ITERATIVE SOLUTIONS OF MAXWELL'S EQUATIONS

by

John Stelios Asvestas

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in
The University of Michigan
1968

Doctoral Committee:
Doctor Ralph E. Kleinman, Co-chairman
Professor Chen-To Tai, Co-chairman
Professor Chiao-Min Chu
Doctor Raymond F. Goodrich
Doctor Thomas B. A. Senior

ITERATIVE SOLUTIONS OF MAXWELL'S EQUATIONS

by

John Stelios Asvestas

Co-Chairmen: Ralph E. Kleinman
Chen-To Tai

The problem of scattering of electromagnetic waves by a closed, bounded, smooth, perfectly conducting surface immersed in vacuum is considered and a method for determining the scattered electric and magnetic field vectors (solutions of the homogeneous Maxwell equations satisfying well known boundary conditions on the surface and the Silver-Müller radiation condition at infinity) everywhere exterior to the surface is presented. Specifically, two integral equations are derived, one for each scattered field vector. These equations are coupled. The kernels of the equations are dyadic functions of position and can be derived from the solutions of standard interior and exterior potential problems. Once these dyadic kernels are determined for a particular surface geometry the integral equations can be solved by iteration for the wave number k being sufficiently small. Alternatively, the scattered fields in the integral equations may be expanded in a power series of the wave number k and recursion formulas be found for the unknown coefficients in the expansions by equating equal powers of k. As a check, the method is applied to the problem of scattering of a plane electromagnetic wave by a perfectly conducting sphere. The first two terms in the low frequency expansions of the electric and magnetic scattered fields are found and are shown to be in complete agreement with known results.

ACKNOWLEDGEMENTS

The author wishes to thank Dr. R. E. Kleinman for suggesting the problem and for his continuous guidance and encouragement.

The author wishes to thank the members of his committee for their support during the execution of this work. In particular, he wishes to express his appreciation to Prof. C-T Tai for his interest in the problem and for his inspired teaching.

TABLE OF CONTENTS

INTRODUCTION

The subject of low frequency electromagnetic scattering dates back to Lord Rayleigh (1897). In his well-known paper Lord Rayleigh examined the scattering of both acoustical and electromagnetic waves by two-dimensional as well as three-dimensional bodies and he showed that in the limit as the wave number k tends to zero the electric and magnetic scattered vectors in the near field can be expressed in terms of solutions of standard potential problems. Furthermore, he was able to continue these solutions into the far field region and arrive at his famous fourth power of frequency law for the scattering cross section of objects whose characteristic dimension is small compared with the wavelength.

Since that time considerable work has been done in this direction. An extensive bibliography for both acoustical and electromagnetic low frequency scattering is given by Kleinman (1965a). Much of the work in deriving higher order terms in the low frequency expansion, however, has depended explicitly on a particular geometry for the scatterer and on a particular type of source with a restricted direction of incidence. Stevenson (1953) overcame these limitations by means of the Stratton-Chu integral representation of the scattered fields. He showed that the scattered fields for a sufficiently smooth three-dimensional scatterer and for arbitrary excitation can be written in the form

$$\vec{E}_m^s = \vec{F}_m + \nabla \phi_m^s ,$$

$$\vec{H}_m^s = \vec{G}_m + \nabla \psi_m^s ,$$

where $\vec{E}_m^s$ and $\vec{H}_m^s$ are the coefficients of k^m in the k-series expansions of the electric and magnetic scattered fields, respectively. The vectors $\vec{F}_m$ and $\vec{G}_m$ are known in terms of the previous coefficients $\vec{E}_o^s, \ldots, \vec{E}_{m-1}^s$ and $\vec{H}_o^s, \ldots, \vec{H}_{m-1}^s$. The sealar functions ϕ_m^s and ψ_m^s are solutions of Laplace's equation satisfying known conditions on the scatterer arising from the properties of the electromagnetic field. They also satisfy the Kellogg regularity conditions at infinity. The vectors $\vec{E}_m^s$ and $\vec{H}_m^s$ can be continued into the far field by substituting them in the Stratton-Chu equations and using the far field approximation for the free space Green's function which is involved in these equations. Kleinman (1965b) showed, however, that Stevenson's method leads to incorrect results after the first few terms and proposed an alternate scheme largely based on Stevenson's. We will undertake this point in Ch. III. In either case, however, the labor for obtaining higher order terms becomes prohibitive at a very early stage.

Inherent to all three-dimensional low frequency techniques is the assumption that the scattered electric and magnetic fields can be written in a power series of the wave number k. Werner (1963) put the whole subject on a rigorous mathematical basis by showing that this assumption was correct. Specifically, he proved that the scattered electric field $\vec{E}^s$ tends, as $k \rightarrow 0$, analytically to a corresponding electrostatic field.

In the present work we propose an alternate low frequency scheme by means of which one can obtain as many terms as desired in the expansions of the scattered fields by operating on potential functions (solutions of Laplace's equation) satisfying certain boundary conditions on the scatterer and the Kellogg regularity conditions at infinity. The advantage of the present method over Stevenson's is that it does away with the determination of 2m potential functions (ϕ_m^s and ψ_m^s). That is, for every $\vec{E}_m^s$ and $\vec{H}_m^s$ we wish to determine we do not have to solve two boundary value problems to determine ϕ_m^s and ψ_m^s. Moreover, we believe (though we do not prove) that the resulting series expansions through the present method represent the scattered fields not only in the near region but everywhere in space. Its disadvantage is that it applies only to perfectly conducting scatterers while Stevenson's applies to scatterers of finite or zero conductivity also.

In order to facilitate reading of this work and to clarify our approach we include an introductory section where the problem, its motivation, and the main results are presented.

The Problem, its Motivation, and the Principal Results

The problem under consideration is the following:

In the three-dimensional free space (vacuum) we have a closed, bounded, perfectly conducting surface S which separates the whole space into two regions: the finite region V_i enclosed by S and the rest of the space V. The surface S is sufficiently smooth to guarantee the existence of a normal at all of its points. A time harmonic source of electromagnetic waves is located in V and its electric and magnetic fields are denoted by $\vec{E}^i$ and $\vec{H}^i$, respectively. The time dependence $e^{-i\omega t}$ is omitted. The presence of the perfectly conducting surface S gives rise to an electromagnetic wave whose electric and magnetic vectors we denote by $\vec{E}^s$ and $\vec{H}^s$, respectively. These two vectors satisfy

1) Maxwell's equations

$$\nabla \times \vec{E}^s = ikZ\vec{H}^s, \quad \nabla \times \vec{H}^s = -ikY\vec{E}^s, \tag{1}$$

$Z = 1/Y$, the free space characteristic impedance,

2) the homogeneous vector wave equation

$$\nabla \times \nabla \times \begin{Bmatrix} \vec{E}^s \\ \vec{H}^s \end{Bmatrix} - k^2 \begin{Bmatrix} \vec{E}^s \\ \vec{H}^s \end{Bmatrix} = 0 \quad \text{in V}, \tag{2}$$

a consequence of Maxwell's equations,

3) the boundary conditions

$$\hat{n} \times \vec{E}^s = -\hat{n} \times \vec{E}^i, \quad \hat{n} \cdot \vec{H}^s = -\hat{n} \cdot \vec{H}^i \quad \text{on S}, \tag{3}$$

where $\hat{n}$ is the unit normal on S directed out of V and into V_i, and

4) the radiation conditions

$$\lim_{R \to \infty} R\left[\hat{R} \times (\nabla \times \vec{E}^s) + ik\vec{E}^s\right] = 0, \qquad \lim_{R \to \infty} R\left[\hat{R} \times (\nabla \times \vec{H}^s) + ik\vec{H}^s\right] = 0, \quad \text{uniformly in } \hat{R}, \tag{4}$$

$\hat{R}$ being the radial unit vector and R the distance from the origin of a coordinate system to a point in V.

Our intention is to determine $\vec{E}^s$ and $\vec{H}^s$ for k "sufficiently" small. The plan is as follows: First, we express the scattered fields in terms of two coupled integral equations. The kernels of these equations are dyadic functions of position which are derivable from solutions of Laplace 's equation. The equations are coupled in the sense that both $\vec{E}^s$ and $\vec{H}^s$ appear in each of them. Secondly, for k "sufficiently" small, we iterate these equations in an alternating manner to produce a Neumann series for each of the scattered fields.

The motivation for such an approach to the problem is a paper by Kleinman (1965c) entitled "The Dirichlet Problem for the Helmholtz Equation." In it the author derives a new integral equation for the regular part of the Dirichlet Green's function for the Helmholtz equation. The "kernel" of this equation is not, as is commonly the case, the free space Green's function but the Dirichlet Green's function for the Laplace equation. The equation can be solved by iteration as a Neumann series to produce the regular part of the Dirichlet Green's function for the Helmholtz equation for the absolute value of k sufficiently small[1]. The practical value of this method lies in the fact that it employs the static Dirichlet Green's function which is known for most coordinate systems of interest and which usually involves special functions whose properties have been studied extensively. What is more important, however, is its conceptual value: for the first time it was rigorously demonstrated that the slowly varying dynamic Green's function can be obtained by suitably perturbing the corresponding static function. That this could be true had long been felt among workers in the field[2]. The same feeling certainly existed regarding the electromagnetic problem and now Kleinman's method suggested a way of attacking it.

---

1 In connection with this work it should be mentioned that the Neumann problem for the Helmholtz equation has been treated in a manner analogous to the Dirichlet by Ar and Kleinman (1966).

2 Cf. M. M. Schiffer's work in 'Lecture Series on Partial Differential Equations," The University of Kansas Press, 1957.

The equivalent concept of a scalar Green's function in the vector case is the dyadic Green's function. After Levine and Schwinger (1950), we define it as follows:[3]

1) $\nabla \times \nabla \times \bar{\bar{G}}(\vec{R}|\vec{R}') - k^2 \bar{\bar{G}}(\vec{R}|\vec{R}') = \bar{\bar{I}}\delta(\vec{R}|\vec{R}')$ in V (5)

2) Either

$$\hat{n} \times \bar{\bar{G}} = 0 \text{ on } S \tag{6}$$

or

$$\hat{n} \times \nabla \times \bar{\bar{G}} = 0 \text{ on } S \tag{7}$$

3) $\lim_{R \to \infty} R(\nabla \times \bar{\bar{G}} - ik\hat{R} \times \bar{\bar{G}}) = 0$. (8)

We have in reality defined two dyadic Green's function depending on whether we choose the boundary condition (6) or the boundary condition (7). (That the Green's function must be a dyadic instead of, say, a vector is necessitated by the fact that we wish to obtain a linear relation between the field vector in V and the field vectors on the scatterer, and the most general linear relation between two vectors is a dyadic.) Using these dyadics (one at a time) in the dyadic form of Green's theorem we can find two sets of two integral equations each for the scattered fields $\vec{E}^s$ and $\vec{H}^s$. The suitable form of Green's theorem in this case is

$$\int_V [(\nabla \times \nabla \times \vec{Q}) \cdot \bar{\bar{P}} - \vec{Q} \cdot (\nabla \times \nabla \times \bar{\bar{P}})] dV = \int_{S+S'+S_\infty} \hat{n} \cdot [\vec{Q} \times (\nabla \times \bar{\bar{P}}) + (\nabla \times \vec{Q}) \times \bar{\bar{P}}] dS, \tag{9}$$

where S' is the surface of a small sphere centered at the singularity at $\bar{R}'$, S_∞ is the surface of a sphere with infinite radius, and $\hat{n}$ is the unit normal always directed out of V. The vector $\vec{Q}$ stands for either of the scattered fields while the dyadic $\bar{\bar{P}}$ usually stands for one of the dyadics defined above. In our case, however, we wish to use dyadics which

---

[3] Arrows (⇀) over letters denote vectors while double bars (=) denote dyadics. Carets (^) over letters denote unit vectors.

are derivable from potential functions. At this point one must exercise care in defining these dyadics. They must have an appropriate singularity to make (9) give a desirable result (i. e. evaluate the field at the singularity) and they must satisfy appropriate conditions on S so that together with the natural boundary conditions of the electromagnetic fields they will make the integral over the scattering surface S in (9) a known term.

The boundary conditions on S are readily determined by inspection of the surface integral in (9). The appropriate singularity was found by noticing that vector wave functions for the equation $\nabla x \nabla x \vec{A} - k^2 \vec{A} = 0$ are formed by letting $\vec{A} = \nabla x (\hat{c}\psi)$, $\hat{c}$ being a constant unit vector and ψ a solution of the Helmholtz equation. In our case we let ψ be the free space Green's function for the Laplace equation. From now on the road is open and we can reach the following result (with respect to the geometry defined at the outset).

If

1) $$\bar{\bar{H}}^{(1)}_e = \nabla x \left[- \frac{\bar{\bar{I}}}{4\pi \cdot |\vec{R} - \vec{R}'|} \right] + \bar{\bar{H}}^{(1)}_{e_r} , \qquad (10)$$

$$\bar{\bar{E}}^{(1)}_m = \nabla x \left[- \frac{\bar{\bar{I}}}{4\pi |\vec{R} - \vec{R}'|} \right] + \bar{\bar{E}}^{(1)}_{m_r} , \quad \text{in V} \qquad (11)$$

2) $$\nabla x \nabla x \begin{Bmatrix} \bar{\bar{H}}^{(1)}_{e_r} \\ \bar{\bar{E}}^{(1)}_{m_r} \end{Bmatrix} = 0 , \quad \text{in V} \qquad (12)$$

3) $$\hat{n} x \bar{\bar{E}}^{(1)}_m = 0, \quad \hat{n} x \nabla x \bar{\bar{H}}^{(1)}_e = 0 , \quad \text{on S} \qquad (13)$$

4) $$|R^2 (\hat{R} x \bar{\bar{A}})| < \infty , \; |R^3 \nabla x \bar{\bar{A}}| < \infty, \; \text{as } R \rightarrow \infty , \qquad (14)$$

where $\bar{\bar{A}}$ stands for either $\bar{\bar{H}}^{(1)}_e$ or $\bar{\bar{E}}^{(1)}_m$,

5) $\vec{E}$ and $\vec{H}$ regular in V (15)

6) $|\nabla \times (R\vec{E})| < \infty, \ |\nabla \times (R\vec{H})| < \infty, \quad \text{as } R \to \infty,$ (16)

then

$$\nabla' \times \vec{E}(\vec{R}') = -\int_V (\nabla \times \nabla \times \vec{E}) \cdot \overline{\overline{E}}{}_m^{(1)} \, dV + \int_S (\hat{n} \times \vec{E}) \cdot (\nabla \times \overline{\overline{E}}{}_m^{(1)}) \, dS, \quad (17)$$

and

$$\nabla' \times \vec{H}(\vec{R}') = -\int_V (\nabla \times \nabla \times \vec{H}) \cdot \overline{\overline{H}}{}_e^{(1)} dV + \int_S \left[\hat{n} \times (\nabla \times \vec{H})\right] \cdot \overline{\overline{H}}{}_e^{(1)} \, dS. \quad (18)$$

This is the principal result of Ch. I. (The notation used for the two dyadics will become clear when we recognize their physical significance.) Wilcox' (1956) expansion theorem makes it obvious that the scattered fields defined in (1) - (4) satisfy condition (16) and we can therefore write

$$\vec{E}^s(\vec{R}') = -ikZ \int_V \vec{H}^s \cdot \overline{\overline{H}}{}_e^{(1)} dV - \int_S (\hat{n} \times \vec{E}^i) \cdot \overline{\overline{H}}{}_e^{(1)} \, dS \quad (19)$$

and

$$ikZ\vec{H}^s(\vec{R}') = -k^2 \int_V \vec{E}^s \cdot \overline{\overline{E}}{}_m^{(1)} dV - \int_S (\hat{n} \times \vec{E}^i) \cdot (\nabla \times \overline{\overline{E}}{}_m^{(1)}) \, dS. \quad (20)$$

The last equation can be written in a better form. To do this we need the explicit representation of $\overline{\overline{E}}{}_m^{(1)}$ in terms of potential functions which leads us to the question of how to determine the dyadics defined by (10) - (14). This problem may be attacked from a purely mathematical point of view. We had the feeling, however, that these functions should be related to the electrostatic and magnetostatic fields of infinitesimal electric and magnetic dipoles. Indeed, we can show (for our geometry) that $\overline{\overline{H}}{}_e^{(1)}$ is the coefficient of k in the low frequency expansion of the total magnetic field of three orthogonally crossed infinitesimal electric dipoles at the point $\vec{R}'$ of V. Similarly, $\overline{\overline{E}}{}_m^{(1)}$ is the coefficient of k in the low frequency expansion of the total electric field of three orthogonally crossed infinitesimal magnetic dipoles at the point

$\vec{R}'$ of V. After this the determination of the two dyadics becomes an easy matter. In Chapter II we derive these dyadics in their explicit form. In Chapter III we start with Eqs. (19) and (20) and modify them in a way that will render them amenable to iteration. Finally, in Chapter IV we apply our method to the sphere.

Chapter I

THE DERIVATION OF TWO INTEGRAL EQUATIONS

In this chapter we proceed to derive in detail the integral equations (17) and (18) of the Introduction. To do this we need the divergence theorem in its dyadic form.[1]

If V is a volume bounded by a regular surface S ($S \in L_2$, where L_p is the class of surfaces whose equations have continuous derivatives up to and including p'th order and whose p'th derivatives satisfy a Hölder condition[2]) and if $\bar{\bar{A}}$ is regular in V and on S ($\bar{\bar{A}} \in C^{(1)}$, where $C^{(n)}$ is the class of functions with continuous derivatives up to and including the n'th order), then

$$\int_V \nabla \cdot \bar{\bar{A}}\, dV = \int_S \hat{n} \cdot \bar{\bar{A}}\, dS\,, \tag{1.1}$$

where $\hat{n}$ is the unit normal directed out of V.

The proof of this theorem follows immediately from the corresponding theorem for vector functions. Attention should be drawn to the fact that the dot product in the surface integral is not commutative.

By writing $\bar{\bar{A}}$ in the form

$$\bar{\bar{A}} = \vec{Q} \times (\nabla \times \bar{\bar{P}}) + (\nabla \times \vec{Q}) \times \bar{\bar{P}} \quad , \tag{1.2}$$

we obtain the following Green's identity

$$\int_V \left[(\nabla \times \nabla \times \vec{Q}) \cdot \bar{\bar{P}} - \vec{Q} \cdot (\nabla \times \nabla \times \bar{\bar{P}})\right] dV = \int_S \hat{n} \cdot \left[\vec{Q} \times (\nabla \times \bar{\bar{P}}) + (\nabla \times \vec{Q}) \times \bar{\bar{P}}\right] dS\,, \tag{1.3}$$

---

[1]All dyadic identities that will be employed subsequently may be found in Van Bladel (1964).

[2]A function $f(\vec{R})$ is said to satisfy a Hölder condition at $\vec{R}_0$ if there are three positive constants A, B, C such that $|f(\vec{R}) - f(\vec{R}_0)| \leq A|\vec{R} - \vec{R}_0|^B$ for all points $\vec{R}$ for which $|\vec{R} - \vec{R}_0| < C$.

where, in arriving at (1.2), we used the dyadic identity

$$\nabla \cdot (\vec{a} \times \bar{\bar{b}}) = (\nabla \times \vec{a}) \cdot \bar{\bar{b}} - \vec{a} \cdot (\nabla \times \bar{\bar{b}}) \,. \tag{1.4}$$

Equation (1.3) is the form of the divergence theorem that we will use to derive the two integral equations.

The First Integral Equation

Let S be a closed, bounded, regular surface. This surface separates the whole space into two regions: the finite region V_i enclosed by S and the rest of the space V. Let $\bar{\bar{E}}_m^{(1)}$ be a function of position defined in V and on S by

$$\bar{\bar{E}}_m^{(1)} = \nabla \times \left[- \frac{\bar{\bar{I}}}{4\pi |\vec{R} - \vec{R}'|} \right] + \bar{\bar{E}}_{m_r}^{(1)} \,, \tag{1.5}$$

where $\bar{\bar{E}}_{m_r}^{(1)}$ is regular in V and on S and satisfies

$$\nabla \times \nabla \times \bar{\bar{E}}_{m_r}^{(1)} = 0, \quad \text{in V and on S} \,. \tag{1.6}$$

Moreover,

$$\hat{n} \times \bar{\bar{E}}_m^{(1)} = 0, \quad \text{on S} \,, \tag{1.7}$$

and

$$\left| R^2 (\hat{R} \times \bar{\bar{E}}_m^{(1)}) \right| < \infty \quad \text{and} \quad \left| R^3 \nabla \times \bar{\bar{E}}_m^{(1)} \right| < \infty \,, \quad \text{as } R \to \infty \,, \tag{1.8}$$

where $\hat{R}$ is the radial unit vector and R the distance from the origin of a rectangular coordinate system to a point in V. For convenience, the origin of the coordinate system is located in V_i . Let, finally, $\vec{E}$ be a regular function of position in V and apply (1.3) in V with $\vec{Q}$ substituted by $\vec{E}$ and $\bar{\bar{P}}$ by $\bar{\bar{E}}_m^{(1)}$.

