THE ATMOSPHERIC PRESSURE WAVE GENERATED BY A NUCLEAR EXPLOSION

by
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DEDICATION

This thesis is dedicated with love to my wife, Jeannine, and to our two children, David, Jr. and Lisa.

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The author wishes to express his gratitude to Professor Herbert Uberall for his guidance and for the interest he showed in this project. He is also indebted to Professor Nicholas Kazarinoff, Dr. Raymond Goodrich, Professor Ziya Akcasu and Professor Byron Roe for many suggestions and contributions which helped to make this presentation possible. He would like to acknowledge the technical assistance provided by the entire staff of the Radiation Laboratory at the University of Michigan.

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ABSTRACT

The problem of describing the atmospheric pressure wave that an observer located on the earth's surface will detect at a great distance from the source has been investigated in considerable detail during the past twenty years. Previous researchers attempted to describe the phase and group velocities of propagation by postulating atmospheric temperature structures of varying complexity. In the course of such analysis, they were forced to rely upon numerical methods in order to determine the dispersion relations and the corresponding wave solutions.

In this presentation we consider the simplified isothermal temperature structure in order to obtain an analytic development of the solution. Using a method presented by Langer, we determine that there is one stable mode of oscillation, which we refer to as the gravity wave and an infinite number of modes, which arise from Langer's turning point analysis. This latter set, as it develops, are spurious modes which damp out quite rapidly due to the viscous nature of the atmosphere. On this basis we can obtain an excellent approximation to the total solution by considering only the gravity wave contribution.

In the analytic development of the solution it is necessary to describe a time-independent Green's function which will make it possible to determine the pressure at the observer's position in terms of the pressure generated by the blast. It is shown that two alternate representations for the Green's function are available. The first of these can be determined as an expansion in spherical harmonics. With this development the time variation of the pressure wave satisfies the causality requirement but converges far too slowly to be useful. The alternate

scheme is obtained by a Watson transformation and does converge rapidly. Since this expression is entirely equivalent to the first representation, this form must also yield a useful solution.

The description of the explosion is treated by considering a surface S_o which encloses the source. On this surface the time variation of the pressure is assumed to be known from empirical data, while in the region exterior to S_o the equations of hydrodynamics can be linearized. The advantage in considering this surface is that it permits the determination of the relationship between the energy of the explosion and the amplitude of the theoretical pulse that an observer might expect to detect. The result of the investigation is that the time independent solutions are in excellent agreement with observed data. In fact, it is possible to determine the energy of the explosion very accurately if the height of burst and the distance between the source and the observer are known.

PREFACE

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INTRODUCTION

The investigation of the atmospheric pressure disturbance produced at a great distance from a large explosion is a relatively new subject in the field of acoustics. Prior to the Krakatoa volcanic eruption of 1883 and the great Siberian meteorite of June 1908, in fact, there had been no recorded observations of large scale disturbances, either natural or artificial, to prompt any interest in this area. It is perhaps surprising, however, that even these two historic phenomena aroused very little scientific curiosity until Whipple[1][2][3] produced several lengthy compilations of what was then known about the Siberian meteorite. A few years later, Pekeris[4][5] provided the first quantitative treatment of the problem in his analysis of the free atmospheric oscillations stimulated by the Krakatoa eruption.

With the invention and subsequent testing of the hydrogen bomb, there has been considerably more effort devoted to the analysis of atmospheric pressure waves. One reason for the increased interest is that there are now a number of extremely sensitive devices located at various stations around the world which can detect very weak signals and determine the nature of the pressure wave train. Yamomoto[6] in his analysis of microbarograph recordings of Soviet and United States thermonuclear detonations reported that such a signal generally consists of a wave of relatively long period followed, after several cycles, by oscillations of higher frequency. Donn and Ewing[7] state that there seem to be waves which begin with highest amplitude and which "....appear to be superimposed on a lower amplitude, long period train of waves...". The exact cause of the higher frequency waves is not completely established, although I ~~suspect they~~ are due to atmospheric ducting and reflection. The problem arises, however, that many of Donn and Ewing's recordings do not indicate these more rapid oscillations, making it difficult to state positively that they are always present in the wave pattern. The only feature which appears to be characteristic of all microbarograph recordings is the presence of a low frequency gravity wave. For this reason, we shall focus our attention on this mode ~~of oscillation~~ and discuss the high frequency, superposed waves only as the need arises.

In the formulation of the problem it is convenient to use the geometric picture presented by Weston (Figure 1) in which the earth is assumed to be a non-rotating sphere of radius a. Winds, arising from a combination of temperature variation and earth's rotation, will not be present in the treatment since we postulate the atmosphere to be isothermal. With this simplification, then, we have symmetry about the z-axis of Figure 1. At some point on this axis above the earth's surface an explosion occurs which, in the initial stages, propagates radially outward, arriving after some time interval at the surface S_o. This surface must be large enough so that on S_o the overpressure will be an order of magnitude less than the ambient atmospheric pressure and yet small enough ~~so that~~ it is still approximately spherical, with radius $R_o \ll a$.

In the region exterior to S_o, Euler's hydrodynamic equations are linearized and made time independent by means of a Laplace transform. We can then obtain from these a single equation for the pressure, where this function depends upon the frequency and two spacial coordinates. The behavior of this

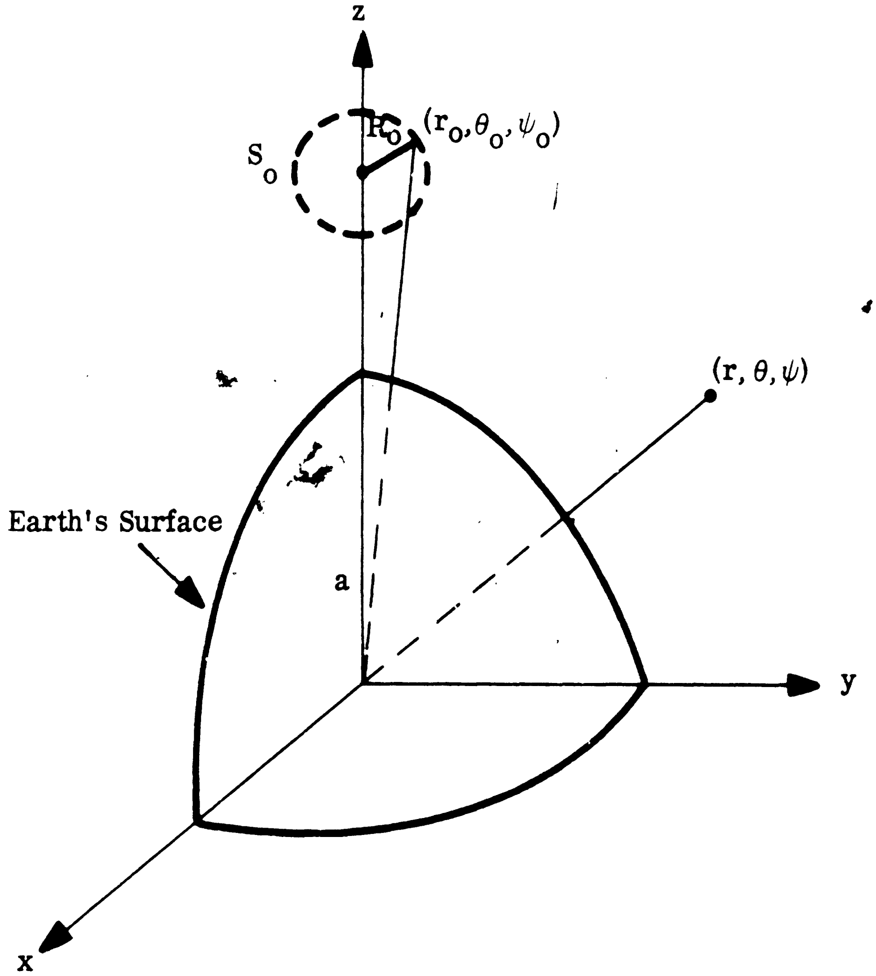

FIG. 1: SCHEMATIC REPRESENTATION OF AN EXPLOSION OCCURRING AT A POINT ON THE Z-AXIS ABOVE THE EARTH'S SURFACE
The spherical coordinates for the observer are (r, θ, ψ) relative to the center of the earth, while the coordinates of a point on the surface S_o are (r_o, θ_o, ψ_o).

pressure at any point must, of course, depend upon the nature of the source. In order to introduce this input into the problem, we describe, using experimental data, the pressure wave form which a nuclear explosion will produce on the surface S_0. This pressure can then be matched with the pressure at any point exterior to S_0 by means of an appropriately chosen Green's function.

For this Green's function, two equivalent representations are available. The first of these satisfies the requirement of causality but is of little use in determining the pressure wave form because of its slow convergence. The alternate representation has the advantage of being rapidly convergent but also has a drawback in that all of the modes have to be analyzed separately. The fact that the two developments are equivalent does indeed mean that the second representation must yield causal solutions. This facet of the problem will be discussed more thoroughly when the two Green's functions are presented.

The procedures we will follow are, in some respects, similar to those of Weston.[8] [9] There are, however, a number of differences which are worth indicating at this time. The first of these is that Weston does not discuss the fact that his wave form appears to violate the condition of causality. The inclusion of this aspect of the problem into our analysis is one of the more significant accomplishments of this presentation. As a matter of fact, Pierce[10], to the writer's knowledge, is the only other investigator to have considered this feature, although he did so for a problem which is much simpler than the one we shall be discussing.

A second difference is that Weston, as well as Scorer[11] and Hunt et al[12] considered atmospheric temperature structures of varying complexity, whereas we will assume an isothermal temperature distribution. The primary advantage gained by postulating this simplified representation is that it allows us to develop an analytic, as opposed to a numerical, solution to the problem. At first glance it might seem that this choice is actually an oversimplification which would lead to physically unrealistic results. This, however, does not appear to be the case, as a comparison of our wave solution with those experimentally obtained by several observers will indicate at the conclusion of this presentation.

The first indication that such a temperature representation might be a valid approximation was afforded by the theoretical calculations of the aforementioned investigators. Weston[9] working with a complicated thermosphere model, and Hunt et al[12], working with a structure consisting of two isothermal layers, computed dispersion curves for the gravity wave mode. The significant feature of their results is that the phase velocity is practically constant over most of the frequency range considered, varying noticeably only at both ends of the interval. Figure 2 presents a comparison of these numerical evaluations with the phase velocity that we obtain from considering the isothermal approximation. The fact that the curves are in good agreement with each other indicates that as far as the gravity wave mode is concerned, this simplifying assumption does not introduce any significant distortion. Whether or not the temperature variations produce any marked effect upon the other modes is something we will not be concerned with at this time. It is unlikely, however, that any significant variation would be introduced into any mode by atmospheric

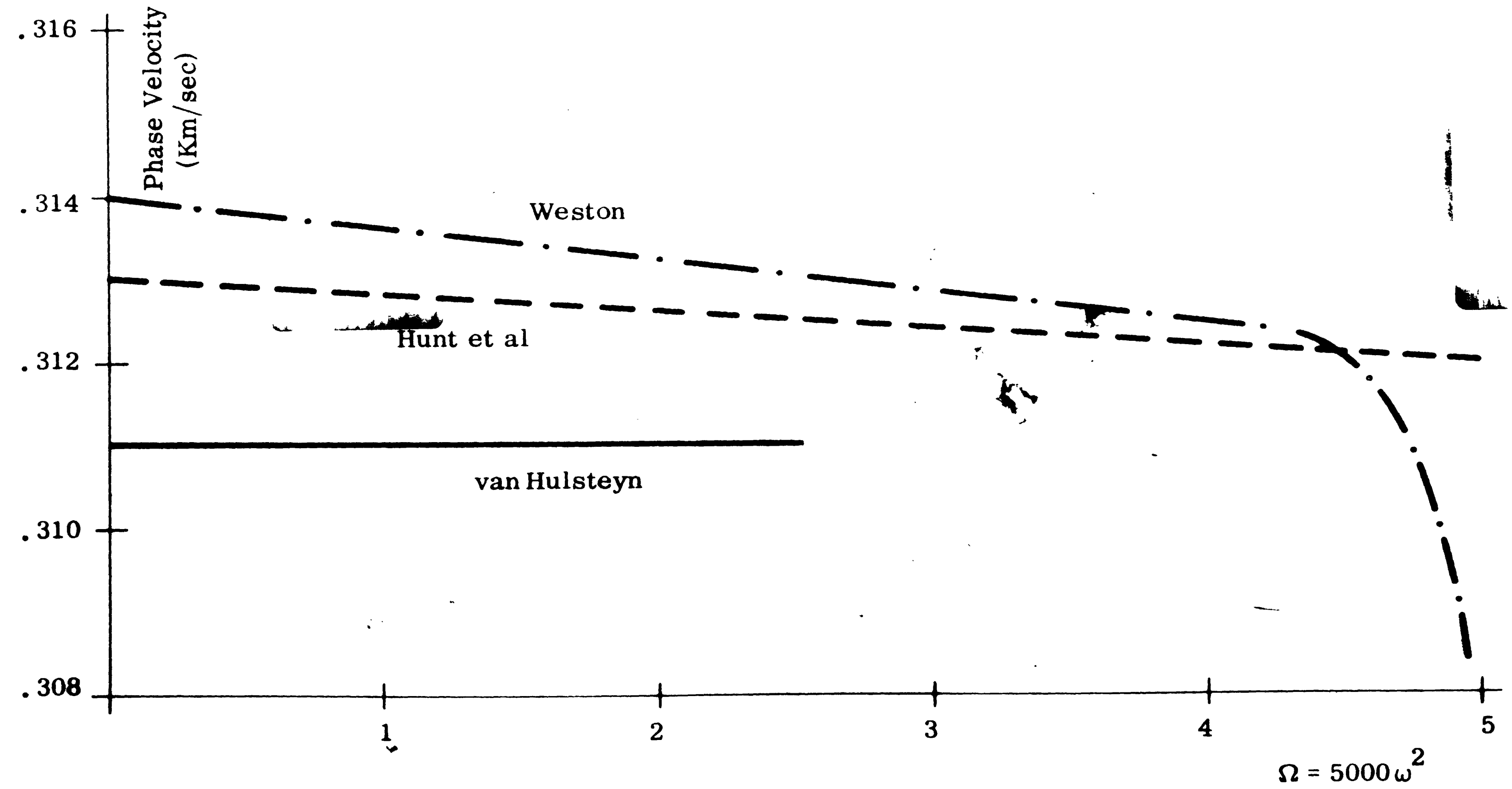

FIG. 2: **COMPARISON OF THE PHASE VELOCITY CURVES FOR THREE ATMOSPHERIC STRUCTURES.** Hunt, et al considered a two-layer model with troposphere at 13.7°C and stratosphere at -43.7°C and obtained an almost linear curve which cuts off at $5000\omega^2 = 14.5\ \sec^{-2}$. Weston's Thermosphere model has an upper cutoff at $5000\omega^2 = 4.95\ \sec^{-2}$. van Hulsteyn's isothermal model has a cuttoff at $5000\omega^2 = 2.53\ \sec^{-2}$.

temperatures at high altitudes when it is recalled that more then 90 per cent of the mass of the atmosphere is in the first 20 Km of elevation.

A third difference between our formulation and that of Weston is that he obtains a Green's Function by means of an eigenfunction expansion, while we will be using a Regge pole development. The two presentations must be equivalent in the final analysis, but our scheme has the advantage of simplicity. Weston[13], using a proof presented by Titchmarsh, was able to show that there exist an infinite number of discrete modes of oscillation and that the eigenfunctions must therefore form a complete set. He apparently believed, however, that the eigenfunctions are real quantities, whereas we will show that they actually lie in the complex plane.

The final difference which should be noted deals with the physical description of the explosive source. Rather than considering the detonation as being equivalent to the introduction of a volume of air into the atmosphere, we assume the pressure disturbance to arise from the flow of energy across a surface enclosing the source. A knowledge of the temporal variation of the pressure on this surface is then sufficient to determine the wave train at points outside. The example on which we focus our attention is that in which an observer is located at a point on the earth's surface several hundred kilometers from a low-altitude explosion. As it develops, the amplitude of the pulse is directly proportional to the energy of the explosion, while the period of the oscillation is independent of this parameter. The fact that we can obtain an extremely good correlation between theoretical and experimental results provides, in the final analysis, the justification for our assumptions and simplifications.

STATEMENT OF THE PROBLEM

The approach to be presented in this section is based upon the formulation provided by Weston[8] with two important differences. Whereas Weston considered vertical atmospheric temperature variations and constant gravitational acceleration, we will assume the temperature to be constant and allow gravity to vary as $1/r^2$. The reason for considering this simple atmospheric model is, as was explained in the Introduction, to allow an analytic development of the solution. It might appear illogical, then, to complicate matters by including the variation of gravity particularly since this acceleration is practically constant at all heights where the atmosphere behaves macroscopically. In Appendix A, however, it is shown that the analytic properties of the wave function are different for the two cases $g = g_0$ and $g = g_0 \frac{a^2}{r^2}$, so that a slight difference is sufficient to alter the form of the solution. At a later point, when the difference in behavior has been established, we will consider the situation where $g = g_0$ as a limiting approximation.

The first step involves the hydrostatic relationships, which are valid in the absence of any disturbance. At a point (r, ϕ, ψ) above the earth's surface, the atmosphere may be described by its ambient pressure p_0, density ρ_0, and temperature T_0, the three parameters being related through the ideal gas law

$$p_0 = R \rho_0 T_0 . \tag{1}$$

Here, R is the ideal gas constant. The speed of sound c_0, is related to these quantities by means of the expression

$$c_0^2 = \gamma R T_0 = \frac{\gamma p_0}{\rho_0} \tag{2}$$

where γ is the ratio of specific heats. At the point in question, the buoyant force produced by the hydrostatic pressure difference gives the relationship

$$\frac{\partial p_0}{\partial r} = -g\rho_0 \tag{3}$$

where, in general, p_0, ρ_0 and T_0 are functions of all three coordinates. From equations (1-3)

$$\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial r} = \frac{-1}{T_0}\frac{\partial T_0}{\partial r} - \frac{g}{RT_0} = -\left[\frac{1}{c_0^2}\frac{\partial c_0^2}{\partial r} + \frac{\gamma g}{c_0^2}\right] . \tag{4}$$

For the special case of the isothermal atmosphere where T_0 is constant, however, $\frac{\partial c_0^2}{\partial r} = 0$, so that integration of equation (4) yields

$$\rho_0(r) = \rho_0(a)\, e^{\left\{\frac{\gamma g}{c_0^2}\frac{r}{a}(a-r)\right\}} \tag{5}$$

and similarly, from equation (1),

$$p_0(r) = p_0(a)\, e^{\left\{\frac{\gamma g}{c_0^2}\frac{r}{a}(a-r)\right\}} . \tag{6}$$

There expressions for $p_0(r)$ and $\rho_0(r)$ are, of course, independent of θ and ψ since rotation and horizontal temperature variations have been neglected. In a more realistic atmospheric representation, these effects would have to be included, along with the fact that at sufficiently great altitudes the macroscopic relationships given in equations (3) and (4) are not valid. From the standpoint of a reasonable mathematical model, however, equations (1) - (6) are sufficient to describe the hydrostatic situation.

When the atmosphere is perturbed by the source as indicated in Figure 1, the disturbance in the region exterior to S_0 may be determined from Euler's hydrodynamic equations which state that

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\bar{\rho}}\nabla \bar{p} - \underline{i}_r g \qquad \text{(momentum)} \qquad (7)$$

$$\frac{D\bar{\rho}}{Dt} + \bar{\rho}\,\nabla \cdot \underline{u} = 0 \qquad \text{(continuity)} \qquad (8)$$

$$\frac{D\bar{p}}{Dt} = \frac{\gamma\bar{p}}{\bar{\rho}}\,\frac{D\bar{\rho}}{Dt} \qquad \text{(adiabatic energy).} \qquad (9)$$

In these expressions, $\underline{u}$ is the velocity of a parcel of air, $\bar{\rho}$ is its total density, and $\bar{p}$ is the total scalar pressure. The differential operator $\frac{D}{Dt}$ is a symbolic notation for the total time derivative $(\underline{u} \cdot \nabla + \frac{\partial}{\partial t})$. For the purpose of linearizing equations (7) - (9) it is convenient to write

$$\bar{p} = p_o + p \quad \text{and} \quad \bar{\rho} = \rho_o + \rho \qquad (10)$$

where p and ρ are the overpressure and excess density, respectively. If, then, $\frac{p}{p_o} \ll 1$, $\frac{\rho}{\rho_o} \ll 1$, and $\left|\frac{1}{c_o}\underline{u}\right| \ll 1$, these equations simplify to

$$\frac{\partial \underline{u}}{\partial t} = -\frac{1}{\rho_o}\nabla p - \underline{i}_r g \frac{\rho}{\rho_o} \qquad (11)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho_o \underline{u}) = 0 \qquad (12)$$

$$\frac{\partial p}{\partial t} - c_o^2 \frac{\partial \rho}{\partial t} + \underline{u} \cdot \left[\nabla p_o - c_o^2 \nabla \rho_o\right] = 0 \qquad (13)$$

once the hydrostatic relationship (3) has been subtracted from equation (11). The fact that $\frac{\partial p}{\partial t}$, for example, is a first order approximation to $\frac{Dp}{Dt}$ is based upon the assumption that $\left|\nabla p\right| = 0(\frac{1}{c_o}\,\frac{\partial p}{\partial t})$. This supposition is not actually warranted at this point, but will be borne out in the section involving the pressure function.

In order to eliminate four of the five dependent variables from the five scalar equations (11) - (13) it is expedient to introduce the Laplace transform

$$X(s) = \int_0^{\infty} X(t) e^{-st}\, dt \tag{14}$$

and its inverse

$$X(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} X(s) e^{st}\, ds \tag{15}$$

where X(t) represents any one of the five time dependent functions appearing in equations (11) - (13). At a later stage in the development it will be more convenient to use the Fourier transform, but this simply involves replacing s appropriately with $-i\omega + c$ so as to satisfy the causality requirement. For the present, the Laplace transform is preferable since it avoids the complications introduced in the event there are any singularities on or to the right of the imaginary s axis.

