THE SCATTERING OF ELECTROMAGNETIC WAVES BY MOVING BODIES

by

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This study considers an analysis of the transient behavior of the total far zone field when an electromagnetic pulse is incident upon a uniformly moving, perfectly conducting wedge. The solution for the total field is obtained using the concepts of the Special Theory of Relativity. An integral solution for the total field in the primed or moving frame is obtained as a contour integral along the well known Sommerfeld contours by means of Laplace transformation techniques. This result is then transformed to the reference frame of a stationary observer by means of the Lorentz Transformations. The total field is determined for three different incident fields; the time harmonic plane wave, the unit step plane pulse and the cylindrical impulse.

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF APPENDICES

Chapter I

INTRODUCTION

The scattering of electromagnetic waves by moving bodies has received some attention by researchers; however, little is known, and to the author's knowledge, nothing has been written concerning transient problems in relativistic electrodynamics prior to this thesis. Sommerfeld *Optics* (1964) was the first to consider the scattering of a time harmonic plane wave by a uniformly moving object when he examined the infinite flat plate or infinite sheet. Cole (1958) considered a similar problem. Not until 1967 when Lee and Mittra (1967) considered the infinite right circular cylinder and Restrick (1967) considered the sphere was any other work published on scattering by a moving object. Just two months ago Ott and Hufford (1968) presented their work on the scattering of a time harmonic spherical wave by an arbitrarily shaped moving object and as recently as last month Tsandoulas (1968) published his work on the edge diffraction field when a plane wave is scattered by a moving wedge. There are two papers in the literature dealing with aperiodic solutions of the wave equation for moving surfaces but both of these are concerned with the acoustic problem. In 1930, Morgans developed Kirchhoff's integral solution of the wave equation for moving surfaces. The application of this technique is hampered by the necessity of knowing the surface fields. In 1964, Lewis presented the Progressive Wave Formalism which leads to an asymptotic series solution of the wave equation. Lewis' work is concerned only with problems involving reflection and refraction. It should be noted here that only Lewis (1964) and Restrick (1967) deal with accelerating objects.

Interest in studying the class of problems defined as electromagnetic scattering by moving bodies has come about as a result of the interest in radar detection and the fact that man-made objects are now capable of much higher velocities. Current radar analysis is made on the basis of stationary objects so the question naturally arises as to what exactly is the effect of a moving object. It has been assumed in the past that the effect for any realizable velocity is negligible.

In this thesis some simple problems involving moving wedges, for which a rigorous solution may be obtained, are investigated in some detail with a view to determining the effect of the velocity.

The problem considered in this research effort concerns finding the total far zone field when a known incident field interacts with a uniformly moving, perfectly conducting wedge. Three incident fields are considered; the plane time harmonic wave, the plane unit-step pulse and the cylindrical impulse.

Chapter Two includes an explicit statement of the problem along with the development of an integral representation for the total field $E_z^T(\mathcal{R}, \Phi, t)$ as seen by a stationary observer. In Chapter Three the time harmonic plane wave is considered and the resulting expression for the total field is given by equation (3.17). For the special case of a half plane the total field may be expressed in closed form in terms of Fresnel Integrals as given in equation (3.20). These results reduce, as they should, to known results for the stationary wedge (3.21) and half plane (3.23) when the velocity is set equal to zero. Sommerfeld's result for the infinite flat plate is also obtained. In Chapter Four, the plane unit-step pulse is considered and the resulting expression for the total field is given by equation (4.19). This is a closed form result which also reduces to known solutions, i.e. the stationary half plane (4.28), when the velocity is set equal to zero. This result, (4.19), is also obtainable in the case of the infinite flat plate as the inverse Laplace transform of the time harmonic solution (3.17) multiplied by the spectral function of the unit step pulse. In Chapter Five the cylindrical impulse is considered and the resulting expression for the total field is given by the sum of equations (5.12) and (5.32). For the special case of the half plane the result may be written in closed form as the sum of equations (5.12) and (5.33). As expected, these results agree with the known results for the respective stationary scattering problems when the velocity is set equal to zero. Chapter Six discusses the salient features of the results which are incorporated at the end of each chapter and lists some recommendations for future work in the area of transient scattering by moving objects.

Chapter II

TRANSIENT RELATIVISTIC ELECTRODYNAMICS IN THE CASE OF UNIFORM MOTION

2.1 Introduction

The determination of the scattered electromagnetic field in cases where the scattering object experiences a uniform motion can be accomplished using the concepts of the Special Theory of Relativity. The concept of the invariance in form of the equations of electrodynamics, when subject to the Lorentz transformations relating reference frames moving with uniform velocities, allows the solution of these equations in whichever reference frame is most convenient. Appendix A gives a brief mathematical summary of these invariant forms and the transformations which relate the respective reference frames. The most convenient reference frame in which to solve the electrodynamic equations is the one which is fixed with respect to the scattering object since then the problem reduces to that of scattering by a stationary object. Following the convention of the Special Theory of Relativity the laboratory or stationary reference frame will be referred to as the unprimed frame while the reference frame fixed with respect to the scatterer will be referred to as the primed frame. The analysis procedure which will be followed is to obtain an expression for the incident field in the primed frame, then to solve the electrodynamic equations for the scattered field in the primed frame and finally to transform this expression back to a retarded reference position in the unprimed frame. It is the purpose of this chapter to explicitly define the problem and then to follow the stated procedure arriving finally at a contour integral representation for the scattered field as seen by a stationary observer in the unprimed reference frame.

Two features of the problem and its solution are worth noting at this point. First; the incident fields in the stationary frame are so chosen that they may be written in terms of a variable which is invariant under the Lorentz transformations. Thus the shape of the incident pulse fronts are retained in the primed frame. Second; when the primed frame result is transformed back to the unprimed frame it is expressed in terms of the retarded reference position described in Appendix B. This latter transformation has the advantage of presenting the result in a form which is more easily interpreted in a physical sense.

2.2 Statement of the Problem

The scattering object studied in this work is a perfectly conducting, infinite wedge which has a interior angle $2\Omega'$ and which is oriented such that the edge coincides with the z' axis. The velocity of the wedge in the laboratory frame is $\bar{v}$. At the moment $t = t' = 0$ the coordinate axis of the primed and unprimed frames coincide and the incident pulse front makes its initial contact with the wedge. For each incident field considered the direction of propagation will be assumed normal to the edge thus, the problem becomes two dimensional in space. The situation at $t = t' = 0$ is shown in Fig. 2-1.

The scattered field will be determined for three different incident fields and in each case the electric field vector will be linearly polarized in the $\hat{z}$ direction. The incident fields are:

1) A Time-Harmonic plane wave

$$\bar{E}^i(R, \phi, t) = E_o e^{-ik\left[ct - R\cos(\phi - \phi_o)\right]}\hat{z} \tag{2.1}$$

where

$$k = \frac{\omega}{c} \, .$$

2) A Unit step plane pulse

$$\bar{E}^i(R, \phi, t) = H\left[ct - R\cos(\phi - \phi_o)\right]\hat{z} \tag{2.2}$$

In both of these cases ϕ_o defines the direction of propagation with respect to the positive $\hat{x}$ axis.

3) A Cylindrical impulse

$$\bar{E}^i(R, \phi, t) = \frac{H\left[c^2\left(t + \frac{R_o}{c}\right)^2 - \mathbb{R}^2\right]}{\sqrt{c^2\left(t + \frac{R_o}{c}\right)^2 - \mathbb{R}^2}} z \tag{2.3}$$

where

$$\mathbb{R}^2 = R^2 + R_o^2 - 2RR_o\cos(\phi - \phi_o) \, . \tag{2.4}$$

The cylindrical impulse is the field radiated from a line source located at (R_o, ϕ_o) which has been excited by an impulse of current $\delta(t + R_o/c)$.

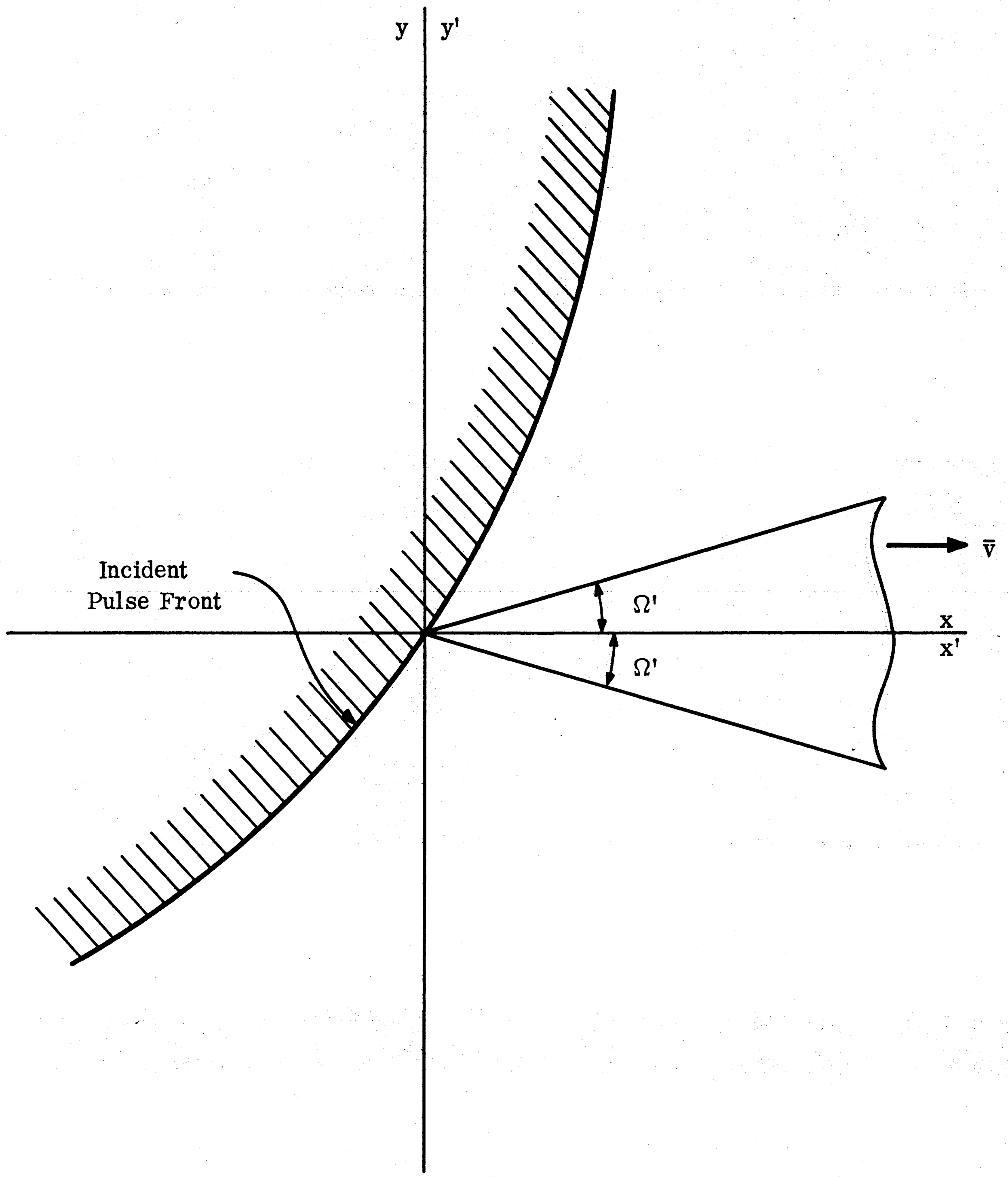


FIG. 2-1: PROBLEM GEOMETRY AT $t = 0$.

2.3 An Integral Representation for the Total Field

A contour integral representation for the total field (incident pluse scattered fields) seen by an observer in the laboratory frame will now be developed. This particular result will be applicable only to the Sommerfeld class (half plane, wedge, infinite sheet) of scatterers, however, the approach may certainly be applied to other scattering objects.

Once the incident field in the unprimed frame is specified $\bar{E}^{i}(R, \phi, t)$ the incident field in the primed frame $\bar{E}^{i'}(R', \phi', t')$ may be found using (A.36)

$$\bar{E}' = \gamma\left[\bar{E} + \bar{v} \times \bar{B}\right]_{\perp} \tag{A.36}$$

where signifies the component of the field perpendicular to the velocity vector, the only field component present in the problem being considered. It will be assumed that only the far zone field is being considered so that if the unit vector $\hat{p}$ represents the direction of propagation of the wave the relationship between $\bar{E}$ and $\bar{B}$ in free space becomes:

$$\bar{B} = c^{-1}(\hat{p} \times \bar{E}) \tag{2.5}$$

Substitution of this into (A.36) leads to the expression

$$\bar{E}'_{\perp} = \gamma\left[1 - \beta(\hat{x} \cdot \hat{p})\right] \bar{E}_{\perp} \tag{2.6}$$

where

$$\beta = \frac{v}{c}, \qquad \gamma = (1 - \beta^2)^{-1/2} .$$

The right hand side of this expression is still in terms of (x, y, z, t) or (R, ϕ, t) and must be transformed to primed variables by means of the Lorentz transformation (B.1) to (B.3).

Note here that for plane waves and pulses the term $\left| ct - R\cos(\phi - \phi_0)\right.$ transforms by means of the Lorentz transformations into $\gamma(1 - \beta\cos\phi_0)\left[ct' - R'\cos(\phi' - \phi_0)\right]$ in the primed frame so that a plane wave transforms to a plane wave and a plane pulse. See Section 1 in Chapter III and Chapter IV respectively.

Note also that for the cylindrical waves the term $c^2t^2 - \mathbb{R}^2 = c^2t^2 - x^2 - y^2 - z^2$ is invariant when subject to the Lorentz transformations. The invariance of this quantity is, in fact, one of the basic properties upon which the Lorentz transformations are based. This means that the cylindrical impulse in the transforms to a cylindrical impulse in the primed frame.

With $\bar{E}^{i'}(R', \phi', t')$ known the scattered field $\bar{E}^{sc'}(R', \phi', t')$ will be sought which satisfies the homogeneous wave equation

$$\nabla'^2 \bar{E}^{sc'}(R', \phi', t') - \frac{1}{c^2} \frac{\partial^2 \bar{E}^{sc'}(R', \phi', t')}{\partial t'} = 0 \tag{2.7}$$

the boundary conditions

$$\hat{n} \times (\bar{E}^{sc'} + \bar{E}^{i'})\Big|_{Bdry} = 0 \tag{2.8}$$

the initial condition

$$\bar{E}^{sc'}\Big|_{t' = 0} = 0 \tag{2.9}$$

as well as the physical radiation and edge conditions. The radiation condition assures that the wave is propagating and, in cases where the sources are confined to finite volumes, diminishing in energy density. The edge condition assures that the energy density at the edge of the wedge is finite. The Dirichlet boundary condition was considered here since we are considering a perfectly conducting body.

The solution for $\bar{E}^{sc'}$ can be found using the Laplace transform. This integral transform is defined as:

$$\mathcal{F}(s) = \int_0^\infty F(t)\, e^{-st}\, dt \tag{2.10}$$

$$F(s) = \frac{1}{2\pi i} \int_{\Delta - i\infty}^{\Delta + i\infty} \mathcal{F}(s)\, e^{st}\, ds \tag{2.11}$$

where Δ is chosen such that $\mathcal{F}(s)$ has no poles to the right of the contour in the complex s plane. With this transform and making use of the initial condition (2.9) Eqs. (.7) and (2.8) become

$$\nabla'^2 \bar{\mathcal{E}}^{sc'}(R', \phi', s) - \frac{s^2}{c^2} \bar{\mathcal{E}}^{sc'}(R', \phi', s) = 0 \tag{2.12}$$

and

$$\hat{n} \times (\bar{\mathcal{E}}^{sc'} + \bar{\mathcal{E}}^{i'})\Big|_{Bdry} = 0 \tag{2.13}$$

$\mathcal{E}^{sc'}(R', \phi', s)$ must also satisfy the physical restrictions corresponding to the edge and radiation conditions.

In the case of time harmonic plane wave incidence $\bar{\mathcal{E}}^{i'}(R', \phi', s)$ becomes:

$$\bar{\mathcal{E}}^{i'}(R', \phi', s) = E_0^{i'} e^{-\frac{sR'}{c}\cos(\phi' - \phi_0')} \delta(s + i\omega') \hat{z} \tag{2.14}$$

where R', ϕ' and ϕ_0' are related to R, ϕ and ϕ_0 by the expressions given in Appendix B. Customarily the function $\delta(s + i\omega)$ is not carried along when discussing the time harmonic problem. The solution for $\bar{\mathcal{E}}^{sc'}(R', \phi', s)$ which satisfies the above stated conditions is, in this time harmonic case, given by: [Tuzhilin (1963)]

$$\bar{\mathcal{E}}^{sc'}(R', \phi', s) = E_0^i \hat{z} \frac{\delta(s+i\omega')}{4\pi\nu' i} \int_{C_1+C_2} e^{-\frac{sR'}{c}\cos\alpha} S_p(\phi', \alpha)\, d\alpha - \bar{\mathcal{E}}^{i'}(R', \phi', s) \tag{2.15}$$

where

$$\nu'\pi = 2\pi - 2\Omega' \tag{2.16}$$

and where

$$S_p(\phi', \alpha) = \cot\left[\frac{2\pi - \alpha - \phi' + \phi_0'}{2\nu'}\right] - \cot\left[\frac{2\Omega' - \alpha - \phi' - \phi_0'}{2\nu'}\right] \tag{2.17}$$

with the subscript p indicating plane waves incident. The contours $C_1 + C_2$ are the Sommerfeld contours shown in Fig. 2-2.

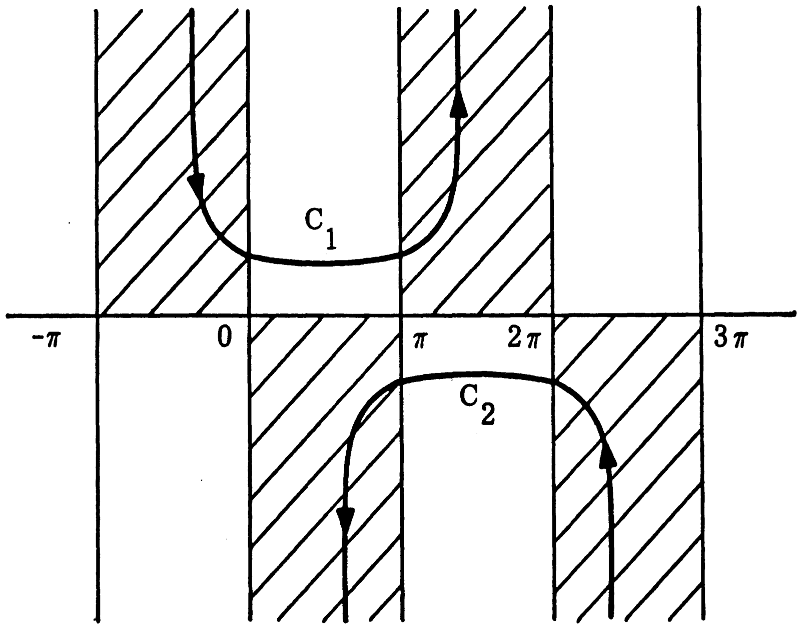


FIG. 2-2: THE SOMMERFELD CONTOURS

If a plane pulse having arbitrary time dependence is being considered, the expression for $\bar{\mathcal{E}}^{i'}$ is given by

$$\bar{\mathcal{E}}^{i'}(R', \phi', s) = e^{-\frac{sR'}{c}\cos(\phi' - \phi'_o)}\, \mathcal{F}(s)\, \hat{z} \tag{2.18}$$

and the corresponding expression for the scattered field becomes:

$$\bar{\mathcal{E}}^{sc'}(R', \phi', s) = \frac{\mathcal{F}(s)}{4\pi\nu' i}\, \hat{z} \int_{C_1 + C_2} e^{-\frac{sR'}{c}\cos\alpha}\, S_p(\phi', \alpha)\, d\alpha - \bar{\mathcal{E}}^{i'}(R', \phi', s) \; . \tag{2.19}$$

The solution for $E^{sc'}(R', \phi', t')$ is given by the inverse Laplace transform of (2.19).

$$\bar{E}^{sc'}(R',\phi',t') = \frac{1}{2\pi i}\int_{\Delta - i\infty}^{\Delta + i\infty} \left[\hat{z}\,\frac{\mathcal{F}(s)}{4\pi\nu' i}\int_{C_1+C_2} e^{-\frac{sR'}{c}\cos\alpha}\, S_p(\phi',\alpha)\, d\alpha \right] e^{st'}\, ds$$

$$- \frac{1}{2\pi i}\int_{\Delta - i\infty}^{\Delta + i\infty} \bar{\mathcal{E}}^{i'}(R', \phi', s)\, e^{st'}\, ds \tag{2.20}$$

Interchanging the order of integration on the first integral and perfomming the second integration obtain:

$$\bar{E}^{sc'}(R',\phi',t') = \frac{1}{4\pi\nu' i}\int_{C_1+C_2} \hat{z}\left[\frac{1}{2\pi i}\int_{\Delta - i\infty}^{\Delta + i\infty} \mathcal{F}(s)\, e^{s(t' - \frac{R}{c}\cos\alpha)}\, ds\right] S_p(\phi',\alpha)\, d\alpha$$

$$- \bar{E}^{i'}(R', \phi', t') \tag{2.21}$$

where the term in the square brackets is recognized as $\bar{E}^{i'}(R', \alpha + \phi'_o, t')$. The solution for the scattered field in the primed frame due to a plane pulse with arbitrary time dependence can thus be written:

$$\bar{E}^{sc'}(R', \phi', t') = \frac{1}{4\pi\nu' i}\int_{C_1+C_2} \bar{E}^{i'}(R', \alpha + \phi'_o, t')\, S_p(\phi',\alpha)\, d\alpha - \bar{E}^{i'}(R',\phi',t') \tag{2.22}$$

A similar argument can be made for cylindrical waves. In this case the time harmonic incident field transforms to:

$$\bar{\mathcal{E}}^{i'}(R', \phi', s) = H_o^{(1)}\left[\frac{is\mathbb{R}}{c}\right] \delta\,(s + i\omega)\,\hat{z} \tag{2.23}$$

and the expression for the scattered field becomes [Tuzhilin (1963)]:

$$\bar{\mathcal{E}}^{sc'}(R', \phi', s) = \frac{\delta(s + i\omega)}{4\pi\nu' i} \int_{C_1 + C_2} \hat{z} H_0^{(1)} \left[\frac{i s \mathbb{R}_1}{c}\right] S_c(\phi', \alpha)\, d\alpha - \bar{\mathcal{E}}^{i'}(R', \phi', s) \tag{2.24}$$

where

$$\mathbb{R}_1^2(\alpha) = R^2 + R_o^2 + 2RR_o \cos\alpha \tag{2.25}$$

and where

$$S_c(\phi', \alpha) = \cot\left[\frac{\pi - \alpha - \phi' + \phi_o'}{2\nu'}\right] - \cot\left[\frac{\pi - \alpha - \phi' - \phi_o' + 2\Omega'}{2\nu'}\right] \tag{2.26}$$

with the subscript c indicating cylindrical waves incident.