Since the dyadic $\bar{\bar{E}}_m^{(1)}$ is singular at $\vec{R}'$, we exclude a small sphere from V centered at $\vec{R}'$ with radius r and surface S' . Eq. (1.3) then becomes,

$$\int_V (\nabla\times\nabla\times\vec{E})\cdot\bar{\bar{E}}_m^{(1)}dV = \int_{S+S'+S_\infty} dS\hat{n}\cdot\left[\vec{E}\times(\nabla\times\bar{\bar{E}}_m^{(1)})+(\nabla\times\vec{E})\times\bar{\bar{E}}_m^{(1)}\right] , \quad (1.9)$$

where S_∞ is the surface of a large sphere, with center at the origin, bounding the volume V at infinity (see Fig. 1). We now proceed to examine the surface integrals one by one. First we examine the integral over S. Using the dyadic identity,

$$(\vec{a}\times\vec{b})\cdot\bar{\bar{A}} = \vec{a}\cdot(\vec{b}\times\bar{\bar{A}}) = -\vec{b}\cdot(\vec{a}\times\bar{\bar{A}}) \quad (1.10)$$

and the boundary condition (1. 7), we have that

$$I_s = \int_S (\hat{n}\times\vec{E})\cdot(\nabla\times\bar{\bar{E}}_m^{(1)})\, dS \ . \quad (1.11)$$

Next we examine the integral over S' , where we take S' to be the surface of a sphere of radius r centered at $\vec{R}'$. We intend first to evaluate the integral and then let the radius r of the sphere go to zero. In the process, the integral involving the regular part of $\bar{\bar{E}}_m^{(1)}$ will go to zero and we are left with the following expression

$$I_{s'} = \lim_{r\to 0}\int_{S'}\hat{n}\cdot\left\{\vec{E}\times\nabla\times\nabla\times\left[-\frac{\bar{\bar{I}}}{4\pi|\vec{R}-\vec{R}'|}\right]+(\nabla\times\vec{E})\times\nabla\times\left[-\frac{\bar{\bar{I}}}{4\pi|\vec{R}-\vec{R}'|}\right]\right\} dS. \quad (1.12)$$

Since on S' $\hat{n}=-\hat{r}$ and since

$$\nabla\times\nabla\times\left[-\frac{\bar{\bar{I}}}{4\pi|\vec{R}-\vec{R}'|}\right]=\nabla\nabla\left[-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right]-\bar{\bar{I}}\nabla^2\left[-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right]$$

$$= \nabla\nabla\left[-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right]-\bar{\bar{I}}\,\delta(\vec{R}|\vec{R}') \quad ,$$

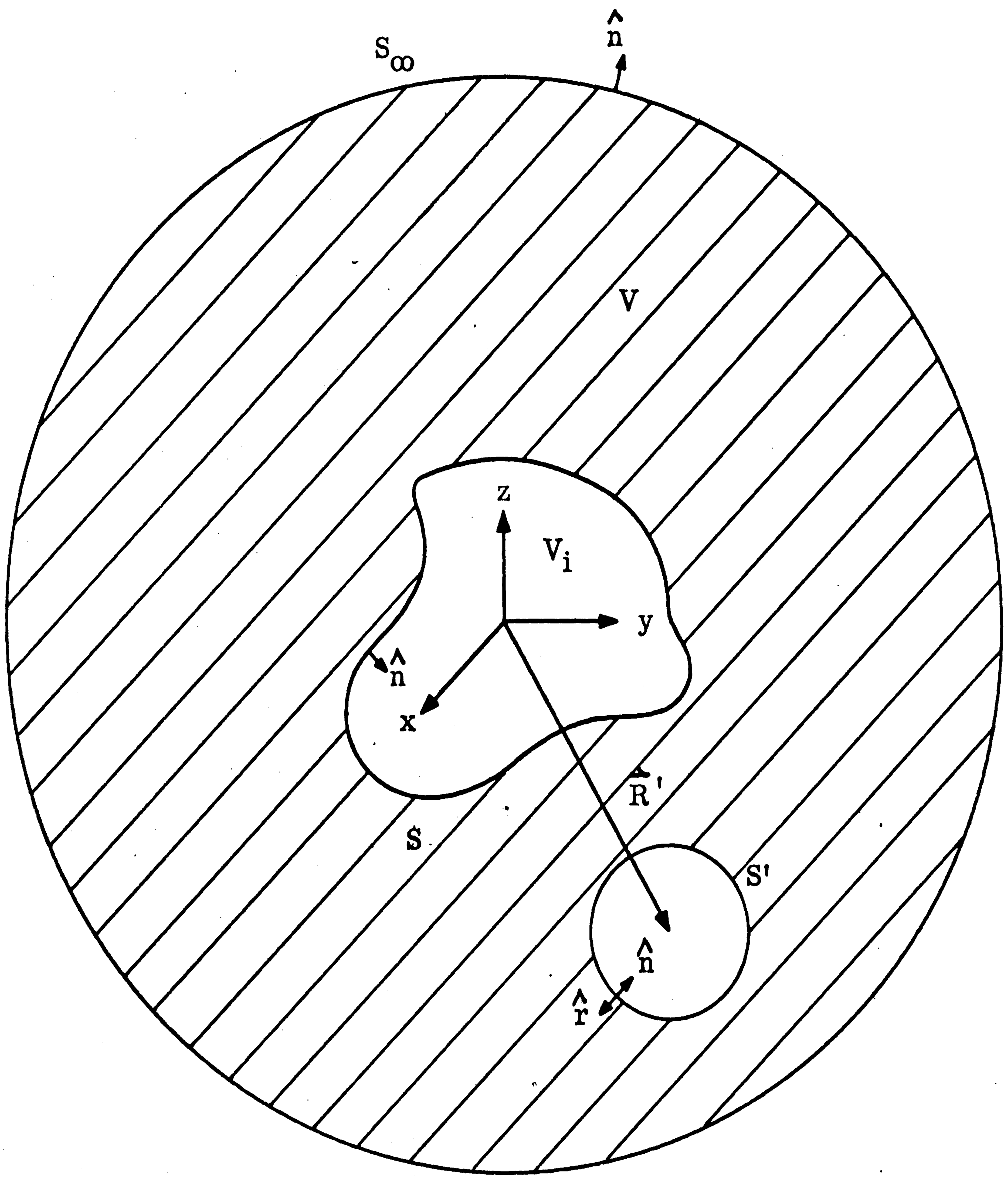

FIG. 1: GEOMETRY FOR THE APPLICATION OF GREEN'S IDENTITY.

we have that

$$I_{S'} = \lim_{r \to o} \int_{S'} (-\hat{r}) \cdot \left[\vec{E} \times \nabla\nabla(-\frac{1}{4\pi r}) + (\nabla \times \vec{E}) \times \nabla \times (-\frac{\bar{\bar{I}}}{4\pi r}) \right] dS \ , \qquad (1.13)$$

where

$$r = |\vec{R} - \vec{R}'| \ , \qquad (1.14)$$

Now

$$\nabla \times (\vec{a}\vec{b}) = (\nabla \times \vec{a})\vec{b} - \vec{a} \times \nabla \vec{b} \ . \qquad (1.15)$$

Substitution of this relation in (1.13) leads to

$$I_{S'} = -\lim_{r \to o} \int_{S'} \hat{r} \cdot \left[(\nabla \times \vec{E}) \nabla(-\frac{1}{4\pi r}) - \nabla \times \left(\vec{E} \nabla(-\frac{1}{4\pi r}) \right) + \right.$$

$$\left. + (\nabla \times \vec{E}) \times \nabla \times (-\frac{\bar{\bar{I}}}{4\pi r}) \right] dS \ . \qquad (1.16)$$

By Stokes' theorem the part of the integral involving $\hat{r} \cdot \nabla \times$ vanishes and by (1.10)

$$\hat{r} \cdot \left[(\nabla \times \vec{E}) \nabla(-\frac{1}{4\pi r}) + (\nabla \times \vec{E}) \times \nabla \times (-\frac{\bar{\bar{I}}}{4\pi r}) \right] =$$

$$= (\hat{r} \cdot \nabla \times \vec{E}) \nabla(-\frac{1}{4\pi r}) - (\nabla \times \vec{E}) \cdot \left[\hat{r} \times \nabla \times (-\frac{\bar{\bar{I}}}{4\pi r}) \right] =$$

$$= (\hat{r} \cdot \nabla \times \vec{E}) \nabla(-\frac{1}{4\pi r}) - (\nabla \times \vec{E}) \cdot \left\{ \hat{r} \times \left[\nabla(-\frac{1}{4\pi r}) \times \bar{\bar{I}} \right] \right\} =$$

$$= (\hat{r} \cdot \nabla \times \vec{E}) \nabla(-\frac{1}{4\pi r}) - (\nabla \times \vec{E}) \cdot \left\{ \nabla(-\frac{1}{4\pi r}) (\hat{r} \cdot \bar{\bar{I}}) - \bar{\bar{I}} \left[\hat{r} \cdot \nabla(-\frac{1}{4\pi r}) \right] \right\} =$$

$$= (\hat{r} \cdot \nabla \times \vec{E}) \nabla(-\frac{1}{4\pi r}) - \left[(\nabla \times \vec{E}) \cdot \nabla(-\frac{1}{4\pi r}) \right] \hat{r} + \hat{r} \cdot \nabla(-\frac{1}{4\pi r}) (\nabla \times \vec{E}) =$$

$$= (\nabla \times \vec{E}) \times \left[\nabla(-\frac{1}{4\pi r}) \times \hat{r} \right] + \hat{r} \cdot \nabla(-\frac{1}{4\pi r}) (\nabla \times \vec{E}) = \hat{r} \cdot \nabla(-\frac{1}{4\pi r}) (\nabla \times \vec{E}) \ , \qquad (1.17)$$

where, above, we made use of the dyadic identity,

$$\vec{a} \times (\vec{b} \times \bar{\bar{A}}) = \vec{b}(\vec{a} \cdot \bar{\bar{A}}) - \bar{\bar{A}}(\vec{a} \cdot \vec{b}) \ . \qquad (1.18)$$

Equation (1.16) then becomes

$$I_{S'} = -\lim_{r\to o} \int_{S'} \hat{r} \cdot \left[\nabla\left(-\frac{1}{4\pi r}\right)(\nabla \times \vec{E})\right] dS$$

$$= -\lim_{r\to o} \int_{S'} dS \, \frac{(\nabla \times \vec{E})}{4\pi r^2} = -\nabla' \times \vec{E}\,(\vec{R}') \ . \tag{1.19}$$

We are now left with the evaluation of the integral over S_∞. To do this we draw a sphere of radius R and center at the origin of the coordinate system. We then let R go to infinity and we require the integral to vanish in the limit. If $\hat{R}$ is the unit vector in the radial direction we have that (see Eq. 1.9)

$$I_{S_\infty} = \int_{S_\infty} \hat{R} \cdot \left[\vec{E} \times (\nabla \times \bar{\bar{E}}_m^{(1)}) + (\nabla \times \vec{E}) \times \bar{\bar{E}}_m^{(1)}\right] dS \ .$$

By (1.10)

$$I_{S_\infty} = \int_0^\pi \int_0^{2\pi} R^2 \sin\theta \, d\theta \, d\phi \left[(\hat{R} \times \vec{E}) \cdot (\nabla \times \bar{\bar{E}}_m^{(1)}) - (\nabla \times \vec{E}) \cdot (\hat{R} \times \bar{\bar{E}}_m^{(1)})\right]$$

from which we can write

$$\lim_{R\to\infty} \left|I_{S_\infty}\right| \leq \lim_{R\to\infty} \int_0^\pi \int_0^{2\pi} \sin\theta \, d\theta \, d\phi \left\{ \left|R^2 (\hat{R} \times \vec{E}) \cdot (\nabla \times \bar{\bar{E}}_m^{(1)})\right| + \right.$$

$$\left. + \left|R^2 (\nabla \times \vec{E}) \cdot (\hat{R} \times \bar{\bar{E}}_m^{(1)})\right| \right\} \ . \tag{1.20}$$

In order that this integral vanish in the limit we must have

$$\lim_{R\to\infty} \left|R^2 (\hat{R} \times \vec{E}) \cdot (\nabla \times \bar{\bar{E}}_m^{(1)})\right| = 0 \ ,$$

and

$$\lim_{R\to\infty} \left|R^2 (\nabla \times \vec{E}) \cdot (\hat{R} \times \bar{\bar{E}}_m^{(1)})\right| = 0 \ ,$$

which, together with (1.8), imply that

$$|\hat{R} \times \vec{E}| < \infty \quad \text{and} \quad |R\, \nabla \times \vec{E}| < \infty, \quad \text{as } R \to \infty . \tag{1.21}$$

Since

$$\nabla \times (R\vec{E}) = \hat{R} \times \vec{E} + R\, \nabla \times \vec{E} ,$$

Eq. (1.21) can be written

$$|\nabla \times (R\vec{E})| < \infty \quad \text{as } R \to \infty . \tag{1.22}$$

This is the regularity condition on $\vec{E}$ if the integral (1.20) is to vanish.

Collecting our results from (1.9), (1.11) and (1.19), we can then state the following theorem:

Theorem A:

If V is the volume exterior to a closed, bounded, regular surface S and $\vec{E}$ is a regular function of position in V and on S satisfying the regularity condition $|\nabla \times (R\vec{E})| < \infty$, as $R \to \infty$, then $\vec{E}$ satisfies the integral equation

$$\nabla' \times \vec{E}(\vec{R}') = -\int_V (\nabla \times \nabla \times \vec{E}) \cdot \bar{\bar{E}}_m^{(1)} dV + \int_S (\hat{n} \times \vec{E}) \cdot (\nabla \times \bar{\bar{E}}_m^{(1)})\, dS , \tag{1.23}$$

where $\bar{\bar{E}}_m^{(1)}$ is defined in (1.5) - (1.8), and $\hat{n}$ is the unit normal on S directed out of V and into the interior of S.

The Second Integral Equation

The derivation of the second integral equation follows closely that of the first and for this reason we shall be brief.

The geometry of the problem remains the same. We define the dyadic $\bar{\bar{H}}_e^{(1)}$ as follows:

$$\bar{\bar{H}}_e^{(1)} = \nabla \times \left[- \frac{\bar{\bar{I}}}{4\pi |\vec{R} - \vec{R}'|} \right] + \bar{\bar{H}}_{e_r}^{(1)} \tag{1.24}$$

where $\overline{\overline{H}}_{e_r}^{(1)}$ is regular in V and on S and satisfies

$$\nabla \times \nabla \times \overline{\overline{H}}_{e_r}^{(1)} = 0, \text{ in V and on S} . \tag{1.25}$$

Moreover,

$$\hat{n} \times \nabla \times \overline{\overline{H}}_{e}^{(1)} = 0 \text{ , on S ,} \tag{1.26}$$

and

$$\left| R^2 (\hat{R} \times \overline{\overline{H}}_{e}^{(1)}) \right| < \infty \text{ and } \left| R^3 \nabla \times \overline{\overline{H}}_{e}^{(1)} \right| < \infty, \text{ as } R \rightarrow \infty . \tag{1.27}$$

Letting in (1.3), $\overline{\overline{P}}$ be $\overline{\overline{H}}_{e}^{(1)}$ and $\vec{Q}$ be $\vec{H}$ ($\vec{H}$ regular in V) we have that

$$\int_V (\nabla \times \nabla \times \vec{H}) \cdot \overline{\overline{H}}_{e}^{(1)} dV = \int_{S+S'+S_\infty} \hat{n} \cdot \left[\vec{H} \times (\nabla \times \overline{\overline{H}}_{e}^{(1)}) + (\nabla \times \vec{H}) \times \overline{\overline{H}}_{e}^{(1)} \right] dS. \tag{1.28}$$

By (1.26) and the dyadic identity (1.10), the integral over S becomes

$$I_S = \int_S \left[\hat{n} \times (\nabla \times \vec{H}) \right] \cdot \overline{\overline{H}}_{e}^{(1)} d S . \tag{1.29}$$

The integral over S' is evaluated as in the previous section yielding

$$I_{S'} = - \nabla' \times \vec{H} (\vec{R}') . \tag{1.30}$$

Similarly, the integral over S_∞ vanishes in the limit as $R \rightarrow \infty$ provided

$$\left| \nabla \times (R \vec{H}) \right| < \infty \quad \text{as } R \rightarrow \infty . \tag{1.31}$$

Collecting these results we can state the following theorem:

Theorem B.

If V is the volume exterior to a closed, bounded, regular surface S and $\vec{H}$ is a regular function of position in V and on S satisfying the regularity condition $\left| \nabla \times (R \vec{H}) \right| < \infty$, as $R \rightarrow \infty$, then $\vec{H}$ satisfies the integral equation

$$\nabla' \times \vec{H}(\vec{R}') = -\int_V (\nabla \times \nabla \times \vec{H}) \cdot \overline{\overline{H}}_e^{(1)} \, dV + \int_S \left[\hat{n} \times (\nabla \times \vec{H})\right] \cdot \overline{\overline{H}}_e^{(1)} \, dS \,, \tag{1.32}$$

where $\overline{\overline{H}}_e^{(1)}$ is defined in (1.24) - (1.27), and $\hat{n}$ is the unit normal on S directed out of V and into the interior of S .

At this point we conclude Chapter I, the main results being Theorems A and B. The integral equations (1.23) and (1.32) will be subsequently employed to find integral representations for the scattered field defined in (1) - (4) of the Introduction. Our immediate concern, however, is the explicit representation of $\overline{\overline{H}}_e^{(1)}$ and $\overline{\overline{E}}_m^{(1)}$, the dyadic kernels of these equations, in terms of potential functions. This we proceed to do in the next chapter.

Chapter II

THE FIELDS OF INFINITESIMAL DIPOLES AND THE SOLUTION OF THE TWO DYADIC PARTIAL DIFFERENTIAL EQUATIONS

As mentioned on p. 7 of the Introduction, the problem of finding explicit solutions in terms of potential functions for the dyadic kernels $\bar{\bar{H}}_e^{(1)}$ and $\bar{\bar{E}}_m^{(1)}$ can be dealt with either from a mathematical point of view (i. e. without taking recourse to the physical significance of the dyadics) or by recognizing the relation between these dyadics and the fields of static electric and magnetic dipoles and proceed to determine them by utilizing the available knowledge on potential theory. We chose the second course of action.

Let S be a closed, bounded, perfectly conducting surface immersed in vacuum. This surface separates the whole three dimensional space into two regions: the finite region V_i enclosed by S and the rest of the space V. It is, moreover, regular in the sense that it satisfies the requirements of Green's theorem: $S \in L_2$, where L_p is the class of surface whose equations have continuous derivatives up to and including pth order and whose pth derivatives satisfy a Hölder condition. For later purposes, erect a rectangular coordinate system x, y, z with origin in V_i. Let $\bar{\bar{J}}$ be the volume dyadic current density in a finite region of V. We let $\bar{\bar{J}}$ have a harmonic time variation $e^{-i\omega t}$ which we suppress throughout this work. Then the electromagnetic fields in dyadic form satisfy

1) the dyadic Maxwell's equations

$$\nabla \times \bar{\bar{E}} = ikZ\bar{\bar{H}} \tag{2.1}$$

$$\nabla \times \bar{\bar{H}} = \bar{\bar{J}} - ikY\bar{\bar{E}} \tag{2.2}$$

$Z = 1/Y$, the free space characteristic impedance,

2) the dyadic wave equations

$$\nabla \times \nabla \times \bar{\bar{E}} - k^2 \bar{\bar{E}} = ikZ\bar{\bar{J}} \tag{2.3}$$

$$\nabla \times \nabla \times \bar{\bar{H}} - k^2 \bar{\bar{H}} = \nabla \times \bar{\bar{J}} \quad , \tag{2.4}$$

3) the boundary conditions

$$\hat{n} \times \bar{\bar{E}} = 0, \quad \hat{n} \cdot \bar{\bar{H}} = 0 \text{ on } S, \tag{2.5}$$

$\hat{n}$ the unit normal on S directed out of V and into V_i.

Moreover, these fields satisfy a radiation condition in which we are not presently interested.

Except for the harmonic time variation we have left the current distribution $\bar{\bar{J}}$ completely unspecified. We now turn our attention to two types of it, namely:

$$\bar{\bar{J}}_e = -ik \bar{\bar{I}} \; \delta(\vec{R}|\vec{R}'), \tag{2.6}$$

and

$$\bar{\bar{J}}_m = -Y \nabla \times \left[\bar{\bar{I}} \, \delta(\vec{R}|\vec{R}')\right], \tag{2.7}$$

where $\bar{\bar{I}}$ is the identity dyadic defined by

$$\bar{\bar{I}} = \hat{a}_1\hat{a}_1 + \hat{a}_2\hat{a}_2 + \hat{a}_3\hat{a}_3 \; ^{1}. \tag{2.8}$$

The current distribution in (2.6) is that of three harmonically oscillating infinitesimal electric dipoles situated, each along one coordinate direction, at the point $\vec{R}'$ of V and of dipole moment

$$\vec{p}_{e_i} = \frac{1}{c}\hat{a}_i, \quad i=1,2,3, \tag{2.9}$$

c being the speed of light in vacuum. Similarly, the current distribution in (2.7) is that of three harmonically oscillating infinitesimal magnetic dipoles situated, each along one coordinate direction, at the point $\vec{R}'$ of V and of dipole moment

$$\vec{p}_{m_i} = -Y \hat{a}_i, \quad i=1,2,3 \; . \tag{2.10}$$

Let now $\bar{\bar{E}}_e$ and $\bar{\bar{H}}_e$ be the fields due to the current distribution (2.6) and expand them in a power series of k:

$$\bar{\bar{E}}_e = \sum_{n=o}^{\infty} (ik)^n \, \bar{\bar{E}}_e^{(n)}, \quad \bar{\bar{H}}_e = \sum_{n=o}^{\infty} (ik)^n \, \bar{\bar{H}}_e^{(n)} \; . \tag{2.11}$$

---

[1] For convenience we use $\hat{a}_1, \hat{a}_2, \hat{a}_3$ instead of $\hat{x}, \hat{y}, \hat{z}$ for the rectangular unit vectors.

Substitution of these equations together with (2. 6) in (2. 1) - (2. 5) leads to the following relations

$$\nabla x \bar{\bar{E}}_e^{(0)} = 0 \tag{2.12a}$$

$$\nabla x \bar{\bar{E}}_e^{(n+1)} = Z\bar{\bar{H}}_e^{(n)}, \quad n=0,1,2,\ldots \tag{2.12b}$$

$$\nabla x \bar{\bar{H}}_e^{(0)} = 0 \tag{2.13a}$$

$$\nabla x \bar{\bar{H}}_e^{(1)} = -\bar{\bar{I}}\delta(\vec{R}\,|\,\vec{R}') - Y\bar{\bar{E}}_e^{(0)} \tag{2.13b}$$

$$\nabla x \bar{\bar{H}}_e^{(n+1)} = -Y\bar{\bar{E}}_e^{(n)}, \quad n=1,2,3,\ldots \tag{2.13c}$$

$$\nabla x \nabla x \bar{\bar{E}}_e^{(0)} = 0 \tag{2.14a}$$

$$\nabla x \nabla x \bar{\bar{E}}_e^{(1)} = 0 \tag{2.14b}$$

$$\nabla x \nabla x \bar{\bar{E}}_e^{(2)} + \bar{\bar{E}}_e^{(0)} = \bar{\bar{I}}\, Z\, \delta(\vec{R}|\vec{R}') \tag{2.14c}$$

$$\nabla x \nabla x \bar{\bar{E}}_e^{(n+2)} + \bar{\bar{E}}_e^{(n)} = 0, \quad n=1,2,3,\ldots \tag{2.14d}$$

$$\nabla x \nabla x \bar{\bar{H}}_e^{(0)} = 0 \tag{2.15a}$$

$$\nabla x \nabla x \bar{\bar{H}}_e^{(1)} = -\nabla x\left[\bar{\bar{I}}\,\delta(\vec{R}|\vec{R}')\right] \tag{2.15b}$$

$$\nabla x \nabla x\, \bar{\bar{H}}_e^{(n+2)} + \bar{\bar{H}}_e^{(n)} = 0, \quad n=0,1,2,\ldots \tag{2.15c}$$

$$\hat{n} x \bar{\bar{E}}_e^{(n)} = 0, \quad \hat{n}\cdot\bar{\bar{H}}_e^{(n)} = 0, \quad n=0,1,2,\ldots, \text{ on } S. \tag{2.16}$$

Repeating the procedure for the fields

$$\bar{\bar{E}}_m = \sum_{n=o}^{\infty} (ik)^n\, \bar{\bar{E}}_m^{(n)}, \quad \bar{\bar{H}}_m = \sum_{n=o}^{\infty} (ik)^n\, \bar{\bar{H}}_m^{(n)}, \tag{2.17}$$

of the current distribution (2. 7) we obtain

$$\nabla x \bar{\bar{E}}_m^{(0)} = 0 \tag{2.18a}$$

$$\nabla x \bar{\bar{E}}_m^{(n+1)} = Z\bar{\bar{H}}_m^{(n)}, \quad n=0,1,2,\ldots \tag{2.18b}$$

$$\nabla x \overline{\overline{H}}_m^{(0)} = -Y \nabla x \left[\overline{\overline{I}} \delta (\vec{R} | \vec{R}')\right] \tag{2.19a}$$

$$\nabla x \overline{\overline{H}}_m^{(n+1)} = Y \overline{\overline{E}}_m^{(n)} \; , \; n = 0, 1, 2, \ldots \tag{2.19b}$$

$$\nabla x \nabla x \overline{\overline{E}}_m^{(0)} = 0 \tag{2.20a}$$

$$\nabla x \nabla x \overline{\overline{E}}_m^{(1)} = - \nabla x \left[\overline{\overline{I}} \delta (\vec{R} | \vec{R}')\right] \tag{2.20b}$$

$$\nabla x \nabla x \overline{\overline{E}}_m^{(n+2)} + \overline{\overline{E}}_m^{(n)} = 0, \; n=0, 1, 2, \ldots \tag{2.20c}$$

$$\nabla x \nabla x \overline{\overline{H}}_m^{(0)} = -Y \nabla x \nabla x \left[\overline{\overline{I}} \delta (\vec{R} | \vec{R}')\right] \tag{2.21a}$$

$$\nabla x \nabla x \overline{\overline{H}}_m^{(1)} = 0 \tag{2.21b}$$

$$\nabla x \nabla x \overline{\overline{H}}_m^{(n)} + \overline{\overline{H}}_m^{(n)} = 0, \; n=0, 1, 2, \ldots \tag{2.21c}$$

$$\hat{n} x \overline{\overline{E}}_m^{(n)} = 0 \; , \; \hat{n} \cdot \overline{\overline{H}}_m^{(n)} = 0, \; n=0, 1, 2, \ldots, \text{ on } S \; . \tag{2.22}$$

At this point we offer some relief to the reader by saying that, of all this multitude of equations, we are only interested in those involving $\overline{\overline{H}}_e^{(1)}$ and $\overline{\overline{E}}_m^{(1)}$. These two dyadics are directly related (if not identifiable) with those defined in Eqs. (10) - (14) of the Introduction: By (2.15b), $\overline{\overline{H}}_e^{(1)}$ can be written in the form (10); by Eq. (2.13b) and the boundary condition (2.16) on $\overline{\overline{E}}_e^{(0)}$, it satisfies (13). Similarly, by (2.20b), $\overline{\overline{E}}_m^{(1)}$ can be written in the form (11); by (2.22), it satisfies (13). Our next step is to find explicit forms for $\overline{\overline{H}}_e^{(1)}$ and $\overline{\overline{E}}_m^{(1)}$ in terms of potential functions. Subsequently, we will verify that these solutions satisfy the regularity condition (14) of the Introduction and, therefore, they qualify as kernels of the integral equations (17) and (18). In effect, we will have shown that the kernels of the integral equations can be obtained from the solutions of electrostatic and magnetostatic problems.

