ELECTROMAGNETISM IN MOVING, CONDUCTING MEDIA

by

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I

PRELIMINARY DISCUSSION

1.1 Introduction

There have been several papers written in recent years on the subject of moving media, most of which deal with lossless media. Nag and Sayied (1956) applied Minkowski's theory of the electrodynamics of moving bodies to the phenomenon of Cerenkov radiation, by considering the problem of a static charge in a moving medium. Sayied (1958) later extended this to the two-medium problem of a charge imbedded in a channel of moving dielectric. Wave-motion in moving media has been discussed by Collier and Tai (1964 and 1965). The more involved problem of harmonic current source has received appreciable attention, notably from Compton and Tai (1964 and 1965), Lee and Papas (1964 and 1965), Tai (1965a and 1965b), and Daly, Lee, and Papas (1965). Unlike the present work, the concern there was with lossless media.

The formulation of field problems involving charge and current distributions as sources in a moving, conducting medium is delicate, and raises certain questions which have not been clearly settled up to now; Pyati (1966) notes this in his thesis. One of the questions raised regards the formulation of Ohm's law for moving media, for which two different forms exist in the literature. Another concerns the relaxation phenomenon and its expression in moving media. In order to discuss the fields set up by charge distributions moving in a medium it becomes necessary to either set up an initial-value ballistic problem, where at a given instant of time the charges have a given velocity, mass, and location, or to postulate impressed currents and charges which maintain their velocity by some unspecified energy source. In order to adequately treat the first problem one should consider the reaction forces and collisions and find the resulting velocity as a function of time. This is an extremely difficult approach to use. The second approach is used in altered form in antenna problems and, in fact, most problems involving the calculation of fields due to a particular source configuration. The second method will be employed here, treating the sources as stationary, and imbedded in a uniformly moving medium. The medium is assumed to have constant permeability, permittivity and conductivity.

In the first chapter Maxwell's equations for moving media are reviewed, and cast in dyadic form. The second chapter is devoted to the formulation of Ohm's law, with a discussion on the two apparently different forms which exist in the literature. Chapter III treats the decomposition of charges and currents into source and response terms, thus making it possible to rigorously approach problems in which sources are present. The relationship of the response charges to the sources is derived. Finally, Chapter IV is devoted to the development of the vector and scalar potentials and their differential equations. Green's functions are found in closed form, allowing the complete solution of field problems in moving, conducting media. Throughout the work, attention is focussed on two classes of problems: the first involves stationary charge distributions, and the second treats harmonic, stationary current distributions. At no time is any low-velocity approximation used; that is, the results are valid for relativistic velocities. Furthermore, it is not necessary to limit the values of conductivity to either low or high values.

1.2 Maxwell's Equations for Moving Media

For the sake of completeness we shall now develop the constitutive relations for an isotropic, linear medium in motion and introduce the dyadic symbolism convenient to discussion of the theory. Minkowski's powerful theory will be used throughout this work, as it provides an elegant framework for the discussion of electrodynamics.

As is well known, Minkowski postulated as his starting point that Maxwell's equations in their indefinite form are to be treated as physical laws, and as such have the same form in any coordinate system in uniform motion relative to the medium, in accord with the postulates of special relativity. The terminology "indefinite" and "definite" forms of Maxwell's equations was explained by Tai (1964). Maxwell's equations in their indefinite form are:

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \text{ (I)}, \quad \nabla \cdot \bar{D} = \rho \text{ (II)},$$

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J} \text{ (III)}, \quad \nabla \cdot \bar{B} = 0 \text{ (IV)}.$$

These along with the constitutive relations comprise the definite form. Denoting the coordinate system of the medium by primes (i.e., that coordinate system with respect to which the medium is stationary), we remark again that the above equations hold for primed quantities; in addition, for linear, isotropic media the following constitutive relations hold:

$$\bar{B}' = \mu' \bar{H}' , \quad \bar{D}' = \epsilon' \bar{E}' . \qquad (1.1)$$

The corresponding constitutive relations in any other system of reference which is moving with respect to the medium are not as simple. To find them, it is first necessary to know the relations between the field quantities of the two reference frames.

In particular, let us choose for the unprimed system one which moves in the negative z-direction with a constant velocity v. This we may do with no loss of generality. The medium then moves with velocity v in the positive z-direction relative to the unprimed, or "stationary", system. The transformation of electric field, for example, is given by

$$\bar{E}'_z = \bar{E}_z = (\bar{E} + \bar{v} \times \bar{B})_z , \quad \bar{E}'_{x,y} = \gamma(\bar{E} + \bar{v} \times \bar{B})_{x,y} , \qquad (1.2a)$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$, and c is the speed of light in vacuo. The development of the transformation of the field quantities is discussed by Sommerfeld (1952), Section 34; the results will be used here. The above transformation relation may be written in dyadic symbolism as

$$\bar{E}' = \bar{\bar{\gamma}} \cdot (\bar{E} + \bar{v} \times \bar{B}) , \qquad (1.2b)$$

where the elements of the dyadic $\bar{\bar{\gamma}}$ are given by the array

$$\begin{bmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

The other field quantities transform in a similar manner:

$$\bar{D}' = \bar{\bar{\gamma}} \cdot (\bar{D} + \frac{1}{c^2} \bar{v} \times \bar{H}) ,$$

$$\bar{B}' = \bar{\bar{\gamma}} \cdot (\bar{B} + \frac{1}{c^2} \bar{v} \times \bar{E}) ,$$

and

$$\bar{H}' = \bar{\bar{\gamma}} \cdot (\bar{H} - \bar{v} \times \bar{D}) . \qquad (1.3)$$

Substituting these relations into the constitutive relations above, we get

$$\bar{D} + \frac{1}{c^2} \bar{v} \times \bar{H} = \epsilon'(\bar{E} + \bar{v} \times \bar{B})$$

and

$$\bar{B} - \frac{1}{c^2} \bar{v} \times \bar{E} = \mu'(\bar{H} - \bar{v} \times \bar{D}) . \qquad (1.4)$$

Combining these eliminates one field quantity. Thus eliminating $\bar{B}$ allows $\bar{D}$ to be expressed in terms of $\bar{E}$ and $\bar{H}$, and eliminating $\bar{D}$ gives $\bar{B}$ in terms of $\bar{E}$ and $\bar{H}$ (Tai, (1965b)):

$$\bar{D} = \epsilon' \bar{\bar{\alpha}} \cdot \bar{E} + \bar{\Omega} \times \bar{H} ,$$

and (1.5)

$$\bar{B} = \mu' \bar{\bar{\alpha}} \cdot \bar{H} - \bar{\Omega} \times \bar{E} ,$$

where

$$\bar{\Omega} = \frac{(n^2 - 1)}{1 - n^2\beta^2} \frac{\bar{v}}{c^2} ,$$

$$\beta = v/c$$

$$n^2 = c^2 \mu' \epsilon' = \mu' \epsilon' / \mu_o \epsilon_o ,$$

and the elements of the dyadic $\bar{\bar{\alpha}}$ are given by

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

where

$$a = \frac{1 - \beta^2}{1 - n^2\beta^2} ,$$

In the stationary system, then, $\bar{D}$ and $\bar{E}$, $\bar{B}$ and $\bar{H}$ no longer are related uniquely as in the case of stationary media. If, in addition, $\bar{J}$ is a known independent function or is related to the field quantities in a known manner, the indefinite form of Maxwell's equations along with the constitutive relations comprise the definite form of Maxwell's equations.

II

OHM'S LAW

2.1 The Forms of Ohm's Law

Ohm's Law for moving media appears in two different forms in the literature: one is isotropic, given by Weyl (1922), p. 195:

$$\bar{J}_c^{(1)} = \gamma\,\sigma'\,\bar{E}^* \quad , \tag{2.1}$$

where the superscript "(1)" indicates the first form of $\bar{J}_c$, the conduction current density, σ' denotes the rest-frame conductivity,

and

$$\bar{E}^* = \bar{E} + \bar{v} \times \bar{B} \quad .$$

The other form is anisotropic and is the one most widely used in the literature (see especially Sommerfeld (1952), p. 283, and Cullwick (1959) p. 92):

$$\bar{J}_c^{(2)} = \frac{\sigma'}{\gamma}\,\bar{\bar{\gamma}} \cdot \bar{\bar{\gamma}} \cdot \bar{E}^* \quad . \tag{2.2}$$

The difference between them,

$$\bar{J}_c^{(1)} - \bar{J}_c^{(2)} = \sigma'\gamma\,(1 - \gamma^{-2})\bar{E}_z = \beta^2\sigma'\,\gamma\,\bar{E}_z \quad , \tag{2.3}$$

is of the order of β^2, and is negligible for velocities significantly less than the speed of light c. It is important to know which, if either, is correct. Before we treat this question, it will be instructive to note how each form arises.

We first note the transformation relations between the current and charge densities, which arise from the Lorentz transformation of special relativity, and relate quantities in two systems in uniform relative motion (see Appendix A, Eq. (A.9)):

$$\left\{\begin{array}{ll} \bar{J}' = \gamma\, \bar{\bar{\gamma}}^{-1} \cdot (\bar{J} - \rho\bar{v}) \\ \rho' = \gamma(\rho - \dfrac{\bar{v}\cdot\bar{J}}{c^2}) \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{ll} \bar{J} = \gamma\, \bar{\bar{\gamma}}^{-1} \cdot (\bar{J}' + \rho'\bar{v}) \\ \rho = \gamma(\rho' + \dfrac{\bar{v}\cdot\bar{J}'}{c^2}) \end{array}\right\}, \qquad (2.4)$$

where the elements of the dyadic $\bar{\bar{\gamma}}^{-1}$ are given by

$$\begin{bmatrix} \gamma^{-1} & 0 & 0 \\ 0 & \gamma^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

The crux of the difference concerns the decomposition of current density into convection and conduction terms. Convection current is associated with free charge in motion, while conduction is associated with electric fields in conducting media. Both formulations of Ohm's law procede from the assumption that in the rest frame system of the medium (indicated by primed quantities) the current is all conduction:

$$\bar{J}' = \sigma' \bar{E}' = \bar{J}'_c . \qquad (2.5)$$

Weyl, on the one hand, uses the relation $\bar{J} = \gamma\, \bar{\bar{\gamma}}^{-1} \cdot (\bar{J}' + \rho'\bar{v})$ to show that

$$\bar{J} = \gamma\, \bar{\bar{\gamma}}^{-1} \cdot \sigma' \bar{E}' + \gamma\rho' \bar{v} = \sigma' \gamma \bar{E}^* + \gamma\rho' \bar{v} , \qquad (2.6)$$

since the transformation of electric field is given by (see Eq. (1.2))

$$\bar{E}' = \bar{\bar{\gamma}} \cdot \bar{E}^* \quad . \tag{2.7}$$

Weyl then calls that part which depends explicitly on the conductivity "conduction current density", denoted by the subscript c, and the remaining part " convection current density", denoted by the subscript v:

$$\bar{J}_c^{(1)} = \sigma' \gamma \bar{E}^* \quad , \quad \bar{J}_v^{(1)} = \gamma \rho' \bar{v} \quad . \tag{2.8}$$

Sommerfeld, on the other hand, uses the transformation relation $\bar{J}' = \gamma \bar{\bar{\gamma}}^{-1} \cdot (\bar{J} - \rho \bar{v})$ to show that

$$\bar{J} = \rho \bar{v} + \frac{\bar{\bar{\gamma}}}{\gamma} \cdot \bar{J}' = \rho \bar{v} + \frac{\sigma'}{\gamma} \bar{\bar{\gamma}} \cdot \bar{\bar{\gamma}} \cdot E^* \quad , \tag{2.9}$$

and calls the second term "conduction current density":

$$\bar{J}_c^{(2)} = \frac{\sigma'}{\gamma} \bar{\bar{\gamma}} \cdot \bar{\bar{\gamma}} \cdot \bar{E}^* \quad , \quad \bar{J}_v^{(2)} = \rho \bar{v} \quad . \tag{2.10}$$

The difference between the two charge densities which appear, in $\bar{J}_v^{(1)}$ and $\bar{J}_v^{(2)}$,

$$\rho - \gamma \rho' = \gamma \dot{\rho}' - \gamma \rho' + \sigma' \gamma \frac{\bar{v} \cdot \bar{E}'}{c^2} = \sigma' \gamma \frac{\bar{v} \cdot \bar{E}}{c^2} \quad , \tag{2.11}$$

is called the "apparent charge density" and arises from the relativistic transformations. In pre-relativistic electrodynamics a moving charge resulted in a current, but a moving current did not give rise to a charge. In relativistic electrodynamics this is not the case, but intuition is of little help in attaching a physical significance to the apparent charge density. Depending on whether it is assigned to the convection term or the conduction term, one or the other of the decompositions above is derived.

We shall show by elementary thermodynamical considerations that the heat loss expression can be derived independently of the form of Ohm's law used, and thus that either form is adequate. Further, the fields arising from charge distributions can also be equally well formulated in either form. While Schlomka (1950) uses an electron-theoretic model to conclude that $J_c^{(2)}$ is "correct", and Cullwick accepts his reasoning, we shall disagree with his argument and conclude that the two forms are interchangeable, and differ only in definitions of "convection" and "conduction" terms.

2.2 Formulation of Joule Heat

The rate at which heat is developed per unit volume is given in the rest frame of the medium by

$$\frac{dQ'}{dV'} = \bar{J}_c' \cdot \bar{E}' = \sigma' \left|\bar{E}'\right|^2 , \tag{2.12}$$

which can be expressed in the stationary (unprimed) system as

$$\frac{dQ'}{dV'} = \sigma' \left|\bar{\bar{\gamma}} \cdot \bar{E}^*\right|^2 \tag{2.13}$$

by use of Eq. (1.2).

A less direct method, but one which demonstrates the balance between stored, radiated, and heat energies, is given as follows: from Maxwell's equations in the rest-frame system of the medium,

$$\nabla' \times \bar{E}' = -\frac{\partial \bar{B}'}{\partial t'} \quad (\mathrm{I}') \quad , \quad \nabla' \cdot \bar{D}' = \rho' \quad (\mathrm{II}') \quad ,$$

$$\nabla' \times \bar{H}' = \frac{\partial \bar{D}'}{\partial t'} + \bar{J}_c' \quad (\mathrm{III}') \quad , \quad \nabla' \cdot \bar{B}' = 0 \quad (\mathrm{IV}') \quad ,$$

assuming no convection current; noting the vector identity

$$\nabla' \cdot (\bar{E}' \times \bar{H}') = \bar{H}' \cdot \nabla' \times \bar{E}' - \bar{E}' \cdot \nabla' \times \bar{H}' , \tag{2.14}$$

then

$$\bar{J}_c' \cdot \bar{E}' = \bar{E}' \cdot \nabla' \times \bar{H}' - \bar{E}' \cdot \frac{\partial \bar{D}'}{\partial t'} = \bar{H}' \cdot \nabla' \times \bar{E}' - \nabla' \cdot (\bar{E}' \times \bar{H}') - \bar{E}' \cdot \frac{\partial \bar{D}'}{\partial t'} ,$$

or

$$\bar{J}'_c \cdot \bar{E}' = -\nabla' \cdot (\bar{E}' \times \bar{H}') - \bar{E}' \cdot \frac{\partial \bar{D}'}{\partial t'} - \bar{H}' \cdot \frac{\partial \bar{B}'}{\partial t'} \quad . \qquad (2.15)$$

Similarly, from the Pauli-Lorentz form of Maxwell's equations in the stationary frame (see Pauli (1958), p. 101),

$$\nabla \times \bar{E}^* = -\frac{D\bar{B}}{Dt} \;(\mathrm{Ip}) \;, \quad \nabla \cdot \bar{D} = \rho \;\; (\mathrm{IIp}) \;,$$

$$\nabla \times \bar{H}^* = \frac{D\bar{D}}{Dt} + \bar{J}_c^{(2)} \;(\mathrm{IIIp}) \;, \quad \nabla \cdot \bar{B} = 0 \;\; (\mathrm{IVp}) \;,$$

where

$$\bar{J}_c^{(2)} = \frac{\sigma'}{\gamma} \gamma \cdot \gamma \cdot E^* \;,$$

$$\bar{H}^* = \bar{H} - \bar{v} \times \bar{D} \;, \qquad (2.16)$$

and

$$\frac{D\bar{A}}{Dt} = \frac{\partial \bar{A}}{\partial t} + \bar{v}(\nabla \cdot \bar{A}) - \nabla \times (\bar{v} \times \bar{A})$$

$$= \frac{\partial \bar{A}}{\partial t} + (\bar{v} \cdot \nabla) \bar{A} \qquad \text{for any vector } \bar{A} \text{, and constant } \bar{v} \;,$$

we obtain

$$\bar{J}_c^{(2)} \cdot \bar{E}^* = -\bar{E}^* \cdot \frac{D\bar{D}}{Dt} + \bar{H}^* \cdot \nabla \times \bar{E}^* - \nabla \cdot (\bar{E}^* \times \bar{H}^*) \;,$$

or

$$\bar{J}_c^{(2)} \cdot \bar{E}^* = -\nabla \cdot (\bar{E}^* \times \bar{H}^*) - \bar{E}^* \cdot \frac{D\bar{D}}{Dt} - \bar{H}^* \cdot \frac{D\bar{B}}{Dt} \quad . \qquad (2.17)$$

In the expressions for $\bar{J}_c^{(2)} \cdot \bar{E}^*$ and $\bar{J}'_c \cdot \bar{E}'$, the term on the left is related to the Joule heat loss per unit time, and the terms on the right are related to energy storage and radiation terms. The following relation holds for uniformly moving systems:

$$\nabla' \cdot (\bar{E}' \times \bar{H}') + \bar{E}' \cdot \frac{\partial \bar{D}'}{\partial t'} + \bar{H}' \cdot \frac{\partial \bar{B}'}{\partial t'} = \gamma \left[\nabla \cdot (\bar{E}^* \times \bar{H}^*) + \bar{E}^* \cdot \frac{D\bar{D}}{Dt} + \bar{H}^* \cdot \frac{D\bar{B}}{Dt} \right]. \tag{2.18}$$

This quantity is an invariant scalar; that is, among systems in uniform relative motion, it has the same value. To show this relation, we first note that from Appendix A, Eq. (A.7),

$$\nabla' = \frac{\bar{\bar{\gamma}} \cdot \nabla}{\gamma} + \frac{\gamma \bar{v}}{c^2} \frac{D}{Dt} \quad \text{and} \quad \frac{\partial}{\partial t'} = \gamma \frac{D}{Dt} , \tag{2.19}$$

and from (1.3)

$$\bar{D}' = \bar{\bar{\gamma}}^{-1} \cdot \bar{D} + \gamma \frac{\bar{v} \times \bar{H}^*}{c^2} , \quad \bar{E}' = \bar{\bar{\gamma}} \cdot \bar{E}^* ,$$

$$\bar{B}' = \bar{\bar{\gamma}}^{-1} \cdot \bar{B} - \gamma \frac{\bar{v} \times \bar{E}^*}{c^2} , \quad \bar{H}' = \bar{\bar{\gamma}} \cdot \bar{H}^* . \tag{2.20}$$

Then

$$\nabla' \cdot (\bar{E}' \times \bar{H}') = \gamma \left[\nabla \cdot (\bar{E}^* \times \bar{H}^*) + \frac{\gamma^2 \bar{v}}{c^2} \cdot \frac{D}{Dt} (\bar{E}^* \times \bar{H}^*) \right] ,$$

$$\bar{E}' \cdot \frac{\partial \bar{D}'}{\partial t'} = \gamma \left[\bar{E}^* \cdot \frac{D\bar{D}}{Dt} + \frac{\gamma^2}{c^2} \bar{E}^* \cdot \frac{D}{Dt} (\bar{v} \times \bar{H}^*) \right] ,$$

and

$$\bar{H}' \cdot \frac{\partial \bar{B}'}{\partial t'} = \gamma \left[\bar{H}^* \cdot \frac{D\bar{B}}{Dt} - \frac{\gamma^2}{c^2} \bar{H}^* \cdot \frac{D}{Dt} (\bar{v} \times \bar{E}^*) \right] . \tag{2.21}$$

Since

$$\bar{v} \cdot \frac{D}{Dt} (\bar{E}^* \times \bar{H}^*) = \bar{v} \cdot \frac{D\bar{E}^*}{Dt} \times \bar{H}^* + \bar{v} \cdot \bar{E}^* \times \frac{D\bar{H}^*}{Dt}$$

$$= \overline{H}^* \cdot \frac{D}{Dt}(\bar{v} \times \overline{E}^*) - \overline{E}^* \cdot \frac{D}{Dt}(\bar{v} \times \overline{H}^*) \quad , \tag{2.22}$$

the right hand terms of (2.21) cancel, and the desired relation (2.18) is obtained. Comparing relations (2.15) and (2.18), the total minus the stored and radiated energy densities is

$$\bar{J}_c^{(2)} \cdot \overline{E}^* = \frac{1}{\gamma} \bar{J}_c' \cdot \overline{E}' \quad , \tag{2.23}$$

or

$$\sigma' \left|\overline{E}'\right|^2 = \gamma \frac{\sigma'}{\gamma} (\overline{\overline{\gamma}} \cdot \overline{\overline{\gamma}} \cdot \overline{E}^*) \cdot \overline{E}^* = \sigma' \left|\overline{\overline{\gamma}} \cdot \overline{E}^*\right|^2 = \frac{dQ'}{dV'} \quad , \tag{2.24}$$

which agrees with the direct result (2.13) for the Joule heat rate per unit volume.

We must, of course, consider the same volume in each system, so that relative to the unprimed system the volume is moving, and in accordance with the results of special relativity, appears shortened, i.e.

$$dV' = \gamma \, dV \quad . \tag{2.25}$$

Borrowing on the results of relativistic thermo-dynamics (Møller (1952), p. 107), the heat developed per unit time transforms as follows:

$$dQ' = \gamma^2 dQ \quad , \tag{2.26}$$

so that the rate of heat, per unit volume seen from the stationary system is given by

$$\frac{dQ}{dV} = \frac{1}{\gamma} \frac{dQ'}{dV'} = \frac{\sigma'}{\gamma} \left|\overline{\overline{\gamma}} \cdot \overline{E}^*\right|^2 \quad , \tag{2.27}$$

or

$$\frac{dQ}{dV} = \bar{J}_c^{(1)} \cdot \frac{\bar{\bar{\gamma}} \cdot \bar{\bar{\gamma}} \cdot \bar{E}^*}{\gamma^2} = \bar{J}_c^{(2)} \cdot \bar{E}^* \quad . \tag{2.28}$$

Thus the Joule heat loss per unit volume per unit time can be readily expressed by either formulation of Ohm's law.

2.3 The Atomistic Model

Schlomka (1950) uses an atomistic, or electron-theoretic model, much like one described by Pauli (1958), p. 106, as a basis for claiming that $\bar{J}_c^{(2)}$ is the correct formulation of Ohm's law in moving media. His argument is briefly the following: conduction current is composed of a flow of electrons which travel on the average with some velocity $\bar{u}'$ relative to the medium, i.e.,

$$\bar{J}_c' = \rho_e' \, \bar{u}' \quad , \tag{2.29}$$

where ρ_e' is the charge density of the electrons ($\rho_e' < 0$). By conservation of charge

$$dq = \rho_e' \, dV' = \rho_e \, dV = \rho_e^o \, dV^o \quad , \tag{2.30}$$

where the superscript zero indicates that frame of reference with respect to which the charge is at rest, i.e. which has a velocity $\bar{u}'$ relative to the medium. Thus, using (2.25) and noting that here the relative velocities are u and u' rather than v,

$$\frac{dq}{dV^o} = \rho_e' \sqrt{1 - \left(\frac{u'}{c}\right)^2} = \rho_e \sqrt{1 - \left(\frac{u}{c}\right)^2} \quad . \tag{2.31}$$

The transformation of velocities is given by Møller (1952), p. 53. In the dyadic notation, they can be condensed to one vector equation:

$$\bar{u} = \frac{\bar{\bar{\gamma}}^{-1} \cdot (\bar{u}' + \bar{v})}{1 + \frac{\bar{u}' \cdot \bar{v}}{c^2}} \quad . \tag{2.32}$$

Now the conduction current in the unprimed system is given by the product of the charge density ρ_e and the relative velocity of the electrons and the medium $\bar{u} - \bar{v}$ as seen from the unprimed coordinate system, or

$$\bar{J}_c = \rho_e (\bar{u} - \bar{v}) \quad , \tag{2.33}$$

which by using the relations above, gives

$$\bar{J}_c = \frac{\bar{\bar{\gamma}} \cdot \bar{J}'_c}{\gamma} = \frac{\sigma'}{\gamma} \bar{\bar{\gamma}} \cdot \bar{\bar{\gamma}} \cdot \bar{E}^* = \bar{J}_c^{(2)} \quad , \tag{2.34}$$

which is the expression used by Sommerfeld.

There are two considerations which cast some doubt on the generality and validity of the reasoning. The first regards the concept of the relative velocity of two bodies as seen by a third (moving) observer. This is an intuitive carry-over from the Newtonian concept of addition of velocities. This being so, it is doubtful whether such an argument can be used in a situation where special relativity holds, to distinguish a second-order effect.