For the arbitrary time dependent cylindrical pulse write the Laplace transformed incident pulse as:

$$\bar{\mathcal{E}}^{i'}(R', \phi', s) = H_o^{(1)} \left[\frac{i s \mathbb{R}}{c}\right] \mathcal{F}(s)\, \hat{z} \tag{2.27}$$

then the corresponding scattered field can be written:

$$\bar{\mathcal{E}}^{sc'}(R', \phi', s) = \frac{\hat{z}\mathcal{F}(s)}{4\pi\nu' i} \int_{C_1 + C_2} H_o^{(1)} \left[\frac{i s \mathbb{R}_1(\alpha)}{c}\right] S_c(\phi', \alpha)\, d\alpha - \bar{\mathcal{E}}^{i'}(R', \phi', s) \tag{2.28}$$

and the time dependent scattered field becomes; after changing the order of integration:

$$\bar{E}^{sc'}(R', \phi', t') - \frac{1}{4\pi\nu' i} \int_{C_1 + C_2} \left[\frac{\hat{z}}{2\pi i} \int_{\Delta - i\infty}^{\Delta + i\infty} \mathcal{F}(s) H_o^{(1)} \left[\frac{i s \mathbb{R}_1(\alpha)}{c}\right] e^{st'} ds\right] S_c(\phi', \alpha)\, d\alpha$$

$$- \bar{E}^{i'}(R', \phi', t') \tag{2.29}$$

Now consider the interior integration. Comparing the integrand with Eq. (2.27) the difference is that $\mathbb{R}_1$ is present instead of $\mathbb{R}$. Since $\bar{E}^{i'}$ is a function of the variable $\mathbb{R}$ it may also be written in terms of $\mathbb{R}_1$ and $\overline{E}^{i'}$ when such a substitution is made will be designated as $\overline{E}_1^{i'}$. With this notation the result of the interior integration is $\overline{E}_1^{i'}(R', \alpha + \phi'_o, t')$. For cylindrical pulses incident on the wedge the scattered field is given by:

$$\overline{E}^{sc'}(R', \phi', t') = \frac{1}{4\pi\nu' i}\int_{C_1+C_2} \overline{E}_1^{i'}(R', \alpha+\phi'_o, t')\, S_c(\phi', \alpha)\, d\alpha - \overline{E}^{i'}(R', \phi', t') \quad . \tag{2.30}$$

In both representations (2.22) and (2.30) for the scattered field the incident field may be combined with the scattered field to give an integral representation for the total field. Equation (2.22) becomes:

$$\overline{E}^{T'} = \overline{E}^{i'} + \overline{E}^{sc'} = \frac{1}{4\pi\nu' i}\int_{C_1+C_2} \overline{E}^{i'}(R', \alpha+\phi'_o, t')\, S_p(\phi', \alpha)\, d\alpha \tag{2.31}$$

A similar expression for incident cylindrical waves involves $\overline{E}_1^{i'}$ and S_c.

The final step in the development of a contour integral representing the field scattered by a uniformly moving wedge is to transform (2.31) back to the unprimed frame. The expression for $\overline{E}^{T'}$ can be transformed to the unprimed frame using the transformation (A.36). Making use of the far zone field relationship (2.5) the far zone field in the unprimed frame may be written as:

$$\overline{E}_\perp^{T} = \gamma\left[1 + \beta(\hat{x}'\cdot\hat{p}')\right]\overline{E}_\perp^{T'} \tag{2.32}$$

where $\hat{p}'$ is the direction of propagation of the field. $E^{T'}$ will be resolved into three components each of which propagate in different directions, so that the expression (2.32) is understood to mean:

$$\overline{E}_\perp^{T} = \gamma\left[1 + \beta(\hat{x}'\cdot\hat{p}'_i)\right]\overline{E}_\perp^{i'} + \gamma\left[1 + \beta(\hat{x}'\cdot\hat{p}'_r)\right]\overline{E}_\perp^{r'} + \gamma\left[1 + \beta(\hat{x}'\cdot\hat{p}'_d)\right]\overline{E}_\perp^{d'} \tag{2.33}$$

where i, r, d stand for incident, reflected and diffracted fields respectively. The scattered field is the sum $\bar{E}^{sc'} = \bar{E}^{r'} + \bar{E}^{d'}$. Since the only electric field component present in the problems being considered is perpendicular to the velocity we may write $\bar{E}^T = E_z^T$.

In transforming a field quantity in the primed frame, $E_z^{T'}(R', \phi', t')$ to the variables (R, ϕ, t), i.e. $E_{z2}^T(R, \phi, t)$ in the unprimed frame experience has shown that the result is unnecessarily complex. In order to circumvent this difficulty this research makes use of the concept of a retarded position for the scatterer. That is, the position from which the scattered field appears to come at that moment. Doing this results in a change of variables by which the field being observed at a fixed position and a given time is described, i.e.

$$E_z^T(\mathcal{R}, \Phi, t) = E_z^T[\mathcal{R}(R, \phi, t), \ \Phi(R, \phi, t), \ t] = E_{z2}^T(R, \phi, t) \ .$$

The change of variables is given by equations (B.7), (B.8) and (B.9) in Appendix B. Our experience shows that $E_z^T(\mathcal{R}, \Phi, t)$ is a simpler form to use in making physical interpretations than the form $E_{z2}^T(R, \phi, t)$. For a further discussion of the reference frames the reader is referred to Appendix B.

Substituting (2.31) in (2.32) and making the change of variables from (R', ϕ', t') to $(\mathcal{R}, \Phi, t)$ defined by Eqs. (B.17) to (B.19) the expression for the total field becomes:

$$E_z^T(\mathcal{R}, \Phi, t) = \frac{\gamma[1 + \beta(\hat{x}' \cdot \hat{p}')^{\sim}]}{4\pi\nu' i} \int_{C_1 + C_2} \tilde{E}_z^i(\mathcal{R}, \Phi, t; \alpha)\, \tilde{S}_p(\Phi, \alpha)\, d\alpha \ . \tag{2.34}$$

The functions denoted by the tilda, $\tilde{E}_z^i$ and $\tilde{S}_p$ represent E_z^i and S_p respectively with the above mentioned change of variables. When dealing with an incident cylindrical pulse the functions under the integral are $\tilde{E}_{z1}^i$ and $\tilde{S}_c$ respectively.

In order to accomplish the transformation of $S_p(\phi', \alpha)$ [and $S_c(\phi', \alpha)$] it is necessary to decompose the cotangent functions and obtain $S_p(\phi', \alpha)$ in terms of $\sin\phi'$, $\cos\phi'$, etc. which can then be transformed. Using the identity

$$\cot a - \cot b = -\frac{\sin(a-b)}{\sin a \sin b} \tag{2.35}$$

along with the identities for the sine or cosine of the sum (difference) of two angles, the expression (2.16), and transformation (B.16) the expression for $S_p(\phi', \alpha)$ becomes $\tilde{S}_p(\Phi, \alpha)$ where:

$$\tilde{S}_p(\Phi, \alpha) = \frac{D_p(\Phi)}{A(\Phi)\cos\frac{\alpha}{\nu'} + B(\Phi)\sin\frac{\alpha}{\nu'} + C_p(\Phi)} \tag{2.36}$$

with

$$A(\Phi) = A_1(\Phi)\sin\frac{\pi}{\nu'} + B_1(\Phi)\cos\frac{\pi}{\nu'} \tag{2.37}$$

$$B(\Phi) = -A_1(\Phi)\cos\frac{\pi}{\nu'} + B_1(\Phi)\sin\frac{\pi}{\nu'} \tag{2.38}$$

since ν, Ω and ϕ_o are constant:

$$A_1(\Phi) = \gamma\left[\sin\left(\frac{\Omega-\Phi}{\nu}\right) - \beta\left(\sin\frac{\Omega}{\nu} - \sin\frac{\Phi}{\nu}\right)\right]\left(1 - \beta\cos\frac{\phi_o}{\nu}\right) \tag{2.39}$$

$$B_1(\Phi) = -\left[\gamma^2\left(\cos\frac{\Phi}{\nu} - \beta\right)\left(\cos\frac{\Omega}{\nu} - \beta\right) + \sin\frac{\Phi}{\nu}\sin\frac{\Omega}{\nu}\right]\left(1 - \beta\cos\frac{\phi_o}{\nu}\right) \tag{2.40}$$

$$C_p(\Phi) = -\gamma\sin\frac{\phi_o}{\nu}\left(1 - \beta\cos\frac{\Phi}{\nu}\right)\left(1 - \beta\cos\frac{\Omega}{\nu}\right) \tag{2.41}$$

$$D_p(\Phi) = -2\gamma^2(\cos\frac{\phi_o}{\nu} - \beta)(1-\beta\cos\frac{\Phi}{\nu})(1-\beta\cos\frac{\Omega}{\nu}) \quad . \tag{2.42}$$

The expression for $S_c(\Phi,\alpha)$ is identical to (2.36) except that $C_p(\Phi)$ and $D_p(\Phi)$ now become $C_c(\Phi)$ and $D_c(\Phi)$ where:

$$C_c(\Phi) = \left[\gamma^2(\cos\frac{\phi_o}{\nu} - \beta)(\cos\frac{\Omega}{\nu} - \beta) + \sin\frac{\phi_o}{\nu}\sin\frac{\Omega}{\nu}\right](1-\beta\cos\frac{\Phi}{\nu}) \tag{2.43}$$

$$D_c(\Phi) = -2\gamma\left[\sin(\frac{\phi_o - \Omega}{\nu}) - \beta(\sin\frac{\phi_o}{\nu} - \sin\frac{\Omega}{\nu})\right](1-\beta\cos\frac{\Phi}{\nu}) \quad . \tag{2.44}$$

The integration variable α and the contours $C_1 + C_2$ are independent of any variables which require transformation and are therefore the same in both the primed and unprimed reference frames.

The contour integral expression for the total far zone field in the vicinity of a uniformly moving, perfectly conducting, infinite wedge is thus:

$$E_z^T(\mathcal{R},\Phi,t) = \frac{\gamma\left[1+\beta(x'\cdot p')^{\sim}\right]}{4\pi\nu' i}\int_{C_1+C_2} \tilde{E}_z^i(\mathcal{R},\Phi,t;\alpha)\,\tilde{S}(\Phi,\alpha)\,d\alpha \tag{2.45}$$

where

$$\tilde{S}(\Phi,\alpha) = \frac{D(\Phi)}{A(\Phi)\cos\frac{\alpha}{\nu'} + B(\Phi)\sin\frac{\alpha}{\nu'} + C(\Phi)}$$

with the proper C, D being chosen depending upon whether the incident field is a plane or cylindrical wave.

The determination of the appropriate $\tilde{E}_z^i$ and the evaluation of the far zone scattered field for the several different incident fields are carried out in the following chapters.

Chapter III

SCATTERING OF A TIME-HARMONIC PLANE WAVE

3.1 The Determination of $\tilde{E}_z^i$

The time-harmonic plane wave is represented in the unprimed frame by expression (2.1)

$$E_z^i(R,\phi,t) = E_o e^{ikR\cos(\phi-\phi_o) - ikct} \quad . \tag{2.1}$$

The incident field seen by an observer stationary with respect to the wedge is found by transforming (2.1) to the primed reference frame using Eq. (2.6) and (B.1) to (B.3). This becomes

$$E_z^{i'}(R',\phi',t') = \gamma(1-\beta\cos\phi_o) E_o e^{ik'R'\cos(\phi'-\phi'_o) - ik'ct'} \tag{3.1}$$

where

$$k' = \gamma(1-\beta\cos\phi_o)k \tag{3.2}$$

and

$$\cos\phi'_o = \frac{\cos\phi_o - \beta}{1-\beta\cos\phi_o} \quad . \tag{3.3}$$

The expression required in the integral (2.22) follows directly from (3.1) when the substitution $\phi' - \phi'_o = \alpha$ is made.

$$E_z^{i'}(R', \alpha+\phi'_o, t') = \gamma(1-\beta\cos\phi_o) E_o e^{ik'R'\cos\alpha - ik'ct'} \tag{3.4}$$

The desired $\tilde{E}_z^i$ can then be found by making the change of variables defined by (B.9) and (B.17) to (B.19). The expression for $\tilde{E}_z^i$ in this case is:

$$\tilde{E}^i_z(\mathcal{R}, \Phi, t ; \alpha) = \gamma(1 - \beta \cos \phi_o)\, E_o\, e^{-ik(a - b\cos\alpha)} \tag{3.5}$$

where

$$a = \left[ct - \beta\gamma^2 \mathcal{R}(\cos\Phi - \beta)\right](1 - \beta\cos\phi_o) \tag{3.6}$$

and

$$b = \gamma^2 \mathcal{R}(1 - \beta\cos\Phi)(1 - \beta\cos\phi_o) \tag{3.7}$$

3.2 <u>The Integral for the Total Field</u>

The contour integral representing the total far zone field in the vicinity of the wedge when the incident field is a time-harmonic plane wave is obtained by substituting Eq. (3.5) into the integral expression (2.45). The result is:

$$E^T_z(\mathcal{R}, \Phi, t) = K \int_{C_1 + C_2} \frac{e^{-ik[a - b\cos\alpha]}}{A\cos\frac{\alpha}{\nu'} + B\sin\frac{\alpha}{\nu'} + C}\, d\alpha \tag{3.8}$$

where

$$K = \frac{\gamma^2\left[1 - \beta\cos\phi_o\right]\left[1 + \beta(\hat{x}'\cdot\hat{p}')\right]}{4\pi\nu' i}\, D(\Phi)\, E_o \quad . \tag{3.9}$$

As a first step in the evaluation of this integral consider the poles of the integrand.

3.3 <u>The Poles in the Complex α Plane</u>

The poles in the complex α plane are located at the roots of $A\cos\alpha/\nu' + B\sin\alpha/\nu' + C$. These roots can be found using the quadratic formula and are given by:

$$\cos\frac{\alpha_p}{\nu'} = \frac{-AC \pm B\sqrt{A^2 + B^2 - C^2}}{A^2 + B^2} \tag{3.10}$$

where α_p indicates the pole location. An examination of (3.10) reveals that the periodicity of the roots is $2\pi\nu$. Within one such period two roots exist which can be shown to lie on the real axis. To see that these poles do, in fact, lie on the real axis note that the functions defined as A, B, and C in Eqs. (2.37), (2.38) and (2.41) (or (2.43)) are all real. The quantity $A^2 + B^2 - C^2$ is always positive and this for either function C as is shown in Appendix D. That is to say $\cos \alpha/\nu' =$ [a real number]. Further, straightforward algebra reveals that

$$-1 < \frac{-AC \pm B\sqrt{A^2 + B^2 - C^2}}{A^2 + B^2} < 1 \tag{3.11}$$

from which it can be concluded that α_p/ν' lies in the interval $[0, 2\pi]$ on the real axis. The location of these poles on the real axis is a function of several parameters: Φ, ϕ_0, Ω and β. Changing any one of these parameters while keeping the others fixed will shift the poles along the real axis. The physical significance of the poles is discussed in association with the evaluation of the integral (3.8).

3.4 Evaluation of the Integral Representing E_z^T

The evaluation of the integral (3.8) representing the total field $E_z^T(\mathcal{R}, \Phi, t)$ will now be considered. Once the locations of the singularities of the integrand have been established the contours of Fig. 2-2 may be continuously deformed into the contours of Fig. 3-1. Depending upon the values of Φ, ϕ_0, Ω, $(\Omega \rightarrow \nu)$ and β the poles indicated in Fig. 3-1 may or may not be present in the interval $[0, 2\pi]$ on the real axis. For purposes of calculation at this point both poles will be assumed to be present in the interval $[0, 2\pi]$. Note also that the integrand has saddle points located at $\alpha = n\pi + i0$ for $n = 0, 1, 2,$ etc. The only saddle points of interest are those at $\alpha = 0 + i0$ and $\alpha = 2\pi + i0$ through which the contours D_1 and D_2 pass.

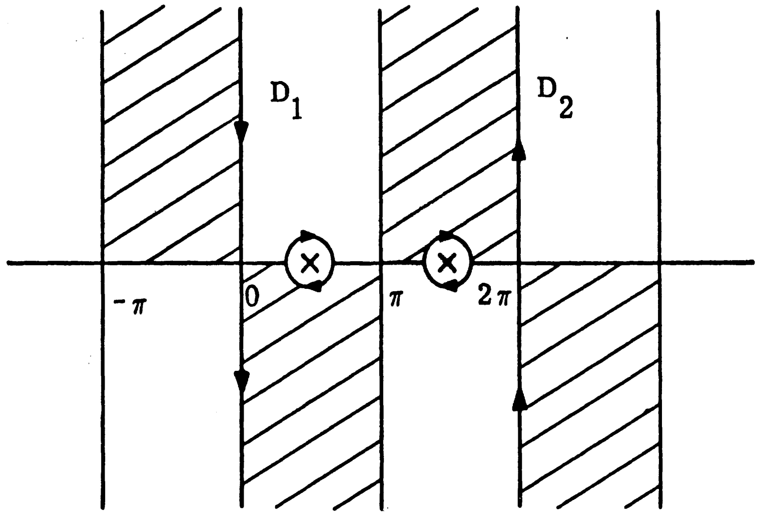


FIG. 3-1: CONTOUR DEFORMATION FOR TIME HARMONIC PLANE WAVE

The contributions due to the integration around the poles are computed using Cauchy's Residue Theorem. The residues at the poles are given by:

$$\text{Residue} = \frac{\pm \nu' e^{-ik\left[a - b\cos\alpha_p\right]}}{\sqrt{A^2 + B^2 - C^2}} \tag{3.12}$$

so that the contributions to the total field made by these integrations are $-2\pi i$ times the residue and the expression for $E_z^T(\mathcal{R}, \Phi, t)$ may be written:

$$E_z^T(\mathcal{R}, \Phi, t) = \frac{\mp 2\pi\nu' i K}{\sqrt{A^2 + B^2 - C^2}} e^{-ik\left[a - b\cos\alpha_p\right]} + K \int_{D_1 + D_2} \frac{e^{-ik\left[a - b\cos\alpha\right]}\, d\alpha}{A\cos\frac{\alpha}{\nu} + B\sin\frac{\alpha}{\nu} + C} \tag{3.13}$$

Examining the expression (3.13) it may be noted that the terms resulting from the residue integrations represent plane waves. The negative sign is associated with α_{p1} and when the expression for $\cos\alpha_{p1}$ is substituted the plane wave is found to represent the incident field. The positive sign is associated with α_{p2} and when the expression for $\cos\alpha_{p2}$ is substituted the plane wave is found to represent a reflected field. Again it is emphasized that either one or both of these plane waves may be absent depending upon where the poles α_{p1} and α_{p2} lie on the real axis. A more detailed discussion of these incident and reflected plane waves will be given in connection with the specific examples to be considered later.

The remaining integral in (3.13) will be evaluated using the saddle point method. In particular, consider the integral I;

$$I = K e^{-ika} \int_{D_1+D_2} \frac{e^{ikb\cos\alpha}\, d\alpha}{A\cos\frac{\alpha}{\nu'} + B\sin\frac{\alpha}{\nu'} + C} . \tag{3.14}$$

Recalling that it is the far zone field which is of interest the integral will be evaluated for large R. Note that kb is given by:

$$kb = \gamma^2 k\mathcal{R}(1-\beta\cos\Phi)(1-\beta\cos\phi_o) \tag{3.15}$$

and even though the terms in parenthesis are small R may be chosen sufficiently large so that kb will be large. In practical situations where β is small R need not be chosen as large to allow the asymptotic evaluation of (3.14). Performing the asymptotic evaluation of I to the first order results in the expression:

$$I \sim K\sqrt{2\pi}\, e^{-i\frac{\pi}{4}} \left\{ \frac{2A_1 \sin\frac{\pi}{\nu'}}{A_1^2\sin^2\frac{\pi}{\nu'} - (B_1\cos\frac{\pi}{\nu'} + C)^2} \right\} \frac{e^{-ik(a-b)}}{\sqrt{kb}} . \tag{3.16}$$

Examining the expression (3.16) it is found that I represents a cylindrical wave emanating from the edge of the wedge - a diffraction field. A detailed discussion of the diffraction field will be left to specific examples.

Finally the expression for the total far zone field when a time-harmonic plane wave is scattered by a uniformly moving, perfectly conducting wedge may be written:

$$E_z^T(\mathcal{R}, \Phi, t) = \frac{\gamma^2(1-\beta\cos\phi_o)\,D(\Phi)\,E_o}{2\sqrt{A^2+B^2-C^2}}\left\{-\left[1+\beta(\hat{x}'\cdot\hat{p}_i')^{\sim}\right]e^{-ik\left[a-b\cos\alpha_{p1}\right]}\right.$$

$$\left.+\left[1+\beta(\hat{x}'\cdot\hat{p}_r')^{\sim}\right]e^{-ik\left[a-b\cos\alpha_{p2}\right]}\right\}$$

$$+\frac{\gamma^2(1-\beta\cos\phi_o)\left[1+\beta(\hat{x}'\cdot\hat{p}_d')^{\sim}\right]D(\Phi)\,E_o A_1\sin\frac{\pi}{\nu'}\,e^{-\frac{3\pi i}{4}}}{\sqrt{2\pi}\,\nu'\left\{A_1^2\sin^2\frac{\pi}{\nu'}-(B_1\cos\frac{\pi}{\nu'}+C)^2\right\}}\;\frac{e^{-ik\left[a-b\right]}}{\sqrt{kb}} \tag{3.17}$$

where, as has been noted, there may or may not be plane wave terms present depending upon the observers location.

