SOME PROBLEMS INVOLVING IMPERFECT HALF PLANES*

by

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Abstract

As part of a continuing study of how the material of a body affects its scattering, we consider here the influence of material properties on edge diffraction. An appropriate canonical problem is the scattering of an electromagnetic wave by a half plane whose boundary condition is chosen to simulate the material in question. The half planes considered fall into two categories: impedance sheets subject to a Leontovich boundary condition on both faces, and resistive and/or conductive sheets characterised by a jump condition. In each case the material parameter (impedance, resistivity or conductivity) may be a tensor, but is assumed to have the same value at all points on a given face, corresponding to a homogeneous, anisotropic sheet. The various forms of the boundary conditions are discussed for a wave incident in a plane perpendicular to the edge and for oblique incidence when the conditions couple two components of the field. We then consider the two basic methods which are available for solving boundary value problems of this type and examine their application to the different half planes. With the advances that have taken place recently, the two methods now have the same capability, but neither works when the material is anisotropic except in those situations where the anisotropy has no effect.

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1. Introduction

The scattering from any body is a function of its material properties as well as its geometry. This fact is exploited when radar absorbing materials are applied to the surface of a target to reduce its scattering, but most RAM are designed to suppress the specular contributions from a body whose electrical dimensions are large. Once these echoes have been eliminated we are left only with non-specular or diffractive sources, and though their magnitudes are much less, they are often the dominant contributors to the backscattering patterns of aerospace targets in the aspect ranges of most interest. If we are to suppress these as well, it is now necessary to examine the effect that the material properties of a body have on such non-specular scatterers as edge, creeping and traveling waves and attempt to tailor the properties of an absorber to maximise the reduction.

The last few years have seen a growing effort directed at this type of scattering. Using the integral equations for the surface fields on two-dimensional bodies subject to an impedance boundary condition, computer programs have been developed (Knott *et al.*, 1973; Knott and Senior, 1974a) to find the impedances which are most effective in suppressing the non-specular contribution to the backscattering from various shapes of target. Resistive (electric current) sheets placed in the vicinity of a metallic edge have also proved effective in reducing edge scattering for carefully chosen variations of the resistivities along the sheets (Knott and Senior, 1974b). Computer programs have been written to obtain the scattering from these types of configuration and to optimise the sheet selection: in contrast to an impedance specification of a surface, the resistivity uniquely defies the sheet, and the computer predictions have been confirmed experimentally. Program RAMVS (Liepa *et al.*, 1974) accepts many resistive and/or conductive sheets in the presence of a body and is now being used to design absorbers to suppress non-specular forms of scattering. In effect, by representing a graded RAM as a number of discrete layers or sheets, this program provides a way of specifying the material parameters of a coating that will realise the surface impedance necessary to minimise the scattering.

Although design problems of this complexity would be about impossible to solve without the power and versatility of a digital computer, high frequency approximations such as GTD can still play a vital role in the investigation. In addition to giving insight into the type of scattering that occurs, the approximations help in pinpointing the material parameters of interest and in interpreting the computer-generated data. This is particularly true in the case of edge scattering, whether produced by the incident field directly or by a creeping or traveling wave on the surface. Edge waves are important in many applications, including the specification of antenna ground planes and the design of acoustic baffles (Rawlins, 1976). They are major contributors to the scattering from the wings and tailplanes of an aircraft, and are almost inevitably introduced when resistive sheets are used to reduce these (and other) forms of scattering. It is therefore of interest to examine the scattering from the edge of a plate or sheet to see how it depends on the material properties, and the present paper is concerned with some aspects of the mathematical analyses involved.

To determine the diffraction coefficient of an edge it is necessary to solve an appropriate canonical problem, and a natural one to choose is a half plane whose boundary conditions are such as to simulate the material in question. It will be assumed that the primary field is a plane wave incident in a plane perpendicular to the edge (normal incidence) or obliquely (skew incidence), and though we shall devote most attention to electromagnetics, it is convenient to preface our discussions with a brief consideration of the analogous but simpler problems in acoustics. The task is then to take some of the half planes relevant to an assessment of material effects in edge scattering, and to explore the various forms of the boundary conditions and the methods which are available for the exact solution of the scattering problem.