If we now use the fact that p_o and ρ_o are functions of r alone and denote all r-derivatives by a prime('), equations (11) - (13) become

$$s\underline{u} = -\frac{1}{\rho_o} \nabla p - \underline{i}_r g \frac{\rho}{\rho_o} \tag{16}$$

$$s\rho + \rho_o' u_r + \rho_o \nabla \cdot \underline{u} = 0 \tag{17}$$

$$sp - c_o^2 s\rho + u_r \left[p_o' - c_o^2 \rho_o' \right] = 0. \tag{18}$$

With the equations in this form, the variable ρ may be eliminated by solving equation (18) for ρ and substituting into equations (16) and (17) respectively. We thus obtain

$$\begin{bmatrix} s^2 + \frac{g}{c_o^2}\frac{1}{\rho_o}(p_o' - c_o^2\rho_o'), & 0, & 0 \\ 0, & s^2, & 0 \\ 0, & 0, & s^2 \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \\ u_\psi \end{bmatrix} = \frac{-s}{\rho_o} \begin{bmatrix} \frac{\partial p}{\partial r} + \frac{g}{c_o^2} p \\ \frac{1}{r}\frac{\partial p}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial p}{\partial \psi} \end{bmatrix} \tag{19}$$

and

$$sp + c_o^2 \rho_o \nabla \cdot \underline{u} + p_o' u_r = 0 . \tag{20}$$

From the expressions for p_o' and ρ_o' as given by equations (3) and (4), equations (19) and (20) may be simplified in appearance to

$$\rho_o \begin{bmatrix} hs^2 & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & s^2 \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \\ u_\psi \end{bmatrix} = -s \begin{bmatrix} \frac{\partial p}{\partial r} + \frac{g}{c_o^2} p \\ \frac{1}{r}\frac{\partial p}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial p}{\partial \psi} \end{bmatrix} \tag{21}$$

and

$$sp + c_o^2 \rho_o \nabla \cdot \underline{u} - g\rho_o u_r = 0 \tag{22}$$

where, in equation (21)

$$hs^2 = s^2 + \frac{g}{\rho_o c_o^2}\left[p_o' - c_o^2 \rho_o'\right] = s^2 + \frac{(\gamma-1)g^2}{c_o^2} . \tag{23}$$

In order to derive an equation for the pressure which resembles the wave equation it is convenient to multiply equations (21) and (22) by $\rho_o^{-1/2}$, from which we obtain

$$\begin{bmatrix} hs^2 & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & s^2 \end{bmatrix} \begin{bmatrix} \rho_o^{1/2} u_r \\ \rho_o^{1/2} u_\theta \\ \rho_o^{1/2} u_\psi \end{bmatrix} = -s \begin{bmatrix} \frac{\partial}{\partial r}(\rho_o^{-1/2} p) + A(\rho_o^{-1/2} p) \\ \frac{1}{r} \frac{\partial(\rho_o^{-1/2} p)}{\partial \theta} \\ \frac{1}{r \sin\theta} \frac{\partial(\rho_o^{-1/2} p)}{\partial \psi} \end{bmatrix} \tag{24}$$

and

$$s(\rho_o^{-1/2} p) + c_o^2 \nabla \cdot (\rho_o^{1/2} \underline{u}) - A c_o^2 \rho_o^{1/2} u_r = 0 \tag{25}$$

respectively, in which **A**, appearing in both equations, is given by

$$A = (1 - \frac{\gamma}{2}) \frac{g}{c_o^2} . \tag{26}$$

The three velocity components may be eliminated by solving equation (24) for $\rho_o^{1/2} \underline{u}$ and substituting into equation (25), with the result that

$$(\nabla \cdot \underline{L}) (\rho_o^{-1/2} p) - \left[\frac{s^2}{c_o^2} + \frac{A^2}{h} - \frac{1}{r^2} \frac{d}{dr} (\frac{Ar^2}{h}) \right] (\rho_o^{-1/2} p) = 0 \tag{27}$$

where

$$\underline{L} = \left[\frac{1}{h} \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{1}{r \sin\theta} \frac{\partial}{\partial \psi} \right] . \tag{28}$$

Since $g = g_o \frac{a^2}{r^2}$, we see from equation (26) that

$$\frac{1}{r^2} \frac{d}{dr} (\frac{Ar^2}{h}) = - \frac{Ah'}{h^2} . \tag{29}$$

With this simplification, we finally obtain the desired form for the time independent pressure equation, namely

$$(\nabla \cdot \underline{L}) (\rho_o^{-1/2} p) - \left[\frac{s^2}{c_o^2} + \frac{A^2}{h} + \frac{Ah'}{h^2} \right] (\rho_o^{-1/2} p) = 0 . \tag{30}$$

Equation (30) represents a logical starting point for treating this problem in acoustic diffraction. In the analysis, the function $(\rho_o^{-1/2} p)$ will play the role of a scalar potential, while the operator $\nabla \cdot \underline{L}$ replaces the simpler Laplacian operator ∇^2. For the purpose of separating the variables at a later stage, we introduce the notation

$$q(r) = r^2 h\left[\frac{s^2}{c_o^2} + \frac{A^2}{h} + \frac{h'}{h^2}\right] \tag{31}$$

whereupon equation (30) becomes

$$(\nabla \cdot \underline{L})(\rho_o^{-1/2} p) - \frac{1}{h}\frac{q(r)}{r^2}(\rho_o^{-1/2} p) = 0 . \tag{32}$$

Since this equation is valid for all points (r, θ, ψ) exterior to the surface S_o of Figure 1, we postulate the existence of a Green's function $G(\underline{r}, \underline{r}_o, s)$ which satisfies

$$(\nabla \cdot \underline{L})G - \frac{1}{h}\frac{q(r)}{r^2} G = \delta(\underline{r} - \underline{r}_o) . \tag{33}$$

Here, $\underline{r}_o$ is any point in space. Upon multiplying equations (33) and (32) by $\rho_o^{-1/2} p$ and G, respectively, and subtracting, we obtain

$$\nabla \cdot \left[(\rho_o^{-1/2} p)\,\underline{L}G - G\,\underline{L}(\rho_o^{-1/2} p)\right] = \rho_o^{-1/2} p\,\delta(\underline{r} - \underline{r}_o) . \tag{34}$$

This equation is applicable in the volume of space V bounded by the earth's surface, the surface at infinity and the surface S_o. If we denote these three surfaces by the symbol $\sum$, integration of equation (34) over $d^3\underline{r}_o$ yields

$$\rho_o^{-1/2} p(\underline{r}, s) = \int_V \nabla \cdot \left[(\rho_o^{-1/2} p)\underline{L}G - G\,\underline{L}(\rho_o^{-1/2} p)\right] dv$$

and hence, by Gauss' theorem

$$\rho_o^{-1/2} p(\underline{r}, s) = \int_{\sum} \left[(\rho_o^{-1/2} p)\underline{L}G - G\,\underline{L}(\rho_o^{-1/2} p)\right] \cdot \underline{n}_{\sum}\, d\sigma . \tag{35}$$

The form presented in this equation is still not the most convenient representation available because it involves an integration over the earth's surface and the surface at infinity. In the following section we simplify the situation by imposing a boundary condition on the function $(\rho_o^{-1/2} p)$ at the earth's surface and another condition at the surface at infinity. By an appropriate development, we can also require that $G(\underline{r}, \underline{r}_o, s)$ satisfy the same conditions. Hence, the integrals at $r = a$ and $r = \infty$ will vanish identically, reducing the integral equation to

$$\rho_o^{-1/2} p(\underline{r}, s) = \int_{S_o} \left[G\, \underline{L}(\rho_o^{-1/2} p) - (\rho_o^{-1/2} p) \underline{L} G \right] \cdot \underline{n}\, d\sigma . \tag{36}$$

The change of sign that arises in going from equation (35) to equation (36) is due to the fact that the unit vector $\underline{n}$ appearing in equation (36) is directed normally out of the surface S_o.

DIFFERENTIAL EQUATIONS, BOUNDARY CONDITIONS AND CAUSALITY

The concepts and arguments set forth in this section are designed to satisfy both the mathematical and physical requirements of the system we have described. In a purely mathematical sense, the pressure equation (30) may be assumed to hold at all $r > a$, and for this case we must allow g to approach zero as r goes to infinity. Physical reasoning, on the other hand, indicates that at sufficiently great altitudes the atmosphere is too rare to permit any wave propagation. This latter case corresponds to the situation in which gravity is essentially given by g_0 at all heights where the macroscopic formulation is valid. The condition which we impose upon the solution at infinity, then, must take both these criteria into account.

An important simplification introduced into the analysis by neglecting the earth's rotation is that we can treat the propagation as being symmetric about the z-axis of Figure 1. Thus, the azimuthal angle ψ does not come into play, and the differential operator $\nabla \cdot \underline{L}$ presented in the previous section may be written as

$$\nabla \cdot \underline{L} = \frac{1}{r^2} \left[D_r + D_\theta \right] , \tag{37}$$

where

$$D_r = \frac{d}{dr} \left[\frac{r^2}{h} \frac{d}{dr} \right] \tag{38}$$

and

$$D_\theta = \frac{1}{\sin\theta} \frac{d}{d\theta} \left[\sin\theta \frac{d}{d\theta} \right] . \tag{39}$$

With this expression for $\nabla \cdot \underline{L}$, equation (32) becomes

$$\left[D_r - \frac{q(r)}{h}\right](\rho_o^{-1/2}p) + D_\theta(\rho_o^{-1/2}p) = 0 \ . \tag{40}$$

This enables us to write $\rho_o^{-1/2}p$ as a separable function

$$\rho_o^{-1/2}p = \phi(r)\,P(\theta) \ . \tag{41}$$

Introducing the separation constant $\ell(\ell + 1)$ we obtain from equations (40) and (41) the formula

$$D_r\phi - \left[\frac{q(r)}{h} + \ell(\ell+1)\right]\phi = 0 \tag{42}$$

and

$$D_\theta P + \ell(\ell+1)P = 0 \ . \tag{43}$$

Since the solutions of equation (43) are Legendre functions, we proceed to develop a Green's function in the form of a series of Legendre polynomials. This scheme has the advantage of retaining the sense of causality but will not be practicable from the standpoint of determining the time dependent solution because of its slow convergence. With this in mind we construct $G(\underline{r}, \underline{r}_o, s)$ from the two functions $g(r, r_o, s, \ell)$ and $\tilde{g}(\theta, \theta_o, \ell)$ which are solutions of

$$D_r g_\ell - \left[\frac{q(r)}{h} + \ell(\ell+1)\right] g_\ell = \delta(r-r_o) \tag{44}$$

and

$$D_\theta \tilde{g} + \ell(\ell+1)\tilde{g}_\ell = 0 \tag{45}$$

respectively. Using the fact that $\sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} P_\ell(\cos\theta)\,P_\ell(\cos\theta_o) = \frac{\delta(\theta-\theta_o)}{\sin\theta}$ we obtain from the eigenfunction expansion the expression

$$G(\underline{r}, \underline{r}_o, s) = \frac{1}{2\pi}\sum_{\ell=0}^{\infty} g_\ell(r, r_o, s)\,\tilde{g}_\ell(\theta, \theta_o) \tag{46}$$

where

$$\tilde{g}_\ell(\theta, \theta_o) = \frac{2\ell+1}{2} P_\ell(\cos\theta)\,P_\ell(\cos\theta_o) \tag{47}$$

and

$$g_\ell(r, r_o, s) = \frac{f_\ell(r_<)v_\ell(r_>)h}{r^2 W(f, v, r)} . \tag{48}$$

In this last expression $f_\ell(r)$ is the solution of equation (42) which satisfies a boundary condition at $r = a$ while $v_\ell(r)$ satisfies a condition at infinity. The notation $r_<$ and $r_>$ indicates that $f_\ell(r_<)$ and $v_\ell(r_>)$ are evaluated at the lesser and greater values of r_o and r respectively. The quantity $\frac{r^2 W(f, v, r)}{h}$ may be shown from the definition of D_r in equation (38) to be independent of r but may obviously depend on both ℓ and s. An alternate method for arriving at the form for $G(\underline{r}, \underline{r}_o, s)$ which is less direct than the eigenfunction expansion technique is to use the contour integration approach presented by Kazarinoff and Ritt.[14] This latter scheme will be discussed more thoroughly in the following section.

The condition imposed upon the function $f_\ell(r)$ is determined from the assumption that the vertical component of the velocity at the earth's surface vanishes, or, in other words, that $u_r(a, \theta, t) = 0.$[1] This implies that $u_r(a, \theta, s) = 0$. From equation (24), then, we have

$$\frac{df_\ell}{dr} + A f_\ell = 0 \qquad \text{at } r = a, \tag{49}$$

where $f_\ell(r)$ is a linear combination of two solutions of equation (42) given by

$$f_\ell(r) = \phi_1(\ell, r) + \alpha \phi_2(\ell, r) . \tag{50}$$

1. In cases where the time dependent forms for p, ρ, and $\underline{u}$ are being used, the variable t will be explicitly mentioned. Otherwise, if no dependence is stated, these quantities may be assumed to be functions of s.

From equations (49) and (50), then, we have

$$(\phi_1' + A\phi_1) + \alpha(\phi_2' + A\phi_2) = 0 \quad \text{at} \quad r = a \cdot \tag{51}$$

The constant α is at this point unspecified but will be determined from the dispersion relations later in this section.

The restriction imposed upon $v_\ell(r)$ requires that it be such that $p(\underline{r}, t)$ behave as an outgoing wave for large values of r. In the mathematical case where $g = \frac{g_0 a^2}{r^2}$, h, from equation (23), goes to unity when r becomes infinite while A, from equation (26) approaches zero as $\frac{1}{r^2}$. Thus, in the limit of large r, equation (42) may be written asymptotically as

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\phi}{dr}\right] - \frac{s^2}{c_o^2}\phi \simeq 0. \tag{52}$$

In the limit as $g = g_o$, on the other hand, both A and h are independent of r, so that the asymptotic form for equation (42) becomes

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\phi}{dr}\right] - \left[\frac{hs^2}{c_o^2} + A^2\right]\phi \simeq 0. \tag{53}$$

The forms presented by both equations (52) and (53) are obvious idealizations, since the atmosphere has a finite thickness beyond which the waves cannot propagate. Nevertheless, since no reflective medium has been assumed to exist at great altitudes, we can choose the solutions to equations (52) and (53) which make $p(\underline{r}, t)$ behave as an outgoing wave. In the former case, then

$$v_\ell(r) = \phi_1(r) \simeq c_1 \frac{e^{-(s/c_o)r}}{r} \tag{54}$$

while, in the latter case

$$v_\ell(r) = \phi_1(r) \simeq \frac{c_1'}{r} \exp\left\{-\left[\frac{hs^2}{c_o^2} + A^2\right]^{1/2} r\right\}. \tag{55}$$

One very important feature is that we can use the fact that $\phi_1(r)$ satisfies the radiation condition to obtain a precise statement of the causal nature of $p(\underline{r}, t)$. The principle of causality simply states that if an event occurs at some time t_o, no signal can be detected at any time prior to t_o. To show how this concept is included in our analysis we use the fact, which will be justified later, that equation (36) can be written in formal notation as

$$\rho_o^{-1/2}(r)\, p(\underline{r}, s) = R(\underline{r}_o, s)\, G(\underline{r}, \underline{r}_o, s) \tag{56}$$

where $R(\underline{r}_o, t)$ represents the source function. From equations (54) or (55) and equation (50), $g_\ell(r, r_o, s)$ in equation (48) becomes

$$g_\ell(r, r_o, s) = \frac{h(r')}{(r')^2 W(\phi_2, \phi_1, r')} \; \frac{\left[\phi_1(r_<) + \alpha\phi_2(r_<)\right]\phi_1(r_>)}{\alpha} \tag{57}$$

where r' is any value of r on (a, ∞). The functions $\phi_1(r)$ and $\phi_2(r)$ are analytic in the right half s-plane as we shall show later. Hence, in performing the inverse Laplace transform of equation (56), the poles of the integrand will be given by the zeros of α and the poles of $R(s)$. From equation (51), however, α can vanish only when $\phi_1'(a) + A\phi_1(a) = 0$. In Appendix A, we show that all these zeros lie to the left of the imaginary s-axis, although there may be singularities on the axis itself. On the assumption (which will be proved later) that $R(s)$ has no singularities in the right half s-plane, we can write

$$\rho_o^{-1/2}(r)\, p(\underline{r}, t) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} R(\underline{r}_o, s)\, G(\underline{r}, \underline{r}_o, s) e^{st}\, ds \tag{58}$$

where ϵ is a small positive number. The problem of causality, then, is one of closing the contour of equation (58) in the appropriate half plane. The choice, in [redacted] determined by the requirement that the contribution to the integral for $|s|$ infinite must vanish. We have, then, to determine $R(\underline{r}_0, s)\,G(\underline{r}, \underline{r}_0, s)$ as $|s|$ becomes very large.

To do this, we note from equation (23) that $h = 1$ as s goes to infinity, so that equation (42) becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) - \frac{s^2}{c_o^2}\phi(r) \underset{s\to\infty}{\simeq} 0 . \tag{59}$$

The solutions are

$$\phi_1(r) = \frac{c_1}{r}\exp\left\{\frac{-s}{c_o}r\right\}$$

and

$$\phi_2(r) = \frac{c_2}{r}\exp\left\{\frac{s}{c_o}r\right\}.$$

Upon substitution of the value

$$\alpha = -\frac{\phi_1'(a) + A\phi_1(a)}{\phi_2'(a) + A\phi_2(a)}$$

Into equation (57), we find that

$$g_\ell(r, r_o, s) \underset{|s|\to\infty}{\simeq} \frac{h(r')}{(r')^2\,\frac{2s}{c_o}\,\phi_1(r')\phi_2(r')} \times$$

$$\phi_1(r_>)\left\{\frac{s/c_o\left[\phi_2(r_<)\phi_1(a)+\phi_2(a)\phi_1(r_<)\right] - (A - 1/a)\left[\phi_2(r_<)\phi_1(a) - \phi_1(r_<)\phi_2(a)\right]}{\phi_1(a)\left[(A - \frac{1}{a}) - \frac{s}{c_o}\right]}\right\}$$

$$= k_1(\ell, s)^{\left[r_< - r_>\right] s/c_o} + k_2(\ell, s)^{\left[2a - r_< - r_>\right] s/c_o} \tag{60}$$

where $k_1(l, s)$ and $k_2(l, s)$ are two functions that do not contain exponentials in s. The first term corresponds to a wave which travels directly between $r_<$ and $r_>$ while the second represents a signal which has been reflected from the earth's surface. Thus, from equations (46) and (60)

$$G(\underline{r}, r_0, s) = e^{\left[r_< - r_>\right] s/c_0} \frac{1}{2\pi} \sum_{l=0}^{\infty} \tilde{g}_l(\theta, \theta_0)\, k_1(l, s)$$

$$+\, e^{\left[2a - r_< - r_>\right] s/c_0} \frac{1}{2\pi} \sum_{l=0}^{\infty} \tilde{g}_l(\theta, \theta_0)\, k_2(l, s). \qquad (61)$$

It will be proved in the section on the source representation that $R(s)$ is a function of s containing no exponentials. We then have for large s that

$$p(\underline{r}, s) = \frac{1}{2\pi} \sum_{l=0}^{\infty} \tilde{g}_l(\theta, \theta_0) \left\{ k_1(l, s) R(s) e^{\left[r_< - r_>\right] s/c_0} + k_2(l, s) R(s) e^{\left[2a - r_< - r_>\right] s/c_0} \right\}$$

$$= p_1(\underline{r}, s) + p_2(\underline{r}, s)$$

where $p_1(\underline{r}, s)$ and $p_2(\underline{r}, s)$ are the Laplace transforms of the direct and reflected pressure waves. The inverse transform applied to $p_1(\underline{r}, s)$ will give a contribution only when

$$t_1 \geq \frac{1}{c_0} \left[r_> - r_<\right],$$

while $p_2(\underline{r}, s)$ contributes only for

$$t_2 \geq \frac{1}{c_0} \left[r_> + r_< - 2a\right].$$

If the observer is located at a level below the height of the source, for example, he can obtain no direct signal before

$$t = t_1 = \frac{1}{c_0} \left[r_0 - r\right]$$

and no reflection prior to

$$t = t_2 = \frac{1}{c_o}\left[(r_o - a) + (r - a)\right].$$

Thus, we have established the fact that $p(\underline{r}, t)$ is indeed a causal solution, but we have done more. From the values for t_1 and t_2, it is evident that for an observer directly above or below the source $p_1(\underline{r}, t)$ and $p_2(\underline{r}, t)$ propagate at the speed of sound. This is, of course, what one would have expected from an intuitive physical standpoint, but it is important to note that we have presented it in a formal mathematical fashion.

In the following sections where we consider the case in which the observer is located on the earth's surface, the direct and reflected signals will, of course, arrive simultaneously. If d is used to represent the distance the wave travels in reaching the observer we extend the above conclusion to postulate that the arrival time t_a will be given by $t_a = \frac{d}{c_o}$. Hence, by introducing the unit step function $H(t - t_a)$ we can write

$$p(\underline{r}, t) = H(t - t_a) \int_{\epsilon - i\infty}^{\epsilon + i\infty} R(\underline{r}_o, s) G(\underline{r}_o, \underline{r}, s) e^{st} ds \tag{62}$$

as a reminder that the pressure wave propagates at the speed of sound and that no signal can be detected prior to $t = t_a$.

GREEN'S FUNCTION

It is convenient from this point on to develop our time dependent solutions as Fourier, rather than Laplace, transforms by replacing s with $-i\omega$. In the event there are any singularities on the real ω axis, the proper path of integration will be dictated by the causality requirement.

The first step is to consider equation (42), which, when written in full becomes

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{h}\frac{d\phi}{dr}\right]+\frac{1}{h}\left[\frac{h\omega^2}{c_o^2}-A^2-\frac{Ah'}{h}-\frac{h\ell(\ell+1)}{r^2}\right]\phi\simeq 0. \qquad (63)$$

Since

$$h\omega^2=\omega^2-\frac{(\gamma-1)g^2}{c_o^2}=\omega^2-\frac{(\gamma-1)g_o^2}{c_o^2}\frac{a^4}{r^4}$$

from equation (23), and

$$A=(1-\frac{\gamma}{2})\frac{g}{c_o^2}=(1-\frac{\gamma}{2})\frac{g_o}{c_o^2}\frac{a^2}{r^2}$$

from equation (26), one possible approach would be to solve equation (63) exactly, allowing for the r variations of A and h. The obvious difficulty in this idea is that there may be as many as three turning points on the interval (a,∞), forcing us to use Langer's[15] theory of two turning points at best or some complicated matching technique at worst.

The alternative is to use the approximation $g=g_o$, which, as we see from Appendix A, alters the analytic properties of $\phi(r,\omega)$ somewhat. The loss in accuracy presented by this simplification, however, cannot be very great from a physical standpoint since equation (63) is valid only in the portion of the atmosphere where the mean free path of molecules is small enough to permit a wave to propagate. If at each step we bear in mind that the solutions are approximate, we can avoid some seeming parodoxes that occur. With

$$h\omega^2 \simeq \omega^2 - \frac{(\gamma-1)g_o^2}{c_o^2} \quad \text{and} \quad A \simeq \left(1 - \frac{\gamma}{2}\right)\frac{g_o}{c_o}$$

we introduce the notation

$$K^2 = \frac{h\omega^2}{c_o^2} - A^2 = \frac{\omega^2 - \omega_c^2}{c_o^2} \qquad \text{(64)}$$

and

$$h = 1 - \frac{(\gamma-1)}{\omega^2}\frac{g_o^2}{c_o^2} = 1 - \frac{\omega_b^2}{\omega^2} \,. \qquad (65)$$

Here, $\omega_c^2 = \left(\frac{\gamma g_o}{2c_o}\right)^2$ and $\omega_b^2 = (\gamma-1)\frac{g_o^2}{c_o^2}$ are numbers introduced for notational simplicity with $\omega_c^2 - \omega_b^2 = c_o^2 A^2 > 0$. We thus obtain from equation (63) an equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) + \left[k^2 - \frac{h\ell(\ell+1)}{r^2}\right]\phi = 0 \qquad (66)$$

whose solutions are spherical Bessel functions. These functions are analytic in the upper and lower half ω-plane for real values of ℓ, although $\pm\omega_c$ and $\pm\sqrt{(\lambda+1/2)(\lambda-1/2)}\,\frac{\omega_b}{\lambda}$ are branch points, as we show in Appendix A.