The second objection involves the phenomenological quality of Maxwell's and Minkowski's equations. The model of a cloud of electrons each traveling with a velocity u is an artificial one, especially since the possibility of fast conduction electrons is ignored, i.e. notions of Newtonian mechanics are again assumed. In view of these objections and the fact that the Joule heat has a unique and consistent expression in either formulation, the question is reduced to one of definition. Schlomka asserts that new formulas would have to be derived in the first formulation, a statement that is not born out by this work. In fact, it will prove more convenient for our purposes to use the first form when discussing problems where sources are present. This will be made clear in the next chapter.

III

SOURCE AND RESPONSE CHARGES AND CURRENTS

3.1 Decomposition of Charges and Currents

It is desirable to be able to treat problems that involve charge particles which obtain their velocities through a medium by other than electrical means. An example is the problem of a charged particle moving through a dielectric; Nag and Sayied (1956) treat this by considering a stationary charge in a moving dielectric. The charge is the source of the fields, and acts as a forcing function in Maxwell's equations. In treating conducting media a peculiar problem arises, that of the relaxation phenomenon: any charge placed in a conducting medium tends to disappear. If the charge is moving, the situation is more complicated. Suppose there is a convection current $\bar{J}'_v$ caused by charges moving through the medium in addition to the conduction current:

$$\bar{J}' = \bar{J}'_v + \sigma' \bar{E}' \quad . \tag{3.1}$$

Taking the divergence of (3.1) and using the relations $\nabla' \cdot \bar{D}' = \rho'$ and $\bar{D}' = \epsilon' \bar{E}'$ along with the equation of continuity, $\nabla' \cdot \bar{J}' + \partial\rho'/\partial t' = 0$, we get

$$\frac{\partial \rho'}{\partial t'} + \frac{\sigma'}{\epsilon'} \rho' = - \nabla' \cdot \bar{J}'_v \quad . \tag{3.2}$$

If we consider a constant charge moving along the z-axis with constant velocity u', and attempt to identify this charge with the total charge, i.e.

$$\bar{J}'_v = \rho' \bar{u}'$$

where

$$\rho' = \rho'(z' - u't') = \text{constant} \quad ,$$

substitution into (3.2) requires that $\rho' = 0$. Thus, we conclude that one cannot arbitrarily assume a given convection current that is compatible with the relaxation condition. This leads us to separate the total charge density ρ' into a source term ρ'_s and a response term ρ'_r, and identify the source term with the moving charge:

$$\rho' = \rho'_s + \rho'_r \quad \text{and} \quad \bar{J}'_v = \rho'_s \bar{u}' \ ;$$

then ρ'_r can be found by requiring that it be consistent with (3.2). Thus (3.2) becomes

$$\frac{\partial \rho'_r}{\partial t'} + \frac{\sigma'}{\epsilon'} \rho'_r = - \nabla' \cdot \left[\rho'_s (z' - u't')\bar{u}'\right] - \frac{\partial \rho'_s}{\partial t'} - \frac{\sigma'}{\epsilon'} \rho'_s$$

$$= - \frac{\sigma'}{\epsilon'} \rho'_s \ , \qquad (3.3)$$

since

$$\nabla' \cdot (\rho'_s \bar{u}') = \bar{u}' \cdot \nabla' \rho'_s = - \frac{\partial \rho'_s (z' - u't')}{\partial t'} \ .$$

Similarly if a current source such as an antenna is placed in a moving medium and considered as an independent forcing function, the total current in the primed system is comprised of conduction current and the source current as seen from the rest frame of the medium:

$$\bar{J}' = \bar{J}'_s + \sigma' \bar{E}' \qquad (3.4)$$

The problem that presents itself is the expression of charge and current densities in the unprimed system.

In this work we shall usually define the stationary or unprimed system as that coordinate system which transforms the source to rest. At this point it is not necessary to restrict consideration only to harmonic current sources, although later discussions will have that limitation. There are two classes of problems that will be dealt with in this work: "static" charge sources and

[illegible]

harmonic current sources in conducting, moving media. We shall now discuss the decomposition of currents and charges in the stationary system.

3.1.1 Case A: Charge Sources

First we will suspend the restriction that the stationary system be that with respect to which the charges are at rest, in order to show the generality of the formulation. Consider a set of charges moving through a conducting medium with constant velocity $\bar{u}'$ relative to the medium, the motion being maintained by an unspecified mechanical force. Suppose the medium moves with velocity $\bar{v} = v\hat{z}$ relative to the stationary coordinate frame. An observer in the stationary frame sees the charge moving with velocity $\bar{u}$, where $\bar{u}$ and $\bar{u}'$ are uniquely related. This relation involves the relativistic addition of velocities, given by Eq. (2.32):

$$\bar{u}' = \frac{\bar{\bar{\gamma}}^{-1} \cdot (\bar{u} - \bar{v})}{1 - \frac{\bar{u} \cdot \bar{v}}{c^2}} , \quad \text{or } \bar{u} = \frac{\bar{\bar{\gamma}}^{-1} \cdot (\bar{u}' + \bar{v})}{1 + \frac{\bar{u}' \cdot \bar{v}}{c^2}} . \tag{3.5}$$

The source charge densities are related by

$$\rho_s \sqrt{1 - \left(\frac{u}{c}\right)^2} = \rho_s' \sqrt{1 - \left(\frac{u'}{c}\right)^2} = \rho_s'' , \tag{3.6}$$

where the double prime indicates that coordinate system which transforms the charge to rest. This relation follows from the principle of the invariance of charge:

$$dq_s = \rho_s dV = \rho_s' dV' = \rho_s'' dV'' \tag{3.7}$$

and

$$\frac{dV}{\sqrt{1 - \left(\frac{u}{c}\right)^2}} = \frac{dV'}{\sqrt{1 - \left(\frac{u'}{c}\right)^2}} = dV'' , \tag{3.8}$$

(see Møller (1952), p. 45), which combine to give the above relation of charge densities.

As stated above, in the rest frame of the medium the total current consists only of conduction current $\bar{J}'_c = \sigma' \bar{E}'$ and convection current $\bar{J}'_v = \rho'_s \bar{u}'$ due to the motion of the source charge:

$$\bar{J}' = \rho'_s \bar{u}' + \sigma' \bar{E}' \quad , \quad \rho' = \rho'_s + \rho'_r \quad . \tag{3.9}$$

In the stationary system, we add $\rho_s \bar{u}$ to the current density expression of the Ohm's law discussion which consists of conduction current and convection current due to the motion of the medium:

$$\bar{J} = \rho_s \bar{u} + \bar{J}_c + \bar{J}_v \; , \quad \rho = \rho_s + \rho_r \quad . \tag{3.10}$$

It will now be shown that the quantity $\bar{J}_c + \bar{J}_v$ does not explicitly depend on ρ_s, and the decomposition into conduction and convection is similar to that previously discussed in Section 2.1.

The transformation law for current density is given by Eq. (A.9) of Appendix A:

$$\bar{J} = \gamma \bar{\bar{\gamma}}^{-1} \cdot (\bar{J}' + \rho' \bar{v}) \; , \tag{3.11}$$

so that

$$\bar{J}_c + \bar{J}_v = \bar{J} - \rho_s \bar{u} = \gamma \bar{\bar{\gamma}}^{-1} \cdot \bar{J}' + \gamma \rho' \bar{v} - \rho_s \bar{u}$$

$$= \rho'_s \gamma \bar{\bar{\gamma}}^{-1} \cdot \bar{u}' + \gamma \sigma' \bar{\bar{\gamma}}^{-1} \cdot \bar{E}' + \gamma \rho'_s \bar{v} + \gamma \rho'_r \bar{v} - \rho_s \bar{u} \; . \tag{3.12}$$

Since $\bar{E}' = \bar{\bar{\gamma}} \cdot \bar{E}^*$, and using Eq. (3.5),

$$\bar{J}_c + \bar{J}_v = \rho'_s \gamma \bar{\bar{\gamma}}^{-1} \cdot (\bar{u}' + \bar{v}) - \rho_s \bar{u} + \gamma \sigma' \bar{E}^* + \gamma \rho'_r \bar{v}$$

$$= \rho_s \left[\frac{\sqrt{1 - \left(\frac{u}{c}\right)^2}}{\sqrt{1 - \left(\frac{u'}{c}\right)^2}} \gamma \bar{\bar{\gamma}}^{-1} : (\bar{u}' + \bar{v}) - \bar{u} \right] + \gamma \sigma' \bar{E}^* + \gamma \rho'_r \bar{v} \quad . \tag{3.13}$$

From Eq. (3.5), it is a matter of simple vector algebra to show that the following identity holds:

$$\gamma \left(1 - \frac{\bar{v} \cdot \bar{u}}{c^2}\right) = \sqrt{\frac{1 - \left(\frac{u}{c}\right)^2}{1 - \left(\frac{u'}{c}\right)^2}} \quad . \tag{3.14}$$

Using this relation along with Eq. (3.5), and substituting them into Eq. (3.13), the bracketed term vanishes, leaving

$$\bar{J}_c + \bar{J}_v = \sigma' \gamma \bar{E}^* + \gamma \rho'_r \bar{v} \quad . \tag{3.15}$$

Here as in the sourceless case, we are free to decompose the convection and conduction terms in two ways:

$$\bar{J}_c^{(1)} = \sigma' \gamma \bar{E}^* \quad , \quad \bar{J}_v^{(1)} = \gamma \rho'_r \bar{v} \quad ,$$

or

$$\bar{J}_c^{(2)} = \sigma' \frac{\bar{\bar{\gamma}} \cdot \bar{\bar{\gamma}}}{\gamma} \cdot \bar{E}^* \quad , \quad \bar{J}_v^{(2)} = \rho_r \bar{v} \quad . \tag{3.16}$$

We will generally use the first form,

$$\bar{J} = \rho_s \bar{u} + \sigma' \gamma \bar{E}^* + \gamma \rho'_r \bar{v} \quad . \tag{3.17}$$

In the case where the charge is at rest in the stationary system, $\bar{u} = 0$, leaving

$$\bar{J} = \sigma' \gamma \bar{E}^* + \gamma \rho'_r \bar{v} \quad . \tag{3.18}$$

Similarly the response charge density ρ_r is related to ρ'_r in a manner similar to Eq. (2.11):

$$\rho_r = \rho - \rho_s = \gamma(\rho' + \frac{\bar{v} \cdot \bar{J}'}{c^2}) - \rho_s = \gamma\rho'_s + \gamma\rho'_r + \gamma\rho'_s \frac{\bar{v} \cdot \bar{u}}{c^2} + \sigma' \gamma \frac{\bar{v} \cdot \bar{E}'}{c^2} - \rho_s$$

$$= \rho'_s \left[\gamma(1 + \frac{\bar{u} \cdot \bar{v}}{c^2}) - \frac{\rho_s}{\rho'_s} \right] + \sigma' \gamma \frac{\bar{v} \cdot \bar{E}}{c^2} + \gamma\rho'_s \quad . \tag{3.19}$$

Substituting Eqs. (3.6) and (3.14) into (3.19) the bracketed term vanishes. Thus we can write the charge density in the stationary frame as

$$\rho = \rho_s + \rho_r = \rho_s + \gamma\rho'_r + \sigma' \gamma \frac{\bar{v} \cdot \bar{E}}{c^2} \quad . \tag{3.20}$$

Equations (3.18) and (3.20) constitute the desired current-charge expressions in the stationary system. Later on in Section 3.2.1, the relationship between the source and response terms will be derived. There it will be shown that $\gamma \rho'_r$ satisfies a first-order partial differential equation, with ρ_s as the forcing function.

3.1.2 Case B: Current Sources

Instead of a convection current $\rho_s \bar{u}$ there is here an impressed current density $\bar{J}_s$ in this class of problems, having an associated charge ρ_s. In the primed system the impressed current density moves, so that a convection term appears in the transformation (Eq. (3.11)):

$$\bar{J}'_s = \gamma \bar{\bar{\gamma}}^{-1} \cdot (\bar{J}_s - \rho_s \bar{v}) \; ; \tag{3.21}$$

also, from the transformation law (Eq. (2.4)),

$$\rho'_s = \gamma\left(\rho_s - \frac{\bar{v}\cdot\bar{J}_s}{c^2}\right) . \qquad (3.22)$$

These quantities, being independent source quantities, do not depend on the parameters of the medium. Equation (3.9) becomes

$$\bar{J}' = \gamma \bar{\bar{\gamma}}^{-1} \cdot (\bar{J}_s - \rho_s \bar{v}) + \sigma' \bar{E}' . \qquad (3.23)$$

Using the transformation relations, we have also

$$\bar{J}' = \gamma \bar{\bar{\gamma}}^{-1} \cdot (\bar{J} - \rho\bar{v}) = \gamma \bar{\bar{\gamma}}^{-1} \cdot \bar{J} - \gamma\rho_s\bar{v} - \gamma\rho_r\bar{v} . \qquad (3.24)$$

Equating these two expressions yields the decomposition of the second form

$$\bar{J} = \bar{J}_s + \rho_r\bar{v} + \sigma' \frac{\bar{\bar{\gamma}} \cdot \bar{\bar{\gamma}}}{\gamma} \cdot \bar{E}^*$$

$$= \bar{J}_s + \bar{J}_v^{(2)} + \bar{J}_c^{(2)} ,$$

or, equivalently, in the first form,

$$\bar{J} = \bar{J}_s + \bar{J}_v^{(1)} + \bar{J}_c^{(1)} = \bar{J}_s + \gamma\rho'_r \bar{v} + \gamma\sigma' \bar{E}^* . \qquad (3.25)$$

As before, the charge density decompositions are given by

$$\rho = \rho_s + \rho_r = \rho_s + \gamma\rho'_r + \gamma\sigma' \frac{\bar{v}\cdot\bar{E}}{c^2} . \qquad (3.26)$$

The relationship of $\gamma\rho'_r$ to ρ_s is discussed in Section 3.2.2.

3.2 Relationship of Response Charge Density to Source Charge and Current Densities

In this section we shall develop the differential equations for the response charge density $\gamma\rho'_r$ in terms of the source terms in the unprimed system. While the resulting expression for $\gamma\rho'_r$ involves only charge and current terms, it does not appear that this is the case for ρ_r, since a term appears involving the electric field intensity $\bar{E}$. Thus the first form of Eq. (3.16), describing the decomposition of current density, will be used here. There is an inherent difficulty in assuming time-independence in the first case; this will be discussed below.

3.2.1 Case A: Charge Sources

We know that the assumption of time-independence in the primed system for a stationary charge is inconsistent with the relaxation phenomenon, since the latter dictates an exponential decay in time. It is not so clear what happens when the charge is moving. An expression will be derived for both the assumption of time-independence in the unprimed system and for the general time-dependent case. Naturally the assumption of time-independence greatly simplifies Maxwell's equations, and the calculation of fields is more manageable. The unprimed system here transforms the source charge to rest.

A. Time-independent Solution

With the condition $\partial/\partial t = 0$, Maxwell's equations for a conducting medium moving with velocity $\bar{v}$ become:

$$\nabla \times \bar{E} = 0 \text{ (Ia)}, \quad \nabla \cdot \bar{D} = \rho \text{ (IIa)}, \quad \nabla \times \bar{H} = \bar{J} \text{ (IIIa)}, \quad \nabla \cdot \bar{B} = 0 \text{ (IVa)},$$

where

$$\bar{D} = \epsilon' \bar{\bar{\alpha}} \cdot \bar{E} + \bar{\Omega} \times \bar{H} ,$$

$$\bar{B} = \mu' \bar{\bar{\alpha}} \cdot \bar{H} - \bar{\Omega} \times \bar{E} ,$$

$$\bar{J} = \sigma(\bar{E} + \bar{v} \times \bar{B}) + \gamma \rho'_r \bar{v} ,$$

$$\sigma = \gamma\sigma' ,$$

and

$$\rho = \rho_s + \gamma\rho_r' + \sigma \frac{\bar{v} \cdot \bar{E}}{c^2} \quad . \tag{3.27}$$

In order to express everything in terms of $\bar{E}$ and $\bar{H}$, the term $\bar{E} + \bar{v} \times \bar{B}$ must be changed as follows: from the constitutive relation for $\bar{B}$,

$$\begin{aligned} \bar{v} \times \bar{B} &= \mu' \, \bar{v} \times (\bar{\bar{\alpha}} \cdot \bar{H}) - \bar{v} \times (\bar{\Omega} \times \bar{E}) \\ &= \mu' \, a\bar{v} \times \bar{H} - \bar{\Omega}(\bar{v} \cdot \bar{E}) + (\bar{v} \cdot \bar{\Omega})\bar{E} \quad , \end{aligned} \tag{3.28}$$

where it is noted that $\bar{v} \times (H_z \hat{z}) = 0$ and the well-known vector identity

$$\bar{a} \times (\bar{b} \times \bar{c}) = \bar{b}(\bar{a} \cdot \bar{c}) - \bar{c}(\bar{a} \cdot \bar{b}) \quad .$$

is used. Thus

$$\begin{aligned} \bar{E} + \bar{v} \times \bar{B} &= (1 + v\Omega)\bar{E} - \bar{\Omega}(\bar{v} \cdot \bar{E}) + \mu' a \bar{v} \times \bar{H} \\ &= a \bar{E}_T + \bar{E}_z + \mu' a \bar{v} \times \bar{H} \\ &= \bar{\bar{\alpha}} \cdot \bar{E} + \mu' a \, \bar{v} \times \bar{H} \quad , \end{aligned} \tag{3.29}$$

where the subscripts T and z represent the transverse and longitudinal components, respectively, and the identity $1 + \Omega v = a$ follows in a straightforward manner from the definitions (see Eq. (1.5)).

Expanding $\nabla \cdot \bar{D} = \rho$ gives, from the constitutive relation for $\bar{D}$,

$$\nabla \cdot \bar{D} = \epsilon' \, \nabla \cdot \bar{\bar{\alpha}} \cdot \bar{E} + \nabla \cdot \bar{\Omega} \times \bar{H} = \epsilon' \nabla \cdot \bar{\bar{\alpha}} \cdot \bar{E} - \bar{\Omega} \cdot \nabla \times \bar{H} \, , \tag{3.30}$$

using the identity $\nabla \cdot (\bar{a} \times \bar{b}) = \bar{b} \cdot \nabla \times \bar{a} - \bar{a} \cdot \nabla \times \bar{b}$ and noting that $\bar{\Omega}$ is a constant. Now

$$\bar{\Omega} \cdot \nabla \times \bar{H} = \bar{\Omega} \cdot \bar{J} = \sigma \bar{\Omega} \cdot (\bar{E} + \bar{v} \times \bar{B}) + \gamma \rho_r' v \Omega$$

$$= \sigma \bar{\Omega} \cdot \bar{E} + \gamma \rho_r' v \Omega , \tag{3.31}$$

or

$$\nabla \cdot \bar{D} = \epsilon' \nabla \cdot \bar{\bar{\alpha}} \cdot \bar{E} - \sigma \bar{\Omega} \cdot \bar{E} - v \Omega \gamma \rho_r' = \rho = \rho_s + \gamma \rho_r' + \sigma \frac{\bar{v} \cdot \bar{E}}{c^2} . \tag{3.32}$$

Again noting that $1 + v\Omega = a$, and using the relation $\bar{\Omega} + \frac{\bar{v}}{c^2} = \mu' \epsilon' a \bar{v}$,

$$\epsilon' \nabla \cdot \bar{\bar{\alpha}} \cdot \bar{E} - \sigma \mu' \epsilon' a \bar{v} \cdot \bar{E} = \rho_s + a \gamma \rho_r' . \tag{3.33}$$

Similarly from (IIIa) it follows that $\nabla \cdot \bar{J} = 0$, so that using the constitutive relation for $\bar{J}$ in (3.27) and noting Eq. (3.29) above,

$$\nabla \cdot \bar{J} = \sigma \nabla \cdot \bar{\bar{\alpha}} \cdot \bar{E} + \sigma \mu' a \nabla \cdot \bar{v} \times \bar{H} + \bar{v} \cdot \nabla \gamma \rho_r' = 0 . \tag{3.34}$$

Similarly to Eq. (3.31),

$$-\nabla \cdot (\bar{v} \times \bar{H}) = \bar{v} \cdot \nabla \times \bar{H} = \bar{v} \cdot \bar{J} = \sigma \bar{v} \cdot \bar{E} + v^2 \gamma \rho_r' , \tag{3.35}$$

so that (3.34) becomes

$$\sigma \nabla \cdot \bar{\bar{\alpha}} \cdot \bar{E} - \sigma^2 \mu' a \bar{v} \cdot \bar{E} = -\bar{v} \cdot \nabla \gamma \rho_r' + \sigma \mu' a v^2 \gamma \rho_r' . \tag{3.36}$$

Comparison of Eqs. (3.32) and (3.36) eliminates all terms involving $\bar{E}$, leaving the desired differential equation

$$\left(\bar{v} \cdot \nabla + \frac{\sigma}{\epsilon' \gamma^2} \right) \gamma \rho_r' = - \frac{\sigma}{\epsilon'} \rho_s . \tag{3.37}$$

This has the form

$$\left(\frac{\partial}{\partial z} + b\right) u(\bar{R}) = - U_o(\bar{R}) \quad , \tag{3.38}$$

where for convenience we have substituted the auxiliary quantities

$$R = x\hat{x} + y\hat{y} + z\hat{z} \quad ,$$

$$b = \sigma/\epsilon'\gamma^2 v \quad ,$$

$$u(\bar{R}) = \gamma\rho'_r \quad , \tag{3.39}$$

$$U_o(\bar{R}) = \sigma\rho_s/\epsilon' v \, ,$$

and

$\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors.

This equation will now be solved by the method of Fourier transforms.

