

## Some Extensions of Babinet's Principle\*

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### Abstract

Two different extensions of Babinet's principle are discussed. In the first of these, an aperture in a soft or hard screen is covered with a membrane characterized by a jump discontinuity in either the normal component of the fluid velocity or the pressure, and the complementary problem is then a membrane in isolation. In the second, a boundary condition is imposed in the aperture, the solution for which can be obtained from the solution for the complementary disk.

## Introduction

In its standard form, Babinet's principle provides an explicit connection between the acoustic velocity potential for an aperture in a soft or hard screen and the velocity potential of the complementary, hard or soft disk, the primary field being the same in both cases. The principle has a precise analogue in electromagnetic theory and over the years there have been a number of attempts to extend it to surfaces which are not perfect, for example, by Neugebauer<sup>[1]</sup> to surfaces which are absorbing and, more recently, by Lang<sup>[2]</sup> to electrically resistive surfaces. Although there are doubts<sup>[3]</sup> whether the equivalence that exists in the latter case is exact, Baum and Singaraju<sup>[4]</sup> have shown that an equivalence is possible for resistive sheets. This was the inspiration for part of the present work.

Some boundary conditions which could serve to simulate an acoustic membrane are discussed in Section 1. These are characterized by jump discontinuities in either the pressure or the normal component of the fluid velocity, and for this type of membrane it is then shown that membrane-covered apertures in soft and hard screens and membranes in isolation are complementary structures as regards Babinet's principle. The second extension of the principle is to hard or soft "apertures" in soft or hard screens. In effect we now have a mixed boundary condition surface and here also the exact solution can be obtained from that of the complementary disk.

### 1. Boundary Conditions

The Dirichlet and Neumann boundary conditions are the classical ones in acoustics, and it is these which are involved in the standard form of Babinet's

principle. If  $\phi$  is the velocity potential, the boundary condition at a soft surface S is then the Dirichlet one

$$\phi = 0 \quad , \quad (1)$$

corresponding to zero excess pressure on S. At a hard or rigid surface, the normal component of the fluid velocity is zero, giving rise to the Neumann condition

$$\partial\phi/\partial\mathbf{n} = 0 \quad . \quad (2)$$

In practice, however, any surface may yield a little under the influence of the pressure, and a boundary condition which simulates this to some degree is

$$\frac{\partial\phi}{\partial\mathbf{n}} + \frac{ik}{\eta} \phi = 0 \quad (3)$$

where  $\mathbf{n}$  is the outward normal to the surface,  $\eta$  is the normal specific impedance of the material relative to the intrinsic impedance of the external medium,  $k$  is the propagation constant, and a time factor  $e^{-i\omega t}$  has been assumed and suppressed.

If (3) is applied at each side of a thin slice of material, then in the limit as the thickness tends to zero, addition and subtraction of the two conditions gives

$$\left[ \frac{\partial\phi}{\partial\mathbf{n}} \right]_{-}^{+} + \frac{ik}{\eta} (\phi_{+} + \phi_{-}) = 0 \quad (4)$$

$$[\phi]_{-}^{+} + \frac{\eta}{ik} \left( \frac{\partial\phi_{+}}{\partial\mathbf{n}} + \frac{\partial\phi_{-}}{\partial\mathbf{n}} \right) = 0 \quad (5)$$

where  $\mathbf{n}$  is outward to the side indicated by the plus sign. In general, (4) and (5) imply discontinuities in  $\partial\phi/\partial\mathbf{n}$  and  $\phi$  across the membrane, but in special cases

one or the other may be continuous. It is these special cases only which characterize our modelling of a membrane. If, for example,  $\phi_+ = \phi_-$ , we have

$$\phi \text{ continuous} \tag{6}$$

$$\left[ \frac{\partial \phi}{\partial \mathbf{n}} \right]_+ + \frac{2ik}{\eta} \phi = 0 .$$

The conditions (6) represent the situation in which the pressure is equal on the two sides of the membrane, producing a proportional jump discontinuity in the normal component of the fluid velocity across it. They are valid conditions from a mathematical viewpoint and are the acoustic counterpart of those at an electrically resistive sheet in which  $\phi$  is then the tangential component of the electric field. We shall refer to (6) as the boundary conditions at a resistive membrane. When  $\eta = 0$  they reduce to the Dirichlet condition (1), and when  $\eta = \infty$  the membrane ceases to exist.