In the special case of a uniformly moving half plane the integral I may be evaluated exactly in terms of Fresnel integrals. For this situation the integral I, since $\nu' = 2$, is given by:

$$I = K e^{-ika}\int_{D_1+D_2}\frac{e^{ikb\cos\alpha}}{A\cos\frac{\alpha}{2}+B\sin\frac{\alpha}{2}+C}\;. \tag{3.18}$$

The evaluation of (3.18) is given in detail in Appendix D. The result of that integration is that I may be expressed as:

$$I = \frac{4\sqrt{\pi}\,K e^{i\frac{\pi}{4}} e^{-ik[a-b]}}{\sqrt{A^2+B^2-C^2}} \left\{ \pm e^{-2ikb\mu_2^2} F_o(\pm\mu_2\sqrt{2kb}) \mp e^{-2ikb\mu_1^2} F_o(\pm\mu_1\sqrt{2kb}) \right\} \tag{3.19}$$

where

$$F_o(z) = \int_z^\infty e^{it^2}\, dt \tag{D.23}$$

and where

$$\mu_1 = \frac{BC + A\sqrt{A^2+B^2-C^2}}{A^2+B^2} \tag{D.12}$$

$$\mu_2 = \frac{BC - A\sqrt{A^2+B^2-C^2}}{A^2+B^2} \tag{D.13}$$

are both real numbers. The upper signs in (3.19) correspond to $\mu_1(\mu_2)$ positive, the lower signs to $\mu_1(\mu_2)$ negative. The expression for the total field when a time-harmonic plane wave is scattered by a uniformly moving half plane is given by

$$E_z^T(\mathcal{R}, \Phi, t) = \frac{\gamma^2(1-\beta\cos\phi_o)\, D(\Phi) E_o}{2\sqrt{A^2+B^2-C^2}} \left\{ -\left[1+\beta(\hat{x}'\cdot\hat{p}_i')\right]^{\sim} e^{-ik\left[a-b\cos\alpha_{p1}\right]} \right.$$

$$+ \left[1+\beta(\hat{x}'\cdot\hat{p}_r')^{\sim}\right] e^{-ik\left[a-b\cos\alpha_{p2}\right]} - \left[1+\beta(\hat{x}'\cdot\hat{p}_d')^{\sim}\right] e^{-ik[a-b]-i\frac{\pi}{4}} (\pi)^{-1/2}$$

$$\left. \left[\mp e^{-2ikb\mu_1^2} F_o(\pm\mu_1\sqrt{2kb}) \pm e^{-2ikb\mu_2^2} F_o(\pm\mu_2\sqrt{2kb}) \right] \right\} \tag{3.20}$$

with the same sign convention as in (3.19). As previously mentioned, μ_1, μ_2, α_{p1} and α_{p2} depend upon the observers position. When α_{p1} and/or α_{p2} lie outside of the interval $[0, 2\pi]$ on the real axis then the corresponding plane wave term(s) are not to be included in the representation (3.20). Correspondingly the values of μ_1 and μ_2 will change and thus determine which set of signs to use in representing the field at that observation point.

3.5 Physical Interpretation of Total Field

In order to gain some insight into the effect of the velocity it will first be noted that when $\beta = 0$ the representation of the total field given by (3.17) reduces to:

$$E_z^T(R, \phi, t) = E_o e^{-ik\left[ct - R\cos(\phi - \phi_o)\right]} - E_o e^{-ik\left[ct - R\cos(2\Omega - \phi - \phi_o)\right]}$$

$$+ \frac{E_o \sin\frac{\pi}{\nu}}{\nu'\sqrt{2\pi}} \left\{ \frac{2\cos\frac{\phi_o}{\nu}\sin\left(\frac{\Omega-\phi}{\nu}\right)}{\left[\sin\frac{\pi}{\nu'}\sin\left(\frac{\Omega-\phi}{\nu}\right)\right]^2 - \left[\cos\frac{\pi}{\nu'}\cos\frac{\Omega-\phi}{\nu} + \sin\frac{\phi_o}{\nu}\right]^2} \right\}$$

$$\cdot \frac{e^{-ik\left[ct - R\right] + i\frac{\pi}{4}}}{\sqrt{kR}} \tag{3.21}$$

which is the well known asymptotic expression for a plane wave scattered by a wedge. The pole locations (already substituted in Eq. (3.21)) are given by

$$\alpha_{p1} = \phi - \phi_o$$
$$\alpha_{p2} = \phi - (2\Omega - \phi_o) \tag{3.22}$$

one representing an incident wave propagating in a direction ϕ_o and the other a reflected wave propagating in the direction $2\Omega - \phi_o$. If the observation point ϕ, where $\Omega < \phi < 2\pi - \Omega$ is such that either α_{p1} and/or α_{p2} lie in the interval $[0, 2\pi]$ then the plane wave term is included, otherwise it is not included. The diffraction term is present for all values of ϕ. The exact expression for the half plane (3.20) reduces for $\beta = 0$ to

$$E_z^T(R, \phi, t) = E_o e^{-ik\left[ct - R\cos(\phi - \phi_o)\right]} - E_o e^{-ik\left[ct - R\cos(\phi + \phi_o)\right]}$$

$$+ E_o \frac{e^{-ikct - i\frac{\pi}{4}}}{\sqrt{\pi}} \left\{ \mp e^{ikR\cos(\phi - \phi_o)} F_o\left[\pm \mu_1 \sqrt{2kR}\right] \right.$$

$$\left. \pm e^{ikR\cos(\phi + \phi_o)} F_o\left[\pm \mu_2 \sqrt{2kR}\right] \right\} \qquad (3.23)$$

where

$$\mu_1 = -\sin\left(\frac{\phi - \phi_o}{2}\right)$$

$$\mu_2 = \sin\left(\frac{\phi - \phi_o}{2}\right) \quad . \qquad (D.32)$$

This is exactly the expression for the stationary half plane as given in Born and Wolf (1964) p. 569 when consideration is given to the locations of the poles and the signs on μ_1 and μ_2. If the asymptotic expression for $F_o(z)$ is used Eq. (3.23) reduces to Eq. (3.21) with ν' set equal to two.

The effect of a scattering object moving with relativistic speeds is most easily seen by considering Eq. (3.17) for the case of an infinite sheet i.e. a wedge with interior half angle $\Omega' = \pi/2$. Setting $\nu' = 1$ in Eq. (3.17) the diffraction term disappears completely due to the term $\sin\pi$. The total field is due entirely to the sum of the incident and the reflected waves and is given by

$$E_z^T = \frac{\gamma^2(1-\beta\cos\phi_o)D(\Phi)E_o}{2\sqrt{A^2+B^2-C^2}}\left\{-\left[1+\beta(\hat{x}'\cdot\hat{p}_i')^\sim\right]e^{-ik\left[a-b\cos\alpha_{p1}\right]}\right.$$

$$\left.+\left[1+\beta(\hat{x}'\cdot\hat{p}_r')^\sim\right]e^{-ik\left[a-b\cos\alpha_{p2}\right]}\right\}. \tag{3.24}$$

In the primed frame the vectors $\hat{p}_i'$ and $\hat{p}_r'$ may easily be determined as

$$\hat{p}_i' = \cos\phi_o'\,\hat{x}' + \sin\phi_o'\,\hat{y}' \tag{3.25}$$

$$\hat{p}_r' = -\cos\phi_o'\,\hat{x}' + \sin\phi_o'\,\hat{y}' \tag{3.26}$$

so that using Eq. (B.17) it is found that

$$\left[1+\beta(\hat{x}'\cdot\hat{p}_i')^\sim\right] = \frac{1}{\gamma^2(1-\beta\cos\phi_o)} \tag{3.27}$$

and

$$\left[1+\beta(\hat{x}'\cdot\hat{p}_r')^\sim\right] = \frac{(1+\beta^2)-2\beta\cos\phi_o}{(1-\beta\cos\phi_o)}\ . \tag{3.28}$$

In order to evaluate the remaining functions note, since $\Omega'/\nu' = \pi/2$, that

$$\cos\frac{\Omega}{\nu} = \beta \quad , \quad \sin\frac{\Omega}{\nu} = \frac{1}{\gamma} \tag{3.29}$$

and the functions A, B, C and D become

$$A = -B_1 = \frac{1}{\gamma}\sin\frac{\Phi}{\nu}\,(1-\beta\cos\frac{\phi_o}{\nu}) \tag{3.30}$$

$$B = A_1 = (\cos\frac{\Phi}{\nu} - \beta)(1 - \beta\cos\frac{\phi_o}{\nu}) \tag{3.31}$$

$$C = -\frac{1}{\gamma}\sin\frac{\phi_o}{\nu}(1 - \beta\cos\frac{\Phi}{\nu}) \tag{3.32}$$

$$D = -2(\cos\frac{\phi_o}{\nu} - \beta)(1 - \beta\cos\frac{\Phi}{\nu}) \quad . \tag{3.33}$$

The pole locations may now be found by substituting these expressions into Eq. (3.10) which yields

$$\cos\alpha_p = \frac{\sin\frac{\Phi}{\nu}\sin\frac{\phi_o}{\nu} \pm \gamma^2(\cos\frac{\Phi}{\nu} - \beta)(\cos\frac{\phi_o}{\nu} - \beta)}{\gamma^2(1 - \beta\cos\frac{\Phi}{\nu})(1 - \beta\cos\frac{\phi_o}{\nu})} \quad . \tag{3.34}$$

The exponential term $\left[a - b\cos\alpha_p\right]$ will now be evaluated and expressed in the unprimed coordinate frame. In order to do this Eqs. (B.7) and (B.9) may be written as

$$R(\cos\Phi - \beta) = x - v\tau \tag{3.35}$$

and used along with (B.8).

$$R\sin\Phi = y \quad . \tag{B.8}$$

Substituting a, b and $\cos\alpha_p$ in $\left[a - b\cos\alpha_p\right]$ and using (3.35) and (B.8) obtain

$$a - b\cos\alpha_{p1} = ct - x\cos\phi_o - y\sin\phi_o = ct - R\cos(\phi - \phi_o) \tag{3.36}$$

and

$$a - b\cos\alpha_{p2} = \left[\gamma^2(1+\beta^2) - 2\beta\gamma^2\cos\phi_o\right]\left\{ct - R\cos(\phi - \phi_r)\right\} \tag{3.37}$$

where

$$\cos\phi_r = \frac{2\beta-(1+\beta^2)\cos\phi_o}{\left[(1+\beta^2) - 2\beta\cos\phi_o\right]} \quad . \tag{3.38}$$

The upper sign in (3.34) determines α_{p1}, the lower sign, α_{p2}. The coefficient in Eq. (3.34) becomes $-\gamma^2(1-\beta\cos\phi_o)E_o$ when the substitution for A, B, C and D is made. Finally then the field scattered by a uniformly moving sheet may be written

$$E_z^T(R, \phi, t) = E_o e^{-ik\left[ct - R\cos(\phi-\phi_o)\right]} - QE_o e^{-ikQ\left[ct - R\cos(\phi-\phi_o)\right]} \tag{3.39}$$

where

$$Q = \gamma^2\left[(1+\beta^2) - 2\beta\cos\phi_o\right] \quad . \tag{3.40}$$

This is the result obtained by Sommerfeld (1964). Examining (3.39) it is found that the effect of the velocity is to

1) Change the frequency of the reflected wave, i.e., the doppler frequency:

$$k_r = Qk \tag{3.41}$$

2) Change the angle of reflection as indicated by Eq. (3.38)

3) Change the amplitude of the reflected wave as determined by the factor Q.

These are well known results and are pointed out here only to serve as basis for comparison when the results of the pulse scattering are analyzed in Chapters IV and V.

The field scattered by a half plane provides a second special case through which the effects of a moving scatterer may be studied. The exact representation of this field is given by Eq. (3.20) when Ω' is set equal to zero and ν' equal to 2. The various terms in (3.20) will now be evaluated. First note that for the half plane

$$\cos\frac{\Omega}{\nu} = 1 \quad , \qquad \sin\frac{\Omega}{\nu} = 0 \tag{3.42}$$

so that the functions A , B , C and D become

$$A = A_1 = -\gamma(1-\beta)\sin\frac{\Phi}{\nu}\left(1-\beta\cos\frac{\phi_o}{\nu}\right) \tag{3.43}$$

$$B = B_1 = -\gamma^2(1-\beta)\left(\cos\frac{\Phi}{\nu}-\beta\right)\left(1-\beta\cos\frac{\phi_o}{\nu}\right) \tag{3.44}$$

$$C = -\gamma(1-\beta)\left(1-\beta\cos\frac{\Phi}{\nu}\right)\sin\frac{\phi_o}{\nu} \tag{3.45}$$

$$D = -2\gamma^2(1-\beta)\left(1-\beta\cos\frac{\Phi}{\nu}\right)\left(\cos\frac{\phi_o}{\nu}-\beta\right) \quad . \tag{3.46}$$

With these expressions for A , B , C and D the various factors in Eq. (3.20) are computed as

$$\frac{\gamma^2(1-\beta\cos\phi_o)D(\Phi)E_o}{2\sqrt{A^2+B^2-C^2}} = -\gamma^2(1-\beta\cos\phi_o)E_o \tag{3.48}$$

$$\left[1+\beta(\hat{x}'\cdot\hat{p}'_i)^{\sim}\right] = \left[1+\beta(\hat{x}'\cdot\hat{p}'_r)^{\sim}\right] = \frac{1}{\gamma^2(1-\beta\cos\phi_o)} \tag{3.49}$$

$$\left[1+\beta(\hat{x}'\cdot\hat{p}'_d)^{\sim}\right] = \frac{1}{\gamma^2(1-\beta\cos\Phi)} \tag{3.50}$$

$$\cos\alpha_p = \frac{\gamma^2(\cos\Phi-\beta)(\cos\phi_o-\beta)\pm\sin\Phi\sin\phi_o}{\gamma^2(1-\beta\cos\Phi)(1-\beta\cos\phi_o)} \tag{3.51}$$

$$\mu_{1,2} = \left[\frac{1-\cos(\phi_o\mp\Phi)}{2\gamma^2(1-\beta\cos\Phi)(1-\beta\cos\phi_o)}\right]^{1/2} \tag{3.52}$$

so that the exact expression for the total field when a time harmonic plane wave is scattered by a uniform velocity half plane becomes

$$E_z^T(\mathcal{R},\Phi,t) = E_o e^{-ik\left[(1-\beta\cos\phi_o)ct-\mathcal{R}(\cos\Phi-\beta)\cos\phi_o-\mathcal{R}\sin\Phi\sin\phi_o\right]}$$

$$- E_o e^{-ik\,(1-\beta\cos\phi_o)ct-\mathcal{R}(\cos\Phi-\beta)\cos\phi_o+\mathcal{R}\sin\Phi\sin\phi_o}$$

$$+ E_o\frac{(1-\beta\cos\phi_o)}{(1-\beta\cos\Phi)}\frac{e^{-i\frac{\pi}{4}}}{\sqrt{\pi}}e^{-ik(1-\beta\cos\phi_o)(ct-\mathcal{R})}$$

$$\left\{\mp e^{-ik\mathcal{R}\left[1-\cos(\Phi-\phi_o)\right]}F_o\left[\pm\sin\left(\frac{\phi_o-\Phi}{2}\right)\sqrt{2k\mathcal{R}}\right]\right.$$

$$\left.\pm e^{-ik\mathcal{R}\left[1-\cos(\Phi-\phi_o)\right]}F_o\left[\pm\sin\left(\frac{\phi_o+\Phi}{2}\right)\sqrt{2k\mathcal{R}}\right]\right\} \tag{3.53}$$

The two plane wave terms take on a more familiar form when written in the regular unprimed variables (R, ϕ) rather than in the retarded variables $(\mathcal{R}, \Phi)$, i.e.

$$E_z^i(R,\phi,t) = E_o e^{-ik\left[ct-R\cos(\phi-\phi_o)\right]} \tag{3.54}$$

and

$$E_z^r(R, \phi, t) = E_o e^{-ik\left[ct - R\cos(\phi + \phi_o)\right]} . \tag{3.55}$$

The incident field appears as expected and the reflected field reveals no difference between reflection by a stationary half plane and that by a moving half plane. This fact was pointed out by Sommerfeld (1964).

Consider the expression for the total field in the shadow region where the plane wave terms disappear and the diffracted field is the total field, i.e.

$$E_z^d(\mathcal{R}, \Phi, t) = E_o \frac{(1-\beta\cos\phi_o)}{(1-\beta\cos\Phi)} \frac{e^{-i\frac{\pi}{4}}}{\sqrt{\pi}} e^{-ik(1-\beta\cos\phi_o)(ct-\mathcal{R})}$$

$$\left\{ e^{-ik\mathcal{R}\left[1-\cos(\Phi+\phi_o)\right]} F_o\left[\sin\left(\frac{\phi+\phi_o}{2}\right)\sqrt{2k\mathcal{R}}\right] \right.$$

$$\left. - e^{-ik\mathcal{R}\left[1-\cos(\Phi-\phi_o)\right]} F_o\left[\sin\left(\frac{\phi_o-\Phi}{2}\right)\sqrt{2k\mathcal{R}}\right] \right\}. \tag{3.56}$$

This expression represents the diffracted field not only in the shadow region but also in the illuminated regions. The expression (3.56) can be written

$$E_z^d(\mathcal{R}, \Phi, t) = \frac{(1-\beta\cos\phi_o)}{(1-\beta\cos\Phi)} e^{ik\beta\cos\phi_o(ct-\mathcal{R})} E_{zo}^d(\mathcal{R}, \Phi, t) \tag{3.57}$$

where $E_{z0}^d(\mathcal{R}, \Phi, t)$ is precisely the expression for the diffraction field in the stationary half plane case provided the edge of the half plane is located at the origin of the retarded reference frame.

The effect of the velocity is included completely in the coefficient of $E_{zo}^d(\mathcal{R}, \Phi, t)$. When β is reduced to zero this coefficient becomes unity. Examining the coefficient

$$\frac{(1-\beta\cos\phi_o)}{(1-\beta\cos\Phi)}\, e^{ik\beta\cos\phi_o(ct-\mathcal{R})} \tag{3.58}$$

it is apparent that both the amplitude and phase of the edge diffracted field are affected by the motion of the half plane. Let $(1-\beta\cos\phi_o)/(1-\beta\cos\Phi)$ be called the "Amplitude Pattern factor". Figure 3-2 shows a sketch of this pattern factor after it has been normalized with respect to the constant $(1-\beta\cos\phi_o)$. The effect of the motion is seen to be to increase the intensity of the diffracted field in the direction of motion and to reduce the intensity of the diffracted field in a direction opposite to that of the motion. The pattern of the field diffracted by a moving half plane can be found by a pattern multiplication of the field pattern of a stationary half plane with the Amplitude Pattern factor. If β is zero the Amplitude pattern factor reduces to a unit circle. The phase change due to motion is seen to be proportional to the factor $(ct-\mathcal{R})$. This means that the equal phase surfaces remain circles, however, the value of the phase angle on a given equal phase surface is different from what it would be in the stationary case. The constant phase circles have their centers at the location of the edge corresponding to the position of the edge at the time at which that phase front was diffracted. Since the edge is moving the constant phase circles are not concentric but offset giving rise to the well known changing doppler frequency.

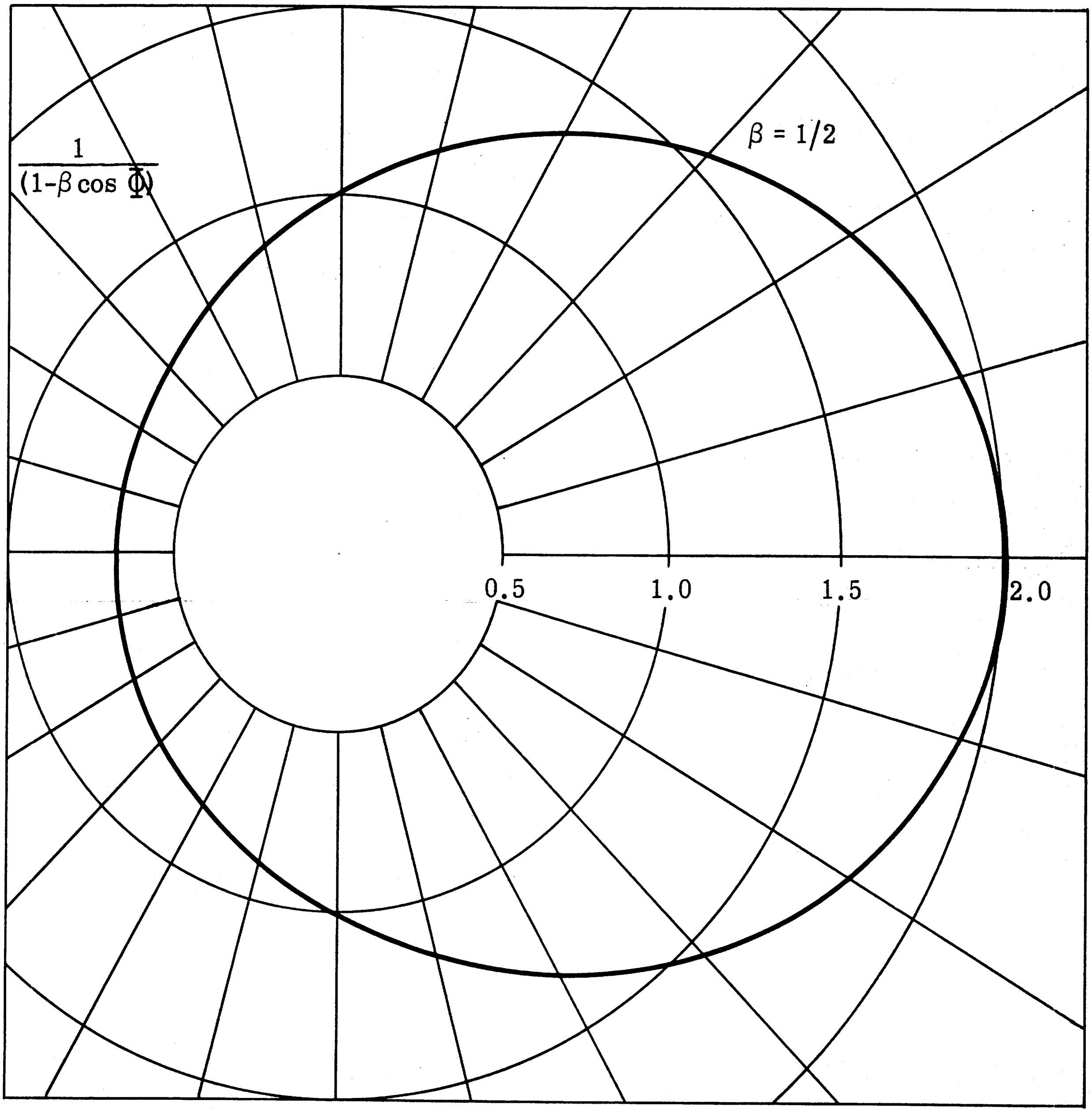


FIG. 3-2: AMPLITUDE PATTERN FACTOR FOR DIFFRACTION FIELD.

Chapter IV

SCATTERING OF A UNIT STEP PLANE PULSE

4.1 Determination of $\tilde{E}_z^i$

The incident field which is a unit step plane pulse is expressed mathematically in the unprimed frame by Eq. (2.2)

$$E_z^i(R, \phi, t) = H\left[ct - R\cos(\phi - \phi_o)\right] \tag{2.2}$$

where $H(x)$ is the Heaviside unit function defined in Appendix E. Figure 4-1 illustrates the conditions which exist at $t = 0$.