Let the Fourier transform in z of a function $f(z)$ be defined by

$$F\{f\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ihz} f(z)\, dz \tag{3.40}$$

where it is assumed that $\int_{-\infty}^{\infty} |f(z)|^2 dz$ is bounded, that is, $f(z)$ is L^2 integrable in $(-\infty, \infty)$. Then we know that the integral $\int_{-\infty}^{\infty} F\{f\}\, dh$ converges to $f(z)$ wherever $f(z)$ is continuous (Morse and Feshbach (1953), p. 458). We first note that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ihz} \frac{\partial f}{\partial z} dz = \frac{1}{2\pi} e^{ihz} f \Big|_{-\infty}^{\infty} - \frac{ih}{2\pi}\int_{-\infty}^{\infty} e^{ihz} f(z) dz$$

$$= -ih F\{f\} \, , \tag{3.41}$$

since $f(\pm z)$ must vanish as z approaches infinity for $f(z)$ in the class L^2. Multiplying Eq. (3.38) by $e^{ihz}/2\pi$ and integrating from $-\infty$ to $+\infty$ gives

$$(-ih + b)\, F\left\{u\right\} = -F\left\{U_0\right\} ,$$

or

$$F\left\{u\right\} = \frac{-i F\left\{U_0\right\}}{h + ib} . \qquad (3.42)$$

Taking the inverse transform by multiplying by e^{-ihz} and integrating over h from $-\infty$ to ∞ gives, at points where $u(z)$ is continuous,

$$u(z) = -i \int_{-\infty}^{\infty} \frac{e^{-ihz} F\left\{U_0\right\}}{h + ib}\, dh . \qquad (3.43)$$

We now make use of a theorem related to the convolution integral, and described in Morse and Feshbach (1953), p. 465, which states, for $g_1(z)$ and $g_2(z)$ L^2 integrable in $(-\infty, \infty)$:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} g_1(\zeta)\, g_2(z-\zeta)\, d\zeta = \int_{-\infty}^{\infty} F\left\{g_1\right\} F\left\{g_2\right\} e^{ihz}\, dh . \qquad (3.44)$$

Letting $g_1(z) = U_0(z)$ and $g_2(z) = -i \int_{-\infty}^{\infty} \frac{e^{-ihz}}{h + ib}\, dh$, , we note that

that $F\left\{g_2\right\} = \frac{-i}{h + ib}$, and that

$$u(z) = \int_{-\infty}^{\infty} e^{-ihz} F\left\{U_0\right\} F\left\{g_2\right\} dh = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_2(z-\zeta)\, U_0(\zeta)\, d\zeta . \qquad (3.45)$$

In order to evaluate $g_2(z)$ we use the technique of contour integration. Referring to Fig. 3-1, it is noted that for $z < 0$, the exponential e^{-ihz}

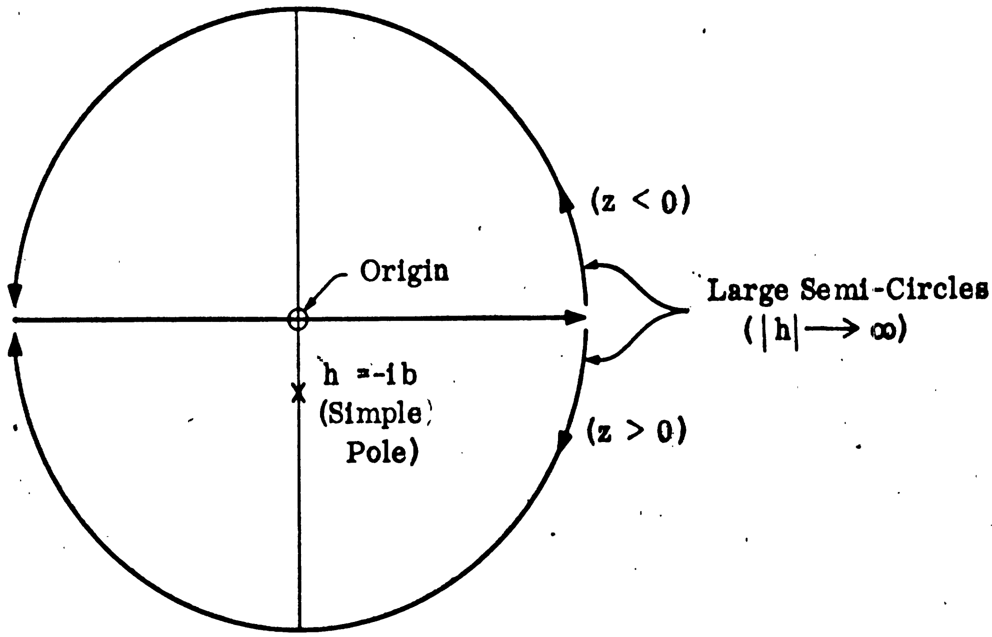

FIG. 3-1: CONTOURS IN THE h-PLANE FOR EVALUATING $g_2(z)$.

approaches zero uniformly in the upper half plane on the semi-circle as the radius approaches infinity. Thus the contribution along the semi-circle contour to the integral is negligible, and from the theory of residues,

$$g_2(z) = -i\int_{-\infty}^{\infty} \frac{e^{-ihz}}{h+ib}\,dh = -i\int_{\text{Semi-Circle}} + 2\pi i \sum \text{Residue} = 0, \; z < 0 , \tag{3.46}$$

since e^{-ihz} has no finite poles, and $(h + ib)^{-1}$ has only one pole, not enclosed by the contour. For $z > 0$, the contour can be closed in the lower half-plane. Then the contribution to the integral along the infinite semi-circle is again zero, and the residue at $h = -ib$ is merely ie^{-bz}, giving

$$g_2(z) = -2\pi e^{-bz}, \quad z > 0 \quad . \tag{3.47}$$

Thus, combining (3.46) and (3.47),

$$g_2(z) = \left\{ \begin{array}{ll} 0, & z < 0 \\ -2\pi e^{-bz}, & z > 0 \end{array} \right\}, \tag{3.48}$$

so that from Eq. (3.45)

$$u(z) = -\int_{-\infty}^{z} e^{-b(z-\zeta)} U_o(\zeta)\, d\zeta \quad , \tag{3.49}$$

or

$$u(z) = -\int_{0}^{\infty} e^{-b\zeta} U_o(z-\zeta)\, d\zeta \quad . \tag{3.50}$$

Written in the original terminology we can now state that the differential Eq. (3.37) has the solution

$$\gamma \rho_r'(z) = -\frac{\sigma}{\epsilon' v} \int_{0}^{\infty} e^{-\frac{\sigma\zeta}{\epsilon'\gamma^2 v}} \rho_s(z-\zeta)\, d\zeta \quad . \tag{3.51}$$

In the important case where ρ_s is a point charge at the origin,

$$\rho_s = q\,\delta(x)\,\delta(y)\,\delta(z) \quad , \tag{3.52}$$

where $\delta(x)$ has the properties that $\delta(x) = 0$ for $x \neq 0$,

$\int_a^b \delta(x)\,dx = 1$ for $a < 0 < b$, and $\int_a^b f(x)\delta(x)\,dx = f(0)$ for $a < 0 < b$.

Then here

$$\gamma \rho_r'(z) = \begin{Bmatrix} 0\,, & & z < 0 \\ -\frac{\sigma}{\epsilon' v}\, q e^{-\frac{\sigma z}{\epsilon' \gamma^2 v}}\, \delta(x)\,\delta(y) & , & z > 0 \end{Bmatrix} . \qquad (3.53)$$

B. Time-dependent Solution

The more general case of time-dependent fields due to a source charge distribution follows the same line of reasoning, but care must be taken in properly defining the transient behavior of the source. We shall assume here that no fields or sources exist before time t_o, and that the source charge density is constant after that; i.e. the source charge density is a step function in time:

$$\rho_s(\overline{R}, t) = \rho_o(\overline{R})\; S_o(t - t_o)\,, \quad t_o > 0\,, \qquad (3.54)$$

where $S_o(t)$ is the unit step function, defined by

$$S_o(t) = \begin{Bmatrix} 0\,, & t < 0 \\ 1\,, & t > 0 \end{Bmatrix} . \qquad (3.55)$$

It is not defined at the origin, but does not behave as $\delta(t)$ in that

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} S_o(t)\, dt = 0. \qquad (3.56)$$

It is first necessary to establish the partial differential equation for the response charge; from the general form of Maxwell's equations we have

$$\nabla \cdot \bar{D} = \rho \text{ (II)}, \quad \nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J} \text{ (III)},$$

where the constitutive relations are given by

$$\bar{D} = \epsilon' \bar{\bar{\alpha}} \cdot \bar{E} + \bar{\Omega} \times \bar{H},$$

and

$$\bar{J} = \sigma \bar{E}^{*} + \gamma \rho_r' \bar{v}, \tag{3.57}$$

where

$$\rho = \rho_s + \gamma \rho_r' + \sigma \frac{\bar{v} \cdot \bar{E}}{c^2}.$$

We note here that only (II) and (III)? are necessary for finding the relations for ρ_r'. The same is true in the time-independent case.

Expanding (II) gives

$$\nabla \cdot \bar{D} = \epsilon' \nabla \cdot \bar{\bar{\alpha}} \cdot \bar{E} - \bar{\Omega} \cdot \nabla \times \bar{H} = \rho = \rho_s + \gamma \rho_r' + \sigma \frac{\bar{v} \cdot \bar{E}}{c^2}. \tag{3.58}$$

Using (III), and noting that

$$\bar{\Omega} \cdot \left(\frac{\partial \bar{D}}{\partial t} + \bar{J} \right) = \epsilon' \bar{\Omega} \cdot \frac{\partial \bar{E}}{\partial t} + \sigma \bar{\Omega} \cdot \bar{E} + \Omega v \gamma \rho_r',$$

$$1 + \Omega v = a,$$

and

$$\bar{\Omega} + \frac{\bar{v}}{c^2} = \frac{\bar{v}}{c^2} (n^2 a),$$

the following expression results:

$$\sigma \nabla \cdot \bar{\bar{\alpha}} \cdot \bar{E} - \sigma \bar{\Omega} \cdot \frac{\partial \bar{E}}{\partial t} - \sigma^2 \mu' a \bar{v} \cdot \bar{E} = \frac{\sigma}{\epsilon'} \rho_s + \frac{\sigma}{\epsilon'} a \gamma \rho_r'. \tag{3.59}$$

Similarly, taking the divergence of (III) and combining with (II) gives the continuity equation

$$\nabla \cdot \bar{J} = -\frac{\partial \rho'}{\partial t} , \tag{3.60}$$

or

$$\sigma \nabla \cdot \bar{\bar{\alpha}} \cdot \bar{E} + \nabla \cdot (\gamma \rho'_r \bar{v}) - \sigma \mu' a \bar{v} \cdot \nabla \times \bar{H} = -\frac{\partial \rho_s}{\partial t} - \frac{\partial \gamma \rho'_r}{\partial t} - \sigma \frac{\bar{v}}{c^2} \cdot \frac{\partial \bar{E}}{\partial t} . \tag{3.61}$$

Noting that

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J} ,$$

$$\bar{v} \cdot \left(\frac{\partial \bar{D}}{\partial t} + \bar{J} \right) = \epsilon' \bar{v} \cdot \frac{\partial \bar{E}}{\partial t} + \sigma \bar{v} \cdot \bar{E} + v^2 \gamma \rho'_r ,$$

$$\nabla \cdot (\gamma \rho'_r \bar{v}) = \bar{v} \cdot \nabla \gamma \rho'_r ,$$

$$\frac{\bar{v}}{c^2} (a n^2 - 1) = \bar{\Omega} ,$$

$$\mu' \epsilon' v^2 = \left(\frac{n^2}{c^2} \right) (\beta^2 c^2) = n^2 \beta^2 ,$$

Eq. (3.61) can be written

$$\sigma \nabla \cdot \bar{\bar{\alpha}} \cdot \bar{E} - \sigma \bar{\Omega} \cdot \frac{\partial \bar{E}}{\partial t} - \sigma^2 \mu' a \bar{v} \cdot \bar{E} = -\frac{\partial \rho_s}{\partial t} - \left(\bar{v} \cdot \nabla + \frac{\partial}{\partial t} - \frac{\sigma a \beta^2}{\epsilon'} \right) \gamma \rho''_r . \tag{3.62}$$

Subtracting (3.59) from (3.62) eliminates the field quantities, leaving the desired differential equation

$$\left(\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial z} + \frac{\sigma}{\epsilon'\gamma^2}\right)\gamma\rho_r' = -\left(\frac{\partial}{\partial t} + \frac{\sigma}{\epsilon'}\right)\rho_s \quad , \tag{3.63}$$

where it is noted that $1 - \beta^2 = 1/\gamma^2$ by definition. Comparison with Eq. (3.37) shows that the time dependency has merely introduced partial time derivations on both sides. This has the form

$$\left(\frac{1}{v}\,\frac{\partial}{\partial t} + \frac{\partial}{\partial z} + b\right) u(\bar{R}, t) = -\left(\frac{1}{v}\,\frac{\partial}{\partial t} + d\right) U_o(\bar{R}, t) \quad , \tag{3.64}$$

where the auxiliary quantities are given here by

$$b = \sigma/\epsilon' v\gamma^2 > 0 \quad ,$$

$$d = \sigma/\epsilon' v \quad > 0 \quad ,$$

$$u = \gamma\rho_r' \quad ,$$

and

$$U_o = \rho_s \quad . \tag{3.65}$$

We are interested in a restricted class of possible sources, namely those given by (3.54). Thus we let

$$U_o(\bar{R}, t) = u_o(\bar{R})\, S_o(t - t_o), \quad t_o > 0 \quad , \tag{3.66}$$

where $S_o(t)$ is given by (3.55). We assume that the response charge is zero for all $t < t_o$, and thus the initial conditions are

$$U_o(\bar{R}, 0) = 0 \tag{3.67}$$

and

$$u(\bar{R}, 0) = 0 \tag{3.68}$$

Since this is evidently an initial value problem, the technique of Laplace transforms will be used in the time domain. The Laplace transform of a function $f(t)$ is given by $L\{f\}$, where

$$L\{f\} = \int_0^\infty e^{-st} f(t)\,dt \quad , \tag{3.69}$$

and the integral is assumed to exist. Note that the derivative has the transform

$$L\left\{\frac{\partial f}{\partial t}\right\} = \int_0^\infty e^{-st}\frac{\partial f}{\partial t}\,dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\,L\{f\} \tag{3.70}$$

In order that the integral in (3.69) exist and (3.70) hold, it is sufficient that $f(t)$ be of exponential order; i.e. for $Re\{s\} > a_o$ for some positive constant a_o, $\left|e^{-st} f(t)\right| < M$ for all $t > 0$, where M is a positive constant independent of time, but which depends on a_o.

With the source charge having the behavior in Eq. (3.66), and with initial conditions (3.67) and (3.68), Eq. (3.64) can be written

$$\left(\frac{1}{v}\frac{\partial}{\partial t} + \frac{\partial}{\partial z} + b\right) u(\bar{R}, t) = -\left(\frac{1}{v}\frac{\partial}{\partial t} + d\right)\left(u_o(\bar{R}) S_o(t - t_o)\right) \quad , t > 0 \quad , \tag{3.71}$$

As in the time-independent case, the Fourier integral technique will be used here in the z - domain. Multiplying Eq. (3.71) by $e^{ihz}/2\pi$ and integrating over all z yields

$$\left(-ih + \frac{1}{v}\,\partial/\partial t + b\right) F\{u\} = -\left(\frac{1}{v}\frac{\partial}{\partial t} + d\right) F\{u_o\}\, S_o(t - t_o) \quad , \tag{3.72}$$

where $F\{u\}$ is the Fourier transform of u, given by Eq. (3.40).

Taking the Laplace transform gives, noting (3.70),

$$(-ihv + s + bv)\ L\left\{F\{u\}\right\} = -(s + dv)\ F\left\{u_o\right\}\ L\left\{\delta(t - t_o)\right\}$$

$$= \frac{-(s + dv)\ F\left\{u_o\right\} e^{-st_o}}{s} \qquad (3.73)$$

since u_o is independent of time; and since for C a constant,

$$\int_{t_o}^{\infty} e^{-st}\ C\ dt = \frac{C}{s}\ e^{-st_o}\ . \qquad (3.74)$$

Equation (3.73) can be written

$$L\left\{F\{u\}\right\} = \frac{-(s + dv)\ F\left\{u_o\right\}}{s(s - ihv + bv)}\ . \qquad (3.75)$$

The inverse Laplace transformation for a function $f(t)$ is given by

$$L^{-1}\ L\{f\} = \frac{1}{2\pi i}\int_{a_1 - i\infty}^{a_1 + i\infty} L\{f\}\ e^{st}\ ds = f(t),\ a_1 > a_o, \qquad (3.76)$$

for all points for which $f(t)$ is continuous. This inverse operation performed on (3.75) gives

$$F\{u\} = -\frac{F\{u_o\}}{2\pi i}\int_{a_1 - i\infty}^{a_1 + i\infty} \frac{e^{s(t - t_o)}(s + dv)\ ds}{s\left[s - (ih - b)v\right]}\ . \qquad (3.77)$$

The contour of integration is shown in Fig. 3-2; for $t < t_o$ the contour can

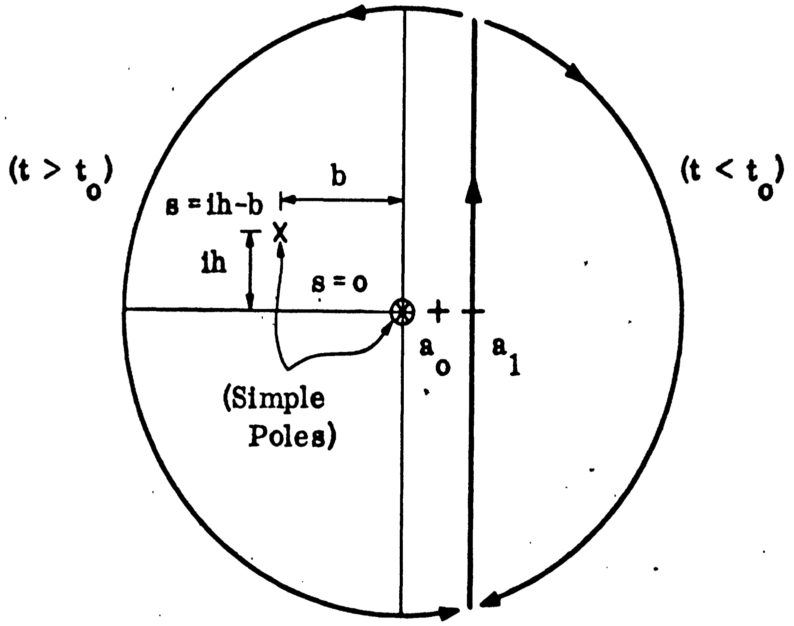

FIG. 3-2: CONTOURS IN THE s-PLANE FOR EVALUATING $F\{u\}$.

be closed in the right-half plane where there are no poles of the integrand. The integrand converges uniformly to zero along the semi-circle, so by Cauchy's theorem, the integral vanishes. For $t > t_o$, the contour is closed in the left-half plane, and the integrand converges uniformly to zero along the semi-circle. Applying the theory of residues to (3.77), and noting that the contour encloses poles at $s = 0$ and $s = ih - b$,

$$F\{u\} = - F\{u_o\} \cdot \sum (\text{Residues})$$

$$= - F\{u_o\} \cdot \left[\frac{dv}{(b - ih)v} - \frac{e^{(ih-b)v(t-t_o)}(ih - b + d)v}{(b - ih)v} \right]$$

$$= - i\, F\{u_o\} \cdot \left[\frac{d - e^{(ih-b)v(t-t_o)}(ih - b + d)}{h + ib} \right] . \qquad (3.78)$$

Let

$$g_2(z) = -i \int_{-\infty}^{\infty} \frac{e^{-ihz}\left[d - e^{(ih - b)v(t - t_o)}(ih - b + d)\right] dh}{h + ib} ; \qquad (3.79)$$

by inspection, the Fourier transform of $g_2(z)$ is

$$F\left\{g_2\right\} = -i \left[\frac{d - e^{(ih - b)v(t - t_o)}(ih - b + d)}{h + ib}\right] . \qquad (3.80)$$

The function $g_2(z)$ can be evaluated by the same method as was used in Eq. (3.45), by first splitting $g_2(z)$ into two parts: i.e., let

$$g_2(z) = g_3(z) + g_4(z) ,$$

where

$$g_3(z) = -id \int_{-\infty}^{\infty} \frac{e^{-ihz}}{h + ib} dh$$

and

$$g_4(z) = ie^{-bv(t - t_o)} \int_{-\infty}^{\infty} \frac{e^{-ih(z - vt + vt_o)}(ih - b + d)}{h + ib} . \qquad (3.81)$$

For $g_3(z)$, the contour can be closed above for $z < 0$, and below for $z > 0$, yielding

$$g_3(z) = -2\pi d\ e^{-bz}\ S_o(z) . \qquad (3.82)$$

Similarly for $g_4(z)$, the contour can be closed above for $z < v(t - t_o)$ and below for $z > v(t - t_o)$, giving

$$g_4(z) = 2\pi d\ e^{-bz}\ S_o\left[z - b(t - t_o)\right] \tag{3.83}$$

Only the pole at $h = -ib$ contributes to the integral.

Adding g_3 and g_4 together, we obtain the following expression for g_2;

$$g_2(z) = \left\{ \begin{array}{l} 0\ ,\ z < 0 \\ -2\pi d e^{-bz},\ 0 < z < v(t - t_o) \\ 0,\ z > v(t - t_o) \end{array} \right\} ,$$

or

$$g_2(z) = -2\pi d\ e^{-bz}\ S_o(z)\ S_o(t - t_o - z/v)\ . \tag{3.84}$$

Multiplying (3.78) by e^{-ihz} and integrating h along the real axis yields the inverse Fourier transform

$$u(z, t) = \int_{-\infty}^{\infty} F\left\{u_o\right\} \left[\frac{-i(d - (ih - b + d)\, e^{(ih - b)\, v(t - t_o)})}{h + ib}\right] dh\ . \tag{3.85}$$

We can again apply the theorem (3.44) by substituting u_o for g_1, and noting (3.80). This gives

$$u(z, t) = -id\left[\int_{z - v(t - t_o)}^{z} e^{-b(z - \zeta)}\ u_o(\zeta)\, d\zeta\right] S_o(t - t_o)\ ,$$

or

$$u(z, t) = -id\left[\int_0^{v(t - t_o)} e^{-b\zeta}\ u_o(z - \zeta)\, d\zeta\right] S_o(t - t_o)\ . \tag{3.86}$$

In the physical terminology, this is

$$\gamma \rho_r'(x, y, z, t) = -\frac{\sigma}{\epsilon' v}\left[\int_0^{v(t-t_o)} e^{-\frac{\sigma \zeta}{\epsilon' \gamma^2 v}} \rho_s(x, y, z-\zeta)\, d\zeta\right] S_o(t-t_o) \quad . \tag{3.87}$$

For a point charge, where $\rho_s = q\,\delta(x)\,\delta(y)\,\delta(z)\,S_o(t)$, this becomes

$$\gamma \rho_r' = -\frac{\sigma q}{\epsilon' v}\,\delta(x)\,\delta(y)\, e^{-\frac{\sigma z}{\epsilon' \gamma^2 v}} S_o(z)\, S_o(t - z/v)\, S_o(t) \quad . \tag{3.88}$$

The z-dependent part of this is plotted in Fig. 3-3. The response charge den-

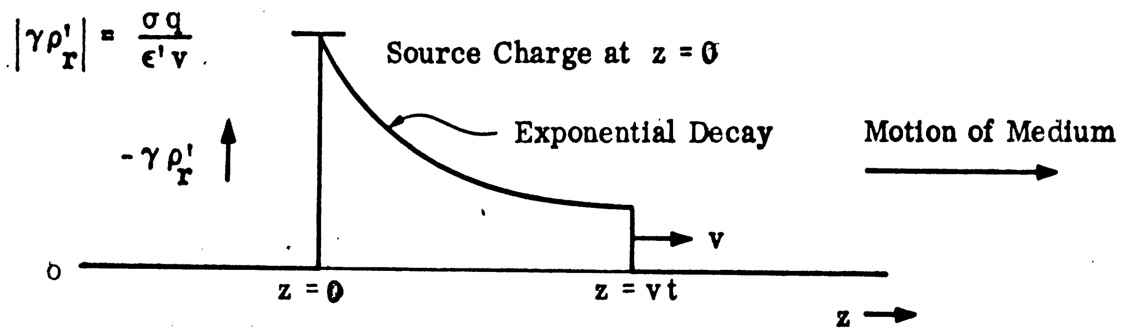

FIG. 3-3: RESPONSE CHARGE DENSITY ALONG THE z-AXIS FOR A POINT SOURCE CHARGE AT THE ORIGIN.

sity is zero everywhere except on part of the positive real z-axis. It jumps at $z = 0$ to a value which depends on the conductivity and the velocity or more accurately, on the relaxation time σ'/ϵ' and the velocity of the medium. The step down at $z = vt$, which might be termed the response charge "front", travels with the velocity of the medium, and decays in amplitude exponentially with time. As t approaches infinity,

$$\gamma \rho'_r \longrightarrow -\frac{\sigma q}{\epsilon' v} \, \delta(x)\, \delta(y)\; e^{-\frac{\sigma z}{\epsilon' \gamma^2 v}} \; S_o(z) \quad , \tag{3.89}$$

which is the same as (3.53). Thus the time-dependent case reduces to the time-independent case as t approaches infinity, and we can speak of a steady-state condition. In this condition the source must have its velocity maintained by some unspecified mechanical force, and its charge maintained by some electrical energy source.