As in the analogous electromagnetic case, equation (66) may have a turning point at some r in the interval (a, ∞). The situation is similar to the one discussed by Fock[16] and by Keller and Magiros[17] except that the eigenvalue $\ell(\ell+1)$ is multiplied by the factor h which is negative for $|\omega| < \omega_b$, while the quantity K^2 can be negative or positive depending upon whether $|\omega| < |\omega_c|$ or $|\omega| > |\omega_c|$. We thus expect the behavior of ϕ to be different in three intervals $0 < |\omega| < \omega_b$, $\omega_b < |\omega| < \omega_c$ and $|\omega| > \omega_c$.

Before proceeding on that subject, however, we determine an alternate representation for $G(\underline{r}, \underline{r}_o, \omega)$ by first introducing

$$\lambda = \ell + \frac{1}{2} \tag{67}$$

and defining

$$\nu^2 = h\ell(\ell+1) + \frac{1}{4} = h(\lambda + \frac{1}{2})(\lambda - \frac{1}{2}) + \frac{1}{4} \tag{68}$$

so that ν^2 is an even function of λ. The two independent solutions to equation (66) which we shall be using here may be written as

$$\phi_1(r) = \frac{1}{\sqrt{r}} H_\nu^{(1)}(Kr) \tag{69a}$$

$$\phi_2(r) = \frac{1}{\sqrt{r}} H_\nu^{(2)}(Kr) . \tag{69b}$$

According to a basic identity for Bessel Functions as given, for example, by Magnus and Oberhettinger[18],

$$H_{-\nu}^{(1)}(Kr) = e^{i\pi\nu} H_\nu^{(1)}(Kr) \tag{70a}$$

and

$$H_{-\nu}^{(2)}(Kr) = e^{-i\pi\nu} H_\nu^{(2)}(Kr) . \tag{70b}$$

The relationships presented in equations (70) may be used to prove that $g_\ell(r, r_o, \omega)$ is an even function of λ, since from equations (48), (50), and (51) it can be expressed as

$$g_{\lambda-1/2}(r, r_o, \omega) = \frac{h}{r^2} \frac{\left\{\left[\phi_2(r_<)\phi_1'(a) - \phi_2'(a)\phi_1(r_<)\right] + A\left[\phi_1(a)\phi_2(r_<) - \phi_2(a)\phi_1(r_<)\right]\right\}}{\phi_2(r)\phi_1'(r) - \phi_1(r)\phi_2'(r)} \times$$

$$\frac{\phi_1(r_>)}{\phi_1'(a) + A\phi_1(a)} . \tag{71}$$

This form for $g_{\lambda-1/2}$ is even in ν, and since ν goes to $\pm\nu$ as $\lambda \longrightarrow -\lambda$, it ~~must then be~~ an even function of λ.

The function $\tilde{g}_{\lambda-1/2}(\theta, \theta_o)$ may similarly be shown to be an odd function in λ by employing some basic properties of Legendre functions as given by Magnus and Oberhettinger.[18] From equation (47),

$$\frac{1}{2\pi}\tilde{g}_{\lambda-1/2}(\theta, \theta_o) = \frac{1}{2\pi}\lambda P'_{\lambda-1/2}(\cos\theta_>)P_{\lambda-1/2}(\cos\theta_<) \tag{72}$$

where $\theta_<$ and $\theta_>$ are the lesser and greater values of θ and θ_o respectively. Since

$$P_{-\lambda-1/2}(\cos\theta) = P_{\lambda-1/2}(\cos\theta), \tag{73}$$

$\tilde{g}_{\lambda-1/2}(\theta, \theta_o) = -\tilde{g}_{-\lambda-1/2}(\theta, \theta_o)$ as stated.

For the following discussion the expression in equation (72) may be written more conveniently as

$$\frac{1}{2\pi}\tilde{g}_{\lambda-1/2}(\theta, \theta_o) = \frac{\lambda}{2\pi}\frac{(-1)^{\lambda-1/2}}{\cos(\lambda-1/2)\pi} P_{\lambda-1/2}(\cos\theta_>)P_{\lambda-1/2}(\cos\theta_<)$$

$$= \frac{P_{\lambda-1/2}(-\cos\theta_>)\, P_{\lambda-1/2}(\cos\theta_<)}{\frac{\partial}{\partial\lambda}\sin(\lambda-1/2)\pi}. \tag{74}$$

If we now simplify the notation by expressing equation (71) as

$$g_{\lambda-1/2}(r, r_o, \omega) = \frac{F(r, r_o, \lambda)}{\phi_1'(a)+A\phi_1(a)}, \tag{75}$$

$G(\underline{r}, \underline{r}_o, \omega)$ from equation (46) can be put in the form

$$G(\underline{r},\underline{r}_o,\omega) = \frac{1}{2\pi i} \int_c \frac{\lambda}{2} \frac{P_{\lambda-1/2}(-\cos\theta_>)P_{\lambda-1/2}(\cos\theta_<)}{\sin(\lambda-1/2)\pi} \frac{F(r,r_o,\lambda)}{\phi_1'(a)+A\phi_1(a)} d\lambda . \tag{76}$$

In this representation, the contour, shown in Figure 3 runs from $\lambda = i\epsilon + \infty$ to $\lambda = i\epsilon$ and from $\lambda = -i\epsilon$ to $\lambda = -i\epsilon + \infty$. Since the integrand is an odd function of λ, the integral from $-i\epsilon$ to $-i\epsilon + \infty$ may be reflected with the path now running from $i\epsilon$ to $i\epsilon - \infty$. With this scheme, the contour can be closed in the upper half λ-plane to enclose all the zeros of $\phi'(a) + A\phi(a)$.[2] In Appendix A, we show that when gravity varies as $\frac{1}{r^2}$, all these zeros are complex and lie in the first quadrant. The degeneracy to $g = g_o$ changes this property by producing some pure real and pure imaginary λ poles, but this difficulty arises only through the approximation. Any λ poles, then, may be assumed to have a positive imaginary part no matter how small. In this manner, we can technically avoid the difficulty of having the poles shown in Figures 3 and 4 coincide. From this scheme, equation (76) may thus be written as

$$G(\underline{r},\underline{r}_o,\omega) = -\sum_{m=o}^{\infty} \frac{\lambda_m}{2} \frac{P_{\lambda_m-1/2}(-\cos\theta_>)\,P_{\lambda_m-1/2}(\cos\theta_>)}{\sin(\lambda_m-1/2)\pi} \left. \frac{F(r,r_o,\lambda_m)}{\frac{\partial}{\partial\lambda}\left[\phi' + A\phi\right]} \right|_{\substack{\lambda=\lambda_m \\ r=a}}$$

$$= \sum_{m=0}^{\infty} G_m(\underline{r},\underline{r}_o,\omega). \tag{77}$$

The negative sign in this equation results from the fact that the contour c in equation (76) encircles the poles in the clockwise direction.

2. Strictly speaking, the problem of closing the contour requires that the observer be in the shadow region so that the exponential in the integrand of equation (76) is positive in λ. This feature is implicit in the assumption that the observer is located on the earth's surface at a great distance from a low altitude explosion.

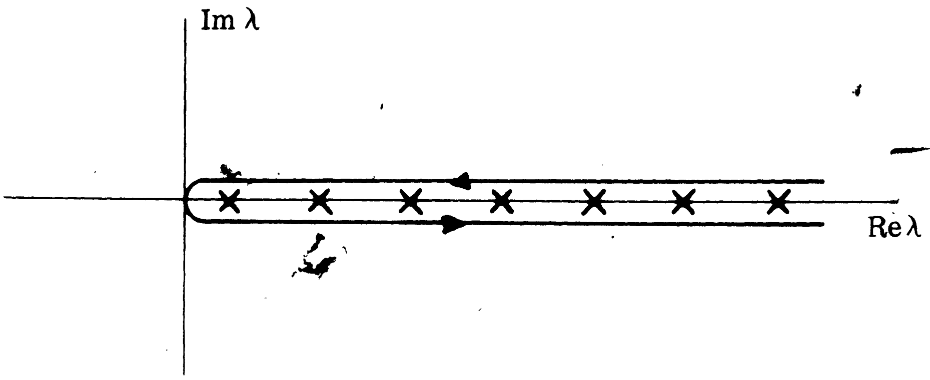

FIG. 3: THE POLES OF $G(\underline{r}, \underline{r}_o, \omega)$ WHICH FROM EQUATION (76), OCCUR WHEN λ IS AN ODD MULTIPLE OF 1/2.

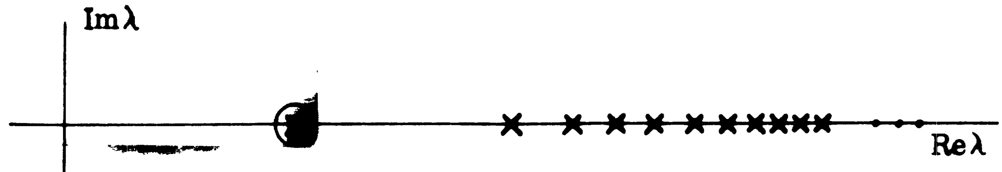

FIG. 4a: A SCHEMATIC REPRESENTATION OF THE POLES OF $g_\lambda(r, r_o)$ WHEN $|\omega| < \omega_b$. The encircled pole is the gravity wave pole and lies in a different half plane from the others of the set.

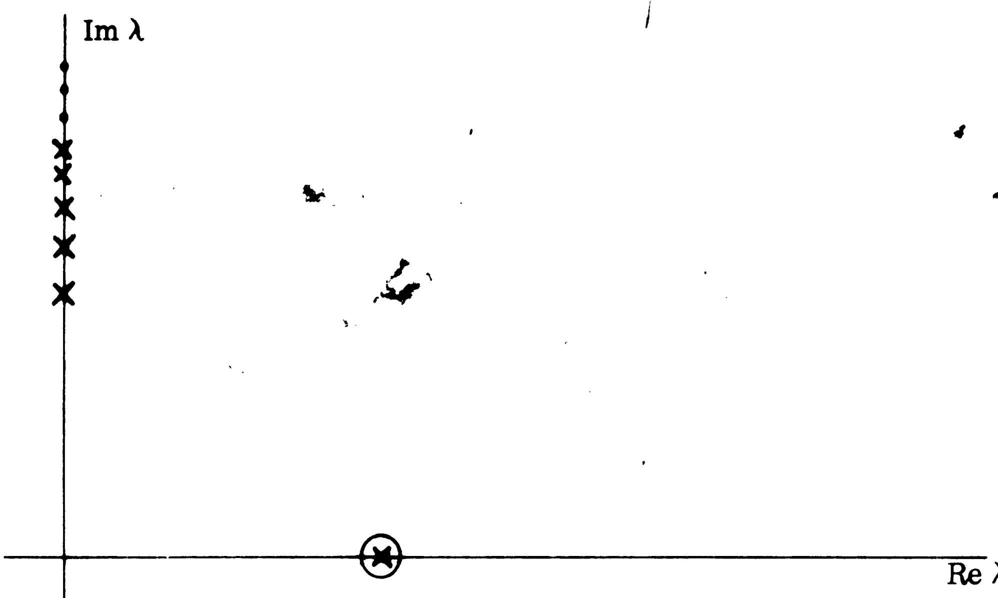

FIG. 4b: A SCHEMATIC REPRESENTATION OF THE POLES OF $g_\lambda(r, r_o)$ WHEN $\omega_b < |\omega| < \omega_c$. The encircled pole is the gravity wave pole which lies in a different half plane from the others of the set.

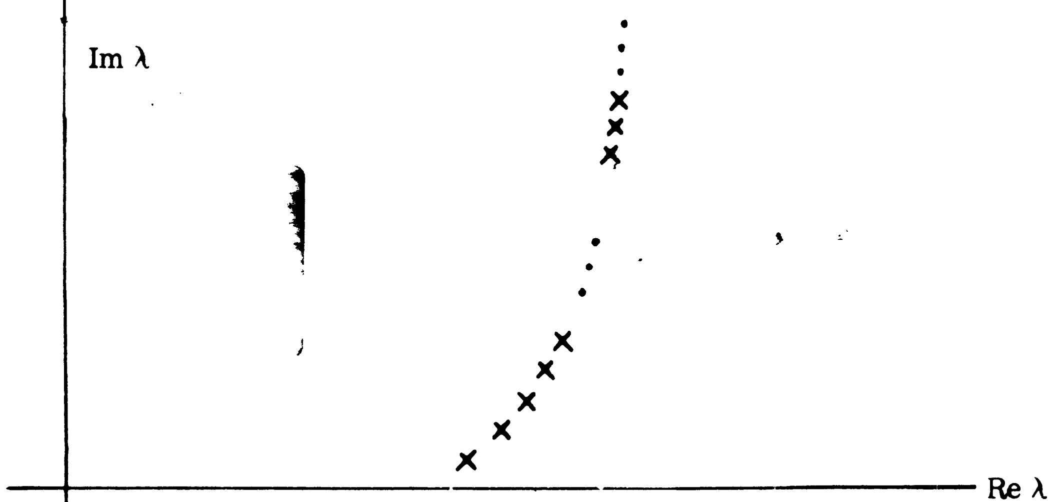

FIG. 4c: SCHEMATIC REPRESENTATION OF THE POLES OF $g_\lambda(r, r_o)$ WHEN $|\omega| > \omega_c$. For this frequency range there is no gravity wave pole.

ANALYSIS OF THE GRAVITY WAVE SOLUTION

There is no reason to presume, a priori, that the behavior of the Green's function is dominated by any particular mode of oscillation. It is nevertheless a striking feature of this problem that of all the functions that satisfy equation (66) and the boundary conditions, only one solution exists which does not have a turning point on the interval $a \leq r \leq \infty$. This concept of a turning point has a direct bearing on the physical stability of the solution in question. To see how this situation arises, we consider, in Appendix B, all the solutions of equation (66) that are obtained as expansions about the turning point. The mathematical interpretation of any one of these solutions is that it gives rise to a wave which propagates in the vertical direction until it reaches its turning point, whereupon it is partially reflected and partially transmitted. A physical consideration which must be included is that if the radiation is of a given wavelength so that it cannot propagate to the height of the turning point in question because of the rarefied nature of the atmosphere at this altitude, the wave will be attenuated. The formal nature of this damping could be obtained by considering the viscous nature of the atmosphere as a function of altitude. This is an unduly difficult approach, however, in that it would lead to an extremely complicated fourth order differential equation. It is preferable, in light of this, to examine the various solutions and their corresponding turning points to determine a qualitative estimate of the damping effect that our simplified analysis has overlooked.

In Appendix B, equation (B-30) we see that the turning point $r_{TP}(m)$ for the m-th mode of oscillation is given by the expression

$$r_{TP}(m) = a\left[1 + \frac{(Ka)^{-2/3}}{2} e^{i\pi/3} t_m\right]$$

where

$$t_m = \left[\frac{3}{4}(4m-1)\pi\right]^{2/3}.$$

The solutions which are of greatest interest to us at this point are those that exist on the frequency interval $|\omega| < \omega_b$. The reason for our concern with these modes at this time is that, in the absence of viscous considerations, our idealized formulation yields a set of real λ-poles. This would lead, as we shall see later, to a set of waves that will propagate around the earth with very little attenuation. We now use the turning point concept to determine how the damping would actually arise. At the lower end of the frequency interval, $r_{TP}(1) - a = 190$ Km while at the higher end, $r_{TP}(1) - a = 362$ Km. The atmospheric densities at these two heights are given by

$$\rho_o(r_{TP}(1)) = \rho_{oo} e^{-26.6}$$

and

$$\rho_o(r_{TP}(1)) = \rho_{oo} e^{-50.4}$$

respectively, where ρ_{oo} is the density at sea level. Since the kinematic coefficient of viscosity varies inversely as the density we find that the damping for the first mode should actually be a significant factor, particularly near $\omega = \omega_b$. For higher modes of oscillation, this effect is even more pronounced since $r_{TP}(m) > r_{TP}(n)$ for $m > n$. Thus, the mathematical results that we obtain in Appendix B must be tempered by the physical consideration of damping.

This turning point analysis applies in a similar fashion to solutions which exist on the frequency interval $|\omega| > \omega_b$. For this situation, however, the problem is not quite so sensitive since the λ poles that we obtain for this case are either pure imaginary or complex. As we shall point out in the following section, this property of these λ's leads to waves which do damp out as they propagate around the earth.

The conclusion that we draw from these remarks, then, is that the solutions which depend upon the existence of a turning point on the interval $a \leq r \leq \infty$ are metastable states. From the mathematical model we have established, these solutions can be rigorously determined. The physics of the viscous atmosphere, however, leads us to conclude that these modes are spurious, particularly when $|\omega| < \omega_b$. The question arises, then, as to whether we can obtain a good approximation to $G(\underline{r}, \underline{r}_0, \omega)$ by considering only the stable solution, or gravity wave mode, which does not depend upon the turning point analysis. The answer to this depends upon the particular problem we wish to solve. For time dependent pressure waves that we might wish to detect at a point above the source, this procedure would probably be unsatisfactory. In the case of waves which travel around the earth, however, this simplification leads to a time dependent solution which is in excellent agreement with observed data insofar as its amplitude and period of oscillation are concerned.

The first step in analyzing the gravity wave mode is to determine whether equation (66) has a solution which does not depend upon the existence of a turning point on the interval (a, ∞). More specifically, we assume that there is no $r_{TP}^2 = \frac{h(\lambda - 1/2)(\lambda + 1/2)}{K^2}$ such that $r_{TP} > a$ and determine the appropriate solution of equation (66) which satisfies the condition at infinity and the requirement that

$\phi_1'(a) + A\phi_1(a) = 0$. The most instructive manner for investigating this case would be to use Langer's theory as we present it in Appendix B. A more direct method, however, which is entirely equivalent is to use Debye's[18] asymptotic expression for $H_\nu^{(1)}(Kr)$. In this, we let $\phi_1(r) = \frac{1}{\sqrt{r}} H_\nu^{(1)}(Kr)$ and define $\sin\sigma = \nu/\xi$ where $\xi = Ka$. Then

$$H_\nu^{(1)}(\xi) \approx \sqrt{\frac{2}{\pi}}\, \frac{e^{-i\pi/4}}{\sqrt{\xi\cos\sigma}} \exp\left\{ i\xi\left[\cos\sigma + \left(\sigma - \frac{\pi}{2}\right)\sin\sigma\right]\right\} \tag{78}$$

where this formulation is valid provided that

$$\left|1 - \frac{\nu}{\xi}\right| > \left|\frac{3}{\xi}\nu^{1/3}\right| . \tag{79}$$

The condition that $\phi_1'(a) + A\phi_1(a) = 0$ is satisfied by this representation if there is a $\nu = \nu_o$ such that

$$\left\{ \frac{-1}{2\xi_a\left[1 - \frac{\nu_o^2}{\xi_a^2}\right]} + i\sqrt{1 - \frac{\nu_o^2}{\xi_a^2}} + \frac{1}{K}\left(A - \frac{1}{a}\right)\right\} = 0 \tag{80}$$

where $\xi_a = Ka$. A first approximation to this value ν_o is obtained by setting

$$1 - \frac{\nu_o^2}{\xi_a^2} = -\frac{A^2}{K^2} , \tag{81a}$$

or, using the definition of ν_o given in equation (68) and the definition of K^2 given by equation (64)

$$\nu_o^2 = h(\lambda_o + \frac{1}{2})(\lambda_o - \frac{1}{2}) + \frac{1}{4} = (A^2 + K^2)a^2 = \frac{h\omega^2 a^2}{c_o^2} . \tag{81b}$$

From the numerical values following Table II we see that $A \gg \frac{1}{a}$, so that the accuracy of the approximation (81a) is determined by comparing the relative magnitudes of $\sqrt{1 - \nu_o^2/\xi_a^2}$ and $\frac{1}{2\xi_a\left[1 - \nu_o^2/\xi_a^2\right]}$. In other words, if

$$\left|K^2a^2 - \nu_o^2\right|^{3/2} \simeq (Aa)^{3/2} \gg \left|Ka\right|^{1/2} \tag{82a}$$

equation (81b) will be sufficient to specify the value of ν_o^2 and hence λ_o. Since the largest value of $|K|$ on $0 \leq \omega \leq \omega_c$ is given at $\omega = 0$ by $|K| = \frac{\sigma_c}{c_o}$, the numerical values following Table II, when substituted into equation (82a), state that

$$2.69 \times 10^2 \gg 2.15. \tag{82b}$$

We see, then, that when $|\omega| < \omega_c$ the magnitude of $\sqrt{1 - \nu_o^2/\xi_a^2}$ is greater than the magnitude of $\frac{1}{2\xi_a} \frac{1}{\left[1 - \nu_o^2/\xi_a^2\right]}$ by the order of 10^2. Hence, in this frequency interval, equation (81b) gives an extremely good approximation to the value of ν_o^2, which in turn enables us to determine λ_o. A singular exception occurs in the neighborhood of the point $\pm\omega_b$ (where $\omega_b < \omega_c$), for at this value, $h \to 0$. Otherwise, we have from equation (81b) that

$$\lambda_o \simeq \pm \frac{\omega a}{c_o}, \tag{83}$$

where $a/c_o = O(10^4)$ according to the numerical values following Table II. The reason for the singular behavior at ω_b is that when $h \to 0$ equation (66) is an oversimplification which has neglected a significant term. In order to take this into account, we have to use equation (63) and retain all the coefficients in h^{-1}. If we consider for this special case that gravity varies as $1/r^2$, we obtain the equation

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{h}\frac{d\phi}{dr}\right] - \left[\frac{A^2}{h} + \frac{Ah'}{h^2}\right]\phi \underset{h \to 0}{\simeq} 0 \tag{84}$$

which has the particularly simple solution

$$\phi_1(r, \omega_b) = c \exp\left\{-\int_a^r A \, d\tau\right\} . \tag{85}$$

This result obviously satisfies the condition $\phi_1' + A\phi_1 = 0$ at $r = a$, but is independent of ω and ν_o and thus leaves the value of $\lambda_o(\omega_b)$ undetermined. This, however, represents a difficulty in the mathematical rather than the physical sense, and for the analysis that follows we lose no accuracy in assuming that equation (83) gives the approximate value of λ_o for all $0 < |\omega| < \omega_c$.