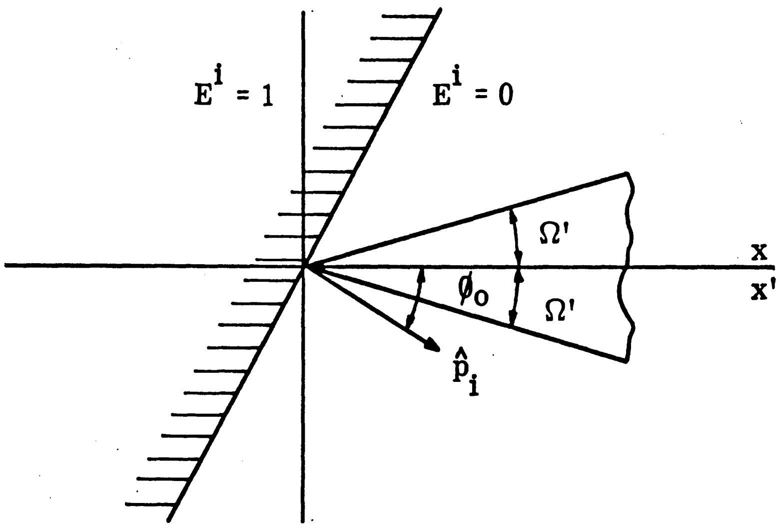


FIG. 4-1: PROBLEM GEOMETRY FOR PLANE PULSE

The incident field as observed in the primed frame is found from (2.2) through the use of Eqs. (2.6) and (B.1) to (B.3). The description of the incident field in the primed frame is thus

$$E_z^{i'}(R', \phi', t') = \gamma(1-\beta\cos\phi_o) H \left\{\gamma(1-\beta\cos\phi_o)\left[ct' - R'\cos(\phi' - \phi'_o)\right]\right\} . \tag{4.1}$$

For use in the contour integral representing the total field in the primed frame set $\phi' - \phi'_o = \alpha$ in (4.1).

$$E_z^{i}(R', \alpha + \phi'_o, t') = \gamma(1-\beta\cos\phi_o) H \left\{\gamma(1-\beta\cos\phi_o)\left[ct' - R'\cos\alpha\right]\right\} . \tag{4.2}$$

Finally, the desired expression for $\tilde{E}_z^i(\mathcal{R}, \Phi, t; \alpha)$ is obtained from (4.2) by the transformation (2.32) and change of variables (B.9) and (B.17) to (B.19). and becomes

$$\tilde{E}_z^i(\mathcal{R}, \Phi, t; \alpha) = \gamma(1-\beta\cos\phi_o) H \left[a - b\cos\alpha\right] \tag{4.3}$$

where

$$a = \left[ct - \beta\gamma^2 \mathcal{R}(\cos\Phi - \beta)\right](1 - \beta\cos\phi_o) \tag{4.4}$$

$$b = \gamma^2 \mathcal{R}(1 - \beta\cos\Phi)(1 - \beta\cos\phi_o) . \tag{4.5}$$

4.2 The Integral for the Total Field

The integral representing the total field when a unit step plane pulse is scattered by a uniformly moving wedge is obtained when (4.3) is inserted into expression (2.45) giving

$$E_z^T(\mathcal{R}, \Phi, t) = K \int_{C_1+C_2} \frac{H[a - b\cos\alpha]\,d\alpha}{A\cos\frac{\alpha}{\nu'} + B\sin\frac{\alpha}{\nu'} + C} \tag{4.6}$$

where

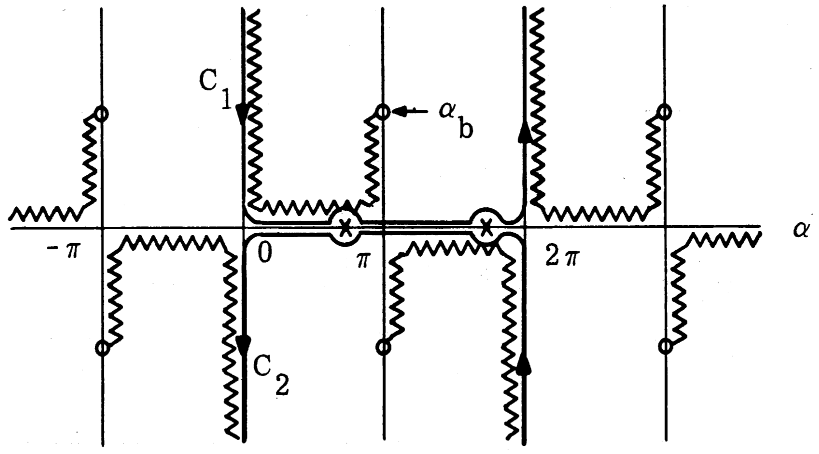
$$K = \frac{\gamma^2(1-\beta\cos\phi_0)\left[1+\beta(\hat{x}'\cdot\hat{\beta}')\gamma^{\sim}\right] D(\Phi)}{4\pi\nu' i} \tag{4.7}$$

4.3 Cuts in the Complex α Plane

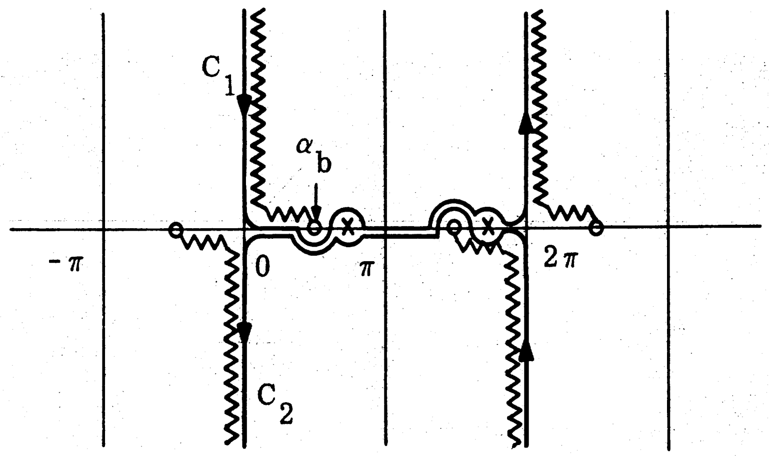
The Heaviside function which appears in the integrand of Eq. (4.6) can be thought of as a function which is analytic in the entire α plane and also single valued provided certain cuts are introduced. Appendix E discusses the analytic representation of $H(z)$ and the cuts introduced in the α plane by the mapping $z = a - b\cos\alpha$. The feature of such a representation which is useful is that $H(z) = 0$ on the cut. The branch points for these cuts in the α plane depend upon the ratio a/b which in turn is a function of $(\mathcal{R}, \Phi, t)$, ϕ_0 and β. This means that even for a stationary wedge ($\beta = 0$) the branch points in the complex α plane will move. When the wedge is moving the motion of these branch points is altered by the influence of β thus introducing one aspect of the effect of the velocity upon the scattering problem. Knowing that the integrand is zero on these cuts facilitates the visualization of what is happening in the evaluation of the integral. Figure E-1 shows the cuts in the complex α plane for different ranges of the ratio a/b.

4.4 Evaluation of the Integral Representing E_z^T

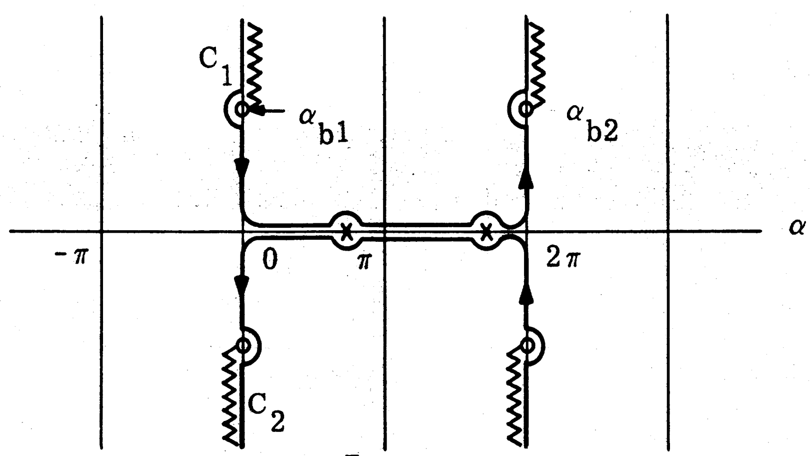
The evaluation of the integral (4.6) involves the same poles of the integrand as were discussed in Section 3.3. There the poles were shown to lie somewhere on the real axis in the complex α plane, With a knowledge of the cuts of the Heaviside function and the location of the poles the contours $C_1 + C_2$ shown in Fig. 2-2 may be deformed to those shown in Figure 4-2.



i - Case 1 $[-\infty < a < -b]$



ii - Case 2 $[-b < a < b]$



iii - Case 3 $[b < a < \infty]$

FIG. 4-2: CONTOUR DEFORMATION FOR THE UNIT STEP PLANE PULSE.

The significance of the intervals corresponding to each of the three cases shown in Fig. 4-2 can be obtained by considering the problem of a stationary wedge. The branch point α_b is given by

$$\cos\alpha_b = \frac{a}{b} \quad . \tag{4.8}$$

For the stationary problem this becomes

$$\cos\alpha_b = \frac{a}{b} = \frac{ct}{R} \quad . \tag{4.9}$$

Case 1 corresponds to large negative time, that is, from $a/b = -\infty$ up to $a/b = -1$. This is a time interval $t = -\infty$ to $t = -R/c$. The time $t = -R/c$ is recognized as the earliest possible time an observer can see the incident field. The branch point when $a/b = -1$ is located at $\alpha_b = n\pi + i0$ where n is odd. Prior to this time ($t = -R/c$) the branch point has been moving along a vertical line in the α plane toward the real axis. This means that prior to $a/b = -1$ the entire contour $C_1 + C_2$ lies on the cut of the Heaviside function where the integrand is known to be zero. Thus case 1 corresponds to the time interval during which the observer does not see even the incident pulse and the evaluation of E_z^T along $C_1 + C_2$ in this interval correspondingly yields zero.

Case 2 includes the interval $-1 < a/b < 1$ or for the stationary problem

$$-\frac{R}{c} < t < \frac{R}{c} \tag{4.10}$$

The time $t = R/c$ is recognized as the earliest possible time at which an observer will see the field diffracted by the edge. The location of the branch point at this time is $\cos\alpha_b = 1$ or $\alpha_b = n\pi + i0$ where n is even. This means that the moving branch points (and thus cuts) have "uncovered" the real axis in the interval $[0, 2\pi]$. If either of the two poles lies in the uncovered interval of the real axis that means that the observer sees that particular contribution to the field during this time interval. However, as mentioned above, the edge diffracted field is not observed during this time interval.

Case 3 corresponds to $1 < a/b < \infty$ or in the stationary problem

$$\frac{R}{c} < t < \infty \quad . \tag{4.11}$$

This is recognized as the time interval during which an observer sees the edge diffracted field and possibly the reflected and/or incident (pole contribution) fields. During this time interval the branch points move along vertical lines in the α plane away from the real axis.

Consider now the evaluation of $E_z^T(\mathcal{R}, \Phi, t)$ as given by (4.6) for each of these three cases. In case one the integrand is zero on the entire contour and thus

$$E_z^T(\mathcal{R}, \Phi, t) = 0 \tag{4.12}$$

for $\left[-\infty < a/b < -1\right]$.

In case 2 the branch points have moved to "uncover" a portion of the contour along the real axis. The two branch points in the interval $\left[0, 2\pi\right]$ may be described by $\alpha_{b1} = x_b + i0$ and $\alpha_{b2} = (2\pi - x_b) + i0$ where $\cos x_b = a/b$ and $0 < x_b < \pi$. If one of the two poles α_p, lie in the interval $\alpha_{b1} < \alpha_p < \alpha_{b2}$ then a contribution is made to the value of E_z^T which is given by $-2\pi i$ times the residue at the pole. Such a contribution is given by

$$E_z^T(\mathcal{R}, \Phi, t) = \mp K \frac{2\pi\nu' i\, H\left[a - b\cos\alpha_p\right]}{\sqrt{A^2 + B^2 - C^2}} \tag{4.13}$$

where the upper sign corresponds to α_{p1} and the lower sign to α_{p2}. If neither pole lies in this interval then the total field remains zero. Examining Eq. (4.13) the pole contributions are seen to be in the form of unit step pulses. One of these pulses corresponds to the incident pulse and the other to the reflected pulse. The total field may be written as

$$E_z^T(\mathcal{R}, \Phi, t) = \frac{\gamma^2(1-\beta\cos\phi_0)D(\Phi)}{2\sqrt{A^2+B^2-C^2}}\left\{-\left[1+\beta(\hat{x}'\cdot\hat{p}_i')^{\sim}\right]H\left[a-b\cos\alpha_{p1}\right]\right.$$

$$\left.+\left[1+\beta(\hat{x}'\cdot\hat{p}_r')^{\sim}\right]H\left[a-b\cos\alpha_{p2}\right]\right\} \tag{4.16}$$

provided $\alpha_{b1} < \alpha_p < \alpha_{b2}$ otherwise

$$E_z^T(\mathcal{R}, \Phi, t) = 0$$

during the interval $[-1 < a/b < 1]$. That is either one or both or neither pulse will be seen depending upon where the poles lie.

In case 3 the real axis portion of the contour is completely uncovered. If the observers position is such that the poles lie in the interval $[0, 2\pi]$ then (4.16) represents pole contribution to the total field seen by the observer and the expression for the total field may be written

$$E_z^T(\mathcal{R}, \Phi, t) = \frac{\gamma^2(1-\beta\cos\phi_0)D(\Phi)}{2\sqrt{A^2+B^2-C^2}}\left\{-\left[1+\beta(\hat{x}'\cdot\hat{p}_i')^{\sim}\right]H\left[a-b\cos\alpha_{p1}\right]\right.$$

$$\left.+\left[1+\beta(\hat{x}'\cdot\hat{p}_r')^{\sim}\right]H\left[a-b\cos\alpha_{p2}\right]\right\}+K\int_{\alpha_{b1}}^{\alpha_{b1}^*}\frac{d\alpha}{A\cos\frac{\alpha}{\nu'}+B\sin\frac{\alpha}{\nu'}+C}$$

$$+K\int_{\alpha_{b2}^*}^{\alpha_{b2}}\frac{d\alpha}{A\cos\frac{\alpha}{\nu'}+B\sin\frac{\alpha}{\nu'}+C}\;. \tag{4.17}$$

where α_{b1} and α_{b2} are shown in Fig. 4-2. The integration around the branch points can be shown to contribute nothing to the value of the integral. The integrals in Eq. (4.17) can be evaluated using an integral identity tabulated by Gradshteyn and Ryzhik (1965) [Eq. 2.558 (4)].

$$\int \frac{dx}{A\cos x + B\sin x + C} = \frac{-2i}{\sqrt{A^2+B^2-C^2}} \tan^{-1}\left[\frac{(C-A)\tan\frac{x}{2}+B}{\sqrt{C^2-A^2-B^2}}\right]. \tag{4.18}$$

The total field when a plane unit step pulse is scattered by a uniformly moving wedge can now be written down as

$$E_z^T(\mathcal{R}, \Phi, t) = \frac{\gamma^2(1-\beta\cos\phi_0)D(\Phi)}{2\sqrt{A^2+B^2-C^2}}\Bigg\{-\left[1+\beta(\hat{x}'\cdot\hat{p}_i')^{\sim}\right] H\left[a-b\cos\alpha_{p1}\right]$$

$$+\left[1+\beta(\hat{x}'\cdot\hat{p}_r')^{\sim}\right] H\left[a-b\cos\alpha_{p2}\right] - \frac{\left[1+\beta(\hat{x}'\cdot\hat{p}_d')^{\sim}\right]}{\pi}$$

$$\cdot\left[\tan^{-1}\left\{\frac{(C-A)\tan\frac{\alpha}{2\nu'}+B}{\sqrt{C^2-A^2-B^2}}\right\}_{\alpha_{b1}}^{\alpha_{b1}^*} + \tan^{-1}\left\{\frac{(C-A)\tan\frac{\alpha}{2\nu'}+B}{\sqrt{C^2-A^2-B^2}}\right\}_{\alpha_{b1}^*+2\pi}^{\alpha_{b1}+2\pi}\right]\Bigg\} \tag{4.19}$$

where

$$\alpha_{b1} = i\cosh^{-1}\left(\frac{a}{b}\right) . \tag{4.20}$$

Recall that this represents the field in the time interval $1 < a/b < \infty$. Setting $a/b = 1$ the range of t is seen to be

$$\frac{\mathcal{R}}{c} < t < \infty \tag{4.21}$$

so that (4.19) is valid for times after the edge diffracted pulse front has passed the observer located at $(\mathcal{R}, \Phi)$. The arctangent terms are always present while the plane unit step pulse terms may or may not be present. The interpretation of (4.19) is given in the following section as it relates to some specific examples.

4.5 Interpretation of Results

In order to gain some insight into the effect of the velocity upon the field scattered by a wedge when the incident field is a unit step pulse, two specific examples will be considered: the half plane and the infinite sheet.

Consider first the infinite uniform velocity sheet for which $\nu' = 1$. Examining Eq. (4.19) which represents the total field it is seen that the arctangent terms representing the diffraction field disappear. This is evident if one first interchanges the limits on the second arctangent and then evaluates the functions at the limits. Since $\tan\left(\frac{\alpha+2\pi}{2}\right) = \tan\frac{\alpha}{2}$ the arctangent terms exactly cancel one another. The total field is thus given by an incident step pulse and a reflected step pulse. Comparing Eq. (4.19), excluding the arctangent terms, with Eq. (3.24) they are seen to be identical except that the exponentials functions are replaced by Heaviside functions. The evaluation of the terms in Eq. (4.19) thus will exactly parallel those of Eq. (3.24) so that Eqs. (3.27), (3.28), (3.36) and (3.37) may be used in the rewritting of Eq. (4.19). As a result the total field when a unit-step plane pulse is scattered by a uniformly moving infinite sheet is given by

$$E_z^T(R, \phi, t) = H\left[ct - R\cos(\phi - \phi_o)\right] - QH\left\{Q\left[ct - R\cos(\phi - \phi_r)\right]\right\} \tag{4.22}$$

where

$$Q = \gamma^2\left[(1+\beta^2) - 2\beta\cos\phi_o\right] \tag{3.40}$$

and

$$\cos\phi_r = \frac{2\beta - (1+\beta^2)\cos\phi_o}{\left[(1+\beta^2) - 2\beta\cos\phi_o\right]} \quad . \tag{3.38}$$

The reflected unit step pulse has an amplitude given by $\bar{Q}$. Note if ϕ_o is zero that $\bar{Q}$ approaches zero as β approaches plus one, i.e., the incident pulse never catches the moving sheet. As β approaches minus one the amplitude of the reflected field becomes infinite. Looking at the argument of the Heaviside function it is seen that when ϕ_o is not zero the reflected angle is the same as in the time harmonic plane wave case. This is reasonable if the Fourier Synthesis of a unit step pulse is considered. That is, when Eq. (3.39) is multiplied by the spectral function of the incident unit step pulse and the inverse Fourier transform taken the result is precisely Eq. (4.22). The factor Q in the argument of the Heaviside function is always positive and thus has no effect upon the reflected pulse.

Consider now the uniform velocity half plane for which $\nu' = 2$. The representation for the total field given by (4.19) may be rewritten for the half plane by making use of the Eqs. (3.42) through (3.51). Recall that the presence of the plane pulse terms depends upon the presence of the poles α_p in the interval $[0, 2\pi]$ of the real axis. With this in mind Eq. (4.19) becomes

$$E_z^T(\mathcal{R}, \bar{\Phi}, t) = H\left[a - b\cos\alpha_{p1}\right] - H\left[(a - b\cos\alpha_{p2})\right]$$

$$- \frac{(1-\beta\cos\phi_o)}{\pi(1-\beta\cos\bar{\Phi})} \tan^{-1}\left[\frac{2X(A\tan\frac{\alpha}{4} + B)}{X^2 + \left[(C+A)\tan\frac{\alpha}{4} + B\right]\left[(C-A)\tan\frac{\alpha}{4} - B\right]}\right]_{\alpha_{b1}}^{\alpha_{b1}^*} \tag{4.23}$$

where

$$X = \sqrt{C^2 - A^2 - B^2} \quad .$$

and α_{b1} is given by Eq. (4.20). Examining the unit step terms with $\cos\alpha_p$ given by Eq. (3.51) it is found that they represent incident and reflected pulses. They may be expressed either in terms of retarded coordinates $(\mathcal{R}, \Phi)$

$$E_z^i(\mathcal{R}, \Phi, t) = H\left[(1-\beta\cos\phi_o)ct - \mathcal{R}(\cos\Phi - \beta)\cos\phi_o - \mathcal{R}\sin\Phi\,\sin\phi_o\right] \tag{4.24}$$

$$E_z^r(\mathcal{R}, \Phi, t) = H\left[(1-\beta\cos\phi_o)ct - \mathcal{R}(\cos\Phi - \beta)\cos\phi_o + \mathcal{R}\sin\Phi\,\sin\phi_o\right] \tag{4.25}$$

or in terms of (R, ϕ)

$$E_z^i(R, \phi, t) = H\left[ct - R\cos(\phi - \phi_o)\right] \tag{4.26}$$

$$E_z^r(R, \phi, t) = H\left[ct - R\cos(\phi + \phi_o)\right] \quad . \tag{4.27}$$

As in the case of the time harmonic plane wave there is no change in the reflected field due to the motion of the half plane. If a wedge were being considered as an example one would find a reflected pulse which has a changed amplitude and a changed angle of reflection such as depicted by the infinite sheet considered earlier (i.e., the wedge with interior half angle $\Omega' = \pi/2$). In order to facilitate the examination of the diffracted field consider the special case where $\phi_o = 0$. That is, the plane pulse front is normal to the surface of the half plane in addition to being normal to the edge. For this condition the pole α_{p2} moves out of the interval $[0, 2\pi]$ and the reflected field disappears. The total field in the region $\mathcal{R} \leq ct$ may then be written

$$E_z^T(\mathcal{R}, \Phi, t) = 1 - \frac{2(1-\beta)}{\pi(1-\beta\cos\Phi)}\tan^{-1}\left[\frac{\sqrt{1-\beta}}{\sin\frac{\Phi}{2}}\sqrt{\frac{ct-\mathcal{R}}{2\mathcal{R}}}\right] \quad . \tag{4.28}$$

To reduce the argument of the arctangent term in (4.23) to that in (4.28) involves the substitution of A, B and C as given by Eqs. (3.43) to (3.45) with ϕ_o set equal to zero. The argument may be transformed to the primed frame for simplification and then transformed back yielding Eq. (4.29) as a representation of the arctangent term in Eq. (4.23).

$$2\tan^{-1}\left[\frac{\sqrt{1-\beta}}{\sin\frac{\Phi}{2}} \; \frac{2\tanh\frac{|\alpha_b|}{4}}{\left(1-\tanh^2\frac{|\alpha_b|}{4}\right)}\right] . \tag{4.29}$$

Using the expression for the sum of two tangents obtain

$$2\tan^{-1}\left[\frac{\sqrt{1-\beta}}{\sin\frac{\Phi}{2}} \sinh\frac{|\alpha_b|}{2}\right] . \tag{4.30}$$

Now

$$\sinh\frac{|\alpha_b|}{2} = \sqrt{\frac{\cosh|\alpha_b| - 1}{2}} \tag{4.31}$$

so that when (4.20), (4.31) and (4.30) are used Eq. (4.23) becomes (4.28). It is convenient to note at this point that when β in Eq. (4.28) is set equal to zero the total field representation is exactly that for the stationary half plane as given by Friedlander (1958). Now consider the diffraction term in Eq. (4.28). Recall that (4.28) describes the field in the retarded reference frame. The effect of the motion is to change the argument of the arctangent by the factor $\sqrt{1-\beta}$ where

$$0 < \sqrt{1-\beta} < \sqrt{2} \quad . \tag{4.32}$$

Figure 4-3 illustrates the arctangent function.

