation obtained in (2.85) will be solved. The integral equation for $A_2(\mu)$ is repeated here with all the necessary details.

$$\int_{-j\infty}^{j\infty} \frac{A_2(\mu)}{2\sin\mu\,\phi_o}\, I_1 \left[\mu^2 \cos\mu\,\phi_p \sin\nu\phi_v + \mu\,\nu\sin\mu\phi_p \cos\nu\phi_v\right] d\mu =$$

$$= -2D\sin\nu\,(2\pi - \phi_o - \phi_s) \tag{3.1}$$

$$\phi_p = \phi_1 - \phi_o \tag{3.2}$$

$$\phi_v = 2\pi - \phi_o - \phi_1 \tag{3.3}$$

$$D = j\omega\mu_o\, J H_\nu^{(2)}(k_1\rho_s) \tag{3.4}$$

$$I_1 = \frac{j e^{-\frac{1}{2}(\mu-\nu)\pi j}}{2\sin\frac{1}{2}(\mu-\nu)\pi}\; \frac{k_2^\mu}{k_1^\mu}\; \frac{\Gamma(\frac{\mu+\nu}{2})}{\Gamma(1-\frac{\mu-\nu}{2})\Gamma(\mu+1)} \; x$$

$$x\ {}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}; \mu+1; \frac{k_2^2}{k_1^2}\right) \qquad \left[\mathrm{Re}\,\mu > \mathrm{Re}\,\nu\right] \tag{3.5}$$

The integral equation in (3.1) is a singular integral equation because of the pole at

$$\mu = \nu \tag{3.6}$$

in I_1 on the contour of integration. No other poles of the integrand lie on the contour of integration. This type of singular integral equation is usually solved by reducing it to an equivalent Fredholm equation. But before reducing it, it is necessary to put (3.1) into the standard singular operator form. For this purpose, (3.1) is rewritten as follows,

$$\int_{-j\infty}^{j\infty} A_2(\mu) \frac{e^{-\frac{1}{2}(\mu-\nu)\pi j}}{\sin\frac{1}{2}(\mu-\nu)\pi} T(\nu,\mu)\, H(\nu,\mu)\, d\mu = -f'(\nu) \qquad (3.7)$$

where

$$T(\nu,\mu) = \frac{j}{2} \frac{k_2^{\mu}}{k_1^{\mu}} \frac{\Gamma(\frac{\mu+\nu}{2})}{\Gamma(1-\frac{\mu-\nu}{2})\Gamma(\mu+1)} \; {}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}; \mu+1; \frac{k_2^2}{k_1^2}\right)$$

$$H(\nu,\mu) = \frac{\mu^2 \cos\mu\phi_p \sin\nu\phi_v + \mu\nu \sin\mu\phi_p \cos\nu\phi_v}{2\sin\mu\phi_o} \qquad (3.9)$$

$$f'(\nu) = 2D \sin\nu(2\pi - \phi_o - \phi_s) \qquad (3.10)$$

In order to express (3.7) in standard operator form used by Muskhelishvili [5], the following transformations are used,

$$\tau = e^{-j\mu\pi} \qquad (3.11)$$

and

$$\tau_o = e^{-j\nu\pi} \qquad (3.12)$$

$$\mu = \frac{j}{\pi} \ln \tau \qquad (3.13)$$

$$\nu = \frac{j}{\pi} \ln \tau_o . \qquad (3.14)$$

$$\mu^2 = -\frac{1}{\pi^2}(\ln \tau)^2 \tag{3.15}$$

$$\mu\nu = -\frac{1}{\pi^2}\ln \tau \ln \tau_0 \tag{3.16}$$

$$\mu + \nu = \frac{j}{\pi}\ln(\tau\tau_0) \tag{3.17}$$

$$\mu - \nu = \frac{j}{\pi}\ln\left(\frac{\tau}{\tau_0}\right) \tag{3.18}$$

$$\frac{d\tau}{\tau - \tau_0} = \frac{-j\pi\, e^{-j\mu\pi}\, d\mu}{e^{-j\mu\pi} - e^{-j\nu\pi}} \tag{3.19}$$

Multiplying the numerator and denominator of R. H. S. of (3.19) by $e^{\frac{1}{2}(\mu+\nu)\pi j}$ gives

$$\frac{d\tau}{\tau - \tau_0} = \frac{-j\pi\, e^{-\frac{1}{2}(\mu-\nu)\pi j}\, d\mu}{e^{-\frac{1}{2}(\mu-\nu)\pi j} - e^{\frac{1}{2}(\mu-\nu)\pi j}} = \frac{\pi}{2}\,\frac{e^{-\frac{1}{2}(\mu-\nu)\pi j}}{\sin\frac{1}{2}(\mu-\nu)\pi}\, d\mu \tag{3.20}$$

Substituting (3.11) through (3.18) and (3.20) into (3.7) gives,

$$\int_0^\infty A_2(\tau)\cdot\frac{2}{\pi}\,\frac{T(\tau_0,\tau)\,M(\tau_0,\tau)}{\tau - \tau_0}\, d\tau = -f(\tau_0) \tag{3.21}$$

where

$$T(\nu,\mu) \rightarrow T(\tau_0,\tau) = \frac{j}{2}\left(\frac{k_2}{k_1}\right)^{\frac{j}{\pi}\ln\tau} \frac{\Gamma\left(\frac{j}{2\pi}\ln(\tau\tau_0)\right)}{\Gamma\left(1 - \frac{j}{2\pi}\ln\left(\frac{\tau}{\tau_0}\right)\right)\Gamma\left(1 + \frac{j}{\pi}\ln(\tau)\right)}\ \times$$

$$\times\ {}_2F_1\left(\frac{j}{2\pi}\ln(\tau\tau_0),\ \frac{j}{2\pi}\ln\left(\frac{\tau}{\tau_0}\right);\ \frac{j}{\pi}\ln(\tau) + 1;\ \frac{k_2^2}{k_1^2}\right) \tag{3.22}$$

$$H(\nu,\mu) \rightarrow H(\tau_o,\tau) = -\frac{1}{2\pi^2 \sinh\left[\frac{\phi_e}{\pi}\ \ln(\tau)\right]}$$

$$\left[(\ln\tau)^2 \cosh\left[\frac{\phi_p}{\pi}\ln\tau\right]\sinh\left[\frac{\phi_v}{\pi}\ln\tau_o\right] + (\ln\tau)(\ln\tau_e)\cosh\left[\frac{\phi_v}{\pi}\ln\tau_o\right]\right.$$
$$\left.\sinh\left[\frac{\phi_p}{\pi}\ln\tau\right]\right] \tag{3.23}$$

$$f'(\nu) \rightarrow f'(\tau_o) = 2j\ D(\tau_e)\ \sinh\left[\left(\frac{2\pi-\phi_e-\phi_s}{\pi}\right)\ln\tau_e\right] \tag{3.24}$$

where

$$D(\nu) = j\omega\mu_o\ J\ H_\nu^{(2)}\ (k_1\rho_s) \rightarrow D(\tau_e). \tag{3.25}$$

Using the integral representation for $H_\nu^{(2)}(k_1\rho_s)$.