3.2.2 Case B: Current Sources

A comparison of (3.18) and (3.20) with (3.25) and (3.26) shows that the introduction of a current source merely adds the term $\bar{J}_s$ to the current charge density expression. The relations (3.58) and (3.61) of Section 3.2.1 are modified as follows: into (3.58) the term $\frac{\bar{v} \cdot \bar{J}_s}{c^2}$, and to (3.60) the term $\nabla \cdot \bar{J}_s$, is added. Noting that the source charges and currents are quantities independent of the system, they are related by the equation of continuity,

$$\nabla \cdot \bar{J}_s = -\frac{\partial \rho_s}{\partial t} , \tag{3.91}$$

and the final differential Eq. (3.63) is modified to

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z} + \frac{\sigma}{\epsilon' \gamma^2} \right) \gamma \rho'_r = -\frac{\sigma}{\epsilon'} \left(\rho_s + \frac{\bar{v} \cdot \bar{J}_s}{c^2} \right) . \tag{3.92}$$

For harmonic current sources, ρ_s, $\gamma \rho'_r$, and $\bar{J}_s$ all have a time behavior given by $e^{-i\omega t}$, so that (3.92) becomes

$$\left[\frac{\partial}{\partial z} + \frac{1}{v} \left(\frac{\sigma}{\epsilon' \gamma^2} - i\omega \right) \right] \gamma \rho'_r = -\frac{\sigma}{\epsilon' v} \left(\rho_s + \frac{\bar{v} \cdot \bar{J}_s}{c^2} \right) . \tag{3.93}$$

This has the form

$$\left[\frac{\partial}{\partial z} + b \right] u(\bar{R}) = -U_o(\bar{R}) , \tag{3.94}$$

where the auxiliary variables are defined by

$$u = \gamma \rho'_r ,$$

$$U_o = (\sigma/\epsilon' v) \left(\rho_s + \frac{\bar{v} \cdot \bar{J}_s}{c^2} \right) ,$$

and

$$b = \sigma/\epsilon' \gamma^2 v - i\omega/v . \tag{3.95}$$

Noting that the Eq. (3.94) has been encountered before in (3.38), we can apply the results of the latter directly. The one difference is that here b is complex rather than real and positive, but since $\mathrm{Re}\{b\} > 0$, it does not affect the validity of using the former results. From (3.49), then, we have

$$u(z) = - \int_{-\infty}^{z} e^{-b(z-\zeta)}\, U_0(\zeta)\, d\zeta \quad , \tag{3.96}$$

which, written in the original variables, becomes

$$\gamma \rho_r'(z) = - \frac{\sigma}{\epsilon' v} \int_{-\infty}^{z} e^{(i\omega - \sigma/\epsilon'\gamma^2)\left(\frac{z-\zeta}{v}\right)} \left[\rho_s(\zeta) + \bar{\Omega} \cdot \bar{J}_s(\zeta)\right] d\zeta \quad . \tag{3.97}$$

As an example, consider a thin wire antenna of length $2l$ oriented in the x-direction, and having a triangular current distribution:

$$\bar{J}_s = \hat{x} \frac{I_0(l - |x|)}{l} \delta(y)\, \delta(z) , \quad |x| \le l \quad . \tag{3.98}$$

Then by the equation of continuity (3.91),

$$\rho_s = \frac{1}{i\omega} \nabla \cdot \bar{J}_s$$

$$= \frac{I_0}{i\omega l} \delta(y)\, \delta(z) \frac{\partial (l - |x|)}{\partial x} , \quad |x| < l ,$$

or

$$\rho_s = \frac{I_0}{i\omega l} \delta(y)\, \delta(z) \cdot \begin{Bmatrix} 1, & -l \le x \le 0 \\ -1, & 0 < x \le l \end{Bmatrix} \quad . \tag{3.99}$$

Then since $\bar{\nabla} \cdot \bar{J}_s = 0$, the response charge density can be written, from (3.97),

$$\gamma \rho_r' = -\frac{\sigma}{\epsilon' v} \int_{-\infty}^{z} e^{(i\omega - \sigma/\epsilon'\gamma^2)\left(\frac{z - \zeta}{v}\right)} \rho_s(\zeta)\, d\zeta \quad ,$$

or

$$\gamma \rho_r' = -\frac{\sigma I_o}{i\epsilon' v \omega \ell}\, \delta(y)\, e^{\left(i\omega - \frac{\sigma}{\epsilon'\gamma^2}\right)\frac{z}{v}}\, S_o(z) \cdot \begin{Bmatrix} 1, & -\ell \le x < 0 \\ -1, & 0 < x \le \ell \end{Bmatrix} . \quad (3.100)$$

This example is indicated schematically in Fig. 3-4:

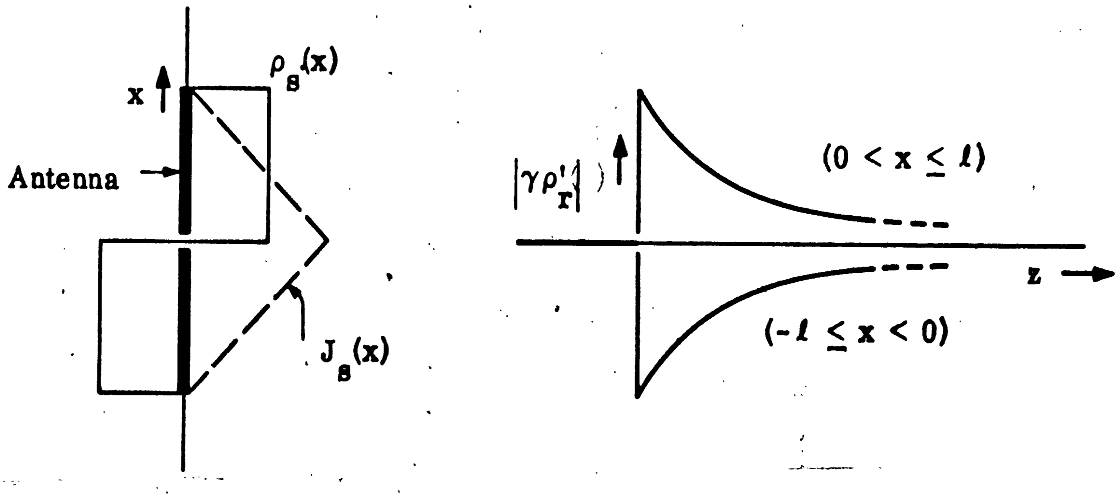

FIG. 3-4: CHARGES AND CURRENTS FOR A THIN-WIRE ANTENNA.

It can now be demonstrated why it was necessary to decompose charges as well as currents into source and response terms. For if, instead, we had begun with the sourceless formulation of Sommerfeld,

$$\bar{J} = \rho \bar{v} + \sigma' \frac{\bar{\bar{\gamma}} \cdot \bar{\bar{\gamma}} \cdot \bar{E}^*}{\gamma} , \quad (2.9)$$

and added an impressed current source $\bar{J}_s$, and written

$$\bar{J} = \bar{J}_s + \rho\bar{v} + \sigma' \frac{\bar{\bar{\gamma}} \cdot \bar{\bar{\gamma}} \cdot \bar{E}^*}{\gamma} , \tag{3.101}$$

the convection term $\rho\bar{v}$ would become meaningless, if ρ is taken as the total charge density. For as the conductivity σ' vanishes, we would then get

$$\bar{J} = \bar{J}_s + \rho\bar{v} . \tag{3.102}$$

But we know that a lossless medium, in motion or not, with a stationary charge and current distribution, gives rise to no convection term, that is,

$$\bar{J} = \bar{J}_s , \tag{3.103}$$

and since in general, $\rho \neq 0$, this contradicts (3.102).

On the other hand, in the formulation of the present work, we have

$$\bar{J} = \bar{J}_s + \gamma\rho_r' \bar{v} + \sigma' \gamma \bar{E}^* . \tag{3.104}$$

Now as the conductivity vanishes, $\gamma\rho_r'$ vanishes by (3.97), and we are left with

$$\bar{J} = \bar{J}_s ,$$

as is required.

IV

VECTOR AND SCALAR POTENTIALS; DEVELOPMENT OF THE GREEN'S FUNCTIONS

In this chapter we shall derive the vector and scalar potentials and the differential equations they satisfy, for the two classes of problems of interest to us. The Green's function approach will be used to find solutions to the linear, inhomogeneous, partial differential equations. In this approach the forcing function is replaced by a point function, or δ-function, in space, and the solution to the resulting differential equation is called a Green's function. The solutions to the differential equations for the vector or scalar potentials are then given by a superposition of Green's functions. The field quantities then follow from the potentials.

The class of problems involving charge sources gives rise to a complicated differential equation in the general time-dependent case, one not readily solved. If steady-state behavior is assumed, that is, $\partial/\partial t = 0$, the equation is greatly simplified, and is amenable to solution. We shall derive the differential equations and present them in their entirety, and find the Green's function solution in closed form for the steady-state case.

The harmonic current source class of problems is treated in a modified way, i.e., the potentials are defined differently than usual. The modified approach gives rise to simpler differential equations. Steady-state behavior is again assumed, and Green's function solutions are found in closed form.

The discussion is limited to consideration of unbounded media. Thus we are primarily interested in the particular solutions to the differential equations. There is thus a unique correspondence between the solutions and their transforms; we will use the method of Hankel transforms in the cylindrical coordinate $\bar{r} = (x^2 + y^2)^{1/2}$, and Fourier transforms in the longitudinal coordinate z. The solutions are valid for all values of conductivity σ, and all velocities v.

4.1 Static Charge Source Distributions

4.1.1 Differential Equations for the Potentials

For a linear, uniformly moving, conducting medium, Maxwell's equations are given by

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad (I) \quad , \qquad \nabla \cdot \bar{D} = \rho \quad (II) \quad ,$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \quad (III) \quad , \qquad \nabla \cdot \bar{B} = 0 \quad (IV) \quad ,$$

where the constitutive relations are, using the definitions of Eq. (1.5)

$$\bar{B} = \mu' \bar{\bar{\alpha}} \cdot \bar{H} - \bar{\Omega} \times \bar{E} \quad ,$$
$$\bar{D} = \epsilon' \bar{\bar{\alpha}} \cdot \bar{E} + \bar{\Omega} \times \bar{H} \quad , \tag{4.1}$$

and charge and current densities are decomposed as follows:

$$\bar{J} = \sigma(\bar{E} + \bar{v} \times \bar{B}) + \gamma \rho_r' \bar{v} = \sigma(\bar{\bar{\alpha}} \cdot \bar{E} + \mu' a \nabla \times \bar{H}) + \gamma \rho_r' \bar{v} \quad ,$$

$$\rho = \rho_s + \gamma \rho_r' + \sigma \frac{\bar{v} \cdot \bar{E}}{c^2} \quad . \tag{4.2}$$

Here we have used Eqs. (3.57) and (3.29). The quantity $\gamma \rho_r'$ is determined by the source density ρ_s; this was discussed in Section 3.2.1. In finding this relationship of response to source, it should be noted that only (II) and (III) of Maxwell's equations were used. In deriving the expressions for the potentials, it is necessary to use (I) and (IV) as well.

For source charge problems, the vector potential $\bar{A}$ is defined in the usual manner, using (IV):

$$\bar{B} = \nabla \times \bar{A} \quad . \tag{4.3}$$

Note that this is only a partial definition, since $\bar{A}$ is not unique. Any other vector potential $\bar{A}$, which differs from $\bar{A}$ by the gradient of some scalar, would also satisfy this relation. From (I),

$$\nabla \times \bar{E} = -\frac{\partial}{\partial t}\nabla \times \bar{A} = -\nabla \times \frac{\partial \bar{A}}{\partial t} ,$$

or

$$\bar{E} = -\frac{\partial \bar{A}}{\partial t} - \nabla \phi , \qquad (4.4)$$

where ϕ is some scalar potential.

We are free to choose $\bar{A}$ to be in the z-direction without losing generality. Thus cross-products of $\bar{A}$ with $\bar{v}$ or $\bar{\Omega}$ will vanish in the following development. The equation $\nabla \cdot \bar{D} = \rho$ has already been expanded in Eq. (3.59)

$$\nabla \cdot \bar{\bar{\alpha}} \cdot \bar{E} - \bar{\Omega} \cdot \frac{\partial \bar{E}}{\partial t} - \sigma\mu' a \nabla \cdot \bar{E} = \frac{\rho_s + a\gamma\rho_r'}{\epsilon'} . \qquad (4.5)$$

This becomes, using (4.4),

$$\nabla \cdot \bar{\bar{\alpha}} \cdot \left(\frac{\partial \bar{A}}{\partial t} + \nabla \phi\right) - \bar{\Omega} \cdot \left(\frac{\partial^2 \bar{A}}{\partial t^2} + \nabla \frac{\partial \phi}{\partial t}\right) - \sigma\mu' a \nabla \cdot \left(\frac{\partial \bar{A}}{\partial t} + \nabla \phi\right) = -\frac{(\rho_s + a\gamma\rho_r')}{\epsilon} . \qquad (4.6)$$

From (4.1)

$$\begin{aligned} \mu'\bar{H} &= \bar{\bar{\alpha}}^{-1} \cdot \bar{B} + \frac{1}{a}\bar{\Omega} \times \bar{E} \\ &= \bar{\bar{\alpha}}^{-1} \cdot (\nabla \times \bar{A}) - \frac{1}{a}\bar{\Omega} \times \left(\frac{\partial \bar{A}}{\partial t} + \nabla \phi\right) \\ &= \frac{1}{a}\nabla \times (\bar{A} + \bar{\Omega}\phi) , \end{aligned} \qquad (4.7)$$

where the elements of $\bar{\bar{\alpha}}^{-1}$ are given simply by

$$\begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Also, from (4.1),

$$\bar{D} = \epsilon' \bar{\bar{\alpha}} \cdot \bar{E} + \bar{\Omega} \times \bar{H}$$

$$= -\epsilon' \frac{\partial \bar{A}}{\partial t} - \epsilon' \bar{\bar{\alpha}} \cdot \nabla \phi + \frac{1}{a\mu'} \bar{\Omega} \times \left[\nabla \times (\bar{A} + \bar{\Omega}\phi) \right] . \quad (4.8)$$

From Eq. (4.2),

$$\bar{J} = \sigma(\bar{E} + \bar{v} \times \bar{B}) + \gamma \rho'_r \bar{v} . \quad (4.9)$$

Thus (III) becomes

$$\nabla \times \bar{H} = \frac{1}{\mu' a} \nabla \times \nabla \times (\bar{A} + \bar{\Omega}\phi) = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

$$= -\sigma \left[\frac{\partial \bar{A}}{\partial t} + \nabla \phi - \bar{v} \times (\nabla \times \bar{A}) \right] - \epsilon' \frac{\partial^2 \bar{A}}{\partial t^2} - \epsilon' \bar{\bar{\alpha}} \cdot \nabla \frac{\partial \phi}{\partial t} + \frac{1}{\mu' a} \bar{\Omega} \times \left[\nabla \times \frac{\partial}{\partial t} (\bar{A} + \bar{\Omega}\phi) \right]$$

$$+ \gamma \rho'_r \bar{v} . \quad (4.10)$$

By choosing a gauge condition which is consistent with the well-known gauge condition for stationary, conducting media, separate partial differential equations may be obtained for $\bar{A}$ and ϕ. The development of the gauge condition is given in Appendix A. It can be written:

$$\nabla \cdot \bar{A} - \sigma \mu' a \bar{v} \cdot \bar{A} - \bar{\Omega} \cdot \frac{\partial \bar{A}}{\partial t} = - \bar{\Omega} \cdot \nabla \phi - \sigma \mu' a \phi - \frac{1}{c^2} \left(\frac{n^2 - \beta^2}{1 - n^2 \beta^2} \right) \frac{\partial \phi}{\partial t} . \quad (4.11)$$

Substituting this into (4.6) yields the differential equation for ϕ:

$$(\nabla \cdot \bar{\bar{\alpha}} \cdot \nabla)\phi - \sigma\mu' a\bar{v} \cdot \nabla\phi - 2\,\bar{\Omega} \cdot \nabla \frac{\partial\phi}{\partial t} - \frac{1}{c^2}\left(\frac{n^2-\beta^2}{1-n^2\beta^2}\right)\frac{\partial^2\phi}{\partial t^2}$$

$$= -\frac{1}{\epsilon'}(\rho_s + a\,\gamma\,\rho_r') \quad . \tag{4.12}$$

To show this, we first note the following relation, which follows from (4.11):

$$\nabla \cdot \bar{\bar{\alpha}} \cdot \left(\frac{\partial \bar{A}}{\partial t} + \nabla\phi\right) = \frac{\partial}{\partial t}\nabla \cdot \bar{A} + (\nabla \cdot \bar{\bar{\alpha}} \cdot \nabla)\phi$$

$$= (\nabla \cdot \bar{\bar{\alpha}} \cdot \nabla)\phi + \sigma\mu' a\bar{v} \cdot \frac{\partial\bar{A}}{\partial t} + \bar{\Omega} \cdot \frac{\partial^2\bar{A}}{\partial t^2} - \bar{\Omega} \cdot \nabla\frac{\partial\phi}{\partial t} - \sigma\mu' a\frac{\partial\phi}{\partial t} - \frac{1}{c^2}\left(\frac{n^2-\beta^2}{1-n^2\beta^2}\right)\frac{\partial^2\phi}{\partial t^2}$$

$$\tag{4.13}$$

Using this relation in (4.6), it can be seen that the terms involving the vector potential $\bar{A}$ drop out, leaving (4.12).

Turning our attention now to (4.10), we first draw upon a vector identity noted by Tai (1965a):

$$\nabla \times (\bar{\bar{\alpha}}^{-1} \cdot (\nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{F}))) = \frac{1}{a^2}\left[(\bar{\bar{\alpha}} \cdot \nabla)(\nabla \cdot \bar{F}) - (\nabla \cdot \bar{\bar{\alpha}} \cdot \nabla)\bar{F}\right] . \tag{4.14}$$

When $\bar{F}$ is the vector potential $\bar{A}$, and it is noted that $\bar{A}$ is in the z-direction only, the left hand side becomes

$$\nabla \times (\bar{\bar{\alpha}}^{-1} \cdot (\nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{A}))) = \nabla \times (\bar{\bar{\alpha}}^{-1} \cdot (\nabla \times \bar{A})) = \frac{1}{a}\nabla \times \nabla \times \bar{A} ,$$

and thus

$$\nabla \times \nabla \times \bar{A} = \frac{1}{a}\left[(\bar{\bar{\alpha}} \cdot \nabla)(\nabla \cdot \bar{A}) - (\nabla \cdot \bar{\bar{\alpha}} \cdot \nabla)\bar{A}\right] . \tag{4.15}$$

When this is substituted into Eq. (4.10), and the terms are regrouped, we get

$$(\nabla \cdot \overline{\overline{\alpha}} \cdot \nabla)\overline{A} - (\overline{\overline{\alpha}} \cdot \nabla)(\nabla \cdot \overline{A}) - \mu' \epsilon' a^2 \frac{\partial^2 \overline{A}}{\partial t^2} - \sigma \mu' a^2 \frac{\partial \overline{A}}{\partial t}$$

$$+ a\,\overline{\Omega} \times (\nabla \times \frac{\partial \overline{A}}{\partial t}) + \sigma \mu' a^2 \overline{v} \times (\nabla \times \overline{A})$$

$$= a \nabla \times \nabla \times (\overline{\Omega}\phi) + \sigma \mu' a^2 \nabla \phi + \mu' \epsilon' a^2 \overline{\overline{\alpha}} \cdot \nabla \frac{\partial \phi}{\partial t} - a\,\overline{\Omega} \times (\nabla \times (\overline{\Omega} \frac{\partial \phi}{\partial t}))$$

$$- \mu' a^2 \gamma \rho_r' \overline{v} \quad . \tag{4.16}$$

The fifth term on the left can be written as follows:

$$a\,\overline{\Omega} \times (\nabla \times \frac{\partial \overline{A}}{\partial t}) = a \nabla(\overline{\Omega} \cdot \overline{A}) - a(\overline{\Omega} \cdot \nabla)\overline{A} \quad , \tag{4.17}$$

where use is made of the vector identity

$$\nabla(\overline{F} \cdot \overline{G}) = \overline{F} \times (\nabla \times \overline{G}) + \overline{G} \times (\nabla \times \overline{F}) + (\overline{F} \cdot \nabla)\overline{G} + (\overline{G} \cdot \nabla)\overline{F}, \tag{4.18}$$

and it is noted that derivatives of $\overline{\Omega}$ are zero since $\overline{\Omega}$ is assumed constant. Similarly, the sixth term on the left becomes

$$\sigma \mu' a^2 \overline{v} \times (\nabla \times \overline{A}) = \sigma \mu' a^2 \nabla(\overline{v} \cdot \overline{A}) - \sigma \mu' a^2 (\overline{v} \cdot \nabla)\overline{A} \quad , \tag{4.19}$$

and the third and fourth terms on the right combine to give

$$\mu' \epsilon' a^2 \overline{\overline{\alpha}} \cdot \nabla \frac{\partial \phi}{\partial t} - a\,\overline{\Omega} \times (\nabla \times (\overline{\Omega} \frac{\partial \phi}{\partial t}))$$

$$= \frac{n^2}{c^2} a^2 \overline{\overline{\alpha}} \cdot \nabla \frac{\partial \phi}{\partial t} - a \nabla (\Omega^2 \frac{\partial \phi}{\partial t}) + a(\overline{\Omega} \cdot \nabla)(\overline{\Omega} \frac{\partial \phi}{\partial t})$$

$$= \frac{n^2}{c^2} a^2 \bar{\bar{\alpha}} \cdot \nabla \frac{\partial \phi}{\partial t} - a\,\Omega^2 \nabla \frac{\partial \phi}{\partial t} + a\bar{\Omega}(\bar{\Omega} \cdot \nabla) \frac{\partial \phi}{\partial t} \quad . \tag{4.20}$$

The transverse component, denoted by the subscript T, can be written

$$a\left(\frac{n^2 a^2}{c^2} - \Omega^2\right) \nabla_T \frac{\partial \phi}{\partial t} = \frac{a}{c^2}\left(\frac{n^2 - \beta^2}{1 - n^2 \beta^2}\right) \nabla_T \frac{\partial \phi}{\partial t} \tag{4.21a}$$

which follows from the definitions of Ω and a. Similarly the z-component is merely that of the first term of (4.20), or

$$\frac{n^2 a^2}{c^2} \frac{\partial^2 \phi}{\partial z\, \partial t} \quad . \tag{4.21b}$$

Also, the first term on the right of (4.16) can be re-written:

$$\begin{aligned} a \nabla \times \nabla \times (\bar{\Omega}\phi) &= a \nabla \nabla \cdot (\bar{\Omega}\phi) - a\, \nabla^2 (\bar{\Omega}\phi) \\ &= a \nabla(\bar{\Omega} \cdot \nabla \phi) - a \bar{\Omega} \nabla^2 \phi \\ &= a \bar{\Omega} \times (\nabla \times \nabla \phi) + a(\bar{\Omega} \cdot \nabla) \nabla \phi - a \bar{\Omega} \nabla^2 \phi \\ &= a(\bar{\Omega} \cdot \nabla) \nabla \phi - a \bar{\Omega} \nabla^2 \phi \quad , \end{aligned} \tag{4.22}$$

where use is made of the following vector identities, in addition to (4.18):

$$\nabla \times \nabla \times \bar{F} = \nabla \nabla \cdot \bar{F} - \nabla^2 \bar{F} \quad , \tag{4.23a}$$

$$\nabla \cdot (\bar{F} \psi) = \bar{F} \cdot \nabla \psi + \psi \nabla \cdot \bar{F} \quad , \tag{4.23b}$$

$$\nabla^2 (\bar{F} \psi) = \bar{F} \nabla^2 \psi \quad \text{for } \bar{F} \text{ a constant vector,} \tag{4.23c}$$

$$\nabla \times (\nabla \psi) = 0 \quad , \tag{4.23d}$$

and it is noted that derivatives of $\bar{\Omega}$ vanish, since $\bar{\Omega}$ is a constant vector. The second term on the left of (4.16) can be rewritten, using the gauge condition of Eq. (4.11). This gives, for the transverse components,

$$\left[-(\bar{\bar{\alpha}}\cdot\nabla)(\nabla\cdot\bar{A})\right]_T = -\sigma\mu' a^2 v \nabla_T A - a\Omega\nabla_T\frac{\partial A}{\partial t} + a\Omega\nabla_T\frac{\partial\phi}{\partial z}$$

$$+\sigma\mu' a^2\nabla_T\phi + \frac{a}{c^2}\left(\frac{n^2-\beta^2}{1-n^2\beta^2}\right)\nabla_T\frac{\partial\phi}{\partial t}\ . \qquad (4.24a)$$

For the z-component,

$$\left[-(\bar{\bar{\alpha}}\cdot\nabla)(\nabla\cdot\bar{A})\right]_z = -\sigma\mu' a v\frac{\partial A}{\partial z} - \Omega\frac{\partial^2 A}{\partial z\,\partial t} + \Omega\frac{\partial^2\phi}{\partial z^2}$$

$$+\sigma\mu' a\frac{\partial\phi}{\partial z} + \frac{1}{c^2}\left(\frac{n^2-\beta^2}{1-n^2\beta^2}\right)\frac{\partial^2\phi}{\partial z\,\partial t}\ . \qquad (4.24b)$$

Using (4.17), (4.19), and (4.24), the transverse part of the left hand side of (4.16) reduces to

$$a\Omega\nabla_T\frac{\partial\phi}{\partial z} + \sigma\mu' a^2\nabla_T\phi + \frac{a}{c^2}\left(\frac{n^2-\beta^2}{1-n^2\beta^2}\right)\nabla_T\frac{\partial\phi}{\partial t}\ . \qquad (4.25a)$$

Similarly the transverse part of the right side of (4.16) after using (4.21) and (4.22) becomes

$$a\Omega\nabla_T\frac{\partial\phi}{\partial z} + \sigma\mu' a^2\nabla_T\phi + \frac{a}{c^2}\left(\frac{n^2-\beta^2}{1-n^2\beta^2}\right)\nabla_T\frac{\partial\phi}{\partial t}\ , \qquad (4.25b)$$

which is the same as (4.25a). Thus the transverse components cancel, and we are left with only longitudinal components. This means that the vector differential equation reduces to a single scalar differential equation, an important result.