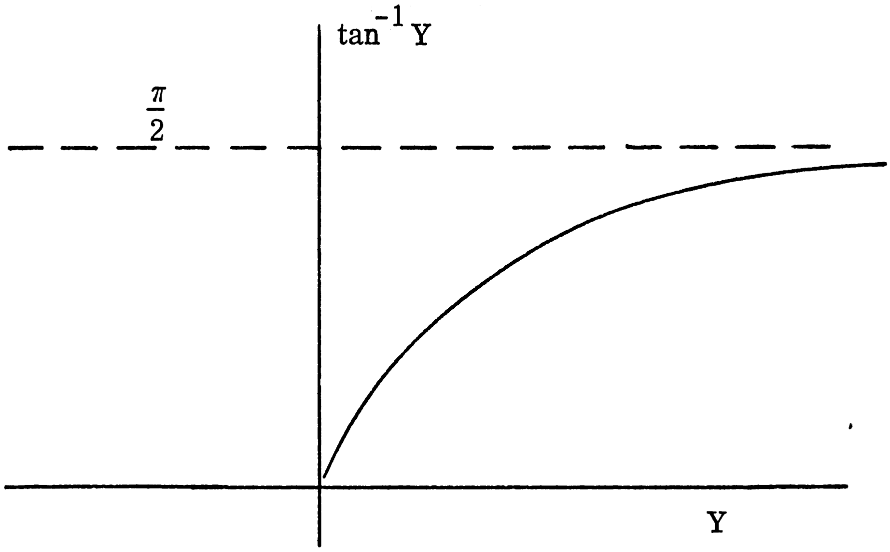


FIG. 4-3: ARCTANGENT FUNCTION.

When the half plane moves to the right β lies in the interval $[0, 1]$ and the effect of the factor $\sqrt{1-\beta}$ is to reduce the argument of the arctangent function and thus reduce the diffraction field in a given direction Φ. The coefficient of the diffraction term $1/(1-\beta\cos\Phi)$ serves to redistribute the field intensity in the direction of motion and is the same normalized amplitude pattern factor which was discussed in Chapter III.

To obtain the transient field seen by a stationary observer recall the definition of the retarded reference frame. At the time the incident pulse front is diffracted τ (location of the origin of the retarded frame) equals zero. As time passes an observer location (R, ϕ) takes on different values $(\mathcal{R}, \Phi)$ due to changing τ. This is illustrated in Fig. 4-4.

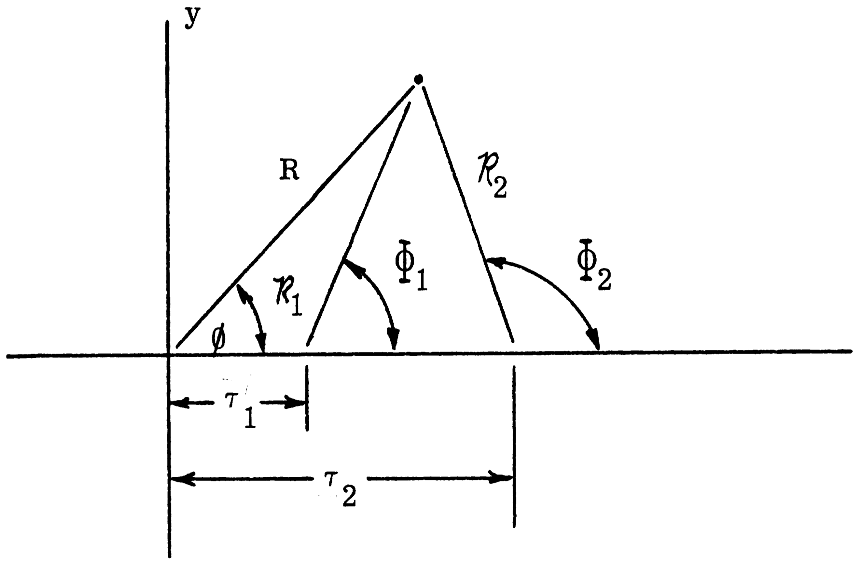


FIG. 4-4: VARIATION IN ($\mathcal{R}$, Φ) WITH TIME.

To evaluate the transient field at (R, ϕ) select a value for τ. Using Eqs. (B.7), (B.8) and (B.9) determine ($\mathcal{R}$, Φ) and t. The Eq. (4.28) may then be evaluated to determine E_z^T (R, ϕ, t). Select a new value for τ and repeat the procedure. The total influence of the motion is thus reflected in the amplitude pattern factor, the factor $\sqrt{1 - \beta}$ and the variation of $\mathcal{R}$ and Φ with time.

Chapter 5

SCATTERING OF A CYLINDRICAL IMPULSE

5.1 Determination of $\tilde{E}_z^i$

The incident field considered in this chapter is due to a line source located at the point (R_o, ϕ_o) and running parallel to the z axis in the stationary frame. The excitation of the line source is an impulse in time $\delta(t + R_o/c)$ so that the field radiated by the line source, the incident field, is given by

$$E_z^i(R, \phi, t) = \frac{H\left[c^2(t-t_o)^2 - \mathbb{R}^2(\phi-\phi_o)\right]}{\sqrt{c^2(t-t_o)^2 - \mathbb{R}^2(\phi-\phi_o)}} \tag{5.1}$$

where $t_o = -R_o/c$ and $\mathbb{R}^2$ is given by Eq. (2.4)

$$\mathbb{R}^2 = R^2 + R_o^2 - 2RR_o\cos(\phi-\phi_o) \, . \tag{2.4}$$

The geometry of this problem is shown for $t = 0$ in Fig. 5-1.

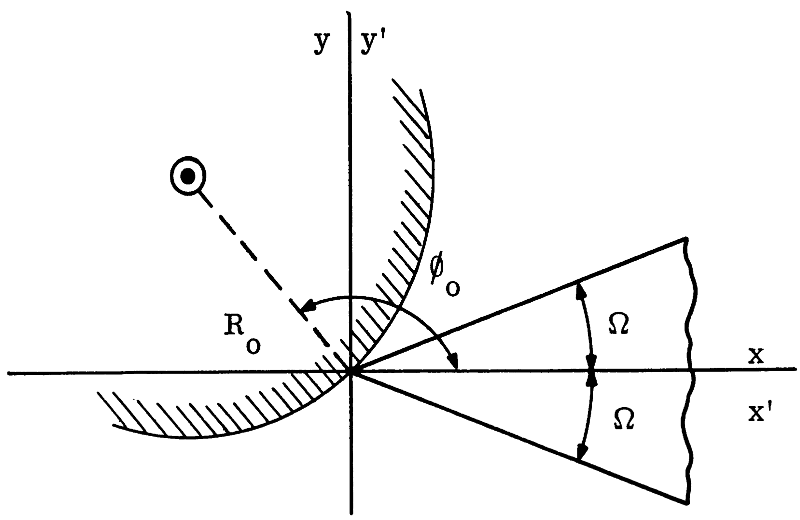


FIG. 5-1: PROBLEM GEOMETRY FOR LINE SOURCE.

Note that ϕ_o here defines the location of the line source rather than the direction of propagation of the pulse front as in the plane wave cases. The incident field in the primed frame is easily found from (5.1) if we recall that $c^2 t^2 - R^2$ is invariant under the Lorentz transformation. With this fact and Eq. (2.6) the incident field in the primed frame becomes

$$E_z^{i'}(R', \phi', t') = \frac{\gamma\left[1-\beta(\hat{p}_i \cdot \hat{x})\right] H\left[c^2(t'-t'_o)^2 - \mathbb{R}'^2(\phi'-\phi'_o)\right]}{\sqrt{c^2(t'-t'_o)^2 - \mathbb{R}'^2(\phi'-\phi'_o)}} . \tag{5.2}$$

For use in the Sommerfeld contour integral this equation is written

$$E_z^{i'}(R', \alpha+\phi'_o, t') = \frac{\gamma\left[1-\beta(\hat{p}_i \cdot \hat{x})\right] H\left[c^2(t'-t'_o)^2 - \mathbb{R}_1'^2(\alpha)\right]}{\sqrt{c^2(t-t_o)^2 - \mathbb{R}_1'^2(\alpha)}} \tag{5.3}$$

where $\mathbb{R}_1^2$ is given by Eq. (2.25). Upon making the transformation of (5.3) back to the unprimed frame (retarded coordinates) the desired expression for $\tilde{E}_z^i(\mathcal{R}, \Phi, t; \alpha)$ is obtained

$$\tilde{E}_z^i(\mathcal{R}, \Phi, t; \alpha) = \frac{\gamma\left[1-\gamma(\hat{p}_i \cdot \hat{x})\right] H\left[a-b\cos\alpha\right]}{\sqrt{a-b\cos\alpha}} \tag{5.4}$$

where

$$\begin{aligned} a = {} & \frac{c^2 T^2}{\gamma^2} - 2\beta ct\mathcal{R}(\cos\Phi-\beta) - \mathcal{R}^2(\cos\Phi-\beta)^2 \\ & - \mathcal{R}^2\sin^2\Phi + 2\,ct\,\mathcal{R}_o(1-\beta\cos\Phi_o) \\ & - 2\beta\gamma^2\mathcal{R}\mathcal{R}_o(\cos\Phi-\beta)(1-\beta\cos\Phi_o) \end{aligned} \tag{5.5}$$

and

$$b = 2\gamma^2 \mathcal{R}\mathcal{R}_o (1-\beta\cos\Phi)(1-\beta\cos\Phi_o) \quad . \tag{5.6}$$

5.2 The Integral for the Total Field

The integral representing the total field when a cylindrical impulse is scattered by a uniformly moving wedge is obtained by substituting Eq. (5.4) into Eq. (2.45). The result is

$$E_z^T(\mathcal{R}, \Phi, t) = K\int_{C_1+C_2} \frac{H[a-b\cos\alpha]\, d\alpha}{(A\cos\frac{\alpha}{\nu'}+B\sin\frac{\alpha}{\nu'}+C)\sqrt{a-b\cos\alpha}} \tag{5.7}$$

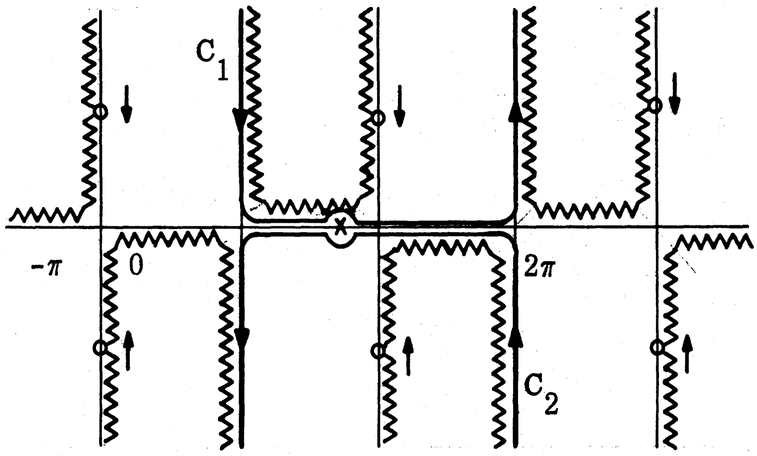
where

$$K = \frac{\gamma^2\left[1-\beta(\hat{x}\cdot\hat{p}_i)\right]\left[1+\beta(\hat{x}'\cdot\hat{p}')^{\sim}\right]D(\Phi)}{4\pi\nu' i} \tag{5.8}$$

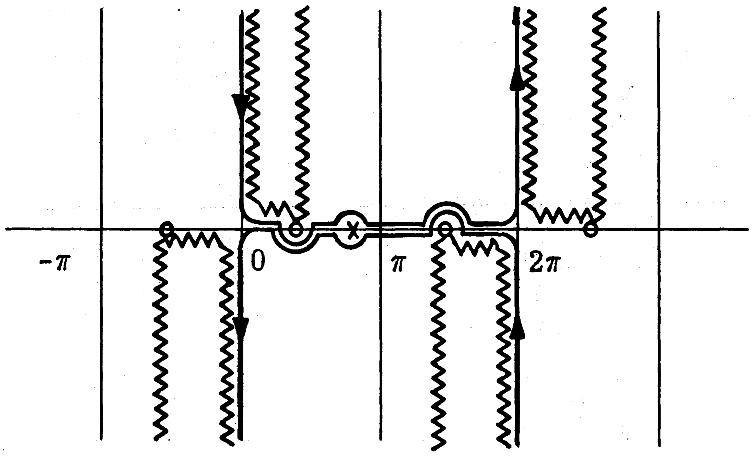
and A, B, C and D are given by Eqs. (2.39), (2.40), (2.43) and (2.44) respectively. The contours $C_1 + C_2$ in the complex α plane may be continuously deformed from those originally given in Fig. 2-2 to those shown in Fig. 5-2. The choice of this contour deformation is based upon the locations of the cuts in the complex α plane.

5.3 More Cuts in the Complex α plane

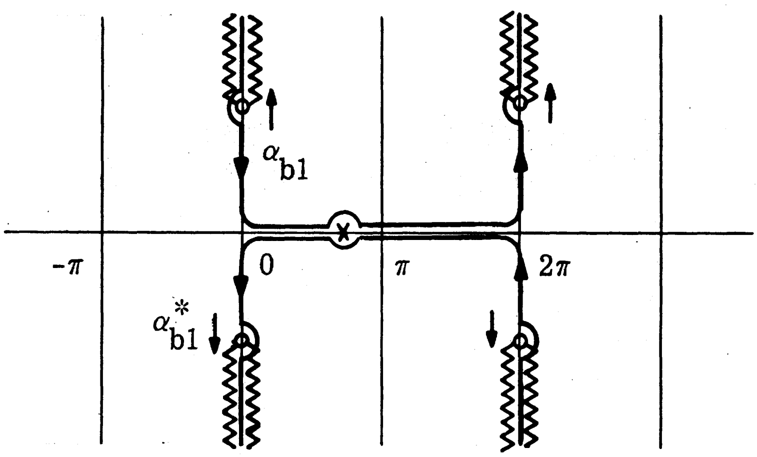
The integrand of the integral (5.7) for the total field exhibits the poles that were discussed in Section 3.3, the cuts of the Heaviside function that were discussed in Section 4.3 and in addition the branch points and cuts of the irrational function $1/\sqrt{a-b\cos\alpha}$. Notice that the branch points of the irrational function, $\cos\alpha_b = a/b$, are precisely the branch points of the Heaviside function. Recall that a is a function of $(\mathcal{R}, \Phi, t)$, ϕ_o and β (different for cylindrical waves than for plane waves) so that the branch points move. The motion of the branch points as a function



(a) $-\infty < a/b < 1$



(b) $-1 < a/b < 1$



(c) $1 < a/b < \infty$

FIG. 5-2: CONTOUR DEFORMATION FOR CYLINDRICAL IMPULSES.

of the ratio a/b was introduced in Appendix E and discussed in detail in Sections 4.3 and 4.4 for plane waves incident. The irrational function will be made single valued by the choice of cuts from the branch points α_b to infinity as shown in Fig. 5-2. The significance of the three different intervals corresponding to the values of a/b as shown in Fig. 5-2 is discussed in the next section.

5.4 Evaluation of the Integral Representing $E_z^T(\mathcal{R}, \Phi, t)$

In this section the evaluation of the integral given by (5.7) will be considered for each of the three intervals of time shown in Fig. 5-2. As an aid in interpreting the resulting integration we will first consider the physical significance of the three intervals for the ratio a/b. This will be done as in the plane wave case discussed in Section 4.4 by examining the ratio a/b when $\beta = 0$. For cylindrical waves incident and $\beta = 0$ the ratio a/b becomes

$$\frac{a}{b} = \frac{c^2 t^2 + 2ct R_o - R^2}{2 R R_o} . \tag{5.9}$$

The two critical points in time correspond to a/b equal -1 and $+1$ respectively. If we set $a/b = -1$ then by (5.9) we have that $t = -R/c$. This is recognized physically as the earliest possible time at which an observer can see the incident field. The motion of the branch points in the interval $-R_o/c < t < -R/c$, $R < R_o$ is along a vertical line $\alpha = n\pi + iy$ (n odd) to the point $\alpha = n\pi + i0$ when $t = -R/c$ or $a/b = -1$. In this time interval the observer does not see even the incident pulse. If we set $a/b = +1$ then by (5.9) we have that $t = R/c$. This is recognized physically as the earliest possible time at which an observer can see an edge diffracted pulse. The motion of the branch points in the interval $-R/c < t < R/c$ or $-1 < a/b < 1$ is along the real axis from $\alpha_b = n\pi + i0$ (n odd) to $\alpha_b = n\pi + i0$ (n even). Prior to $t = R/c$ only incident and reflected fields (pole contributions) may

may be observed. Subsequent to $t = R/c$ the diffracted field will be observed. Also subsequent to $t = R/c$ the branch points move away from the real axis along the lines $\alpha = n\pi + iy$ (n even).

Consider now the evaluation of the integral representing $E_z^T(\mathcal{R}, \Phi, t)$ in case (a) where $a/b < -1$. In this interval the branch points are located such that the entire contour $C_1 + C_2$ lies on the cut of the Heaviside function. Recalling that the integrand is zero on the cut as discussed in Appendix E we have that the total field during this time interval is zero; i.e:

$$E_z^T(\mathcal{R}, \Phi, t) = 0 \tag{5.10}$$

for $a/b < -1$.

The value of the integral in case (b) is due entirely to the contributions of the poles on the real axis. In the interval $-1 < a/b < 1$ the branch points have moved such that a portion of the contour along the real axis has been "uncovered" with regard to the cut of the Heaviside function. The value of the integral is determined by the integration around the branch points and the poles. The branch point integrations contribute nothing while the pole contributions are easily shown to yield

$$E_z^T(\mathcal{R}, \Phi, t) = \frac{\mp \gamma^2 \left[1 - \beta(\hat{x} \cdot \hat{p}_i)\right] \left[1 - \beta(\hat{x}' \cdot \hat{p}')^{\sim}\right] D(\Phi)\, H\left[a - b\cos\alpha_p\right]}{2\sqrt{A^2 + B^2 - C^2}\,\sqrt{a - b\cos\alpha_p}} \tag{5.11}$$

in the interval $-1 < a/b < 1$.

Recall that the presence of either one or both of these pole contributions depends upon the presence of the pole in the uncovered interval of the contour. The function $\cos\alpha_p$ is given in the case of cylindrical waves incident also by Eq. (3.10) where now we realize that the functions C and D are given by Eqs. (2.43) and (2.44). Equation (5.11) may be written as

$$E_z^T(\mathcal{R}, \Phi, t) = \sqrt{\frac{\gamma^2\, 1 - \beta(\hat{p}_i \cdot \hat{x})\, D(\Phi)}{2\sqrt{A^2 + B^2 - C^2}}}$$

$$\left\{ -\left[1 - \beta(p_i' \cdot x')^{\sim}\right] \frac{H\left[a - b\cos\alpha_{p1}\right]}{\sqrt{a - b\cos\alpha_{p1}}} \right.$$

$$\left. + \left[1 - \beta(p_r' \cdot x')^{\sim}\right] \frac{H\left[a - b\cos\alpha_{p2}\right]}{\sqrt{a - b\cos\alpha_{p2}}} \right\} \tag{5.12}$$

in order to more clearly illustrate the fact of incident and reflected pulses. An interpretation of Eq. (5.12) will be given in Section (5.5) for two specific values of wedge angle.

In case (c) the branch points have moved away from the real axis so that the Heaviside cut now uncovers some vertical portions of the contour in addition to the portion of the contour along the real axis. The result of this is that the integral for the total field (5.7) may be written as the sum of the pole contributions given by Eq. (5.11) and an integral representing the contribution given by the integration along the vertical portion of the contours. This latter integral represents the edge diffraction field $E_z^d(\mathcal{R}, \Phi, t)$

$$E_z^d(\mathcal{R}, \Phi, t) = K \int_{\alpha_{b1}}^{\alpha_{b1}^*} \frac{d\alpha}{\sqrt{a - b\cos\alpha}\left[A\cos\frac{\alpha}{\nu'} + B\sin\frac{\alpha}{\nu'} + C\right]}$$

$$+ K \int_{\alpha_{b1}^* + 2\pi}^{\alpha_{b1} + 2\pi} \frac{d\alpha}{\sqrt{a - b\cos\alpha}\left[A\cos\frac{\alpha}{\nu'} + B\sin\frac{\alpha}{\nu'} + C\right]}$$

(5.13)

where K is given by (5. 3) and

$$\alpha_{b1} = iY_b = i\cosh^{-1}\left(\frac{a}{b}\right) \; . \tag{5.14}$$

Several attempts at a closed form result for the integral in (5. 13) met with no success so the following series approach was used.