$$H_\nu^{(2)}(k_1\rho_s) = \frac{j\ e^{j\frac{\pi}{2}\nu}}{\pi}\int_{-\infty}^{+\infty} e^{-jk_1\rho_s\cosh\eta-\nu\eta}\ d\eta \tag{3.26}$$

and substituting (3.12), gives,

$$D(\nu) \rightarrow D(\tau_o) = -\frac{\omega\mu_e J}{\pi\sqrt{\tau_e}}\int_{-\infty}^{+\infty} e^{-jk_1\rho_s\cosh\eta}\ (\tau_e)^{-\frac{j\eta}{\pi}}\ d\eta \tag{3.27}$$

Now in order to obtain finite limit of integration in (3.21), the following transformations are used;

$$\tau = \frac{t}{1-t} \tag{3.28}$$

$$\tau_o = \frac{t_e}{1-t_o} \tag{3.29}$$

$$d\tau = \frac{dt}{(1-t)^2} \tag{3.30}$$

$$\frac{d\tau}{\tau - \tau_0} = \frac{(1-t_0)\,dt}{(t-t_0)(1-t)} \tag{3.31}$$

Substituting (3.28), (3.29) and (3.31) into (3.21) through (3.24) and (3.27), gives,

$$\int_0^1 A_2(t)\ \frac{2}{\pi}\ \frac{(1-t_0)}{(t-t_0)\,(1-t)}\ T(t_0, t)\, H(t_0, t)\, dt = -\, f'(t_0) \tag{3.32}$$

where

$$T(\tau_0, \tau) \rightarrow T(t_0, t) = \frac{j}{2}\left(\frac{k_2}{k_1}\right)^{\frac{j}{\pi}\ln\frac{t}{(1-t)}} \frac{\Gamma\left(\frac{j}{2\pi}\ln\frac{t\,t_0}{(1-t)(1-t_0)}\right)}{\Gamma\left[1-\frac{j}{2\pi}\ln\frac{t(1-t_0)}{t_0(1-t)}\right]\Gamma\left(1+\frac{j}{\pi}\ln\frac{t}{1-t}\right)}$$

$$\times\ {}_2F_1\left(\frac{j}{2\pi}\ln\frac{t\,t_0}{(1-t)(1-t_0)},\ \frac{j}{2\pi}\ln\frac{t(1-t_0)}{t_0(1-t)};\ \frac{j}{\pi}\ln\frac{t}{1-t}+1;\ \frac{k_2^2}{k_1^2}\right)$$

(3.33)

$$H(\tau_0\ \tau) \rightarrow H(t_0, t) = -\,\frac{1}{2\pi^2 \sinh\left[\frac{\beta_s}{\pi}\ln\frac{t}{1-t}\right]}$$

$$\left[\left(\ln\frac{t}{1-t}\right)^2 \cosh\left[\frac{\beta_2}{\pi}\ \ln\frac{t}{1-t}\right]\ \sinh\left[\frac{\beta_v}{\pi}\ln\ \frac{t}{1-t}\right]\right] \tag{3.34}$$

$$f'(\tau_0) \rightarrow f'(t_0) = 2j\ D(t_0)\ \sinh\left[\left(\frac{2\pi-\beta_0-\beta_s}{\pi}\right)\ln\ \frac{t_0}{1-t_0}\right] \tag{3.35}$$

and

$$D(\tau_0) \rightarrow D(t_0) = -\,\frac{\omega\mu_0 I\sqrt{(1-t_0)}}{\pi\sqrt{t_0}} \int_{-\infty}^{+\infty} e^{-jk_1\rho_s\cosh\eta}\left(\frac{t_0}{1-t_0}\right)^{-\frac{j\eta}{\pi}} d\eta.$$

(3.36)

The equation (3.32) can now be put into the standard operator form,

$$\frac{1}{\pi j}\int_0^1 \frac{K(t_o, t)\,A_2(t)\,dt}{t-t_o} = -f'(t_o) \tag{3.37}$$

where

$$K(t_o, t) = \frac{2j\,(1-t_o)}{(1-t)}\;T(t_o, t)\,H(t_o, t). \tag{3.38}$$

The following method is used by Muskhelisvili [5] for reducing a singular integral equation of the first kind to an equivalent Fredholm equation.

$$B(t_o) = K(t_o, t_o) = 2j\,T(t_o, t_o)\,H(t_o, t_o) \tag{3.39}$$

$$T(t_o, t_o) = \frac{\pi}{2\ln\frac{t_o}{1-t_o}}\left(\frac{k_2}{k_1}\right)^{\frac{j}{\pi}\ln\frac{t_o}{1-t_o}} \tag{3.40}$$

$$H(t_o, t_o) = -\frac{\ln\left(\frac{t_o}{1-t_o}\right)^2}{2\pi^2\sinh\left[\frac{\beta_p}{\pi}\ln\frac{t_o}{1-t_o}\right]}\left[\sinh\left\{\left(\frac{\beta_p+\beta_v}{\pi}\right)\ln\frac{t_o}{1-t_o}\right\}\right] \tag{3.41}$$

$$B(t_o) = -\frac{j\ln\left(\frac{t_o}{1-t_o}\right)}{2\pi\sinh\left[\text{[illegible]}\ln\frac{t_o}{1-t_o}\right]}\left(\frac{k_2}{k_1}\right)^{\frac{j}{\pi}\ln\frac{t_o}{1-t_o}}\left[\sinh\left\{\left(2-\frac{2\beta_0}{\pi}\right)\ln\frac{t_o}{1-t_o}\right\}\right]$$

(3.42)

$$k(t_o, t) = \frac{K(t_o, t) - K(t_o, t_o)}{t - t_o} \qquad (3.43)$$

In terms of $B(t_o)$ and $k(t_o, t)$, equation (3.37) can be expressed as,

$$\frac{B(t_o)}{\pi j} \int_0^1 \frac{A_2(t)}{t - t_o}\, dt + \frac{1}{\pi j} \int_0^1 k(t_o, t)\, A_2(t)\, dt = - f'(t_o) \qquad (3.44)$$

Equation (3.44) is now reduced to an equivalent Fredholm equation. By the equivalence of the given singular equation to the reduced Fredholm equation is meant that the solutions belonging to the class h are being sought [6].

For a singular integral equation of the first type given by (3.37) or equivalently by (3.44), all the ends of curves of integration are non-special ends. Thus the total number of non-special ends for this equation is,

$$2p = 2. \qquad (3.45)$$

Since the transformations (3.28) and (3.29) introduce an extra singularity at $t = 1$ in the equation, it is desirable to seek the solution belonging to a class h (1) that will be bounded at $t = 1$. For this class,

$$q = 1. \qquad (3.46)$$

Thus the index of the problem is,

$$p - q = \chi = 0 \qquad (3.47)$$

The fundamental function of the class h (1) corresponding to (3.44) is given by

$$Z(t) = \frac{c_o \sqrt{t-1}}{\sqrt{t}} \quad B(t) \tag{3.48}$$

Define operaters

$$K^*\phi \equiv \frac{\sqrt{t_o - 1}}{\pi j \sqrt{t_o}} \int_0^1 \frac{\sqrt{t}\, \phi(t)\, dt}{\sqrt{t-1}\, (t-t_o) B(t)} \tag{3.49}$$

and

$$k\phi \equiv \frac{1}{\pi j} \int_0^1 k(t_o, t)\, \phi(t)\, dt\ . \tag{3.50}$$

Then the Fredholm equation equivalent to (3.44) is given by,

$$A_2(t_o) + K^* k A_2 = - K^* f' \tag{3.51}$$

In (3.51), the function $- K^* f'$ and operator $K^* k A_2$ are evaluated as follows:

$$- K^* f' = f(t_o) = - \frac{\sqrt{t_o - 1}}{\pi j \sqrt{t_o}} \int_0^1 \frac{\sqrt{t}\; 2j\, D(t) \sinh\left[\frac{\theta}{\pi} \ln \frac{t}{1-t}\right]}{\sqrt{t-1}\; (t-t_o)\, B(t)}\, dt \tag{3.52}$$

where,

$$\theta = 2\pi - \phi_o - \phi_s. \tag{3.53}$$

The integral,

$$I_4 = \int_0^1 \frac{\sqrt{t}\, 4\pi D(t) \sinh\left[\frac{\theta}{\pi} \ln \frac{t}{1-t}\right] \sinh\left[\frac{\phi_o}{\pi} \ln \frac{t}{1-t}\right]}{\sqrt{t-1}\, (t-t_o) \ln\left(\frac{t}{1-t}\right) \text{[illegible]} \sinh\left[\left(\frac{\text{[illegible]}}{\pi}\right) \ln \frac{t}{1-t}\right]}\, dt$$

(3.54)

appearing in (3.52) is more easily evaluated in the μ - space.

$$D(t) \longrightarrow D(\mu) = j\omega\mu_o J\, H_\mu^{(2)}(k_1\rho_s) \tag{3.55}$$

$$j \sinh\left(\frac{\theta}{\pi}\ \ell n \frac{t}{1-t}\right) \rightarrow \sin \mu\theta . \tag{3.56}$$

$$j \sinh\left(\frac{\phi_o}{\pi}\ \ell n \frac{t}{1-t}\right) \rightarrow \sin \mu \phi_o \tag{3.57}$$

$$j \sinh\left[\left(\frac{2\pi - 2\phi_o}{\pi}\right)\ell n \frac{t}{1-t}\right] \rightarrow \sin \mu\,(2\pi - 2\phi_o) \tag{3.58}$$

$$\left(\frac{k_2}{k_1}\right)^{\frac{j}{\pi}\,\ell n \frac{t}{1-t}} \rightarrow \left(\frac{k_2}{k_1}\right)^{\mu} \tag{3.59}$$

$$\frac{\pi}{j\ell n\left(\frac{t}{1-t}\right)} \rightarrow \frac{1}{\mu} \tag{3.60}$$

$$\frac{dt}{t - t_o} \rightarrow \frac{(1 - t)}{(1-t_o)}\,\frac{d\tau}{\tau - \tau_o} \rightarrow \frac{(1-t)}{(1-t_o)}\,\frac{\pi}{2}\,\frac{e^{-\frac{1}{2}(\mu - \nu)\pi j}}{\sin \frac{1}{2}(\mu - \nu)\pi}\, d\mu \tag{3.61}$$

$$-j\sqrt{t}\,\sqrt{1-t} \rightarrow \frac{-j\tau}{1+\tau} \rightarrow \frac{-j\, e^{-j\frac{\mu\pi}{2}}}{1 + e^{-j\mu\pi}} \tag{3.62}$$

Substituting (3.55) through (3.62) into (3.54) gives,

$$I_4 = \frac{2\pi\omega\mu_0 J e^{j\frac{\pi\nu}{2}}}{1 - t_0} \int_{-j\infty}^{j\infty} \frac{H_\mu^{(2)}(k_1\rho_s) \sin\mu\theta \sin\mu\phi_0 \, e^{-\mu\pi j}}{\mu\left(\frac{k_2}{k_1}\right)^\mu \sin\mu(2\pi - 2\phi_0) \sin\frac{1}{2}(\mu - \nu)\pi\left(1 + e^{-j\mu\pi}\right)} \, d\mu \tag{3.63}$$

$$I_4 = A \, e^{j\frac{\pi\nu}{2}} \, I_5 \tag{3.64}$$

where

$$A = \frac{2\pi\omega\mu_0 J}{1 - t_0} \tag{3.65}$$

and

$$I_5 = \int_{-j\infty}^{j\infty} \frac{H_\mu^{(2)}(k_1\rho_s) \sin\mu\theta \sin\mu\phi_0 \, e^{-\mu\pi j}}{\mu\left(\frac{k_2}{k_1}\right)^\mu \sin\mu(2\pi - 2\phi_0) \sin\frac{1}{2}(\mu - \nu)\pi\left(1 + e^{-j\mu\pi}\right)} \, d\mu \tag{3.66}$$

Using,

$$H_\mu^{(2)}(k_1\rho_s) = -\frac{j e^{j\mu\pi} J_\mu(k_1\rho_s)}{\sin\mu\pi} + \frac{j J_{-\mu}(k_1\rho_s)}{\sin\mu\pi} \tag{3.67}$$

the integral I_5 can be broken up into two integrals,

$$I_5 = I_6 + I_7 \tag{3.68}$$

where,

$$I_6 = \int_{-j\infty}^{j\infty} - \frac{j \sin\mu\theta \cdot \sin\mu\phi_0 \, J_\mu(k_1\rho_s)}{\mu\left(\frac{k_2}{k_1}\right)^\mu \sin\mu\pi \sin\mu(2\pi - 2\phi_0) \sin\frac{1}{2}(\mu - \nu)\pi\left(1 + e^{-j\mu\pi}\right)} \, d\mu \tag{3.69}$$

and

$$I_7 = \int_{-j\infty}^{j\infty} \frac{j \sin \mu \theta \cdot \sin \mu \phi_e \, J_{-\mu}(k_1 \rho_s)}{\mu \left(\frac{k_2}{k_1}\right)^{\mu} \sin \mu \pi \sin \mu (2\pi - 2\phi_e) \sin \frac{1}{2}(\mu - \nu)\pi (1 + e^{j\mu\pi})} \, d\mu \qquad (3.70)$$

The integral I_5 is evaluated in Appendix C.

Substituting (3.54), (3.64) and (3.65) into (3.52) gives,

$$f(t_o) = \frac{2\omega\mu_o J \, e^{j\frac{\pi\nu}{2}}}{\sqrt{t_o}\,\sqrt{1-t_o}} \, I_5 \qquad (3.71)$$

This function can be in terms of the variable t_o, or more conveniently in terms of the variable ν, by means of transformations (3.12) and (3.20). Thus,

$$f(t_o) \rightarrow f(\nu) = 2\omega\mu_o J (1 + e^{j\nu\pi}) \, I_5 . \qquad (3.72)$$

It is observed that there is a pole in $f(\nu)$ at $\nu = 0$, but this can be avoided by using the line of integration indented at $\nu = 0$, as shown in Appendix C. The solutions for $\bar{\psi}_1$ and $\bar{\psi}_2$ given by (2.49) and (2.53), however do not hold for $\nu = 0$, and this point may be considered a singular point of the problem. Moreover, $\nu = 0$ generally indicates no variation in the ϕ domain, and with the geometry of the problem, no variation in the ϕ domain is not admissible.