The next task is to show that the representation $H_\nu^{(1)}(\xi)$ given by equation (78) is the correct solution only when $|\omega| < \omega_c$. The simplest way to do this is to consider the definitions of K^2, h, and ν^2 given by equations (64), (65) and (68), respectively, and then locate the turning point. Since r_{TP} occurs for this mode when

$$\begin{aligned} r_{TP}^2 &= \frac{h(\lambda + 1/2)(\lambda - 1/2)}{K^2} \simeq \frac{\nu_o^2}{K^2} = \frac{(K^2 + A^2)}{K^2}\, a^2 \\ &= \frac{h\omega^2}{K^2}\,\frac{a^2}{c_o^2} = \frac{\omega^2 - \omega_b^2}{\omega^2 - \omega_c^2}\, a^2 , \end{aligned} \tag{86}$$

we see that for the three frequency intervals,

$$r_{TP}^2 \begin{cases} < a^2 & \text{for } |\omega| < \omega_b \\ \text{is negative for} & \omega_b < |\omega| < \omega_c \\ > a^2 & \text{for } |\omega| > \omega_c . \end{cases} \tag{87}$$

Since the representation for $H_{\nu}^{(1)}(\xi)$ as given by equation (78) was presented on the assumption that there is no turning point on the interval (a, ∞), this expression is appropriate only when $|\omega| < \omega_c$. It is then evident that since $K^2 < 0$ for this frequency interval, $H_{\nu}^{(1)}(\xi)$ is a decaying exponential function of r. The physical reason for this behavior is that gravity serves to trap this mode of oscillation thus tending to make it radiate in the horizontal direction.

As a final note on this solution we must prove that condition (79) is satisfied. Upon using the value of $|\nu_o|$ from equation (81b), a straightforward substitution requires that

$$\left| 1 - \sqrt{\frac{\omega^2 - \omega_b^2}{\omega^2 - \omega_c^2}} \right| >> 3\left(\frac{c_o}{a}\right)^{2/3} \frac{\left|\omega^2 - \omega_b^2\right|^{1/6}}{\left|\sqrt{\omega^2 - \omega_c^2}\right|} . \tag{88}$$

According to the numerical values following Table II, $(c_o/a)^{2/3} \simeq 1.35 \times 10^{-4} \sec^{-2/3}$ which is an extremely small number in comparison with $\omega_c^{2/3} \simeq 7.95 \times 10^{-2} \sec^{-2/3}$. Thus, condition (79) is satisfied for all $|\omega| < \omega_c$ but does not hold in the strict sense at the precise value $|\omega| = \omega_c$.

THE ASYMPTOTIC DEVELOPMENT OF THE GREEN'S FUNCTION

In this section, we develop an asymptotic expression for $G_0(\underline{r}, \underline{r}_0, \omega)$ which can be conveniently used in the inverse Fourier transform. The parallel scheme for the other $G_m(\underline{r}, \underline{r}_0, \omega)$ is presented in Appendix C.

In order to evaluate $G(\underline{r}, \underline{r}_0, \omega)$ for a meaningful case, we proceed to simplify equation (77) with the assumption that the observer is located at some point $r = a$ on the earth's surface. For this situation, we let $r_> = r_0$, $r_< = a$ and evaluate $(r^2/h)W(\phi_1, \phi_2, r)$ at $r = a$, so that the expression for $g_{\lambda-1/2}(r, r_0, \omega)$ in equation (71) becomes

$$g_{\lambda-1/2}(a, r_0, \omega) = \frac{h}{a^2} \frac{\phi_1(r_0)}{\phi_1'(a) + A\phi_1(a)} \quad . \tag{89}$$

We then have from equation (77) the simplified result that

$$G(\underline{a}, \underline{r}_0, \omega) = +\sum_{m=0}^{\infty} G_m(\underline{a}, \underline{r}_0, \omega)$$

$$\doteq -\sum_{m=0}^{\infty} \frac{\lambda_m}{2} \frac{P_{\lambda_m-1/2}(-\cos\theta_>)\, P_{\lambda_m-1/2}(\cos\theta_<)}{\sin \pi (\lambda_m - 1/2)} \frac{h}{a^2} \left. \frac{\phi_1(r_0)}{\frac{\partial}{\partial\lambda}\left[\phi_1' + A\phi_1\right]} \right|_{\substack{\lambda=\lambda_m \\ r=a}} \tag{90}$$

where $\underline{a} = (a, \theta, \psi)$ denotes the observer's position. Still using $m = 0$ to designate the gravity wave pole, we now use equation (78) to determine $\left.\frac{\partial}{\partial\lambda}(\phi_1' + A\phi_1)\right|_{\substack{\lambda=\lambda_0 \\ r=a}}$, which works out to be

$$\left.\frac{\partial}{\partial\lambda}\left[\phi_1' + A\phi_1\right]\right|_{\substack{\lambda=\lambda_0 \\ r=a}} \simeq -\frac{1}{a^{3/2}} \frac{1}{\sqrt{K^2 a^2 - \nu_0^2}} \left. \frac{i}{2} \frac{d\nu^2}{d\lambda} \right|_{\lambda=\lambda_0} H_{\nu_0}^{(1)}(Ka) \, . \tag{91}$$

From equation (81), then,

$$\frac{\partial}{\partial\lambda}(\phi_1' + A\phi_1)\Big|_{\substack{\lambda=\lambda_o \\ r=a}} \simeq \frac{-1}{2}\,\frac{1}{Aa^2}\,\frac{d\nu^2}{d\lambda}\Big|_{\lambda_o} \phi_1(a,\lambda_o) = \frac{-h\lambda_o}{Aa^2}\,\phi_1(a,\lambda_o). \tag{92}$$

For the zeroth, or gravity wave pole, then, we obtain[3]

$$G_o(\underline{a},\underline{r}_o,\omega) = \frac{+P_{\lambda_o-1/2}(-\cos\theta_>)P_{\lambda_o-1/2}(\cos\theta_<)}{2\sin\pi(\lambda_o-1/2)}\;\frac{A\phi_1(r_o,\lambda_o)}{\phi_1(a,\lambda_o)}. \tag{93}$$

In order to present $G_o(\underline{a},\underline{r}_o,\omega)$ in a form that is convenient for performing the inverse Fourier transform, it is necessary to develop an asymptotic expression for

$$\frac{P_{\lambda_o-1/2}(-\cos\theta_>)\,P_{\lambda_o-1/2}(\cos\theta_<)}{2\sin\pi(\lambda_o-1/2)}. \tag{94}$$

We first use the assumption made in the introduction that the radius R_o of the surface S_o in Figure 1 is small enough that $R_o << a$. In this case, $\cos\theta_o \simeq 1$ for all points on S_o so that $P_{\lambda_o-1/2}(\cos\theta_<) \simeq 1$.

For an observer located at some $\theta > \theta_o$, the expression (94) becomes

$$\frac{P_{\lambda_o-1/2}(-\cos\theta)}{2\sin\pi(\lambda_o-1/2)}. \tag{95}$$

The representation for this function depends upon the observers position with respect to the source. If he is located in the vicinity of either the explosion

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3. In strictly formal notation we should write

$$H(\omega_c+\omega)\,H(\omega_c-\omega)\,G_o(\underline{a},\underline{r}_o,\omega)$$

(where H is the unit step function) to point out that $G_o(\underline{a},\underline{r}_o,\omega)$ exists only for $|\omega| < \omega_c$.

or the antipode, an asymptotic development for $P_{\lambda_o - 1/2}(-\cos\theta)$ in terms of Bessel functions must be used. Otherwise, when he is at a position remote from both of these points (which is the case we are considering), $P_{\lambda_o - 1/2}(-\cos\theta)$ can be expressed asymptotically as a trigonometric function.

From equation (83), $|\lambda_o| >> 1$ for all $|\omega| > 5 \times 10^{-5} \sec^{-1}$. We can then write

$$P_{\lambda_o - 1/2}(-\cos\theta) \simeq \sqrt{\frac{2}{\pi\lambda_o \sin\theta}} \cos\left[\lambda_o(\pi - \theta) - \frac{\pi}{4}\right] \tag{96}$$

provided that $\epsilon < \theta < \pi - \epsilon$ and $|\lambda_o| >> \epsilon$. The choice of the sign of λ_o is now a point that must be considered. We recall from earlier statements that all λ poles must have a positive imaginary part no matter how small. In the process of considering the constant gravity simplification as we note in Appendix A, this quantity was lost, but must nevertheless be considered to exist. Hence, the positive sign $\lambda_o = \frac{a\omega}{c_o}$ is the appropriate choice, since it states that this λ-pole is situated in the first quadrant and is thus enclosed by the path of integration. We can now use this feature to develop a representation for

$$\frac{1}{2\sin\pi(\lambda_o - 1/2)} = \frac{i}{e^{i\pi(\lambda_o - 1/2)} - e^{-i\pi(\lambda_o - 1/2)}} . \tag{97}$$

Since λ_o is assumed to have a small positive imaginary part, $e^{-i\pi(\lambda_o - 1/2)}$ is dominant in this expression and so

$$\frac{1}{2\sin\pi(\lambda_o - 1/2)} = \frac{-i\,e^{i\pi(\lambda_o - 1/2)}}{1 - e^{2i\pi(\lambda_o - 1/2)}} = -e^{i\pi\lambda_o} \sum_{n=0}^{\infty} e^{2i\pi n(\lambda_o - 1/2)} \tag{98}$$

Upon combining equations (96) and (98) we obtain

$$\frac{P_{\lambda_o - 1/2}(-\cos\theta)}{2\sin\pi(\lambda_o - 1/2)} = \frac{-1}{2}\sqrt{\frac{2}{\pi\lambda_o \sin\theta}}\left\{e^{i\left[\lambda_o\theta + \frac{\pi}{4}\right]} + e^{i\left[\lambda_o(2\pi-\theta) - \frac{\pi}{4}\right]}\right\}\sum_{n=0}^{\infty} e^{2i\pi n(\lambda_o - \frac{1}{2})} \tag{99}$$

Substitution of this expression into equation (93) thus yields

$$G_o(a, r_o, \omega) \simeq \frac{-1}{2}\sqrt{\frac{1}{2\pi\lambda_o \sin\theta}}\;\frac{A\phi_1(r_o, \lambda_o)}{\phi_1(a, \lambda_o)} \cdot \left\{e^{i\left[\lambda_o\theta + \frac{\pi}{4}\right]} + e^{i\left[\lambda_o(2\pi-\theta) - \frac{\pi}{4}\right]}\right\}\sum_{n=0}^{\infty} e^{2i\pi n(\lambda_o - \frac{1}{2})} \tag{100}$$

The various terms in this expansion can now be interpreted by using the fact that $\lambda_o \simeq \frac{a\omega}{c_o}$. From the first exponential in the brackets, the quantity $\lambda_o\theta \simeq \frac{\omega}{c_o} a\theta$ indicates that $a\theta$ is the great circle distance between the observer and the source. Hence, $\frac{\omega}{c_o}$ is the wave number for this mode of propagation and c_o is the corresponding phase velocity.

This first term clearly represents the directly received portion of the pulse, while the second exponential represents the antipodal "reflection." Values of $n > 0$ in the sum correspond to signals that have made n complete circulations around the earth. The rate at which these signals have been observed to damp after a few circuits gives an estimate of the imaginary part of λ_o that we have neglected. Araskog[19], for one, was able to detect a distinguishable signal on the second complete circulation which bears out our assumption that the imaginary part must be extremely small.

Introducing now the form for $G_m(\underline{a}, r_o, \omega)$ developed in Appendix C, the complete expression for the directly received portion of the Green's function is given by

$$G(\underline{a}, \underline{r}_o, \omega) = -\frac{1}{2}\sqrt{\frac{c_o}{2\pi a\omega \sin\theta}}\ \frac{A\phi_1(r_o)}{\phi_1(a)}\ e^{i\left[\frac{\omega}{c_o}a\theta + \frac{\pi}{4}\right]}$$
$$+ \sum_{m=1}^{\infty} \lambda_m \frac{\Gamma(\lambda_m + 1/2)}{\Gamma(\lambda_m + 1)}\ \frac{e^{i\left[\lambda_m\theta + \frac{\pi}{4}\right]}}{\sqrt{2\pi \sin\theta}}\ \frac{h}{a^2}\ \frac{\phi_1(r_o)}{\frac{\partial}{\partial\lambda}\left[\phi_1' + A\phi_1\right]_{\substack{r=a\\ \lambda=\lambda_m}}} \tag{101}$$

In the process of neglecting the effects of viscosity we have determined in Appendix B that for the first few m modes,

$$\lambda_m \simeq \sqrt{\frac{\omega^2 - \omega_c^2}{\omega^2 - \omega_b^2}}\ \frac{a\omega}{c_o}\left[1 + \frac{1}{2}(Ka)^{-2/3} e^{i\pi/3} t_m\right] \tag{102}$$

where

$$t_m \simeq \left[\frac{3}{4}(4m-1)\pi\right]^{2/3}. \tag{103}$$

As we pointed out at the beginning of the previous section, however, the viscous effects of the real atmosphere introduces a significant degree of damping that our idealized atmosphere overlooks. This is particularly important in the consideration of the λ poles for $|\omega| < \omega_b$, which, from equation (102), appear to be real quantities. When the physical consideration of viscosity is included, we know that the solutions on this frequency interval are actually damped, and that the degree of attenuation is more pronounced with each succeeding mode. When $\omega_b < |\omega| < \omega_c$, the effects of viscosity should also be taken into account but this case does not cause as much concern since the λ_m poles given by equation (102) are pure imaginary quantities. For this frequency range, then,

$\exp\left[i(\lambda_m \theta + \pi/4)\right]$ decays exponentially and thus contributes essentially nothing to $G(\underline{a}, \underline{r}_0, \omega)$. From the physical and mathematical considerations, then, the behavior of $G(\underline{a}, \underline{r}_0, \omega)$ is dominated by $G_0(\underline{a}, \underline{r}_0, \omega)$ when $0 < |\omega| < \omega_c$.

When $|\omega| > \omega_c$ we have a more interesting situation in some respects in that $G_0(\underline{a}, \underline{r}_0, \omega)$ does not exist on this interval. The set of λ_m that are obtained for this case give rise to what have been referred to in electromagnetic theory as creeping waves. If we neglect the effects of viscosity, the imaginary part of λ_m is estimated quite accurately from equation (102) to be

$$\mathrm{Im}(\lambda_m) \simeq \sqrt{\frac{\omega^2 - \omega_c^2}{\omega^2 - \omega_b^2}} \; \frac{a\omega}{c_o} \; \frac{\sqrt{3}}{4} (Ka)^{-2/3} \, t_m \, . \tag{104}$$

As a numerical example, if $\omega = 1.1\,\omega_c$,

$$\mathrm{Im}\,\lambda_1 \simeq 34.$$

According to equation (101), this means that the contribution from $\omega = 1.1\,\omega_c$ to the signal will have practically vanished for θ on the order of one radian. For higher frequencies the rate of decay is not quite so pronounced, but it is nevertheless true that

$$\mathrm{Im}\,\lambda_1 \underset{\omega \to \infty}{\simeq} 12 .$$

Thus, all the creeping wave solutions do damp out quite rapidly although the actual rate of damping is somewhat different since we have neglected viscosity.

In view of these arguments, a first order approximation to $G(\underline{a}, \underline{r}_0, \omega)$ is given by

$$G(\underline{a}, \underline{r}_0, \omega) \simeq H(\omega_c + \omega)\, H(\omega_c - \omega)\, G_0(\underline{a}, \underline{r}_0, \omega) . \tag{105}$$

The ultimate test as to the accuracy of this approximation depends upon how well our time dependent pressure function agrees with observed data.

THE SOURCE REPRESENTATION

Having determined the Green's function, we now turn to the problem of developing a description of the nuclear explosion which will involve several parameters inherent in such a source. The essence of the approach is similar to that presented by Weston[9] in that on the surface S_o, the initial form of the pressure pulse is known from empirical data. This surface, which is assumed to be a sphere of radius R_o, is large enough so that $p(R,t)$ has an acoustic representation at $R = R_o$ and satisfies the Euler equations exterior to S_o. Weston's idea was to develop an equivalent acoustic source which placed at $R=0$ would generate the same pulse form at $R = R_o$ as is produced by the explosion itself. This concept involves the process of 'shrinking' the surface to zero and has the advantage of simplicity in handling. It also has a drawback in that it causes terms of fairly large magnitude to be discarded.

As an alternative we avoid the equivalent source representation and deal strictly with the surface S_o itself, so that in a sense this becomes a Huygen's principle approach. The only restrictions to be imposed on S_o are that the variations in atmospheric parameters over this surface be slight and that its radius, R_o, be much less than the earth's radius. Other than this, it is obvious that once the pulse form has been characterized on S_o, it is unnecessary to have any knowledge of the source which generated this pulse.

The problem of obtaining a mathematical representation for the pressure which describes both the positive and negative phases, is delicate. An approximate version obtained by Brode[20] and used by Weston[9], agrees with the results published by the Department of Defense[21] and for this reason is our choice.

In this scheme

$$p(R,t) = \begin{cases} p_1 \dfrac{R_o}{R}\left[1+\dfrac{R-c_o t}{\alpha L}\right]\exp\dfrac{R-c_o t}{\alpha L} & \text{for } R < R_o \text{ and } t > \dfrac{R_o}{c_o} \\ 0 & \text{otherwise} \end{cases} \tag{106}$$

where p_1 is the peak overpressure on S_o, α is a scaling factor characteristic of the explosion, and L is the dimensionless length of the positive phase. One criticism of this representation is that the pressure at $R=0$ would appear to remain infinite for all finite time. This shortcoming is of only academic interest, however, since our concern will only be with what takes place on S_o. A more complicated possibility which was considered avoided the singular behavior of $p(0,t)$ by using a trigonometric expression for the negative duration of the pulse. The algebra involved was onerous and gave results which differed from those to be presented here by only a small factor. For these reasons, the representation of equation (106) is entirely satisfactory as far as characterizing the behavior of the pressure at R_o is concerned.

Turning now to equation (36) as our starting, we see that the form of the integral may be made simpler by the fact that from equation (24)

$$\underline{L}(\rho_o^{-1/2}p) = -s(\rho_o^{1/2}\underline{u}) - \underline{i}_r \frac{A}{h}(\rho_o^{-1/2}p), \tag{107}$$

and therefore

$$\rho_o^{-1/2}(a)p(\underline{a},s) = -s\int_{S_o}\rho_o^{1/2}G\underline{u}\cdot\underline{n}\,d\sigma - \int_{S_o}\rho_o^{-1/2}p\left[\underline{L}G+\underline{i}_r\frac{AG}{h}\right]\cdot\underline{n}\,d\sigma \tag{108}$$

where $\underline{n}$ (see Figure 5) is the unit vector directed normally away from the surface S_o and $\underline{i}_r$ is a unit vector directed normally away from the earth's

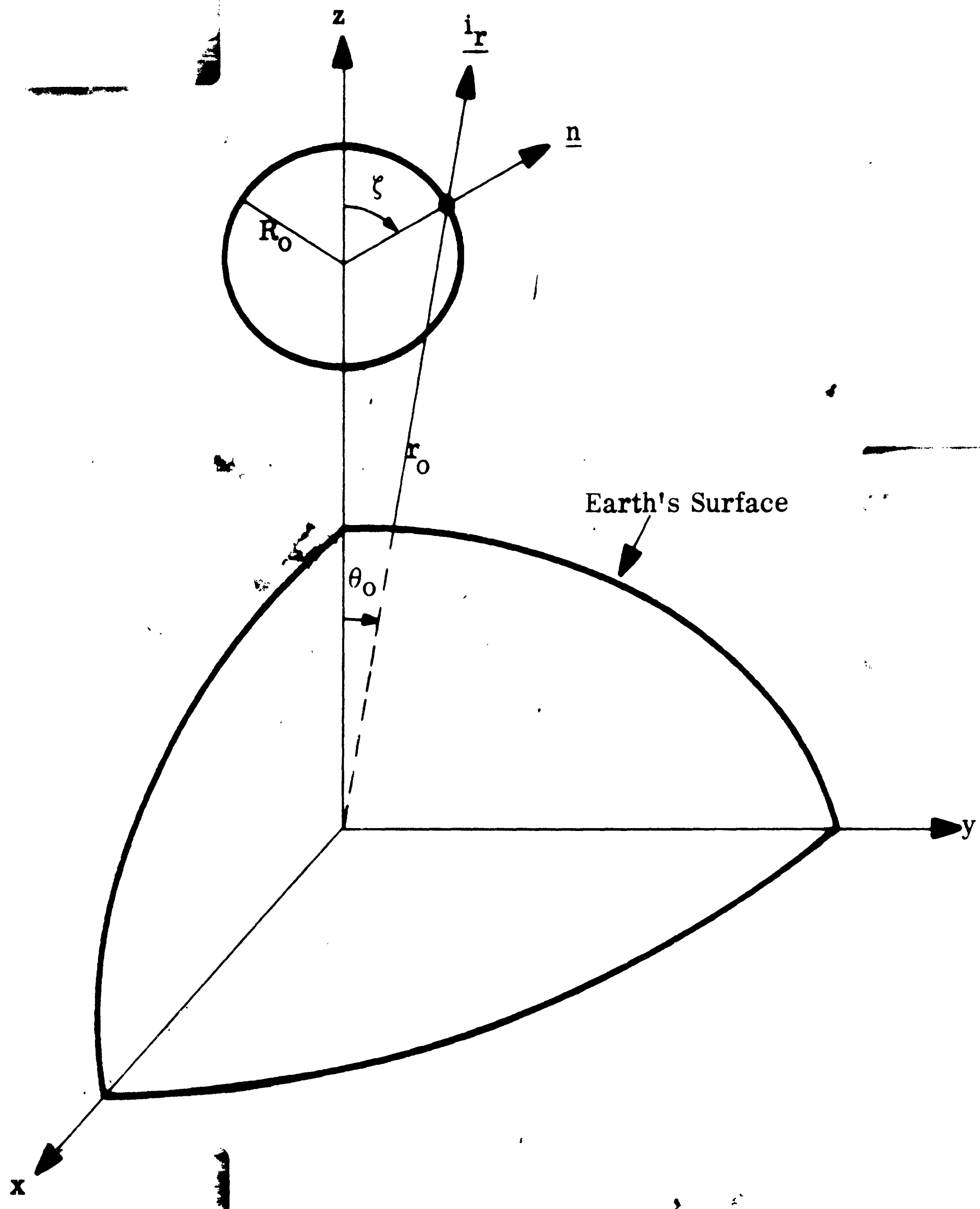

FIG. 5: SAME GEOMETRIC CONFIGURATION AS FIG. 1, REDRAWN TO INDICATE ANGLES AND UNIT VECTORS USED IN SIMPLIFYING EQUATION (108).

surface. Since we have postulated that $R_o \ll a$, it is evident from Figure 5 that

$$\underline{i}_r \cdot \underline{n} \simeq \cos\zeta \qquad \text{and} \qquad \underline{i}_\theta \cdot \underline{n} \simeq \sin\zeta$$

for points on S_o. Hence, in the integrand of the second term on the right hand side of equation (108) we have

$$\underline{n} \cdot \left[\underline{L}G + \underline{i}_r \frac{A}{h} G\right] = \cos\zeta \left[\frac{1}{h}\frac{\partial G}{\partial r_o} + \frac{A}{h} G\right] + \sin\zeta \frac{1}{r_o}\frac{\partial G}{\partial \theta_o} . \quad (109)$$

Since $d\sigma = 2\pi R_o^2 \sin\zeta d\zeta$ and since $p(R_o, s)$ is independent of the angle ζ, this second integral becomes

$$2\pi R_o^2 p(R_o) \int_0^\pi \rho_o^{-1/2}(r_o) \left\{\cos\zeta \left[\frac{1}{h}\frac{\partial G}{\partial r_o} + \frac{A}{h} G\right] + \sin\zeta \frac{1}{r_o}\frac{\partial G}{\partial \theta_o}\right\} \sin\zeta d\zeta . \quad (110)$$

Variations in $\rho_o^{-1/2}(r_o) G(\underline{a}, \underline{r}_o, s)$ and $\rho_o^{-1/2}(r_o) \frac{\partial G}{\partial r_o}$ over the surface S_o are as $\rho_o^{-1/2}(r_o) \phi_1(r_o, \lambda_o)$ and $\rho_o^{-1/2}(r_o) \frac{\partial \phi_1}{\partial r_o}$ respectively. Since these functions are monotonic over S_o and do not have a zero, the r_o dependence can be removed from the expression (110) by applying the mean value theorem for integrals. The angular variation of $G(\underline{a}, \underline{r}_o, s)$ on the other hand is as $P_{\lambda_o - 1/2}(\cos\theta_o)$, where θ_o is approximately zero. For the $\cos\zeta$ term in the integral of equation (110), then, we have

$$\rho_o^{-1/2}(\bar{r}_o) \int_0^\pi \cos\zeta \left[\frac{1}{h}\frac{\partial G}{\partial r_o} + \frac{A}{h} G(\underline{a}, \bar{r}_o, s)\right] \sin\zeta d\zeta$$

$$= \rho_o^{-1/2}(\bar{r}_o) \left[\frac{1}{h}\frac{\partial G}{\partial r_o}(\underline{a}, \bar{r}_o, s) + \frac{A}{h} G(\underline{a}, \bar{r}_o, s)\right] \int_0^\pi \sin\zeta \cos\zeta \, d\zeta = 0 . \quad (111)$$

Here, $\bar{r}_o = (\bar{r}_o, \bar{\theta}_o)$ is some point in the volume enclosed by S_o.