Consider the function f(α) where

$$f(\alpha) = \frac{1}{A\cos\frac{\alpha}{\nu'} + B\sin\frac{\alpha}{\nu'} + C} \; . \tag{5.15}$$

This function may also be written in the form

$$f(\alpha) = \frac{1}{\mathcal{A}}\left[\frac{M}{e^{i\frac{\alpha}{\nu'}} - e^{i\frac{\alpha_1}{\nu'}}} + \frac{N}{e^{i\frac{\alpha}{\nu'}} - e^{i\frac{\alpha_2}{\nu'}}}\right] \tag{5.16}$$

where

$$\mathcal{A} = \frac{1}{2}\left[A - iB\right] \quad , \tag{5.17}$$

$$M = \frac{1}{1 - e^{i\left(\frac{\alpha_1 - \alpha_2}{\nu'}\right)}} \quad , \tag{5.18}$$

and

$$N = \frac{1}{1 - e^{i\left(\frac{\alpha_2 - \alpha_1}{\nu'}\right)}} \tag{5.19}$$

and where α_1 and α_2 are the roots of the denominator function . Since the poles of f(α) are known to lie on the real axis the following series form is also valid.

$$f(\alpha) = \frac{e^{i\pi}}{\mathcal{A}}\left[M e^{-i\frac{\alpha_1}{\nu'}} \sum_{m=o}^{\infty} e^{i\frac{m}{\nu'}(\alpha-\alpha_1)} + N e^{-i\frac{\alpha_2}{\nu'}} \sum_{m=o}^{\infty} e^{i\frac{m}{\nu'}(\alpha-\alpha_2)}\right] . \tag{5.20}$$

With this expression for f(α) the integral (5. 13) may be rewritten as

$$E_z^d(\mathcal{R}, \Phi, t) = \frac{Ke^{i\pi}}{\mathcal{a}} \left\{ Me^{-i\frac{\alpha_1}{\nu'}} \sum_{m=o}^{\infty} e^{-i\frac{m}{\nu'}\alpha_1} \int_{\alpha_{b1}}^{\alpha_{b1}^*} \frac{e^{i\frac{m}{\nu'}\alpha} d\alpha}{\sqrt{a-b\cos\alpha}} \right.$$

$$+ Ne^{-i\frac{\alpha_2}{\nu'}} \sum_{m=o}^{\infty} e^{-i\frac{m}{\nu'}\alpha_2} \int_{\alpha_{b1}}^{\alpha_{b1}^*} \frac{e^{i\frac{m}{\nu'}\alpha} d\alpha}{\sqrt{a-b\cos\alpha}}$$

$$+ Me^{-i\frac{\alpha_1}{\nu'}} \sum_{m=o}^{\infty} e^{-i\frac{m}{\nu'}\alpha_1} \int_{\alpha_{b1}^*+2\pi}^{\alpha_{b1}+2\pi} \frac{e^{i\frac{m}{\nu'}\alpha} d\alpha}{\sqrt{a-b\cos\alpha}}$$

$$\left. + Ne^{-i\frac{\alpha_2}{\nu'}} \sum_{m=o}^{\infty} e^{-i\frac{m}{\nu'}\alpha_2} \int_{\alpha_{b1}^*+2\pi}^{\alpha_{b1}+2\pi} \frac{e^{i\frac{m}{\nu'}\alpha} d\alpha}{\sqrt{a-b\cos\alpha}} \right\} . \quad (5.21)$$

The integrals in (5.21) can be evaluated as follows. Along the contour from α_{b1} to α_{b1}^* we have $\alpha = 0 + iy$ so that

$$I = \int_{\alpha_{b1}}^{\alpha_{b1}^*} \frac{e^{i\frac{m}{\nu'}\alpha} d\alpha}{\sqrt{a-b\cos\alpha}} = \frac{i}{\sqrt{b}} \int_{Y_b}^{-Y_b} \frac{e^{-\frac{my}{\nu'}} dy}{\sqrt{\cosh Y_b - \cosh y}} \quad . \quad (5.22)$$

The irrational function g(y) where

$$g(y) = \frac{1}{\sqrt{\cosh Y_b - \cosh y}} \quad (5.23)$$

may be expanded in a Fourier cosine series assuming a periodicity of $2Y_b$. Thus

$$g(y) = \sum_{n=o}^{\infty} a_n \cos \frac{n\pi}{Y_b} y \quad (5.24)$$

where the a_n's are given by

$$a_n = \frac{2}{Y_b} \int_0^{Y_b} \frac{\cos \frac{n\pi}{Y_b} y \, dy}{\sqrt{\cosh Y_b - \cosh y}} \quad . \tag{5.25}$$

Recognizing the integral representation for the Legendre function $P_{-\frac{1}{2}+\lambda i}(\cosh Y_b)$ which is

$$P_{-\frac{1}{2}+\lambda i}(\cosh Y_b) = \frac{\sqrt{2}}{\pi} \int_0^{Y_b} \frac{\cos \lambda y \, dy}{\sqrt{\cosh Y_b - \cosh y}} \tag{5.26}$$

the series for g(y) becomes

$$g(y) = \frac{\sqrt{2}\,\pi}{Y_b} \sum_{n=o}^{\infty} P_{-\frac{1}{2}+\frac{n\pi i}{Y_b}} \left(\frac{a}{b}\right) \cos \frac{n\pi y}{Y_b} \quad . \tag{5.27}$$

Using (5.27) the expression for the integral I given by (5.22) may now be written as

$$I = \frac{\sqrt{2}\,\pi i}{\sqrt{b}\, Y_b} \sum_{n=o}^{\infty} P_{-\frac{1}{2}+\frac{n\pi i}{Y_b}} \left(\frac{a}{b}\right) \int_{Y_b}^{-Y_b} e^{-\frac{my}{\nu'}} \cos \frac{n\pi y}{Y_b} \, dy \quad . \tag{5.28}$$

The remaining integral is easily evaluated yielding

$$I = \frac{\sqrt{2}\,\pi i \, e^{i\pi} 2m}{\sqrt{b}\, Y_b \, \nu'} \sum_{n=o}^{\infty} P_{-\frac{1}{2}+\frac{n\pi i}{Y_b}} \left(\frac{a}{b}\right) \frac{\cos n\pi \sinh \frac{mY_b}{\nu'}}{\left[\left(\frac{m}{\nu'}\right)^2 + \left(\frac{n\pi}{Y_b}\right)^2\right]} \quad . \tag{5.29}$$

The two integrals in (5.21) which have limits $\alpha^*_{b1}+2\pi$ to $\alpha_{b1}+2\pi$ have a value of $-e^{\frac{2\pi mi}{\nu'}}\, I$. Finally, the diffraction field given by (5.21) may be written

$$E_z^d(\mathcal{R}, \Phi, t) = \frac{2^{3/2}\, m e^{i\frac{\pi}{2}}\, \pi K}{\sqrt{b}\, Y_b\, \mathcal{a}\, \nu'} \times$$

$$\Bigg\{ M e^{-i\frac{\alpha_1}{\nu'}} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} e^{-i\frac{m}{\nu'}\alpha_1} P_{-\frac{1}{2}+\frac{n\pi i}{Y_b}}\left(\frac{a}{b}\right) L_{mn}(Y_b, \nu')$$

$$+ N e^{-i\frac{\alpha_2}{\nu'}} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} e^{-i\frac{m}{\nu'}\alpha_2} P_{-\frac{1}{2}+\frac{n\pi i}{Y_b}}\left(\frac{a}{b}\right) L_{mn}(Y_b, \nu')$$

$$- M e^{-i\frac{\alpha_1}{\nu'}} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} e^{-i\frac{m}{\nu'}(\alpha_1 - 2\pi)} P_{-\frac{1}{2}+\frac{n\pi i}{Y_b}}\left(\frac{a}{b}\right) L_{mn}(Y_b, \nu')$$

$$- N e^{-i\frac{\alpha_2}{\nu'}} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} e^{-i\frac{m}{\nu'}(\alpha_2 - 2\pi)} P_{-\frac{1}{2}+\frac{n\pi i}{Y_b}}\left(\frac{a}{b}\right) L_{mn}(Y_b, \nu') \Bigg\}. \qquad (5.30)$$

where

$$L_{mn}(Y_b, \nu') = \frac{\left[\cos n\pi \sinh \frac{mY_b}{\nu'}\right]\left(\frac{m}{\nu'}\right)}{\left(\frac{m}{\nu'}\right)^2 + \left(\frac{n\pi}{Y_b}\right)^2} \, . \qquad (5.31)$$

The four series in (5. 30) may be combined yielding for the edge diffraction field when a cylindrical impulse is incident

$$E_z^d(\mathcal{R}, \Phi, t) = \frac{2^{3/2}\, \pi K e^{i\frac{\pi}{2}}}{\sqrt{b}\, Y_b\, \mathcal{a}} \times$$

$$\sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \left[M e^{-i\frac{\alpha_1}{\nu'}(m+1)} + N e^{-i\frac{\alpha_2}{\nu'}(m+1)} \right] \left(1 - e^{\frac{2\pi m i}{\nu'}}\right) L_{mn}(Y_b, \nu') P_{-\frac{1}{2}+\frac{n\pi i}{Y_b}}\left(\frac{a}{b}\right) .$$

(5. 32)

The total field in case (c) is given by the sum of (5. 12) and (5. 32). The expression (5. 32) is too unwieldy to use directly in making any physical statements regarding the behavior of the scattered field. For this reason we shall attempt to obtain a simpler representation for the two special cases of $\nu' = 1$ and $\nu' = 2$.

Consider equation (5. 13) for the edge diffraction field when $\nu' = 1$. In this case the periodicity of the integrand is 2π so that when the limits are reversed on the second integral the two integrals exactly cancel one another. This is also apparent from the factor

$$(1-e^{\frac{2\pi mi}{\nu'}})$$

in equation (5. 32). Since $\nu' = 1$ represents a flat sheet one would expect that the edge diffraction field would disappear.

Consider the equation (5. 13) in the case of a half plane where $\nu' = 2$. The evaluation of this integral parallels that of the evaluation of (3. 14) for the half plane and thus is included in Appendix D with that discussion. The result of integrating (5. 13) for the half plane is

$$E_z^d(\mathcal{R}, \Phi\, t) = \frac{2\pi\, iK}{\sqrt{A^2+B^2-C^2}} \left\{ \frac{1}{\sqrt{a-b+2bu_2^2}} - \frac{1}{\sqrt{a-b+2b\, u_1^2}} \right\} \tag{5. 33}$$

where u_1 and u_2 are given by (D. 12) and (D. 13) respectively. The total field when a cylindrical impulse is scattered by a uniformly moving half plane is given by the sum of equations (5. 12) and (5. 33) where again it is recalled that the contributions given by (5. 12) are only included if the poles lie in the interval $[0, 2\pi]$ on the real axis. The interpretation of these results are given in the next section.

5.5 Interpretation of Results

In this chapter as in the previous two the results for the total field will be examined for the special cases of the infinite flat sheet ($\nu' = 1$) and the half plane ($\nu' = 2$). The far zone total field will be considered for times $t > \mathcal{R}/C$ since then it is possible for an observer to see incident, reflected and diffracted fields provided he is in the correct position.

Consider first the infinite flat sheet. As discussed in the previous section, the integral (5.13) representing the edge diffracted field is zero so that the total electric field is given by equation (5. 12).

Evaluating Eq. (5.12) for the case $\nu' = 1$ we find that

$$\sin\frac{\Omega}{\nu} = \frac{1}{\gamma} \; ; \qquad \cos\frac{\Omega}{\nu} = \beta \tag{5.34}$$

$$A = \frac{1}{\gamma}\sin\frac{\Phi}{\nu}\left(1 - \beta\cos\frac{\Phi_o}{\nu}\right) \tag{5.35}$$

$$B = \left(\cos\frac{\Phi}{\nu} - \beta\right)\left(1 - \beta\cos\frac{\Phi_o}{\nu}\right) \tag{5.36}$$

$$C_c = \frac{1}{\gamma}\sin\frac{\Phi_o}{\nu}\left(1 - \beta\cos\frac{\Phi}{\nu}\right) \tag{5.37}$$

$$D_c = 2\left(\cos\frac{\Phi_o}{\nu} - \beta\right)\left(1 - \beta\cos\frac{\Phi}{\nu}\right) \tag{5.38}$$

$$\cos\alpha_{p1,2} = \frac{-\sin\Phi\sin\Phi_o \mp \gamma^2(\cos\Phi - \beta)(\cos\Phi_o - \beta)}{\gamma^2(1 - \beta\cos\Phi)(1 - \beta\cos\Phi_o)} \; . \tag{5.39}$$

The functions a and b are given by (5.5) and (5.6). Setting $\beta = 0$ in these equations and substituting in (5.12) the total field for a stationary infinite plate is obtained

$$E_z^T(R, \phi, t) = \frac{H\left[c^2\left(t + \frac{R_o}{c}\right)^2 - \mathbb{R}^2(\phi - \phi_o)\right]}{\sqrt{c^2\left(t + \frac{R_o}{c}\right)^2 - \mathbb{R}^2(\phi - \phi_o)}} - \frac{H\left[c^2\left(t + \frac{R_o}{c}\right)^2 - \mathbb{R}^2(\phi - \pi + \phi_o)\right]}{\sqrt{c^2\left(t + \frac{R_o}{c}\right)^2 - \mathbb{R}^2(\phi - \pi + \phi_o)}} \tag{5.40}$$

where $\mathbb{R}$ is given by Eq. (2.4). The Eq. (5.40) is recognized as representing the field due to a line source (the incident field) and the field due to an image of the line source. This is exactly what one would expect to find.

Consider the case of a stationary half plane for which $\nu' = 2$. In this case we find that

$$\sin\frac{\Omega}{\nu} = 0 \quad ; \qquad \cos\frac{\Omega}{\nu} = 1 \tag{5.41}$$

$$A = -\gamma \sin\frac{\Phi}{\nu}(1-\beta)(1-\beta\cos\frac{\Phi_o}{\nu}) \tag{5.41}$$

$$B = -\gamma^2(\cos\frac{\Phi}{\nu} - \beta)(1-\beta)(1-\beta\cos\frac{\Phi_o}{\nu}) \tag{5.42}$$

$$C_c = \gamma^2(\cos\frac{\Phi_o}{\nu} - \beta)(1-\beta)(1-\beta\cos\frac{\Phi}{\nu}) \tag{5.43}$$

$$D_c = -2\gamma\sin\frac{\Phi_o}{\nu}(1-\beta)(1-\beta\cos\frac{\Phi}{\nu}) \tag{5.44}$$

$$\cos\alpha_{p1,2} = \frac{-\gamma^2(\cos\Phi-\beta)(\cos\Phi_o-\beta) \mp \sin\Phi\sin\Phi_o}{\gamma^2(1-\beta\cos\Phi)(1-\beta\cos\Phi_o)} \tag{5.45}$$

$$\mu^2_{1,2} = \frac{1}{2}\left[1 + \frac{\gamma^2(\cos\Phi-\beta)(\cos\Phi_o-\beta) \pm \sin\Phi\sin\Phi_o}{\gamma^2(1-\beta\cos\Phi)(1-\beta\cos\Phi_o)}\right]. \tag{5.46}$$

Setting $\beta = 0$ and substituting in Eqs. (5.12) and (5.33) we obtain

$$E_z^T(R, \phi, t) = \frac{H\left[c^2(t+\frac{R_o}{c})^2 - \mathbb{R}^2(\phi-\phi_o)\right]}{\sqrt{c^2(t+\frac{R_o}{c})^2 - \mathbb{R}^2(\phi-\phi_o)}}$$

$$- \frac{H\left[c^2(t+\frac{R_o}{c})^2 - \mathbb{R}^2(\phi+\phi_o)\right]}{\sqrt{c^2(t+\frac{R_o}{c})^2 - \mathbb{R}^2(\phi+\phi_o)}}$$

$$+ \frac{1}{2}\left\{\frac{1}{\sqrt{c^2(t+\frac{R_o}{c})^2 - \mathbb{R}^2(\phi-\phi_o)}} - \frac{1}{\sqrt{c^2(t+\frac{R_o}{c})^2 - \mathbb{R}^2(\phi+\phi_o)}}\right\} \tag{5.47}$$

which is the result obtained by Friedlander (1958) as Green's function for the stationary half plane.

The effect of the velocity will first be noted by considering the infinite flat sheet. Using Eqs. (5.35) through (5.39) note that

$$\frac{D(\Phi)}{2\sqrt{A^2 + B^2 - C^2}} = -1 \quad . \tag{5.48}$$

The total field may now be written

$$E_z^T(\mathcal{R}, \Phi, t) = \gamma^2\left[1 - \beta(\hat{p}_i \cdot \hat{x})\right]\left\{\left[1+\beta(\hat{p}_i' \cdot \hat{x}')^\sim\right]\frac{H\left[a - b\cos\alpha_{p1}\right]}{\sqrt{a - b\cos\alpha_{p1}}}\right.$$

$$\left. - \left[1+\beta(\hat{p}_r' \cdot \hat{x}')^\sim\right]\frac{H\left[a - b\cos\alpha_{p2}\right]}{\sqrt{a - b\cos\alpha_{p2}}}\right\} \quad . \tag{5.49}$$

The unit vectors $\hat{p}$ represent the direction of propagation of the far zone field and are given by

$$\hat{p}_i = \frac{(R\cos\phi - R_o\cos\phi_o)\,\hat{x} + (R\sin\phi - R_o\sin\phi_o)\,\hat{y}}{\mathbb{R}(\phi-\phi_o)} \tag{5.50}$$

$$\hat{p}_i = \frac{(R'\cos\phi' - R'_o\cos\phi'_o)\hat{x} + (R'\sin\phi' - R'_o\sin\phi'_o)\hat{y}'}{\mathbb{R}'(\phi'-\phi'_o)} \tag{5.51}$$

$$\hat{p}'_r = \frac{\left[R'\cos\phi' - R'_o\cos(\pi-\phi'_o)\right]\hat{x}' + \left[R'\sin\phi' - R'_o\sin(\pi-\phi'_o)\right]\hat{y}'}{\mathbb{R}'(\phi'-\pi+\phi'_o)} \,. \tag{5.52}$$

These three equations along with Eqs. (5.5), (5.6), (5.39) and the transformations (B.17) and (B.18) completely specify the total field (5.49) when a cylindrical impulse is scattered by a uniformly moving infinite flat plate. The first term in (5.49) represents the incident field the second term the reflected field. Finally consider the effect of motion upon the half plane solution. We note that (5.48) holds for the half plane also so that the expression for the total field becomes

$$\begin{aligned} E_z^T(\mathcal{R}, \Phi, t) = \gamma^2\left[1-\beta(\hat{p}_i\cdot\hat{x})\right]\Bigg\{&\left[1+\beta(\hat{p}'_i\cdot\hat{x}')^{\sim}\right]\frac{H\left[a-b\cos\alpha_{p1}\right]}{\sqrt{a-b\cos\alpha_{p1}}} \\ &-\left[1+\beta(\hat{p}'_r\cdot\hat{x}')^{\sim}\right]\frac{H\left[a-b\cos\alpha_{p2}\right]}{\sqrt{a-b\cos\alpha_{p2}}} \\ &+\left[1+\beta(\hat{p}'_d\cdot\hat{x}')^{\sim}\right]\left[\frac{1}{\sqrt{a-b+2b\mu_1^2}}-\frac{1}{\sqrt{a-b+2b\mu_2^2}}\right]\Bigg\} \end{aligned} \tag{5.53}$$

where now

$$\hat{p}'_r = \frac{(R'\cos\phi' - R'_o\cos\phi'_o)\hat{x}' + (R'\sin\phi' + R'\sin\phi'_o)\hat{y}'}{\mathbb{R}(\phi' + \phi'_o)} \tag{5.54}$$

$$\hat{p}'_d = \cos\phi'\,\hat{x}' + \sin\phi'\,\hat{y}' \tag{5.55}$$

and $\hat{p}'_i$ is the same as previously given (5.51). These three equations along with Eqs. (5.5), (5.6), (5.45) and (5.46) completely specify the total field (5.53) when a cylindrical impulse is scattered by a moving half plane. The first term in (5.53) represents the incident field, the second the reflected field and the third the edge diffracted field.

Chapter VI

SUMMARY AND RECOMMENDATIONS FOR FUTURE WORK

A discussion of the physical significance of the total field which results when each of the incident pulses interacts with the moving wedge is given at the end of the respective chapter which deals with that pulse. In each instance the interpretation has been made for the two special cases of an infinite flat plane and a semi-infinite or half plane. What we wish to do here is to summarize those results for the special case of the moving half plane.

For the half plane we have considered the incident field, the reflected field and the edge diffracted field. The motion of the scatterer should have no effect upon the incident field and (fortunately) our results indicate this. The field reflected by a moving half plane is seen to be the same as the field reflected by a stationary half plane so that the motion of the scatter would seem to have no effect here either. However, an observer in a fixed position will see a transient in the reflected field as the shadow boundary crosses the observers position. The edge diffraction field is affected by the motion of the half plane. A stationary half plane exhibits a certain diffraction pattern for the edge diffracted field. As the half plane moves a stationary observer might expect to see this pattern move past him as is true for the reflected field. We find, however, that the shape of the edge diffraction field pattern is changed and it is this new pattern which a stationary observer sees moving past him. Considering the edge diffraction field for the case of the time harmonic plane wave we find that

$$E_z^d(\mathcal{R}, \Phi, t) = \frac{(1 - \beta \cos \phi_o)}{(1 - \beta \cos \Phi)} \, e^{ik\beta \cos \phi_o (ct - \mathcal{R})} \, E_{zo}^d(\mathcal{R}, \Phi, t) \qquad (3.57)$$

where $E_{zo}^d(\mathcal{R}, \Phi, t)$ is the edge diffraction pattern of a stationary half plane. Examining this for a given ϕ_o one finds the lobes of stationary pattern increased in the direction of the velocity and reduced in the opposite direction. The exact shape of the new diffraction pattern can be found by a pattern multiplication of $\left|E_{zo}^d\right|$ with the pattern of Fig. 3-2.

This is precisely the effect which Tsandoulos (1968) reports. It should be noted that Tsandoulos considered only the asymptotic form of the edge diffracted field of a moving half plane. He states in his conclusion that to extract any meaningful results from the exact Fresnel integral formulation would be very difficult. We note this here only to point out the fact that by using the retarded reference position meaningful results (Eq. (3.56) or (3.57)) have been obtained from the exact Fresnel integral formulation for the moving half plane.

A similar effect is seen in the diffraction field when a plane unit step pulse is incident. In this case the edge diffraction field is given

$$E_z^d(\mathcal{R}, \Phi, t) = \frac{2(1-\beta)}{\pi(1-\beta\cos\Phi)} \tan^{-1}\left[\frac{\sqrt{1-\beta}}{\sin\frac{\Phi}{2}}\sqrt{\frac{ct-\mathcal{R}}{2\mathcal{R}}}\right] . \qquad (4.28)$$

Here the factor $\sqrt{1-\beta}$ in the argument does not change the shape of the diffraction pattern but the factor $(1-\beta\cos\Phi)$ does change its shape and, in fact, in exactly the same way as in the plane wave case in Eq. (3.57).

The results for the cylindrical impulse incident cannot be written in as simple a form as the above two cases and thus their interpretation is not readily apparent. Further analysis of the result for cylindrical waves will be required before any conclusive statements can be made.

The present results may be used as a first step in developing a ray-optic technique for computing the fields scattered by two dimensional bodies moving with uniform velocity. By considering an arbitrary body as a composite structure of planes and edges and the incident field locally as a plane wave the scattered field may be synthesized. The reflection by moving planes has previously been established and the effect of the moving edge is contributed by this work. The sum of these contributions should yield an engineering approximation to the value of the field at a given point. The validity of such an engineering approximation, however, should be tested by experimentation.

Part of the original objective of this study was to investigate the effect upon the scattered field of more general motions of the scattering body such as acceleration.

However, due to the lack of a plausible physical explanation of the boundary conditions which exist on an accelerating surface no detailed investigation was carried out except as was discussed in Appendix B. An engineering approximation to the field scattered by an accelerating two dimensional body could be obtained using the above described ray-optics technique in connection with the assumption that the problem may be considered locally in time as involving only uniform motion.

Extensions of this work might include a study of the problem for the case where the incident field is described as a magnetic vector polarized parallel to the edge. Such an investigation would follow closely the procedures used in this study. Another possibility is the investigation of scattering by uniformly moving three-dimensional bodies. This would require first an analysis of the effect of a moving corner; i.e., the tip of a uniformly moving cone. Following this a three-dimensional body could be considered a composite of planes, edges and corners and an approximate engineering solution synthesized using a ray-optics technique.

REFERENCES

Born, M. and E. Wolf (1964), Principles of Optics, Pergamon Press.

Cole, C. F. (1958), "The Characteristics of an Electromagnetic Wave Reflected From a Moving Object," J. Franklin Inst., 265, No. 6, pp. 463-472.

Friedlander, F. G. (1958), Sound Pulses, Cambridge University Press.

Gradshteyn, I. S. and I. M. Ryzhik (1965), Table of Integrals, Series and Products, Academic Press.

Gröebner, W. and N. Hofreiter (1950), Integraltafel - Bestimmte, Integral, Springer-Verlag.