Summing up the longitudinal components on each side, we get

$$(\nabla \cdot \bar{\bar{\alpha}} \cdot \nabla)\bar{A} - \sigma\mu' a v \frac{\partial \bar{A}}{\partial z} - \Omega \frac{\partial^2 \bar{A}}{\partial z\, \partial t} + \bar{\Omega} \frac{\partial^2 \phi}{\partial z^2} + \hat{z}\sigma\mu' a \frac{\partial \phi}{\partial z} + \frac{\hat{z}}{c^2} \frac{(n^2 - \beta^2)}{(1 - n^2\beta^2)} \frac{\partial^2 \phi}{\partial z\, \partial t}$$

$$- \frac{n^2 a^2}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2} - \sigma\mu' a^2 \frac{\partial \bar{A}}{\partial t}$$

$$= a\hat{z}(\bar{\Omega} \cdot \nabla) \frac{\partial \phi}{\partial z} - a\bar{\Omega} \nabla^2 \phi + \hat{z}\sigma\mu' a^2 \frac{\partial \phi}{\partial z} + \hat{z} \frac{n^2}{c^2} a^2 \frac{\partial^2 \phi}{\partial z\, \partial t}$$

$$- \mu' a^2 \gamma \rho_r' \bar{v} \quad . \tag{4.26}$$

Examining those terms involving $\frac{\partial^2 \phi}{\partial z^2}$ and $\nabla^2 \phi$, we note that

$$\hat{z}\, a(\bar{\Omega} \cdot \nabla) \frac{\partial \phi}{\partial z} - a\bar{\Omega} \nabla^2 \phi - \bar{\Omega} \frac{\partial^2 \phi}{\partial z^2} = -a\bar{\Omega} \nabla_T^2 \phi - \bar{\Omega} \frac{\partial^2 \phi}{\partial z^2}$$

$$= -\bar{\Omega}(\nabla \cdot \bar{\bar{\alpha}} \cdot \nabla)\phi \quad . \tag{4.27}$$

Since the quantities are parallel vectors, we can drop the vector notation. After multiplication by Ω, Eq. (4.6) becomes

$$-\Omega(\nabla \cdot \bar{\bar{\alpha}} \cdot \nabla)\phi = \Omega \frac{\partial^2 A}{\partial z\, \partial t} - \Omega^2 \frac{\partial^2 A}{\partial t^2} - \sigma\mu' a v\, \Omega \frac{\partial A}{\partial t}$$

$$- \Omega^2 \frac{\partial^2 \phi}{\partial z\, \partial t} - \sigma\mu' a v \Omega \frac{\partial \phi}{\partial z} + \Omega \frac{\rho_s}{\epsilon'} + a \frac{\Omega}{\epsilon'} \gamma \rho_r' \quad . \tag{4.28}$$

Using (4.27) and (4.28), Eq. (4.26) becomes

$$(\nabla \cdot \bar{\bar{\alpha}} \cdot \nabla)A - \sigma\mu' a v \frac{\partial A}{\partial z} - 2\Omega \frac{\partial^2 A}{\partial z\, \partial t} - \sigma\mu' a (a - v\Omega) \frac{\partial A}{\partial t} - \left(\frac{n^2 a^2}{c^2} - \Omega^2\right) \frac{\partial^2 A}{\partial t^2}$$

$$= - \sigma\mu' a(1 - a + v\Omega) \frac{\partial \phi}{\partial z} - \left[\frac{1}{c^2} \frac{(n^2 - \beta^2)}{(1 - n^2\beta^2)} - \frac{n^2 a^2}{c^2} + \Omega^2\right] \frac{\partial^2 \phi}{\partial z\, \partial t} + \Omega \frac{\rho_s}{\epsilon'} - (\mu' a v - \frac{\Omega}{\epsilon'}) a\, \gamma \rho_r' \, . \tag{4.29}$$

It can be seen from the definitions of a and Ω that the following relations hold:

$$a - v\Omega = 1 \;,$$

$$\frac{n^2 a^2}{c^2} - \Omega^2 = \frac{1}{c^2} \frac{n^2 - \beta^2}{1 - n^2\beta^2} \;,$$

and

$$\mu' a v - \frac{\Omega}{\epsilon'} = \frac{1}{\epsilon'} \cdot \frac{a v}{c^2} \;, \tag{4.30}$$

so that (4.29) becomes, finally,

$$(\nabla \cdot \bar{\bar{\alpha}} \cdot \nabla)A - \sigma\mu' a v \frac{\partial A}{\partial z} - 2\Omega \frac{\partial^2 A}{\partial z\, \partial t} - \sigma\mu' a \frac{\partial A}{\partial t} - \frac{1}{c^2}\left(\frac{n^2 - \beta^2}{1 - n^2\beta^2}\right) \frac{\partial^2 A}{\partial t^2}$$

$$= \frac{1}{\epsilon'}\left(\Omega \rho_s - \frac{a v}{c^2} \gamma \rho_r'\right) \, . \tag{4.31}$$

Written in scalar form, the corresponding expression (4.12) for the scalar potential ϕ becomes

$$(\nabla \cdot \bar{\bar{\alpha}} \cdot \nabla)\phi - \sigma\mu' a v \frac{\partial \phi}{\partial z} - 2\Omega \frac{\partial^2 \phi}{\partial z \, \partial t} - \sigma\mu' a \frac{\partial \phi}{\partial t} - \frac{1}{c^2}\left(\frac{n^2 - \beta^2}{1 - n^2\beta^2}\right)\frac{\partial^2 \phi}{\partial t^2}$$

$$= -\frac{1}{\epsilon'}(\rho_s + a\gamma\rho_r') \quad . \tag{4.32}$$

Comparison of (4.31) and (4.32) reveals that the two differential equations are identical, except for the source terms.

Let ϕ_s be the solution to the differential equation when only the first term appears on the right of (4.32), and ϕ_r the solution when only the second term appears. Then

$$\phi = \phi_s + \phi_r \quad . \tag{4.33}$$

If $\bar{A}_s$ and $\bar{A}_r$ are similarly defined from Eq. (4.31), the particular solutions are related by constant quantities:

$$\bar{A}_s = -\bar{\Omega}\phi_s$$

and

$$\bar{A}_r = \frac{\bar{v}}{c^2}\phi_r \quad . \tag{4.34}$$

From the transformation relations for A' and ϕ' in the rest system of the medium (see Appendix A, Eq. (A.8)),

$$A' = \gamma\left(A - \frac{v\phi}{c^2}\right) = \gamma\left(-\Omega\phi_s + \frac{v}{c^2}\phi_r - \frac{v}{c^2}\phi_r - \frac{v}{c^2}\phi_s\right) ,$$

or

$$A' = -\gamma\mu'\epsilon' a v \phi_s , \tag{4.35}$$

and thus A' depends only on the source term ρ_s and not on the response charge density ρ'_r. On the other hand,

$$\phi' = \gamma(\phi - vA) = \gamma(\phi_s + \phi_r + v\Omega\phi_s - \frac{v^2}{c^2}\phi_r) ,$$

or

$$\phi' = \gamma a \phi_s + \frac{1}{\gamma}\phi_r . \tag{4.36}$$

It is also of interest to express the fields in terms of the scalar potentials; from (4.3), (4.4), (4.7), and (4.8) we get

$$\bar{B} = \nabla \times \bar{A} = (\nabla\phi \times \hat{z}) = \bar{\Omega} \times \nabla\phi_s - \frac{\bar{v}}{c^2} \times \nabla\phi_r ,$$

$$\bar{E} = -\frac{\partial \bar{A}}{\partial t} - \nabla\phi$$

$$= \bar{\Omega}\frac{\partial \phi_s}{\partial t} - \nabla\phi_s - \frac{\bar{v}}{c^2}\frac{\partial \phi_r}{\partial t} - \nabla\phi_r ,$$

$$\bar{H} = \frac{1}{\mu' a} \nabla \times (\bar{A} + \bar{\Omega}\phi)$$

$$= \frac{n^2 a}{\mu' a c^2} (\nabla\phi_r \times \bar{v})$$

$$= -\epsilon' \bar{v} \times \nabla\phi_r ,$$

and

$$\bar{D} = -\epsilon' \frac{\partial \bar{A}}{\partial t} - \epsilon' \bar{\bar{\alpha}} \cdot \nabla\phi + \frac{1}{\mu' a}\bar{\Omega} \times \left[\nabla \times (\bar{A} + \bar{\Omega}\phi)\right]$$

$$= \epsilon' \Omega \frac{\partial \phi_s}{\partial t} - \epsilon' \bar{\bar{\alpha}} \cdot \nabla \phi_s - \frac{\epsilon' v}{c^2} \frac{\partial \phi_r}{\partial t} - \epsilon' \bar{\bar{\alpha}} \cdot \nabla \phi_r$$

$$- \epsilon' \bar{\Omega} \times (\bar{v} \times \nabla \phi_r) \quad . \tag{4.37}$$

For this case it can be seen that for zero conductivity, $\gamma \rho_r'$ vanishes, and so does ϕ_r and $\bar{H}$. Thus for lossless moving media when A and ϕ have the same boundary conditions, $\bar{H} = 0$. In this case we have a static configuration in space, so the result that $\bar{E} \times \bar{H} = 0$ is to be expected, since there is no radiation at all.

4.1.2 Green's Function Solution

The system of equations for the potentials given in equations (4.31) and (4.32) is quite complicated, involving, as it does, three variables. It was seen in Section 3.2.1 that there is a steady-state behavior for the currents and charges for large t which is found either by letting t approach infinity or setting $\partial/\partial t = 0$ from the start. If the assumption is made that $\partial/\partial t = 0$, the differential equations simplify to

$$(\nabla \cdot \bar{\bar{\alpha}} \cdot \nabla) A - \sigma \mu' a v \frac{\partial A}{\partial z} = \frac{1}{\epsilon'} (\Omega \rho_s - a \frac{v}{c^2} \gamma \rho_r')$$

and

$$(\nabla \cdot \bar{\bar{\alpha}} \cdot \nabla) \phi - \sigma \mu' a v \frac{\partial \phi}{\partial z} = - \frac{1}{\epsilon'} (\rho_s + a \gamma \rho_r') \quad . \tag{4.38}$$

The Green's function method utilizes a function $G(\bar{R} | \bar{R}_0)$ which is the solution to a given differential equation when the source term is a point source in space at $\bar{R}_0$; that is, $G(\bar{R} | \bar{R}_0)$ satisfies the equation

$$\left[(\nabla \cdot \bar{\bar{\alpha}} \cdot \nabla) - \sigma \mu' a v \frac{\partial}{\partial z} \right] G(\bar{R} | \bar{R}_0) = - \delta (\bar{R} | \bar{R}_0) \tag{4.39}$$

where $\bar{R}$ is the vector from the origin to the field point, $\bar{R}_o$ is the vector from the origin to the source point, and the derivatives operate on the field coordinates. The symbol $\delta(\bar{R}|\bar{R}_o)$ denotes a quantity which vanishes for $\bar{R} \neq \bar{R}_o$, and has the property that

$$\int_V f(\bar{R})\,\delta(\bar{R}|\bar{R}_o)\;dV = f(\bar{R}_o)$$

where the volume V encloses the point $\bar{R}_o$. In all of the problems in this work we shall be dealing with unbounded media, so that for all fields, potentials, and Green's functions, the boundary conditions will be the radiation condition, namely that only functions which do not increase away from the source are allowed; also, that for unbounded media, the homogeneous solutions vanish. This means that no sources exist for the fields other than the given sources, which are assumed to occupy a finite region.

It will now be shown that the vector and scalar potentials are related to the Green's function in the following way:

$$\bar{A}(\bar{R}) = -\frac{1}{\epsilon'}\iiint_{V_o} G(\bar{R}|\bar{R}_o)\left[\bar{\Omega}\rho_s(\bar{R}_o) - \frac{a\bar{v}}{c^2}\gamma\rho_r'(\bar{R}_o)\right]dV_o \;,$$

and

$$\phi(\bar{R}) = \frac{1}{\epsilon'}\iiint_{V_o} G(\bar{R}|\bar{R}_o)\left[\rho_s(\bar{R}_o) + a\gamma\rho_r'(\bar{R}_o)\right]dV_o \,. \qquad (4.40)$$

where V_o indicates a volume enclosing the sources. To show this, we shall define three-dimensional Fourier transforms and use the relation (3.44), extended to three-dimensions. Let the Fourier transform $\mathcal{F}$ of a function $F(\bar{R})$ be defined as follows:

$$\tilde{F}(\bar{h}) = (2\pi)^{-3} \iiint_{-\infty}^{\infty} e^{i\bar{h}\cdot\bar{R}_o} F(\bar{R}_o)\, d^3R_o \;, \qquad (4.41)$$

where $d^3R_o = dx_o\, dy_o\, dz_o$. Then if $F(\bar{R}_o)$ is class L^2 in each variable x_o, y_o, and z_o for all real values of x_o, y_o, and z_o, the inverse transform is given by

$$F(\bar{R}) = \iiint_{-\infty}^{\infty} e^{-i\bar{h}\cdot\bar{R}} \tilde{F}(\bar{h})\, d^3h \;, \qquad (4.42)$$

where $d^3h = dh_x\, dh_y\, dh_z$. Taking the transform of both sides of (4.38a) and (4.39) gives:

$$(-\bar{h}\cdot\bar{\bar{\alpha}}\cdot\bar{h} + i\,\sigma\mu'\, a\bar{v}\cdot\bar{h})\tilde{A} = \tilde{J} \;,$$

and

$$(-\bar{h}\cdot\bar{\bar{\alpha}}\cdot\bar{h} + i\,\sigma\mu' a\bar{v}\cdot\bar{h})\,\tilde{G}(\bar{h}|\bar{R}_o) = (2\pi)^{-3} \iiint_{-\infty}^{\infty} \delta(\bar{R}|\bar{R}_o) e^{i\bar{h}\cdot\bar{R}}\, d^3R$$

$$= (2\pi)^{-3} e^{i\bar{h}\cdot\bar{R}_o} \;, \qquad (4.43)$$

where J represents the term on the right of (4.38a). From these relations,

$$\tilde{A}(\bar{h}) = (2\pi)^3 e^{-i\bar{h}\cdot\bar{R}_o} \tilde{J}(h)\, \tilde{G}(\bar{h}|\bar{R}_o) \;. \qquad (4.44)$$

The three-dimensional version of (3.44) follows from Parseval's theorem:

$$\iiint_{-\infty}^{\infty} \tilde{F}\tilde{H}^*\, d^3h = (2\pi)^{-3} \iiint_{-\infty}^{\infty} F(\bar{R}_o)\, H^*(\bar{R}_o)\, d^3R_o \;, \qquad (4.45)$$

where the asterisk (*) indicates the complex conjugate; this is subject to the conditions stated above. Let $F(\bar{R}_o) = J(\bar{R}_o)$ so that $\tilde{F}(\bar{h}) = \tilde{J}(\bar{h})$. Similarly, let

$$\tilde{H}^*(\bar{h}) = e^{-i\bar{h}\cdot(\bar{R}+\bar{R}_o)}\,\tilde{G}(\bar{h}|\bar{R}_o) \quad . \tag{4.46a}$$

Then

$$\tilde{H}(h) = e^{i\bar{h}\cdot(\bar{R}+\bar{R}_o)}\,\tilde{G}^*(\bar{h}|\bar{R}_o)$$

and

$$H(\bar{R}_1) = \iiint_{-\infty}^{\infty} e^{-i\bar{h}\cdot(\bar{R}_1-\bar{R}-\bar{R}_o)}\,\tilde{G}^*(\bar{h}|\bar{R}_o)\,d^3h \quad ,$$

or

$$H^*(\bar{R}_1) = \iiint_{-\infty}^{\infty} e^{-i\bar{h}\cdot(\bar{R}+\bar{R}_o-\bar{R}_1)}\,\tilde{G}(\bar{h}|\bar{R}_o)\,d^3h$$

$$= G(\bar{R}+\bar{R}_o-\bar{R}_1|\bar{R}_o) \quad . \tag{4.46b}$$

Noting that $H^*(\bar{R}_o) = G(\bar{R}|\bar{R}_o)$ and using (4.42), (4.44), (4.45), and (4.46) to find $A(\bar{R})$,

$$A(\bar{R}) = \iiint_{-\infty}^{\infty} e^{-i\bar{h}\cdot\bar{R}}\,\tilde{A}(\bar{h})\,d^3h = (2\pi)\iiint_{-\infty}^{\infty} e^{-i\bar{h}\cdot(\bar{R}+\bar{R}_o)}\,\tilde{J}(\bar{h})\tilde{G}(\bar{h}|\bar{R}_o)\,d^3h$$

$$= \iiint_{-\infty}^{\infty} J(\bar{R}_o)\,G(\bar{R}|\bar{R}_o)\,d^3R_o \quad , \tag{4.47}$$

which was to be proved. Substitution of the appropriate term for J yields Eq. (4.40). Similar results hold for $\phi(\bar{R})$. It should also be noted that this discussion holds independently of the operator form; this will be useful later on.

We turn our attention now to finding the Green's function that satisfies Eq. (4.39), which in cylindrical components can be written:

$$\frac{a}{r}\frac{\partial}{\partial r} r \frac{\partial G}{\partial r} + \frac{\partial^2 G}{\partial z^2} - \sigma\mu' a v \frac{\partial G}{\partial z} = -\delta(\bar{R}|\bar{R}_o) = -\frac{\delta(r - r_o)\,\delta(z - z_o)}{2\pi r} . \tag{4.48}$$

The parameter "a" can be either positive or negative, depending on the value of $n\beta$; for low velocities $n\beta < 1$, and $a > 0$. For velocities which are very high, $n\beta > 1$, and $a < 0$; in this case the velocity $v = \beta c$ of the medium is greater than the speed of light in the medium, c/n, and the Cerenkov radiation condition is met. We shall treat both conditions in this work.

The method of solving the differential equations is straightforward: by taking appropriate transforms, the differential equation can be transformed into an algebraic expression like (4.43). The transformed unknown can then be expressed as a ratio of polynomials. Upon taking the inverse transforms, the solution can be expressed as a multiple integral. If we are fortunate, the integrals may be reduced to a closed form. This will prove to be the case in the present work.

Case A. Low Velocities: $v < c/n$, and $a > 0$. Here we let $\alpha^2 = a$, and $b = \sigma\mu'\alpha^2 v/2$, and without loss of generality, we may choose $\bar{R}_o = 0$ temporarily. Then (4.48) becomes

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial G}{\partial r} + \frac{1}{\alpha^2}\frac{\partial^2 G}{\partial z^2} - \frac{2b}{\alpha^2}\frac{\partial G}{\partial z} = -\frac{\delta(r)\,\delta(z)}{2\pi r\,\alpha^2} \tag{4.49}$$

The Hankel transform technique is well suited to this problem. Given a function $f(r)$, the Hankel transform $H\{f\}$ is defined by

$$H\left\{f(r)\right\} = \int_0^\infty J_o(\lambda r)\, f(r)\, r\, dr \quad , \tag{4.50}$$

for all functions $f(r)$ of class L^1, i.e. such that $\int_0^\infty |f(r)| dr$ is bounded. From the well-known theory of Hankel transforms, the function $f(r)$ is related to its transform by

$$f(r) = \int_0^\infty J_o(\lambda r)\, H\{f\}\, \lambda\, d\lambda \quad . \tag{4.51}$$

It follows from the definition (4.50) that

$$H\left\{\frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial f}{\partial r}\right\} = -\lambda^2 H\{f\} \quad . \tag{4.52}$$

This can be shown as follows:

$$H\left\{\frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial f}{\partial r}\right\} = \int_0^\infty J_o(\lambda r)\frac{\partial}{\partial r}\left(r\,\frac{\partial f}{\partial r}\right) dr$$

$$= J_o(\lambda r)\, r\, \frac{\partial f}{\partial r}\Big|_0^\infty - \int_0^\infty r\,\frac{\partial f}{\partial r}\,\partial\,\frac{J_o(\lambda r)}{\partial r}\, dr$$

$$= 0 - f\, r\,\partial\,\frac{J_o(\lambda r)}{\partial r}\Big|_0^\infty + \int_0^\infty f(r)\left(\frac{1}{r}\frac{\partial}{\partial r}\, r\,\partial\,\frac{J_o(\lambda r)}{\partial r}\right) r\, dr$$

$$= -\lambda^2 H\{f\} \, , \tag{4.53}$$

where it is noted that from the definition of the Bessel function that

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial J_o(\lambda r)}{\partial r} + \lambda^2 J_o(\lambda r) = 0 \quad ,$$

and it is assumed that $f(r)$ has a behavior such that:

$$\lim_{r\to 0} \frac{\partial f}{\partial r} r J_o(\lambda r) = \lim_{r\to 0} r \frac{\partial f}{\partial r} = 0 \quad ,$$

$$\lim_{r\to 0} f(r)\, r \frac{\partial J_o(\lambda r)}{\partial r} = \lim_{r\to 0} r f(r)(-\lambda^2 r) = -\lambda^2 \lim_{r\to 0} r^2 f(r) = 0 \quad ,$$

$$\lim_{r\to\infty} \frac{\partial f}{\partial r} r J_o(\lambda r) = \lim_{r\to\infty} \sqrt{\frac{2}{\pi}} \sqrt{r} \frac{\partial f}{\partial r} = 0 \quad ,$$

and

$$\lim_{r\to\infty} r f(r) \frac{\partial J_o(\lambda r)}{\partial r} = \lim_{r\to\infty} \sqrt{\frac{2}{\pi}} \sqrt{r}\, f(r) = 0 \quad . \tag{4.54}$$

Taking the Hankel transform of Eq. (4.49) gives

$$\left(\lambda^2 - \frac{1}{\alpha^2}\frac{\partial^2}{\partial z^2} + \frac{2b}{\alpha^2}\right) H\{G\} = \frac{\delta(z)}{2\pi\alpha^2} \quad . \tag{4.55}$$

Further, taking the Fourier transform in z gives

$$\left(\lambda^2 + \frac{h^2 - 2ibh}{\alpha^2}\right) F\{H\{G\}\} = \frac{1}{4\pi^2\alpha^2} \quad , \tag{4.56}$$

or

$$F\{H\{G\}\} = \frac{1/4\pi^2}{(h-ib)^2 + \alpha^2\lambda^2 + b^2} \quad . \tag{4.57}$$

The roots of the denominator are at

$$h_{1,2} = ib \pm i\sqrt{\alpha^2\lambda^2 + b^2} \quad ; \tag{4.58}$$

one is in the upper half-plane, one in the lower for all λ. The inverse Fourier

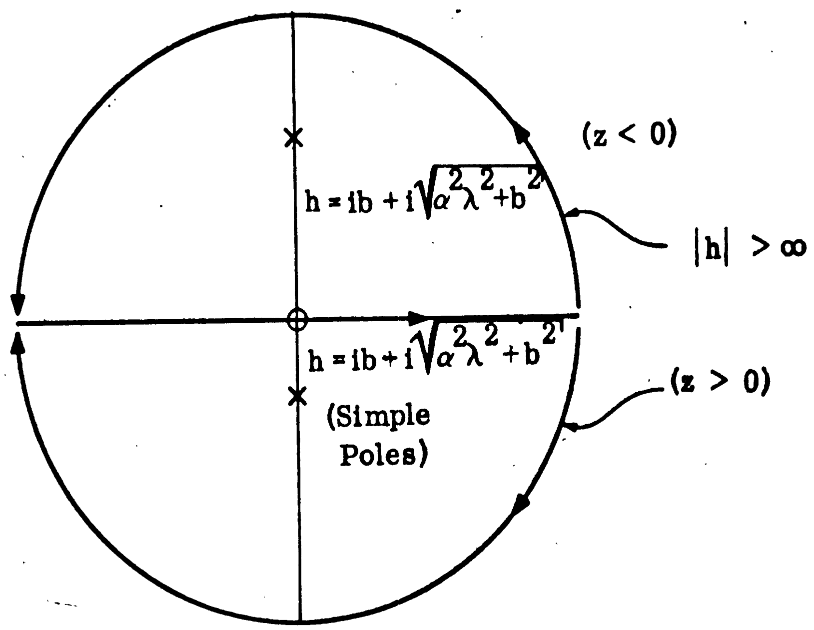

FIG. 4-1: CONTOURS IN THE h-PLANE FOR EVALUATING H $\{G\}$.

transform of (4.57) in h is given by

$$H\left\{G\right\} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{e^{-ihz}\,dh}{(h-h_1)(h-h_2)} \quad . \tag{4.59}$$

From the exponential it is evident that for $z < 0$ the contour can be closed in the upper half plane, and for $z > 0$ closed in the lower half plane, and the theory of residues applied. For $z < 0$, the residue is

$$\frac{e^{bz}\, e^{z\sqrt{\alpha^2\lambda^2 + b^2}}}{2i\sqrt{\alpha^2\lambda^2 + b^2}} ,$$

and for $z > 0$, it is

$$\frac{e^{bz}\, e^{-z\sqrt{\alpha^2\lambda^2 + b^2}}}{-2i\sqrt{\alpha^2\lambda^2 + b^2}} .$$

Noting that the contour in the lower half-plane is counterclockwise, (4.59) becomes

$$H\{G\} = \frac{1}{4\pi^2}\,(2\pi i)\,\frac{e^{bz}\, e^{-|z|\sqrt{\alpha^2\lambda^2 + b^2}}}{2i\sqrt{\alpha^2\lambda^2 + b^2}}$$

$$= \frac{1}{4\pi}\, e^{bz}\, \frac{e^{-|z|\sqrt{\alpha^2\lambda^2 + b^2}}}{\sqrt{\alpha^2\lambda^2 + b^2}} . \qquad (4.60)$$

The inverse Hankel transform, from Eq. (4.51), gives the integral

$$G = \frac{e^{bz}}{4\pi\alpha}\int_0^{\infty} J_o(\lambda r)\,\frac{e^{-\alpha|z|\sqrt{\lambda^2 + b^2/\alpha^2}}}{\sqrt{\lambda^2 + b^2/\alpha^2}}\,\lambda d\lambda . \qquad (4.61)$$

This can be solved by a change of variables: let $\xi^2 = \lambda^2 + b^2/\alpha^2$; then $\xi\, d\xi = \lambda\, d\lambda$, and the positive real axis in λ maps into the straight line contour $b/\alpha < \xi < \infty$ in ξ. Then Eq. (4.61) can be written

$$G = \frac{e^{bz}}{4\pi\alpha}\int_{b/\alpha}^{\infty} e^{-\alpha|z|\xi}\, J_o\left(r\sqrt{\xi^2 - b^2/\alpha^2}\right) d\xi . \qquad (4.62)$$