For the remaining integral in equation (110), we use the fact that

$$\frac{\partial P_{\lambda_o - 1/2}(\cos\theta_o)}{\partial\theta_o} = -\sin\theta_o \frac{(\lambda_o + 1/2)(\lambda_o - 1/2)}{2} P_{\lambda_o - 1/2}(\cos\theta_o) \qquad (112)$$

to simplify the $\frac{\partial G}{\partial\theta_o}$ term. In addition, from Figure 5,

$$\sin\theta_o = \frac{R_o}{r_o}\sin\zeta$$

so that

$$\frac{\partial G}{\partial\theta_o} = \frac{-(\lambda_o + 1/2)(\lambda_o - 1/2)}{2}\frac{R_o}{r_o}\sin\zeta\, G(\underline{a}, \underline{r}_o, s) . \qquad (113)$$

Hence, for the expression (110) we obtain

$$-2\pi R_o^2 p(R_o)\int_o^\pi \rho_o^{-1/2}(r_o)\frac{(\lambda_o + 1/2)(\lambda_o - 1/2)}{2}\frac{R_o}{r_o^2} G(\underline{a}, \underline{r}_o, s)\sin^3\zeta d\zeta$$

$$= -\frac{4}{3}\pi R_o^3 \frac{(\lambda_o + 1/2)(\lambda_o - 1/2)}{(\bar{r}_o)^2} G(\underline{a}, \bar{\underline{r}}_o, s)\,\rho_o^{-1/2}(\bar{r}_o)\,p(R_o) . \qquad (114)$$

From equation (108), then,

$$\rho_o^{-1/2}(a)p(\underline{a}, s) = -s\rho_o^{1/2}(\bar{r}_o)G(\underline{a}, \bar{\underline{r}}_o, s)\int_{S_o}\underline{u}\cdot\underline{n}\, d\sigma +$$

$$\frac{4}{3}\pi R_o^3 \frac{(\lambda_o + 1/2)(\lambda_o - 1/2)}{(\bar{r}_o)^2} p(R_o)\rho_o^{-1/2}(\bar{r}_o)G(\underline{a}, \bar{\underline{r}}_o, s) , \qquad (115)$$

where the quantities $p(R_o)$ and $\int_{S_o}\underline{u}\cdot\underline{n}\, d\sigma$ are functions of s. The form of these variables may be derived from equation (106) in the following manner.

On the assumption that the disturbance on S_o propagates radially outward as equation (106) implies, and that at $R=R_o$ the pressure and velocity behave acoustically, we have

$$p(R_o, t) = \frac{1}{R} f'(t - \frac{R_o}{c_o}) = \frac{1}{R_o} f'(t_1) . \qquad (116)$$

The explicit forms for $f(t_1)$ and $f'(t_1)$ are presented in equations (118) and (119) below. For small values of R_o, variations in $p_o(r_o)$ and $\rho_o(r_o)$ can be neglected without introducing any appreciable error. For larger R_o, on the order of several Km, the acoustic assumption will introduce some distortion since it neglects the atmospheric changes as ζ varies. This objection can be overcome, however, by selecting the average values of $p_o(r_o)$ and $\rho_o(r_o)$ on the surface S_o. From the acoustic relations, then,

$$\underline{n} \cdot \underline{u}(R_o, t_1) = \frac{1}{\rho_o(r_o)} \left\{ \frac{1}{c_o R_o} f'(t_1) + \frac{1}{R_o^2} f(t_1) \right\} . \qquad (117)$$

Upon substituting the explicit form for $p(R_o, t)$ presented in equation (106), we obtain

$$f'(t_1) = \begin{cases} p_1 R_o (1 - t_1/T) e^{-t_1/T} & t_1 \geq 0 \\ 0 & t_1 < 0 \end{cases} \qquad (118)$$

and

$$f(t_1) = \begin{cases} p_1 R_o t_1 e^{-t_1/T} & t_1 \geq 0 \\ 0 & t_1 < 0 \end{cases} \qquad (119)$$

where

$$T = \frac{\alpha L}{c_o} \qquad (120)$$

is the time duration of the positive phase. In equations (116-119), $t_1 = t - \frac{R_o}{c_o}$ represents the time at which the pulse appears to arrive at S_o. Thus, for our purposes it represents the initial time, and we choose the time interval to be

$0 \leq t_1 \leq \infty$. From equations (117-119)

$$\underline{n} \cdot \underline{u}(R_o, t_1) = \frac{p_1}{\rho_o(\bar{r}_o)} \left\{ \frac{1}{c_o} \left(1 - \frac{t_1}{T}\right) e^{-t_1/T} + \frac{1}{R_o} t_1 e^{-t_1/T} \right\} . \tag{121}$$

We now introduce, for future reference, the so-called "equivalent volume of gas," which is given by

$$\int_o^\infty dt_1 \int_{S_o} \underline{n} \cdot \underline{u}(R_o, t_1) d\sigma = V \tag{122}$$

where $d\sigma = 4\pi R_o^2$. Thus, from equation (121),

$$V = \frac{4\pi p_1 R_o T^2}{\rho_o(\bar{r}_o)} . \tag{123}$$

Upon taking the Laplace transform of $\underline{n} \cdot \underline{u}(R_o, t)$ we have

$$\underline{n} \cdot \underline{u}(R_o, s) = \int_o^\infty \underline{n} \cdot \underline{u}(R_o, t_1) e^{-st_1} dt_1$$

$$= \frac{p_1 T^2}{\rho_o(\bar{r}_o) R_o} \frac{(1 + s\frac{R_o}{c_o})}{(1+sT)^2} = \frac{V}{4\pi R_o^2} \frac{(1 + s\frac{R_o}{c_o})}{(1+sT)^2} . \tag{124}$$

Thus,

$$\int_{S_o} \underline{n} \cdot \underline{u}(R_o, s) d\sigma = V \frac{(1 + sR_o/c_o)}{(1+sT)^2} . \tag{125}$$

In a similar fashion, we obtain from equations (116) and (118) that

$$p(R_o, s) = \frac{V\rho_o(\bar{r}_o)}{4\pi R_o} \frac{s}{(1+sT)^2} . \tag{126}$$

If we now use the fact that $(\lambda_o + 1/2)(\lambda_o - 1/2) \simeq \frac{-a^2 s^2}{c_o^2}$, equation (108) yields

$$\rho_o^{-1/2}(a)\, p(a, s) = -\rho_o^{1/2}(\bar{r}_o)\, V\, G(\underline{a}, \underline{\bar{r}}_o, s)\, \frac{s}{(1+sT)^2} \times$$

$$\left[1 + \frac{sR_o}{c_o} + \frac{1}{3}\frac{a^2}{(\bar{r}_o)^2}\frac{s^2 R_o^2}{c_o^2}\right]. \qquad (127)$$

As a final remark upon the nature of the source, it is worth noting that the quantity V is rather ineffective in characterizing the source since nuclear explosions are not customarily expressed in terms of their equivalent volumes. It is more convenient, then, to define the parameter E_c as that portion of the of the explosive energy that produces the compressive effect. Then

$$E_c = \int_0^{\infty} dt_1 \int_{S_o} (p_o + p)\underline{u} \cdot \underline{n} d\sigma = p_o(\bar{r}_o)V + \int_0^{\infty} dt_1 \int_{S_o} p(R_o, t_1)\underline{n} \cdot \underline{u}(R_o, t_1) d\sigma. \qquad (128)$$

The integral on the right hand side of equation (128) is found from equations (116) and (121) to be

$$W = \frac{\pi}{c_o \rho_o(\bar{r}_o)}\, p_1^2 R_o^2 T \qquad (129)$$

and thus, from equation (123),

$$W = \frac{1}{4}\, p_1 \frac{R_o}{c_o T}\, V. \qquad (130)$$

With this expression we can replace the equivalent volume of air with the more tenable concept of the compression energy, to wit

$$V = \frac{E_c}{p_o(\bar{r}_o)\left[1 + \frac{1}{4}\frac{p_1}{p_o(\bar{r}_o)}\frac{R_o}{c_o T}\right]}. \qquad (131)$$

Under average conditions, E_c will be approximately 50% of the total energy E, according to the Department of Defense[21].

From equations (127) and (131), then, we have as our result

$$p(\underline{a}, s) = \frac{-\rho_o^{1/2}(\bar{r}_o)\rho_o^{1/2}(a)\ E_c}{p_o(\bar{r}_o)\left[1+\frac{1}{4}\ \frac{p_1}{p_o(\bar{r}_o)}\ \frac{R_o}{c_oT}\right]}\ \frac{s}{(1+sT)^2}\left[1+\frac{sR_o}{c_o}+\frac{1}{3}\frac{a^2}{(\bar{r}_o)^2}\frac{s^2R_o^2}{c_o^2}\right]G(\underline{a}, \underline{\bar{r}}_o, s). \tag{132}$$

In other words,

$$\rho_o^{-1/2}(\underline{a})\, p(\underline{a}, s) = R(\underline{\bar{r}}_o, s)\ G(\underline{a}, \underline{\bar{r}}_o, s)\ .$$

By retracing our steps, we can show, in a similar fashion, that

$$\rho_o^{-1/2}(r)\ p(\underline{r}, s) = R(\underline{\bar{r}}_o, s)\ G(\underline{r}, \underline{\bar{r}}_o, s)\ , \tag{133}$$

which substantiates the statement made in equation (56).

Here, we also note that $R(\underline{\bar{r}}_o, s)$ does not have an exponential term in s and that it contains no singularities in the right half s-plane. Replacing s with $-i\omega+\epsilon$ and taking the limit as $\epsilon \to 0$, we have from equations (101) and (132) that

$$p(\underline{a}, \omega) = \frac{1}{2}\sqrt{\frac{\rho_o(\bar{r}_o)\rho_o(a)\, c_o}{\pi a \sin\theta}}\ \frac{A}{p_o(\bar{r}_o)}\ \frac{E_c}{\left[1+\frac{1}{4}\ \frac{p_1}{p_o(\bar{r}_o)}\ \frac{R_o}{c_oT}\right]}\ I(\omega) \tag{134}$$

where

$$I(\omega) = -H(\omega_c+\omega)\, H(\omega_c-\omega)\ \frac{i\omega}{(1-i\omega T)^2}\left[1-i\frac{\omega R_o}{c_o}-\frac{1}{3}\frac{a^2}{(\bar{r}_o)^2}\frac{\omega^2R_o^2}{c_o^2}\right]\frac{\phi_1(\bar{r}_o)}{\phi_1(a)}\ \frac{e^{i\left[\frac{\omega a\theta}{c_o}+\frac{\pi}{4}\right]}}{\sqrt{\omega}}\ . \tag{135}$$

Since

$$p(\underline{a}, t) = H(t - t_a) \frac{1}{2\pi} \int_{-\infty}^{\infty} p(\underline{a}, \omega) e^{-i\omega t} d\omega \quad (136)$$

from equation (62), the final problem is to evaluate

$$I(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I(\omega) e^{-i\omega t} d\omega , \quad (137)$$

THE PRESSURE FUNCTION

In evaluating I(ω) we consider the simple case in which the explosion is near the earth's surface ($r_o \simeq a$) so that the radio $\frac{\phi_1(\bar{r}_o)}{\phi_1(a)}$ in the expression for I(ω) is approximately one. This particular case is important both because of its simplicity and its importance in application. For the situation in which the source is at a higher altitude, $\frac{\phi_1(\bar{r}_o)}{\phi_1(a)}$ is a negative exponential function which will reduce the amplitude of I(ω) accordingly. In addition, we would have to worry about the branch points of $\phi_1(\bar{r}_o)$ and whether or not the observer is in the shadow region.

If we introduce

$$\beta = \frac{R_o}{c_o T} \tag{138}$$

and write

$$\frac{1}{(1 - i\omega T)}\left[1 - i\frac{\omega R_o}{c_o} - \frac{1}{3}\frac{\omega^2 R_o^2}{c_o^2}\right] = \eta(\beta, \omega) e^{i\psi(\beta,\omega)} \tag{139}$$

the form of I(ω) may be simplified. In this relationship,

$$\eta(\beta, \omega) = \frac{\left[1 + \frac{\beta^2\omega^2T^2}{3} + \frac{1}{9}\beta^4\omega^4T^4\right]^{1/2}}{(1+\omega^2T^2)} \tag{140}$$

and

$$\psi = \psi_1 + \psi_2 , \tag{141}$$

where

$$\tan\psi_1 = \frac{-\beta\omega T}{1 - \frac{1}{3}\beta^2\omega^2T^2} ; \quad \tan\psi_2 = \frac{2\omega T}{1 - \omega^2T^2} . \tag{142}$$

With this notation,

$$I(\omega) = H(\omega_c + \omega) H(\omega_c - \omega) \sqrt{\omega}\; \eta(\beta, \omega)\, e^{i\psi(\beta, \omega)} \exp\left\{ i \left[\frac{\omega a \theta}{c_o} - \frac{\pi}{4}\right]\right\} . \tag{143}$$

It is evident that since the phase and group velocities are both given by c_o,

$$t_a = \frac{a\theta}{c_o} \tag{144}$$

is the total time it takes the pulse to reach the observer. Thus, denoting the observer's time by

$$\tau = t - t_a \tag{145}$$

$$I(\omega)e^{-i\omega t} = \eta(\beta, \omega)e^{i\psi(\beta,\omega)} \sqrt{\omega}\; e^{-i\left[\omega\tau + \pi/4\right]} H(\omega_c + \omega) H(\omega_c - \omega)\,. \tag{146}$$

In this expression, there is a branch point at $\omega = 0$ so that some caution must be observed in performing the inverse Fourier Transform. We recall, now, from earlier discussions that in the event there are any singularities on the real ω-axis the path of integration must actually be above these points. Thus, for example, $\sqrt{-\omega} = e^{i\pi/2}\sqrt{\omega}$ and not $e^{-i\pi/2}\sqrt{\omega}$. With this in mind, we write

$$I(t) = \frac{1}{2\pi}\oint_{-\infty}^{\infty} I(\omega)e^{-i\omega t}d\omega = \frac{1}{2\pi}\oint_{-\omega_c}^{\omega_c} I(\omega)e^{-i\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_0^{\omega_c} I(\omega)e^{-i\omega t}d\omega + \frac{1}{2\pi}\int_0^{\omega_c} \bar{I}(\omega)e^{i\omega t}d\omega$$

$$= \frac{1}{\pi} R_e \left[\int_0^{\omega_c} I(\omega)e^{-i\omega t}\,d\omega\right]. \tag{147}$$

In this next to the last step we have used the relationship $\bar{I}(-\omega) = I(\omega)$ which is most easily verified from equation (135). Thus,

$$I(t) = \frac{1}{\pi}\int_0^{\omega_c} \sqrt{\omega}\,\frac{\sqrt{1 + \dfrac{\beta^2 T^2 \omega^2}{3} + \dfrac{\beta^4 T^4 \omega^4}{9}}}{(1 + T^2\omega^2)} \cos\left[\omega\tau - \psi(\beta,\omega) + \frac{\pi}{4}\right] d\omega \tag{148}$$

and so, from equation (136)

$$p(\underline{a}, t) = \frac{1}{2}\sqrt{\frac{\rho_o(\bar{r}_o)\rho_o(a)c_o}{2\pi a \sin\theta}}\;\frac{A}{p_o(\bar{r}_o)}\;\frac{E_c}{\left[1 + \dfrac{1}{4}\dfrac{p_1}{p_o(\bar{r}_o)}\beta\right]} H(\tau)\, I(t)\,. \tag{149}$$

After a tedious process, then, we have arrived at a relatively simple result. In the following section we will perform the numerical integration of I(t) but it is worth mentioning in advance that the ratio

$$\frac{\sqrt{1+\frac{\beta^2T^2}{3}\omega^2+\frac{1}{9}\beta^4T^4\omega^4}}{(1+T^2\omega^2)}$$

varies only slightly over a wide range of energies. The behavior of this quantity is presented in Table I, where the values of β and T are those given in Table II.

TABLE I

Values of the Ratio $$\frac{\sqrt{1+\frac{\beta^2T^2\omega^2}{3}+\frac{\beta^4T^4\omega^4}{9}}}{(1+\omega^2T^2)}$$

as a function of the explosive energy E and the frequency ω.

E(KT) \ ω/ω_c	0	.2	.4	.6	.8	1.0
1	1.00	1.00	1.00	1.00	1.00	1.00
250	1.00	1.00	1.00	1.01	1.02	1.02
500	1.00	1.00	1.01	1.02	1.03	1.04
1000	1.00	1.00	1.02	1.03	1.05	1.07
5000	1.00	1.01	1.03	1.07	1.13	1.24
10000	1.00	1.01	1.04	1.10	1.21	1.38
15000	1.00	1.02	1.05	1.15	1.29	1.55
25000	1.00	1.03	1.07	1.28	1.50	1.86

Thus, $I_{max}(t)$ will be roughly proportional to $\omega_c^{3/2}$ sec$^{-3/2}$, indicating that the cut-off frequency has a very great effect in determining the amplitude of the pressure wave train.

Two remarks on the time dependent solution are worthwhile at this point since they were implicit in the formulation of problem. We note first from equations (148) and (149) that since $\tau = t - \frac{a\theta}{c_o}$, we have

$$\frac{\partial p}{\partial t} = -c_o \frac{\partial p}{\partial (a\theta)} = -c_o \frac{\partial p}{\partial d} \tag{150}$$

where d is the great circle distance the pulse has traveled. An assumption of this expression was made in the paragraph following equation (13) in postulating that $\frac{Dp}{Dt} \simeq \frac{\partial p}{\partial t}$.

A second point is that in equation (10) the perturbation scheme $\tilde{p} = p_o + p$ was presented on the assumption that $\frac{p}{p_o} \ll 1$. We will see in the following section that this condition is indeed satisfied when the observer is located on the ground. It must also hold, however, for large values of r. To indicate how this develops for the approximate solution we have obtained, we consider the case in which the source is located on the ground and the observer is at some high altitude within the shadow region. Then, from equations (105) and (101),

$$G(\underline{r}, \underline{a}, \omega) = H(\omega_c + \omega)\, H(\omega_c - \omega) \left\{ -\frac{1}{2} \sqrt{\frac{c_o}{2\pi \acute{a} \omega \sin\theta}}\ \frac{A\phi_1(r)}{\phi_1(a)}\ e^{i\left[\frac{\omega a \theta}{c_o} + \frac{\pi}{4}\right]} \right\}. \tag{151}$$

For the case where gravity is constant, we have, from equations (64) and (66) that

$$\lim_{r \to \infty} \phi_1(r) \simeq \frac{1}{r} e^{-\frac{r}{c_o}\left(\omega_c^2 - \omega^2\right)^{1/2}} . \tag{152}$$

In a manner identical to the way in which we obtained equation (134) it develops that

$$p(\underline{r},\omega) \sim \frac{1}{2}\sqrt{\frac{\rho_o(a)\ldots}{2\pi a \sin\theta}}\;\frac{A}{p_o(a)}\;\frac{E_c}{\left[1+\frac{1}{4}\frac{p_1}{p_o(a)}\frac{R_o}{c_oT}\right]}\;J(\omega)\rho_o^{1/2}(r)\;\frac{e^{\frac{-r}{c_o}(\omega_c^2-\omega^2)^{1/2}}}{r} \tag{153}$$

where

$$J(\omega) = -H(\omega_c+\omega)\,H(\omega_c-\omega)\,\frac{i\omega}{(1-i\omega T)^2}\left[1-\frac{i\omega R_o}{c_o}-\frac{\omega^2 R_o^2}{3c_o^2}\right]\frac{e^{i\left[\frac{\omega a\theta}{c_o}+\frac{\pi}{4}\right]}}{\sqrt{\omega}\;\phi_1(a)} . \tag{154}$$

Hence,

$$p(\underline{r},t) \underset{r\to\infty}{\simeq} \frac{\rho_o^{1/2}(r)}{r}\,C\,H(t-t_a)\int_{-\omega_c}^{\omega_c} J(\omega)e^{\frac{-r}{c_o}(\omega_c^2-\omega^2)}\,e^{-i\omega t}d\omega, \tag{155}$$

C denoting all the constant terms appearing in equation (153). Since $(\omega_c^2-\omega^2)>0$ on the interval $(-\omega_c,\omega_c)$ equation (155) may be evaluated asymptotically by applying Laplace's method.[22] With $\frac{r}{c_o}$ as a large parameter,

$$\int_{-\omega_c}^{\omega_c} J(\omega)e^{-\frac{r}{c_o}(\omega_c^2-\omega^2)^{1/2}}\,e^{-i\omega t}d\omega \simeq e^{\frac{-r\omega_c}{c_o}}\int_{-\omega_c}^{\omega_c} J(\omega)\exp\left[\frac{r}{c_o}\frac{\omega^2}{2\omega_c}\right]e^{-i\omega t}d\omega. \tag{156}$$

From equation (154), $J(\omega)e^{-i\omega t}$ is a well behaved function on the interval in question, so that the order of magnitude of the integral is given by $e^{-r\omega_c/c_o}$.

Thus,

$$p(\underline{r},t) \underset{r\to\infty}{\simeq} C\,H(t-t_a)\,\frac{\rho_o^{1/2}(r)}{r}\,0\left\{\exp\left[-\frac{r}{c_o}\,\omega_c\right]\right\}. \tag{157}$$

For this constant gravity case, $\rho_o^{1/2}(r) = \rho_{oo}e^{-(r/c_o)\omega_c}$ and so

$$p(\underline{r},t) \underset{r\to\infty}{\simeq} C\,H(t-t_a)\frac{1}{r}\,0\left[\rho_o(r)\right] = \frac{C}{r}\,H(t-t_a)\,0\left[p_o(r)\right].$$

Thus, $\frac{p(\underline{r},t)}{p_o(r)}$ varies asymptotically as $\frac{1}{r}$ for large r which justifies our having used the perturbation scheme in the first section.