Now the operator $K^* k\, A_2$ is evaluated.

$$K^* k\, A_2 = \frac{1}{\pi j} \int_0^1 N(t_o, t)\, A_2(t)\, dt . \tag{3.73}$$

where,

$$N(t_o, t) = \frac{\sqrt{t_o - 1}}{\pi j \sqrt{t_o}} \int_0^1 \frac{\sqrt{t_1}\; k(t_1, t)}{\sqrt{t_1 - 1}\; (t_1 - t_o)\, B(t_1)}\, dt_1 \tag{3.74}$$

$$k(t_1, t) = \frac{K(t_1, t) - K(t_1, t_1)}{t - t_1} \tag{3.75}$$

$$K(t_1, t_1) = B(t_1) \tag{3.76}$$

$$K(t_1, t) = \frac{2j(1 - t_1)}{1 - t}\; T(t_1, t)\, H(t_1, t) \tag{3.77}$$

Substituting (3.76) and (3.77) into (3.75) gives,

$$k(t_1, t) = \frac{2j(1 - t_1)\, T(t_1, t)\, H(t_1, t) - (1 - t)\, B(t_1)}{(1 - t)\,(t - t_1)} \tag{3.78}$$

Substituting (3.78) into (3.74) gives,

$$N(t_o, t) = \frac{\sqrt{t_o - 1}}{\pi j \sqrt{t_o}\; (1 - t)}\; I_8 \tag{3.79}$$

where

$$I_8 = \int_0^1 \frac{\sqrt{t_1}\ \left[2j(1-t_1)\,T(t_1,t)\,H(t_1,t) - (1-t)\,B(t_1)\right]}{\sqrt{t_1-1}\ (t_1-t_o)\,(t-t_1)\,B(t_1)}\, dt_1 \tag{3.80}$$

This integral can be more easily evaluated in the η plane by defining transformations connecting t_1 to η, similar to those connecting t and t_o to μ and ν respectively.

$$\tau_1 = \frac{t_1}{1-t_1} = e^{-j\eta\pi} \tag{3.81}$$

$$\frac{dt_1}{t_1 - t_o} \rightarrow \frac{(1-t_1)}{(1-t_o)}\ \frac{d\tau_1}{\tau_1 - \tau_o} \rightarrow \frac{(1-t_1)}{(1-t_o)}\ \frac{\pi}{2}\ \frac{e^{-\frac{1}{2}(\eta-\nu)\pi j}}{\sin(\eta-\nu)\frac{\pi}{2}}\, d\eta \tag{3.82}$$

$$\frac{1}{t-t_1} \rightarrow \frac{(1+\tau)(1+\tau_1)}{\tau - \tau_1} \rightarrow \frac{(1+e^{-j\mu\pi})(1+e^{-j\eta\pi})}{e^{-j\mu\pi} - e^{-j\eta\pi}} \tag{3.83}$$

$$\frac{\sqrt{t_1}\ \sqrt{1-t_1}}{j(1-t_o)} \rightarrow \frac{1}{j(1-t_o)}\ \frac{\sqrt{\tau_1}}{(1+\tau_1)} \rightarrow \frac{1}{j(1-t_o)}\ \frac{e^{-j\frac{\pi}{2}\eta}}{(1+e^{-j\eta\pi})} \tag{3.84}$$

$$2j(1-t_1)\,T(t_1,t)\,H(t_1,t) \rightarrow \frac{2j}{1+e^{-j\eta\pi}}\ T(\eta,\mu)\,H(\eta,\mu) \tag{3.85}$$

$$T(\eta,\mu) = \frac{1}{2}\left(\frac{k_2}{k_1}\right)^{\mu} \frac{\Gamma\left(\frac{\mu+\eta}{2}\right)}{\Gamma\left(1-\frac{\mu+\eta}{2}\right)\Gamma(\mu+1)}\ {}_2F_1\left(\frac{\mu+\eta}{2}, \frac{\mu-\eta}{2}; \mu+1; \frac{k_2^2}{k_1^2}\right)$$

(3.86)

$$H(\eta,\mu) = \frac{\mu^2 \cos\mu\phi_p \sin\eta\phi_v + \mu\eta\sin\mu\phi_p \cos\eta\phi_v}{2\sin\mu\phi_o} \tag{3.87}$$

$$(1-t)\,B(t_1) \rightarrow \frac{1}{1+e^{-j\mu\pi}}\,B(\eta) = -\frac{1}{1+e^{-j\mu\pi}}\,\frac{\eta}{2}\left(\frac{k_2}{k_1}\right)^{\eta} \frac{\sin\eta(2\pi-2\phi_o)}{\sin\eta\phi_o} \tag{3.88}$$

Substituting (3.82) through (3.88) into (3.80) gives,

$$I_8 = \frac{\pi e^{j\frac{\pi\nu}{2}}}{1-t_o} \int_{-j\infty}^{j\infty} \left[\frac{e^{-j\pi\eta}}{(e^{-j\pi\mu} - e^{-j\eta\pi})\sin(\eta-\nu)\frac{\pi}{2}} \right. \times$$

$$\left. \left\{ \frac{(1+e^{-j\mu\pi})\,T(\eta,\mu)\,H(\eta,\mu)}{(1+e^{-j\eta\pi})\,B(\eta)} + \frac{1}{2} \right\} \right] d\eta \tag{3.89}$$

Substituting (3.89) into (3.79), gives,

$$N(t_o, t) = \frac{e^{\frac{j\pi\nu}{2}}}{\sqrt{t_o}\,\sqrt{1-t_o}\,(1-t)}\, I_9 \tag{3.90}$$

where

$$I_9 = \int_{-j\infty}^{j\infty} (F_1 + F_2)\, d\eta \tag{3.91}$$

$$F_1 = \frac{(1+e^{-j\mu\pi})\ T(\eta,\mu)\ H(\eta,\mu)}{(1+e^{j\eta\pi})(e^{-j\mu\pi} - e^{-j\eta\pi})\sin(\eta-\nu)\frac{\pi}{2}\ B(\eta)} \tag{3.92}$$

$$F_2 = \frac{j\ e^{-j\pi\eta}}{2(e^{-j\mu\pi} - e^{-j\eta\pi})\sin(\eta-\nu)\frac{\pi}{2}} \tag{3.93}$$

In order to evaluate I_9, the following contour is taken.

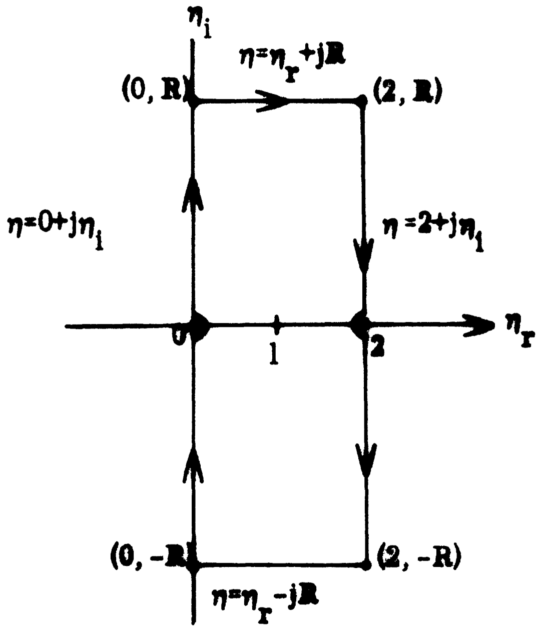

Figure 2. Contour to help evaluation of I_9 on the $\eta=\eta_r+j\eta_i$ plane

Set up a contour integral,

$$I_{10} = \int_C (F_1 + F_2)\, d\eta \tag{3.94}$$

where C is the contour shown in the Figure 2.