The Dyadic $\bar{\bar{H}}_e^{(1)}$ in Terms of Potential Functions

Let

$$\bar{\bar{H}}_e^{(1)} = \nabla x \left[- \frac{\bar{\bar{I}}}{4\pi|\vec{R}-\vec{R}'|} \right] + \bar{\bar{H}}_{e_r}^{(1)} \tag{2.23}$$

where

$$\nabla x \nabla x \bar{\bar{H}}_{e_r}^{(1)} = 0 . \tag{2.24}$$

The dyadic $\bar{\bar{H}}_e^{(1)}$ satisfies Eq. (2.15b), namely

$$\nabla x \nabla x \bar{\bar{H}}_e^{(1)} = - \nabla x \left[\bar{\bar{I}} \delta \; (\vec{R}|\vec{R}') \right] , \tag{2.25}$$

and is related to $\bar{\bar{E}}_e^{(0)}$ by (2.13b). Equation (2.12a) permits us to write $\bar{\bar{E}}_e^{(0)}$ in the form

$$\bar{\bar{E}}_e^{(0)} = -Z_\circ \sum_{i=1}^{3} \left(\nabla \phi_{ei}^{(0)} \right) \hat{a}_i . \tag{2.26}$$

Substituting this expression in (2.13b) and taking the divergence of the resulting equation we have

$$\sum_{i=1}^{3} \nabla^2 \phi_{ei}^{(0)} \; \hat{a}_i = \bar{\bar{I}} \cdot \nabla \delta (\vec{R}|\vec{R}') \quad . \tag{2.27}$$

We wish then to find solutions to the problem

$$\nabla^2 \phi_{ei}^{(0)} = \hat{a}_i \cdot \nabla \delta (\vec{R}|\vec{R}') , \quad i=1, 2, 3. \tag{2.28}$$

By virtue of (2.16), these scalar functions satisfy the boundary condition

$$\hat{n} x \nabla \phi_{ei}^{(0)} = 0 \text{ on S}, \; i=1, 2, 3. \tag{2.29}$$

For the total field $\phi_{ei}^{(0)}$ we can write

$$\phi_{ei}^{(0)} = \hat{a}_i \cdot \nabla\left[-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right] + \phi_{ei_r}^{(0)}(\vec{R}|\vec{R}') \quad , \tag{2.30}$$

where

$$\nabla^2 \phi_{ei_r}^{(0)}(\vec{R}|\vec{R}') = 0 \, . \tag{2.31}$$

Stokes' theorem together with the boundary condition (2. 29) implies that $\phi_{ei}^{(0)}$ is a constant on the surface S. From (2. 30) we can then write

$$\phi_{ei_r}^{(0)}(\vec{R}_s|\vec{R}') = -\hat{a}_i \cdot \nabla_s\left[-\frac{1}{4\pi|\vec{R}_s-\vec{R}'|}\right] + C_{ei}, \quad \vec{R}_s \in S, \tag{2.32}$$

where C_{ei}, a constant, is the value of $\phi_{ei}^{(0)}$ on S. We now employ Green's theorem and write

$$\phi_{ei_r}^{(0)}(\vec{R}|\vec{R}') = \int_S \phi_{ei_r}^{(0)}(\vec{R}_s|\vec{R}') \frac{\partial}{\partial n_s} G^{(e)}(\vec{R}_s|\vec{R})\, dS \, , \tag{2.33}$$

where $G^{(e)}$ is the exterior Dirichlet Green's function for S:

$$G^{(e)}(\vec{R}|\vec{R}') = -\frac{1}{4\pi|\vec{R}-\vec{R}'|} + G_r^{(e)}(\vec{R}|\vec{R}') \quad , \tag{2.34}$$

with

$$\nabla^2 G_r^{(e)}(\vec{R}|\vec{R}') = 0, \quad \text{in } V \, , \tag{2.35}$$

and

$$G^{(e)}(\vec{R}_s|\vec{R}') = 0 \, , \quad \vec{R}_s \in S \, . \tag{2.36}$$

Substituting (2. 32) in (2. 33) we have

$$\phi_{ei_r}^{(0)}(\vec{R}|\vec{R}')=(\hat{a}_i\cdot\nabla')\int_S\left[-\frac{1}{4\pi|\vec{R}_s-\vec{R}'|}\right]\frac{\partial}{\partial n_s}G^{(e)}(\vec{R}_s|\vec{R})\,dS+$$

$$+C_{ei}\int_S\frac{\partial}{\partial n_s}G^{(e)}(\vec{R}_s|\vec{R})dS$$

$$=-(\hat{a}_i\cdot\nabla')G_r^{(e)}(\vec{R}|\vec{R}')+C_{ei}\int_S\frac{\partial}{\partial n_s}G^{(e)}(\vec{R}_s|\vec{R})\,dS\,. \qquad (2.37)$$

By a simple application of the divergence theorem we find that

$$\int_S\frac{\partial}{\partial n_s}\left[-\frac{1}{4\pi|\vec{R}_s-\vec{R}|}\right]dS=0\,; \qquad (2.38)$$

therefore,

$$\phi_{ei_r}^{(0)}(\vec{R}|\vec{R}')=-(\hat{a}_i\cdot\nabla')G_r^{(e)}(\vec{R}|\vec{R}')+C_{ei}\int_S\frac{\partial}{\partial n_s}G_r^{(e)}(\vec{R}_s|\vec{R})\,dS\;. \qquad (2.39)$$

Substituting (2.39) in (2.30) and taking (2.34) into account we have

$$\phi_{ei}^{(0)}=-(\hat{a}_i\cdot\nabla')G^{(e)}(\vec{R}|\vec{R}')+C_{ei}\int_S\frac{\partial}{\partial n_s}G_r^{(e)}(\vec{R}_s|\vec{R})\,dS\;. \qquad (2.40)$$

In order to determine the constant C_{ei} we employ the relation

$$\int_S\hat{n}_s\cdot\nabla_s\,\phi_{ei_r}^{(0)}(\vec{R}_s|\vec{R}')\,dS=0\,. \qquad (2.41)$$

This is a consequence of (2. 13b) and Stokes' theorem. It is a mathematical statement of the physical fact that the total induced static charge on the perfectly conducting surface must be zero. Substituting (2. 39) in this expression we have that

$$C_{ei} = \frac{(\hat{a}_i \cdot \nabla') \int_S \frac{\partial}{\partial n_s} G_r^{(e)}(\vec{R}_s | \vec{R}') d\,S}{\int_S dT \frac{\partial}{\partial n_T} \int_S dS \frac{\partial}{\partial n_s} G_r^{(e)}(\vec{R}_s | \vec{R}_T)} = (\hat{a}_i \cdot \nabla') D_e(\vec{R}') \tag{2.42}$$

where

$$D_e(\vec{R}') = \frac{\int_S \frac{\partial}{\partial n_s} G_r^{(e)} (\vec{R}_s | \vec{R}') \, dS}{\int_S dT \frac{\partial}{\partial n_T} \int_S dS \frac{\partial}{\partial n_s} G_r^{(e)}(\vec{R}_s | \vec{R}_T)} \; . \tag{2.43}$$

The electric field dyadic (2.26) can then be written

$$\bar{\bar{E}}_e^{(0)}(\vec{R}|\vec{R}') = Z \sum_{i=1}^{3} \nabla \left[(\hat{a}_i \cdot \nabla') G^{(e)}(\vec{R}|\vec{R}') - C_{ei} \int_S dS \frac{\partial}{\partial n_s} G_r^{(e)}(\vec{R}_s | \vec{R}) \right] \hat{a}_i \, , \tag{2.44}$$

or by (2.42)

$$\bar{\bar{E}}_e^{(0)}(\vec{R}|\vec{R}') = Z \, \nabla\nabla' \left[G^{(e)}(\vec{R}|\vec{R}') - D_e(\bar{R}') \int_S dS \frac{\partial}{\partial n_s} G_r^{(e)} (\vec{R}_s | \vec{R}) \right] \, , \tag{2.45}$$

and if we let

$$\psi_e^{(0)}(\vec{R}|\vec{R}') = G^{(e)}(\vec{R}|\vec{R}') - D_e(\vec{R}') \int_S dS \frac{\partial}{\partial n_s} G_r^{(e)}(\vec{R}_s | \vec{R}) \, , \tag{2.46}$$

then

$$\bar{\bar{E}}_e^{(0)}(\vec{R}|\vec{R}') = Z \, \nabla\nabla' \, \psi_e^{(0)} (\vec{R}|\vec{R}') \quad . \tag{2.47}$$

In passing we note that this is the electric field due to three orthogonally crossed electric dipoles with moments defined by (2.9). Now that we have determined $\bar{\bar{E}}_e^{(0)}$ we return to (2.13b) and substitute our result in it:

$$\nabla \times \bar{\bar{H}}_e^{(1)} = - \bar{\bar{I}} \; \delta \, (\vec{R}|\vec{R}') - \nabla\nabla' \; \psi_e^{(0)} \quad . \tag{2.48}$$

Substituting (2.23) and (2.46) in this expression and employing the identity dyadic

$$\nabla \times \nabla \times (\bar{\bar{I}}\,\psi) = \nabla\nabla\psi - \bar{\bar{I}}\,\nabla^2\psi \tag{2.49}$$

we obtain

$$\nabla\nabla\left[-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right] = \bar{\bar{I}}\,\nabla^2\left[-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right] + \nabla\times\bar{\bar{H}}^{(1)}_{e_r} =$$

$$= -\bar{\bar{I}}\,\delta(\vec{R}\;\vec{R}') - \nabla\nabla'\left[G^{(e)}(\vec{R}|\vec{R}') - D_e(\vec{R}')\int_S dS\,\frac{\partial}{\partial n_s}\,G^{(e)}_r(\vec{R}_s|\vec{R})\right]. \tag{2.50}$$

By taking into account the definition (2.34) of $G^{(e)}$ and that

$$\nabla^2\left[-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right] = \delta(\vec{R}|\vec{R}')\,, \tag{2.51}$$

and

$$\nabla\left[-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right] = -\nabla'\left[-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right], \tag{2.52}$$

Eq. (2.50) reduces to

$$\nabla\times\bar{\bar{H}}^{(1)}_{e_r} = -\nabla\nabla'\left[G^{(e)}_r(\vec{R}|\vec{R}') - D_e(\vec{R}')\int_S dS\,\frac{\partial}{\partial n_s}\,G^{(e)}_r(\vec{R}_s|\vec{R})\right]\,, \tag{2.53}$$

$$= -\nabla\nabla'\,\psi^{(0)}_{e_r}\,, \tag{2.54}$$

where $\psi^{(0)}_{e_r}(\vec{R}|\vec{R}')$ is the regular part of $\psi^{(0)}_e(\vec{R}|\vec{R}')$. This dyadic equation can be broken into three vector equations of the form

$$\nabla\times\vec{H}^{(1)}_{ei_r} = -(\hat{a}_i\cdot\nabla')\,\nabla\psi^{(0)}_{e_r}\,. \tag{2.55}$$

From (2.40), (2.42) and (2.46) we can relate $\phi^{(0)}_{ei}$ to $\psi^{(0)}_e$ as follows:

$$\phi^{(0)}_{ei} = -(\hat{a}_i\cdot\nabla')\,\psi^{(0)}_e\,, \tag{2.56}$$

and consequently,

$$\phi^{(0)}_{ei_r} = -(\hat{a}_i \cdot \nabla') \psi^{(0)}_{e_r} \quad . \tag{2.57}$$

Equation (2.55) can then be written

$$\nabla x \vec{H}^{(1)}_{ei_r} = \nabla \phi^{(0)}_{ei_r} \quad . \tag{2.58}$$

Stevenson (1954) has shown that the necessary and sufficient conditions for (2.58) to have a solution are

$$\nabla^2 \phi^{(0)}_{ei_r} = 0 \text{ in V} \tag{2.59}$$

and

$$\int_S \hat{n}_s \cdot \nabla_s \phi^{(0)}_{ei_r} \, dS = 0 \; . \tag{2.60}$$

But $\phi^{(0)}_{ei_r}$ was constructed to satisfy Laplace's equation and (2.60) is automatically satisfied if C_{ei} is chosen according to (2.42). According to Kleinman (1965b), a particular solution, $\vec{H}^{(1)p}_{ei_r}$, of (2.60) may be cast in either of the following forms:

$$\vec{H}^{(1)p}_{ei_r}(\vec{R}|\vec{R}') = \frac{1}{4\pi} \nabla x \left\{ \int_V \frac{\nabla_V \phi^{(0)}_{ei_r}(\vec{R}_V|\vec{R}')}{|\vec{R}_V - \vec{R}|} dV + \int_{V_i} \frac{\nabla_V N^{(i)}_{ei}(\vec{R}_V)}{|\vec{R}_V - \vec{R}|} dV \right\} , \tag{2.61}$$

where V_i is the volume interior to S, or

$$\vec{H}^{(1)p}_{ei_r}(\vec{R}|\vec{R}') = \frac{1}{4\pi} \nabla x \int_S dS \, \frac{\left[\phi^{(0)}_{ei_r}(\vec{R}_s|\vec{R}') - N^{(i)}_{ei}(\vec{R}_s)\right] \hat{n}_s}{|\vec{R}_s - \vec{R}|} \quad . \tag{2.62}$$

The function $N_{ei}^{(i)}(\vec{R})$, $\vec{R}$ interior to S, is a potential function satisfying

$$\nabla^2 N_{ei}^{(i)}(\vec{R}) = 0 \,, \quad \vec{R} \text{ interior to S} \tag{2.63}$$

with

$$\hat{n}_s \cdot \nabla_s N_{ei}^{(i)}(\vec{R}_s) = \hat{n}_s \cdot \nabla_s \phi_{ei_r}^{(0)}(\vec{R}_s | \vec{R}'), \quad \vec{R}_s \in S \,. \tag{2.64}$$

This is a standard interior Neumann problem and has a solution provided that

$$\int_S \hat{n}_S \cdot \nabla_S N_{ei}^{(i)}(\vec{R}_S)\, dS = 0 \,, \tag{2.65}$$

which is satisfied by virtue of (2.64) and (2.60) .

The complete solution of $\bar{\bar{H}}_e^{(1)}$ can then be written:

$$\bar{\bar{H}}_e^{(1)} = \nabla x \left[- \frac{\bar{\bar{I}}}{4\pi | \vec{R} - \vec{R}' |} \right] + \bar{\bar{H}}_{e_r}^{(1)p} + \sum_{i=1}^{3} \nabla N_{ei}^{(e)} \hat{a}_i \,, \tag{2.66}$$

where $\nabla N_{ei}^{(e)}$ is a solution of the homogeneous part of (2.58). The functions $N_{ei}^{(e)}$ are exterior Neumann functions and can be uniquely determined from the boundary condition (2.16) on $\bar{\bar{H}}_e^{(1)}$:

$$\left.\begin{aligned} &\nabla^2 N_{ei}^{(e)} = 0 \ \text{ in V} \\ &N_{ei}^{(e)} \text{ regular at infinity} \\ &\hat{n}_s \cdot \nabla_s N_{ei}^{(e)} = -\hat{n}_s \cdot \left[\nabla_s \left(- \frac{1}{4\pi | \vec{R}_s - \vec{R}' |} \right) x \hat{a}_i \right] - \hat{n}_s \cdot \vec{H}_{ei_r}^{(1)p} \,, \text{ on S} \,. \end{aligned}\right\} \tag{2.67}$$

<u>The Dyadic $\overline{\overline{E}}_m^{(1)}$ in Terms of Potential Functions</u>

Let

$$\overline{\overline{E}}_m^{(1)} = \nabla \times \left[-\frac{\overline{\overline{I}}}{4\pi |\vec{R}-\vec{R}'|} \right] + \overline{\overline{E}}_{m_r}^{(1)} , \tag{2.68}$$

with

$$\nabla \times \nabla \times \overline{\overline{E}}_{m_r}^{(1)} = 0 . \tag{2.69}$$

The dyadic $\overline{\overline{E}}_m^{(1)}$ satisfies (2.20b), namely

$$\nabla \times \nabla \times \overline{\overline{E}}_m^{(1)} = -\nabla \times \left[\overline{\overline{I}} \delta (\vec{R}|\vec{R}') \right] , \tag{2.70}$$

and is related to $\overline{\overline{H}}_m^{(0)}$ by (2.18b). From (2.19a) and (2.21a) we can write $\overline{\overline{H}}_m^{(0)}$ in either of two forms: Either

$$\overline{\overline{H}}_m^{(0)} = -Y \overline{\overline{I}} \delta (\vec{R}|\vec{R}') + \overline{\overline{H}}_m^{(0)} \tag{2.71}$$

or

$$\overline{\overline{H}}_m^{(0)} = Y \nabla \times \nabla \times \left[-\frac{\overline{\overline{I}}}{4\pi |\vec{R}-\vec{R}'|} \right] + \overline{\overline{H}}_{m_r}^{(0)} , \tag{2.72}$$

where

$$\nabla \times \overline{\overline{H}}_{m_r}^{(0)} = 0 \text{ and } \nabla \times \nabla \times \overline{\overline{H}}_{m_r}^{(0)} = 0 . \tag{2.73}$$

From (2.18b) we see, however, that if we are to be consistent with (2.68) we must choose $\overline{\overline{H}}_m^{(0)}$ according to (2.72). Thus substituting (2.68) and (2.73) in (2.18b) we obtain

$$\nabla \times \overline{\overline{E}}_{m_r}^{(1)} = Z \overline{\overline{H}}_{m_r}^{(0)} . \tag{2.74}$$

The first of (2.73) permits us to write $\overline{\overline{H}}_{m_r}^{(0)}$ in the form

$$\overline{\overline{H}}_{m_r}^{(0)} = -Y \sum_{i=1}^{3} (\nabla \phi_{mi_r}^{(0)}) \hat{a}_i \tag{2.75}$$

and by taking the divergence of (2.74) we see that

$$\nabla^2 \phi_{mi_r}^{(0)} = 0 , \; i=1,2,3 . \tag{2.76}$$

From the boundary condition (2. 22) on $\overline{\overline{H}}^{(0)}_{m}$ and (2. 72) we can write

$$Y\hat{n}_s \cdot \nabla_s \times \nabla_s \times \left[-\frac{\overline{\overline{I}}}{4\pi |\vec{R}_s - \vec{R}'|} \right] + \hat{n}_s \cdot \overline{\overline{H}}^{(0)}_{m_r} = 0 \,, \text{ on } S \,. \tag{2.77}$$

But

$$\nabla \times \nabla \times \left[-\frac{\overline{\overline{I}}}{4\pi |\vec{R} - \vec{R}'|} \right] = \nabla\nabla \left[-\frac{1}{4\pi |\vec{R} - \vec{R}'|} \right] - \overline{\overline{I}} \nabla^2 \left[-\frac{1}{4\pi |\vec{R} - \vec{R}'|} \right] . \tag{2.78}$$

By (2. 75) and (2. 78) the boundary condition (2. 77) becomes

$$\hat{n}_s \cdot \nabla_s \nabla_s \left[-\frac{1}{4\pi |\vec{R}_s - \vec{R}'|} \right] - \hat{n}_s \cdot \nabla_s \sum_{i=1}^{3} \phi^{(0)}_{mi_r} \hat{a}_i = 0 \,, \tag{2.79}$$

or

$$\hat{n}_s \cdot \nabla_s \left[\nabla_s \left(-\frac{1}{4\pi |\vec{R}_s - \vec{R}'|} \right) - \sum_{i=1}^{3} \phi^{(0)}_{mi_r} \right] \hat{a}_i = 0 \,, \tag{2.80}$$

or

$$\sum_{i=1}^{3} \hat{n}_s \cdot \nabla_s \left[(\hat{a}_i \cdot \nabla_s) \left(-\frac{1}{4\pi |\vec{R}_s - \vec{R}'|} \right) - \phi^{(0)}_{mi_r} \right] \hat{a}_i = 0 \,. \tag{2.81}$$

The last relation gives us

$$\frac{\partial}{\partial n_s} \phi^{(0)}_{mi_r} (\vec{R}_s | \vec{R}') = \frac{\partial}{\partial n_s} \left[(\hat{a}_i \cdot \nabla_s) \left(-\frac{1}{4\pi |\vec{R}_s - \vec{R}'|} \right) \right], \quad i=1, 2, 3. \tag{2.82}$$

We now employ the scalar Green's theorem and write

$$\phi^{(0)}_{mi_r} (\vec{R} | \vec{R}') = -\int_S N^{(e)} (\vec{R}_s | \vec{R}) \frac{\partial \phi^{(0)}_{mi_r} (\vec{R}_s | \vec{R}')}{\partial n_s} dS \,, \tag{2.83}$$

where $N^{(e)} (\vec{R} | \vec{R}')$ is the exterior Neumann Green's function for S :

$$N^{(e)} (\vec{R} | \vec{R}') = -\frac{1}{4\pi |\vec{R} - \vec{R}'|} + N^{(e)}_r (\vec{R} | \vec{R}') \,, \tag{2.84}$$

with

$$\nabla^2 N_r^{(e)}(\vec{R}|\vec{R}') = 0 \text{ , in V ,} \tag{2.85}$$

and

$$\frac{\partial}{\partial n_s} N^{(e)}(\vec{R}_s|\vec{R}') = 0 \text{ , } \vec{R}_s \in S \text{ .} \tag{2.86}$$

Substituting (2.82) in (2.83) we have

$$\phi^{(0)}_{mi_r}(\vec{R}|\vec{R}')=(\hat{a}_i\cdot\nabla')\int_S N^{(e)}(\vec{R}_s|\vec{R})\frac{\partial}{\partial n_s}\left[-\frac{1}{4\pi|\vec{R}_s-\vec{R}'|}\right]dS$$

$$= (\hat{a}_i\cdot\nabla')\, N_r^{(e)}(\vec{R}|\vec{R}') \text{ .}^{(2)} \tag{2.87}$$

Equation (2.73) then becomes

$$\nabla \text{x}\, \bar{\bar{E}}^{(1)}_{m_r} = -\sum_{i=1}^{3} (\nabla\phi^{(0)}_{mi_r})\, \hat{a}_i = -\nabla\nabla'\, N_r^{(e)}(\vec{R}|\vec{R}') \text{ .} \tag{2.88}$$

The necessary and sufficient conditions (2.59) and (2.60) for the equation

$$\nabla \text{x}\, \vec{E}^{(1)}_{mi_r} = -\nabla\phi^{(0)}_{mi_r} \tag{2.89}$$

to have a solution can be seen to be readily satisifed: $\phi^{(0)}_{mi_r}$ satisfies Laplace's equation (2.76) and it also satisfies the condition

$$\int_S \hat{n}_s \cdot \nabla_s \phi^{(0)}_{mi_r}\, dS = 0 \tag{2.90}$$

by virtue of (2.82) and (2.38). According to Kleinman (1965b) a particular solution, $\vec{E}^{(1)p}_{mi_r}$, is

---

[2] It is interesting to note that substitution of this expression in (2.76) and then in (2.71) gives the total magnetic field of three orthogonally crossed static magnetic dipoles of moments defined by (2.10) as

$$\bar{\bar{H}}^{(0)}_m = -Y\, \nabla\nabla' N^{(e)} - Y\,\bar{\bar{I}}\,\delta(\vec{R}|\vec{R}') \text{ .}$$

$$\vec{E}^{(1)p}_{mi_r}(\vec{R}|\vec{R}') = -\frac{1}{4\pi}\nabla x\int_S dS\frac{\left[\phi^{(0)}_{mi_r}(\vec{R}_s|\vec{R}')-N^{(i)}_{mi}\right]\hat{n}_s}{|\vec{R}_s-\vec{R}|} \quad . \tag{2.91}$$

The function $N^{(i)}_{mi}(\vec{R})$, $\vec{R}$ interior to S , is an interior Neumann function satisfying

$$\nabla^2 N^{(i)}_{mi}(\vec{R}) = 0\ ,\ \vec{R}\text{ interior to S} \tag{2.92}$$

with

$$\hat{n}_s\cdot\nabla_s N^{(i)}_{mi}(\vec{R}_s)=\hat{n}_s\cdot\nabla_s\phi^{(0)}_{mi_r}(\vec{R}_s|\vec{R}')\ ,\ \vec{R}_s\in S\ . \tag{2.93}$$

This is a standard interior Neumann problem and has a solution provided

$$\int_S \hat{n}_s\cdot\nabla_s N^{(i)}_{ei}(\vec{R}_s)\, dS = 0\ , \tag{2.94}$$

a condition guaranteed by (2.90).