The half planes fall into two distinct categories: impedance sheets subject to a Leontovich boundary condition, and resistive and/or conductive sheets characterised by a jump condition. In each case the material parameter (impedance, resistivity, etc) will be allowed to be a tensor corresponding to a non-isotropic sheet with the tensor constant at all points on a given face, though different on the two faces. Note that the assumption of homogeneity excludes problems which can be solved in some particular coordinate system by virtue of a surface impedance having a specific variation matching that of the metric coefficients. Though it has proved essential for some absorber applications to have the properties of a sheet vary as a function of the distance from the edge, under most circumstances edge diffraction is still a local phenomenon determined by the properties within a small fraction of a wavelength of the edge.

In the first part of the paper we describe the various sheets and their associated boundary conditions, emphasising those situations where the problem can be scalarised. We then turn to the methods available for their solution. There are, in effect, two basic methods. Each has a number of variations and/or extensions, and there are in addition methods which are effective in special cases and which may then provide a rather elementary approach, but one or other of the two methods is applicable in all circumstances for which solutions have been obtained so far.

2. Boundary Conditions

A generalisation of the standard boundary conditions for soft (yielding) and hard (rigid) surfaces in acoustics is the impedance boundary condition

$$\hat{n} \cdot \underline{u} + p/Z = 0 \tag{1}$$

(Morse and Ingard, 1961) where $\underline{u}$ is the perturbation velocity, p is the excess pressure, $\hat{n}$ is the unit outward normal to the surface and Z is the acoustic impedance. When expressed in terms of the velocity potential V for a time harmonic field, (1) becomes

$$\frac{\partial V}{\partial n} + \frac{ik}{\eta} V = 0 \tag{2}$$

where η (proportional to Z) is the complex specific acoustic impedance of the surface, and reduces to the Dirichlet boundary condition for a soft surface when $\eta = 0$ and to the Neumann boundary condition for a hard surface when $\eta = \infty$. A time dependence $e^{-i\omega t}$ has been assumed and suppressed.

The condition (2) is widely employed in studies of acoustically imperfect surfaces, for example, in the design of absorbing baffles. Its form is the same regardless of the incident field and for curved as well as planar surfaces, and though a rigorous justification may require that the radii of curvature of the surface are everywhere large compared to the wavelength, the condition has nevertheless found useful application even to edged structures such as an absorbing sheet or wedge. In the case of an absorbing half plane occupying the region $x \geqslant 0$ of the plane $y = 0$ in a Cartesian coordinate system, (2) implies

$$\frac{\partial V}{\partial y} \pm \frac{ik}{\eta} V = 0 \qquad (y = \pm 0) \tag{3}$$

where η may differ on the two faces; and for an absorbing wedge whose surfaces are defined as $\rho \geqslant 0$, $\phi = \Omega$ and $2\pi - \Omega$, in a cylindrical polar coordinate system where 2Ω is the closed angle of the wedge,

$$\frac{1}{\rho}\frac{\partial V}{\partial \phi} \pm \frac{ik}{\eta} V = 0 \quad (\phi = \Omega, \ 2\pi - \Omega) \quad . \tag{4}$$

There are two extensions of the boundary conditions that should be mentioned. The first arises when there is a uniform main stream flow past an absorbing wedge or half plane (Rawlins, 1974). The result is to add a tangential derivative to (2), but since this does not affect the applicability of the methods available for the solution of the problems, we shall not discuss it further. The second extension is rather more profound. If (2) is applied on each side of a thin slice of material having η the same on both sides, then in the limit as the thickness tends to zero, addition and subtraction of the two conditions gives

$$\begin{aligned} &\left[\frac{\partial V}{\partial n}\right]_-^+ + \frac{ik}{\eta}(V_+ + V_-) = 0 \\ &\left[V\right]_-^+ + \frac{\eta}{ik}\left(\frac{\partial V_+}{\partial n} + \frac{\partial V_-}{\partial n}\right) = 0 \end{aligned} \tag{5}$$

where $\hat{n}$ is outward to the side indicated by the plus sign. In general, the eqs. (5) imply discontinuities in $\frac{\partial V}{\partial n}$ and V across the sheet, but we can conceive of situations in which one or other is continuous. This additional constraint imposed in place of one of the conditions (5) converts the boundary value problem into a transition problem. If, for example, $V_+ = V_-$, then