On the horizontal portions of C,

$$\eta = \eta_r \pm j R \tag{3.95}$$

where η_r is the real part of η, and the positive and negative signs refer to the top and bottom portions of C respectively.

Consider the functions F_1 and F_2 as $R \to \infty$. Using the asymptotic behavior of $T(\eta, \mu)$ given by (B-2), gives,

$$F_1 \Big|_{\eta_r + jR} \sim \frac{C_1 \, e^{R(\frac{\sigma}{2} + \phi_v + \phi_o \mp \theta_k)}}{(\eta_r + jR)^{3/2} \, e^{R(2\pi - 2\phi_o + \frac{3\pi}{2})}}$$

$$= 0 \, e^{-R(2\pi - 2\phi_o + \frac{3\pi}{2} + \theta_k - \frac{\sigma}{2} - 2\pi + \phi_1)} \quad \text{as } R \to \infty \tag{3.96}$$

where,

$$C_1 = j \sqrt{\frac{2}{\pi}} \; \frac{(1+e^{-j\mu\pi})(1-e^{-\zeta})^{-\mu-\frac{1}{2}} (1+e^{-\zeta})^{\frac{1}{2}} e^{\pi j(\mu+\frac{1}{2})} e^{-\frac{\mu\zeta}{2}} \cdot \mu \sin \mu \phi_p}{\sin \mu \phi_o \, (\frac{k_1}{2k_2})^{\mu}} \times X \tag{3.97}$$

X being some function of η_r and $\left| \frac{k_1}{k_2} \right|$; ζ and σ are defined in (A-5); and,

$$\theta_k = \text{Arg.} \, \frac{k_1}{k_2} \tag{3.98}$$

$$F_1 \Big|_{\eta_r - jR} \sim \frac{C_2\, e^{R(\frac{\sigma}{2} + \phi_v + \phi_o \pm \theta_k)}}{(\eta_r - jR)^{3/2}\, e^{R(2\pi - 2\phi_o + \frac{3\pi}{2})}}$$

$$= 0\, e^{-R(2\pi - 2\phi_o + \frac{3\pi}{2} - \theta_k - \frac{\sigma}{2} - 2\pi + \phi_1)} \quad \text{as } R \rightarrow \infty \tag{3.99}$$

where,

$$C_2 = C_1 e^{-\pi j(\mu + \frac{1}{2})} \tag{3.100}$$

Although in (A-5), positive as well as negative values of σ are admitted, the geometry of the problem requires both σ and θ_k to be positive. This was taken into consideration while obtaining the asymptotic behavior of F_1 in (3.96) and (3.99).

Thus F_1 will tend to zero on the top and bottom portions of C, as $R \rightarrow \infty$, provided,

$$\frac{3\pi}{2} \pm \theta_k + \phi_1 > 2\phi_o + \frac{\sigma}{2} \tag{3.101}$$

and η_r is finite.

The class of solution obtained in this chapter is valid only for the geometry for which (3.101) holds.

$$F_2\Big|_{\eta_r \pm jR} = -\frac{e^{-j\pi(\eta_r \pm jR)}}{\left(e^{-j\mu\pi} - e^{-j\pi(\eta_r \pm jR)}\right)\left(e^{j\frac{\pi}{2}(\eta_r \pm jR-\nu)} - e^{-j\frac{\pi}{2}(\eta_r \pm jR-\nu)}\right)} \tag{3.102}$$

which tends to zero as R tends to ∞.

Thus for,

$$0 \le \eta_r \le 2 \tag{3.103}$$

the integrand of I_{10} vanishes on the horizontal portions of C, when (3.101) holds, and I_{10} can be evaluated for only the vertical portions of C.

On the left vertical side of C,

$$\eta = 0 + j\,\eta_i \tag{3.104}$$

where η_i is the imaginary part of η.

On the right vertical side of C,

$$\eta = 2 + j\,\eta_i. \tag{3.105}$$

Then, taking into account that F_1 and F_2 tend to zero as R tends to infinity on the horizontal portions of C, I_{10} can be written down as,

$$I_{10} = \int_{-j\infty}^{j\infty} \left[F_1 + F_2\right]\Big|_{\eta = 0 + j\eta_1} d\eta + \int_{j\infty}^{-j\infty} \left[F_1 + F_2\right]\Big|_{\eta = 2 + j\eta_1} d\eta =$$

$$= 2\pi j \left[\text{Sum of residues within C}\right] \tag{3.106}$$

The numbers of poles of F_1 within C, due to the factor of $\sin \eta(2\pi - 2\phi_o)$ in the denominator will vary with ϕ_o. For the sake of concreteness, let,

$$\frac{\pi}{2} > \phi_o > \frac{\pi}{4} \tag{3.107}$$

in which case, the number of poles of F_1 due to the above factor will be 2. The number of poles of F_1 within C are thus,

$$\eta = 1 \tag{3.108}$$

$$\eta = \nu \tag{3.109}$$

$$\eta = \eta_\ell \; (\ell = 1, 2) \tag{3.110}$$

where

$$\eta_\ell = \frac{\ell \pi}{2\pi - 2\phi_o} \tag{3.111}$$

The pole of F_2 within C is,

$$\eta = \nu . \tag{3.112}$$

All these poles are simple poles, and the residues at these poles are as follows:

$$R_1 = -\frac{j}{\pi}\,\frac{T(1,\mu)\,H(1,\mu)}{\cos\frac{\pi\nu}{2}\,B(1)} \tag{3.113}$$

$$R_\nu = \frac{1}{\pi(e^{-j\mu\pi}-e^{-j\nu\pi})}\left[\frac{2(1+e^{-j\mu\pi})\,T(\nu,\mu)\,H(\nu,\mu)}{(1+e^{j\nu\pi})\,B(\nu)} + j\,e^{-j\nu\pi}\right] \tag{3.114}$$

$$R_{\eta_\ell} = \frac{2(-1)^\ell\,(1+e^{-j\mu\pi})\,T(\eta_\ell,\mu)\,H(\eta_\ell,\mu)\sin\eta_\ell\phi_0}{\ell\pi(1+e^{j\eta_\ell\pi})(e^{-j\mu\pi}-e^{-j\eta_\ell\pi})\sin(\eta_\ell-\nu)\frac{\pi}{2}}\left(\frac{k_1}{k_2}\right)^{\eta_\ell}(\ell=1,2) \tag{3.115}$$

The first integral on the L. H. S. of (3.106) is equal to I_0, and in order to evaluate the second integral on the L. H. S. of (3.106), F_1 and F_2 are evaluated at (3.105).

$$F_1\Big|_{\eta=2+j\eta_i} = -\frac{(1+e^{-j\mu\pi})\,T(\eta+2,\mu)\,H(\eta+2,\mu)}{(1+e^{j\eta\pi})(e^{-j\mu\pi}-e^{-j\eta\pi})\sin(\eta-\nu)\frac{\pi}{2}\,B(\eta+2)} \equiv F_{11} \tag{3.116}$$

$$F_2\Big|_{\eta=2+j\eta_i} = -\frac{j\,e^{-j\eta\pi}}{2(e^{-j\mu\pi}-e^{-j\eta\pi})\sin(\eta-\nu)\frac{\pi}{2}} \equiv F_{21} \tag{3.117}$$