The complete solution of $\overline{\overline{E}}^{(1)}_m$ can then be written

$$\overline{\overline{E}}^{(1)}_m = \nabla x\left[-\frac{\overline{\overline{I}}}{4\pi|\vec{R}-\vec{R}'|}\right]+\overline{\overline{E}}^{(1)p}_{m_r}+\sum_{i=1}^{3}\nabla G^{(e)}_{mi}\hat{a}_i\quad , \tag{2.95}$$

where $\nabla G^{(e)}_{mi}$ is a solution of the homogeneous part of (2.89). The functions $G^{(e)}_{mi}$ are exterior Dirichlet functions that can be partly determined from the boundary condition (2.22) on $\overline{\overline{E}}^{(1)}_m$:

$$\left.\begin{aligned}&\nabla^2 G^{(e)}_{mi} = 0 \text{ in V}\\ &G^{(e)}_{mi}\text{ regular at infinity}\\ &\hat{n}_s\, x\,\nabla_s G^{(e)}_{mi} = -\hat{n}_s\, x\left[\nabla_s\left(-\frac{1}{4\pi|\vec{R}_s-\vec{R}'|}\right)x\,\hat{a}_i\right]-\hat{n}_s\, x\,\vec{E}^{(1)p}_{mi_r}\ ,\ \text{on S}\end{aligned}\right\} \tag{2.96}$$

To completely determine $G^{(e)}_{mi}$ we employ the additional condition

$$\int_S \hat{n}_s\cdot\left[\vec{E}^{(1)p}_{mi_r}+\nabla_s G^{(e)}_{mi}\right]dS = 0\ ,\ i=1,2,3\ . \tag{2.97}$$

This condition arises from (2. 19b) and Stokes' theorem over a closed surface. Since $\hat{E}^{(1)p}_{mi_r}$ is the curl of another vector (see 2. 91), by Stokes' theorem again, its normal component integrated over a closed surface is zero. Equation (2. 97) then becomes

$$\int_S \hat{n}_s \cdot \nabla_s G^{(e)}_{mi}\, dS = 0, \quad i=1, 2, 3. \tag{2.98}$$

At this point we conclude Chapter II. In summary, we have derived explicit expressions for the dyadics $\bar{\bar{H}}^{(1)}_e$ and $\bar{\bar{E}}^{(1)}_m$, defined at the beginning of the chapter, in terms of potential functions. These dyadics satisfy Eqs. (10) - (13) of the Introduction. They also satisfy the regularity condition (14) as shown in Appendix A. It is of interest to note that in order to determine the two dyadics we employed more boundary conditions than those specified in Eq. (13). Specifically, for $\bar{\bar{H}}^{(1)}_e$ we used (2. 16) ($\hat{n} \cdot \bar{\bar{H}}^{(1)}_e = 0$) and (2. 41); for $\bar{\bar{E}}^{(1)}_m$ we used (2. 22) ($\hat{n} \cdot \bar{\bar{H}}^{(0)}_m = 0$) and (2. 98). The question then arises whether the boundary conditions (13) together with the regularity condition (14) of the Introduction determine uniquely the two dyadics, as specified there. If the dyadics are determined uniquely, one should be able to show that the two boundary conditions, together with the regularity conditions, imply the additional ones mentioned above. This question has not been answered as yet. The only statement we can make (from Appendix A) is that the boundary condition (2. 41) implies the regularity condition

$$\left| R^3 \left(\nabla \times \bar{\bar{H}}^{(1)}_e\right) \right| < \infty \ , \quad \text{as } R \to \infty \ .$$

Chapter III

INTEGRAL REPRESENTATIONS OF THE ELECTROMAGNETIC SCATTERED FIELDS

In this chapter we employ theorems A and B of Chapter I to find integral representations for the electric and magnetic fields scattered by a perfectly conducting surface. We start by defining the geometry of the problem and the properties of the scattered fields.

In the three-dimensional free space (vacuum) we have a closed, bounded, perfectly conducting, regular surface S which separates the whole space into two regions: the finite region V_i enclosed by S and the rest of space V. A time harmonic source of electromagnetic waves is located in V and its electric and magnetic fields are denoted by $\vec{E}^i$ and $\vec{H}^i$, respectively. The time dependence $e^{-i\omega t}$ is omitted. The presence of the perfectly conducting surface S gives rise to an electromagnetic wave whose electric and magnetic vectors we denote by $\vec{E}^s$ and $\vec{H}^s$, respectively. These vectors satisfy

1) Maxwell's equations

$$\nabla \times \vec{E}^s = ik\,Z\,\vec{H}^s \ , \quad \nabla \times \vec{H}^s = -ik\,Y\,\vec{E}^s \ , \tag{3.1}$$

$Z = \frac{1}{Y}$, the free space characteristic impedance,

2) the homogeneous vector wave equation

$$\nabla \times \nabla \times \begin{Bmatrix} \vec{E}^s \\ \vec{H}^s \end{Bmatrix} - k^2 \begin{Bmatrix} \vec{E}^s \\ \vec{H}^s \end{Bmatrix} = 0 \ , \quad \text{in } V \ , \tag{3.2}$$

a consequence of Maxwell's equations,

3) the boundary conditions

$$\hat{n} \times \vec{E}^s = -\hat{n} \times \vec{E}^i \ , \quad \hat{n} \cdot \vec{H}^s = -\hat{n} \cdot \vec{H}^i \ \text{on } S \ , \tag{3.3}$$

where $\hat{n}$ is the unit normal on S directed out of V and into V_i, and

4) the radiation conditions

$$\lim_{R\to\infty} R\left[\hat{R}\times(\nabla\times\vec{E}^s)+ik\vec{E}^s\right]=0\,,$$

uniformly in R , (3.4)

$$\lim_{R\to\infty} R\left[\hat{R}\times(\nabla\times\vec{H}^s)+ik\vec{H}^s\right]=0\,,$$

$\hat{R}$ being the radial unit vector and R the distance from the origin of a rectangular coordinate system with origin in V_i to a point in V.

Our intention is to substitute $\vec{E}^s$ and $\vec{H}^s$ in the integral equations (1.23) and (1.32) of Theorems A and B, respectively. In order to do this, however, we must show that $|\nabla\times(R\vec{E}^s)|<\infty$ and $|\nabla\times(R\vec{H}^s)|<\infty$, as $R\to\infty$. This follows from the following expansion theorem by Wilcox (1956):

Theorem C

Let $\vec{A}(\vec{R})$ be a vector radiation function[1] for a region $R > c$ where (R, θ, ϕ) are spherical coordinates. Then $\vec{A}(\vec{R})$ has an expansion

$$\vec{A}(\vec{R}) = \frac{e^{ikR}}{R}\sum_{n=0}^{\infty}\frac{\vec{A}_n(\theta,\phi)}{R^n} \tag{3.5}$$

which is valid for $R > c$ and which converges absolutely and uniformly in the parameters R, θ, ϕ in any region $R \geq c + \epsilon > c$. The series can be differentiated term by term with respect to R, θ, and ϕ any number of times and the resulting series all converge absolutely and uniformly.

It immediately follows from this theorem that

$$|\nabla\times(R\vec{E}^s)|<\infty \quad\text{and}\quad |\nabla\times(R\vec{H}^s)|<\infty,\quad \text{as } R\to\infty\,. \tag{3.6}$$

Letting then $\vec{E}^s$ and $\vec{H}^s$ stand for $\vec{E}$ and $\vec{H}$ of Eqs. (1.23) and (1.32), respectively, and at the same time, employing (3.1) - (3.3), we obtain

[1] A vector radiation function is one that satisfies Eqs. (3.2) and (3.4).

$$ikZ\vec{H}^s(\vec{R}') = -k^2 \int_V \vec{E}^s \cdot \overline{\overline{E}}_m^{(1)} dV - \int_S (\hat{n} \times \vec{E}^i) \cdot (\nabla \times \overline{\overline{E}}_m^{(1)}) \, dS , \tag{3.7}$$

and

$$\vec{E}^s(\vec{R}') = -ikZ \int_V \vec{H}^s \cdot \overline{\overline{H}}_e^{(1)} dV - \int_S (\hat{n} \times \vec{E}^i) \cdot \overline{\overline{H}}_e^{(1)} \, d\, S. \tag{3.8}$$

Equation (3.7) can be written in a better form. By Eq. (2.68)

$$\nabla \times \overline{\overline{E}}_m^{(1)} = \nabla\nabla\left(-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right) + \nabla \times \overline{\overline{E}}_{m_r}^{(1)} , \quad \vec{R} \neq \vec{R}' , \tag{3.9}$$

and by (2.81)

$$\nabla \times \overline{\overline{E}}_m^{(1)} = \nabla\nabla\left(-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right) - \nabla\nabla' N_r^{(e)} (\vec{R}|\vec{R}')$$

$$= -\nabla\nabla' N^{(e)} (\vec{R}|\vec{R}') , \qquad \vec{R} \neq \vec{R}' , \tag{3.10}$$

where $N^{(e)}$ is the exterior Neumann's function for S defined in (2.84). Employing the identities

$$(\vec{a} \times \vec{b}) \cdot \overline{\overline{A}} = \vec{a} \cdot (\vec{b} \times \overline{\overline{A}})$$

and

$$\nabla \times (\vec{a}\vec{b}) = (\nabla \times \vec{a})\vec{b} - \vec{a} \times \nabla \vec{b}$$

we have

$$(\hat{n} \times \vec{E}^i) \cdot (\nabla \times \overline{\overline{E}}_m^{(1)}) = \hat{n} \cdot \left[\vec{E}^i \times (\nabla \times \overline{\overline{E}}_m^{(1)})\right] = -\hat{n} \cdot \left[\vec{E}^i \times \nabla\nabla' N^{(e)}\right] =$$

$$= -\hat{n} \cdot \left[(\nabla \times \vec{E}^i)\nabla' N^{(e)} - \nabla \times (\vec{E}\, \nabla' N^{(e)}\right] . \tag{3.11}$$

The second term of this last expression vanishes by Stokes' theorem when integrated over the closed surface S. Employing (3.1) in the first part we have that

$$\int_S (\hat{n} \times \vec{E}^i) \cdot (\nabla \times \overline{\overline{E}}_m^{(1)}) dS = -ikZ \int_S (\hat{n} \cdot \vec{H}^i) \nabla' N^{(e)} dS . \tag{3.12}$$

Substituting this result in (3. 7) we have that

$$\vec{H}^s(\vec{R}')=ikY\int_V \vec{E}^s\cdot\bar{\bar{E}}_m^{(1)}\,dV+\nabla'\int_S(\hat{n}\cdot\vec{H}^i)N^{(e)}(\vec{R}_s|\vec{R}')\,dS\,. \qquad (3.13)$$

We can then state the following theorem.

Theorem D

The fields $\vec{E}^s$ and $\vec{H}^s$ defined in (3. 1) - (3. 4) satisfy the integral equations

$$\vec{H}^s(\vec{R}') = ikY\int_V \vec{E}^s\cdot\bar{\bar{E}}_m^{(1)}\,dV+\nabla'\int_S(\hat{n}\cdot\vec{H}^i)N^{(e)}(\vec{R}_s|\vec{R}')\,dS \qquad (3.14)$$

and

$$\vec{E}^s(\vec{R}')= -ikZ\int_V \vec{H}^s\cdot\bar{\bar{H}}_e^{(1)}dV-\int_S(\hat{n}\times\vec{E}^i)\cdot\bar{\bar{H}}_e^{(1)}\,dS\,, \qquad (3.15)$$

where $\bar{\bar{E}}_m^{(1)}$ is defined by (2.95), $\bar{\bar{H}}_e^{(1)}$ by (2. 66) and $N^{(e)}$ by (2. 84)-(2. 86).

Equations (3. 14) and (3. 15) constitute a system of two coupled integral equations for the scattered fields $\vec{E}^s$ and $\vec{H}^s$. They can be written in operator form by defining

$$L_1=ikY\int_V[\;\;]\cdot\bar{\bar{E}}_m^{(1)}(\vec{R}|\vec{R}')dV,\quad L_2= -ikZ\int_V[\;\;]\cdot\bar{\bar{H}}_e^{(1)}(\vec{R}|\vec{R}')\,dV \qquad (3.16)$$

and

$$\vec{H}^s_{(o)}=\nabla'\int_S(\hat{n}\cdot\vec{H}^i)N^{(e)}(\vec{R}_s|\vec{R}')dS,\quad \vec{E}^s_{(o)}= -\int_S(\hat{n}\times\vec{E}^i)\cdot\bar{\bar{H}}_e^{(1)}(\vec{R}_s|\vec{R}')\,dS\,. \qquad (3.17)$$

With these definitions, (3. 14) and (3. 15) becomes

$$\vec{H}^s = L_1\vec{E}^s + \vec{H}^s_{(o)}\,, \qquad (3.18)$$

and

$$\vec{E}^s= L_2\vec{H}^s+\vec{E}^s_{(o)}\;. \qquad (3.19)$$

For the wave number k sufficiently small we can solve these two equations by the method of successive approximations. We let $\vec{H}^s_{(0)}$ and $\vec{E}^s_{(0)}$ be the first approximation to $\vec{H}^s$ and $\vec{E}^s$, respectively. The first correction to this solution is

$$\vec{H}^s_{(1)} = L_1 \vec{E}^s_{(0)} + \vec{H}^s_{(0)} , \tag{3.20}$$

$$\vec{E}^s_{(1)} = L_2 \vec{H}^s_{(0)} + \vec{E}^s_{(0)} , \tag{3.21}$$

and the second one

$$\vec{H}^s_{(2)} = L_1 \vec{E}^s_{(1)} + \vec{H}^s_{(0)} = L_1 L_2 \vec{H}^s_{(0)} + L_1 \vec{E}^s_{(0)} + \vec{H}^s_{(0)} , \tag{3.22}$$

$$\vec{E}^s_{(2)} = L_2 \vec{H}^s_{(1)} + \vec{E}^s_{(0)} = L_2 L_1 \vec{E}^s_{(0)} + L_2 \vec{H}^s_{(0)} + \vec{E}^s_{(0)} , \tag{3.23}$$

and so on. In this manner we generate two sequences of functions $\left\{\vec{H}^s_{(N)}\right\}$ and $\left\{\vec{E}^s_{(N)}\right\}$, which we must show to be convergent for a certain range of values of k and, also, that they converge to the desired solutions, i. e.

$$\vec{H}^s = \lim_{N\to\infty} \vec{H}^s_{(N)} , \quad \vec{E}^s = \lim_{N\to\infty} \vec{E}^s_{(N)} . \tag{3.24}$$

We strongly suspect, however, that this approach would lead to divergent volume integrals quite early in the process. The reason for this is the following:

The incident fields of the zeroth order iterates (3. 17), be they dipole fields or plane waves, are independent of the primed coordinates. This fact together with our knowledge of the nature of $N^{(e)}$ and $\overline{\overline{H}}^{(1)}_e$ from Chapter II leads us to the conclusion that $\vec{H}^s_{(0)}(\vec{R}')$ and $\vec{E}^s_{(0)}(\vec{R}')$ do not contain the exponential $e^{ikR'}$. In fact, by a simple inductive argument it can be shown that none of the iterates contain the $e^{ikR'}$ as a factor. From (3. 5), however, we know that the scattered fields should contain this factor. We are then led to the conclusion that $e^{ikR'}$ appears in the iterates expanded in a power series in k. As the iteration proceeds there will come a point when positive powers of R will appear in the volume integrals of the operator and these integrals will diverge and this is precisely what happens when applying Stevenson's method.

To avoid this difficulty, we must remove the troublesome e^{ikR} factor from $\vec{E}^s$ and $\vec{H}^s$. Accordingly, we propose to use the following vector functions in the integral equations

$$\vec{e} = e^{-ikR}\,\vec{E}^s \quad \text{and} \quad \vec{h} = e^{-ikR}\,\vec{H}^s \; . \tag{3.25}$$

The motivation for doing so lies in (3. 5) of Wilcox' expansion theorem. From this equation we see that, at least in the region where the expansion holds, the new fields $\vec{e}$ and $\vec{h}$ do not contain the troublesome factor e^{ikR}. Our next step is to rewrite the integral equations (1. 23) and (1. 32) in terms of $\vec{e}$ and $\vec{h}$. That these two fields satisfy the regularity conditions

$$\left| \nabla x (R\vec{e}) \right| < \infty \quad \text{and} \quad \left| \nabla x (R\vec{h}) \right| < \infty, \text{ as } R \rightarrow \infty \; , \tag{3.26}$$

so as to be admissible in the integral equations, is obvious from their definition and Eq. (3. 5).

In terms of $\vec{e}$ and $\vec{h}$ Eqs. (1. 23) and (1. 32) are written

$$\nabla' x \vec{e}(\vec{R}') = -\int_V (\nabla x \nabla x \vec{e}) \cdot \bar{\bar{E}}_m^{(1)} \, dV + \int_S (\hat{n} x \vec{e}) \cdot (\nabla x \bar{\bar{E}}_m^{(1)}) dS \, , \tag{3.27}$$

and

$$\nabla' x \vec{h}(\vec{R}') = -\int_V (\nabla x \nabla x \vec{h}) \cdot \bar{\bar{H}}_e^{(1)} dV + \int_S \left[\hat{n} x (\nabla x \vec{h}) \right] \cdot \bar{\bar{H}}_e^{(1)} \, dS \, . \tag{3.28}$$

From the definition (3. 25) the functions $\vec{e}$ and $\vec{h}$ satisfy:

By (3. 1)

$$\nabla x \vec{e} = ik\,(Z\vec{h} - \hat{R} x \vec{e}) \; , \tag{3.29}$$

$$\nabla x \vec{h} = -ik\,(Y\vec{e} + \hat{R} x \vec{h}) \; ; \tag{3.30}$$

and by (3. 2)

$$\nabla x \nabla x \vec{e} = k^2(\vec{e} + Z\hat{R} x \vec{h}) - ik \nabla x (\hat{R} x \vec{e}) \; , \tag{3.31}$$

$$\nabla x \nabla x \vec{h} = k^2(\vec{h} - Y\hat{R} x \vec{e}) - ik \nabla x (\hat{R} x \vec{h}) \; . \tag{3.32}$$

Substituting these expression in the integral equations we obtain

$$ik\left[Z\vec{h}(\vec{R}')-\hat{R}'\times\vec{e}(\vec{R}')\right]=-k^2\int_V(\vec{e}+Z\hat{R}\times\vec{h})\cdot\bar{\bar{E}}_m^{(1)}\,dV+$$

$$+ik\int_V\left[\nabla\times(\hat{R}\times\vec{e})\right]\cdot\bar{\bar{E}}_m^{(1)}\,dV+\int_S(\hat{n}\times\vec{e})\cdot(\nabla\times\bar{\bar{E}}_m^{(1)})\,dS\,, \qquad (3.33)$$

and

$$-ik\left[Y\vec{e}(\vec{R}')+\hat{R}'\times\vec{h}(\vec{R}')\right]=-k^2\int_V(\vec{h}-Y\hat{R}\times\vec{e})\cdot\bar{\bar{H}}_e^{(1)}dV+$$

$$+ik\int_V\left[\nabla\times(\hat{R}\times\vec{h})\right]\cdot\bar{\bar{H}}_e^{(1)}dV-ik\int_S\left[\hat{n}\times(Y\vec{e}+\hat{R}\times\vec{h})\right]\cdot\bar{\bar{H}}_e^{(1)}\,dS\,. \qquad (3.34)$$

The second of the volume integrals in each of these expressions may be written in a different form. We start with that of (3. 33):

By the dyadic identity

$$\nabla\cdot(\vec{a}\times\bar{\bar{A}})=(\nabla\times\vec{a})\cdot\bar{\bar{A}}-\vec{a}\cdot(\nabla\times\bar{\bar{A}}) \qquad (3.35)$$

we have

$$\int_V\left[\nabla\times(\hat{R}\times\vec{e})\right]\cdot\bar{\bar{E}}_m^{(1)}\,dV=\int_V\nabla\cdot\left[(\hat{R}\times\vec{e})\times\bar{\bar{E}}_m^{(1)}\right]dV+\int_V(\hat{R}\times\vec{e})\cdot(\nabla\times\bar{\bar{E}}_m^{(1)})dV\,. \qquad (3.36)$$

[2]

By (2. 66) and (2. 88) and application of the divergence theorem to the first integral on the right, (3. 36) becomes:

$$\int_V\left[\nabla\times(\hat{R}\times\vec{e})\right]\cdot\bar{\bar{E}}_m^{(1)}dV=\int_{S+S'}\hat{n}\cdot\left[(\hat{R}\times\vec{e})\times\bar{\bar{E}}_m^{(1)}\right]dS-\int_V(\hat{R}\times\vec{e})\cdot\nabla\nabla'N^{(e)}dV. \qquad (3.37)$$

Using the boundary condition $\hat{n}\times\bar{\bar{E}}_m^{(1)}=0$ on S and the identity

$$\nabla\cdot(\vec{a}\,\vec{b})=(\nabla\cdot\vec{a})\vec{b}+\vec{a}\cdot\nabla\vec{b} \qquad (3.38)$$

---

[2] In the discussion that follows we will use the notation of Chapter I: S' denotes a spherical surface over the singularity at $\vec{R}'$ and the volume enclosed by this surface is excluded from V. Integrals over the surface at infinity (S_∞), will be omitted if it is clear that they vanish. A knowledge of the properties of the dyadics and the notation of Chapter II will also be assumed.

we can write (3. 37) as follows:

$$\int_V \left[\nabla x(\hat{R}x\vec{e})\right]\cdot \overline{\overline{E}}_m^{(1)} dV = \int_{S'} \hat{n}\cdot\left[(\hat{R}x\vec{e})x\overline{\overline{E}}_m^{(1)}\right]dS - \int_V \nabla\cdot\left[(\hat{R}x\vec{e})\nabla' N^{(e)}\right]dV + \int_V \nabla\cdot(\hat{R}x\vec{e})\nabla' N^{(e)} dV =$$

$$= \int_{S'} \hat{n}\cdot\left[(\hat{R}x\vec{e})x\overline{\overline{E}}_m^{(1)}\right]dS - \int_{S+S'} \hat{n}\cdot\left[(\hat{R}x\vec{e})\nabla' N^{(e)}\right]dS + \int_V \left[\vec{e}\cdot\nabla x\hat{R} - \hat{R}\cdot\nabla x\vec{e}\right]\nabla' N^{(e)} dV =$$

$$= \int_{S'} \hat{n}\cdot\left[(\hat{R}x\vec{e})x\overline{\overline{E}}_m^{(1)} - (\hat{R}x\vec{e})\nabla' N^{(e)}\right]dS + \int_S \hat{R}\cdot\left[(\hat{n}x\vec{e})\nabla' N^{(e)}\right]dS -$$

$$-ik\int_V \hat{R}\cdot\left[Z\vec{h} - \hat{R}x\vec{e}\right]\nabla' N^{(e)}\, dV \,, \qquad (3.39)$$

where, above, we made use of (3. 29). Notice now that the part of the integral over S' involving the regular parts of $\overline{\overline{E}}_m^{(1)}$ and $N^{(e)}$ will vanish in the limit. We then write

$$\int_V \left[\nabla x(\hat{R}x\vec{e})\right]\cdot \overline{\overline{E}}_m^{(1)} dV = \int_{S'} \hat{n}\cdot\left[(\hat{R}x\vec{e})x\left\{\nabla\left(-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right)x\,\overline{\overline{I}}\right\} -\right.$$

$$\left. - (\hat{R}x\vec{e})\,\nabla'\left(-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right)\right] dS + \int_S \hat{R}\cdot\left[(\hat{n}x\vec{e})\,\nabla' N^{(e)}\right]dS -$$

$$- ikZ\int_V \hat{R}\cdot\vec{h}\,\nabla' N^{(e)}\, dV \;. \qquad (3.40)$$

The evaluation of the surface integral over the singularity proceeds in the same manner as the corresponding one of (1. 16) of Chapter I and for this reason we omit it. The form of (3. 40) with the integral over S' evaluated then is

$$\int_V \left[\nabla x(\hat{R}x\vec{e})\right]\cdot \overline{\overline{E}}_m^{(1)} dV = -\hat{R}'x\vec{e}(\vec{R}') + \int_S \hat{R}\cdot\left[(\hat{n}x\vec{e})\nabla' N^{(e)}\right]dS - ikZ\int_V \hat{R}\cdot\vec{h}\;\nabla' N^{(e)} dS. \qquad (3.41)$$

Substituting this expression in (3. 33) we have that

$$ikZ\vec{h}(\vec{R}') = -k^2\int_V (\vec{e}+Z\hat{R}x\vec{h})\cdot \overline{\overline{E}}_m^{(1)}\,dV + k^2 Z\nabla'\int_V \hat{R}\cdot \vec{h}N^{(e)}dV +$$

$$+ik\int_S \hat{R}\cdot\left[(\hat{n}x\vec{e})\nabla' N^{(e)}\right]dS + \int_S (\hat{n}x\vec{e})\cdot \nabla x\,\overline{\overline{E}}_m^{(1)}dS\,. \tag{3. 42}$$

According to (3. 11) the last integral above can be written

$$\int_S (\hat{n}x\vec{e})\cdot \nabla x\,\overline{\overline{E}}_m^{(1)}dS = -\int_S (\hat{n}\cdot \nabla x\,\vec{e})\ \nabla' N^{(e)}dS \tag{3. 43}$$

and by (3. 29)

$$\int_S (\hat{n}x\vec{e})\cdot \nabla x\,\overline{\overline{E}}_m^{(1)}\,dS = -ik\int_S \hat{n}\cdot(Z\vec{h}-\hat{R}x\vec{e})\,\nabla' N^{(e)}dS = -ikZ\int_S (\hat{n}\cdot\vec{h})\,\nabla' N^{(e)}dS\ -$$

$$-ik\int_S \hat{R}\cdot\left[(\hat{n}x\vec{e})\,\nabla' N^{(e)}\right]dS\,. \tag{3. 44}$$

Substituting this last expression in (3. 42) we have that

$$\vec{h}(\vec{R}')| = ik\int_V (Y\vec{e}+\hat{R}\,x\,\vec{h})\cdot \overline{\overline{E}}_m^{(1)}\,dV - ik\nabla'\int_V \hat{R}\cdot \vec{h}N^{(e)}dV\ -\ \nabla'\int_S (\hat{n}\cdot\vec{h})N^{(e)}dS\,. \tag{3. 45}$$

This is the first integral equation for $\vec{h}$ and $\vec{e}$.