This is a tabulated Laplace transform given, for example, in Magnus and Oberhettinger (1954), p. 132. Finally, the Green's function can be written as

$$G(\bar{R}\,|\,0) = \frac{e^{bz}}{4\pi\alpha}\;\frac{e^{-\frac{b}{\alpha}\sqrt{r^2+\alpha^2 z^2}}}{\sqrt{r^2+\alpha^2 z^2}} . \tag{4.63}$$

Replacing $\bar{R}$ by $\bar{R} - \bar{R}_o$ and substituting for α and b gives

$$G(\bar{R}\,|\,\bar{R}_o) = \frac{e^{\sigma\frac{\mu' a v}{2}(z - z_o)}}{4\pi a^{1/2}}\;\frac{e^{-\sigma\frac{\mu' a^{1/2} v}{2} R_1}}{R_1} , \tag{4.64}$$

where $R_1 = \sqrt{(r - r_o)^2 + a(z - z_o)^2}$. This is the form desired. Note that as $\sigma \to 0$, this becomes simply

$$G(\bar{R}\,|\,\bar{R}_o) = \frac{1}{4\pi a^{1/2} R_1} . \tag{4.65}$$

<u>Case B. High Velocities:</u> $v > c/n$, and $a < 0$. Here we let $\alpha^2 = -a$ and define b as before, i.e. $b = \sigma\mu'\alpha^2 v/2$. Again letting $\bar{R}_o = 0$ Eq. (4.48) now becomes

$$\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial G}{\partial r} - \frac{1}{\alpha^2}\frac{\partial^2 G}{\partial z^2} - \frac{2b}{\alpha^2}\frac{\partial G}{\partial z} = \frac{\delta(r)\,\delta(z)}{2\pi r\,\alpha^2} . \tag{4.66}$$

Taking the Fourier transform first this time gives

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial}{\partial r} + \frac{h^2 + 2ihb}{\alpha^2}\right) F\left\{G\right\} = \frac{\delta(r)}{4\pi^2 r\,\alpha^2} . \tag{4.67}$$

Taking the Hankel transform and using (4.52) yields the algebraic expression

$$\left[\lambda^2 - \frac{h^2 + 2ihb}{\alpha^2}\right] \quad H\left\{F\left\{G\right\}\right\} = -\frac{1}{4\pi^2\alpha^2}\,, \tag{4.68}$$

or

$$H\left\{F\left\{G\right\}\right\} = \frac{-1/4\pi^2\alpha^2}{\lambda^2 - \left(\dfrac{h^2 + 2ihb}{\alpha^2}\right)}\,. \tag{4.69}$$

The inverse Hankel transform of (4.69), using Eq. (4.51) is then

$$F\left\{G\right\} = -\frac{1}{4\pi^2\alpha^2}\int_0^\infty \frac{J_o(\lambda r)\,\lambda\, d\lambda}{\lambda^2 - \left(\dfrac{h^2 + 2ihb}{\alpha^2}\right)}\,. \tag{4.70}$$

Now $J_o(\lambda r) = \frac{1}{2} H_o^{(1)}(\lambda r) + \frac{1}{2} H_o^{(2)}(\lambda r)$. If $R(\lambda^2)$ denotes a rational function in λ^2, then

$$\int_0^{-\infty} H_o^{(2)}(\lambda r)\, R(\lambda^2)\,\lambda\, d\lambda = \int_0^{-\infty} H_o^{(2)}(e^{i\pi}\lambda r)\, R(\lambda^2)\,(-\lambda)\,(-d\lambda)$$

$$= -\int_{-\infty}^{0} H_o^{(2)}(e^{i\pi}\lambda r)\, R(\lambda^2)\,\lambda\, d\lambda$$

$$= \int_{-\infty}^{0} H_o^{(1)}(\lambda r)\, R(\lambda^2)\,\lambda\, d\lambda\,, \tag{4.71}$$

since

$$H_o^{(2)}(e^{i\pi} z) = -H_o^{(1)}(z)\,,$$

from the circuit relations for the Hankel functions. (See, for example, Sommerfeld (1949), p. 315, (11)). Thus (4.70) can be written

$$F\left\{G\right\} = -\frac{1}{8\pi^2\alpha^2}\int_C \frac{H_0^{(1)}(\lambda r)\,\lambda d\lambda}{\lambda^2 - \left(\frac{h^2 + 2ihb}{\alpha^2}\right)}, \tag{4.72}$$

where the contour C is given in Fig. 4-2, and the branch cut must not be taken in the upper half-plane. (Otherwise, the circuit relation above could not hold).

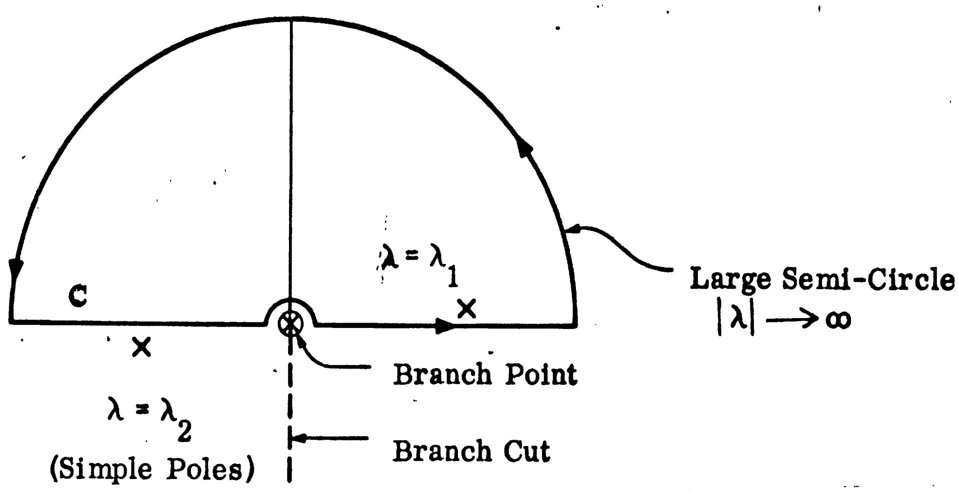

FIG. 4-2: CONTOURS IN THE λ-PLANE FOR EVALUATING F {G}.

It is well known that the asymptotic behavior of the Hankel function is given by

$$H_0^{(1)}(\lambda r) \sim \sqrt{\frac{2}{\pi\lambda r}}\; e^{i\lambda r}\, e^{-i\pi/4}$$

for large amplitudes of λ, and since $r > 0$, the contour can be closed in the upper half-plane, enclosing only the pole at $\lambda_1 = \frac{1}{\alpha}\sqrt{h^2 + 2ihb}$. By the theory of residues, (4.72) becomes

$$F\left\{G\right\} = -\frac{1}{8\pi^2\alpha^2} \cdot 2\pi i \cdot \frac{H_o^{(1)}\left[\frac{r}{\alpha}\sqrt{h^2 + 2ihb}\right]}{2\lambda_1} \lambda_1$$

$$= -\frac{i}{8\pi\alpha^2} H_o^{(1)} \left[\frac{r}{\alpha}\sqrt{h^2 + 2ihb}\right] . \qquad (4.73)$$

The inverse Fourier transform in h gives

$$G(\bar{R}|0) = -\frac{i}{8\pi\alpha^2}\int_{-\infty}^{\infty} e^{-ihz} H_o^{(1)}\left[\frac{r}{\alpha}\sqrt{h^2 + 2ihb}\right] dh . \qquad (4.74)$$

The argument of the Hankel function vanishes at $h = 0$ and $h = -2ib$, the Hankel function itself behaves logarithmically at these points, so that these points are branch points, and the branch cuts must extend to infinity. Thus it is appropriate to choose the branch cut so that it lies along the negative imaginary axis, as in Fig. 4-3.

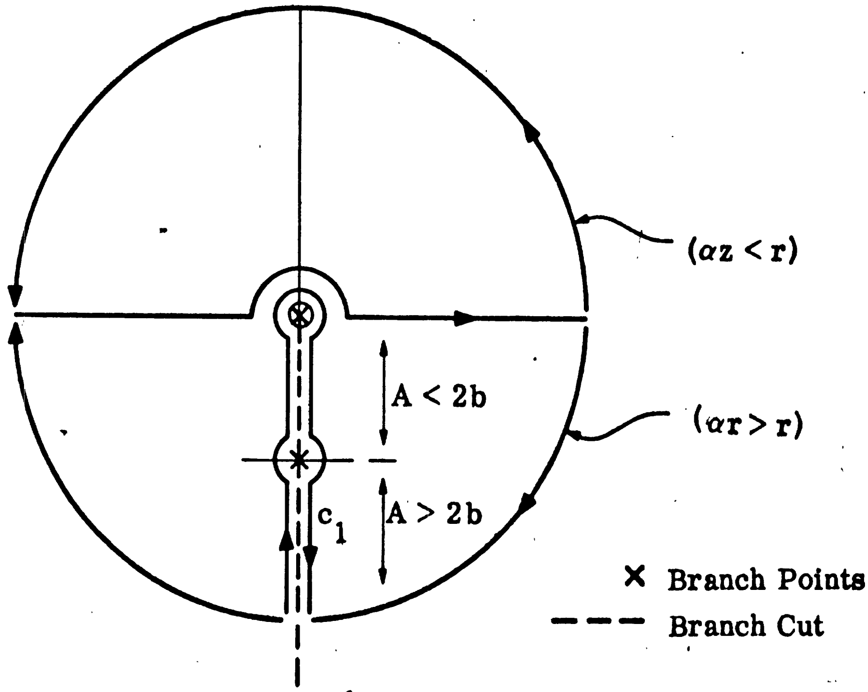

FIG. 4-3: CONTOURS IN THE h-PLANE FOR EVALUATING G ($\bar{R}$|0) .

For large amplitudes of h, the integrand behaves as

$$\exp\left[-ih\left(z - \frac{r}{\alpha}\right)\right] .$$

Thus for $z < r/\alpha$, the contour can be closed in the upper-half plane, and for $z > r/\alpha$ it can be closed in the lower half-plane.

For $z < r/\alpha$ the contour in the upper-half plane encloses no poles and encounters no branch cuts; furthermore, since the Hankel function behaves logarithmically near the branch points, the integral around the branch points points vanishes, leaving

$$G(\bar{R}|0) = 0 \quad \text{for} \quad z < r/\alpha . \tag{4.75}$$

For $z > r/\alpha$, the presence of the branch cut dictates that the imaginary axis cannot be crossed, and since no poles are enclosed,

$$\int_C = \int_{C_1} , \tag{4.76}$$

where the contours are indicated in Fig. 4-7.

It is necessary to examine the argument of the Hankel function with some care. Assuming the radical is taken as positive, the arguments of the radical can be obtained for the contour C_1 in the following manner: for $\arg h = -\pi/2$, or $h = -iA$, where $A = |h|$,

$$\arg\sqrt{h^2 + 2ihb} = \arg\sqrt{(-i)^2 A^2 + 2Ab} = \begin{cases} \arg\left[\sqrt{2Ab - A^2}\right], & A < 2b \\ \arg\left[-i\sqrt{A^2 - 2Ab}\right], & A > 2b \end{cases}$$

$$= \begin{cases} 0, & A < 2b \\ -\pi/2, & A > 2b \end{cases} . \tag{4.77}$$

Similarly for $\arg h = 3\pi/2$, or $h = i^3 A$,

$$\arg\sqrt{h^2 + 2ihb} = \arg\sqrt{i^6 A^2 + 2i^4 Ab}$$

$$= \arg\ i^2\sqrt{2Ab + i^2 A^2}$$

$$= \begin{cases} \arg\left(i^2\sqrt{2Ab - A^2}\right), & A < 2b \\ \arg\left(i^3\sqrt{A^2 - 2Ab}\right) & A > 2b \end{cases}$$

$$= \begin{cases} \pi, & A < 2b \\ 3\pi/2, & A > 2b \end{cases} . \tag{4.78}$$

Comparing (4.77) with (4.78), it can be seen that for $A < 2b$, the argument of the radical along the left side of the contour C_1 differs from that along the right by π, and for $A > 2b$, this difference is 2π.

Thus for $z > r/\alpha$, (4.74) can be written

$$G(\bar{R}|0) = -\frac{i}{8\pi\alpha^2}\left\{\int_0^{-i2b} - \int_0^{i^3 2b} + \int_{-i2b}^{-i\infty} - \int_{i^3 2b}^{i^3\infty}\right\}$$

$$= -\frac{i}{8\pi\alpha^2}\left\{\int_0^{-i2b} e^{-ihz}\left[H_0^{(1)}\left(\frac{r}{\alpha}\sqrt{h^2 + 2ibh}\right)\right.\right.$$

$$\left. - H_0^{(1)}\left(\frac{r}{\alpha}e^{i\pi}\sqrt{h^2 + 2ibh}\right)\right] dh$$

$$\left. + \int_{-i2b}^{-i\infty} e^{-ihz}\left[H_0^{(1)}\left(\frac{r}{\alpha}\sqrt{h^2 + 2ibh}\right) - H_0^{(1)}\left(\frac{r}{\alpha}e^{i2\pi}\sqrt{h^2 + 2ibh}\right)\right] dh\right\}$$

(4.79)

Substituting $u = i(h + ib)$ in the first integral and $v = i(h + ib)$ in the second, this becomes

$$G(\bar{R}|0) = -\frac{e^{-bz}}{8\pi\alpha^2}\left\{\int_{-b}^{b} e^{-uz}\left[H_0^{(1)}\left(\frac{r}{\alpha}\sqrt{b^2-u^2}\right) - H_0^{(1)}\left(\frac{r}{\alpha}e^{i\pi}\sqrt{b^2-u^2}\right)\right]du\right.$$

$$\left. + \int_b^{\infty} e^{-vz}\, H_0^{(1)}\left(-\frac{ir}{\alpha}\sqrt{v^2-b^2}\right) - H_0^{(1)}\left(-\frac{ir}{\alpha}e^{i2\pi}\sqrt{v^2-b^2}\right)dv\right\}. \tag{4.80}$$

From the circuit relations given, for example, in Sommerfeld, (1949) p. 314, we have

$$H_0^{(1)}(e^{i\pi}z) = -H_0^{(2)}(z)$$

and

$$H_0^{(1)}(e^{i2\pi}z) = -2H_0^{(2)}(z) - H_0^{(1)}(z) , \tag{4.81}$$

which when substituted into (4.80) gives

$$G(\bar{R}|0) = -\frac{e^{-bz}}{8\pi\alpha^2}\left\{2\int_{-b}^{b} e^{-uz}J_0\left(\frac{r}{\alpha}\sqrt{b^2-u^2}\right)du + 4\int_b^{\infty} e^{-vz}J_0\left(-\frac{ir}{\alpha}\sqrt{v^2-b^2}\right)dv\right\}. \tag{4.82}$$

Since $J_0(-iz) = J_0(iz) = I_0(z)$, where I_0 is the modified Bessel function, this can be written

$$G(\bar{R}|0) = -\frac{e^{-bz}}{2\pi\alpha^2}\left\{\frac{1}{2}\int_{-b}^{b} e^{-uz}J_0\left(\frac{r}{\alpha}\sqrt{b^2-u^2}\right)du + \int_b^{\infty} e^{-vz}I_0\left(\frac{r}{\alpha}\sqrt{v^2-b^2}\right)dv\right\}. \tag{4.83}$$

The right-hand integral is a tabulated Laplace transform given, for example, in Magnus and Oberhettinger (1954), p. 134:

$$\int_b^{\infty} e^{-vz} I_o\left(\frac{r}{\alpha}\sqrt{v^2 - b^2}\right) dv = \frac{e^{-b\sqrt{z^2 - \frac{r^2}{\alpha^2}}}}{\sqrt{z^2 - r^2/\alpha^2}} \ . \tag{4.84}$$

The finite integral of (4.83) is more involved. First of all, we note that J_o is an even function in u, so that

$$\frac{1}{2}\int_{-b}^{b} e^{-uz} J_o\left(\frac{r}{\alpha}\sqrt{b^2 - u^2}\right) du = \frac{1}{2}\int_{-b}^{b} \cosh(uz)\, J_o\left(\frac{r}{\alpha}\sqrt{b^2 - u^2}\right) du$$

$$= \int_o^b \cosh uz\, J_o\left(\frac{r}{\alpha}\sqrt{b^2 - u^2}\right) du = \int_o^b \cos(iuz)\, J_o\left(\frac{r}{\alpha}\sqrt{b^2 - u^2}\right) du$$

$$= \frac{\pi i z}{2}\int_o^b J_{-1/2}(iuz)\, J_o\left(\frac{r}{\alpha}\sqrt{b^2 - u^2}\right) \sqrt{u}\, du \ ,$$

since $J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z$. Let $u = b \sin\theta$: then $\sqrt{b^2 - u^2} = b\cos\theta$, and this becomes

$$\sqrt{\frac{\pi i z b^3}{2}} \int_o^{\pi/2} J_{-1/2}(ibz\sin\theta)\, J_o\left(\frac{rb}{\alpha}\cos\theta\right) \sin^{1/2}\theta \cos\theta\, d\theta \ . \tag{4.85}$$

Now Sonine's second finite integral can be written (see Watson (1922), p. 376)

$$\int_o^{\pi/2} J_{\mu}(z_1 \sin\theta)\, J_{\nu}(Z\cos\theta) \sin^{\mu+1}\theta \cos^{\nu+1}\theta\, d\theta$$

$$= z_1^{\mu} \frac{Z^{\nu} J_{\mu+\nu+1}\left(\sqrt{Z^2 + z_1^2}\right)}{(Z^2 + z_1^2)^{1/2(\nu+\mu+1)}} \ . \tag{4.86}$$

Thus by letting $\mu = -1/2$, $\nu = 0$, $z_1 = ibz$, $Z = rb/\alpha$, (4.85) becomes

$$\sqrt{\frac{\pi i z b^3}{2}} \cdot (ibz)^{-1/2} \frac{J_{1/2}\left(b\sqrt{\frac{r^2}{\alpha^2} - z^2}\right)}{b^{1/2}\sqrt[4]{\frac{r^2}{\alpha^2} - z^2}} = \frac{\sin\left(ib\sqrt{z^2 - r^2/\alpha^2}\right)}{i\sqrt{z^2 - r^2/\alpha^2}}$$

$$= \frac{\sinh\left(b\sqrt{z^2 - r^2/\alpha^2}\right)}{\sqrt{z^2 - r^2/\alpha^2}} . \tag{4.87}$$

Thus, using (4.83), (4.84), and (4.87), we find that

$$G(\bar{R}|0) = -\frac{e^{-bz}}{2\pi\alpha^2}\left[\frac{\sinh\left(b\sqrt{z^2 - r^2/\alpha^2}\right) + e^{-b\sqrt{z^2 - r^2/\alpha^2}}}{\sqrt{z^2 - r^2/\alpha^2}}\right]$$

$$= -\frac{e^{-bz}}{2\pi\alpha} \frac{\cosh\left(\frac{b}{\alpha}\sqrt{z^2\alpha^2 - r^2}\right)}{\sqrt{z^2\alpha^2 - r^2}} . \tag{4.88}$$

Letting R be replaced by R_o and noting the definitions of α and b, we have the desired solution for the Green's function:

$$G(\bar{R}|\bar{R}_o) = \begin{cases} 0, & |a|(z - z_o) < (r - r_o) \\ -\dfrac{e^{-\sigma\frac{\mu'|a|v}{2}(z - z_o)}}{2\pi|a|^{1/2}} \dfrac{\cosh\left[\frac{1}{2}\sigma\mu'|a|^{1/2} v R_2\right]}{R_2}, & |a|(z - z_o) > (r - r_o) \end{cases} , \tag{4.89}$$

where $R_2 = \sqrt{(z - z_o)^2|a| - (r - r_o)^2}$. The Cerenkov cone is defined by

$$|a|^{1/2}(z - z_0) = r - r_0 \quad ,$$

or

$$(z - z_0) = \gamma\sqrt{n^2\beta^2 -}\;(r - r_0) \quad . \tag{4.90}$$

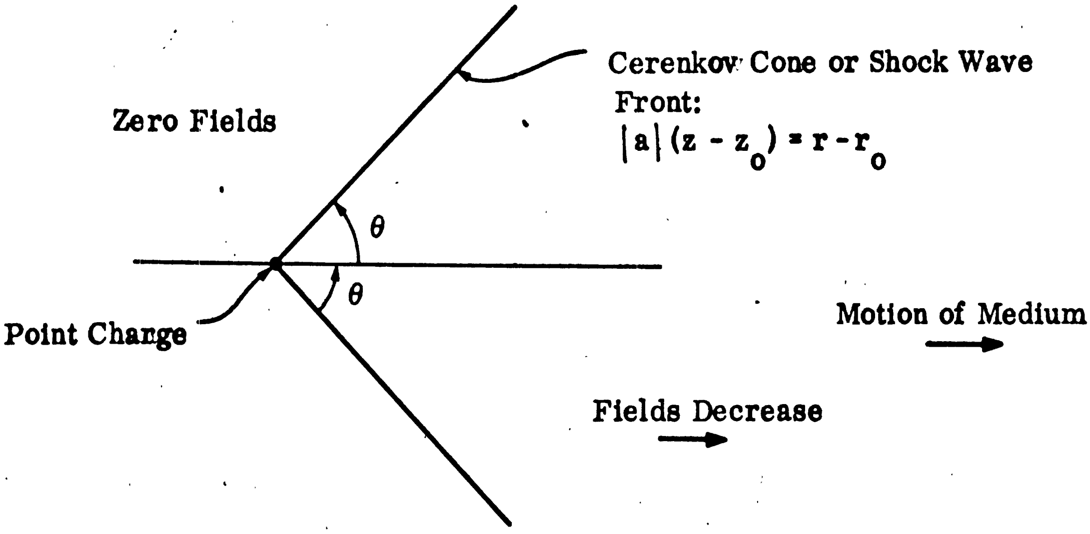

FIG. 4-4: CERENKOV CONE GEOMETRY FOR HIGH VELOCITIES.

In Fig. 4-4,

$$\theta = \cot^{-1}\left(\gamma\sqrt{n^2\beta^2 - 1}\right) = \cos^{-1}\left(\frac{\sqrt{n^2\beta^2 - 1}}{\sqrt{n^2\beta^2 - \beta^2}}\right) , \tag{4.91}$$

which for β small, while $n\beta > 1$, approaches the familiar shock wave formula

$$\theta \cong \sin^{-1}(1/n\beta) \quad . \tag{4.92}$$

From the solution (4.89) it can be seen that as $z - z_0$ increases, the solution decays exponentially, since the exponential dominates over the hyperbolic cosine function of large values of $z - z_o$. For lossless media, this approaches the well-known result

$$G(\bar{R}|\bar{R}_o) \doteq \begin{Bmatrix} 0\,, & z < r/|a|^{1/2} \\ -\dfrac{1}{2\pi|a|^{1/2}R_2}\,, & z > r/|a|^{1/2} \end{Bmatrix} . \qquad (4.93)$$

4.1.3 Summary

Let us now summarize the results for static charge distributions, where $\partial/\partial t = 0$: given a static source charge distribution $\rho_s(\bar{R}_o)$ in a moving conducting medium, the fields are related to the vector and scalar potentials by

$$\bar{B} = \nabla \times \bar{A}, \quad \bar{E} = -\nabla\phi ,$$

$$\bar{H} = \frac{1}{\mu' a} \nabla \times (\bar{A} + \bar{\Omega}\phi) ,$$

$$\bar{D} = \epsilon' \bar{\bar{\alpha}} \cdot \nabla\phi + \frac{1}{\mu' a} \bar{\Omega} \times \left[\nabla \times (\bar{A} + \bar{\Omega}\phi)\right] , \qquad (4.94)$$

and the potentials are related to the sources by

$$\bar{A}(\bar{R}) = -\frac{1}{\epsilon'} \iiint_{V_o} G(\bar{R}|\bar{R}_o)\left[\bar{\Omega}\rho_s(\bar{R}_o) - \frac{a\bar{v}}{c^2}\gamma\rho_r'(\bar{R}_o)\right] dV_o ,$$

$$\phi(\bar{R}) = \frac{1}{\epsilon'} \iiint_{V_o} G(\bar{R}|\bar{R}_o)\left[\rho_s(\bar{R}_o) + a\gamma\rho_r'(\bar{R}_o)\right] dV_o , \qquad (4.95)$$

where the volume V_o encloses the sources; the response charge is related to the source charge density by

$$\gamma \rho_r'(\bar{R}_o) = -\frac{\sigma}{\epsilon' v} \int_{-\infty}^{z_o} \exp\left[-\frac{\sigma}{\epsilon' \gamma_\perp^2 v}(z_o - \zeta)\right] \rho_s(\bar{r}_o, \zeta)\, d\zeta \,, \qquad (4.96)$$

and the Green's function is given by

$$G(\bar{R}|\bar{R}_o) = \frac{e^{\sigma \frac{\mu' a v}{2}(z - z_o)}\, e^{-\sigma \frac{\mu' a^{1/2} v}{2} R_1}}{4\pi a^{1/2} R_1} \,, \quad n\beta < 1$$

or

$$G(\bar{R}|\bar{R}_o) = \left\{ \begin{array}{ll} 0 \,, & , |a|(z - z_o) < (r - r_o) \\ -\dfrac{e^{-\sigma \frac{\mu'|a| v}{2}(z - z_o)} \cosh\left(\sigma \frac{\mu'|a|^{1/2} v}{2} R_2\right)}{2\pi |a|^{1/2} R_2} & , |a|(z - z_o) > (r - r_o) \end{array} \right\} ,$$

$$n\beta > 1 \,,$$

where

$$R_1^2 = (r - r_o)^2 + a(z - z_o)^2$$

and (4.97)

$$R_2^2 = |a|(z - z_o)^2 - (r - r_o)^2 \,,$$

4.2 Harmonic Current Source Distributions

4.2.1 Differential Equations for the Potentials

One way to approach the problem of harmonic current source distributions would be to develop a differential equation of the form (4.31) for current sources, and make the substitution $\partial/\partial t = -i\omega$. It turns out that there is another approach which develops a Green's function equation that is considerably simpler. This development follows that of Tai (1965b) for lossless media, and involves the introduction of a set of potentials differing from $\bar{A}$ and ϕ.