Substituting (3.113) through (3.117) into (3.106) gives,

$$I_9 = 2\pi j \left[R_1 + R_\nu + R_{\eta_\ell} \right] + Q_1 (\nu, \mu) \qquad (\ell = 1, 2) \qquad (3.118)$$

where

$$Q_1 (\nu, \mu) = \int_{-j\infty}^{j\infty} \left[F_{11} + F_{21} \right] d\eta . \qquad (3.119)$$

Substituting (3.118) into (3.90) gives,

$$N(t_0, t) = \frac{e^{j\frac{\pi\nu}{2}}}{\sqrt{t_0}\sqrt{1-t_0}\,(1-t)} \left[2\pi j \left\{ R_1 + R_\nu + R_{\eta_\ell} \right\} + Q_1(\nu, \mu) \right] \qquad (3.120)$$

In terms of $N(t_0, t)$ and $f(t_0)$ given by (3.120) and (3.71) respectively, the Fredholm equation equivalent to the original singular equation (3.1), in which a class of solution that remains bounded, subject at the condition (3.101), is sought, is given by,

$$A_2(t_0) + \frac{1}{\pi j} \int_0^1 N(t_0, t)\, A_2(t)\, dt = f(t_0) . \qquad (3.121)$$

The solution of this equation is given by, [9].

$$A_2(t_0) = f(t_0) - \sum_{n=1}^{\infty} \int_0^1 \frac{1}{\pi j} \left[N_n(t_0, t)\, f(t) \right] dt . \qquad (3.122)$$

where,

$$N_1(t_o, t) = N(t_o, t) \tag{3.123}$$

$$N_n(t_o, t) = \int_0^1 N_1(t_o, t_1)\, N_{n-1}(t_1, t)\, dt_1 \tag{3.124}$$

It is more convenient to use the (ν, μ) variables instead of (t_o, t) variables by means of transformations, (3.11), (3.12), (3.28) and (3.29). The variable of integration t_1 may also be expressed in terms of the variable η_n by means of transformation similar to (3.81). Thus, (3.122) expressed in terms of (ν, μ) variables, reads,

$$A_2(\nu) = f(\nu) - \sum_{n=1}^{\infty} \int_{-j\infty}^{j\infty} \frac{1}{\pi j} \left[N_n(\nu, \mu)\, f(\mu) \right] d\mu \tag{3.125}$$

where $f(\nu)$ is given by (3.72), and

$$N_1(\nu, \mu) = N(\nu, \mu) = (1+e^{-j\mu\pi})(1+e^{j\nu\pi}) \left[2\pi j \left\{ R_1 + R_\nu + R_{\eta_l} \right\} + Q_1(\nu, \mu) \right] \tag{3.126}$$

$$N_n(\nu, \mu) = \int_{-j\infty}^{j\infty} N_1(\nu, \eta_n)\, N_{n-1}(\eta_n, \mu)\, d\eta_n\,. \tag{3.127}$$

CHAPTER IV

INVESTIGATION OF DIFFERENT CLASSES OF SOLUTIONS

There are in general four classes of solutions to the singular integral equation (2.85). In Chapter III, the class of solution remaining bounded, subject to the condition (3.101), has been obtained. This class of solution has been obtained as class h(1) pertaining to (3.44) that is bounded at the non-special end $t = 1$, and having an index given by (3.47). The other three classes of solutions are,

a) h(0, 1) which is obtained by considering solution that remains bounded at both $t = 0$ and $t = 1$;

b) h(0) which remains bounded at $t = 0$;

c) h_0 which is not necessarily required to be bounded at either $t = 0$ or $t = 1$; this is the highest class of solutions.

The Class h(0, 1)

The class h(0, 1) is the lowest class of solutions; its index is equal to

$$\chi = p - q = -1 \tag{4.1}$$

since for this class,

$$q = 2 \tag{4.2}$$

and p is given by (3.45).

The fundamental function of class h(0, 1) is,

$$Z(t) = C_0 \sqrt{t}\,\sqrt{t-1}\;\; B(t). \tag{4.3}$$

The operators similar to those defined in (3.49) and (3.50) are now defined for this class as follows.

$$K^* \phi \equiv \frac{1}{\pi j}\; (\sqrt{t_0}\,)\;(\sqrt{t_0 - 1}\,) \int_0^1 \frac{\phi(t)\, dt}{\sqrt{t}\,\sqrt{t-1}\;(t - t_0)\; B(t)} \tag{4.4}$$

$$k \phi \equiv \frac{1}{\pi j} \int_0^1 k(t_0, t)\, \phi(t)\, dt. \tag{4.5}$$

where B(t) and $k(t_0, t)$ are defined by (3.42) and (3.43) respectively.

The Fredholm equation, giving a solution belonging to the class h(0, 1), equivalent to the singular integral equation (3.44) is given by,

$$A_2(t_0) + K^* k\, A_2 = - K^* f' \tag{4.6}$$

The existence of this class of solution is subject to the necessary and sufficient condition [5],

$$\int_0^1 \frac{f'(t)\, dt}{\sqrt{t}\,\sqrt{t-1}\;\; B(t)} = I_{11} = 0. \tag{4.7}$$

Transforming I_{11} to the μ space, by the use of (3.11) and (3.28), gives,

$$I_{11} = 2j\pi\,\omega\mu_o\,J \int_{-j\infty}^{j\infty} \frac{H_\mu(k_1\rho_s)\sin\mu\theta\,\sin\mu\phi_o}{\mu\sin\mu(2\pi-2\phi_o)\cos\frac{\mu\pi}{2}} \left(\frac{k_1}{k_2}\right)^\mu d\mu \tag{4.8}$$

Expressing I_{11} as a sum of two integrals,

$$I_{11} = 2\pi\omega\mu_o\,J\left[I_{12} + I_{13}\right] \tag{4.9}$$

where,

$$I_{12} = \int_{-j\infty}^{j\infty} \frac{e^{j\mu\pi}\sin\mu\theta\,\sin\mu\phi_o\,J_\mu(k_1\rho_s)}{\mu\sin\mu\pi\cos\mu\frac{\pi}{2}\sin\mu(2\pi-2\phi_o)} \left(\frac{k_1}{k_2}\right)^\mu d\mu \tag{4.10}$$

$$I_{13} = -\int_{-j\infty}^{j\infty} \frac{\sin\mu\theta\cdot\sin\mu\phi_o\,J_{-\mu}(k_1\rho_s)}{\mu\sin\mu\pi\cos\mu\frac{\pi}{2}\sin\mu(2\pi-2\phi_o)} \left(\frac{k_1}{k_2}\right)^\mu d\mu \tag{4.11}$$