V continuous

$$\left[\frac{\partial V}{\partial n}\right]_-^+ + \frac{2ik}{\eta} V = 0 \tag{6}$$

and these conditions in which the pressure is equal on the two sides, but creates a proportional jump discontinuity in the normal component of the fluid velocity, have been used (Senior, 1975a) to characterise an acoustically resistive membrane. We can likewise conceive of a 'conductive' membrane having

$$\frac{\partial V}{\partial n} \text{ continuous}$$

$$\left[V\right]_{-}^{+} + \frac{2\eta}{ik}\frac{\partial V}{\partial n} = 0 \tag{7}$$

and these two membranes are complementary in the sense required for the existence of a Babinet principle (Senior, 1975a). For present purposes, however, an important point is the following: although (6) and (7) have obvious similarities to (5), the mathematical problems are quite distinct, and a method which is applicable for the solution of the boundary value problem for an impedance half plane may be ineffective when applied to a half plane membrane.

The electromagnetic analogue of (1) is the surface impedance (or Leontovich) boundary condition

$$\underline{E} - (\hat{n} \cdot \underline{E})\,\hat{n} = Z\,\bar{\bar{\eta}} \cdot (\hat{n}_\wedge \underline{H}) \tag{8}$$

where $\bar{\bar{\eta}}$ is a tensor surface impedance relative to the impedance Z of the free space medium above. Since (8) can be written as

$$\hat{n}_\wedge (\hat{n}_\wedge \underline{E}) = -Z\,\bar{\bar{\eta}} \cdot \hat{n}_\wedge \underline{H} \;,$$

the surface magnetic current density $\underline{K}^* = -\hat{n}_\wedge \underline{E}$ is directly related to the surface electric current density $\underline{K} = \hat{n}_\wedge \underline{H}$ via

$$\hat{n}_\wedge \underline{K}^* = -Z\,\bar{\bar{\eta}} \cdot \underline{K}$$

implying

$$\underline{K}^* = - Z\, \underline{K} \cdot (\bar{\bar{\eta}}_\wedge \hat{n}) \ , \tag{9}$$

and this is particularly convenient if Hertz vectors are used to represent the unknown scattered field.

The boundary condition (9) is intrinsically a vector one in that each component involves two components of the field. Thus, for a locally plane surface with $\hat{n} = \hat{y}$

$$E_x = \eta_1 Z H_z \ , \qquad E_z = - \eta_3 Z H_x \tag{10}$$

where (x, y, z) are Cartesian coordinates and $\bar{\bar{\eta}} = \eta_1 \hat{x}\hat{x} + \eta_3 \hat{z}\hat{z}$, but in certain special cases (8) can be expressed in terms of a single field component. In particular, if the surface and the incident field are both independent of the z coordinate, the problem is two-dimensional, and for an incident field such that $\underline{E} = \hat{z} E_z$ (E polarization), (8) reduces to

$$\frac{\partial E_z}{\partial n} + \frac{ik}{\eta_3} E_z = 0 \ , \tag{11}$$

whereas for a field having $\underline{H} = \hat{z} H_z$ (H polarization)

$$\frac{\partial H_z}{\partial n} + ik\eta_1 H_z = 0 \ . \tag{12}$$

These are scalar conditions directly analogous to (2).