We now proceed in the same manner as above to modify the second volume integral of (3. 34). The result is

$$\int_V \left[\nabla x(\hat{R}x\vec{h})\right]\cdot \overline{\overline{H}}_e^{(1)}dV = -\hat{R}'x\vec{h}(\vec{R}') + \int_S \hat{n}\cdot\left[(\hat{R}x\vec{h})x\overline{\overline{H}}_e^{(1)} - (\hat{R}x\vec{h})\,\nabla\psi_e^{(0)}\right]dS +$$

$$+ikY\,\nabla'\int_V (\hat{R}\cdot\vec{e})\,\psi_e^{(0)}dV\,, \tag{3. 46}$$

where $\psi_e^{(0)}$ is defined in (2.46). Substituting this expression in (3.34) we have

$$-ikY\vec{e}(\vec{R}')=-k^2\int_V(\vec{h}-Y\hat{R}x\vec{e})\cdot\bar{\bar{H}}_e^{(1)}dV-k^2Y\nabla'\int_V(\hat{R}\cdot\vec{e})\psi_e^{(0)}\,dV-$$

$$-ik\nabla'\int_S\hat{n}\cdot(\hat{R}x\vec{h})\psi_e^{(0)}dS-ikY\int_S(\hat{n}x\vec{e})\cdot\bar{\bar{H}}_e^{(1)}\,dS\,. \qquad (3.47)$$

But on the surface S $\psi_e^{(0)}=-D_e(\vec{R}')$ so that

$$\int_S\hat{n}\cdot(\hat{R}x\vec{h})\psi_e^{(0)}dS=-D_e(\vec{R}')\int_S\hat{n}\cdot(\hat{R}x\vec{h})dS=-D_e(\vec{R}')\int_V\nabla\cdot(\hat{R}x\vec{h})dV=$$

$$=D_e(\vec{R}')\int_V\hat{R}\cdot\nabla x\vec{h}\,dV=-ikY\,D_e(\vec{R}')\int_V\hat{R}\cdot\vec{e}\,dV\,. \qquad (3.48)$$

Substituting this result in (3.47) we have

$$\vec{e}(\vec{R}')=-ik\int_V(Z\vec{h}-\hat{R}x\vec{e})\cdot\bar{\bar{H}}_e^{(1)}dV-ik\,\nabla'\int_V(\hat{R}\cdot\vec{e})\left[\psi_e^{(0)}+D_e\right]dV+\int_S(\hat{n}x\vec{e})\cdot\bar{\bar{H}}_e^{(1)}\,dS\,.$$

(3.49)

This is the second integral equation for $\vec{e}$ and $\vec{h}$.

Employing then the relations in (3.25) we can state the following theorem:

Theorem E

The scattered fields $\vec{H}^s$ and $\vec{E}^s$ defined in (3.1)-(3.4) satisfy the integral equations

$$\vec{H}^s(\vec{R}')=ikYe^{ikR'}\int_V e^{-ikR}\,\vec{E}^s(\vec{R})\cdot\bar{\bar{E}}_m^{(1)}(\vec{R}|\vec{R}')dV+$$

$$+ike^{ikR'}\int_V e^{-ikR}\left\{\left[\hat{R}x\vec{H}^s(\vec{R})\right]\cdot\bar{\bar{E}}_m^{(1)}(\vec{R}|\vec{R}')-\left[\hat{R}\cdot\vec{H}^s(\vec{R})\right]\nabla'N^{(e)}(\vec{R}|\vec{R}')\right\}dV$$

$$+e^{ikR'}\nabla'\int_S e^{-ikR_s}\left[\hat{n}\cdot\vec{H}^i(\vec{R}_s)\right]N^{(e)}(\vec{R}_s|\vec{R}')\,dS\,, \qquad (3.50)$$

and

$$\vec{E}^s(\vec{R}') = -ikZe^{ikR'}\int_V e^{-ikR}\,\vec{H}^s(\vec{R})\cdot\overline{\overline{H}}_e^{(1)}(\vec{R}|\vec{R}')\,dV +$$

$$+ike^{ikR'}\int_V e^{-ikR}\Big\{\big[\hat{R}\times\vec{E}^s(\vec{R})\big]\cdot\overline{\overline{H}}_e^{(1)}(\vec{R}|\vec{R}') - \big[\hat{R}\cdot\vec{E}^s(\vec{R})\big]\nabla'\big[\psi_e^{(0)}(\vec{R}|\vec{R}') + D_e(\vec{R}')\big]\Big\}\,dV$$

$$-e^{ikR'}\int_S e^{-ikR_s}\big[\hat{n}\times\vec{E}^i(\vec{R}_s)\big]\cdot\overline{\overline{H}}_e^{(1)}(\vec{R}_s|\vec{R}')\,dS\,, \qquad (3.51)$$

where $\overline{\overline{E}}_m^{(1)}$ is defined by (2.93), $\overline{\overline{H}}_e^{(1)}$ by (2.64), $N^{(e)}$ by (2.84)-(2.86), $\psi_e^{(0)}$ by (2.46) and D_e by (2.43).

Though these equations give directly the scattered fields it may prove more convenient to work with Eqs. (3.45) and (3.46) for the vectors $\vec{h}$ and $\vec{e}$ rather than the above. These equations may be solved in either of two ways for small values of k. We can either iterate the equations to form two sequences of functions which, hopefully, converge to the actual functions $\vec{e}$ and $\vec{h}$ or we can expand these fields in powers of k of the form

$$\vec{e} = \sum_{n=o}^{\infty}(ik)^n\,\vec{e}_n, \qquad \vec{h} = \sum_{n=o}^{\infty}(ik)^n\,\vec{h}_n \qquad (3.52)$$

and then substitute them in (3.45) and (3.46) to find a recursion formula for the coefficients. That $\vec{e}$ and $\vec{h}$ can be written in the form (3.52) follows from (3.25) and a result by Werner (1963), who showed that the scattered electric field $\vec{E}^S$ tends, as $k \to 0$, analytically to a corresponding electrostatic field.

When we use iteration to find $\vec{e}$ and $\vec{h}$ we let

$$\vec{h}_{(0)} = -\nabla'\int_S(\hat{n}\cdot\vec{h})N^{(e)}dS, \quad \vec{e}_{(0)} = \int_S(\hat{n}\times\vec{e})\cdot\overline{\overline{H}}_e^{(1)}dS\,, \qquad (3.53)$$

be the first approximation to $\vec{h}$ and $\vec{e}$, respectively. The first correction to this solution is

$$\vec{h}_{(1)}=ik\int_V (Y\vec{e}_{(0)}+\hat{R}x\vec{h}_{(0)})\cdot\overline{\overline{E}}_m^{(1)}\,dV-ik\nabla'\int_V \hat{R}\cdot\vec{h}_{(0)}N^{(e)}dV+\vec{h}_{(0)}, \quad (3.54)$$

$$\vec{e}_{(1)}=-ik\int_V (Z\vec{h}_{(0)}-\hat{R}x\vec{e}_{(0)})\cdot\overline{\overline{H}}_e^{(1)}\,dV-ik\nabla'\int_V (\hat{R}\cdot\vec{e}_{(0)})\left[\psi_e^{(0)}+D_e\right]dV+\vec{e}_{(0)}, \quad (3.55)$$

and the second

$$\vec{h}_{(2)}=ik\int_V (Y\vec{e}_{(1)}+\hat{R}x\vec{h}_{(1)})\cdot\overline{\overline{E}}_m^{(1)}\,dV-ik\,\nabla'\int_V \hat{R}\cdot\vec{h}_{(1)}N^{(e)}dV+\vec{h}_{(0)} \quad (3.56)$$

$$\vec{e}_{(2)}=-ik\int_V (Z\vec{h}_{(1)}-\hat{R}x\vec{e}_{(1)})\cdot\overline{\overline{H}}_e^{(1)}dV-ik\,\nabla'\int_V (\hat{R}\cdot\vec{e}_{(1)})\left[\psi_e^{(0)}+D_e\right]dV+\vec{e}_{(0)}, \quad (3.57)$$

and so on. In this way we generate two sequences of functions, $\left\{\vec{h}_{(N)}\right\}$ and $\left\{\vec{e}_{(N)}\right\}$, which, we hope, for a certain range of values of k, converge to $\vec{h}$ and $\vec{e}$ respectively as $N\rightarrow\infty$.

On the other hand, when we use the low frequency expansions (3.52) we proceed as follows: First we expand the known surface integrals in power series of k

$$-\nabla'\int_S (\hat{n}\cdot\vec{h})N^{(e)}dS=\sum_{n=o}^{\infty}(ik)^n\vec{f}_n,\quad \int_S(\hat{n}\,x\,\vec{e})\cdot\overline{\overline{H}}_e^{(1)}dS=\sum_{n=o}^{\infty}(ik)^n\,\vec{g}_n \quad (3.58)$$

and then substitute (3.52) in (3.45) and (3.46) and collect coefficients of equal powers of k to obtain the result

$$\vec{h}_0(\vec{R}')=\vec{f}_0, \quad (3.59)$$

$$\vec{h}_{n+1}(\vec{R}')=\int_V (Y\vec{e}_n+\hat{R}x\vec{h}_n)\cdot\overline{\overline{E}}_m^{(1)}\,dV-\nabla'\int_V \hat{R}\cdot\vec{h}_nN^{(e)}dV+\vec{f}_{n+1},\quad n=0,1,\ldots \quad (3.60)$$

$$\vec{e}_0(\vec{R}') = \vec{g}_0 \tag{3.61}$$

$$\vec{e}_{n+1}(\vec{R}') = -\int_V (Z\vec{h}_n - \hat{R}\times\vec{e}_n)\cdot \overline{\overline{H}}_e^{(1)} dV - \nabla' \int_V (\hat{R}\cdot\vec{e}_n)\left[\psi_e^{(0)} + D_e\right] dV + \vec{g}_{n+1},$$

$$n=0, 1, \ldots \tag{3.62}$$

Chapter IV

AN EXAMPLE: THE SPHERE

In this chapter we apply our results to the problem of scattering of a plane electromagnetic wave by a perfectly conducting sphere. First we will employ the results of Chapter II to determine the dyadics $\bar{\bar{H}}_e^{(1)}$ and $\bar{\bar{E}}_m^{(1)}$ and then the results of Chapter III to determine the first two terms in the low frequency expansion of the scattered fields.

The sphere is of radius a and its center coincides with the origin of a rectangular coordinate system (x, y, z) (see Fig. 2). Using the notation of the previous chapters, V_i denotes the volume of the sphere while V the rest of space. The surface of the sphere is denoted by S and the unit normal, $\hat{n}$, on it is directed out of V and into V_i. The plane wave propagates along the negative z-axis with its electric vector polarized along the positive x-axis. We shall use both the above rectangular coordinate system and its related spherical coordinate system (R, θ, ϕ).

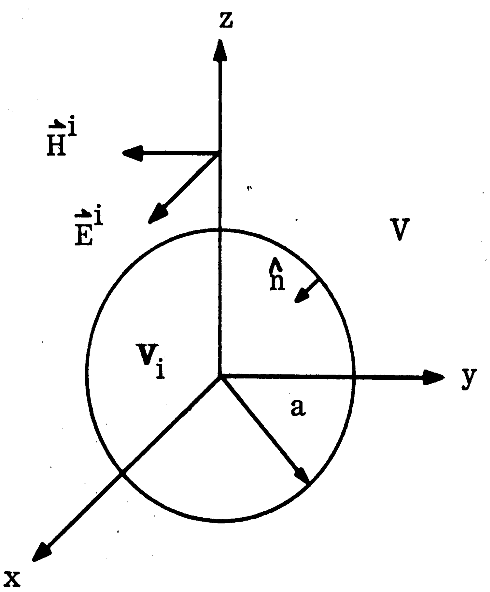

FIG. 2: GEOMETRY FOR THE SPHERE PROBLEM.

We start with some expressions we shall be using rather often: The expansion of the free space static Green's function in spherical harmonics is

$$-\frac{1}{4\pi|\vec{R}-\vec{R}'|} = -\frac{1}{4\pi}\sum_{n=o}^{\infty}\sum_{m=o}^{n}\epsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta') P_n^m(\cos\theta)\cos m(\phi-\phi') \frac{R_<^n}{R_>^{n+1}} , \tag{4.1}$$

where $R_<=\min(R, R')$, $R_>=\max(R, R')$, and ϵ_m is the Neumann factor: $\epsilon_0=1$, $\epsilon_m=2$ for $m=1, 2, \ldots$. The functions P_n^m are the associated Legendre functions defined by

$$P_n^m(x) = \frac{(-1)^m}{2^m} \frac{(n+m)!}{m!(n-m)!} (1-x^2)^{m/2} \, {}_2F_1(1+m+n, m-n; 1+m; \frac{1-x}{2}), \quad -1 \le x \le 1. \tag{4.2}$$

This definition is according to Magnus et al (1966) and all the contiguous relations for these functions that we shall subsequently use can be found there (p. 171). The regular part of the exterior static Dirichlet Green's function for the sphere as defined in (2.34) - (2.36) is given by

$$G_r^{(e)} = \frac{1}{4\pi}\sum_{n=o}^{\infty}\sum_{m=o}^{n}\epsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\theta)\cos m(\phi-\phi') \frac{a^{2n+1}}{R'^{n+1}R^{n+1}} , \tag{4.3}$$

while the regular part of the corresponding Neumann function as defined in (2.84)-(2.86) is given by

$$N_r^{(e)} = -\frac{1}{4\pi}\sum_{n=o}^{\infty}\sum_{m=o}^{n}\epsilon_m \frac{n}{n+1}\frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta') P_n^m(\cos\theta)\cos m(\phi-\phi') \frac{a^{2n+1}}{R'^{n+1}R^{n+1}} . \tag{4.4}$$

We are now ready to proceed with the determination of the two dyadics.

The Explicit Form of $\overline{\overline{H}}_e^{(1)}$

First we determine the constant $D_e(\vec{R}')$ of (2.43). From (4.3)

$$\int_S \frac{\partial}{\partial n_s} G_r^{(e)}(\vec{R}_s|\vec{R}')dS = -\int_S \frac{\partial}{\partial R} G_r^{(e)}(\vec{R}_s|\vec{R}')\, dS =$$

$$= \frac{1}{4\pi}\sum_{n=o}^{\infty}\sum_{m=o}^{n}\epsilon_m(n+1)\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta')\frac{a^{n-1}}{R'^{n+1}}\int_0^{\pi}\int_0^{2\pi} a^2\sin\theta_s d\theta_s d\phi_s \times$$

$$\times\ P_n^m(\cos\theta_s)\cos m(\phi_s-\phi') = \frac{a}{R'}\ , \tag{4.5}$$

$$\int_S dT \frac{\partial}{\partial n_T}\int_S dS \frac{\partial}{\partial n_s} G_r^{(e)}(\vec{R}_s|\vec{R}_T) = 4\pi\ a. \tag{4.6}$$

Substituting these two results in (2.43) we have that

$$D_e(\vec{R}') = \frac{1}{4\pi R'}\ . \tag{4.7}$$

By (4.5) the regular part of $\psi_e^{(0)}$ as defined in (2.46) is

$$\psi_{e_r}^{(0)}(\vec{R}|\vec{R}') = G_r^{(e)}(\vec{R}|\vec{R}') - \frac{a}{4\pi R'R}\ , \tag{4.8}$$

where $G_r^{(e)}$ is defined in (4.3). By (2.57)

$$\phi_{ei_r}^{(0)} = -(\hat{a}_i\cdot\nabla')\left[G_r^{(e)}(\vec{R}|\vec{R}') - \frac{a}{4\pi R'R}\right],\ i=1,2,3\ , \tag{4.9}$$

where $\hat{a}_1, \hat{a}_2, \hat{a}_3$ stand for the rectangular unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$.

In order to determine the particular solution (2.62) of (2.58) we need to determine first the interior Neumann function defined in (2.63) - (2.65).

By (2.64)

$$\hat{n}_s \cdot \nabla N_{ei}^{(i)}(\vec{R}_s) = -\frac{\partial}{\partial R}\phi_{ei_r}^{(0)}\Big|_{R=a} = -(\hat{a}_i \cdot \nabla') \frac{1}{4\pi} \sum_{n=1}^{\infty} \sum_{m=o}^{n} \epsilon_m (n+1) \ x$$

$$x \ \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta') P_n^m(\cos\theta) \cos m(\phi-\phi') \frac{a^{n-1}}{R'^{n+1}}, \ i = 1, 2, 3 . \qquad (4.10)$$

The interior Neumann function that satisfies this boundary condition is

$$N_{ei}^{(i)}(\vec{R}) = \frac{1}{4\pi} (\hat{a}_i \cdot \nabla') \sum_{n=1}^{\infty} \sum_{m=o}^{n} \epsilon_m \frac{n+1}{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) \cos m(\phi-\phi') \frac{R^n}{R'^{n+1}} + \nu_{ei},$$
$$\vec{R} \in V_i, \ i=1, 2, 3 , \qquad (4.11)$$

where ν_{ei} is an arbitrary constant. We now form the difference

$$\phi_{ei_r}^{(0)}(\vec{R}_s | \vec{R}') - N_{ei}^{(i)}(\vec{R}_s) = -\frac{1}{4\pi} (\hat{a}_i \cdot \nabla') \sum_{n=1}^{\infty} \sum_{m=o}^{n} \epsilon_m \left(\frac{2n+1}{n}\right) \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta') \ x$$

$$x \ P_n^m(\cos\theta) \cos m(\phi-\phi') \frac{a^n}{R'^{n+1}} - \nu_{ei}, \ i = 1, 2, 3 . \qquad (4.12)$$

Before proceeding to evaluate (2.62) we note that the fact that ν_{ei} has not been determined is not disturbing. Since, by Stokes' theorem,

$$\nabla x \int_S dS \frac{\hat{n}_s}{|\vec{R}_s - \vec{R}|} = \int_S dS \, \nabla \left(\frac{1}{|\vec{R}_s - \vec{R}|} \right) x \, \hat{n}_s = 0 , \qquad (4.13)$$

we see that the part of (2.62) involving the constant ν_{ei} vanishes. In writing (2.62), therefore, we shall omit ν_{ei}. From (4.1), (4.12) and (2.62) we have that

$$\vec{H}^{(1)p}_{ei_r}(\vec{R}|\vec{R}') = \frac{1}{(4\pi)^2}(\hat{a}_i \cdot \nabla')\nabla x \sum_{n=1}^{\infty}\sum_{m=o}^{n}\sum_{\ell=o}^{\infty}\sum_{t=o}^{\ell}\epsilon_m \epsilon_t \left(\frac{2n+1}{n}\right)\frac{(n-m)!}{(n+m)!}\frac{(\ell-t)!}{(\ell+t)!} \; x$$

$$x \; P_n^m(\cos\theta')P_\ell^t(\cos\theta)\frac{a^{n+\ell}}{R'^{n+1}R^{\ell+1}}\int_0^{\pi}\int_0^{2\pi} a^2 \sin\theta_s d\theta_s d\phi_s \hat{R}_s P_n^m(\cos\theta_s) \; x$$

$$x \; P_\ell^t(\cos\theta_s)\cos m(\phi_s - \phi')\cos t(\phi_s - \phi), \quad i=1,2,3 \; . \qquad (4.14)$$

In order to evaluate the integral we write the radial unit vector on the surface of the sphere, $\hat{R}_s$, in terms of its rectangular components,

$$\hat{R}_s = \hat{x}\sin\theta_s \cos\phi_s + \hat{y}\sin\theta_s \sin\phi_s + \hat{z}\cos\theta_s \; , \qquad (4.15)$$

and then we employ the orthogonality properties of the trigonometric and Legendre functions involved in the integration. The final result is rather simple in form:

$$\vec{H}^{(1)p}_{ei_r}(\vec{R}|\vec{R}') = -\frac{1}{4\pi}(\hat{a}_i \cdot \nabla')\left\{\hat{\theta}\frac{2}{\sin\theta}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{m}{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta') \; x\right.$$

$$x \; P_n^m(\cos\theta)\sin m(\phi-\phi')\frac{a^{2n+1}}{R'^{n+1}R^{n+1}} + \hat{\phi}\sum_{n=1}^{\infty}\sum_{m=o}^{n}\epsilon_m \frac{1}{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta') \; x$$

$$\left. x \; \frac{d}{d\theta}P_n^m(\cos\theta)\cos m(\phi-\phi')\frac{a^{2n+1}}{R'^{n+1}R^{n+1}}\right\}, \quad i=1,2,3 \; . \qquad (4.16)$$

We notice here that this vector is transverse to the radial direction. The labor involved in obtaining this result is substantial and it is rather fortunate that the operations in (4.14) have to be performed only once and not three times (one for each i).