In the same way that the Dirichlet boundary condition has the Neumann one as its dual, so the resistive conditions (6) have their dual, namely

$$\frac{\partial \phi}{\partial \mathbf{n}} \text{ continuous} \tag{7}$$

$$[\phi]_+ + \frac{2\eta}{ik} \frac{\partial \phi}{\partial \mathbf{n}} = 0 .$$

We shall refer to these as the boundary conditions at a conductive membrane, and for (7) to be the dual of (6), it is necessary that  $\eta$  be the same in both cases. When  $\eta = \infty$  they reduce to the Neumann condition (2) and when  $\eta = 0$  the membrane disappears. If  $\eta$  is finite and non-zero, the conditions (7) imply that the normal component of the fluid velocity is continuous whereas the pressure is not, and a membrane satisfying these conditions has the same difficulty of practical realization

that a "magnetically conductive" sheet has in electromagnetic theory. Nevertheless, the conditions (7) are permissible from a mathematical viewpoint and are essential for the subsequent analysis.

## 2. Apertures with Membranes

Babinet's principle is a consequence of the symmetry of fields radiated by planar distributions analogous to the single and double layer distributions in statics. This property is exemplified by the Rayleigh-Sommerfeld formulae for diffraction by an aperture in a soft or hard screen, and in order to demonstrate the effect of covering the aperture with a membrane, it is convenient to begin by outlining a proof of Babinet's principle in the classical case of an open aperture. For a more rigorous derivation, the reader is referred to Bouwkamp<sup>[5]</sup>.

We consider first a soft screen  $S$  containing an aperture  $A$  and lying in the plane  $z = 0$  of a Cartesian coordinate system  $x, y, z$ . The sources are entirely in  $z < 0$  and generate a primary field  $\phi_0(x, y, z)$ . To find the total field<sup>[6]</sup>  $\phi_1(x, y, z)$ , Green's theorem is applied to the half space  $z \geq 0$  using the function

$$g_1(\underline{r}|\underline{r}') = g(\underline{r}|\underline{r}') - g(\tilde{\underline{r}}|\underline{r}') \quad ,$$

where

$$g(\underline{r}|\underline{r}') = \frac{1}{4\pi} \frac{e^{ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}$$

is the free space Green's function and  $\tilde{\underline{r}}$  is the image of the observation point  $\underline{r}$  in the plane  $z = 0$ . When the surface of integration is the plane  $z = 0^+$  just to the right of the screen, the fact that

$$\frac{\partial g_1}{\partial z'} = -2 \frac{\partial g}{\partial z}$$

for  $z' = 0$ , and the boundary condition  $\phi_1 = 0$  on  $S$ , lead to the following expression for the field in  $z \geq 0$ :

$$\phi_1(x, y, z) = -2 \iint_A \phi_1(x', y') \frac{\partial g}{\partial z} dx' dy' .$$

Since  $\phi_1$  must be symmetrical about the plane  $z = 0$ , extension to the whole space gives

$$\phi_1(x, y, z) = \phi_0(x, y, z) - \phi_0(x, y, \mp z) \pm 2 \iint_A \phi_1(x', y') \frac{\partial g}{\partial z} dx' dy' \quad (8)$$

where the upper and lower signs apply to the regions  $z < 0$  and  $z > 0$  respectively.

The only condition still to be enforced is the continuity of  $\partial\phi_1/\partial z$  in the aperture, and this requires that  $\phi_1$  satisfy the integral equation

$$\lim_{z \rightarrow 0} 2 \iint_A \phi_1(x', y') \frac{\partial^2 g}{\partial z^2} dx' dy' + \frac{\partial\phi_0}{\partial z} = 0 \quad (9)$$

for  $x, y$  in  $A$ .