For the gravity variation $g = g_o \frac{a^2}{r^2}$, the situation is even simpler. We find that

$$p(\underline{r}, t) \underset{r \to \infty}{\simeq} C \frac{\rho_o^{1/2}(r)}{r} H(t-t_a) \int_{-\omega_c}^{\omega_c} J(\omega) e^{i\omega(\frac{r}{c_o} - t)} d\omega \tag{159}$$

where $\rho_o^{1/2}(r)$ approaches a constant as r becomes infinite. Thus,

$$\frac{p(\underline{r}, t)}{p_o(r)} \underset{r \to \infty}{\simeq} \frac{\gamma C}{c_o^2} \frac{\rho_o^{1/2}(r)}{r} H(t-t_a) \int_{-\omega_c}^{\omega_c} J(\omega) e^{i\omega(\frac{r}{c_o} - t)} d\omega \tag{160}$$

behaves like an outgoing spherical wave which diminishes in amplitude with increasing r.

NUMERICAL RESULTS AND CONCLUSIONS

In determining the appropriate values for the parameters R_o, T, and β, data from the charts on pages 135 and 143 of the Effects of Nuclear Weapons[(21)] (Department of Defense, 1962) were used which correspond to a peak overpressure of 1.47 psi. This choice of pressure is somewhat arbitrary, its selection being based upon the fact that it is the largest value which will permit the acoustic approximation of the previous section to be made with any accuracy. A smaller value of p_1 would be satisfactory from this standpoint, but due to the inhomogeneity of the atmosphere this would force us to consider a larger surface which is beginning to lose its spherical shape. With this value for p_1, then, the following table indicates the salient blast characteristics corresponding to various values of the explosive energy. It is worth noting that from a basic scaling law, both R_o and T increase as $E^{1/3}$, with the result that β is independent of this quantity for the particular choice of p_1.

TABLE II: CHARACTERISTIC BLAST PARAMETERS FOR ATMOSPHERIC NUCLEAR EXPLOSIONS

E(KT)	p_1/p_o	$10^{-3}R_o$(ft)	T(sec)	β
1	.1	3.000	.450	6.62
250	.1	18.899	2.835	6.62
500	.1	23.811	3.572	6.62
1000	.1	30.000	4.500	6.62
5000	.1	51.300	7.695	6.62
10000	.1	63.462	9.519	6.62
15000	.1	73.980	11.097	6.62
25000	.1	87.720	13.158	6.62

For each of these eight cases the following numerical values are assumed:

c_o = 3.11 x 10^4 cm/sec = speed of sound,

g = 9.8 x 10^2 cm/sec^2 = acceleration due to gravity,

γ = 1.4 = ratio of specific heats,

a = 6.36 x 10^8 cm = radius of the earth,

A = 3.039 x 10^{-7} cm^{-1} = $(1 - \gamma/2)\frac{g_o}{c_o^2}$,

ω_c = 2.25 x 10^{-2} sec^{-1} = $(\frac{\gamma}{2}\frac{g_o}{c_o})$,

E_c = δE = 4.2 x 10^{19} $\delta\epsilon$ ergs, where 4.2 x 10^{19} ergs is the energy equivalent of 1KT of TNT, ϵ is the number of kilotons in the explosion, and δ is the fraction of yield that goes into creating the atmospheric pressure disturbance.

For a pure air burst, δ will be approximately equal to 1/2, while for an explosion on or just above the earth's surface a certain portion of the shock wave goes into cratering and the creation of seismic waves, with the result that δ is correspondingly smaller. Weston[9] took this into account by assuming that a surface burst has half as much surface area as an air burst and thus releases half as much energy into the atmosphere.

Thus with this description, substitution of the numerical values listed above leads to

$$p(\underline{a}, t) = 7.2797 \frac{\delta\epsilon}{\sqrt{\sin\theta}} I(t) H(t) \ \mu\text{-bars} \tag{161}$$

which is a very useful expression insofar as investigating nuclear explosions is concerned. As an example, we will determine $(\delta\epsilon)$ by considering three different sets of microbarograph recordings of the Soviet detonation on 30 October 1961 of a low altitude nuclear device at Novaya Zemlya. These observations, reported by Rose et al,[23] Araskog et al,[19] and Wexler and Haas[24] indicated peak pressures of 1 m-bar at 1160 Km, .8 m-bar at 2150 Km and .5 m-bar at 7000 Km, respectively. Since the peak value of I(t) is found to be on the order of 3×10^{-3}, irrespective of the explosive energy we determine the following table from these data.

TABLE III: VALUES OF $\delta\epsilon$ AS DETERMINED FROM DATA OF THREE INDEPENDENT OBSERVERS

Observer	Distance (Km)	Sin θ	$\delta\epsilon$ (KT)
Rose et al	1160	.1813	23.3976×10^{3}
Araskog et al	2150	.3316	25.2936×10^{3}
Wexler and Haas	7000	.8914	25.9344×10^{3}

In the case of a low altitude air burst ($\delta = 1/2$), ϵ for these three cases is approximately 5×10^{4} KT which corresponds to an explosive of 50 MT. Since it has been stated by Wexler and Haas[24] for example, that the yield was actually on the order of 58 MT, the correlation with amplitude is excellent and certainly within the range of experimental error. It is also worth noting that the values of ϵ computed from the data of the three observers are in good agreement with each other. This indicates that the variation in amplitude with distance is definitely on the order of $\sin^{-1/2}\theta$.

Figure 6 displays the time variation of the pressure function $p(\underline{a}, t)$ that one could expect to observe as a result of a one KT low altitude air burst. The left most column gives the amplitude in μ-bars that would be recorded at a station 2300 Km from the source, while the remaining two columns give the same information at distances of 3600 and 6300 Km, respectively, from the explosion.

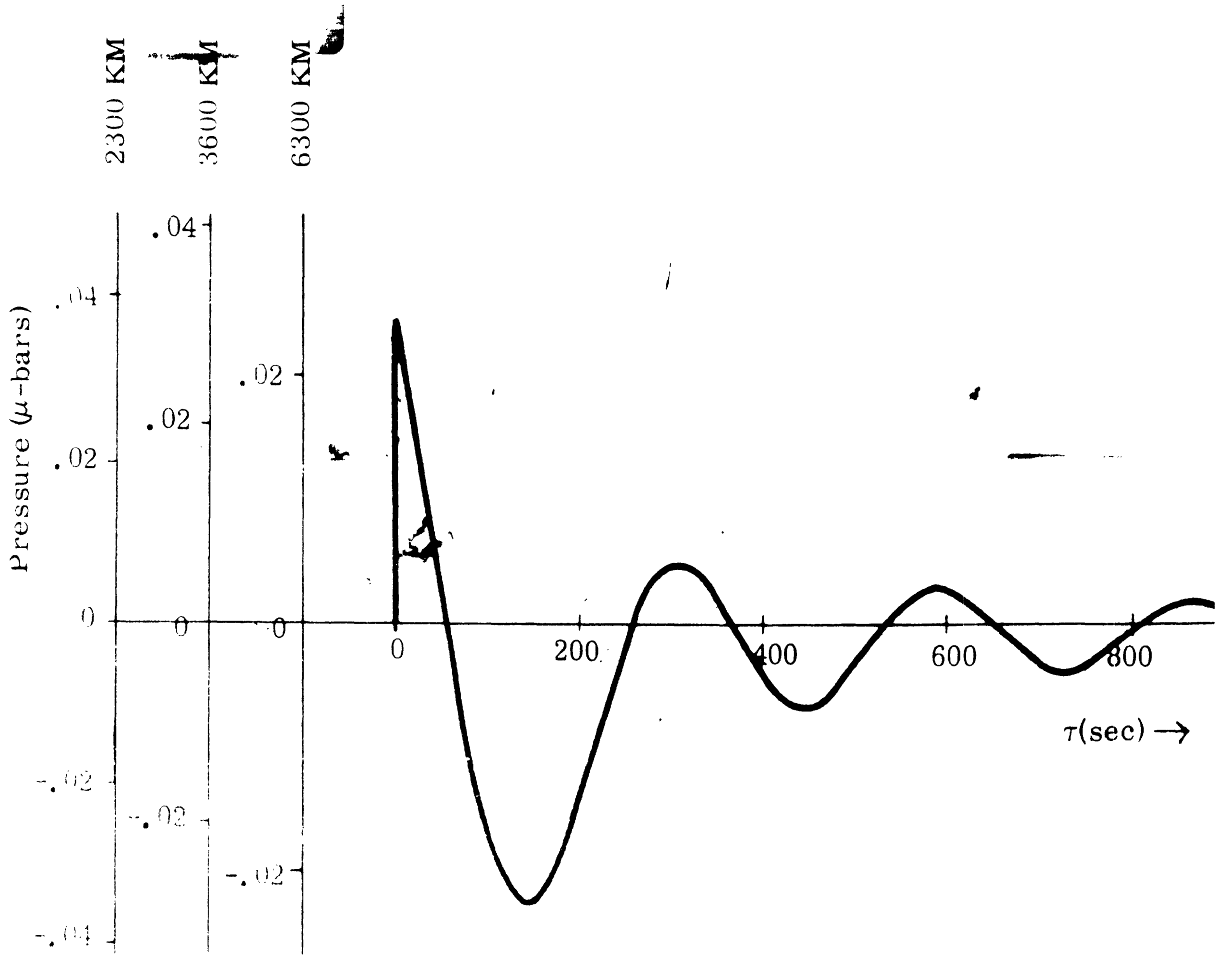

FIG. 6. PRESSURE PULSE AS A FUNCTION OF TIME FOR OBSERVERS AT DISTANCES OF 2300, 3600, and 6300 KM FROM A 1 KT EXPLOSION. A low altitude air burst corresponding to $\delta = 1/2$ is assumed.

In Figure 7 the same distances are considered, except that here the input is assumed to be a low altitude air burst of 15 MT. A comparison of the wave trains for these two cases indicates that the period of the gravity wave is unaffected by the yield of the explosion, the time between successive maxima in both cases being on the order of 300-350 seconds. This is in good agreement with the recordings presented by Araskog et al[19] and somewhat less than the figure of approximately eight minutes indicated by Wexler and Haas[24] and by Rose et al.[23]

One perplexing feature of observed pressure recordings is that the phase of the wave train does not follow any set pattern. Samples of various data are presented in Figure 8, and in some instances the first oscillation is negative while in others it is positive. It is not possible on these bases, either to justify or refute the causality requirement which was introduced in equation (62). We can state, however, that this condition was imposed on both mathematically and physically logical grounds and that the result so obtained is definitely reasonable in all respects.

This then completes the discussion of the pressure pulse produced by low altitude explosion as detected by an observer located on the earth's surface. We have, admittedly, considered the simplest case, but the results can easily be extended to include any tropospheric level burst by retracing the steps and retaining the appropriate value of r_o. There are, in addition, a number of other more complicated problems that could be solved using this approach but which would eventually require a numerical rather than an analytic solution. For example, the case of the high altitude stratospheric burst is similar in principle

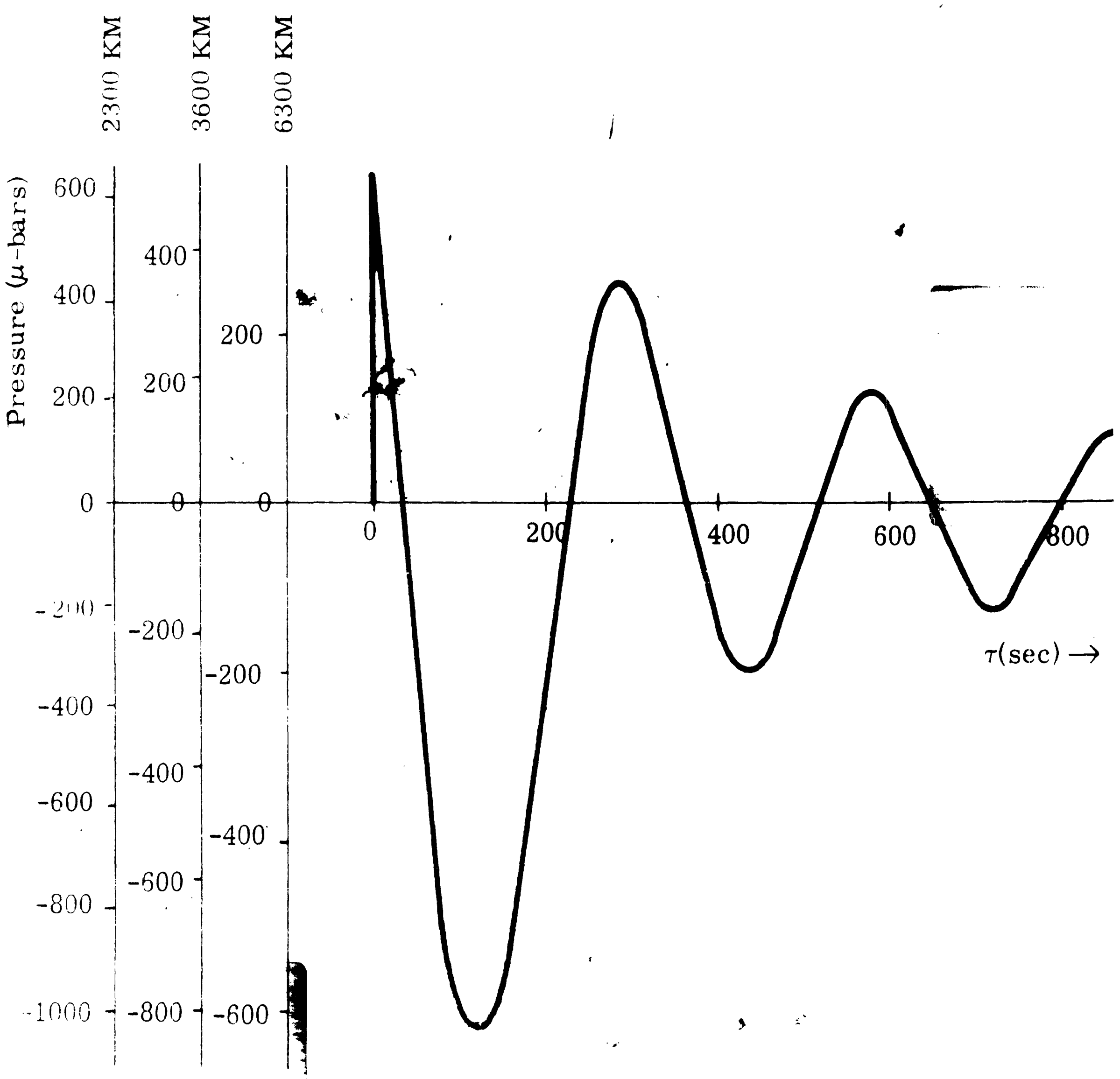

FIG. 7: PRESSURE PULSE AS A FUNCTION OF TIME FOR OBSERVERS AT DISTANCES OF 2300, 3600, AND 6300 KM FROM A 15 MT EXPLOSION. A low altitude air burst corresponding to $\delta = 1/2$ is assumed.

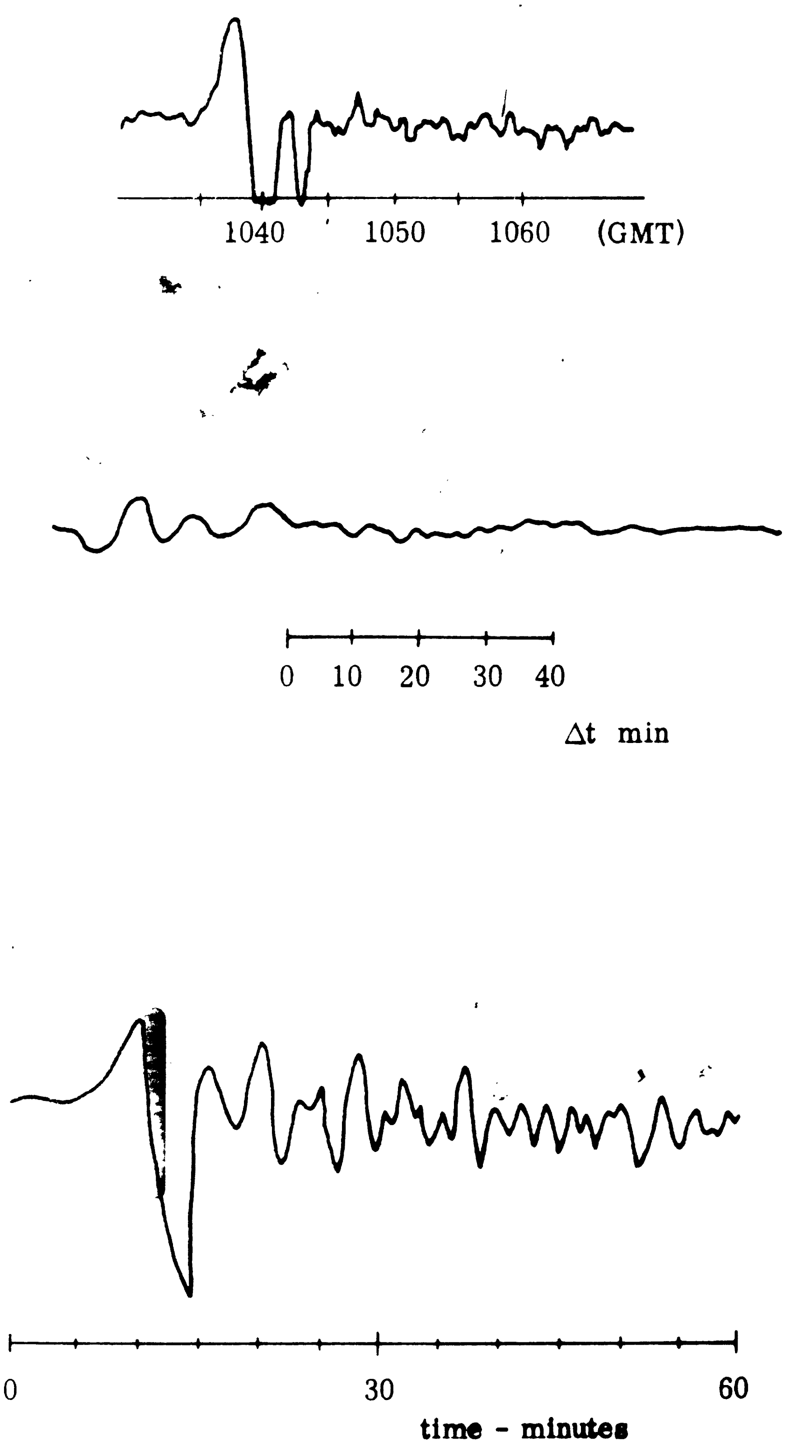

FIG. 8: SAMPLE MICROBAROGRAPH RECORDS. These data from top to bottom were presented by Araskog,[19] Rose et al[23] and Donn and Ewing.[7]

but far more complicated in practice. In the first place, the assumption of an isothermal temperature distribution for this situation is totally unrealistic. To make matters worse, the effect of viscosity, which our quantitative analysis has overlooked, becomes a significant, or even dominant, factor in the problem. We could anticipate, then, a marked filtering of the short wavelengths which might significantly alter the analytic properties of the solution. There is, then, a definite restriction on the type of situation to which our treatment may be applied. In general, however, these methods, when applied to any tropospheric explosion, may be expected to yield good quantitative results.

APPENDIX A

ANALYTIC PROPERTIES OF THE RADIAL WAVE FUNCTIONS

In this appendix, the analytic properties of the solution to the equation

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{h}\frac{d\phi_1}{dr}\right] + \left[\frac{\omega^2}{c_o^2} - \frac{A^2}{h} - \frac{Ah'}{h^2} - \frac{(\lambda + \frac{1}{2})(\lambda - \frac{1}{2})}{r^2}\right]\phi_1 = 0 \qquad \text{(A-1)}$$

are investigated for the two gravitational cases $g = g_o \frac{a^2}{r^2}$ and $g = g_o$. The function $\phi_1(r, \lambda, \omega)$ is assumed to satisfy the radiation condition and the boundary condition

$$\frac{d\phi_1}{dr} + A\phi_1 = 0 \qquad \text{(A-2)}$$

at the earth's surface. The corresponding results for the Laplace transformed equation may be made upon replacing ω with $s = -i\omega$.

The following proofs are an extension of a method presented by Beck and Nussenzweig[(25)] in which we show that

1) If gravity varies as $g = g_o \frac{a^2}{r^2}$ and λ is a real number, all solutions of equation (A-1) that satisfy the two boundary conditions yield complex ω-poles with negative imaginary parts and are analytic in the upper half ω-plane.

2) If gravity varies as $g = g_o \frac{a^2}{r^2}$ and ω is a real number, all solutions of equation (A-1) that satisfy the two boundary conditions yield complex λ-poles that lie in either the first or third quadrants.

3) If gravity is constant $(g = g_o)$ and λ is a real number, all solutions of equation (A1) that satisfy the two boundary conditions yield complex ω-poles with negative imaginary parts and are analytic in the upper half ω-plane.

4) If gravity is constant $(g = g_o)$ and ω is a real number, all solutions

of equation (A-1) that satisfy the two boundary conditions yield

a) Complex λ-poles when $|\omega| > \omega_c$

b) Real or imaginary λ-poles when $|\omega| < \omega_c$.