We now turn to the determination of the exterior Neumann functions defined in (2.67). From the nature of the boundary condition these functions have to be found one at a time. We will show how to find the first one and will just give the results for the other two. From (2.67) we write

$$\hat{n}_s \bullet \nabla N_{e1}^{(e)} = \hat{R} \bullet \left[\nabla_s \left(- \frac{1}{4\pi |\vec{R}_s - \vec{R}'|} \right) x \; \hat{x} \right] + \hat{R} \bullet \vec{H}_{ei_r}^{(1)p} =$$

$$= \hat{R} \bullet \left[\nabla_s \left(- \frac{1}{4\pi |\vec{R}_s - \vec{R}'|} \right) x \; (\hat{R} \sin\theta_s \cos\phi_s + \hat{\theta} \cos\theta_s \cos\phi_s - \hat{\phi} \sin\phi_s) \right] =$$

$$= \frac{1}{8\pi} \left\{ \sum_{n=1}^{\infty} \sum_{m=o}^{n-1} \epsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta') P_n^{m+1}(\cos\theta_s) \sin\left[m(\phi_s - \phi') + \phi_s\right] \right.$$

$$x \; \frac{a^{n-1}}{R'^{n+1}} + \sum_{n=1}^{\infty} \sum_{m=o}^{n} \epsilon_m \frac{(n-m+1)!}{(n+m-1)!} P_n^m(\cos\theta') \; x$$

$$\left. x \; P_n^{m-1} (\cos\theta_s) \sin\left[m(\phi_s - \phi') - \phi_s\right] \frac{a^{n-1}}{R'^{n+1}} \right\} = \frac{1}{4\pi} \; x$$

$$x \left\{ \sum_{n=1}^{\infty} \sum_{m=o}^{n-1} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta') P_n^m(\cos\theta_s) \sin\left[m(\phi_s - \phi') + \phi_s\right] \frac{a^{n-1}}{R'^{n+1}} + \right.$$

$$\left. + \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{(n-m+1)!}{(n+m-1)!} P_n^m(\cos\theta') P_n^{m-1}(\cos\theta_s) \sin\left[m(\phi_s - \phi') - \phi_s\right] \frac{a^{n-1}}{R'^{n+1}} \right\}. \tag{4.17}$$

From this boundary condition $N_{e1}^{(e)}$ can be determined everywhere in V either by inspection or by formal use of Green's theorem:

$$N_{e1}^{(e)}(\vec{R}|\vec{R}') = - \int_S N^{(e)} (\vec{R}_s | \vec{R}) \; \frac{\partial N_{e1}^{(e)}}{\partial n_s} \; dS \; . \tag{4.18}$$

The result is

$$N_{e1}^{(e)}(\vec{R}|\vec{R}') = \frac{1}{4\pi}\left\{ \sum_{n=1}^{\infty}\sum_{m=o}^{n-1} \frac{1}{n+1}\frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta')P_n^m(\cos\theta)\sin\left[(m+1)\phi - m\phi'\right] x \right.$$

$$x \frac{a^{2n+1}}{R'^{n+1}R^{n+1}} + \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{1}{n+1}\frac{(n-m+1)!}{(n+m-1)!} P_n^m(\cos\theta')P_n^{m-1}(\cos\theta) \; x$$

$$\left. x \sin\left[(m-1)\phi - m\phi'\right] \frac{a^{2n+1}}{R'^{n+1}R^{n+1}} \right\}. \qquad (4.19)$$

In a similar manner we find that

$$N_{e2}^{(e)}(\vec{R}|\vec{R}') = \frac{1}{4\pi}\left\{ - \sum_{n=1}^{\infty}\sum_{m=o}^{n-1} \frac{1}{n+1}\frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta')P_n^{m+1}(\cos\theta) \; x \right.$$

$$x \; \cos\left[(m+1)\phi - m\phi'\right] \frac{a^{2n+1}}{R'^{n+1}R^{n+1}} + \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{1}{n+1}\frac{(n-m+1)!}{(n+m-1)!} P_n^m(\cos\theta') \; x$$

$$\left. x \; P_n^{m-1}(\cos\theta)\cos\left[(m-1)\phi - m\phi'\right] \frac{a^{2n+1}}{R'^{n+1}R^{n+1}} \right\}, \qquad (4.20)$$

$$N_{e3}^{(e)}(\vec{R}|\vec{R}') = \frac{1}{2\pi} \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{m}{n+1}\frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta')P_n^m(\cos\theta)\sin m(\phi-\phi') \frac{a^{2n+1}}{R'^{n+1}R^{n+1}}. \qquad (4.21)$$

Equations (4. 1), (4. 16) and (4. 19)-(4. 21) completely determine the dyadic $\overline{\overline{H}}_e^{(1)}$ defined in (2. 66)

The Explicit Form of $\overline{\overline{E}}_m^{(1)}$

First we determine the interior Neumann functions defined in (2.92) - (2.94). By (2.93) and (2.87) and (4.4) we have

$$\hat{n}_s \cdot \nabla_s N_{mi}^{(i)}(\vec{R}_s) = -\frac{\partial}{\partial R} \phi_{mi_r}^{(0)}(\vec{R}_s|\vec{R}')\Big|_{R=a} =$$

$$= -\frac{1}{4\pi}(\hat{a}_i \cdot \nabla') \sum_{n=o}^{\infty} \sum_{m=o}^{n} \epsilon_m \, n \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta') P_n^m(\cos\theta_s) \cos m(\phi_s - \phi') \times$$

$$\times \frac{a^{n-1}}{R'^{n+1}}, \quad i=1, 2, 3, \qquad (4.22)$$

from which we can write

$$N_{mi}^{(i)}(\vec{R}) = \frac{1}{4\pi}(\hat{a}_i \cdot \nabla') \sum_{n=o}^{\infty} \sum_{m=o}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta') P_n^m(\cos\theta) \cos m(\phi - \phi') \times$$

$$\times \frac{R^n}{R'^{n+1}} + \nu_{mi}, \quad \vec{R} \in V_i, \quad i=1, 2, 3,$$

$$(4.23)$$

where ν_{mi} is an arbitrary constant. From Eqs. (2.87), (4.4) and the above result we form the difference

$$\phi_{mi_r}^{(0)}(\vec{R}_s|\vec{R}') - N_{mi}^{(i)}(\vec{R}_s) = -\frac{1}{4\pi}(\hat{a}_i \cdot \nabla') \sum_{n=o}^{\infty} \sum_{m=o}^{n} \epsilon_m \frac{2n+1}{n+1} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta') P_n^m(\cos\theta_s) \times$$

$$\times \cos m(\phi_s - \phi') \frac{a^n}{R'^{n+1}} - \nu_{mi}, \quad i = 1, 2, 3. \qquad (4.24)$$

Substituting this result together with (4.1) in (2.91) we have that

$$\vec{E}^{(1)p}_{mi_r}(\vec{R}|\vec{R}') = -\frac{1}{(4\pi)^2}(\hat{a}_i\cdot\nabla')\nabla\times\sum_{n=o}^{\infty}\sum_{m=o}^{n}\sum_{\ell=o}^{\infty}\sum_{t=o}^{\ell}\epsilon_m\epsilon_t\left(\frac{2n+1}{n+1}\right)\frac{(n-m)!}{(n+m)!}\frac{(\ell-t)!}{(\ell+t)!}\times$$

$$\times P_n^m(\cos\theta')P_\ell^t(\cos\theta)\frac{a^{n+\ell}}{R'^{n+1}R^{\ell+1}}\int_0^{\pi}\int_0^{2\pi}a^2\sin\theta_s d\theta_s d\phi_s\,\hat{R}_s P_n^m(\cos\theta_s)\times$$

$$\times\cos m(\phi_s-\phi')\cos t(\phi_s-\phi),\quad i=1,2,3\,. \tag{4.25}$$

In this expression we have omitted the constant ν_{mi} of (4.24) since, as we explained through (4.13), it does not contribute to the integration. Equation (4.25) is evaluated in the same manner as (4.14) yielding the result

$$\vec{E}^{(1)p}_{mi_r}(\vec{R}|\vec{R}') = \frac{1}{4\pi}(\hat{a}_i\cdot\nabla')\left\{\hat{\theta}\,\frac{2}{\sin\theta}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{m}{n+1}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta')\times\right.$$

$$\times P_n^m(\cos\theta)\sin m(\phi-\phi')\frac{a^{2n+1}}{R'^{n+1}R^{n+1}}+\hat{\phi}\sum_{n=1}^{\infty}\sum_{m=o}^{n}\epsilon_m\frac{1}{n+1}\times$$

$$\left.\times\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta')\frac{d}{d\theta}P_n^m(\cos\theta)\cos m(\phi-\phi')\frac{a^{2n+1}}{R'^{n+1}R^{n+1}}\right\},$$

$$i=1,2,3\,. \tag{4.26}$$

Note the similarity between this expression and (4.16).

We now turn to the determination of the exterior Dirichlet function defined by (2.96). These functions have to be found one at a time. From the boundary condition in (2.96) and the preceding results we have that on the surface of the sphere

$$\hat{n}_s \times \nabla_s G^{(e)}_{m1} = -\hat{n}_s \times \left[\nabla_s \left(-\frac{1}{4\pi|\vec{R}_s - \vec{R}'|}\right) \times \hat{x}\right] - \hat{n}_s \times \vec{E}^{(1)p}_{mi_r} =$$

$$= -\frac{1}{4\pi}\hat{R} \times \Bigg\{ \hat{\theta} \Bigg[\frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=o}^{n-1} \epsilon_m (n-m) \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta') P_n^m(\cos\theta_s) \sin\left[(m+1)\phi - m\phi'\right] \times$$

$$\times \frac{a^{n-1}}{R'^{n+1}} -$$

$$-\frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=o}^{n} \epsilon_m \frac{(n-m)!}{(n+m-1)!} P_n^m(\cos\theta') P_n^m(\cos\theta_s) \sin\left[(m-1)\phi_s - m\phi'\right] \frac{a^{n-1}}{R'^{n+1}} -$$

$$-\frac{2}{\sin\theta_s} (\hat{x} \cdot \nabla') \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{m}{n+1} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta') P_n^m(\cos\theta_s) \sin m(\phi_s - \phi') \frac{a^n}{R'^{n+1}} \Bigg] +$$

$$+\hat{\phi} \frac{1}{2} \Bigg[\sum_{n=1}^{\infty} \sum_{m=o}^{n-1} \epsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta') P_{n-1}^m(\cos\theta_s) \cos\left[(m+1)\phi_s - m\phi'\right] \frac{a^{n-1}}{R'^{n+1}} +$$

$$+\frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=o}^{n-1} \epsilon_m \frac{(n-m)!}{(n+m-1)!} P_n^m(\cos\theta') P_{n-1}^m(\cos\theta_s) \cos\left[(m-1)\phi_s - m\phi'\right] \frac{a^{n-1}}{R'^{n+1}} -$$

$$-(\hat{x} \cdot \nabla') \sum_{n=1}^{\infty} \sum_{m=o}^{n-1} \epsilon_m \frac{1}{n+1} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta') \frac{d}{d\theta_s} P_n^m(\cos\theta_s) \cos m(\phi - \phi') \times$$

$$\times \frac{a^n}{R'^{n+1}} \Bigg] \Bigg\} . \tag{4.27}$$

A theorem by Kellogg (1953, p. 143) allows us to expand $G^{(e)}_{m1}$ in V in a series of spherical harmonics of the form

$$G^{(e)}_{m1} = \frac{1}{4\pi} \sum_{n=o}^{\infty} \sum_{m=o}^{n} \epsilon_m \frac{P_n^m(\cos\theta)}{R^{n+1}} \left[A_{n,m} \cos m\phi + B_{n,m} \sin m\phi\right], \tag{4.28}$$

from which we write

$$\hat{n}_s \times \nabla_s G^{(e)}_{m1} = -\frac{1}{4\pi}\hat{R}\times\left\{\hat{\theta}\sum_{n=1}^{\infty}\sum_{m=o}^{n}\epsilon_m \frac{d}{d\theta_s}P_n^m(\cos\theta_s)\frac{1}{a^{n+2}}\left[A_{n,m}\cos m\phi_s + B_{n,m}\sin m\phi_s\right] + \right.$$

$$\left. + \hat{\phi}\frac{2}{\sin\theta_s}\sum_{n=1}^{\infty}\sum_{m=1}^{n} mP_n^m(\cos\theta_s)\frac{1}{a^{n+2}}\left[-A_{n,m}\sin m\phi_s + B_{n,m}\cos m\phi_s\right]\right\}. \tag{4.29}$$

We now equate corresponding vector components in (4.27) and (4.29), thus obtaining two expressions for the unknown coefficients $A_{n,m}$ and $B_{n,m}$. Using the orthogonality properties of the trigonometric functions involved there we obtain the following.

$$A_{n,m} = \frac{1}{2n}\frac{(n-m)!}{(n+m)!}P_n^{m+1}(\cos\theta')\frac{a^{2n+1}}{R'^{n+1}}\sin(m+1)\phi' +$$

$$+\frac{1}{2n}\frac{(n-m+1)!}{(n+m-1)!}P_n^{m-1}(\cos\theta')\frac{a^{2n+1}}{R'^{n+1}}\sin(m-1)\phi', \quad n\geq 1,\ 0\leq m\leq n, \tag{4.30}$$

$$B_{n,m} = -\frac{1}{2n}\frac{(n-m)!}{(n+m)!}P_n^{m-1}(\cos\theta')\frac{a^{2n+1}}{R'^{n+1}}\cos(m+1)\phi' -$$

$$-\frac{1}{2n}\frac{(n-m+1)!}{(n+m-1)!}P_n^{m-1}(\cos\theta')\frac{a^{2n+1}}{R'^{n+1}}\cos(m-1)\phi', \quad n\geq 1,\ 1\leq m\leq n. \tag{4.31}$$

These expressions determine all the constants except $A_{0,0}$. In order to determine this constant we employ the condition (2.98). Transforming the integral over the surface to an integral over a spherical surface at infinity (by the divergence theorem) we have

$$\lim_{R\to\infty}\int_S \frac{\partial}{\partial R}G^{(e)}_{mi}\, dS = 0, \quad i=1,2,3. \tag{4.32}$$

Substituting (4.28) in this expression we find that if the integral is to vanish, we must have

$$A_{0,0} = 0 \,. \tag{4.33}$$

From this result and (4.30) and (4.31), Eq. (4.28) can be put in the form

$$G^{(e)}_{m1}(\vec{R}|\vec{R}') = -\frac{1}{4\pi}\left\{ \sum_{n=1}^{\infty}\sum_{m=o}^{n-1} \frac{1}{n}\,\frac{(n-m)!}{(n+m)!}\,P_n^m(\cos\theta')\,P_n^{m+1}(\cos\theta) \;\times \right.$$
$$\times \sin\left[(m+1)\phi - m\phi'\right]\frac{a^{2n+1}}{R'^{\,n+1}} \; +$$
$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{1}{n}\,\frac{(n-m+1)!}{(n+m-1)!}\,P_n^m(\cos\theta')\,P_n^{m-1}(\cos\theta) \;\times$$
$$\left. \times \sin\left[(m-1)\phi - m\phi'\right]\frac{a^{2n+1}}{R'^{\,n+1}R^{n+1}} \right\}. \tag{4.34}$$

In a similar manner we find

$$G^{(e)}_{m2}(\vec{R}|\vec{R}') = \frac{1}{4\pi}\left\{ \sum_{n=1}^{\infty}\sum_{m=o}^{n-1} \frac{1}{n}\,\frac{(n-m)!}{(n+m)!}\,P_n^m(\cos\theta')\,P_n^{m+1}(\cos\theta) \;\times \right.$$
$$\times \cos\left[(m+1)\phi - m\phi'\right]\frac{a^{2n+1}}{R'^{\,n+1}R^{n+1}} \; -$$
$$- \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{1}{n}\,\frac{(n-m+1)!}{(n+m-1)!}\,P_n^m(\cos\theta')\,P_n^{m-1}(\cos\theta) \;\times$$
$$\left. \times \cos\left[(m-1)\phi - m\phi'\right]\frac{a^{2n+1}}{R'^{\,n+1}R^{n+1}} \right\}, \tag{4.35}$$

$$G^{(e)}_{m3}(\vec{R}|\vec{R}') = -\frac{1}{2\pi}\sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{m}{n}\,\frac{(n-m)!}{(n+m)!}\,P_n^m(\cos\theta')\,P_n^m(\cos\theta)\,\sin m(\phi-\phi') \;\times$$
$$\times \frac{a^{2n+1}}{R'^{\,n+1}R^{n+1}} \,. \tag{4.36}$$

Notice the similarity between these three expressions and the corresponding ones (4.19) - (4.21) .

Equations (4.1), (4.26) and (4.34) - (4.36) completely determine the dyadic $\overline{\overline{E}}_m^{(1)}$ defined in (2.95). We are now ready to proceed with the scattering problem.

Scattering of a Plane Wave by a Perfectly Conducting Sphere at Low Frequencies

As shown in Fig. 2, the incident plane wave propagates along the negative z-axis with the electric field vector polarized along the positive x-axis. Taking the amplitude of the electric field to be equal to one we write

$$\vec{E}^i = \hat{x}\, e^{-ikz}, \qquad \vec{H}^i = -\hat{y}\, Y e^{-ikz}, \tag{4.37}$$

where Y is the free space characteristic admittance. We shall first determine the zeroth order iterates given by (3.53). By (3.25) and the boundary conditions (3.3) we have that on the surface of the sphere

$$\hat{n} \times \vec{e}(\vec{R}_s) = -e^{-ika}\hat{n} \times \vec{E}^i(\vec{R}_s), \quad \hat{n} \cdot \vec{h}(\vec{R}_s) = -e^{-ika}\hat{n} \cdot \vec{H}^i(\vec{R}_s), \quad \vec{R}_s \in S. \tag{4.38}$$

Substitution of these boundary conditions in (3.53) leads to the following expressions for the zeroth order iterates

$$\vec{h}_{(0)}(\vec{R}') = Y e^{-ika}\, \nabla' \int_S (\hat{n} \cdot \vec{H}^i) N^{(e)}(\vec{R}_s | \vec{R}') dS, \tag{4.39}$$

and

$$\vec{e}_{(0)}(\vec{R}') = -e^{-ika} \int_S (\hat{n} \times \vec{E}^i) \cdot \overline{\overline{H}}_e^{(1)}(\vec{R}_s | \vec{R}')\, dS. \tag{4.40}$$

By (4.1), (4.4) and (4.37), the integral of (4.39) can be written

$$\int_S (\hat{n} \cdot \vec{H}^i) N^{(e)}(\vec{R}_s | \vec{R}') dS = -\frac{1}{4\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \epsilon_m \frac{2n+1}{n+1} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta') \frac{a^{n+2}}{R'^{n+1}} \times$$

$$\times \int_0^{\pi} \int_0^{2\pi} \sin\theta_s\, d\theta_s\, d\phi_s\, e^{-ika\cos\theta_s} P_n^m(\cos\theta_s) \sin\theta_s \cos m(\phi_s - \phi'). \tag{4.41}$$

The indicated integrations are performed by using the expansion (Magnus, et al, 1966, p. 108)

$$e^{-ika\cos\theta} = \sqrt{\frac{\pi}{2ka}} \sum_{m=o}^{\infty} (-i)^m (2m+1) J_{m+\frac{1}{2}}(ka) P_m(\cos\theta), \qquad (4.42)$$

and the orthogonality properties of the trigonometric and Legendre functions. The function $J_{m+1/2}$ is a Bessel function of half order. The resulting expression is

$$\int_S (\hat{n}\cdot\vec{H}^i) N^{(e)}(\vec{R}_s|\vec{R}')dS = \frac{i}{ka}\sqrt{\frac{\pi}{2ka}} \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n+1} J_{n+\frac{1}{2}}(ka) P'_n(\cos\theta') \times$$

$$\times \sin\phi' \frac{a^{n+2}}{R'^{n+1}} . \qquad (4.43)$$

Performing the remaining operation in (4. 39) we obtain

$$\vec{h}_{(0)}(\vec{R}) = \frac{iYe^{-ika}}{ka}\sqrt{\frac{\pi}{2ka}} \left\{ -\hat{R} \sum_{n=1}^{\infty} (-i)^n (2n+1) J_{n+\frac{1}{2}}(ka) P_n^1(\cos\theta)\sin\phi \frac{a^{n+2}}{R^{n+2}} + \right.$$

$$+\hat{\theta} \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n+1} J_{n+\frac{1}{2}}(ka) \frac{d}{d\theta} P_n^1(\cos\theta)\sin\phi \frac{a^{n+2}}{R^{n+2}} +$$

$$\left. + \frac{\hat{\phi}}{\sin\theta} \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n+1} J_{n+\frac{1}{2}}(ka) P_n^1(\cos\theta)\cos\phi \frac{a^{n+2}}{R^{n+2}} \right\} . \qquad (4.44)$$

We now turn to (4. 40) which, by (4. 37), can be written

$$\vec{e}_{(0)}(\vec{R}') = e^{-ika} \int_S e^{-ikz_s} (\hat{\phi}\cos\theta_s \cos\phi_s + \hat{\theta}\sin\phi_s)\cdot \overline{\overline{H}}_e^{(1)}(\vec{R}_s|\vec{R}')\, dS . \qquad (4.45)$$

In order to perform this integration we split the vector integrand into its rectangular components and integrate component by component. The dyadic $\overline{\overline{H}}_e^{(1)}$ is defined in (2. 66) and, for the sphere problem, it can be written in terms of (4.1), (4.16) and (4.19) - (4.21). Performing the dot product indicated in (4.45) results in expressions similar to those in (4.41) which can be integrated readily. We omit all these operations because of their great length and we only give the resulting expressions; thus,

$$
\begin{aligned}
e^{ika}\,\vec{e}_{(0)}(\vec{R})\cdot\hat{x} = & -\frac{3}{2}\sqrt{\frac{\pi}{2ka}}\sum_{n=1}^{\infty}(-i)^n nJ_{n+\frac{1}{2}}(ka)P_n(\cos\theta)\frac{a^{n+1}}{R^{n+1}} + \\
& + \frac{1}{2ka}\sqrt{\frac{\pi}{2ka}}\sum_{n=2}^{\infty}(-i)^n n(n-1)J_{n-\frac{1}{2}}(ka)P_n(\cos\theta)\frac{a^{n+1}}{R^{n+1}} + \\
& - \frac{1}{2}\sqrt{\frac{\pi}{2ka}}\sum_{n=2}^{\infty}(-i)^n\frac{1}{n+1}J_{n+\frac{1}{2}}(ka)P_n^2(\cos\theta)\cos 2\phi\,\frac{a^{n+1}}{R^{n+1}} - \\
& - \frac{1}{2ka}\sqrt{\frac{\pi}{2ka}}\sum_{n=2}^{\infty}(-i)^n J_{n-\frac{1}{2}}(ka)P_n^2(\cos\theta)\cos 2\phi\,\frac{a^{n+1}}{R^{n+1}} + \\
& + i\sqrt{\frac{\pi}{2ka}}\frac{\partial}{\partial x}\sum_{n=1}^{\infty}(-i)^n\frac{1}{n}J_{n-\frac{1}{2}}(ka)P_n^1(\cos\theta)\cos\phi\,\frac{a^{n+2}}{R^{n+1}} - \\
& - \frac{i}{ka}\sqrt{\frac{\pi}{2ka}}\frac{\partial}{\partial x}\sum_{n=1}^{\infty}(-i)^n J_{n+\frac{1}{2}}(ka)P_n^1(\cos\theta)\cos\phi\,\frac{a^{n+2}}{R^{n+1}}\,, \qquad (4.46)
\end{aligned}
$$

$$
\begin{aligned}
e^{ika}\,\vec{e}_{(0)}(\vec{R})\cdot\hat{y} = & -\frac{1}{2}\sqrt{\frac{\pi}{2ka}}\sum_{n=2}^{\infty}(-i)^n\frac{1}{n+1}J_{n+\frac{1}{2}}(ka)P_n^2(\cos\theta)\sin 2\phi\,\frac{a^{n+1}}{R^{n+1}} - \\
& - \frac{1}{2ka}\sqrt{\frac{\pi}{2ka}}\sum_{n=2}^{\infty}(-i)^n J_{n-\frac{1}{2}}(ka)P_n^2(\cos\theta)\sin 2\phi\,\frac{a^{n+1}}{R^{n+1}} + \\
& + i\sqrt{\frac{\pi}{2ka}}\frac{\partial}{\partial y}\sum_{n=1}^{\infty}(-i)^n\frac{1}{n}J_{n-\frac{1}{2}}(ka)P_n^1(\cos\theta)\cos\phi\,\frac{a^{n+2}}{R^{n+1}} - \\
& - \frac{i}{ka}\sqrt{\frac{\pi}{2ka}}\frac{\partial}{\partial y}\sum_{n=1}^{\infty}(-i)^n J_{n+\frac{1}{2}}(ka)P_n^1(\cos\theta)\cos\phi\,\frac{a^{n+2}}{R^{n+1}}\,, \qquad (4.47)
\end{aligned}
$$

$$e^{ika}\,\vec{e}_{(0)}(\vec{R})\cdot\hat{z} = -\sqrt{\frac{\pi}{2ka}}\sum_{n=1}^{\infty}(-i)^n\frac{1}{n+1}J_{n+\frac{1}{2}}(ka)P_n^1(\cos\theta)\cos\phi\frac{a^{n+1}}{R^{n+1}} -$$

$$-\frac{1}{ka}\sqrt{\frac{\pi}{2ka}}\sum_{n=1}^{\infty}(-i)^n\,n\,J_{n-\frac{1}{2}}(ka)P_n^1(\cos\theta)\cos\phi\frac{a^{n+1}}{R^{n+1}} +$$

$$+\sqrt{\frac{\pi}{2ka}}\sum_{n=1}^{\infty}(-i)^n J_{n-\frac{3}{2}}(ka)P_n^1(\cos\theta)\cos\phi\,\frac{a^{n+1}}{R^{n+1}} +$$

$$+\,i\sqrt{\frac{\pi}{2ka}}\,\frac{\partial}{\partial z}\sum_{n=1}^{\infty}(-i)^n\frac{1}{n}J_{n-\frac{1}{2}}(ka)\,P_n^1(\cos\theta)\cos\phi\frac{a^{n+2}}{R^{n+1}} -$$

$$-\frac{i}{ka}\sqrt{\frac{\pi}{2ka}}\,\frac{\partial}{\partial z}\sum_{n=1}^{\infty}(-i)^n J_{n+\frac{1}{2}}(ka)P_n^1(\cos\theta)\,\cos\phi\,\frac{a^{n+2}}{R^{n+1}}\ . \qquad (4.48)$$

Certain simplifications take place in the above three expressions, when the indicated differentiations are performed, by employing the properties of the Bessel functions involved. Nothing nearly as simple as (4.44) results, however, either in rectangular or spherical coordinates.