Evaluating I_{12} and I_{13} by the method of residues using the contours similar to those in Appendix C, gives,

$$I_{12} = -2\pi j\sum_{k=1}^{\infty}\left[R^{(12)}_{2k} + R^{(12)}_{2k-1} + R^{(12)}_{\mu_k}\right] \tag{4.12}$$

$$I_{13} = 2\pi j\sum_{k=1}^{\infty}\left[R^{(13)}_{-2k} + R^{(13)}_{1-2k} + R^{(13)}_{-\mu_k} + R^{(13)}_{o}\right] \tag{4.13}$$

where,

$$R_{2k}^{(12)} = - \frac{(-1)^k \sin 2k\theta \, J_{2k}(k_1\rho_s)}{2k\pi \cos 2k\phi_o} \left(\frac{k_1}{k_2}\right)^{2k} \tag{4.14}$$

$$R_{\mu_k}^{(12)} = \frac{(-1)^k e^{j\mu_k\pi} \sin \mu_k \theta \sin \mu_k \phi_o \, J_{\mu_k}(k_1\rho_s)}{k\pi \sin \mu_k \pi \cos \mu_k \frac{\pi}{2}} \left(\frac{k_1}{k_2}\right)^{\mu_k} \tag{4.15}$$

$$R_{-2k}^{(13)} = \frac{(-1)^k \sin 2k\theta \, J_{2k}(k_1\rho_s)}{2k\pi \cos 2k\phi_o} \left(\frac{k_2}{k_1}\right)^{2k} \tag{4.16}$$

$$R_{-\mu_k}^{(13)} = - \frac{(-1)^k \sin \mu_k \theta \cdot \sin \mu_k \phi_o \, J_{\mu_k}(k_1\rho_s)}{k\pi \sin \mu_k \pi \cos \mu_k \frac{\pi}{2}} \left(\frac{k_2}{k_1}\right)^{\mu_k} \tag{4.17}$$

where μ_k is defined by (C.13).

$$R_{2k-1}^{(12)} = \frac{(-1)^k \sin(2k-1)\theta \, J_{2k-1}(k_1\rho_s)}{(2k-1)\pi^2 \cos(2k-1)\phi_o} \left(\frac{k_1}{k_2}\right)^{2k-1} \Bigg[j\pi + \theta \cot(2k-1)\theta +$$

$$+ \phi_o \cot(2k-1)\phi_o + \frac{\partial_\mu J_\mu(k_1\rho_s)\big|_{\mu=2k-1}}{J_{2k-1}(k_1\rho_s)} + \log\left(\frac{k_1}{k_2}\right) + \frac{1}{2k-1} -$$

$$- (2\pi - 2\phi_o) \cot 2(2k-1)\phi_o \Bigg] \tag{4.18}$$

$$R_{1-2k}^{(13)} = \frac{(-1)^k \sin(2k-1)\theta \cdot J_{2k-1}(k_1\rho_s)}{(2k-1)\pi^2 \cos(2k-1)\phi_0} \left(\frac{k_2}{k_1}\right)^{2k-1} \Bigg[-\theta \cot(2k-1)\theta - \phi_0 \cot(2k-1)\phi_0 +$$

$$+ \frac{\partial_\mu J_{-\mu}(k_1\rho_s)\Big|_{\mu=1-2k}}{J_{2k-1}(k_1\rho_s)} + \log\left(\frac{k_1}{k_2}\right) + \frac{1}{2k-1} + (2\pi - 2\phi_0)\cot 2(2k-1)\phi_0 \Bigg] \quad (4.19)$$

$$R_o^{(13)} = -\frac{\theta \phi_0 J_0(k_1\rho_s)}{\pi(2\pi - 2\phi_0)} \quad (4.20)$$

Substituting (4.12) and (4.13) into (4.9) gives,

$$I_{11} = 4j\pi^2 \omega \mu_o J \sum_{k=1}^{\infty} \left[R_{-2k}^{(13)} - R_{2k}^{(12)} + R_{1-2k}^{(13)} - R_{2k-1}^{(12)} + R_{-\mu_k}^{(13)} - R_{\mu_k}^{(12)} + R_o^{(13)} \right] \quad (4.21)$$

The necessary and sufficient condition (4.7) for the existence of this class of solution yields the values of θ and ρ_s, giving the location of the exciting source, necessary for the existence of this class of solution, by equating separately to zero, the real and imaginary parts of

$$\mathrm{Re} \sum_{k=1}^{\infty} \left[R_{-2k}^{(13)} - R_{2k}^{(12)} + R_{1-2k}^{(13)} - R_{2k-1}^{(12)} + R_{-\mu_k}^{(13)} - R_{\mu_k}^{(12)} + R_o^{(13)} \right] = 0 \quad (4.22)$$

$$\mathrm{Im} \sum_{k=1}^{\infty} \left[R_{-2k}^{(13)} - R_{2k}^{(12)} + R_{1-2k}^{(13)} - R_{2k-1}^{(12)} + R_{-\mu_k}^{(13)} - R_{\mu_k}^{(12)} + R_o^{(13)} \right] = 0\,. \quad (4.23)$$

For the location of the exciting source given from θ and ρ_s by (4. 22) and (4. 23), this class of solution is given from the solution of the Fredholm equation (4. 6) equivalent to the singular equation (3. 44), pertaining to which the two operators $K^* \phi$ and $k\phi$ are defined in (4. 4) and (4. 5) respectively.

Evaluating the function $-K^* f'$ in (4. 6) gives,

$$-K^* f' = f(t_o) = \frac{1}{\pi j} \sqrt{t_o} \sqrt{t_o - 1} \; I_{14} \tag{4. 24}$$

where

$$I_{14} = \int_0^1 \frac{4\pi D(t) \sinh\left[\frac{\theta}{\pi} \ln \frac{t}{1-t}\right] \sinh\left[\frac{\phi_o}{\pi} \ln \frac{t}{1-t}\right]}{\sqrt{t}\sqrt{t-1}\,(t-t_o)\, \ln\left(\frac{t}{1-t}\right) \sinh\left[\frac{2\pi - 2\phi_o}{\pi} \ln \frac{t}{1-t}\right]} \left(\frac{k_1}{k_2}\right)^{\frac{j}{\pi} \ln \frac{t}{1-t}} dt \tag{4. 25}$$

Transforming I_{14} in the μ-space just as in the case of I_4, gives,

$$I_{14} = A\, e^{j\frac{\pi\nu}{2}} \cdot I_{15} \tag{4. 26}$$

where

$$I_{15} = \int_{-j\infty}^{j\infty} \frac{H_\mu^{(2)}(k_1\rho_s) \sin\mu\theta \sin\mu\phi_o}{\mu \sin\mu(2\pi - 2\phi_o) \sin\frac{1}{2}(\mu - \nu)\pi} \left(\frac{k_1}{k_2}\right)^\mu d\mu \tag{4.27}$$

and A is defined in (3. 65).

Expressing I_{15} as the sum of two integrals gives,

$$I_{15} = I_{16} + I_{17} \tag{4.28}$$

where

$$I_{16} = \int_{-j\infty}^{j\infty} - \frac{j e^{j\mu\pi} \sin \mu\theta \sin \mu \phi_{o} J_{\mu}(k_1\rho_s)}{\mu \sin \mu\pi \sin \mu(2\pi - 2\phi_o) \sin \frac{1}{2}(\mu - \nu)\pi} \left(\frac{k_1}{k_2}\right)^{\mu} d\mu \tag{4.29}$$

$$I_{17} = \int_{-j\infty}^{j\infty} \frac{j \sin \mu\theta \sin \mu \phi_{o} J_{-\mu}(k_1\rho_s)}{\mu \sin \mu\pi \sin \mu(2\pi - 2\phi_o) \sin \frac{1}{2}(\mu - \nu)\pi} \left(\frac{k_1}{k_2}\right)^{\mu} d\mu \, . \tag{4.30}$$

Using the method of Appendix C to evaluate I_{16} and I_{17}, except that now all the poles are simple poles, gives,

$$I_{16} = -2\pi j \left[R_{\nu}^{(16)} + \sum_{k=1}^{\infty} R_{2k}^{(16)} + R_{2k-1}^{(16)} + R_{\mu_k}^{(16)} + R_{2k+\nu}^{(16)} \right] \tag{4.31}$$

$$I_{17} = 2\pi j \left[R_{o}^{(17)} + \sum_{k=1}^{\infty} R_{-2k}^{(17)} + R_{1-2k}^{(17)} + R_{-\mu_k}^{(17)} + R_{\nu-2k}^{(17)} \right] \tag{4.32}$$

in which (C.10) is assumed to hold.