Kline, M. and I. Kay (1965), Electromagnetic Theory and Geometrical Optics, Interscience Publishers.

Lee, S.W. and R. Mittra (1967), "Scattering of Electromagnetic Waves by a Moving Cylinder in Free Space," Canadian J. Physics, 45, No. 9, pp. 2999-3007.

Lewis, R. M. (1964), "Progressing Wave Formalism," Proceedings of the Symposium on Quasi-Optics, Polytechnic Press.

Morgans, W. R. (1930), "The Kirchhoff Formula Extended to a Moving Surface," London Philos. Mag., 9, Seventh Series, No. 55, pp. 141-161.

Ott, R. H. and G. Hufford (1968), "Scattering by an Arbitrarily Shaped Conductor in Uniform Motion Relative to the Source of an Incident Spherical Wave," Radio Science, 3, No. 8, pp. 857-861.

Post, E. J. (1963), The Formal Structure of Electromagnetics, North-Holland Publishers.

Restrick, R. C. (1967), "Electromagnetic Scattering by Moving Bodies," Ph. D Thesis, The University of Michigan.

Sommerfeld, A. (1964), Optics , Academic Press, Ch. 13, pp. 72-75.

Tsandoulas G. N. (1968), "Electromagnetic Diffraction/by a Moving Wedge," Radio Science, 3, No. 9, pp. 887-893.

Tuzhilin, A. A. (1963), "New Representations of Diffraction Fields in Wedge-Shaped Regions with Ideal Boundaries," Soviet Physics, Acoustics, 9, No. 2, pp. 168-172.

APPENDIX A

THE NATURAL INVARIANCE OF MAXWELL'S EQUATIONS

Maxwell's equations in free space may be written in terms of a stationary reference frame as

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \qquad (A.1)$$

$$\nabla \cdot \bar{B} = 0 \qquad (A.2)$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \qquad (A.3)$$

$$\nabla \cdot \bar{D} = \rho \qquad (A.4)$$

where $\bar{H}$ and $\bar{D}$ are related to $\bar{B}$ and $\bar{E}$ in free space by the constitutive relations

$$\bar{H} = \frac{1}{\mu_o} \bar{B} \qquad (A.5)$$

$$\bar{D} = \epsilon_o \bar{E} \quad . \qquad (A.6)$$

The Theory of Relativity postulates that the laws governing natural phenomena are invariant in form with respect to the coordinate system in which the laws are described. This holds for reference frames in uniform or nonuniform translational motion. A formalism exists by which Maxwell's equations may be rewritten in a four-dimensional form and transformed by a set of coordinate transformations defined such that the forms of these equations are invariant. The following discussion is a summary of that given by Post (1962). The four-dimensional tensor form of Maxwell's equations, i.e., the Maxwell-Minkowski equations are:

$$\frac{\partial F_{\beta\gamma}}{\partial x^{\alpha}} + \frac{\partial F_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial F_{\gamma\alpha}}{\partial x^{\beta}} = 0 \tag{A.7}$$

$$\frac{\partial f^{\alpha\beta}}{\partial x^{\beta}} = C^{\alpha} \tag{A.8}$$

$$f^{\alpha\beta} = \frac{1}{2}\,\chi^{\alpha\beta\gamma\delta}\,F_{\gamma\delta} \tag{A.9}$$

where

$$F_{\alpha\beta} = \begin{pmatrix} 0 & B_z & -B_y & E_x \\ -B_z & 0 & B_x & E_y \\ B_y & -B_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} \tag{A.10}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & H_z & -H_y & -D_x \\ -H_z & 0 & H_x & -D_y \\ H_y & -H_x & 0 & -D_z \\ D_x & D_y & D_z & 0 \end{pmatrix} \tag{A.11}$$

$$C^{\alpha} = (J_x, J_y, J_z, \rho) \tag{A.12}$$

with

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = t$$

and

$$\alpha, \beta, \gamma, \delta = 1, 2, 3, 4 \quad .$$

Equations (A.1) and (A.2) are included in (A.7), Eqs. (A.3) and (A.4) in (A.8) and the constitutive relations in (A.9). The nonzero terms in the constitutive tensor are

$$\chi^{1212} = \chi^{1313} = \chi^{2323} = \frac{1}{\mu_o} \tag{A.13}$$

$$\chi^{1414} = \chi^{2424} = \chi^{3434} = -\epsilon_o \tag{A.14}$$

and

$$\chi^{\alpha\beta\gamma\delta} = -\chi^{\alpha\beta\delta\gamma} = -\chi^{\beta\alpha\gamma\delta} = \chi^{\gamma\delta\alpha\beta} \quad . \tag{A.15}$$

A new coordinate system may now be defined such that the coordinates of a point χ^{α} are found by the transformation

$$x^{\alpha'} = x^{\alpha'}(x^{\alpha}) \quad . \tag{A.16}$$

Consider only transformations such that

$$\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial x^{\alpha'}}{\partial x^{\beta}}\right) = \frac{\partial}{\partial x^{\beta}}\left(\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\right) \; . \tag{A.17}$$

Now, let $F_{\alpha\beta}$, $f^{\alpha\beta}$, C^{α} and $\chi^{\alpha\beta\delta\gamma}$ transform according to the transformations

$$F_{\alpha'\beta'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} F_{\alpha\beta} \tag{A.18}$$

$$f^{\alpha'\beta'} = |\Delta|^{-1} \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta'}}{\partial x^{\beta}} f^{\alpha\beta} \tag{A.19}$$

$$C^{\alpha'} = |\Delta|^{-1} \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} C^{\alpha} \tag{A.20}$$

$$\chi^{\alpha'\beta'\delta'\gamma'} = \Delta^{-1} \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta'}}{\partial x^{\beta}} \frac{\partial x^{\gamma'}}{\partial x^{\gamma}} \frac{\partial x^{\delta'}}{\partial x^{\delta}} \chi^{\alpha\beta\gamma\delta} \tag{A.21}$$

where Δ is the determinant of the Jacobian matrix. Substituting (A.10), (A.11) and (A.12) into (A.18), (A.19) and (A.20) and making use of Eq. (A.17) obtain the Maxwell - Minkowski equations in the primed frame:

$$\frac{\partial F_{\beta'\gamma'}}{\partial x^{\alpha'}} + \frac{\partial F_{\alpha'\beta'}}{\partial x^{\gamma'}} + \frac{\partial F_{\gamma'\alpha'}}{\partial x^{\beta'}} = 0 \tag{A.22}$$

$$\frac{\partial f^{\alpha'\beta'}}{\partial x^{\beta'}} = C^{\alpha'} \tag{A.23}$$

and

$$f^{\alpha'\beta'} = \frac{1}{2} \chi^{\alpha'\beta'\delta'\gamma'} F_{\gamma'\delta'} \tag{A.24}$$

where $\chi^{\alpha'\beta'\delta'\gamma'}$ is defined by Eq. (A.21). The elements of the primed tensors are functions of the transformation and the field quantities in the unprimed frame. If the elements in the tensors (A.22), (A.23) and (A.24) are defined by Eqs. (A.10), (A.11) and (A.12), where every term would now have to carry a prime, the equations of electrodynamics in the primed frame become

$$\nabla' \times \bar{E}' = -\frac{\partial \bar{B}'}{\partial t'} \tag{A.25}$$

$$\nabla' \cdot \bar{B}' = 0 \tag{A.26}$$

$$\nabla' \times \bar{H}' = \bar{J}' + \frac{\partial \bar{D}'}{\partial t'} \tag{A.27}$$

$$\nabla' \cdot \bar{D}' = \rho' \tag{A.28}$$

with constitutive relations given by

$$\bar{H}' = \bar{H}'(\bar{B}', \bar{E}') \tag{A.29}$$

$$\bar{D}' = \bar{D}'(\bar{B}', \bar{E}') \quad . \tag{A.30}$$

At this point it is worthwhile to note that the expressions $\bar{E}'$, $\bar{B}'$, $\bar{H}'$, $\bar{D}'$ were introduced on a purely mathematical basis and may or may not physically represent the fields which can be experimentally determined. For cases of uniform translational motion of coordinate reference frames the transformations (A.16) are linear representing rotations of the space time coordinate axis. The resulting expressions for $\bar{E}'$, $\bar{B}'$ etc. predict precisely the fields which can be experimentally determined. For cases of non-uniform motion such physical corroboration does not exist and therefore the quantities $\bar{E}'$, $\bar{B}'$ etc. are treated as mathematical quantities which may or may not have the same physical signifigance as in the uniform velocity case.

The problems considered in this research involve uniform motion in the positive x direction with a velocity $v = \beta c$. The transformations (A.16) for this case are known as the Lorentz transformations

$$x' = \gamma(x - \beta ct) \tag{A.31}$$

$$y' = y \tag{A.32}$$

$$z' = z \tag{A.33}$$

$$t' = \gamma(t - \frac{\beta}{c} x) \tag{A.34}$$

where

$$\gamma = (1 - \beta^2)^{-1/2} \quad .$$

The transformation of the field tensors by the Lorentz transformation results in the vector field quantities in the primed frame being given as

$$\bar{E}'_{||} = (\bar{E} + \bar{v} \times \bar{B})_{||} \tag{A.35}$$

$$\bar{E}'_{\perp} = \gamma (\bar{E} + \bar{v} \times \bar{B})_{\perp} \tag{A.36}$$

$$\bar{B}'_{||} = (\bar{B} - \frac{1}{C^2} \bar{v} \times \bar{E})_{||} \tag{A.37}$$

$$\bar{B}'_{\perp} = \gamma (\bar{B} - \frac{1}{C^2} \bar{v} \times \bar{E})_{\perp} \tag{A.38}$$

$$\bar{D}' = \epsilon_o \bar{E}' \tag{A.39}$$

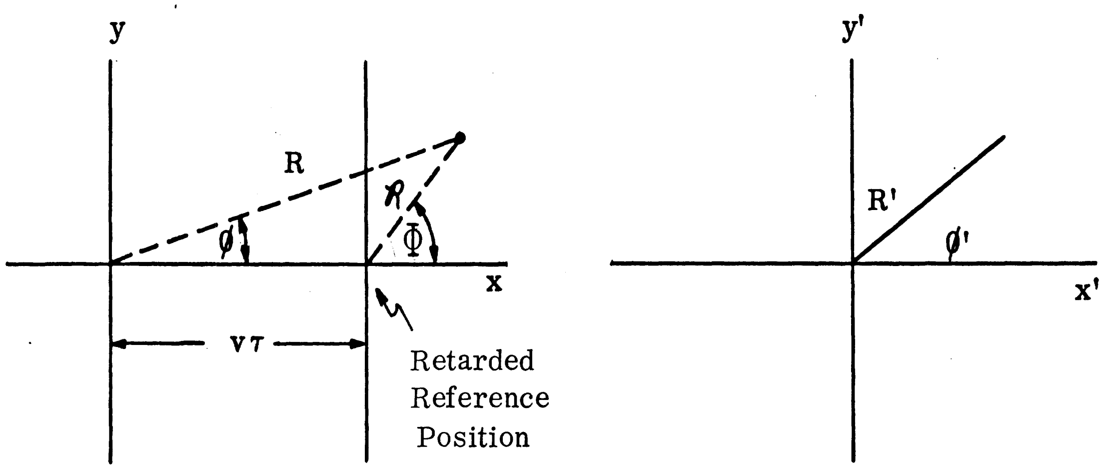
$$\bar{H}' = \frac{1}{\mu_o} \bar{B}' \tag{A.40}$$

where $\perp$ and $||$ indicate the field components perpendicular and parallel to the velocity $\bar{v}$.

APPENDIX B

TABLE OF COORDINATE TRANSFORMATIONS

The coordinate reference frames used throughout this work are illustrated in Fig. B-1.



Unprimed or Laboratory Frame

Primed or Moving Frame

FIG. B-1: COORDINATE REFERENCE FRAMES

The Lorentz transformations relate the location of a world point (x, y, t) in the unprimed frame to a work point (x', y', t') in the primed frame. These transformations are

$$y = y' \tag{B.1}$$

$$x = \gamma\ (x' + \beta\, ct') \tag{B.2}$$

$$t = \gamma\ (t' + \frac{\beta}{c}\, x') \tag{B.3}$$

where $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$. Conversely;

$$y' = y \tag{B.4}$$

$$x' = \gamma(x - \beta ct) \tag{B.5}$$

$$t' = \gamma(t - \frac{\beta}{c}x) \quad . \tag{B.6}$$

In order to be able to transform from the primed frame directly to the unprimed retarded reference position we note from Fig. B-1 that:

$$x = \mathcal{R}\cos\Phi + v\tau \tag{B.7}$$

$$y = \mathcal{R}\sin\Phi \tag{B.8}$$

$$t = \tau + \frac{\mathcal{R}}{c} \tag{B.9}$$

(B.9) indicates the time t at which a field diffracted at $(v\tau, 0, \tau)$ is observed at $(\mathcal{R}, \Phi)$.

In the primed frame obtain

$$x' = R'\cos\phi' \tag{B.10}$$

$$y' = R'\sin\phi' \tag{B.11}$$

$$t' = t' \tag{B.12}$$

Substitution of (B.7), (B.8), (B.9) and (B.10), (B.11), (B.12) into (B.1), (B.2) and (B.3) results in the coordinate transformations from the primed to the unprimed retarded coordinates.

$$\cos\Phi = \frac{\cos\phi' + \beta}{1 + \beta\cos\phi'} \tag{B.13}$$

$$\mathcal{R} = \gamma\,(1+\beta\cos\phi')\,R' \tag{B.14}$$

$$\tau = \gamma\,(t' - \frac{R'}{c}) \tag{B.15}$$

or

$$t = \gamma\left[t' + \frac{\beta}{c}R'\cos\phi'\right] . \tag{B.16}$$

And the inverse

$$\cos\phi' = \frac{\cos\Phi - \beta}{1 - \beta\cos\Phi} \tag{B.17}$$

$$R' = \gamma\,(1 - \beta\cos\Phi)\mathcal{R} \tag{B.18}$$

$$t' = \frac{\tau}{\gamma} + \frac{\gamma}{c}\,(1 - \beta\cos\Phi)\mathcal{R} \tag{B.19}$$

or

$$t' = \frac{t}{\gamma} - \frac{\beta\gamma\mathcal{R}}{c}\,(\cos\Phi - \beta)\ . \tag{B.20}$$

Physically the retarded reference position represents the position of the scatterer at the time the currently observed wave was initially scattered. By using a retarded reference to express the results it becomes easier to obtain their physical interpretation. The reason for this is that the variable t does not appear in as many places as it otherwise would.

APPENDIX C

POLE LOCATIONS IN THE COMPLEX α PLANE

Consider the pole locations given by Eq. (3.10)

$$\cos\frac{\alpha}{\nu} = \frac{-AC \pm B\sqrt{A^2+B^2-C^2}}{A^2+B^2} \quad . \tag{3.10}$$

It will first be shown that $A^2+B^2-C^2>0$. By virtue of Eq. (2.37) and (2.38), $A^2+B^2=A_1^2+B_1^2$. Now:

$$A_1^2 = \gamma^2(1-\beta\cos\frac{\phi_o}{\nu})^2\left[\sin\frac{\Omega}{\nu}(\cos\frac{\Phi}{\nu}-\beta) - \sin\frac{\Phi}{\nu}(\cos\frac{\Omega}{\nu}-\beta)\right]^2 \tag{C.1}$$

$$B_1^2 = (1-\beta\cos\frac{\phi_o}{\nu})^2\left[\gamma^2(\cos\frac{\phi}{\nu}-\beta)(\cos\frac{\Omega}{\nu}-\beta) + \sin\frac{\Phi}{\nu}\sin\frac{\Omega}{\nu}\right]^2 \quad . \tag{C.2}$$

Adding these together and making use of the identity

$$\gamma^2(1-\beta\cos\phi)^2 = \gamma^2(\cos\phi-\beta)^2 + \sin^2\phi \tag{C.3}$$

obtain

$$A^2+B^2 = \gamma^4(1-\beta\cos\frac{\phi_o}{\nu})^2(1-\beta\cos\frac{\Omega}{\nu})^2(1-\beta\cos\frac{\Phi}{\nu})^2 \quad . \tag{C.4}$$

In the plane wave case C^2 is given by:

$$C_p^2 = \gamma^2\sin^2\frac{\phi_o}{\nu}(1-\beta\cos\frac{\Phi}{\nu})^2(1-\beta\cos\frac{\Omega}{\nu})^2 \quad . \tag{C.5}$$

Subtracting and using identity (C.3) obtain:

$$A^2 + B^2 - C^2 = \gamma^4 (1 - \beta\cos\frac{\Phi}{\nu})^2 (1 - \beta\cos\frac{\Omega}{\nu})^2 (\cos\frac{\phi_o}{\nu} - \beta)^2 \qquad (C.6)$$

which verifies the original assertion in the plane wave case.
In the cylindrical wave case C^2 is given by:

$$C_c^2 = (1 - \beta\cos\frac{\Phi}{\nu})^2 \left[\gamma^2 (\cos\frac{\phi_o}{\nu} - \beta)(\cos\frac{\Omega}{\nu} - \beta) + \sin\frac{\phi_o}{\nu}\sin\frac{\Omega}{\nu}\right]^2 . \qquad (C.7)$$

Multiplying this out, subtracting from (C.4) and making use of the identity (C.3) it is found that $A^2 + B^2 - C^2$ can be represented as:

$$A^2 + B^2 - C^2 = \gamma^2 (1 - \beta\cos\frac{\Phi}{\nu})^2 \left[\sin\frac{\phi_o}{\nu}(\cos\frac{\Omega}{\nu} - \beta) - \sin\frac{\Omega}{\nu}(\cos\frac{\phi_o}{\nu} - \beta)\right]^2 \qquad (C.8)$$

which verifies the assertion in the cylindrical wave case

Since $A^2 + B^2 - C^2 > 0$, then

$$\cos\frac{\alpha}{\nu'} = \left[\text{real number}\right] .$$

Straightforward algebra shows this number to lie in the interval $[-1, 1]$ and hence it is concluded that α/ν' lies on the real axis in the α plane and in the interval $[0, 2\pi]$. Depending upon ν', α itself may lie outside of this interval.

APPENDIX D

THE DIFFRACTION FIELD OF A MOVING HALF PLANE

Consider the evaluation of

$$I = K e^{-ika} \int_{D_1+D_2} \frac{e^{ikb\cos\alpha}\, d\alpha}{A\cos\frac{\alpha}{2} + B\sin\frac{\alpha}{2} + C} \tag{3.18}$$

where $D_1 + D_2$ are the contours of Fig. 3-1 repeated here for convenience.

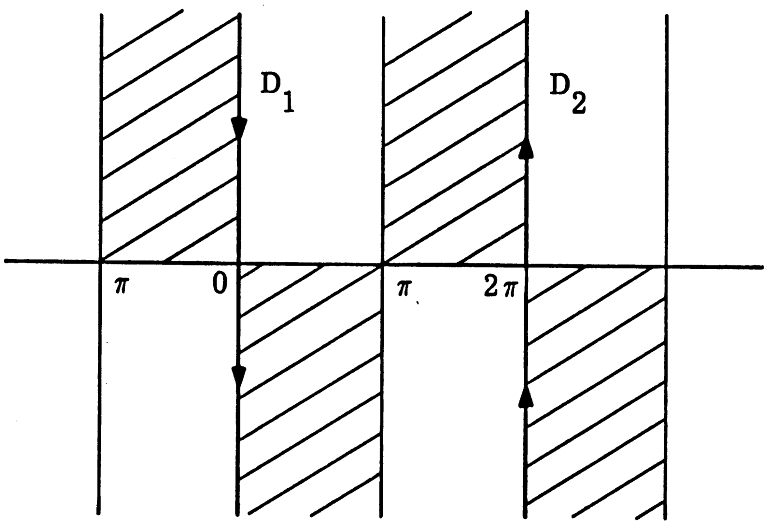


FIG. 3-1: CONTOUR DEFORMATION FOR TIME HARMONIC PLANE WAVE

Along these contours

$$\cos\alpha = \cos(0+iy) = \cos(2\pi + iy) = \cosh y \quad . \tag{D.1}$$

On D_1

$$\cos\frac{\alpha}{2} = \cosh\frac{y}{2} \qquad \text{(D.2)}$$

$$\sin\frac{\alpha}{2} = i\sinh\frac{y}{2} \qquad \text{(D.2)}$$

and on D_2

$$\cos\frac{\alpha}{2} = -\cosh\frac{y}{2} \qquad \text{(D.3)}$$

$$\sin\frac{\alpha}{2} = -i\sinh\frac{y}{2} \quad . \qquad \text{(D.3)}$$

Making the substitution $\alpha = 0+iy$ on D_1 and $\alpha = 2\pi + iy$ on D_2 the integral I, using identities E1 - E3, becomes:

$$I = -iKe^{-ika}\int_{-\infty}^{\infty} dy\, e^{ikb\cosh y}\left\{\frac{1}{A\cosh\frac{y}{2}+iB\sinh\frac{y}{2}+C} + \frac{1}{A\cosh\frac{y}{2}+iB\sinh\frac{y}{2}-C}\right\} . \qquad \text{(D.4)}$$

Decompose the integral $(-\infty, \infty)$ into two integrals $(-\infty, 0)$, $(0, \infty)$. In the integral $(-\infty, 0)$ make the substitution $y = -y$ thus transforming the limits to $(0, \infty)$. Finally substitute

$$\cosh y = 2\cosh^2\frac{y}{2} - 1 \qquad \text{(D.5)}$$

and obtain for I:

$$I = -iK\,e^{-ik[a+b]} \int_0^{\infty} dy\, e^{2ikb\cosh^2 \frac{y}{2}} \left\{ \frac{1}{A\cosh\frac{y}{2} + iB\sin\frac{y}{2} + C} \right.$$

$$\left. + \frac{1}{A\cosh\frac{y}{2} - iB\sinh\frac{y}{2} + C} + \frac{1}{A\cosh\frac{y}{2} + iB\sinh\frac{y}{2} - C} + \frac{1}{A\cosh\frac{y}{2} - iB\sinh\frac{y}{2} - C} \right\}. \tag{D.6}$$

Combine the first and fourth terms in the braces and the second and third terms to obtain the following two integrals

$$I = -2i\,AK\,e^{-ik[a+b]} \int_0^{\infty} \frac{dy\, e^{2ikb\cosh^2\frac{y}{2}} \cosh\frac{y}{2}}{(A\cosh\frac{y}{2})^2 - (iB\sinh\frac{y}{2} + C)^2}$$

$$-2i\,AK\,e^{-ik[a+b]} \int_0^{\infty} \frac{dy\, e^{2ikb\cosh^2\frac{y}{2}} \cosh\frac{y}{2}}{(A\cosh\frac{y}{2})^2 - (iB\sinh\frac{y}{2} - C)^2} \quad . \tag{D.7}$$