For harmonically oscillating fields, Maxwell's equations can be written, where all quantities have the time dependence $e^{-i\omega t}$,

$$\nabla \times \bar{E} = i\omega\bar{B} \text{ (Ih)}, \quad \nabla \cdot \bar{D} = \rho \text{ (IIh)}, \quad \nabla \times \bar{H} = \bar{J} - i\omega\bar{D} \text{ (IIIh)}, \quad \nabla \cdot \bar{B} = 0 \text{ (IVh)}.$$

The constitutive relations are given by

$$\bar{B} = \mu' \bar{\bar{\alpha}} \cdot \bar{H} - \bar{\Omega} \times \bar{E} ,$$

$$\bar{D} = \epsilon' \bar{\bar{\alpha}} \cdot \bar{E} + \bar{\Omega} \times \bar{H} ,$$

and

$$\bar{J} = \bar{J}_s + \gamma \rho_r' \bar{v} + \sigma(\bar{\bar{\alpha}} \cdot \bar{E} + \mu' a\bar{v} \times \bar{H}) ,$$

where

$$\rho = \rho_s + \gamma \rho_r' + \sigma \frac{\bar{v} \cdot \bar{E}}{c^2} . \tag{4.98}$$

Substituting the constitutive relations into Maxwell's equations (I) - (IV) and eliminating $\bar{B}$ and $\bar{D}$, we get for (Ih) and (IIIh),

$$(\nabla + i\omega\bar{\Omega}) \times \bar{E} = i\omega\mu'\bar{\bar{\alpha}} \cdot \bar{H} ,$$

and

$$(\nabla + i\omega\bar{\Omega} - \sigma\mu' a\bar{v}) \times \bar{H} = -i\omega\epsilon' \bar{\bar{\alpha}} \cdot \bar{E} + \bar{J}_s + \gamma\rho_r' \bar{v} . \tag{4.99}$$

Let

$$\bar{p} = \sigma\mu' a\bar{v} .$$

Equation (4.99) can be simplified by introducing two auxiliary field vectors $\bar{E}_1$ and $\bar{H}_1$ defined as follows:

$$\bar{E}_1 = e^{i\omega\Omega z}\,\bar{E} \ ,$$

and

$$\bar{H}_1 = e^{(i\omega\Omega - p)z}\,\bar{H} \ . \tag{4.100}$$

Then

$$\nabla \times \bar{E} = \nabla \times (e^{-i\omega\Omega z}\bar{E}_1) = e^{-i\omega\Omega z}\,\nabla \times \bar{E}_1 + \nabla(e^{-i\omega\Omega z}) \times \bar{E}_1$$

$$= e^{-i\omega\Omega z}\left[\nabla \times \bar{E}_1 - i\omega\bar{\Omega} \times \bar{E}_1\right] \ . \tag{4.101}$$

Similarly

$$\nabla \times \bar{H} = e^{(-i\omega\Omega + p)z}\left[\nabla \times \bar{H}_1 - i\omega\bar{\Omega} \times \bar{H}_1 + \bar{p} \times \bar{H}_1\right] \ . \quad (4.102$$

Substituting these relations into (4.99) gives

$$\nabla \times \bar{E}_1 = i\omega\mu'\, e^{pz}\,\bar{\bar{\alpha}} \cdot \bar{H}_1 \ , \tag{4.103a}$$

and

$$\nabla \times \bar{H}_1 = (-i\omega\epsilon' + \sigma)\, e^{-pz}\,\bar{\bar{\alpha}} \cdot \bar{E}_1 + (\bar{J}_s + \gamma\rho_r'\bar{v})\, e^{(i\omega\Omega - p)z} \ . \tag{4.103b}$$

By taking the divergence of the first relation (4.103a),

$$\nabla \cdot \nabla \times \bar{E}_1 = 0 = i\omega\mu'\,\nabla \cdot (e^{pz}\,\bar{\bar{\alpha}} \cdot \bar{H}_1) \ , \tag{4.104}$$

so that we can partially define a vector potential $\bar{A}_1$ by

$$\mu' e^{pz}\,\bar{\bar{\alpha}} \cdot \bar{H}_1 = \nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{A}_1) \ ,$$

or

$$\mu'\bar{H}_1 = e^{-pz}\,\bar{\bar{\alpha}}^{-1} \cdot \left[\nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{A}_1)\right] \ . \tag{4.105}$$

From Eq. (4.103), $\bar{E}_1$ is then related by

$$\bar{E}_1 = i\omega \bar{\bar{\alpha}}^{-1} \cdot \bar{A}_1 - \nabla \phi_1 , \tag{4.106}$$

where ϕ_1 is a scalar potential.

Substituting the potentials into the second relation (4.103b), we get

$$\nabla \times \left[e^{-pz} \bar{\bar{\alpha}}^{-1} \cdot \left[\nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{A}_1) \right] \right] = \mu' (-i\omega\epsilon' + \sigma) e^{-pz} (i\omega \bar{A}_1 - \bar{\bar{\alpha}} \cdot \nabla \phi_1)$$

$$+ \mu' (\bar{J}_s + \gamma \rho'_r \bar{v}) e^{(i\omega\Omega - p) z} \quad . \tag{4.107}$$

Using the vector identity

$$\nabla \times (\psi \bar{a}) = \psi \nabla \times \bar{a} + \nabla \psi \times \bar{a} \quad ,$$

Eq. (4.107) can be written, after regrouping terms, as

$$\nabla \times \left[\bar{\bar{\alpha}}^{-1} \cdot \left[\nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{A}_1) \right] \right] - \frac{\bar{p}}{a} \times \left[\nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{A}_1) \right] - k^2 \bar{A}_1$$

$$= (i\omega\mu'\epsilon' - \sigma\mu') \bar{\bar{\alpha}} \cdot \nabla \phi_1 + \mu' (\bar{J}_s + \gamma \rho'_r \bar{v}) e^{i\omega\Omega z} \quad , \tag{4.108}$$

where $k^2 = \omega^2 \mu' \epsilon' + i\omega\sigma\mu'$.

Similarly, using (IIh), another equation can be found:

$$\nabla \cdot \bar{D} = \epsilon' \nabla \cdot \bar{\bar{\alpha}} \cdot \bar{E} + \nabla \cdot (\bar{\Omega} \times \bar{H}) = \epsilon' \nabla \cdot \bar{\bar{\alpha}} \cdot \bar{E} - \bar{\Omega} \cdot \nabla \times \bar{H}$$

$$= \epsilon' \nabla \cdot \bar{\bar{\alpha}} \cdot (e^{-i\omega\Omega z} \bar{E}_1) - \bar{\Omega} \cdot \nabla \times (e^{-(i\omega\Omega - p) z} \bar{H}_1)$$

$$= e^{-i\omega\Omega z} \epsilon' (\nabla \cdot \bar{\bar{\alpha}} \cdot \bar{E}_1 - i\omega \bar{\Omega} \cdot \bar{E}_1)$$

$$- e^{-(i\omega\Omega - p) z} \bar{\Omega} \cdot \left[\nabla \times \bar{H}_1 - (i\omega \bar{\Omega} - \bar{p}) \times \bar{H}_1 \right]$$

$$= e^{-i\omega\Omega z} \epsilon' (\nabla \cdot \bar{\bar{\alpha}} \cdot \bar{E}_1 - i\omega \bar{\Omega} \cdot \bar{E}_1)$$

$$- e^{-(i\omega\Omega - p) z} \bar{\Omega} \cdot \left[e^{-pz} (-i\omega\epsilon' + \sigma) \bar{\bar{\alpha}} \cdot \bar{E}_1 + e^{(i\omega\Omega - p) z} (\bar{J}_s + \gamma \rho'_r \bar{v}) \right]$$

$$= e^{-i\omega\Omega z}\left[\epsilon' \nabla\cdot\bar{\bar{\alpha}}\cdot\bar{E}_1 - \sigma\bar{\Omega}\cdot\bar{E}_1\right] - \bar{\Omega}\cdot\bar{J}_s - \gamma\rho_r' v\Omega$$

$$= \rho_s + \gamma\rho_r' + \sigma\frac{\bar{v}\cdot\bar{E}_1}{c^2}$$

$$= \rho_s + \gamma\rho_r' + \sigma\frac{\bar{v}\cdot\bar{E}_1}{c^2} e^{-i\omega\Omega z} \quad . \tag{4.109}$$

Substituting the potentials through the relation (4.106), and grouping terms, we obtain

$$\nabla\cdot\bar{\bar{\alpha}}\cdot\nabla\phi_1 - \frac{\sigma}{\epsilon'}\left(\frac{\bar{v}}{c^2}+\bar{\Omega}\right)\cdot\nabla\phi_1 = i\omega\nabla\cdot\bar{A}_1 - \frac{i\omega\sigma}{\epsilon'}\left(\frac{\bar{v}}{c^2}+\bar{\Omega}\right)\cdot\bar{A}_1$$

$$-\frac{1}{\epsilon'}\left[\bar{\Omega}\cdot\bar{J}_s + (1+v\Omega)\gamma\rho_r' + \rho_s\right]e^{i\omega\Omega z} \quad . \tag{4.110}$$

Since $1+v\Omega = a$, and $\frac{v}{c^2}+\Omega = \mu'\epsilon' a v$, thus can be written as

$$\nabla\cdot\bar{\bar{\alpha}}\cdot\nabla\phi_1 - \bar{p}\cdot\nabla\phi_1 = i\omega\nabla\cdot\bar{A}_1 - i\omega\bar{p}\cdot\bar{A}_1 - \frac{1}{\epsilon'}\left[\bar{\Omega}\cdot\bar{J}_s + \rho_s + a\gamma\rho_r'\right]e^{i\omega\Omega z} \quad . \tag{4.111}$$

Equations (4.108) and (4.111) are two coupled equations for $\bar{A}_1$ and ϕ_1. We are free to further define the potentials by a gauge condition, which we choose to be=

$$\nabla\cdot\bar{A}_1 - \bar{p}\cdot\bar{A}_1 = (i\omega\epsilon' - \sigma)\mu' a^2\phi_1 \quad . \tag{4.112}$$

We have immediately then, from (4.111),

$$\nabla\cdot\bar{\bar{\alpha}}\cdot\nabla\phi_1 - \bar{p}\cdot\nabla\phi_1 + k^2a^2\phi_1 = -\frac{1}{\epsilon'}\left[\bar{\Omega}\cdot\bar{J}_s + \rho_s + a\gamma\rho_r'\right]e^{i\omega\Omega z} \quad . \tag{4.113}$$

Turning our attention to Eq. (4.108), it can be seen that (4.14) can be applied directly to the first term. Noting the vector identity

$$\nabla(\bar{c} \cdot \bar{F}) = \bar{c} \times (\nabla \times \bar{F}) + (\bar{c} \cdot \nabla) \bar{F}$$

for constant $\bar{c}$, the second term of (4.108) becomes

$$\frac{1}{a} \bar{p} \times \left[\nabla \times (\bar{\bar{\alpha}}^{-1} \cdot \bar{A}_1)\right] = \frac{1}{a} \nabla(\bar{p} \cdot \bar{A}_1) - \frac{1}{a} (\bar{p} \cdot \nabla) \bar{\bar{\alpha}}^{-1} \cdot \bar{A}_1 . \tag{4.114}$$

Using (4.14) and (4.114), Eq. (4.108) becomes

$$(\nabla \cdot \bar{\bar{\alpha}} \cdot \nabla)\bar{A}_1 - (\bar{\bar{\alpha}} \cdot \nabla)(\nabla \cdot \bar{A}_1) + a \nabla(\bar{p} \cdot \bar{A}_1) - a (\bar{p} \cdot \nabla)\bar{\bar{\alpha}}^{-1} \cdot \bar{A}_1 + k^2 a^2 \bar{A}_1$$

$$= -(i\omega\epsilon' - \sigma)\mu' a^2 \bar{\bar{\alpha}} \cdot \nabla\phi_1 - \mu' a^2 (\bar{J}_s + \gamma \rho_r' \bar{v}) e^{i\omega\Omega z} . \tag{4.115}$$

By breaking the terms up into components it can be shown that

$$a \nabla(\bar{p} \cdot \bar{A}_1) - a (\bar{p} \cdot \nabla)(\bar{\bar{\alpha}}^{-1} \cdot \bar{A}_1) = -(\bar{p} \cdot \nabla) \bar{A}_1 + (\bar{\bar{\alpha}} \cdot \nabla)(\bar{p} \cdot \bar{A}_1) . \tag{4.116}$$

By substituting (4.116) and the gauge condition (4.112) into (4.115), we have, finally,

$$(\nabla \cdot \bar{\bar{\alpha}} \cdot \nabla)\bar{A}_1 - (\bar{p} \cdot \nabla)\bar{A}_1 + k^2 a^2 \bar{A}_1 = -\mu' a^2 (\bar{J}_s + \gamma \rho_r' \bar{v}) e^{i\omega\Omega z} , \tag{4.117}$$

which involves no terms in ϕ_1. Comparison with Eq. (4.113) shows that the vector and scalar potentials satisfy the same differential equation, except for the source terms, and thus can be found from the same Green's function, ignoring the homogeneous solutions. Furthermore, by comparing this expression with (4.31), it is evident that this formulation is considerably simpler.

4.2.2 Green's Function Solution

From an inspection of Eqs. (4.113) and (4.117), it is evident that the appropriate Green's function for the problem satisfies the following differential equation:

$$(\nabla \cdot \bar{\bar{\alpha}} \cdot \nabla) G - \bar{p} \cdot \nabla G + k^2 a^2 G = -\delta(\bar{R}|\bar{R}_0) . \tag{4.118}$$

It was remarked before that the discussion of Section 4.1.2 does not depend on the form of the operator, so that the results of that section may be applied directly to this case. Thus

$$\bar{A}(\bar{R}) = \mu' a^2 \iiint_{V_o} G(\bar{R}|\bar{R}_o)\left[\bar{J}_s(\bar{R}_o) + \bar{v}\,\gamma\,\rho_r'(\bar{R}_o)\right] e^{i\omega\Omega z_o}\, dV_o$$

and

$$\phi(\bar{R}) = \frac{1}{\epsilon'} \iiint_{V_o} G(\bar{R}|\bar{R}_o)\left[\bar{\Omega}\cdot \bar{J}_s(R_o) + \rho_s(\bar{R}_o) + a\,\gamma\rho_r'(\bar{R}_o)\right] e^{i\omega\Omega z_o}\, dV_o \,, \tag{4.119}$$

where V_o encloses the sources, and $dV_o = d^3R_o = dx_o\, dy_o\, dz_o$.

In cylindrical components, Eq. (4.118)

$$\left(\frac{\alpha}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - p\frac{\partial}{\partial z} + k^2 a^2\right) G(\bar{R}|\bar{R}_o) = -\,\delta\,(\bar{R}|\bar{R}_o) \,. \tag{4.120}$$

As with the static charge distribution, there are two conditions which give rise to different solutions: for low velocities such that $v > c/n$, and $a > 0$, and for high velocities such that $v > c/n$, or $a > 0$.

<u>Case A. Low Velocities:</u> $v < c/n$. Let $\alpha^2 = a$, and $b = p/2$. Again without loss of generality we may take $\bar{R}_o = 0$ for the time being. Then Eq. (4.120) can be written as

$$\left(\frac{\alpha^2}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - 2b\frac{\partial}{\partial z} + k^2\alpha^4\right) G(\bar{R}|0) = -\,\frac{\delta(r)\,\delta(z)}{2\pi r} \,. \tag{4.121}$$

Taking the Hankel transform in r and the Fourier transform in z yields the algebraic equation

$$(\lambda^2\alpha^2 + h^2 - 2ihb - k^2\alpha^4)\;F\left\{H\;\{G\}\right\} = 1/4\pi^2 \,, \tag{4.122}$$

or

$$F\left\{H\left\{G\right\}\right\} = \frac{1}{4\pi^2}\,\frac{1}{h^2 - 2ihb + \lambda^2\alpha^2 - k^2\alpha^4} \tag{4.123}$$

Taking the inverse Fourier transform in h, we obtain the integral

$$H\left\{G\right\} = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{e^{-ihz}\,dh}{h^2 - 2ihb + \lambda^2\alpha^2 - k^2\alpha^4} \quad . \tag{4.124}$$

It is necessary to carefully examine the location of the roots of the denominator of the integrand. Expanded, the denominator is

$$(h - ib)^2 + \lambda^2\alpha^2 - \frac{\omega^2 n^2\alpha^4}{c^2} + b^2 - i\,\frac{\sigma\omega n^2\alpha^4}{\epsilon' c^2} \quad . \tag{4.125}$$

By inspection, the roots are given by

$$h_{1,2} = ib \pm ib\sqrt{1 + \frac{\lambda^2\alpha^2}{b^2} - \frac{\omega^2 n^2\alpha^4}{b^2c^2} - i\,\frac{\sigma\omega n^2\alpha^4}{\epsilon' c^2}} \quad . \tag{4.126}$$

If the real part of the radical is greater than one, then there will be one root in the upper-half plane, and one in the lower. We are thus interested in the range of ω for which this is true for all real λ. The worst case is obviously for $\lambda = 0$; thus we let λ vanish and consider the radical

$$u + iv = \sqrt{1 - \frac{\omega^2 n^2\alpha^2}{b^2c^2} - i\,\frac{\sigma\omega n^2\alpha^2}{\epsilon' c^2}} \quad , \tag{4.127}$$

where u and v are real. Substituting $x = \omega\epsilon'/\sigma$, this simplifies to

$$u + iv = \frac{1}{n\beta}\sqrt{(1 - 2ix)^2 - (1 - n^2\beta^2)} \quad . \tag{4.128}$$

We first note that at $x = 0$, $u = 1$, and at large values of $|x|$,

$$u + iv \sim \frac{(1-2ix)}{n\beta}$$

or

$$u \sim \frac{1}{n\beta} > 1 \quad . \tag{4.129}$$

This suggests that u has a minimum value for some value or values of x. Thus let us set $du/dx = 0$ and take the derivative of (4.128) with respect to x. Then we obtain

$$i\frac{dv}{dx} = \frac{1}{n\beta} \cdot \frac{(1-2ix)(-2i)}{\sqrt{(1-2ix)^2 - (1-n^2\beta^2)}} \cdot \frac{\sqrt{(1+2ix)^2 - (1-n^2\beta^2)}}{\sqrt{(1+2ix)^2 - (1-n^2\beta^2)}} \ , \tag{4.130}$$

or

$$\frac{dv}{dx} = -\frac{2}{n\beta} \frac{\sqrt{(1-2ix)^2\left[(1+2ix)^2 - (1-n^2\beta^2)\right]}}{\sqrt{\left[(1-2ix)^2 - (1-n^2\beta^2)\right]\left[(1+2ix)^2 - (1-n^2\beta^2)\right]}} \ . \tag{4.131}$$

The denominator is real and non-negative, since it involves the product of a quantity and its conjugate. In order for dv/dx to be real, it is necessary that the imaginary part of the numerator be zero; i.e., that

$$\mathrm{Re}\left\{(1-2ix)^2\left[(1+2ix)^2 - (1-n^2\beta^2)\right]\right\} > 0$$

and

$$\mathrm{Im}\left\{(1-2ix)^2\left[(1+2ix)^2 - (1-n^2\beta^2)\right]\right\} = 0 \ . \tag{4.132}$$

The second conditions holds only if $x = 0$, and this also satisfies the first condition. Thus the minimum value of u is 1, and occurs at $x = 0$, or $\omega = 0$, and the roots of expression (4.125) lie one in the upper half plane, and one in the lower, for all $\omega > 0$.

Thus (4.124) can be evaluated by closing the contour in the upper half-plane for $z < 0$, and in the lower half-plane for $z > 0$, and applying the theory of residues; then (4.124) becomes

$$H\left\{G\right\} = \frac{2\pi i}{4\pi^2}\left\{\begin{array}{ll} \dfrac{e^{-ih_1 z}}{(h_1 - h_2)}, & z < 0 \\ \dfrac{e^{-ih_2 z}}{-(h_2 - h_1)}, & z > 0 \end{array}\right\},$$

or

$$H\left\{G\right\} = \frac{e^{bz}\, e^{-\alpha|z|\sqrt{\lambda^2 - k^2\alpha^2 + \frac{b^2}{\alpha^2}}}}{4\pi\alpha\sqrt{\lambda^2 - k^2\alpha^2 + \frac{b^2}{\alpha^2}}} . \tag{4.133}$$

Taking the inverse Hankel transform of (4.133) results in the integral

$$G(\bar{R}|0) = \frac{e^{bz}}{4\pi\alpha}\int_0^\infty \frac{J_0(\lambda r)\, e^{-\alpha|z|\sqrt{\lambda^2 - k^2\alpha^2 + \frac{b^2}{\alpha^2}}}\lambda d\lambda}{\sqrt{\lambda^2 - k^2\alpha^2 + \frac{b^2}{\alpha^2}}} . \tag{4.134}$$

This can be written in closed form by making use of Sommerfeld's formula, given, for example, by Magnus and Oberhettinger, (1954), p. 34. It is first necessary to examine the argument of the radical. First of all the quantity

$$k^2\alpha^2 - \frac{b^2}{\alpha^2} = \frac{\omega^2 n^2 \alpha^2}{c^2} + i\,\frac{\omega\sigma n^2\alpha^2}{\epsilon' c^2} - \frac{b^2}{\alpha^2}$$

lies in the first or second quadrant, for $\omega > 0$, and thus

$$0 \leq \arg\sqrt{k^2\alpha^2 - \frac{b^2}{\alpha^2}} \leq \pi/2 \quad . \tag{4.135}$$

Also, the quantity

$$\lambda^2 - k^2\alpha^2 + \frac{b^2}{\alpha^2} = \lambda^2 - \frac{\omega^2 n^2 \alpha^2}{c^2} - i\,\frac{\omega \sigma n^2 \alpha^2}{\epsilon' c^2} + \frac{b^2}{\alpha^2}$$

lies in the third or fourth quadrant, so that

$$-\pi/2 < \arg\sqrt{\lambda^2 - k^2\alpha^2 + \frac{b^2}{\alpha^2}} \leq 0 \quad . \tag{4.136}$$

Thus Sommerfeld's formula applies, and we get the expression

$$G(\bar{R}|0) = \frac{e^{bz}}{4\pi\alpha}\,\frac{e^{i\sqrt{k^2\alpha^2 - \frac{b^2}{\alpha^2}}\sqrt{\alpha^2 z^2 + r^2}}}{\sqrt{\alpha^2 z^2 + r^2}} \quad . \tag{4.137}$$

Replacing $\bar{R}$ by $\bar{R} - \bar{R}_0$, and using the definitions of b and α, the final solution is obtained:

$$G(\bar{R}|\bar{R}_0) = \frac{e^{\sigma\frac{\mu' a v}{2}(z - z_0)}\; e^{ik_1 a^{1/2} R_1}}{4\pi a^{1/2} R_1} , \tag{4.138}$$

where

$$R_1 = \sqrt{a(z - z_0)^2 + (r - r_0)^2}$$

and

$$k_1 = \sqrt{k^2 - b^2/a^2} = \frac{\omega n}{c}\sqrt{\left(1 + i\,\frac{\sigma}{2\omega\epsilon'}\right)^2 + \left(\frac{\sigma}{2\omega\epsilon' a}\right)^2} \quad .$$

For frequencies in the range $\omega < \sigma/2\epsilon'$, it is more appropriate to write this in ascending form in ω; noting (4.135), Eq. (4.138) can be written

$$G(\bar{R}|\bar{R}_o) = \frac{e^{\sigma\frac{\mu' a v}{2}(z - z_o)} e^{-\alpha_1 a^{1/2} R_1}}{4\pi a^{1/2} R_1}, \tag{4.139}$$

where

$$\alpha_1 = \sqrt{\frac{b^2}{a^2} - k^2} = \frac{\sigma n}{2\epsilon' c}\sqrt{\left(1 - i\,\frac{2\omega\epsilon'}{\sigma}\right)^2 - \frac{1}{a^2\gamma^2}}\,.$$

The Green's function does not increase indefinitely for large positive values of z in spite of the presence of the term e^{bz} in (4.137). Consider the numerator of the expression for large positive values of z; we have, approximately, noting (4.135),

$$\exp\left[\left(b - \alpha\sqrt{\frac{b^2}{\alpha^2} - k^2\alpha^2}\right) z\right] = \exp\left[b\left(1 - \sqrt{1 - k^2\alpha^4/b^2}\right) z\right]. \tag{4.140}$$

The radical is exactly (4.127), whose real part has a minimum value of unity at $\omega = 0$. Thus the exponential has an argument which is not positive, and does not increase indefinitely for large z.