The complementary problem is that of a rigid disk having the same location and configuration as the aperture  $A$ , and illuminated by the same primary field  $\phi_0$ . Let the total field be  $\Phi_1(x, y, z)$ . Using Green's theorem with the free space Green's function  $g$ , the scattered field can be expressed as a surface integral of the discontinuities in  $\Phi_1$  and  $\partial\Phi_1/\partial z$  across the disk, and since  $\partial\Phi_1/\partial z$  is continuous,

$$\Phi_1(x, y, z) = \phi_0(x, y, z) - \iint_A [\Phi_1(x', y')]_{-}^{+} \frac{\partial g}{\partial z} dx' dy' \quad . \quad (10)$$

On applying the Neumann boundary condition (2) at the surface, the integral equation that results is

$$\lim_{z \rightarrow 0} \iint_A [\Phi_1(x', y')]_{-}^{+} \frac{\partial^2 g}{\partial z^2} dx' dy' - \frac{\partial \phi_0}{\partial z} = 0 \quad (11)$$

for  $x, y$  on  $A$ . Comparison of eqs. (9) and (11) shows

$$[\Phi_1(x', y')]_{-}^{+} = -2\phi_1(x', y') \quad (12)$$

and hence, from (8) and (10),

$$\Phi_1(x, y, z) \mp \phi_1(x, y, z) = \phi_0(x, y, \mp z) \quad (13)$$

where, as always, the upper and lower signs are for  $z < 0$  and  $z > 0$  respectively.

Equation (13) provides an explicit connection between the total fields for the aperture and its complementary disk, but each structure has a dual, consideration of which leads to the second of the two standard forms of Babinet's principle. The soft screen is now replaced by a hard one, in which case the complementary disk is soft. If the total fields in the two problems are  $\phi_2(x, y, z)$  and  $\Phi_2(x, y, z)$  respectively, a similar analysis to the above gives

$$\Phi_2(x, y, z) \mp \phi_2(x, y, z) = \mp \phi_0(x, y, \mp z) \quad . \quad (14)$$

We now examine the effect of covering the aperture with a membrane.

If, in the first problem where the screen is soft, we assume a resistive membrane

in the aperture, the total field is still given by eq. (8). The only condition remaining to be satisfied is the jump condition (6) across the membrane, and when this is imposed we have

$$\lim_{z \rightarrow 0} 2 \iint_A \phi_1(x', y') \frac{\partial^2 \mathbf{g}}{\partial z^2} dx' dy' - \frac{ik}{\eta} \phi_1(x', y') + \frac{\partial \phi_0}{\partial z} = 0 \quad (15)$$

for  $x, y$  in  $A$ . The complementary problem is that of a conductive membrane (disk), and here also the analysis parallels our previous one. Application of the jump condition (7) to the total field  $\Phi_1$  of eq. (10) gives

$$\lim_{z \rightarrow 0} \iint_A [\Phi_1(x', y')]_{-}^{+} \frac{\partial \mathbf{g}}{\partial z} dx' dy' - \frac{ik}{2\eta} [\Phi_1(x', y')]_{-}^{+} - \frac{\partial \phi_0}{\partial z} = 0, \quad (16)$$

and provided the impedance  $\eta$  is the same as that of the resistive membrane, the two solutions satisfy eq. (12) as before. It therefore follows that the total fields in the two problems are connected by eq. (13) just as they were when the aperture was open. Indeed, the open aperture is merely the special case  $\eta = \infty$  of our new and more general result, and when  $\eta = 0$  the aperture no longer exists.

The dual of a soft screen with a resistive membrane in the aperture is a hard screen with a conductive membrane. The complementary structure is then a resistive membrane (disk), and provided the impedance  $\eta$  is the same in both cases, the total fields of the two problems satisfy the same relation (14) as when the aperture was open. The open aperture is only the special case  $\eta = 0$  of the general result, and when  $\eta = \infty$  the screen is without an aperture.