Using the identity $\cosh^2 y/2 - \sinh^2 y/2 = 1$ and making the change of variable $\mu = \sinh\, y/2$ obtain for I:

$$I = -4i\,AK\,e^{-ik[a-b]} \left\{ \int_0^{\infty} \frac{e^{2ikb\mu^2}\, d\mu}{(A^2+B^2)\mu^2 - 2i\,BC\mu + i^2(C^2-A^2)} \right.$$

$$\left. + \int_0^{\infty} \frac{e^{2ikb\mu^2}\, d\mu}{(A^2+B^2)\mu^2 + 2i\,BC\mu + i^2(C^2-A^2)} \right\} \tag{D.8}$$

or

$$I = -4i\,A\,K\,e^{-ik[a-b]}\left\{I_1 + I_2\right\} \tag{D.9}$$

I_1 may be written

$$I_1 = a_1 \int_0^\infty d\mu\, e^{2ikb\mu^2} \left[\frac{1}{\mu - i\mu_1} - \frac{1}{\mu - i\mu_2}\right] \tag{D.10}$$

where

$$a_1 = \frac{-i}{(\mu_1 - \mu_2)(A^2 + B^2)} \tag{D.11}$$

and

$$\mu_1 = \frac{BC + A\sqrt{A^2 + B^2 - C^2}}{A^2 + B^2} \tag{D.12}$$

$$\mu_2 = \frac{BC - A\sqrt{A^2 + B^2 - C^2}}{A^2 + B^2} \tag{D.13}$$

and I_2 may be written

$$I_2 = a_1 \int_0^\infty d\mu\, e^{2ikb\mu^2} \left[\frac{1}{\mu + i\mu_2} - \frac{1}{\mu + i\mu_1}\right]. \tag{D.14}$$

Adding $I_1 + I_2$ obtain

$$I_1 + I_2 = a_1 \int_0^\infty d\mu\, e^{2ikb\mu^2} \left[\frac{2i\mu_1}{\mu^2 + \mu_1^2} - \frac{2i\mu_2}{\mu^2 + \mu_2^2}\right]. \tag{D.15}$$

Now I may be written:

$$I = \frac{4iKe^{-ik[a-b]}}{\sqrt{A^2+B^2-C^2}} \int_0^\infty d\mu\, e^{2ikb\mu^2} \left[\frac{\mu_2}{\mu^2+\mu_2^2} - \frac{\mu_1}{\mu^2+\mu_1^2} \right] . \qquad (D.16)$$

Set the coefficient of the integral equal to K_1, i.e:

$$K_1 = \frac{4iKe^{-ik[a-b]}}{\sqrt{A^2+B^2-C^2}} \qquad (D.17)$$

then

$$I = K_1 \left[\mu_2 \int_0^\infty \frac{e^{2ikb\mu^2} d\mu}{\mu^2+\mu_2^2} - \mu_1 \int_0^\infty \frac{e^{2ikb\mu^2} d\mu}{\mu^2+\mu_1^2} \right] . \qquad (D.18)$$

Now consider the integral

$$I_3 = \mu_1 \int_0^\infty \frac{e^{2ikb\mu^2} d\mu}{\mu^2+\mu_1^2} . \qquad (D.19)$$

This integral may be evaluated in terms of Fresnel integrals defined by:

$$F(z) = \sqrt{\frac{2}{\pi}} \int_0^z e^{it^2} dt . \qquad (D.20)$$

The asymptotic series for $F(z)$

$$F(z) \sim F(\infty) + \frac{e^{iz^2}}{i\sqrt{2\pi}\, z} + O\left(\frac{1}{z^2}\right) \qquad (D.21)$$

and the relations

$$F(z) + F(-z) = \sqrt{2}\, e^{i\frac{\pi}{4}}$$

$$F(z) + \sqrt{\frac{2}{\pi}}\, F_o(z) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}} \tag{D.22}$$

where

$$F_o(z) = \int_z^{\infty} e^{it^2}\, dt \tag{D.23}$$

will be useful later on. Born and Wolf (1964) p. 568 develop the identity

$$\pm 2\sqrt{\pi}\, e^{-ikb\eta^2}\, F_o(\pm \eta\sqrt{kb}) = \eta \int_{-\infty}^{\infty} \frac{e^{-kb\tau^2}}{\tau^2 - i\eta^2}\, d\tau \tag{D.24}$$

where the positive sign applies for $\eta > 0$ and the negative sign for $\eta < 0$.

Equation (D.19) will be put in the form (D.24).

Let

$$\tau = \sqrt{2}\, e^{-i\frac{\pi}{4}}\, \mu \tag{D.25}$$

then

$$I_3 = \lim_{h \to 0} \left\{ \frac{\mu_1}{2} \int_{he^{-\frac{3\pi i}{4}}}^{he^{i\frac{\pi}{4}}} \frac{e^{-kb\tau^2}}{\frac{i\tau^2}{2} + \mu_1^2}\, \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}\, d\tau \right\} \tag{D.26}$$

or if $\mu_1 = \eta / \sqrt{2}$ (D.27)

$$I_3 = \frac{e^{-i\frac{\pi}{4}}}{2}\,\eta \int_{-\infty}^{\infty} \frac{e^{-kb\tau^2}\,d\tau}{\tau^2 - i\eta^2} \quad . \tag{D.28}$$

Now by (D.24) for $\eta > 0$

$$I_3 = \pm\sqrt{\pi}\,e^{-i\frac{\pi}{4}}\;e^{-ikb\eta^2}\;F_o\left[\pm\,\eta\sqrt{kb}\right] \tag{D.29}$$

or by virtue of (D.22)

$$I_3 = \pm\,\pi\,e^{-ikb\eta^2}\left[\frac{1}{2} - \frac{e^{-i\frac{\pi}{4}}}{2}F(\pm\eta\sqrt{kb})\right] \quad . \tag{D.30}$$

Taking into account the relation (D.27) the evaluation of (D.19) is given by

$$I = \pm\,\pi\,K_1\left\{e^{-2ikb\mu_1^2}\left[\frac{1}{2} - \frac{e^{-i\frac{\pi}{4}}}{2}\,F(\pm\mu_1\sqrt{2kb})\right]\right\} \tag{D.31}$$

where μ_1 may be either positive or negative depending upon the observers position.

For the stationary half plane μ_1 and μ_2 are given by

$$\begin{aligned}\mu_1 &= -\sin\left(\frac{\phi - \phi_o}{2}\right)\\ \mu_2 &= \sin\left(\frac{\phi + \phi_o}{2}\right)\end{aligned} \tag{D.32}$$

so that in the shadow region $\mu_1 > 0$ and $\mu_2 > 0$. In the illuminated region $\mu_1 < 0$ while $\mu_2 > 0$ in region II and $\mu_2 < 0$ in region I as shown in Fig. D-1.

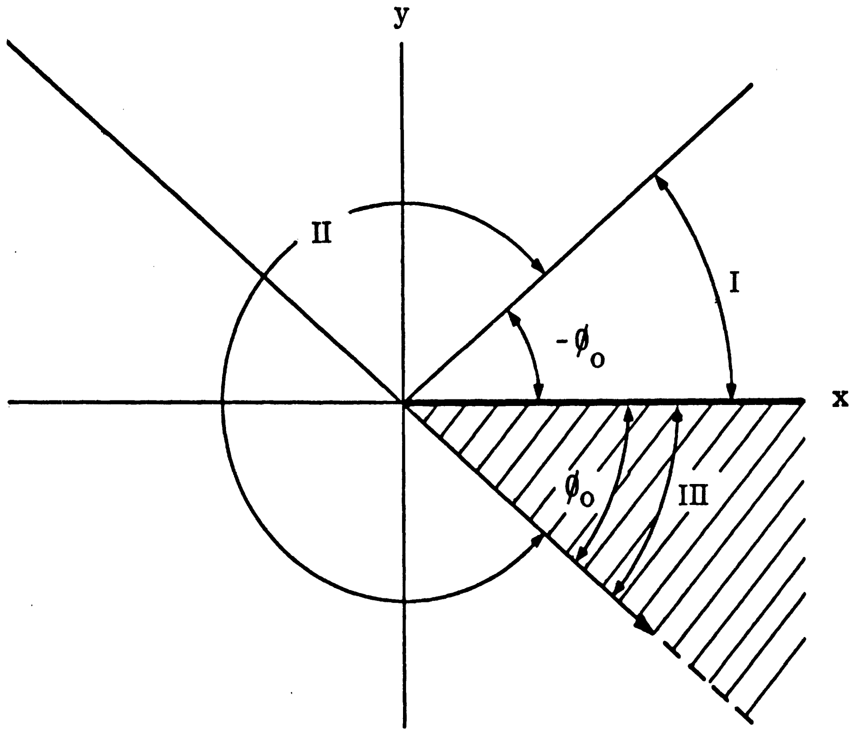


FIG. D-1: STATIONARY HALF PLANE

Since the field representation for the moving half plane must reduce to that of the stationary half plane the evaluation of (3.18) may be written down as follows. In region I

$$I_I = -\pi K_1 \left\{ e^{-2ikb\mu_2^2} \left[\frac{1}{2} - \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}} F(-\mu_2 \sqrt{2kb}) \right] \right.$$

$$\left. - e^{-2ikb\mu_1^2} \left[\frac{1}{2} - \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}} F(-\mu_1 \sqrt{2kb}) \right] \right\} . \qquad (D.33)$$

In region II

$$I_{II} = \pi K_1 \left\{ e^{-2ikb\mu_2^2} \left[\frac{1}{2} - \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}} F(\mu_2 \sqrt{2kb}) \right] \right.$$

$$\left. + \; e^{-2ikb\mu_1^2} \left[\frac{1}{2} - \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}} F(-\mu_1 \sqrt{2kb}) \right] \right\} . \qquad \text{(D.34)}$$

In region III

$$I_{III} = \pi K_1 \left\{ e^{-2ikb\mu_2^2} \left[\frac{1}{2} - \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}} F(\mu_2 \sqrt{2kb}) \right] \right.$$

$$\left. + \; e^{-2ikb\mu_1^2} \left[\frac{1}{2} - \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}} F(\mu_1 \sqrt{2kb}) \right] \right\} . \qquad \text{(D.35)}$$

Consider now the expression for the total field in terms of Fresnel integrals. As given in Eq. (3.13) the total field is the sum of the pole contributions representing incident and reflected fields and the above evaluated Fresnel integrals representing the diffracted fields. Since the poles make no contribution to the total field there is given by Eq. (D. 35). Rewritten in terms of $F_o(z)$ the total field is given by

$$E_z^T(\eta, \Phi, t) = \frac{4\sqrt{\pi}\, K e^{i\frac{\pi}{4}} e^{-ik[a-b]}}{\sqrt{A^2 + B^2 - C^2}} \left\{ e^{-2ikb\mu_2^2} F_o(\mu_2 \sqrt{2kb}) - e^{-2ikb\mu_1^2} F_o(\mu_1 \sqrt{2kb}) \right\} .$$

(D. 36)

Considering the region I where both poles are present the diffracted field is given by (D. 33). Substituting (D. 33) into (3. 13) and making use of the identities (D. 22) the total field in the illuminated region may be written as

$$E_z^T(\mathcal{R}, \Phi, t) = \frac{4\pi i K}{\sqrt{A^2 + B^2 - C^2}} \left\{ - e^{-ik\left[a - b\cos\alpha_{p1}\right]} + e^{-ik\left[a - b\cos\alpha_{p2}\right]} \right.$$

$$\left. - \frac{e^{-ik\left[a-b\right] - i\frac{\pi}{4}}}{\sqrt{\pi}} \left[e^{-2ikb\mu_2^2} F_o(-\mu_2\sqrt{2kb}) - e^{-2ikb\mu_1^2} F_o(-\mu_1\sqrt{2kb}) \right] \right\}$$

(D. 37)

In expression (D. 37) it is understood that the term $\left[1 + \beta\,(\hat{x}'\cdot\hat{p}')^{\sim}\right]$ in the coefficient K has a different form depending which component of the total field is being operated upon. The plane wave terms cannot be combined in the Fresnel integrals for the moving half plane because of the difference in this operator term. In the stationary half plane problem ($\beta = 0$) all of these terms are the same and a Fresnel integral representation of the total field can be obtained which is valid everwhere. Equation (D. 37) is rewritten as Eq. (3. 20) with the above mentioned distinction made explicit.

Consider the evaluation of the integral (5. 13) with $\nu' = 2$.

$$I = K \left\{ \int_{\alpha_{b1}}^{\alpha_{b1}^*} \frac{dx}{\sqrt{a - b\cos\alpha}\left[A\cos\frac{\alpha}{2} + B\sin\frac{\alpha}{2} + C\right]} \right.$$

$$\left. + \int_{\alpha_{b1}^*+2\pi}^{\alpha_{b1}+2\pi} \frac{dx}{\sqrt{a - b\cos\alpha}\left[A\cos\frac{\alpha}{2} + B\sin\frac{\alpha}{2} + C\right]} \right\} \qquad \text{(D. 38)}$$

Along the contour between α_{b1} and α_{b1}^* we have $\alpha = 0 + iy$ and similarly between $\alpha_{b1}^* + 2\pi$ and $\alpha_{b1}^* + 2\pi$ we have $\alpha = 2\pi + iy$. When these substitutions are made, the limits on the second integral inverted and the two integrals are combined we obtain

$$I = -ik \int_{-Y_b}^{Y_b} \frac{dy}{\sqrt{a-b\cosh y}} \left[\frac{1}{A\cosh\frac{y}{2} + B\sinh\frac{y}{2} + C} + \right.$$

$$\left. + \frac{1}{A\cosh\frac{y}{2} + B\sinh\frac{y}{2} - C} \right] \tag{D.39}$$

Comparing this integral with that given in (D. 4) we see that the limits are now finite and the exponential function has been replaced by the square root function. Following the same steps used to reduce (D. 4) to (D. 16) the integral given by (D. 39) may be reduced to

$$I = \frac{4ik}{\sqrt{A^2+B^2-C^2}} \frac{1}{\sqrt{2b}} \int_0^{\sinh\frac{Y_b}{2}} \frac{du}{\sqrt{\sinh^2\frac{Y_b}{2} - u^2}} \left[\frac{u_2}{u^2+u_2^2} - \frac{u_1}{u^2+u_1^2} \right] \tag{D.40}$$

This integral may now be evaluated in closed form [Gröebner and Hofreiter (1950) Eq. 216 (8a)] as

$$I = \frac{4\pi ik}{2\sqrt{2b}\sqrt{A^2+B^2-C^2}} \left\{ \frac{1}{\sqrt{\sinh^2\frac{Y_b}{2} + u_2^2}} - \frac{1}{\sqrt{\sinh^2\frac{Y_b}{2} + u_1^2}} \right\} \tag{D.41}$$

where

$$Y_b = \cosh^{-1}\left(\frac{a}{b}\right) \tag{D.42}$$

and u_1, u_2 are given by equations (D. 12) and (D. 13) respectively.

APPENDIX E

THE HEAVISIDE FUNCTION

The Heaviside [or unit step] function is defined for real x as

$$\begin{aligned} H(x) &= 0 \qquad && x < 0 \\ H(x) &= 1 && x > 0 \end{aligned} \tag{E.1}$$

and is called a generating, generalized or symbolic function. The distribution generated by the Heaviside function is given by

$$< H, \phi > = \int_{-\infty}^{\infty} H(x)\, \phi(x)\, dx \quad . \tag{E.2}$$

In cases where the argument of the Heaviside function becomes complex it is necessary to establish the meaning of $\hat{H}(z)$ where $\hat{H}(z)$ is an analytic representation of the Heaviside function for all points in the complex z plane except the real axis $(y = 0)$. On the real axis we have

$$H(x) = \lim_{\epsilon \to 0_+} \left[\hat{H}(x + i\epsilon) - \hat{H}(x - i\epsilon)\right] , \quad x \neq 0 \tag{E.3}$$

for all x. [Bremermann (1965), p. 48].

The definition given by Bremermann for the analytic representation of a generating function $T(t)$ is:

$$\hat{T}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{T(t)}{t - z}\, dt \quad ; \qquad Im(z) \neq 0 \tag{E.4}$$

or

$$\frac{\partial \hat{T}(z)}{\partial z} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{T(t)}{(t - z)^2}\, dt \quad . \tag{E.5}$$

For the Heaviside function

$$\frac{d\hat{H}(z)}{dz} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{H(t)}{(t-z)^2}\, dt$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dt}{(t-z)^2} \tag{E.6}$$

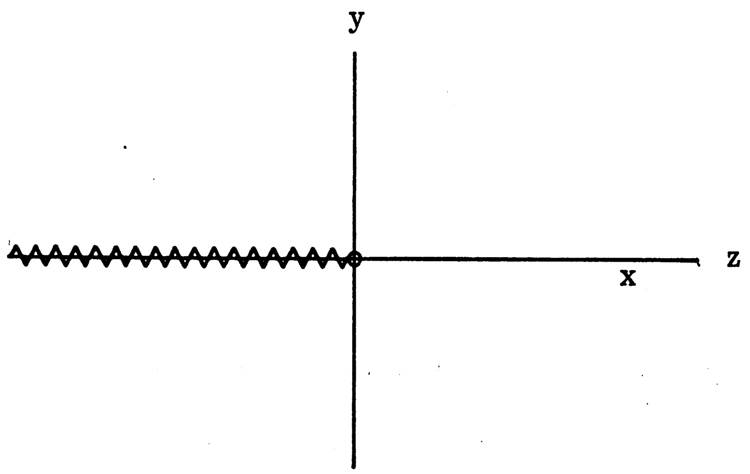
$$\frac{d\hat{H}(z)}{dz} = -\frac{1}{2\pi i}\left[\frac{1}{t-z}\right]_0^{\infty} = -\frac{1}{2\pi i z}$$

Thus

$$\hat{H}(z) = -\frac{1}{2\pi i}\log z + C \quad . \tag{E.7}$$

We define $C = 1/2$, $y > 0$ and $C = -1/2$, $y < 0$.

In order that $\hat{H}(z)$ be single valued as well as analytic we introduce a cut in the complex z plane along the negative real axis.



In this plane the Heaviside function is given by:

$$\hat{H}(z) = -\frac{1}{2\pi i}\log z + \frac{1}{2}\,, \qquad y > 0$$

$$\hat{H}(z) = -\frac{1}{2\pi i}\log z - \frac{1}{2}\,, \qquad y < 0\,. \tag{E.8}$$

Near the real axis we find.

$$z = x + i\epsilon = re^{i\delta}$$

then for $x > 0$

$$H(x) = \lim_{\delta \to 0}\left[-\frac{1}{2\pi i}\log r - \frac{\delta}{2\pi} + \frac{1}{2}\right] - \left[-\frac{1}{2\pi i}\log r - \frac{\delta}{2\pi} - \frac{1}{2}\right]$$

$$H(x) = 1 \tag{E.9}$$

and for $x < 0$

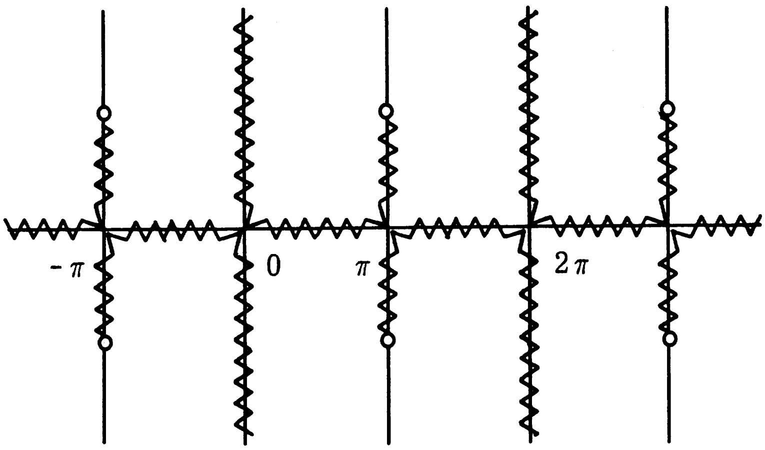
$$H(x) = \lim_{\delta \to 0}\left[-\frac{1}{2\pi i}\log r - \frac{\pi - \delta}{2\pi} + \frac{1}{2}\right] - \left[-\frac{1}{2\pi i}\log r + \frac{\pi - \delta}{2\pi} - \frac{1}{2}\right]$$

$$H(x) = 0\,. \tag{E.10}$$

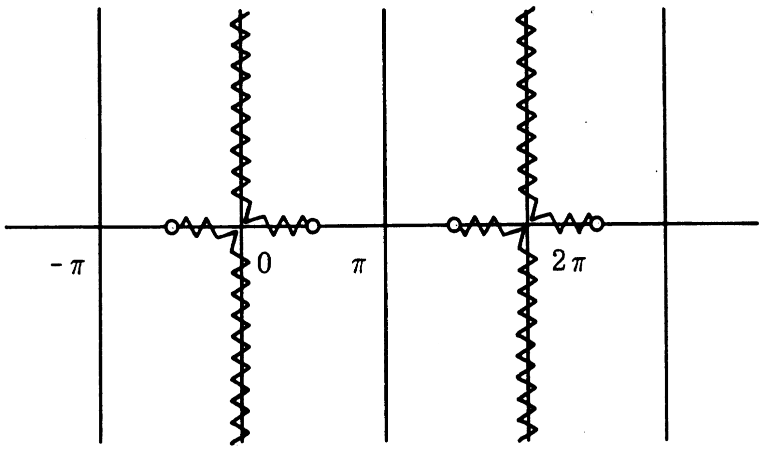
Thus in the complex plane we have a representation for the Heaviside function which is analytic for all points $y \neq 0$ and which corresponds to the usual definition when Eq. (E.3) is taken into account.

The particular feature of this representation which will be useful is the cut in the complex plane and the fact that $H = 0$ on the cut.

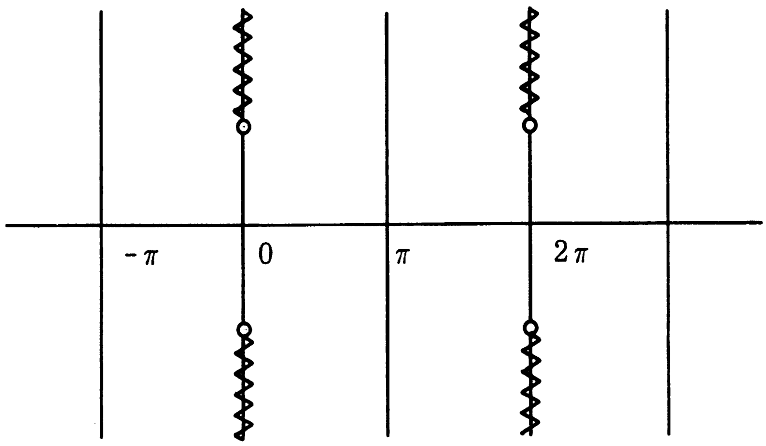
Consider the mapping $z = a - b\cos\alpha$ to determine the position of this cut in the α plane. Figure E-1 shows the location of the cut for different values of the parameters a and b.



Case 1 $\left[-\infty < a < -b\right]$



Case 2 $\left[-b < a < b\right]$



Case 3 $\left[b < a < \infty\right]$

FIG. E-1: CUTS OF $H\left[a - b\cos\right]$ IN COMPLEX α PLANE