From the zeroth order iterates $\vec{h}_{(0)}$ and $\vec{e}_{(0)}$ we can obtain the zeroth order coefficients $\vec{h}_o$ and $\vec{e}_o$,respectively, of the low frequency expansions (3.52). By (3.53) and (3.58)

$$\vec{f}_o(\vec{R}) = \vec{h}_o(\vec{R}) = \lim_{k\to o}\ \vec{h}_{(0)}(\vec{R})\ , \qquad (4.49)$$

$$\vec{g}_o(\vec{R}) = \vec{e}_o(\vec{R}) = \lim_{k\to o}\ \vec{e}_{(0)}(\vec{R})\ . \qquad (4.50)$$

In order to calculate these limits we expand the Bessel functions involved in (4.44) and (4.46) - (4.48) in a power series of ka of the form (Magnus, et al, 1966, p. 65)

$$J_{n+\frac{1}{2}}(ka) = \sqrt{\frac{ka}{2}}\ (ka)^n\sum_{\ell=o}^{\infty}\frac{(-1)^{\ell}\,(ka)^{2\ell}}{\ell!\,2^{2\ell}\,\Gamma(n+\ell+\frac{3}{2})}\ . \qquad (4.51)$$

Proceeding next to the limit we find that

$$\vec{h}_o(\vec{R}) = \hat{R}\frac{a^3}{R^3}\sin\theta\sin\phi - \hat{\theta}\frac{a^3}{2R^3}\cos\theta\sin\phi - \hat{\phi}\frac{a^3}{2R^3}\cos\phi \tag{4.52}$$

$$\vec{e}_o(\vec{R}) = \hat{R}\frac{2a^3}{R^3}\sin\theta\cos\phi - \hat{\theta}\frac{a^3}{R^3}\cos\theta\cos\phi + \hat{\phi}\frac{a^3}{R^3}\sin\phi \quad . \tag{4.53}$$

By (3.25) these are also the zeroth order terms in the low frequency expansions of the scattered fields $\vec{H}^s$ and $\vec{E}^s$, respectively, and are in complete agreement with Kleinman's (1965b) results as derived using the modified Stevenson method. They can also be obtained from Rayleigh's (1897) theory.

We now turn to the calculation of the first order terms in the low frequency expansions for $\vec{h}$ and $\vec{e}$. From (3.60).

$$\vec{h}_1(\vec{R}') \int_V (Y\vec{e}_o + \hat{R}\times\vec{h}_o) \cdot \overline{\overline{E}}_m^{(1)} dV - \nabla' \int_V (\hat{R}\cdot\vec{h}_o) N^{(e)} dV + \vec{f}_1(\vec{R}') \; . \tag{4.54}$$

By (4.52) and (4.53)

$$Y\vec{e}_o + \hat{R}\times\vec{h}_o = Y\left\{\hat{R}\frac{2a^3}{R^3}\sin\theta\cos\phi + \hat{\theta}\frac{a^3}{R^3}\left(\frac{1}{2} - \cos\theta\right)\cos\phi + \hat{\phi}\frac{a^3}{R^3}\left(1 - \frac{1}{2}\cos\theta\right)\sin\phi\right\} . \tag{4.55}$$

By (2.59) the dyadic $\overline{\overline{E}}_m^{(1)}$ can be written

$$\overline{\overline{E}}_m^{(1)} = \nabla\left(-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right)\times\overline{\overline{I}} + \overline{\overline{E}}_{m_r}^{(1)p} + \sum_{i=1}^{3}\nabla G_{mi}^{(e)}\, a_i \; . \tag{4.56}$$

We now define the vector $\vec{A}$ as follows

$$\vec{A} = (Y\vec{e}_o + \hat{R}\times\vec{h}_o)\cdot\left[\nabla\left(-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right)\times\overline{\overline{I}}\right] = (Y\vec{e}_o + \hat{R}\times\vec{h}_o)\times\nabla\left(-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right) , \tag{4.57}$$

and we proceed to integrate it over V as indicated in (4.55). In doing so we use the expansion (4.1). The integration is performed in a standard manner: we split $\vec{A}$ into its rectangular components and we integrate using the orthogonality properties of the trigonometric and Legendre functions. We will just give the results for the x and y components of $\vec{A}_0$, while we show part of the integration for the z component:

$$\int_V (\vec{A}\cdot\hat{x})dV = -\frac{Ya^3}{12}P_2^2(\cos\theta')\sin 2\phi' \frac{R'-a}{R'^3} , \tag{4.58}$$

$$\int_V (\vec{A}\cdot\hat{y})dV = -\frac{Ya^3}{6}\frac{1}{R'^2} + \frac{Ya^3}{3}P_1(\cos\theta')\frac{1}{R'^2} + \frac{Ya^3}{6}P_2(\cos\theta')\frac{R'-a}{R'^3} + $$
$$+\frac{Ya^3}{12}P_2^2(\cos\theta')\cos 2\phi' \frac{R'-a}{R'^3} . \tag{4.59}$$

After performing the angular integrations for the z component of $\vec{A}$ we are left with (the Legendre functions are functions of $\cos\theta'$)

$$\int_V (\vec{A}\cdot\hat{z})dV = -\frac{Ya^3}{3}P_1^1\sin\phi'\int_a^\infty \frac{dR}{R}\frac{\partial}{\partial R}\left(\frac{R_<}{R_>^2}\right) + \frac{Ya^3}{30}P_2^1\sin\phi'\int_a^\infty \frac{dR}{R}\frac{\partial}{\partial R}\left(\frac{R_<^2}{R_>^3}\right) +$$
$$+\frac{Ya^3}{3}P_1^1\sin\phi'\int_a^\infty \frac{dR}{R^2}\frac{R_<}{R_>^2} + \frac{Ya^3}{10}P_2^1\sin\phi'\int_a^\infty \frac{dR}{R^2}\frac{R_<^2}{R_>^3} =$$
$$= -\frac{Ya^3}{3}P_1^1\sin\phi'\int_a^{R'}\frac{dR}{R}\frac{1}{R'^2} + \frac{2Ya^3}{3}P_1^1\sin\phi'\int_{R'}^\infty \frac{dR}{R^4}R' + \frac{Ya^3}{15}P_2^1\sin\phi'\int_a^{R'} dR\frac{1}{R'^3} -$$
$$-\frac{Ya^3}{10}P_2^1\sin\phi'\int_{R'}^\infty \frac{dR}{R^5}R'^2 + \frac{Ya^3}{3}P_1^1\sin\phi'\int_a^{R'}\frac{dR}{R}\frac{1}{R'^2} + \frac{Ya^3}{3}P_1^1\sin\phi\int_{R'}^\infty \frac{dR}{R^4}R'^2 +$$
$$+\frac{Ya^3}{10}P_2^1\sin\phi'\int_a^{R'} dR\frac{1}{R'^3} + \frac{Ya^3}{10}P_2^1\sin\phi'\int_{R'}^\infty \frac{dR}{R^5}R'^2 . \tag{4.60}$$

The reason we present this expression is to show that it contains two terms that behave in a very undesirable manner, namely $\log R'/R'^2$, and that these terms cancel each other. This is an indication (farfetched, perhaps) that the volume integrals of the equations (3.60) and (3.62) involving the dyadics $\bar{\bar{E}}_m^{(1)}$ and $\bar{\bar{H}}_e^{(1)}$ will converge for all n . Terms of the same kind also appeared in the integration of the y component of $\vec{A}$ and they also cancelled out. Performing the indicated integrations in (4.60) we obtain

$$\int_V (\vec{A}\cdot\hat{z})dV = \frac{Ya^3}{3} P_1^1(\cos\theta')\sin\phi' \frac{1}{R'^2} + \frac{Ya^3}{6} P_2^1(\cos\theta')\sin\phi' \frac{R'-a}{R'^3} . \tag{4.61}$$

Collecting our results from (4.58), (4.59) and (4.61) we have that

$$\begin{aligned}\int_V (Y\vec{e}_o+\hat{R}x\vec{h})\cdot\left[\nabla\left(-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right)x\,\bar{\bar{I}}\right]dV = & -\frac{Ya^3}{12} P_2^2(\cos\theta')\sin 2\phi' \frac{R'-a}{R'^3}\hat{x} + \\ & +\frac{Ya^3}{12}\left[-\frac{2}{R'^2}+4P_1(\cos\theta')\frac{1}{R'^2}+2P_2(\cos\theta')\frac{R'-a}{R'^3}+P_2^2(\cos\theta')\cos 2\phi'\frac{R'-a}{R'^3}\right]\hat{y} + \\ & +\frac{Ya^3}{6}\left[2P_1^1(\cos\theta')\sin\phi'\frac{1}{R'^2}+P_2^1(\cos\theta')\sin\phi'\frac{R'-a}{R'^3}\right]\hat{z} .\end{aligned} \tag{4.62}$$

We now turn to the second term of (4.56). By (4.26) we have that

$$(Y\vec{e}_o+\hat{R}x\vec{h}_o)\cdot \bar{\bar{E}}^{(1)p}_{m_r} = \nabla'\left[(Y\vec{e}_o+\hat{R}x\vec{h}_o)\cdot\vec{\mathcal{E}}\right], \tag{4.63}$$

where $\vec{\mathcal{E}}$ is the vector of (4.26), i.e.

$$\begin{aligned}\vec{\mathcal{E}} = \frac{1}{4\pi}\Bigg\{ & \hat{\theta}\frac{2}{\sin\theta}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{m}{n+1}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta')P_n^m(\cos\theta)\sin m(\phi-\phi')\frac{a^{2n+1}}{R'^{n+1}R^{n+1}} + \\ & +\hat{\phi}\sum_{n=1}^{\infty}\sum_{m=o}^{n}\epsilon_m\frac{1}{n+1}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta')\frac{d}{d\theta}P_n^m(\cos\theta)\cos m(\phi-\phi')\frac{a^{2n+1}}{R'^{n+1}R^{n+1}}\Bigg\} .\end{aligned} \tag{4.64}$$

From this expression and (4.55) we form the dot product indicated in (4.63) and then we integrate over V to find

$$\int_V (Y\vec{e}_o+\hat{R}x\vec{h}_o)\cdot\vec{\mathcal{E}}\,dV = \frac{Ya^4}{12}P_1^1(\cos\theta')\sin\phi'\frac{1}{R'^2} . \tag{4.65}$$

Employing now (4.63) we have that

$$\int_V (Y\vec{e}_o+\hat{R}x\vec{h}_o)\cdot \bar{\bar{E}}^{(1)p}_{m_r}\,dV=-\frac{Ya^4}{12}\,\nabla'\left(\frac{\sin\theta'\sin\phi'}{R'^2}\right)=\frac{a}{6}\,\vec{h}_o(\vec{R}')\,, \qquad (4.66)$$

where $\vec{h}_o$ is given by (4.52).

We now turn to the last terms of the dyadic (4.56). The exterior Dirichlet functions $G^{(e)}_{mi}$ are given by (4.34) - (4.36). With relatively little labor we find that

$$\sum_{i=1}^{3}\hat{a}_i\int_V (Y\vec{e}_o+\hat{R}x\vec{h}_o)\cdot \nabla G^{(e)}_{mi}\,dV=\frac{2Ya^3}{3}P_1(\cos\theta')\frac{1}{R'^2}\,\hat{y}+\frac{2Ya^3}{3}P^1_1(\cos\theta')\sin\phi\frac{1}{R'^2}\,\hat{z}\,. \qquad (4.67)$$

This calculation completes the evaluation of the first integral of (4.54). To evaluate the second integral we employ the definition of $N^{(e)}$ in (2.84) - (2.86) and the expansions (4.1) and (4.4). The resulting expression is

$$\int_V (\hat{R}\cdot\vec{h}_o)N^{(e)}dV=\frac{Ya^3}{2}P^1_1(\cos\theta')\sin\phi'\frac{1}{R'}-\frac{Ya^4}{4}P^1_1(\cos\theta')\sin\phi'\frac{1}{R'^2}\,. \qquad (4.68)$$

Taking the gradient of this expression we obtain

$$\nabla'\int_V (\hat{R}\cdot\vec{h}_o)N^{(e)}dV=\frac{Ya^3}{2R'^2}(\hat{R}'\sin\theta'\sin\phi'-\hat{\theta}'\cos\theta'\sin\phi'-\hat{\phi}'\cos\phi')-\frac{a}{2}\vec{h}_o(\vec{R}')\,. \qquad (4.69)$$

To conclude the evaluation of $\vec{h}_1$ in (4.54) we need $\vec{f}_1$. From (3.58) and (4.44) we have that

$$ik\vec{f}_1(\vec{R})=-ika\vec{h}_o(\vec{R})+\frac{iY}{ka}\sqrt{\frac{\pi}{2ka}}\left\{\hat{R}\,5\,J_{\frac{5}{2}}(ka)P^1_2(\cos\theta)\sin\phi\,\frac{a^4}{R^4}-\hat{\theta}\frac{5}{3}\frac{d}{d\theta}P^1_2(\cos\theta)\sin\phi\,\frac{a^4}{R^4}-\right.$$

$$\left.-\hat{\phi}\frac{5}{3}\frac{1}{\sin\theta}J_{\frac{5}{2}}(ka)P^1_2(\cos\theta)\cos\phi\,\frac{a^4}{R^4}\right\}. \qquad (4.70)$$

From (4.51)

$$J_{\frac{5}{2}}(ka) = \sqrt{\frac{ka}{2}}\left(\frac{ka}{2}\right)^2 \left[\frac{8}{15\sqrt{\pi}} + O\,(k^2a^2)\right]. \tag{4.71}$$

Substituting this expression in (4.70) we find that

$$\vec{f}_1(\vec{R}) = -a\vec{h}_o(\vec{R}) + \frac{Ya^5}{3R^4}\left(-\hat{R}\frac{3}{2}\sin 2\theta\sin\phi + \hat{\theta}\cos 2\theta\sin\phi + \hat{\phi}\cos\theta\cos\phi\right). \tag{4.72}$$

Before collecting our results, we write (4.62) and (4.67) in spherical coordinates:

$$\int_V (Y\vec{e}_o + \hat{R}x\vec{h}_o)\cdot\left[\nabla\left(-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right) x\bar{\bar{I}}\right] dV + \sum_{n=1}^{3} a_i \int_V (Y\vec{e}_o + \hat{R}x\vec{h}_o)\cdot \nabla G^{(e)}_{mi}\, dV =$$

$$= \frac{Ya^3}{2R'^2}\left\{-\hat{R}'\sin\theta'\sin\phi' + \hat{\theta}'2\sin\phi' + \hat{\phi}'2\cos\theta'\cos\phi'\right\} + \frac{a}{3}\vec{h}_o(\vec{R}'). \tag{4.73}$$

Collecting our results from (4.66), (4.69), (4.71) and (4.73) and substituting them in (4.54) we obtain the following expression for $\vec{h}_1$:

$$\vec{h}_1(\vec{R}) = \frac{Ya^3}{2R^2}\left\{-\hat{R}2\sin\theta\sin\phi + \hat{\theta}(2+\cos\theta)\sin\phi + \hat{\phi}(2\cos\theta+1)\cos\phi\right\} + $$
$$+ \frac{Ya^5}{3R^4}\left\{-\hat{R}\frac{3}{2}\sin 2\theta\sin\phi + \hat{\theta}\cos 2\theta\sin\phi + \hat{\phi}\cos\theta\cos\phi\right\}. \tag{4.74}$$

Our next step is to determine $\vec{e}_1$. From (3.62)

$$\vec{e}_1(\vec{R}') = -\int_V (Z\vec{h}_o - \hat{R}x\vec{e}_o)\cdot \bar{\bar{H}}^{(1)}_e dV - \nabla'\int_V (\hat{R}\cdot\vec{e}_o)\left[\psi^{(0)}_e + D_e\right] dV + \vec{g}_1(\vec{R}'). \tag{4.75}$$

The procedure for finding $\vec{e}_1$ is analogous to that followed for $\vec{h}_1$ of (4.54). For this reason we will be brief. From (4.52) and (4.53)

$$\hat{R}x\vec{e}_o - Z\vec{h}_o = \frac{a^3}{R^3}\left\{-\hat{R}\sin\theta\sin\phi + \hat{\theta}\left(\frac{1}{2}\cos\theta - 1\right)\sin\phi + \hat{\phi}\left(\frac{1}{2} - \cos\theta\right)\cos\phi\right\}. \tag{4.76}$$

The dyadic $\overline{\overline{H}}_e^{(1)}$ is given by (2.66):

$$\overline{\overline{H}}_e^{(1)} = \nabla\left(-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right)x\overline{\overline{I}}+\overline{\overline{H}}_{e_r}^{(1)p}+\sum_{i=1}^{3}\nabla N_{ei}^{(e)}\hat{a}_i \ . \qquad (4.77)$$

Proceeding as in (4.57) - (4.61) we find that

$$\int_V (R\hat{x}\vec{e}_o-Z\vec{h}_o)\cdot\left[\nabla\left(-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right)x\overline{\overline{I}}\right]dV=\left[-\frac{a^3}{3R'^2}+\frac{a^3}{6}P_1(\cos\theta')\frac{1}{R'^2}+\right.$$

$$+\frac{a^3}{3}P_2(\cos\theta')\frac{R'-a}{R'^3}-\frac{a^3}{6}P_2^2(\cos\theta')\cos2\phi'\frac{R'-a}{R'^3}\Bigg]\hat{x}+$$

$$+\left[-\frac{a^3}{6}P_2^2(\cos\theta')\sin2\phi'\frac{R'-a}{R'^3}\right]\hat{y}+\left[\frac{a^3}{6}P_1^1(\cos\theta')\cos\phi'\frac{1}{R'^2}+\right.$$

$$\left.+\frac{a^3}{3}P_2^1(\cos\theta')\cos\phi'\frac{R'-a}{R'^3}\right]\hat{z} \ . \qquad (4.78)$$

We mention that in deriving this expression undesirable terms $(\log R'/R'^2)$ arose but cancelled out.

The dyadic $\overline{\overline{H}}_{e_r}^{(1)p}$ is given by (4.16)and if we define

$$\mathcal{H} = -\frac{1}{4\pi}\left\{\hat{\theta}\frac{2}{\sin\theta}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{m}{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta')P_n^m(\cos\theta)\sin m(\phi-\phi')\frac{a^{2n+1}}{R'^{n+1}R^{n+1}}+\right.$$

$$\left.+\hat{\phi}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\epsilon_m\frac{1}{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta')\frac{d}{d\theta}P_n^m(\cos\theta)\cos m(\phi-\phi')\frac{a^{2n+1}}{R'^{n+1}R^{n+1}}\right\},$$

(4.79)

we readily find that

$$\int_V (R\hat{x}\vec{e}_o-Z\vec{h}_o)\cdot\mathcal{H}\,dV = -\frac{a^4}{3}P_1^1(\cos\theta')\cos\phi'\frac{1}{R'^2}\ , \qquad (4.80)$$

from which

$$\nabla'\int_V (R\hat{x}\vec{e}_o-Z\vec{h}_o)\cdot\mathcal{H}\,dV = \frac{a^4}{3}\nabla'\left(\frac{\sin\theta'\cos\phi'}{R'^2}\right) = -\frac{a}{3}\vec{e}_o(\vec{R}') \ . \qquad (4.81)$$

The exterior Neumann functions, $N_{ei}^{(e)}$, of (4. 77) are given by (4. 19) - (4. 21). Performing the integration we find that

$$\sum_{i=1}^{3} \hat{a}_i \int_V (\hat{R}x\vec{e}_o - Z\vec{h}_o)\cdot\nabla N_{ei}^{(e)} dV = -\frac{a^3}{6} P_1(\cos\theta')\frac{1}{R'^2}\hat{x} - \frac{a^3}{6} P_1^1(\cos\theta')\cos\phi'\frac{1}{R'^2}\hat{z} . \tag{4.82}$$

From (4. 78) and (4. 82) we find that in spherical coordinates

$$\int_V (\hat{R}x\vec{e}_o - Z\vec{h}_o)\cdot\left[\nabla\left(-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right) x\,\bar{\bar{I}}\right] dV + \sum_{i=1}^{3}\hat{a}_i \int_V (\hat{R}x\vec{e}_o - Z\vec{h}_o)\cdot\nabla N_{ei}^{(e)} dV =$$

$$= -\hat{R}' a^3 \sin\theta'\cos\phi'\frac{1}{R'^2} + \frac{a}{3}\vec{e}_o(\vec{R}') . \tag{4.83}$$

Proceeding to the next volume integral in (4. 75) we have from (4. 7), (4. 8) and (4. 53)

$$\int_V (\hat{R}\cdot\vec{e}_o)\left[\psi_e^{(0)} + D_e\right] dV = 2a^3\int_V \frac{\sin\theta\cos\phi}{R^3} G^{(e)} dV - \frac{2a^4}{4\pi R'}\int_V \frac{\sin\theta\cos\phi}{R^4} dV + \frac{2a^3}{4\pi R'}\int_V \frac{\sin\theta\cos\phi}{R^3} dV . \tag{4.84}$$

Clearly, the last integral vanishes due to the $\cos\phi$ term. Employing (4. 1) and (4. 3) for $G^{(e)}$ we readily find

$$\int_V (\hat{R}\cdot\vec{e}_o)\left[\psi_e^{(0)} + D_e\right] dV = a^3 P_1^1(\cos\theta')\cos\phi'\frac{1}{R'} - a^4 P_1^1(\cos\theta')\cos\phi'\frac{1}{R'^2} , \tag{4.85}$$

from which

$$\nabla'\int_V (\hat{R}\cdot\vec{e}_o)\left[\psi_e^{(0)} + D_e\right] dV = \frac{a^3}{R'^2}(\hat{R}\sin\theta'\cos\phi' - \hat{\theta}'\cos\theta'\cos\phi' + \hat{\phi}'\sin\phi') - a\vec{e}_o(\vec{R}') . \tag{4.86}$$

We are now left with the determination of $\vec{g}_1$. From (3.58) and (4.46) - (4.48) we have that

$$ik\vec{g}_1(\vec{R})=-ika\vec{e}_0+i\sqrt{\frac{\pi}{2ka}}\left\{\frac{3}{2}J_{\frac{3}{2}}(ka)P_1(\cos\theta)\frac{a^2}{R^2}+\frac{3}{ka}J_{\frac{5}{2}}(ka)P_3(\cos\theta)\frac{a^4}{R^4}-\right.$$

$$\left.-\frac{1}{2ka}J_{\frac{5}{2}}(ka)P_3^2(\cos\theta)\cos 2\phi\frac{a^4}{R^4}\right\}\hat{n}-\frac{1}{2ka}\sqrt{\frac{\pi}{2ka}}J_{\frac{5}{2}}(ka)P_3^2(\cos\theta)\sin 2\phi\frac{a^4}{R^4}\hat{y}+$$

$$+i\sqrt{\frac{\pi}{2ka}}\left\{\frac{3}{2}J_{\frac{3}{2}}(ka)P_1^1(\cos\theta)\cos\phi\frac{a^2}{R^2}+\frac{2}{5}J_{\frac{3}{2}}(ka)P_3^1(\cos\theta)\cos\phi\frac{a^4}{R^4}\right\}\hat{z}+$$

$$+i\sqrt{\frac{\pi}{2ka}}\nabla'\left[-\frac{1}{2}J_{\frac{3}{2}}(ka)P_2^1(\cos\theta)\cos\phi\frac{a^4}{R^3}\right]-$$

$$-\frac{i}{ka}\sqrt{\frac{\pi}{2ka}}\nabla'\left[-J_{\frac{5}{2}}(ka)P_3^1(\cos\theta)\cos\phi\frac{a^4}{R^3}\right]. \tag{4.87}$$