1. Proof of Statement 1.

$$\text{Let} \quad \phi_1 = e^{-\int_a^r A\,d\sigma}\,\psi_1(r)\,. \tag{A-3}$$

Then equation (A-1) becomes

$$\frac{d^2\psi_1}{dr^2} + \left[\frac{2}{r} - 2A - \frac{h'}{h}\right]\psi_1' + h\left[\frac{\omega^2}{c_o^2} - \frac{(\lambda + \frac{1}{2})(\lambda - \frac{1}{2})}{r^2}\right]\psi_1 = 0\,. \tag{A-4}$$

If we introduce the factor

$$j(r) = \frac{r^2}{h}\,e^{-2\int_a^r A\,d\sigma} \tag{A-5}$$

equation (A-4) simplifies to

$$\frac{d}{dr}\left[j\,\frac{d\psi_1}{dr}\right] + hj\left[\frac{\omega^2}{c_o^2} - \frac{(\lambda + \frac{1}{2})(\lambda - \frac{1}{2})}{r^2}\right]\psi_1 = 0\,. \tag{A-6}$$

The complex conjugate of equation (A-6) is

$$\frac{d}{dr}\left[\bar{j}\,\frac{d\bar{\psi}_1}{dr}\right] + \bar{h}\bar{j}\left[\frac{\bar{\omega}^2}{c_o^2} - \frac{(\bar{\lambda} + \frac{1}{2})(\bar{\lambda} - \frac{1}{2})}{r^2}\right]\bar{\psi}_1 = 0 \tag{A-7}$$

where $\psi_1(r)$ and $\bar{\psi}_1(r)$ from equation (A-2) satisfy the conditions

$$\frac{\partial\psi_1}{\partial r} = 0 \quad \text{at } r = a \qquad \text{and} \qquad \frac{\partial\bar{\psi}_1}{\partial r} = 0 \quad \text{at } r = 0 \tag{A-8}$$

respectively. Upon multiplying equations (A-6) and (A-7) by $\bar{\psi}_1(r)$ and $\psi_1(r)$

respectively and subtracting, we obtain

$$\frac{d}{dr}\left\{\bar{\psi}_1 j \psi_1' - \psi_1 \bar{j} \bar{\psi}_1'\right\} - (j - \bar{j})\bar{\psi}_1' \psi_1' + r^2 e^{-2\int_a^r A\,d\sigma} \frac{\left[\omega^2 - \bar{\omega}^2\right]}{c_o^2} \psi_1 \bar{\psi}_1 = 0 . \tag{A-9}$$

From the definition of h given in equation (23) plus definition of j(r) from equation (A-5)

$$(j - \bar{j}) = r^2 e^{-2\int_a^r A\,d\sigma} \left[\frac{1}{h} - \frac{1}{\bar{h}}\right] = - \frac{\omega_b^2 a^4}{r^4} \frac{(\omega^2 - \bar{\omega}^2)}{|h\omega^2|^2} r^2 \exp\left\{-2 \int_a^r A\,d\sigma\right\} \tag{A-10}$$

Hence (A-9) can be written

$$\frac{d}{dr}\left[\bar{\psi} j \psi_1' - \psi_1 \bar{j} \bar{\psi}_1'\right] + (\omega^2 - \bar{\omega}^2) r^2 e^{-2\int_a^r A\,d\sigma} \cdot \left\{\frac{\omega_b^2 a^4}{r^4} \frac{\bar{\psi}_1' \psi_1'}{|h\omega^2|^2} + \frac{1}{c_o^2} \bar{\psi}_1 \psi_1\right\} = 0 . \tag{A-11}$$

Upon integrating this expression from a to some r and imposing the boundary conditions of equation (A-8), we have

$$\left\{\bar{\psi}(r) j \psi_1'(r) - \psi_1(r) \bar{j} \bar{\psi}_1'(r)\right\} + (\omega^2 - \bar{\omega}^2) \int_a^r \tau^2 e^{-2\int_a^\tau A\,d\sigma} \left\{\frac{\omega_b^2 a^4}{\tau^4} \frac{\bar{\psi}_1' \psi_1'}{|h\omega^2|^2} + \frac{1}{c_o^2} \bar{\psi}_1 \psi_1\right\} d\tau = 0 . \tag{A-12}$$

For large values of r, h(r) and $\bar{h}(r)$ both approach unity. The asymptotic form for $\psi_1(r)$ is given by

$$\psi_1(r) \simeq c_1(\lambda)\, e^{\int_a^r A\,d\sigma} \frac{1}{r} e^{i\frac{\omega}{c_o} r} \tag{A-13}$$

while

$$\bar{\psi}_1(r) \simeq \overline{c_1(\lambda)}\, e^{\int_a^r A\,d\sigma} \frac{1}{r} e^{-i\frac{\bar{\omega}}{c_o} r} .$$

From the bracketed term of equation (A-12),

$$\left\{\bar{\psi}(r)\, j\psi_1'(r) - \psi(r)\, j\bar{\psi}_1'(r)\right\} \underset{r\to\infty}{\simeq} \left|c_1(\lambda)\right|^2 \frac{i(\omega+\bar{\omega})}{c_o}\, e^{\frac{i(\omega-\bar{\omega})r}{c_o}} \,. \tag{A-14}$$

If we assume that

$$\begin{aligned} \omega &= \omega_1 + i\omega_2 \\ \bar{\omega} &= \omega_1 - i\omega_2 \end{aligned} \tag{A-15}$$

equation (A-12) becomes

$$\left|c_1\right|^2 \frac{1}{c_o} e^{i(\omega-\bar{\omega})r/c_o} + 2\omega_2 \int_a^r \tau^2 e^{-2\int_a^\tau A\, d\sigma} \left\{ \frac{\omega_b^2 a^4}{\tau^4} \frac{|\psi'|^2}{|h\omega^2|^2} + \frac{1}{c_o^2} |\psi_1|^2 \right\} d\tau = 0 \,. \tag{A-16}$$

The integrand in this equation is a positive real quantity, as is the exponential $e^{i(\omega-\bar{\omega})r/c_o}$. Thus,

$$\omega_2 = - \frac{\frac{1}{2}\frac{1}{c_o}\, e^{i(\omega-\bar{\omega})\frac{r}{c_o}} \left|c_1\right|^2}{\int_a^r \tau^2 e^{-2\int_a^\tau A\, d\sigma} \left\{ \frac{\omega_b^2 a^4}{\tau^4} \frac{|\psi'|^2}{|h\omega^2|^2} + \frac{1}{c_o^2} |\psi_1|^2 \right\} d\tau} \tag{A-17}$$

is negative, which proves the statement.

2. Proof of Statement 2

Equation (A-1) and its complex conjugate for real ω and complex λ are given by

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{h}\frac{d\phi_1}{dr}\right] + \left[\frac{\omega^2}{c_o^2} - \frac{A^2}{h} - \frac{Ah'}{h^2} - \frac{(\lambda+1/2)(\lambda-1/2)}{r^2}\right]\phi_1 = 0 \tag{A-18}$$

and

$$\frac{1}{2}\frac{d}{dr}\left[\frac{r^2}{h}\frac{d\bar{\phi}_1}{dr}\right]+\left[\frac{\omega^2}{c_o^2}-\frac{A^2}{h}-\frac{Ah'}{h^2}-\frac{(\bar{\lambda}+1/2)(\bar{\lambda}-1/2)}{r^2}\right]\bar{\phi}_1=0$$

respectively. Upon multiplying the first of these by $\bar{\phi}_1(r)$, the second by $\phi_1(r)$ and subtracting, we obtain

$$\frac{d}{dr}\left[\frac{r^2}{h}(\phi_1\bar{\phi}_1'-\phi_1'\bar{\phi}_1)\right]=\left[(\bar{\lambda}+1/2)(\bar{\lambda}-1/2)-(\lambda+1/2)(\lambda-1/2)\right]\phi_1\bar{\phi}_1. \qquad \text{(A-19)}$$

Since

$$\phi_1'(r)+A\phi_1(r)=0 \qquad \text{at } r=a$$

and

$$\bar{\phi}_1'(r)+A\bar{\phi}_1(r)=0 \qquad \text{at } r=a$$

integration of equation (A-19) from a to r yields

$$\frac{r^2}{h}(\phi_1\bar{\phi}_1'-\bar{\phi}_1\phi_1')=\left[(\bar{\lambda}+1/2)(\bar{\lambda}-1/2)-(\lambda+1/2)(\lambda-1/2)\right]\int_a^r|\phi_1(\tau)|^2\,d\tau. \qquad \text{(A-20)}$$

For positive ω,

$$\phi_1(r)\underset{r\to\infty}{\simeq}\sqrt{\frac{2c_o}{\pi\omega}}\frac{1}{r}e^{i\left[\frac{\omega}{c_o}r-\lambda\frac{\pi}{2}-\frac{\pi}{4}\right]} \qquad \text{(A-21)}$$

and

$$\bar{\phi}_1(r)\simeq\sqrt{\frac{2c_o}{\pi\omega}}\frac{1}{r}e^{-i\left[\frac{\omega}{c_o}r-\bar{\lambda}\frac{\pi}{2}-\frac{\pi}{4}\right]}$$

Thus, from equations (A-20) and (A-21),

$$\frac{-2i\omega}{c_o}r^2|\phi_1|^2=\left[(\bar{\lambda}+1/2)(\bar{\lambda}-1/2)-(\lambda+1/2)(\lambda-1/2)\right]\int_a^r|\phi_1(\tau)|^2\,d\tau. \qquad \text{(A-22)}$$

If we now define

$$\lambda=\lambda_1+i\lambda_2 \qquad \text{(A-23)}$$

we obtain, from equation (A-22)

$$\frac{-2i\omega}{c_0} r^2 |\phi_1|^2 = -4i\lambda_1\lambda_2 \int_a^r |\phi_1|^2 \, d\tau$$

or

$$2\lambda_1\lambda_2 = \frac{\omega r^2 |\phi_1|^2}{\int_a^r |\phi_1|^2 \, d\tau} . \tag{A-24}$$

The right hand side of this is a positive real quantity so that $\lambda_1\lambda_2$ is also positive. The same statement holds when ω is negative, since in ~~that case~~ the asymptotic forms for $\phi_1(r)$ and $\bar{\phi}_1(r)$ are given by

$$\phi_1(r) \simeq \sqrt{\frac{2c_0}{\pi\omega}} \frac{1}{r} e^{-i\left[\frac{\omega}{c_0} r + \frac{\lambda\pi}{2} - \frac{\pi}{4}\right]} \tag{A-25}$$

and

$$\bar{\phi}_1(r) \simeq \sqrt{\frac{2c_0}{\pi\omega}} \frac{1}{r} e^{i\left[\frac{\omega}{c_0} r + \bar{\lambda}\frac{\pi}{2} - \frac{\pi}{4}\right]}$$

In place of equation (A-24) then we have

$$2\lambda_1\lambda_2 = \frac{-\omega r^2 |\phi_1|^2}{\int_a^r |\phi_1|^2 \, d\tau} \tag{A-26}$$

so that $\lambda_1\lambda_2$ must again be positive.

Thus, for all real ω, λ must lie in either the first or third quadrants.

3. Proof of Statement 3.

If $g = g_0$ and λ is real, the scheme presented in proving the first statement can be applied, but this idea is rather awkward. It is more direct to use the fact that h and A are independent of r when gravity is constant.

In this situation equation (A-1) becomes

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\phi_1}{dr}\right]+\left[\frac{h\omega^2}{c_o^2}-A^2-\frac{h(\lambda+1/2)(\lambda-1/2)}{r^2}\right]\phi_1=0 \qquad \text{(A-27)}$$

while its complex conjugate is

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\bar{\phi}_1}{dr}\right]+\left[\frac{\bar{h}\bar{\omega}^2}{c_o^2}-A^2-\bar{h}\frac{(\lambda+1/2)(\lambda-1/2)}{r^2}\right]\bar{\phi}_1=0\,. \qquad \text{(A-28)}$$

As in the first two proofs, we multiply (A-27) by $\bar{\phi}_1$, (A-28) by ϕ_1 and subtract to obtain

$$\frac{d}{dr}\left[r^2(\phi_1\bar{\phi}_1'-\bar{\phi}_1\phi_1')\right]=\left\{\left(\frac{h\omega^2-\bar{h}\bar{\omega}^2}{c_o^2}\right)r^2-(h-\bar{h})(\lambda+1/2)(\lambda-1/2)\right\}\phi_1\bar{\phi}_1 \qquad \text{(A-29)}$$

where

$$\begin{aligned}\phi_1'+A\phi_1&=0 &&\text{at } r=a\\ \bar{\phi}_1'+A\bar{\phi}_1&=0 &&\text{at } r=a\,.\end{aligned} \qquad \text{(A-30)}$$

Upon integrating from a to r, we have

$$r^2\left[\phi_1\bar{\phi}_1'-\bar{\phi}_1\phi_1'\right]=\frac{h\omega^2-\bar{h}\bar{\omega}^2}{c_o^2}\int_a^r\tau^2|\phi_1(\tau)|^2d\tau-(\lambda+1/2)(\lambda-1/2)(h-\bar{h})\int_a^r|\phi_1(\tau)|^2d\tau\,. \qquad \text{(A-31)}$$

In equation (A-27) we let $\frac{h\omega^2}{c_0^2}-A^2=\frac{\omega^2-\omega_c^2}{c_0^2}$ from equation (64) and $\nu^2=h(\lambda+1/2)(\lambda-1/2)+\frac{1}{4}$ from equation (68). It is evident that since $h(\omega)$ changes sign at $\omega=\pm\omega_b$, while $\omega^2-\omega_c^2$ changes at $\pm\omega_c$, $\omega=\pm\omega_c$ and $\omega=\pm\sqrt{(\lambda+1/2)(\lambda-1/2)}\,\frac{\omega_b}{\lambda}$ are branch points. We define Ω about $\omega=\omega_c$ to be

$$\Omega^2=\omega^2-\omega_c^2 \qquad \text{(A-32)}$$

and

$$\Omega=|\Omega|\,e^{-i\alpha}\,. \qquad \text{(A-33)}$$

For large values of r, then

$$\phi_1(r) \approx \sqrt{\frac{2c_o}{\pi\Omega}}\,\frac{1}{r}\,e^{i\left[\frac{\Omega}{c_o}r - \frac{\nu\pi}{2} - \frac{\pi}{4}\right]} \tag{A-34}$$

and

$$\bar{\phi}_1(r) \simeq \sqrt{\frac{2c_o}{\pi\bar{\Omega}}}\,\frac{1}{r}\,e^{-i\left[\frac{\bar{\Omega}}{c_o}r - \frac{\bar{\nu}\pi}{2} - \frac{\pi}{4}\right]}$$

where $\bar{\nu}$ is complex since it depends upon ω. From equation (A-34),

$$r^2\left[\phi_1\bar{\phi}'_1 - \bar{\phi}_1\phi'_1\right] = \frac{-i(\Omega+\bar{\Omega})}{c_o}\,r^2|\phi_1|^2$$

$$= \frac{-2i|\Omega|}{c_o}\,|r\phi_1|^2\cos\alpha. \tag{A-35}$$

In these expressions, Ω and $\bar{\Omega}$ are defined as the principle values

$$\Omega = \sqrt{\Omega^2} = |\Omega|\,e^{-i\alpha} \tag{A-36}$$

and

$$\bar{\Omega} = \sqrt{(\bar{\Omega})^2} = |\Omega|\,e^{i\alpha}$$

From equation (A-31), then,

$$\frac{-2i|\Omega|}{c_o}|r\phi_1|^2\cos\alpha \simeq \frac{h\omega^2 - \bar{h}\bar{\omega}^2}{c_o^2}\int_a^r \tau\,|\phi_1|^2\,d\tau - (\lambda+1/2)(\lambda-1/2)(h-\bar{h})\int_a^r |\phi_1|^2\,d\tau$$

$$= (\omega^2 - \bar{\omega}^2)\left\{\int_a^r \tau^2|\phi_1|^2\,d\tau - \frac{(\lambda+1/2)(\lambda-1/2)\omega_b^2}{|\omega|^4}\int_a^r |\phi_1|^2\,d\tau\right\}. \tag{A-37}$$

No matter how large $|\lambda|$ or how small $|\omega| \neq 0$, we can choose r great enough so that

$$\int_a^r \tau^2|\phi_1|^2\,d\tau > \frac{(\lambda+1/2)(\lambda-1/2)\omega_b^2}{|\omega|^4}\int_a^r |\phi_1|^2\,d\tau.$$

Hence, for sufficiently large r, the bracketed term on the right-hand side of equation (A-37), which we denote by $F(r)$, will be positive.

Since $\omega^2 - \bar{\omega}^2 = \Omega^2 - \bar{\Omega}^2 = -2i\,|\Omega|^2 \sin 2\alpha$, we have, from equation (A-37) that

$$\frac{-2i|\Omega|}{c_o} |r\phi_1|^2 \cos\alpha = -2i|\Omega|^2 \sin 2\alpha\, F(r) \tag{A-38}$$

or

$$\sin\alpha = \frac{1}{2} \frac{1}{|\Omega|\, c_o} \frac{|r\phi_1|^2}{F(r)} > 0 . \tag{A-39}$$

With $\omega = \omega_1 + i\omega_2$, we assume that ω_1 is positive in the expansion (A-32) about $\omega = \omega_c$. Hence, from equation (A-33),

$$\omega_1 \omega_2 = -|\Omega|^2 \sin 2\alpha < 0$$

which means that $\omega_2 < 0$.

In a similar expansion about $\omega = -\omega_c$, we let

$$\Omega = |\Omega|\, e^{i(\pi + \alpha')} \tag{A-40}$$

and obtain

$$\omega_1\omega_2 = |\Omega|^2 \sin 2\alpha' = |\Omega|^2 \sin 2\alpha .$$

For $\omega_1 < 0$, then, ω_2 is negative also. We note that since

$$\omega_1\omega_2 = -|\Omega|^2 \sin 2\alpha \qquad \omega_1 > 0$$

and

$$\omega_1\omega_2 = |\Omega|^2 \sin 2\alpha \qquad \omega_1 < 0$$

there are two ω poles for a fixed λ given by

$$\omega(\lambda) = -\bar{\omega}(\lambda) . \tag{A-41}$$

For both of these, ω_2 is negative which proves statement 3.

4. Proof of Statement 4.

From equation (A-1) and its complex conjugate we have

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\phi_1}{dr}\right] + \left[\frac{h\omega^2}{c_o^2} - A^2 - \frac{h(\lambda+1/2)(\lambda-1/2)}{r^2}\right]\phi_1 = 0 \tag{A-42}$$

and

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\bar{\phi}_1}{dr}\right] + \left[\frac{h\omega^2}{c_o^2} - A^2 - \frac{h(\bar{\lambda}+1/2)(\bar{\lambda}-1/2)}{r^2}\right]\bar{\phi}_1 = 0$$

respectively. In the same manner as before,

$$\frac{d}{dr}\left[r^2(\phi_1\bar{\phi}_1' - \bar{\phi}_1\phi_1')\right] = h\left[(\bar{\lambda}+1/2)(\bar{\lambda}-1/2) - (\lambda+1/2)(\lambda-1/2)\right]\phi_1\bar{\phi}_1 = 0 . \tag{A-43}$$

We again define

$$K^2 = \frac{\omega^2 - \omega_c^2}{c_o^2}$$

and

$$\nu^2 = h(\lambda+1/2)(\lambda-1/2) + \frac{1}{4}$$

and consider the three frequency ranges $|\omega| > \omega_c$, $\omega_c > |\omega| > \omega_b$ and $\omega_b > |\omega| > 0$ in order to ascertain the behavior of the λ-poles when ω is real.

a) For $|\omega| > \omega_c$, $K^2 > 0$.

When ω is positive, the asymptotic forms for $\phi_1(r)$ and $\bar{\phi}_1(r)$ are given by

$$\phi_1(r) \simeq \sqrt{\frac{2}{\pi K}}\frac{1}{r}\, e^{i\left[Kr - \nu\pi/2 - \pi/4\right]} \tag{A-44}$$

and

$$\bar{\phi}_1(r) \simeq \sqrt{\frac{2}{\pi K}}\,\frac{1}{r}\, e^{-i\left[Kr - \bar{\nu}\pi/2 - \pi/4\right]} .$$

Then

$$r^2(\phi_1\bar{\phi}_1' - \bar{\phi}_1\phi_1') \simeq -2iKr^2\left|\phi_1\right|^2 . \tag{A-45}$$

Upon integrating equation (A-43) and applying the condition (A-2) at $r = a$, we obtain,

$$r^2(\phi_1\bar{\phi}_1' - \bar{\phi}_1\phi_1') = h\left[(\bar{\lambda}+1/2)(\bar{\lambda}-1/2) - (\lambda+1/2)(\lambda-1/2)\right]\int_a^r \left|\phi_1(\tau)\right|^2 d\tau \tag{A-46}$$

or, by using equation (A-45),

$$-2iK\left|r\phi_1\right|^2 \simeq h\left[(\bar{\lambda}+1/2)(\bar{\lambda}-1/2) - (\lambda+1/2)(\lambda-1/2)\right]\int_a^r \left|\phi_1(\tau)\right|^2 d\tau. \tag{A-47}$$

With the definition $\lambda = \lambda_1 + i\lambda_2$, we obtain from equation (A-47)

$$2\lambda_1\lambda_2 \simeq \frac{Kr^2\left|\phi_1(r)\right|^2}{h\int_a^r \left|\phi_1(r)\right|^2 dr} . \tag{A-48}$$

Since $h > 0$ for $\omega > \omega_c$, the right hand side of equation (A-48) is positive so that λ must lie in either the first or third quandrants.

When $\omega < -\omega_c$, the same result may be established, since in this range of the frequency interval,

$$\phi(r) \simeq \sqrt{\frac{2}{\pi K}}\,\frac{1}{r}\, e^{-i\left[Kr + \nu\pi/2 - \pi/4\right]} \tag{A-49}$$

and

$$\bar{\phi}_1(r) \simeq \sqrt{\frac{2}{\pi K}}\,\frac{1}{r}\, e^{i\left[Kr + \bar{\nu}\pi/2 - \pi/4\right]}$$

where K denotes the positive square root

$$K = \frac{\sqrt{\omega^2 - \omega_c^2}}{c_o} .$$

b) For $\omega_c > |\omega| > \omega_b$, $K^2 < 0$, $h > 0$. We let $K^2 = -k^2$

The asymptotic form for $\phi_1(r)$ in this frequency interval is

$$\phi_1(r) \simeq \sqrt{\frac{2}{\pi K}}\, \frac{1}{r}\, e^{-kr - i(\nu\pi/2 + \pi/4)} \tag{A-50}$$

while

$$\bar{\phi}_1(r) \simeq \sqrt{\frac{2}{\pi K}}\, \frac{1}{r}\, e^{-kr + i(\bar{\nu}\pi/2 + \pi/4)} .$$

Hence,

$$r^2(\phi_1 \bar{\phi}_1' - \bar{\phi}_1 \phi_1') \underset{r \to \infty}{\simeq} 0 .$$

From equation (A-43), then,

$$h\left[(\bar{\lambda} + 1/2)(\bar{\lambda} - 1/2) - (\lambda + 1/2)(\lambda - 1/2)\right] \int_a^r |\phi_1|^2 \, d\tau \underset{r \to \infty}{\simeq} 0 . \tag{A-51}$$

The quantity $h \int_a^r |\phi_1|^2 d\tau$ is a positive real function of r, and so

$$\left[(\bar{\lambda} + 1/2)(\bar{\lambda} - 1/2) - (\lambda + 1/2)(\lambda - 1/2)\right] = -4i\lambda_1\lambda_2 \simeq 0 . \tag{A-52}$$

Thus, either $\lambda_1 = 0$ or $\lambda_2 = 0$ which means that the λ poles on the frequency interval $\omega_b < |\omega| < \omega_c$ are either real or imaginary.

c) For $\omega_b > |\omega|$, $K^2 < 0$, $h < 0$. Again let $K^2 = -k^2$.

The asymptotic forms for $\phi_1(r)$ and $\bar{\phi}_1(r)$ still go exponentially to zero as e^{-kr}. Thus,

$$\left[(\bar{\lambda} + 1/2)(\bar{\lambda} - 1/2) - (\lambda + 1/2)(\lambda - 1/2)\right] = -4i\lambda_1\lambda_2 \simeq 0 \tag{A-53}$$

which again means that the λ poles must be either real or pure imaginary.

For the two gravity cases discussed, then, the analytic properties of the solution $\phi_1(r)$ are obviously different. In the limit of constant gravity, the ω (or s) integration of equation (58) must take into consideration the fact that $\omega = \pm \sqrt{(\lambda + 1/2)(\lambda - 1/2)} \frac{\omega_b}{\lambda}$ and $\omega = \pm \omega_c$ are branch points. Also, for this case, in deforming the contour in the λ plane to obtain the alternate representation for $G(\underline{r}, \underline{r}_o, \omega)$ in equation (76), we must be aware that there may exist a set of real λ poles. This latter, however, is a mathematical rather than physical objection and the problem may be satisfactorily treated by assuming that the case $g = g_o$ is actually a simplification of the more realistic situation where $g = g_o a^2/r^2$.