The various residues are given as follows:

$$R_{\nu}^{(16)} = -\frac{2j \quad e^{j\nu\pi} \sin\nu\theta \sin\nu\phi_o J_\nu(k_1\rho_s)}{\pi \quad \nu \sin\nu\pi \sin\nu(2\pi - 2\phi_o)} \left(\frac{k_1}{k_2}\right)^{\nu} \tag{4.33}$$

$$R_{2k}^{(16)} = -\frac{j(-1)^k \sin 2k\theta J_{2k}(k_1\rho_s)}{2k\pi \cos 2k\phi_o \sin \frac{\pi\nu}{2}} \left(\frac{k_1}{k_2}\right)^{2k} \tag{4.34}$$

$$R_{2k-1}^{(16)} = -\frac{j(-1)^k \sin(2k-1)\theta J_{2k-1}(k_1\rho_s)}{(2k-1)\pi \cos(2k-1)\phi_o \cos\frac{\pi\nu}{2}} \left(\frac{k_1}{k_2}\right)^{2k-1} \tag{4.35}$$

$$R_{\mu_k}^{(16)} = -\frac{j(-1)^k e^{j\mu_k\pi} \sin\mu_k\theta \sin\mu_k\phi_o J_{\mu_k}(k_1\rho_s)}{k\pi \sin\mu_k\pi \sin(\mu_k - \nu)\frac{\pi}{2}} \left(\frac{k_1}{k_2}\right)^{\mu_k} \tag{4.36}$$

$$R_{2k+\nu}^{(16)} = -\frac{2j(-1)^k e^{j\nu\pi} \sin(2k+\nu)\theta \sin(2k+\nu)\phi_o J_{2k+\nu}(k_1\rho_s)}{(2k+\nu)\pi \sin\nu\pi \sin(2k+\nu)(2\pi - 2\phi_o)} \left(\frac{k_1}{k_2}\right)^{2k+\nu} \tag{4.37}$$

$$R_{o}^{(17)} = -\frac{j\theta\phi_o J_o(k_1\rho_s)}{\pi(2\pi - 2\phi_o) \sin\frac{\nu\pi}{2}} \tag{4.38}$$

$$R_{-2k}^{(17)} = \frac{j(-1)^k \sin 2k\theta J_{2k}(k_1\rho_s)}{2k\pi \cos 2k\phi_o \sin\frac{\pi\nu}{2}} \left(\frac{k_2}{k_1}\right)^{2k} \tag{4.39}$$

$$R_{1-2k}^{(17)} = \frac{j(-1)^k \sin(2k-1)\theta J_{2k-1}(k_1\rho_s)}{(2k-1)\pi \cos(2k-1)\phi_o \cos\frac{\pi\nu}{2}} \left(\frac{k_2}{k_1}\right)^{2k-1} \tag{4.40}$$

$$R^{(17)}_{-\mu_k} = - \frac{j(-1)^k \sin \mu_k \theta \sin \mu_k \phi_o J_{\mu_k}(k_1 \rho_s)}{k\pi \sin \mu_k \pi \sin(\mu_k + \nu) \frac{\pi}{2}} \left(\frac{k_2}{k_1}\right)^{\mu_k} \tag{4.41}$$

$$R^{(17)}_{\nu - 2k} = - \frac{j(-1)^k \sin(2k-\nu)\theta \sin(2k-\nu)\phi_o J_{2k-\nu}(k_1\rho_s)}{(2k-\nu)\pi \sin \nu\pi \sin(2k-\nu)(2\pi - 2\phi_o)} \left(\frac{k_2}{k_1}\right)^{2k-\nu} \tag{4.42}$$

Substituting (4.31) and (4.32) into (4.28) gives,

$$I_{15} = 2\pi j \left[R^{(17)}_{o} - R^{(16)}_{\nu} + \sum_{k=1}^{\infty} R^{(17)}_{-2k} - R^{(16)}_{2k} + R^{(17)}_{1-2k} - R^{(16)}_{2k-1} + R^{(17)}_{-\mu_k} - R^{(16)}_{\mu_k} + R^{(17)}_{\nu - 2k} - R^{(16)}_{2k+\nu} \right] \tag{4.43}$$

in which the various residues are given by (4.33) through (4.42).

Substituting (4.28) into (4.24), gives,

$$f(t_o) = \frac{2\omega\mu_o J \sqrt{t_o}}{\sqrt{1 - t_o}} e^{j\frac{\pi\nu}{2}} I_{15} . \tag{4.44}$$

Expressing $f(t_o)$ in the variable ν,

$$f(t_o) \rightarrow f(\nu) = 2\omega\mu_o J I_{15}. \tag{4.45}$$

Now the operator $K^* k. A_2$ in (4.6) is evaluated

$$K^* k A_2 = \frac{1}{\pi j} \int_0^1 N(t_o, t) A_2(t) dt \tag{4.46}$$

where

$$N(t_o, t) = \frac{\sqrt{t_o}\sqrt{t_o - 1}}{\pi j} \int_0^1 \frac{k(t_1, t)}{\sqrt{t_1}\sqrt{t_1 - 1}\,(t_1 - t_o)\,B(t_1)}\, dt_1 \tag{4.47}$$

in which $k(t_1, t)$ and $B(t_1)$ are defined in (3.75) and (3.76) respectively.

Substituting (3.78) into (4.47) gives,

$$N(t_o, t) = \frac{\sqrt{t_o}\sqrt{t_o - 1}}{\pi j (1-t)}\, I_{18} \tag{4.48}$$

where

$$I_{18} = \int_0^1 \frac{\left[2j(1-t_1)\,T(t_1, t)\,H(t_1, t) - (1-t)\,B(t_1)\right]}{\sqrt{t_1}\sqrt{t_1 - 1}\,(t_1 - t_o)(t - t_1)\,B(t_1)}\, dt_1 \tag{4.49}$$

Transforming the variable of integration into η by means of (3.81), gives,

$$I_{18} = \frac{\pi\, e^{j\frac{\pi\nu}{2}}}{1 - t_o} \int_{-j\infty}^{j\infty} \left[\frac{(1 + e^{-j\eta\pi})}{(e^{-j\mu\pi} - e^{-j\eta\pi})\sin(\eta - \nu)\frac{\pi}{2}} \right. \times$$

$$\left. \left\{ \frac{(1 + e^{-j\mu\pi})\,T(\eta, \mu)\,H(\eta, \mu)}{(1 + e^{-j\eta\pi})\,B(\eta)} + \frac{j}{2} \right\} \right] d\eta \tag{4.50}$$

Substituting (4.50) into (4.48) gives,

$$N(t_o, t) = \frac{\sqrt{t_o}\; e^{j\frac{\pi\nu}{2}}}{\sqrt{1 - t_o}\,(1-t)}\, I_{19} \tag{4.51}$$