By (4.51)

$$J_{\frac{3}{2}}(ka)=\sqrt{\frac{ka}{2}}\left(\frac{ka}{2}\right)\left[\frac{4}{3\sqrt{\pi}}+O(k^2a^2)\right]. \tag{4.88}$$

Substitution of (4.71) together with (4.88) in (4.87) and a transformation from rectangular to spherical coordinates results in

$$\vec{g}_1(\vec{R})=-a\vec{e}_0+\frac{a^3}{2R^2}(\hat{\theta}\cos\phi-\hat{\phi}\cos\theta\sin\phi)+\frac{a^5}{2R^4}(-\hat{R}\frac{3}{2}\sin 2\theta\cos\phi$$

$$+\hat{\theta}\cos 2\theta\cos\phi-\hat{\phi}\cos\theta\sin\phi). \tag{4.89}$$

Collecting our results from (4.81), (4.83), (4.86) and (4.89) and substituting them in (4.75) we obtain

$$\vec{e}_1(\vec{R})=\frac{a^3}{R^2}\left\{-\hat{R}2\sin\theta\cos\phi+\hat{\theta}(\frac{1}{2}+\cos\theta)\cos\phi-\hat{\phi}(1+\frac{1}{2}\cos\theta)\sin\phi\right\}+$$

$$+\frac{a^5}{2R^4}\left\{-\hat{R}\frac{3}{2}\sin 2\theta\cos\phi+\hat{\theta}\cos 2\theta\cos\phi-\hat{\phi}\cos\theta\sin\phi\right\}. \tag{4.90}$$

In order to check the correctness of our expressions for $\vec{h}_1$ and $\vec{e}_1$ we go back to (3. 25) and expand the scattered fields in power series of ik. The first order terms are given by

$$\vec{H}_1^s = R\vec{h}_o + \vec{h}_1 , \qquad \vec{E}_1^s = R\,\vec{e}_o + \vec{e}_1 . \tag{4. 91}$$

Substitution of (4. 52), (4. 53), (4. 75) and (4. 90) in these two expressions leads to

$$\vec{H}_1^s(\vec{R}) = \frac{Ya^3}{R^2}(\hat{\theta}\sin\phi + \hat{\phi}\cos\theta\cos\phi) + \frac{Ya^5}{3R^4}(-\hat{R}\frac{3}{2}\sin 2\theta\sin\phi + \hat{\theta}\cos 2\theta\sin\phi + \\ + \hat{\phi}\cos\theta\cos\phi) , \tag{4. 92}$$

$$\vec{E}_1^s(\vec{R}) = \frac{a^3}{2R^2}(\hat{\theta}\cos\phi - \hat{\phi}\cos\theta\sin\phi) + \frac{a^5}{2\pi^4}(-\hat{R}\frac{3}{2}\sin 2\theta\cos\phi + \hat{\theta}\cos 2\theta\cos\phi - \\ - \hat{\phi}\cos\theta\sin\phi) . \tag{4. 93}$$

Both expressions agree with Kleinman's (1965b) results as derived using the modified Stevenson method.

We will conclude the example by finding the terms of $\vec{h}_2$ and $\vec{e}_2$ that behave as $1/R$. From these terms we can find the first term in the low frequency expansion of the far field for both the electric and the magnetic scattered vectors. By (3. 60)

$$\vec{h}_2(\vec{R}') = \int_V (Y\vec{e}_1 + \hat{R}\times\vec{h}_1)\cdot\overline{\overline{E}}_m^{(1)}dV - \nabla'\int_V \hat{R}\cdot\vec{h}_1 N^{(e)}dV + \vec{f}_2(\vec{R}') . \tag{4. 94}$$

Employing (4. 74) and (4. 90) we have that

$$Y\vec{e}_1+\hat{R}x\vec{h}_1=-\hat{R}\frac{2Ya^3}{R^2}\sin\theta\cos\phi+\frac{Ya^5}{R^4}\left\{-\hat{R}\frac{3}{2}\sin2\theta\cos\phi+\right.$$

$$\left.+\hat{\theta}(\cos2\theta-\frac{1}{3}\cos\theta)\cos\phi+\hat{\phi}(\frac{1}{3}\cos2\theta-\cos\theta)\sin\phi\right\}. \qquad (4.95)$$

Performing the first volume integration in (4.94) as we did in the previous cases we find that

$$\int_V (Y\vec{e}_1+\hat{R}x\vec{h}_1)\cdot\overline{\overline{E}}_m^{(1)}dV=-\hat{y}\frac{Ya^3}{R'}P_1(\cos\theta')-\hat{z}\frac{Ya^3}{R'}P_1^1(\cos\theta')\sin\phi'+O(\frac{1}{R'^2}). \qquad (4.96)$$

Similarly,

$$\hat{R}\cdot\vec{h}_1=-\frac{Ya^3}{R^2}\sin\theta\sin\phi-\frac{Ya^5}{R^4}\sin\theta\cos\theta\sin\phi\ , \qquad (4.97)$$

and

$$\int_V \hat{R}\cdot\vec{h}_1 N^{(e)}\,dV=-\frac{Ya^3}{2}P_1^1(\cos\theta')\sin\phi'+O(\frac{1}{R'})\ , \qquad (4.98)$$

from which we have that

$$\nabla'\int_V \hat{R}\cdot\vec{h}_1 N^{(e)}dV=\frac{Ya^3}{2}\nabla'(\sin\theta'\sin\phi')+O(\frac{1}{R'^2})=$$

$$=\frac{Ya^3}{2R'}(\hat{\theta}'\cos\theta'\sin\phi'+\hat{\phi}'\cos\phi')+O(\frac{1}{R'^2})\ . \qquad (4.99)$$

From Eq. (4.44) it is clear that none of the $\vec{f}_n$'s contribute a $1/R$ term. Transforming then (4.96) into spherical coordinates and substituting the result together with (4.99) in (4.94) we have that

$$\vec{h}_2(\vec{R})=-\frac{Ya^3}{R}(\hat{\theta}\sin\phi+\hat{\phi}\cos\theta\cos\phi)-\frac{Ya^3}{2R}(\hat{\theta}\cos\theta\sin\phi+\hat{\phi}\cos\phi)+O(\frac{1}{R^2})=$$

$$=-\frac{Ya^3}{R}\left\{\hat{\theta}(1+\frac{1}{2}\cos\theta)\sin\phi+\hat{\phi}(\frac{1}{2}+\cos\theta)\cos\phi\right\}+O(\frac{1}{R^2})\ . \qquad (4.100)$$

In order to determine the corresponding terms of $\vec{e}_2$ we substitute (3.52) in (3.30) to obtain

$$\nabla \times \vec{h}_{n+1} = -Y\,\vec{e}_n - \hat{R} \times \vec{h}_n \;, \quad n = 0, 1, 2, \ldots \tag{4.101}$$

But, by (4.100),

$$\nabla \times \vec{h}_2 = O\left(\frac{1}{R^2}\right) \;. \tag{4.102}$$

Substituting then (4.100) in (4.101) we have that

$$\vec{e}_2(\vec{R}) = \frac{a^3}{R}\left\{\hat{\phi}\,(1+\tfrac{1}{2}\cos\theta)\sin\phi - \hat{\theta}\,(\tfrac{1}{2}+\cos\theta)\cos\phi\right\} + O\left(\frac{1}{R^2}\right) \;. \tag{4.103}$$

To check the results we obtained for $\vec{e}_2$ and $\vec{h}_2$ we substitute them in (3.25):

$$\vec{H}^s(\vec{R}) = e^{ikR}\left\{(ik)^2\,\vec{h}_2(\vec{R}) + O(k^3)\right\} =$$

$$= \frac{Ye^{ikR}}{kR}\left\{(ka)^3\left[\hat{\theta}(1+\tfrac{1}{2}\cos\theta)\sin\phi + \hat{\phi}(\tfrac{1}{2}+\cos\theta)\cos\phi\right] + O(k^4)\right\} + O\left(\frac{1}{R^2}\right) \;; \tag{4.104}$$

similarly,

$$\vec{E}^s(\vec{R}) = e^{ikR}\left\{(ik)^2\,\vec{e}_2(\vec{R}) + O(k^3)\right\} =$$

$$= \frac{e^{ikR}}{kR}\left\{(ka)^3\left[\hat{\theta}(\tfrac{1}{2}+\cos\theta)\cos\phi - \hat{\phi}(1+\tfrac{1}{2}\cos\theta)\sin\phi\right] + O(k^4)\right\} + O\left(\frac{1}{R^2}\right) \;. \tag{4.105}$$

These two results for the scattered fields are in complete agreement with Lord Rayleigh's (1897) results.

V
CONCLUSIONS

In this work we have developed a technique for determining the electromagnetic fields scattered by a perfectly conducting surface in three space when the characteristic dimension of the scatterer is small compared with the wavelength of the excitation fields. Though the method appears to work well there are two significant questions that were left unanswered: first the identification of the dyadic kernels of Chapter I with the dyadic dipole fields of Chapter II; second, the convergence of the volume integrals in the higher order approximations and in the higher order terms of the low frequency expansions for $\vec{e}$ and $\vec{h}$ of Chapter III. These two questions will be part of the work we plan to do in the near future. This work will also involve the question of convergence of the sequence of the iterates when we solve the integral equations for $\vec{e}$ and $\vec{h}$ by the method of successive approximations as well as an example more complicated than the sphere. The most probable candidate is the prolate spheroid which has only two degrees of symmetry. From the results for the prolate spheroid one can also obtain the corresponding results for the sphere, the oblate spheroid and the disc by simple transformations.

REFERENCES

Ar, Ergun and R. E. Kleinman (1966), "The Exterior Neumann Problem for the Three-dimensional Helmholtz Equation," Arch. Rational Mech. Analysis, 23 (218-236).

Kellogg, O. D. (1953), Foundations of Potential Theory, Dover Publications, New York.

Kleinman, R. E. (1965a), "The Rayleigh Region," Proc. IEEE, 53 (848-856).

Kleinman, R. E. (1965b), "Low Frequency Solutions of Three-dimensional Scattering Problems," The University of Michigan Radiation Laboratory Report 7133-4-T, AD 623 852.

Kleinman, R. E. (1965c), "The Dirichlet Problem for the Helmholtz Equation," Arch. Rational Mech., Analysis, 18 (205-229).

Levine H. and J. Schwinger (1950), "On the Theory of Electromagnetic Wave Diffraction by an Aperture in an Infinite Conducting Screen," Comm. Pure and Appl. Math., 3 (355-391).

Lord Rayleigh (1897), "On the Incidence of Aerial and Electric Waves upon Small Obstacles in the Form of Ellipsoids or Elliptic Cylinders and on the Passage of Electric Waves through a Circular Aperture in a Conducting Screen," Philo. Mag., XLIV (28-52).

Magnus, D., F. Oberhettinger and R. P. Soni (1966), Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag, Inc. New York.

Stevenson, A. F. (1953), "Solution of Electromagnetic Scattering Problems as a Power Series in the Ratio (dimension of scatterer)/Wavelength," J.Appl. Phys., 24 (1134-1142).

Stevenson, A. F. (1954), "Note on the Existence and Determination of a Vector Potential," Q. Appl. Math., XII (194-197).

Van Bladel, J. (1964) Electromagnetic Fields, McGraw-Hill Book Company, New York.

Werner, P. (1963), "On the Exterior Boundary Value Problem of Perfect Reflection for Stationary Electromagnetic Wave Fields," J. Math. Anal. and Applic., 7 (348-396).

Wilcox, C. H. (1956), "An Expansion Theorem for Electromagnetic Fields," Comm. Pure and Appl. Math., IX (115-134).

APPENDIX A

THE BEHAVIOR AT INFINITY OF THE DYADICS $\bar{\bar{H}}_e^{(1)}$ AND $\bar{\bar{E}}_m^{(1)}$

In this appendix we prove that the dyadics $\bar{\bar{H}}_e^{(1)}$ and $\bar{\bar{E}}_m^{(1)}$ as defined by Eqs. (2.66) and (2.95), respectively, satisfy the regularity conditions

$$\left| R^2 (R \times \bar{\bar{A}}) \right| < \infty \quad \text{and} \quad \left| R^3 \nabla \times \bar{\bar{A}} \right| < \infty, \text{ as } R \to \infty , \tag{A.1}$$

where $\bar{\bar{A}}$ stands for either $\bar{\bar{H}}_e^{(1)}$ or $\bar{\bar{E}}_m^{(1)}$.

<u>The Regularity of $\bar{\bar{H}}_e^{(1)}$</u>

The expression for $\bar{\bar{H}}_e^{(1)}$ is given by (2.66) which we repeat here for convenience

$$\bar{\bar{H}}_e^{(1)} = \nabla\left(\frac{1}{4\pi\left|\vec{R}-\vec{R}'\right|}\right) \times \bar{\bar{I}} + \bar{\bar{H}}_{e_r}^{(1)p} + \sum_{i=1}^{3} \nabla N_{ei}^{(e)} \hat{a}_i . \tag{A.2}$$

We will now examine the behavior of each of the three terms for large R. First we expand the distance function $1/\left|\vec{R}-\vec{R}'\right|$ in spherical harmonics for $R > R'$:

$$\frac{1}{\left|\vec{R}-\vec{R}'\right|} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') \frac{R'^n}{R^{n+1}} \cos m(\phi-\phi') , \tag{A.3}$$

where ϵ_m is the Neumann factor; $\epsilon_m = 1$, for m=1, $\epsilon_m = 2$ for m = 2, 3, ... The gradient of this expression is

$$\nabla\left(\frac{1}{|\vec{R}-\vec{R}'|}\right) = -\hat{R}\sum_{n=o}^{\infty}\sum_{m=o}^{n}\epsilon_m (n+1)\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\frac{R'^n}{R^{n+2}}\cos m(\phi-\phi') -$$

$$-\hat{\theta}\sin\theta\sum_{n=1}^{\infty}\sum_{m=o}^{n}\epsilon_m\frac{(n-m)!}{(n+m)!}\frac{d}{d\theta}P_n^m(\cos\theta)P_n^m(\cos\theta')\frac{R'^n}{R^{n+2}}\cos m(\phi-\phi') -$$

$$-\hat{\phi}\frac{2}{\sin\theta}\sum_{n=1}^{\infty}\sum_{m=1}^{n} m\,\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\frac{R'^n}{R^{n+2}}\sin m(\phi-\phi') \,. \tag{A. 4}$$

At large distances ($R \rightarrow \infty$) this expression becomes

$$\nabla\left(\frac{1}{|\vec{R}-\vec{R}'|}\right) = -\frac{\hat{R}}{R^2} + O\left(\frac{1}{R^3}\right), \quad R \rightarrow \infty \,. \tag{A. 5}$$

We now turn to the second term in (A. 2) and write it in terms of its vector components:

$$\overline{\overline{H}}_{e_r}^{(1)p} = \sum_{i=1}^{3} \vec{H}_{e_r}^{(1)p}\,\hat{a}_i \,. \tag{A. 6}$$

By (2. 62)

$$\vec{H}_{ei_r}^{(1)p}(\vec{R}|\vec{R}') = \frac{1}{4\pi}\nabla \times \int_S dS\,\frac{\left[\phi_{ei_r}^{(0)}(\vec{R}_s) - N_{ei}^{(i)}(\vec{R}_s)\right]}{|\vec{R}_s-\vec{R}|}\,\hat{n}_s =$$

$$= \frac{1}{4\pi}\int_S dS\left[\phi_{ei_r}^{(0)}(\vec{R}_s) - N_{ei}^{(i)}(\vec{R}_s)\right]\nabla\left(\frac{1}{|\vec{R}_s-\vec{R}|}\right) \times \hat{n}_s . \tag{A. 7}$$

For R large we can employ (A. 5) in this expression to get

$$\vec{H}_{ei_r}^{(1)p} = -\frac{1}{4\pi R^2}\hat{R}\times\int_S \hat{n}_s\left[\phi_{ei_r}^{(0)}(\vec{R}_s) - N_{ei}^{(i)}(\vec{R}_s)\right]dS + O\left(\frac{1}{R^3}\right), \quad R \rightarrow \infty \,. \tag{A. 8}$$

In order to evaluate the last term in (A. 2) for large R we employ a theorem by Kellogg (1953, p. 143) which says that if a function satisfies Laplace's equation then it can be written in a series of spherical harmonics of the form

$$N_{ei}^{(e)} = \sum_{n=o}^{\infty} \frac{Y_{i_n}(\theta, \phi)}{R^{n+1}} \tag{A. 9}$$

where Y_{i_n} is a nth order spherical harmonic:

$$Y_{i_n} = \sum_{m=-n}^{n} A_{i_{mn}} P_n^m (\cos\theta) e^{im\phi} . \tag{A. 10}$$

The series (A. 9) is uniformly and absolutely convergent outside a sphere enclosing all sources. Clearly,

$$\nabla N_{ei}^{(e)} = -\hat{R} \frac{A_{i_{oo}}}{R^2} + O\left(\frac{1}{R^3}\right), \quad R \rightarrow \infty . \tag{A. 11}$$

Collecting our results we have that

$$\bar{\bar{H}}_e^{(1)} = \frac{1}{4\pi R^2} (\hat{R} x \bar{\bar{I}}) - \frac{1}{4\pi R^2} \sum_{i=1}^{3} \left\{ \hat{R} x \int_S \hat{n}_S \left[\phi_{ei_r}^{(0)}(\vec{R}_S) - N_{ei}^{(i)}(\vec{R}_S) \right] dS \right\} \hat{a}_i -$$

$$- \frac{\hat{R}}{R^2} \sum_{i=1}^{3} A_{i_{oo}} \hat{a}_i , \quad R \rightarrow \infty . \tag{A. 12}$$

From this last expression we conclude that

$$\left| R^2 (\hat{R} x \bar{\bar{H}}_e^{(1)}) \right| < \infty , \qquad \text{as } R \rightarrow \infty . \tag{A. 13}$$

In order to prove the second statement in (A. 1) we turn to Eq. (2. 23) and write

$$\nabla \times \overline{\overline{H}}_e^{(1)} = \nabla\nabla\left(-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right) + \nabla \times \overline{\overline{H}}_{e_r}^{(1)}\ , \quad \vec{R} \neq \vec{R}'\ . \tag{A. 14}$$

Splitting the dyadics into their vector components and employing (2. 58) we have that

$$\nabla \times \vec{H}_{ei}^{(1)} = -(\hat{a}_i \cdot \nabla')\ \nabla\left(-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right) + \nabla \phi_{ei_r}^{(0)}\ , \quad i=1, 2, 3\ . \tag{A. 15}$$

We now examine (A. 4) and we see that the n=0 term of the $\hat{R}$-component (which is the only term that behaves as $1/R^2$) is independent of the primed coordinates. The rest of the terms in (A. 4) behave at least as $1/R^3$. We then conclude that

$$(\hat{a}_i \cdot \nabla')\ \nabla\left(-\frac{1}{4\pi|\vec{R}-\vec{R}'|}\right) = O\left(\frac{1}{R^3}\right)\ , \quad R \to \infty\ . \tag{A. 16}$$

To show that the second term of (A. 15) behaves similarly we employ (2. 41):

$$\int_S \hat{n}_s \cdot \nabla_s\, \phi_{ei_r}^{(0)\,\prime}(\vec{R}_s|\vec{R}')dS = 0\ . \tag{A. 17}$$

This condition together with the divergence theorem imply that

$$\lim_{R\to\infty} \int_{S_\infty} \hat{R} \cdot \nabla\phi_{ei_r}^{(0)}\ (\vec{R}|\vec{R}')\ dS = 0 \tag{A. 18}$$

or

$$\lim_{R\to\infty} \int_{S_\infty} d\Omega\, R^2\ \frac{\partial\phi_{ei_r}^{(0)}}{\partial R} = 0\ . \tag{A. 19}$$

But $\phi_{ei_r}^{(0)}$ can be written in the form (A. 9). If condition (A. 19) is to hold, however, the constant $A_{i_{o\,o}}$ in (A. 10) must be zero. We therefore have

$$\phi_{ei_r}^{(0)} = \sum_{n=1}^{\infty} \frac{Y_{i_n}(\theta, \phi)}{R^{n+1}}\ , \tag{A. 20}$$

which implies that

$$\nabla \phi^{(0)}_{ei_r} = O\left(\frac{1}{R^3}\right) , \quad \text{as } R \to \infty \quad . \tag{A.21}$$

From (A. 16) and (A. 21) we can readily conclude that

$$\left| R^3 \, \nabla \times \bar{\bar{H}}^{(1)}_e \right| < \infty , \quad \text{as } R \to \infty . \tag{A.22}$$

The Regularity of $\bar{\bar{E}}^{(1)}_m$

To prove that $\left| R^2 (\hat{R} \times \bar{\bar{E}}^{(1)}_m) \right| < \infty$, as $R \to \infty$, we start with (2. 95), which reads

$$\bar{\bar{E}}^{(1)}_m = \nabla \left(- \frac{1}{4\pi \left| \vec{R} - \vec{R}' \right|} \right) \times \bar{\bar{I}} + \bar{\bar{E}}^{(1)p}_{m_r} + \sum_{i=1}^{3} \nabla G^{(e)}_{mi} \, \hat{a}_i , \tag{A.23}$$

and we proceed in exactly the same manner as we did above for $\bar{\bar{H}}^{(1)}_e$. The behavior of the functions involved being the same as those for $\bar{\bar{H}}^{(1)}_e$, we omit the proof.

To prove that $\left| R^3 \nabla \times \bar{\bar{E}}^{(1)}_m \right| < \infty$, as $R \to \infty$, we start with (2. 68):

$$\nabla \times \bar{\bar{E}}^{(1)}_m = \nabla \nabla \left(- \frac{1}{4\pi \left| \vec{R} - \vec{R}' \right|} \right) + \nabla \times \bar{\bar{E}}^{(1)}_{m_r} , \quad \vec{R} \neq \vec{R}' . \tag{A.24}$$

Employing (2. 88) in this expression we get

$$\nabla \times \bar{\bar{E}}^{(1)}_m = - \nabla \nabla' \left(- \frac{1}{4\pi \left| \vec{R} - \vec{R}' \right|} \right) - \nabla \nabla' N^{(e)}_r (\vec{R} | \vec{R}'), \quad \vec{R} \neq \vec{R}' , \tag{A.25}$$

or, in terms of the vector component of the dyadics,

$$\nabla \times \bar{E}^{(1)}_{mi} = - (\hat{a}_i \cdot \nabla') \nabla \left(- \frac{1}{4\pi \left| \vec{R} - \vec{R}' \right|} \right) - (\hat{a}_i \cdot \nabla') N^{(e)}_r (\vec{R} | \vec{R}') , \quad i = 1, 2, 3 . \tag{A.26}$$

The function $N^{(e)}_r$, however, is the regular part of the exterior Neumann Green's function for the surface and it, therefore, has the property that

$$\int_S \hat{n}_s \cdot \nabla_s N_r^{(e)} dS = 0 . \qquad \text{(A. 27)}$$

Transforming this integral by the divergence theorem to an integral over a surface at infinity and writing $N_r^{(e)}$ in a series of spherical harmonics we conclude that

$$N_r^{(e)} = O\left(\frac{1}{R^2}\right), \; \nabla N_r^{(e)} = O\left(\frac{1}{R^3}\right), \text{ as } R \rightarrow \infty . \qquad \text{(A. 28)}$$

This result together with (A. 16) leads us to the conclusion that

$$|R^3 \nabla x \, \bar{\bar{E}}_m^{(1)}| < \infty , \text{ as } R \rightarrow \infty . \qquad \text{(A. 29)}$$