Note that as the conductivity σ vanishes, the Green's function becomes that for the lossless case, reported by Tai (1965a):

$$G(\bar{R}|\bar{R}_o) = \frac{e^{ika^{1/2}R_1}}{4\pi a^{1/2} R_1}, \tag{4.141}$$

where here $k = \omega n/c$.

Note also that if we let $\omega = 0$ in (4.138), we get the static charge source Green's function of (4.64):

$$G(\bar{R}|\bar{R}_o) = \frac{e^{\sigma \frac{\mu' a v}{2}(z - z_o)} \, e^{-\sigma \frac{\mu' a^{1/2} v}{2} R_1}}{4\pi a^{1/2} R_1} . \quad (4.142)$$

(4.138) is probably the most useful Green's function obtained in this work, since it can be readily applied to the problem of an antenna in a moving, conducting medium.

Case B. High Velocities: $v > c/n$, $a < 0$. Let $\alpha^2 = -a$ and $b = -p/2$. As before, we may take $\bar{R}_o = 0$ temporarily without loss of generality. Then Eq. (4.120) becomes

$$\left(\frac{\alpha^2}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{\partial^2}{\partial z^2} - 2b \frac{\partial}{\partial z} - k^2 \alpha^4\right) G(\bar{R}|0) = \frac{\delta(r)\,\delta(z)}{2\pi r} . \quad (4.143)$$

Taking the Hankel transform in r, followed by the Fourier transform in z, yields the algebraic expression:

$$(\lambda^2 \alpha^2 - h^2 - 2ibh + k^2 \alpha^4)\, F\left\{H\{G\}\right\} = -\frac{1}{4\pi^2} , \quad (4.144)$$

or

$$F\left\{H\{G\}\right\} = \frac{1}{4\pi^2} \frac{1}{(h^2 + 2ibh - \lambda^2 \alpha^2 - k^2 \alpha^4)} . \quad (4.145)$$

Taking the inverse Fourier transform in h gives the integral

$$H\{G\} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{e^{-ihz}\, dh}{(h^2 + 2ibh - \lambda^2 \alpha^2 - k^2 \alpha^4)} . \quad (4.146)$$

The roots of the denominator are given by

$$h_{1,2} = -ib \pm \sqrt{\lambda^2\alpha^2 + \frac{\omega^2 n^2 \alpha^4}{c^2}\left(1 + i\,\frac{\sigma}{\omega\epsilon'}\right) - b^2}$$

$$= -ib \pm \sqrt{1 - \frac{\lambda^2\alpha^2}{b^2} - \frac{\omega^2 n^2 \alpha^4}{b^2 c^2}\left(1 + i\,\frac{\sigma}{\omega\epsilon'}\right)} \quad . \tag{4.147}$$

Since we anticipate a shock wave behavior as with static charge sources, we ask the question, for what range of ω are both poles in the lower half plane, or equivalently, for what range of ω is the real value of the radical less than unity? It is evident that the worst case is for $\lambda = 0$. Thus we want to examine the expression

$$\sqrt{1 - \frac{\omega^2 n^2 \alpha^2}{b^2 c^2}\left(1 + i\,\frac{\sigma}{\omega\epsilon'}\right)}\,, \tag{4.148}$$

which is identical to (4.127). The discussion that followed (4.127) applies here as well, with one modification: since now $n\beta > 1$, relations (4.29) now become

$$u + iv \sim \frac{(1 + 2ix)}{n\beta} \quad ,$$

or

$$u \sim \frac{1}{n\beta} < 1 \quad . \tag{4.149}$$

and as before $u = 1$ at $x = 0$. Thus u is maximized rather than minimized, at some finite value of x; as in (4.132), this turns out to be at $x = 0$, or $\omega = 0$. Thus for positive ω, the roots of h are both in the lower half-plane, rather than one in each half-plane as in Fig. 4-5.

For $z < 0$, the contour in (4.146) can be closed in the upper half-plane, and since it encloses no poles of the integrand of (4.146),

$$H\left\{G\right\} = 0 \quad , \quad z < 0 \quad . \tag{4.150}$$

For $z > 0$, the contour can be closed in the lower half-plane, and encloses both poles of the integrand. Thus by the theory of residues,

$$H\{G\} = -\frac{2\pi i}{4\pi^2}\left[\frac{e^{-ih_1 z}}{(h_1 - h_2)} + \frac{e^{-ih_2 z}}{(h_2 - h_1)}\right]$$

$$= -\frac{e^{-bz}}{2\pi\alpha}\,\frac{\sin\left(\alpha z\sqrt{\lambda^2 + \frac{\omega^2 n^2\alpha^2}{c^2}\left(1 + i\frac{\sigma}{\omega\epsilon'}\right) - \frac{b^2}{\alpha^2}}\right)}{\sqrt{\lambda^2 + \frac{\omega^2 n^2\alpha^2}{c^2}\left(1 + i\frac{\sigma}{\omega\epsilon'}\right) - \frac{b^2}{\alpha^2}}} \quad (4.151)$$

Taking the inverse Hankel transform in λ gives the integral

$$G(\bar{R}|0) = -\frac{e^{-bz}}{2\pi\alpha}\int_0^\infty \frac{J_0(\lambda r)\sin\left(\alpha z\sqrt{\lambda^2 + k^2\alpha^2 - \frac{b^2}{\alpha^2}}\right)\lambda\, d\lambda}{\sqrt{\lambda^2 + k^2\alpha^2 - \frac{b^2}{\alpha^2}}} . \quad (4.152)$$

This can be reduced into closed form by means of the Sonine-Gegenbauer formula (see Watson (1922), p. 414), which states:

$$\int_0^\infty J_\mu(\lambda r)\, J_\nu\left(\alpha z\sqrt{\lambda^2 + x^2}\right)(\lambda^2 + k_1^2\alpha^2)^{-\nu/2}\,\lambda^{\mu+1}\, d\lambda$$

$$= \begin{cases} 0, & \alpha z < r \\ \dfrac{r^\mu}{\alpha^\nu z^\nu}\left[\dfrac{\sqrt{\alpha^2 z^2 - r^2}}{k_1\alpha}\right]^{\nu-\mu-1} J_{\nu-\mu-1}\left(k_1\alpha\sqrt{\alpha^2 z^2 - r^2}\right), & \alpha z > r \end{cases} \quad (4.153)$$

for $\mathrm{Re}\,\nu > \mathrm{Re}\,\mu > -1$; αz, r real and non-negative. If we substitute $\mu = 0$, $\nu = 1/2$, and $k_1^2\alpha^2 = k^2\alpha^2 - b^2/\alpha^2$, and use the well-known relations

$$J_{1/2}(Z) = \sqrt{\frac{2}{\pi Z}} \sin Z \ ,$$

and (4.154)

$$J_{-1/2}(Z) = \sqrt{\frac{2}{\pi Z}} \cos Z \ ,$$

then the Sonine - Gegenbauer formula becomes

$$\frac{1}{\sqrt{\alpha z}} \int_0^\infty \frac{J_0(\lambda r) \sin\left(\alpha z \sqrt{\lambda^2 + k^2\alpha^2 - b^2/\alpha^2}\right) \lambda d\lambda}{\sqrt{\lambda^2 + k^2\alpha^2 - b^2/\alpha^2}}$$

$$= \left\{ \begin{array}{l} 0, \quad \alpha z < r \\ \dfrac{1}{\sqrt{\alpha z}} \dfrac{k_1^{1/2} \alpha^{1/2}}{(\alpha^2 z^2 - r^2)^{1/4}} \dfrac{\cos\left(k_1 \alpha \sqrt{z^2\alpha^2 - r^2}\right)}{k_1^{1/2}\alpha^{1/2}(\alpha^2 z^2 - r^2)^{1/4}} \end{array} \right\},$$

$$= \frac{1}{\sqrt{\alpha z}} \left\{ \begin{array}{ll} 0, & \alpha z < r \\ \dfrac{\cos\left(k_1 \alpha \sqrt{\alpha^2 z^2 - r^2}\right)}{\sqrt{\alpha^2 z^2 - r^2}}, & \alpha z > r \end{array} \right\}. \qquad (4.155)$$

Thus the Green's function solution can be written, for $z > 0$,

$$G(\bar{R}|0) = -\frac{e^{-bz}}{2\pi\alpha} \left\{ \begin{array}{ll} 0, & \alpha z < r \\ \dfrac{\cos\left(k_1 \alpha \sqrt{\alpha^2 z^2 - r^2}\right)}{\sqrt{\alpha^2 z^2 - r^2}}, & \alpha z > r \end{array} \right\}. \qquad (4.156)$$

Using (4.150), replacing $\bar{R}$ by $\bar{R} - \bar{R}_0$, and using the definitions of b and α, we have, finally,

$$G(\bar{R}|\bar{R}_o) = \begin{Bmatrix} 0\,, & |a|^{1/2}(z-z_o)<(r-r_o) \\ -\dfrac{e^{-\sigma \frac{\mu'|a|v}{2}(z-z_o)} \cos(k_1|a|^{1/2}R_2)}{2\pi|a|^{1/2}R_2}\,, & |a|^{1/2}(z-z_o)>(r-r_o) \end{Bmatrix}.$$

(4.157)

where

$$R_2 = \sqrt{|a|\,(z-z_o)^2 - (r-r_o)^2}\,.$$

and

$$k_1 = \sqrt{k^2 - b^2/a^2} = \frac{\omega n}{c}\sqrt{(1 + i\,\frac{\sigma}{2\omega\epsilon'})^2 + (\frac{\sigma}{2\omega\epsilon' a\gamma})^2}$$

For large conductivity σ or low frequencies such that $\sigma > 2\omega\epsilon'$, it is more appropriate to write this in terms of an attenuation factor α_1:

$$G(\bar{R}|\bar{R}_o) = \begin{Bmatrix} 0\,, & |a|(z-z_o)<(r-r_o) \\ -\dfrac{e^{-\sigma \frac{\mu'|a|v}{2}(z-z_o)} \cosh(\alpha_1|a|^{1/2}R_2)}{2\pi|a|^{1/2}R_2}\,, & |a|(z-z_o)>(r-r_o) \end{Bmatrix},$$

where (4.158)

$$\alpha_1 = \frac{1}{a}\sqrt{b^2 - k^2a^2} = \sigma\,\frac{n}{2\epsilon' c}\sqrt{\left(1 - \frac{2i\omega\epsilon'}{\sigma}\right)^2 - \frac{1}{a^2\gamma^2}}\ .$$

For lossless media $\sigma = 0$, and the Green's function reduces to:

$$G(\bar{R}|\bar{R}_o) = \begin{Bmatrix} 0\,, & a^{1/2}(z-z_o) < (r-r_o) \\ -\dfrac{\cos(k|a|^{1/2}R_2)}{2\pi|a|^{1/2}R_2}\,, & a^{1/2}(z-z_o) > (r-r_o) \end{Bmatrix}, \quad (4.159)$$

where here $k = \omega n/c$. This agrees with the result obtained by Tai (1965a). Furthermore, if ω is allowed to vanish, the Green's function becomes that for static charge sources, Eq. (4.89):

$$G(\bar{R}|\bar{R}_o) = \begin{Bmatrix} 0, \\ -\dfrac{e^{-\sigma \frac{\mu'|a|v}{2}(z-z_o)} \cosh\left[\frac{1}{2}\sigma\mu'|a|^{1/2} v R_2\right]}{2\pi|a|^{1/2} R_2} \end{Bmatrix} . \tag{4.160}$$

While the hyperbolic cosine term in (4.158) involves a rising exponential, the decaying exponential term dominates. This can be seen by considering the numerator of (4.158) for large values of positive $(z - z_o)$:

$$\exp\left[-b\left(1 - \sqrt{1 - k^2a^2/b^2}\right)(z - z_o)\right] .$$

The radical is exactly (4.148). In the discussion of this quantity it was shown that its real part is less than unity for all positive ω. Thus for large values of positive $(z - z_o)$, the solution decreases with increasing $(z - z_o)$.

4.2.3 Summary

Summarizing the results of Section 4.2, we can say that for harmonically varying current sources, the fields are related to the potentials $\bar{A}_1$ and ϕ_1, by

$$\bar{E} = e^{-i\omega\Omega z}(i\omega\bar{\bar{\alpha}}^{-1}\cdot\bar{A}_1 - \nabla\phi_1) ,$$

$$\bar{H} = \frac{1}{\mu'} e^{-i\omega\Omega z}\bar{\bar{\alpha}}^{-1}\cdot\left[\nabla \times (\bar{\bar{\alpha}}^{-1}\cdot\bar{A}_1)\right] ,$$

$$\bar{B} = e^{-i\omega\Omega z}\left[(\nabla - i\omega\bar{\Omega}) \times (\bar{\bar{\alpha}}^{-1}\cdot\bar{A}_1) + \bar{\Omega}\times\nabla\phi_1\right] ,$$

and

$$\bar{D} = e^{-i\omega\Omega z}\left[\epsilon'(i\omega\bar{A}_1 - \bar{\bar{\alpha}}\cdot\nabla\phi_1) + \frac{\bar{\Omega}}{a\mu'}\times(\nabla\times(\bar{\bar{\alpha}}^{-1}\cdot\bar{A}_1))\right] , \tag{4.161}$$

where the time dependence of all field and charge-current quantities is understood to be $e^{-i\omega t}$. The potentials are given by

$$\bar{A}_1(\bar{R}) = \mu' a^2 \iiint_{V_o} G(\bar{R}|\bar{R}_o)\left[\bar{J}_s(\bar{R}_o) + \bar{v}\,\gamma\rho_r'(\bar{R}_o)\right] e^{i\omega\Omega z_o}\, dV_o$$

and

$$\phi_1(\bar{R}) = \frac{1}{\epsilon'}\iiint_{V_o} G(\bar{R}|\bar{R}_o)\left[\bar{\Omega}\cdot\bar{J}_s(\bar{R}_o) + \rho_s(\bar{R}_o) + a\,\gamma\rho_r'(\bar{R}_o)\right] e^{i\omega\Omega z_o}\, dV_o , \tag{4.162}$$

where the volume V_o encloses the sources. The response charge density is related to the source currents and charges by

$$\gamma\rho_r'(\bar{R}_o) = -\frac{\sigma}{\epsilon' v}\int_{-\infty}^{z_o} e^{-\left(i\omega + \frac{\sigma}{\epsilon'\gamma^2}\right)\left(\frac{(z_o-\zeta)}{v}\right)}\left[\rho_s(\bar{r}_o,\zeta) - \frac{\bar{v}}{c^2}\cdot\bar{J}_s(\bar{r}_o,\zeta)\right] d\zeta , \tag{4.163}$$

and it is noted that $i\omega\rho_s = \nabla\cdot\bar{J}_s$, and $\bar{r}_o = \bar{R}_o - \bar{z}_o$.

The Green's function necessary to find the potentials in (4.162) is, for $v < c/n$, or $a > 0$:

$$G(\bar{R}|\bar{R}_o) = \frac{e^{\sigma\frac{\mu' a v}{2}(z - z_o)}\, e^{ik_1 a^{1/2} R_1}}{4\pi a^{1/2} R_1} , \tag{4.164}$$

where

$$R_1 = \sqrt{a(z-z_o)^2 + (r-r_o)^2}$$

and

$$k_1 = \sqrt{k^2 - b^2/a^2} ;$$

or for $v > c/n_o$ or $a < 0$:

$$G(\bar{R}|\bar{R}_o) = \begin{cases} 0, & |a|^{1/2}(z-z_o) < (r-r_o) \\ -\dfrac{e^{-\sigma \frac{\mu'|a|v}{2}(z-z_o)} \cos\left(k_1|a|^{1/2}R_2\right)}{2\pi|a|^{1/2}R_2}, & |a|^{1/2}(z-z_o) > (r-r_o) \end{cases}, \tag{4.165}$$

where

$$R_2 = \sqrt{|a|\ (z-z_o)^2 - (r-r_o)^2}\ .$$

V

SUMMARY AND CONCLUSIONS

Two classes of problems have been solved in the area of moving, conducting media: static and radiation fields of static charges, and radiation fields of harmonic current sources. No limitation is put either on the range of conductivities and frequencies, or on the velocities. For the limiting case of vanishing conductivity, the solutions here reduce to already published solutions.

The results of the first class of problems find application to the fields of particle beams permeating matter, including the Cerenkov radiation effect. The second class can be applied to antenna problems involving radiating elements in a moving, conducting fluid.

There are several areas and problems to which it would be interesting and useful to extend the methods developed here. The two-dimensional counterpart of both classes of problems can be readily solved, from the differential equations of the Green's functions. The fields of stationary currents as well as stationary charges could be developed. Boundary value problems are also of interest, for example, the fields in a filled circular waveguide excited by charges of high velocities. The application of the methods to the problem of a short dipole in a moving, conducting medium is an important application on which the author is presently working.

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APPENDIX A

TRANSFORMATION RELATIONS FOR THE POTENTIALS AND THE GAUGE CONDITION

Minkowski's theory of the electrodynamics of moving bodies is based on the covariant formulation of electromagnetism with respect to coordinate systems in uniform relative motion. This in turn is based on the Lorentz transformation of coordinates, where in addition to the space coordinates x, y, and z, time is considered as a fourth coordinate ict, where c is the velocity of light in vacuo. This and the following discussion are taken from Sommerfeld, "Electrodynamics", (1952) Section 27. If the primed system coordinates moves with a velocity v in the positive z- direction with respect to the unprimed system, they are related under the Lorentz transformation by

$$x' = x, \quad y' = y, \quad z' = \gamma\,(z - vt) \quad ,$$

$$t' = \gamma\,(t - \frac{vz}{c^2}) \quad . \tag{A.1}$$

In the dyadic symbolism, using the definition

$$\bar{R} = x\hat{x} + y\hat{y} + z\hat{z} \quad ,$$

for primed and unprimed systems, (A.1) can be written as

$$\bar{R}' = \gamma\,\bar{\bar{\gamma}}^{-1} \cdot (\bar{R} - \bar{v}t)$$

and

$$t' = \gamma\,(t - \frac{\bar{v} \cdot \bar{R}}{c^2}) \quad . \tag{A.2}$$

These can be inverted straight forwardly to give

$$\bar{R} = \gamma\,\bar{\bar{\gamma}}^{-1} \cdot (\bar{R}' + \bar{v}t')$$

and

$$t = \gamma\left(t' + \frac{\bar{v}\cdot\bar{R}}{c^2}\right) \quad . \tag{A.3}$$

The "del" operator ∇' with the time derivative $\partial/\partial t'$ can be shown to follow a similar set of relations:

$$\nabla' = \hat{x}\frac{\partial}{\partial x'} + \hat{y}\frac{\partial}{\partial y'} + \hat{z}\frac{\partial}{\partial z'}$$

$$= \hat{x}\frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \hat{y}\frac{\partial y}{\partial y'}\frac{\partial}{\partial y} + \hat{z}\left(\frac{\partial z}{\partial z'}\frac{\partial}{\partial z} + \frac{\partial t}{\partial z'}\frac{\partial}{\partial t}\right) ,$$

by using (A.3), giving

$$\nabla' = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\,\gamma\left(\frac{\partial}{\partial z} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)$$

$$= \gamma\,\bar{\bar{\gamma}}^{-1}\cdot\left(\nabla + \frac{\bar{v}}{c^2}\frac{\partial}{\partial t}\right) \quad . \tag{A.4}$$

Similarly,

$$\frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial z}{\partial t'}\frac{\partial}{\partial z}$$

$$= \gamma\left(\frac{\partial}{\partial t} + \bar{v}\cdot\nabla\right) \quad . \tag{A.5}$$

Sometimes it is convenient to use what is sometimes called the total time derivative, given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{v}\cdot\nabla \quad . \tag{A.6}$$

Then (A.4) and (A.5) become

$$\nabla' = \frac{\bar{\bar{\gamma}}}{\gamma}\cdot\left(\nabla + \frac{\bar{v}}{c^2}\frac{D}{Dt}\right)$$

and

$$\frac{\partial}{\partial t'} = \gamma \frac{D}{Dt} \quad . \tag{A.7}$$

Four-vectors are quantities having four components which obey the Lorentz transformation, and are thus said to transform like the coordinates. In the covariant formulation of electromagnetism, there are two important four-vectors: the four-potential $(\bar{A},\ i\phi/c)$, and the four-current density $(\bar{J},\ i c\rho)$, where the notation used here means that the vector corresponds to the space coordinate $\bar{R}$ and the scalar corresponds to the time coordinate, ict. Thus since the components of these four-vectors transform as the coordinates, from (A. 2) we have

$$\left\{ \begin{aligned} \bar{A}' &= \gamma \bar{\bar{\gamma}}^{-1} \cdot (\bar{A} - \frac{\bar{v}}{c^2} \phi) \\ \phi' &= \gamma (\phi - \bar{v} \cdot \bar{A}) \end{aligned} \right\}, \tag{A.8}$$

and

$$\left\{ \begin{aligned} \bar{J}' &= \gamma \bar{\bar{\gamma}}^{-1} \cdot (\bar{J} - \bar{v} \rho) \\ \rho' &= \gamma (\rho - \frac{\bar{v} \cdot \bar{J}}{c^2}) \end{aligned} \right\} . \tag{A.9}$$

In this theory, then, a moving current produces a charge, although for small velocities it is negligible.

Turning our attention now to the gauge condition, we note that if the primed system is that coordinate system which transforms the medium to rest, then the vector and scalar potentials in that system are related by the familiar gauge condition

$$\nabla' \cdot \bar{A}' + \mu' \epsilon' \frac{\partial \phi'}{\partial t'} + \sigma' \mu' \phi' = 0 \quad . \tag{A.10}$$

Noting that $\sigma = \gamma \sigma'$, using (A.4), (A.5), and (A.8), we obtain

$$\nabla' \cdot \bar{A}' = \gamma^2 (\bar{\bar{\gamma}}^{-1} \cdot \nabla + \frac{\bar{v}}{c^2} \frac{\partial}{\partial t}) \cdot (\bar{A} - \frac{\bar{v}\phi}{c^2})$$

$$= \gamma^2 \left[\nabla \cdot \bar{\bar{\gamma}}^{-1} \cdot \bar{A} + \frac{\bar{v}}{c^2} \cdot \frac{\partial \bar{A}}{\partial t} - \nabla \cdot (\frac{\bar{v}\phi}{c^2}) - \frac{\beta^2}{c^2} \frac{\partial \phi}{\partial t} \right] ,$$

$$\mu' \epsilon' \frac{\partial \phi'}{\partial t'} = \gamma^2 \frac{n^2}{c^2} (\frac{\partial}{\partial t} + \bar{v} \cdot \nabla)(\phi - \bar{v} \cdot \bar{A})$$

$$= \gamma^2 \frac{n^2}{c^2} \left[-\bar{v} \cdot \frac{\partial \bar{A}}{\partial t} - (\bar{v} \cdot \nabla)(\bar{v} \cdot \bar{A}) + \frac{\partial \phi}{\partial t} + \bar{v} \cdot \nabla \phi \right] ,$$

and

$$\sigma' \mu' \phi' = \sigma \mu' (\phi - \bar{v} \cdot \bar{A}) .$$

Noting that $\nabla \cdot (\bar{v}\phi) = \bar{v} \cdot \nabla \phi$, and collecting terms, we get

$$\nabla \cdot \bar{\bar{\gamma}}^{-1} \cdot \bar{A} - n^2 \beta^2 \frac{\partial A_z}{\partial z} - (n^2 - 1) \frac{\bar{v}}{c^2} \cdot \frac{\partial \bar{A}}{\partial t} - \sigma \frac{\mu'}{\gamma^2} \bar{v} \cdot \bar{A}$$

$$= -(n^2 - 1) \frac{\bar{v}}{c^2} \cdot \nabla \phi - \frac{1}{c^2} (n^2 - \beta^2) \frac{\partial \phi}{\partial t} - \sigma \frac{\mu' \phi}{\gamma^2} ,$$

After dividing through by $(1 - n^2 \beta^2)$ and noting that $a = \left[\gamma^2 (1 - n^2 \beta^2) \right]^{-1}$ and $\bar{\Omega} = \bar{v}(n^2 - 1)/(c^2(1 - n^2\beta^2))$, this can be written as

$$\nabla \cdot \bar{\bar{\gamma}} \cdot \bar{\bar{\alpha}} \cdot \bar{A} - \bar{\Omega} \cdot \frac{\partial \bar{A}}{\partial t} - \sigma \mu' a \bar{v} \cdot \bar{A}$$

$$= -\bar{\Omega} \cdot \nabla \phi - \sigma \mu' a \phi - \frac{1}{c^2} \frac{(n^2 - \beta^2)}{(1 - n^2 \beta^2)} \frac{\partial \phi}{\partial t} . \qquad \text{(A.11)}$$

For $\bar{A}$ in the z-direction, the first term becomes $\nabla \cdot \bar{A}$, and (A.11) is exactly Eq. (4.11) of the text.