APPENDIX B

THE DETERMINATION OF THE REGGE POLES FOR THE CONSTANT GRAVITY CASE

In this Appendix the solutions of equation (66) are considered which satisfy the radiation condition and the boundary condition

$$\frac{d\phi_1}{dr} + A\phi_1 = 0 \quad \text{at} \quad r = a. \tag{B-1}$$

The analysis assumes that $g = g_o$ so that

$$h\omega^2 = \omega^2 - \omega_b^2 \qquad \text{and} \qquad A = (1 - \frac{\gamma}{2}) \frac{g_o}{c_o^2}$$

are constant. The first step is to introduce

$$\phi_1(r) = \frac{\mu_1(r)}{r}. \tag{B-2}$$

so that equation (66) becomes

$$\frac{d^2\mu_1}{dr^2} + \left[K^2 - \frac{h(\lambda + 1/2)(\lambda - 1/2)}{r^2}\right] \mu_1 = 0. \tag{B-3}$$

As was mentioned in the section on the gravity wave there is one solution which does not have a turning point on the interval $a \leq r \leq \infty$. This solution, which corresponds to a bound state, has already been discussed in some detail. Here, we shall be concerned with the solutions that do have a turning point on (a, ∞) to determine their properties more precisely. We assume, then, that there exists a point

$$r_{TP} = \left[\frac{h(\lambda + 1/2)(\lambda - 1/2)}{K^2}\right]^{1/2} \tag{B-4}$$

on the interval and examine the solutions to equation (B-3) for the three frequency ranges $|\omega| < \omega_b$, $\omega_b < |\omega| < \omega_c$ and $|\omega| > \omega_c$.

For this type of analysis, the theory of R. E. Langer[26] is preferable to the WKB method, since the solutions will be valid in the neighborhood of the turning point as well as in the asymptotic region. In addition, Langer's method has the advantage of presenting the solutions which connect across the turning point. What we will seek, then, is a function which satisfies the radiation condition above the turning point so that we can choose the appropriate solution in the region below. As it will develop, the solutions in the lower region will correspond to a combination of incoming and outgoing waves. The physical interpretation of the turning point, then, is that at the height r_{TP} given in equation (B-4), some radiation is transmitted while the rest is reflected.

One point to bear in mind in this regard is that as r_{TP} becomes more remote from the earth's surface it enters a region where our simplified formulation does not apply. At a height of 100 Km, for example, the atmospheric density is smaller than at the earth's surface by a factor of e^{-14} or so. In this rarefied medium, the effects of viscosity become more pronounced than they are at lower altitudes and will thus produce a wave which is more attenuated than the solution we shall derive. The degree of attenuation, of course, depends upon the given wavelength and upon the value of r_{TP}. The fact that we have not included the mechanism of viscosity into our formulation will yield a set of solutions whose rate of damping is not determined. It is more convenient, instead to evaluate r_{TP} and determine whether or not it is in a region where the attenuation effects will be significant.

1) When $|\omega| < \omega_b$, K_Γ^2 and h are negative

We then let

$$K_\Gamma^2 = -k^2 \qquad \text{and} \qquad h = -H$$

so that equation (B-3) becomes

$$\frac{d^2\mu}{dr^2} - \left[k^2 - \frac{H(\lambda + 1/2)(\lambda - 1/2)}{r^2}\right] \mu = 0 \,. \tag{B-5}$$

Upon introducing the notation

$$\xi = kr$$

$$\nu^2 = H(\lambda + 1/2)(\lambda - 1/2) + \frac{1}{4} \simeq H(\lambda + 1/2)(\lambda - 1/2)$$

and

$$y = \frac{\xi}{\nu}$$

we obtain, from equation (B-5)

$$\frac{d^2\mu_1}{dy^2} + (i\nu)^2\left[1 - \frac{1}{y^2}\right] \mu_1 = 0 \,. \tag{B-6}$$

We now define the functions

$$\eta(y) = i\nu \int_1^y \left[1 - \frac{1}{y^2}\right]^{1/2} dy \tag{B-7}$$

and

$$\psi(y) = \frac{\left\{\int_1^y \left[1 - \frac{1}{y^2}\right]^{1/2}\right\}^{1/6}}{\left[1 - \frac{1}{y^2}\right]^{1/4}} \,. \tag{B-8}$$

As y becomes very large, $\eta(y) \simeq ikr$. The two solutions to equation (B-6) which apply when $|y| > 1$ are

$$\mu_1(y) \simeq -e^{i\pi/3}\psi(y)\,\eta^{1/3}\,H^{(1)}_{1/3}(\eta) \simeq -\sqrt{\frac{2}{\pi}}\,e^{-i\pi/12}\psi(y)\,e^{i\eta} \tag{B-9}$$

and

$$\mu_2(y) = e^{-i\pi/3}\psi(y)\,\eta^{1/3}\,H^{(2)}_{1/3}(\eta) \simeq \sqrt{\frac{2}{\pi}}\,e^{i\pi/12}\psi(y)\eta^{-1/6}\left[e^{-i\eta} + ie^{i\eta}\right]. \tag{B-10}$$

Of these two solutions (B-10) is not bounded and does not satisfy the radiation condition at large values of r. The function $\mu_1(y)$, on the other hand, satisfies both these criteria and is thus the appropriate choice.

The form for $\mu_1(y)$ which applies at $|y| < 1$ is shown by Langer to be

$$\mu_1(y) = \psi(y)\left[e^{i\pi/3}H^{(1)}_{1/3}(\eta\,e^{-2i\pi}) + H^{(2)}_{1/3}(\eta\,e^{-2i\pi})\right]. \tag{B-11}$$

The asymptotic form for this expression is

$$\mu_1(y) \simeq -\sqrt{\frac{2}{\pi}}\;e^{5i\pi/12}\,\psi(y)\,\eta^{-1/6}\left[e^{-i\eta} - i\,e^{i\eta}\right]. \tag{B-12}$$

This form applies when $|\eta|$ is sufficiently large which, in turn, from equation (B-7) depends upon the value of $|\nu|$'s being large.

The boundary condition which $\mu_1(y)$ must satisfy is found from equations (B-1) and (B-2) to be

$$\frac{d\mu_1(y)}{dy}\,\frac{dy}{dr} + (A - \frac{1}{a})\mu_1 = 0 \qquad \text{at } r = a\,. \tag{B-13}$$

Here, as before, $A >> \frac{1}{a}$. From equations (B-12) and (B-13), then, the λ poles are obtained from the expression

$$i\left[\nu^2 - k^2a^2\right]^{1/2}\left[e^{-i\eta} + ie^{i\eta}\right] + Aa\left[e^{-i\eta} - ie^{i\eta}\right] \simeq 0\,. \tag{B-14}$$

The value of η is determined from equation (B-7) to be

$$\eta = -\nu\left\{\sqrt{1-y^2} - \log\left[\frac{1+\sqrt{1-y^2}}{y}\right]\right\} \simeq -\nu\left\{\frac{\sqrt{\nu^2-k^2a^2}}{\nu} - \log\left[\frac{\nu+\sqrt{\nu^2-k^2a^2}}{ka}\right]\right\}. \qquad \text{(B-15)}$$

a) The first few poles can be found by defining $\epsilon << 1$ such that

$$\nu = ka(1+\epsilon). \qquad \text{(B-16)}$$

Substitution of this value into equation (B-15) yields

$$\eta \simeq \frac{1}{3}(ka)(2\epsilon)^{3/2} \qquad \text{(B-17)}$$

Since ϵ is a small number, equation (B-14) is satisfied approximately by the condition

$$e^{-i\eta} - ie^{i\eta} \simeq 0. \qquad \text{(B-18)}$$

Hence,

$$\eta \simeq (4m-1)\frac{\pi}{4}. \qquad \text{(B-19)}$$

Upon combining equations (B-16), (B-17), and (B-19) we obtain

$$\nu_m \simeq ka + \frac{(ka)^{1/3}}{2} t_m \qquad \text{(B-20)}$$

where

$$t_m = \left[\frac{3}{4}(4m-1)\pi\right]^{2/3}. \qquad \text{(B-21)}$$

From the definition $\nu^2 = H(\lambda+1/2)(\lambda-1/2) + \frac{1}{4}$, the value of λ_m is given approximately by

$$\lambda_m = \sqrt{\frac{\omega_c^2-\omega^2}{\omega_b^2-\omega^2}}\ \frac{a\omega}{c_o}\left[\frac{1+(ka)^{-2/3}}{2}t_m\right]. \qquad \text{(B-22)}$$

b) For large values of $\nu(|\nu| >> ka)$, equation (B-15) shows that

$$\eta = -\nu\left[1 - \log\frac{2\nu}{ka}\right]. \qquad \text{(B-23)}$$

Equation (B-14) is then satisfied approximately when

$$\left[e^{-i\eta} + ie^{i\eta}\right] \simeq 0 . \qquad \text{(B-24)}$$

Hence,

$$\eta \simeq (m + 1/4)\pi . \qquad \text{(B-25)}$$

We now let $\nu = \rho e^{i\phi}$. From equations (B-23), (B-24), and (B-25) we obtain

$$-\rho\left\{\cos\phi\left[1 - \log\frac{2\rho}{ka}\right] + \phi\sin\phi\right\}$$

$$-i\rho\left\{\sin\phi\left[1 - \log\frac{2\rho}{ka}\right] - \phi\cos\phi\right\} \simeq (m + 1/4)\pi. \qquad \text{(B-26)}$$

The imaginary term vanishes identically when $\phi = 0$. Therefore, equation (B-26) is satisfied by

$$\rho_m\left[\log\frac{2\rho_m}{ka} - 1\right] \simeq (m + 1/4)\pi \qquad \text{(B-27)}$$

or

$$\rho_m \simeq \frac{(m + 1/4)\pi}{\log\left[\frac{2(m + 1/4)\pi}{ka}\right]} . \qquad \text{(B-28)}$$

Hence, for large values of m,

$$\lambda_m \simeq \frac{1}{\sqrt{H}}\nu_m = \frac{\rho_m}{\sqrt{\omega_b^2 - \omega^2}}\,\omega . \qquad \text{(B-29)}$$

We now note from equations (B-22) and (B-29) that all λ_m are real quantities which agree with the statement 4 of Appendix A.

The value of r_{TP} for the lower m values is determined from equations (B-4) and (B-22) to be

$$r_{TP}^2 \simeq a^2\left[1+\frac{(ka)^{-2/3}}{2} t_m\right]^2 . \qquad \text{(B-30)}$$

For the first (m = 1) mode, for example, $t_1 = 3.75$ so that

$$r_{TP}^{(1)} \simeq a\left\{1+1.88\left[\frac{\left(\omega_c^2-\omega^2\right)}{c_o^2} a^2\right]^{-1/3}\right\} . \qquad \text{(B-31)}$$

At $\omega \simeq 0$, $r_{TP}^{(1)}$ occurs at an altitude of 190 Km, as a substitution of numerical values from Table II shows. Similarly, near $\omega = \omega_b$, $r_{TP}^{(1)}$ is a point approximately 362 Km above the earth's surface. The atmospheric densities at these two heights are less than their ground level counterparts by factors of $e^{-26.6}$ and $e^{-50.4}$ respectively.

It is evident, then, that even for the first mode, the turning point is in a very rarefied medium. The effect of viscosity, which we have neglected is actually a significant factor, particularly near the higher end of the interval $|\omega| < \omega_b$. For higher modes (m > 1) the turning point is even more remote from the earth's surface so that the viscous damping effect becomes still more pronounced.

The essence of these remarks, then, is that although equations (B-22) and (B-29) appear to yield a set of modes which will propagate with no attenuation, this is a result of our simplified analysis. We have established that the damping is actually quite prominent, and for this reason we will not be concerned with the contribution to the wave function of these idealized modes.

2. When $\omega_b < |\omega| < \omega_c$, $K^2 < 0$, $h > 0$, and we let $k^2 = -K^2$.

Thus, equation (B-3) becomes

$$\frac{d^2\mu}{dr^2} - \left[k^2 + \frac{h(\lambda + 1/2)(\lambda - 1/2)}{r^2}\right] \mu = 0. \tag{B-32}$$

We now introduce

$$\xi = kr$$

$$\nu^2 = h(\lambda + 1/2)(\lambda - 1/2) + \frac{1}{4} \simeq h(\lambda + 1/2)(\lambda - 1/2)$$

$$z = \frac{ikr}{\nu}$$

which simplifies equation (B-32) to

$$\frac{d^2\mu}{dz^2} + \nu^2\left[1 - \frac{1}{z^2}\right] \mu = 0 . \tag{B-33}$$

With

$$\eta = \nu \int_1^z \left[1 - \frac{1}{z^2}\right]^{1/2} dz \tag{B-34}$$

and

$$\psi(z) = \frac{\left\{\int_1^z (1 - \frac{1}{z^2})^{1/2}\right\}^{1/6}}{(1 - \frac{1}{z^2})^{1/4}} \tag{B-35}$$

the asymptotic form for the appropriate solution to equation (B-33) at $|z| < 1$

is given by

$$\mu(z) \simeq -\sqrt{\frac{2}{\pi}}\, e^{-5i\pi/12}\, \psi(z)\, \eta^{-1/6} \left[e^{i\eta} + ie^{-i\eta}\right] . \tag{B-36}$$

By the same method we used in the previous section, the λ-poles are determined

from the expression

$$\left[\nu^2 + k^2a^2\right]^{1/2}\left[e^{i\eta} - ie^{-i\eta}\right] - Aa\left[e^{i\eta} + ie^{-i\eta}\right] \simeq 0 \,. \tag{B-37}$$

The first few poles are given by

$$\lambda_m = i\left\{\sqrt{\frac{\omega_c^2 - \omega^2}{\omega_b^2 - \omega^2}}\ \frac{a\omega}{c_o}\left[1 + \frac{(ka)^{-2/3}}{2}\ t_m\right]\right\} \tag{B-38}$$

where, again, $t_m = \left[\frac{3}{4}(4m-1)\pi\right]^{2/3}$. (B-39)

Similarly, the values of λ_m for large m are found to be

$$\lambda_m = \frac{i\rho_m}{\sqrt{\omega_b^2 - \omega^2}}\ \omega \tag{B-40}$$

where $\rho_m = \dfrac{(m+1/4)\pi}{\log\left[\dfrac{2(m+1/4)\pi}{ka}\right]}$. B-41)

Hence, all λ poles on the interval $\omega_b < |\omega| < \omega_c$ are pure imaginary quantities. We use this feature in the asymptotic representation of the Green's function to show that these waves damp out extremely rapidly. For this reason, it is not necessary to make the turning point investigations that we had to make in the preceding part of this appendix.

3. When $|\omega| > \omega_c$, K^2 and h are both positive, and equation (B-3) may be used as it stands. We again introduce

$$\xi = Kr$$

$$\nu^2 = h(\lambda+1/2)(\lambda-1/2)+1/4 \simeq h(\lambda+1/2)(\lambda-1/2)$$

$$x = \xi/\nu$$

so that equation (B-3) becomes

$$\frac{d^2\mu}{dx^2} + \nu^2\left[1 - \frac{1}{x^2}\right]\mu = 0. \tag{B-42}$$

With

$$\eta(x) = \nu \int_1^x \left[1 - \frac{1}{x^2}\right]^{1/2} dx \tag{B-43}$$

and

$$\psi(x) = \frac{\left\{\int_1^x \left[1 - \frac{1}{x^2}\right]^{1/2} dx\right\}^{1/6}}{\left[1 - \frac{1}{x^2}\right]^{1/4}}, \tag{B-44}$$

the appropriate solution to equation (B-41) has the asymptotic form

$$\mu_1(x) \simeq -\sqrt{\frac{2}{\pi}}\, e^{-5i\pi/12}\, \psi(x)\, \eta^{-1/6} \left[e^{i\eta} + ie^{-i\eta}\right]. \tag{B-45}$$

The λ-poles are then obtained from the expression

$$\sqrt{\nu^2 - K^2 a^2}\left[e^{i\eta} - ie^{-i\eta}\right] - Aa\left[e^{i\eta} + ie^{-i\eta}\right] = 0. \tag{B-46}$$

Using the technique presented in the first part, λ_m for small m is given by

$$\lambda_m \simeq \sqrt{\frac{\omega^2 - \omega_c^2}{\omega^2 - \omega_b^2}}\, \frac{a\omega}{c_o} \left\{1 + \frac{|ka|^{-2/3}}{2}\, e^{i\pi/3}\, t_m\right\} \tag{B-47}$$

where

$$t_m = \left[\frac{3}{4}(4m-1)\pi\right]^{2/3}. \tag{B-48}$$

The λ poles for large m are determined, as in the first part by defining $\nu = \rho e^{i\phi}$. It develops by application of the same procedure that

$$\phi \simeq \frac{\pi}{2}\left[1 - \frac{1}{\log \frac{2\rho}{Ka}}\right] \tag{B-49}$$

and

$$\rho_m \simeq \frac{(m + 1/4)\pi}{\log\left[\frac{2(m+1/4)}{Ka}\pi\right]} . \tag{B-50}$$

Therefore,

$$\lambda_m \simeq \frac{\rho_m \omega}{\sqrt{\omega^2 - \omega_b^2}} \exp\left\{\frac{i\pi}{2}\left[1 - \frac{1}{\log \frac{2\rho_m}{Ka}}\right]\right\} . \tag{B-51}$$

All the λ poles for the frequency interval $|\omega| > \omega_c$ are thus complex quantities whose behavior is depicted graphically in Figure 4. The form of λ_m given by equations (B-47) and (B-51) is very suggestive of the creeping wave solutions obtained in electromagnetic diffraction problems. For this reason, we retain the same terminology.

Since these poles will yield waves which do damp out, it will not be necessary to make the quantitative investigation of r_{TP} that was performed in the first part. Instead, it is sufficient to note that equations (B-47) and (B-51) give good approximations in the absence of viscous considerations, but that as this mechanism is included, the degree of damping will be more severe than what we obtain in this idealized case.

APPENDIX C

THE ASYMPTOTIC DEVELOPMENT OF THE GREEN'S FUNCTION FOR THE HIGHER MODES OF OSCILLATION

In this Appendix the asymptotic expression for $G_m(\underline{a}, \underline{r}_o, \omega)$ which appears in equation (77) is derived for the case $m > 1$. The development is similar to the one for $G_o(\underline{a}, \underline{r}_o, \omega)$ except that the residues are not explicitly evaluated here.

The λ-poles for the first few m-modes as determined in Appendix B may be summarized by the expression

$$\lambda_m \simeq \sqrt{\frac{\omega^2 - \omega_c^2}{\omega^2 - \omega_b^2}}\, \frac{a\omega}{c_o} \left[1 + \frac{1}{2}(Ka)^{-2/3} e^{i\pi/3}\, t_m\right] \tag{C-1}$$

where

$$t_m = \left[\frac{3}{4}(4m - 1)\pi\right]^{2/3}. \tag{C-2}$$

We have already shown that λ_m is real for $0 < |\omega| < \omega_b$, imaginary when $\omega_b < |\omega| < \omega_c$ and complex for $|\omega| > \omega_c$.

Beginning with equation (90),

$$G_m(\underline{a}, \underline{r}_o, \omega) = \frac{-\lambda_m}{2}\, \frac{P_{\lambda_m - 1/2}(-\cos\theta)}{\sin\pi(\lambda_m - 1/2)\pi}\, \frac{h}{a^2}\, \left.\frac{\phi_1(r_o)}{\frac{\partial}{\partial\lambda}(\phi_1' + A\phi_1)}\right|_{\substack{\lambda=\lambda_m \\ r=a}} \tag{C-3}$$

We note, from Magnus and Oberhettinger[18], that the asymptotic form for $P_{\lambda_m - 1/2}(-\cos\theta)$ is given by

$$P_{\lambda_m - 1/2}(-\cos\theta) \approx \frac{2}{\sqrt{\pi}}\, \frac{\Gamma(\lambda_m + 1/2)}{\Gamma(\lambda_m + 1)}\, \frac{\cos\left[\lambda_m(\pi - \theta) - \frac{\pi}{4}\right]}{\sqrt{2\sin\theta}} \tag{C-4}$$

provided that $\epsilon \leq \theta \leq \pi - \epsilon$, $\epsilon > 0$ and $\left|\lambda_m - 1/2\right| >> \frac{1}{\epsilon}$. Also, since the λ_m

are assumed to lie in the first quadrant, there is the convergent expansion

$$\frac{1}{2}\,\frac{1}{\sin(\lambda_m - 1/2)\pi} = \frac{i}{e^{i(\lambda_m - 1/2)\pi} - ie^{-i(\lambda_m - 1/2)\pi}} = -e^{i\lambda_m \pi} \sum_{n=0}^{\infty} e^{i(\lambda_m - 1/2)2n\pi} . \tag{C-5}$$

Then

$$\frac{\lambda_m}{2}\,\frac{P_{\lambda_m - 1/2}(-\cos\theta)}{\sin(\lambda_m - 1/2)\pi} = -\lambda_m e^{i\lambda_m \pi}\,\frac{\Gamma(\lambda_m + 1/2)}{\Gamma(\lambda_m + 1)}\ \times$$

$$\frac{1}{\sqrt{2\pi\sin\theta}}\left\{ e^{i\left[\lambda_m(\pi-\theta) - \frac{\pi}{4}\right]} + e^{-i\left[\lambda_m(\pi-\theta) - \frac{\pi}{4}\right]}\right\} \sum_{n=0}^{\infty} e^{i(\lambda_m - 1/2)2n\pi}$$

$$= -\lambda_m \frac{\Gamma(\lambda_m + 1/2)}{\Gamma(\lambda_m + 1)}\,\frac{1}{\sqrt{2\pi\sin\theta}}\left\{ e^{i\left[\lambda_m\theta + \frac{\pi}{4}\right]} + e^{i\left[(\lambda_m(2\pi-\theta) - \frac{\pi}{4}\right]}\right\} \sum_{n=0}^{\infty} e^{i(\lambda_m - \frac{1}{2})2n\pi} . \tag{C-6}$$

As in the gravity wave case, the first term in the brackets corresponds to the directly received portion of the signal while the second represents the antipodal reflection. Terms in the sum indicate waves that have made n complete circulations around the earth. If we focus our attention on the directly received wave, we have finally, that

$$G_m(\underline{a}, \underline{r}_0, \omega) = +\lambda_m \frac{\Gamma(\lambda_m + 1/2)}{\Gamma(\lambda_m + 1)}\,\frac{e^{i\left[\lambda_m\theta + \frac{\pi}{4}\right]}}{\sqrt{2\pi\sin\theta}}\,\frac{h}{a^2}\,\frac{\phi_1(r_0)}{\frac{\partial}{\partial\lambda}(\phi_1' + A\phi_1)}\Bigg|_{\substack{\lambda=\lambda_m \\ r=a}} . \tag{C-7}$$

The values of the residue at $\lambda = \lambda_m$ can be evaluated explicitly by substituting the appropriate forms for $\phi_1(r)$ from Appendix B. This, however, will not be done here since we will not be concerned with the actual contribution to $G(\underline{a}, \underline{r}_0, \omega)$ of these higher modes.

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