We note in passing that all of our results are unaffected if  $\eta$  is a variable function of position, but it is essential that the soft screen have a resistive membrane and the hard screen a conductive one. If the membranes are reversed, the procedure we have used is no longer valid and it seems unlikely that a Babinet principle exists in this case. The same is true if both  $\phi$  and  $\partial\phi/\partial z$  are discontinuous in A.

### 3. Filled Apertures

The insertion of a membrane in the aperture is a simple generalization of the classical geometry of Babinet's principle, but there is also another extension<sup>[7]</sup> of the principle which has, apparently, been overlooked. In this extension an actual boundary condition is imposed in the aperture—Neumann in the case of a soft screen, Dirichlet in the case of a hard—to create a two-part boundary value problem. The total field is then trivially related to the total field existing when the aperture is open.

To see this, consider a soft screen S containing an aperture A at which the Neumann boundary condition (2) is imposed. From eq. (8) the total field  $\phi_3(x, y, z)$  in  $z \leq 0$  is<sup>[8]</sup>

$$\phi_3(x, y, z) = \phi_0(x, y, z) - \phi_0(x, y, -z) + 2 \iint_A \phi_3(x', y') \frac{\partial g}{\partial z} dx' dy' \quad (17)$$

and on enforcing the boundary condition in A, the following integral equation results

$$\lim_{z \rightarrow 0} \iint_A \phi_3(x', y') \frac{\partial^2 g}{\partial z^2} dx' dy' + \frac{\partial \phi_0}{\partial z} = 0 \quad (18)$$

Comparison with eq. (9) now shows

$$\phi_3(x', y') = 2\phi_1(x', y') \quad (19)$$

and hence, for  $z \leq 0$ ,

$$\phi_3(x, y, z) - 2\phi_1(x, y, z) = \phi_0(x, y, -z) - \phi_0(x, y, z) \quad (20)$$

where  $\phi_1$  is the total field when the aperture is open. A more symmetric and suggestive form is obtained on using the Babinet result (13), and is

$$\phi_3(x, y, z) = \Phi_1(x, y, z) + \phi_1(x, y, z) - \phi_0(x, y, z) \quad (21)$$

where  $\Phi_1$  is the solution for the complementary problem, i.e., a hard disk.

The dual of a hard aperture in a soft screen is, of course, a soft aperture in a hard screen, and eqs. (20) and (21) have their counterparts here. If  $\phi_4(x, y, z)$  is now the total field<sup>[8]</sup> in  $z \leq 0$ , an analysis similar to the above shows

$$\phi_4(x, y, z) - 2\phi_2(x, y, z) = -\phi_0(x, y, -z) - \phi_0(x, y, z) \quad (22)$$

and hence, from (14),

$$\phi_4(x, y, z) = \Phi_2(x, y, z) + \phi_2(x, y, z) - \phi_0(x, y, z) \quad (23)$$

#### 4. Conclusions

These generalizations of Babinet's principle may have practical as well as mathematical significance. The assumption of a jump discontinuity in the aperture would seem to provide a reasonable simulation of the presence of a membrane, and depending on the choice of boundary condition (6) or (7), the solution can be obtained from that of the complementary disk using eq. (13) or (14) respectively. When an actual boundary condition is imposed in A, the surface becomes a two-part

one analogous to that appropriate in (say) radio propagation over an inhomogeneous flat earth consisting of an island in an idealized sea. And the equivalence presented in eq. (20) or (22) now gives the solution in terms of that of the complementary island or disk. In the particular case when the incident field and the aperture are both independent of one of the transverse coordinates, all of our results are directly applicable to electromagnetic fields.

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- [6] The velocity potential is hereafter referred to as a field.
- [7] The author is indebted to Professor R. E. Kleinman for bringing this extension to his notice.
- [8] The fields  $\phi_3$  and  $\phi_4$  are, of course, zero in  $z > 0$ .