In the more general case of a two-dimensional surface with an incident field which is not E- or H-polarized (e.g. a plane wave at skew incidence) or a three-dimensional surface under any illumination, it would seem that the only field component which could satisfy a scalar boundary condition is the normal component

E_n or H_n. To explore this possibility, consider a surface u_2 = constant where (u_1, u_2, u_3) form a right-handed orthogonal curvilinear coordinate system with metric coefficients h_1, h_2, h_3. If now $\bar{\bar{\eta}} = \eta_1 \hat{u}_1\hat{u}_1 + \eta_3 \hat{u}_3\hat{u}_3$, Maxwell's equations and the divergence conditions in conjunction with (8) yield

$$
\begin{aligned}
&\frac{\partial}{\partial u_2}(h_1 h_3 E_2) + ik\eta_1 h_1 h_2 h_3 E_2 = -\frac{h_3}{\eta_1} E_1 \frac{\partial}{\partial u_1}(h_2\eta_1) - \frac{h_1}{h_3} E_3 \frac{\partial}{\partial u_3}(h_2\eta_3) - h_2(\eta_3 - \eta_1)\frac{\partial}{\partial u_3}\left(\frac{h_1}{\eta_3} E_3\right) \\
&\frac{\partial}{\partial u_2}(h_1 h_3 H_2) + \frac{ik}{\eta_3} h_1 h_2 h_3 H_2 = -\eta_3 h_3 H_1 \frac{\partial}{\partial u_1}\left(\frac{h_2}{\eta_3}\right) - \eta_1 h_1 H_3 \frac{\partial}{\partial u_3}\left(\frac{h_2}{\eta_1}\right) - h_2\left(\frac{1}{\eta_1} - \frac{1}{\eta_3}\right)\frac{\partial}{\partial u_3}\left(h_1 \eta_1 H_3\right)
\end{aligned}
\tag{13}
$$

and the right-hand sides are zero only if $\eta_3 = \eta_1$, i.e., the impedance is isotropic, with $h_2\eta_1$ and h_2/η_3 independent of u_1 and $h_2\eta_3$ and h_2/η_1 independent of u_3. This is not even true for a planar surface ϕ = constant in cylindrical polar coordinates, and we further comment that the normal components E_ϕ and H_ϕ do not satisfy the scalar wave equation. No scalarisation has yet been found for the problem of an impedance wedge except when the conditions (11) and (12) are applicable, and thus the problem of a plane wave at skew incidence is still unsolved.

As in the acoustic case, the simplest situation is a surface in the plane $y = 0$ (say) of a Cartesian coordinate system. If the surface impedance is isotropic, ie, $\eta_3 = \eta_1$, simple manipulation of (10) gives

$$
\begin{aligned}
&\frac{\partial E_y}{\partial y} + ik\eta_1 E_y = 0 \\
&\frac{\partial H_y}{\partial y} + \frac{ik}{\eta_1} H_y = 0
\end{aligned}
\tag{14}
$$

in which the (scalar) surface impedance is proportional to the logarithmic normal derivatives of the normal components. The equations (14) are scalar boundary conditions which are the direct counterpart of the acoustic condition (3). They are valid for any type of field, but if $\eta_3 \neq \eta_1$ it follows from (13) that

$$\frac{\partial E_y}{\partial y} + ik\eta_1 E_y = \left(\frac{\eta_1}{\eta_3} - 1\right)\frac{\partial E_z}{\partial z}$$

$$\frac{\partial H_y}{\partial y} + \frac{ik}{\eta_3} H_y = \left(\frac{\eta_1}{\eta_3} - 1\right)\frac{\partial H_z}{\partial z} \qquad .$$

Even for an infinitely extended planar surface in Cartesian coordinates, no scalar boundary condition has been found that is valid for an arbitrary incident field and a general anisotropic impedance, and it is extremely unlikely that one exists. This can be seen by considering an incident plane wave. The total field then consists of the incident and reflected waves, and it is easily verified that only for an isotropic impedance or for a plane wave incident in a principal direction of the impedance tensor is there a linear combination of E_y (or H_y) and its derivatives which is zero on the surface.

The above remarks are obviously applicable to an impedance half plane occupying the region $x \geqslant 0$ of the plane $y = 0$. The most general case is that in which the impedance is anisotropic and differs on the two sides of the half plane, and the boundary conditions can then be written as

$$\begin{aligned} E_x &= \eta_1 Z H_z, \qquad & E_z &= -\eta_3 Z H_x \qquad & (y = 0+) \\ E_x &= -\eta_2 Z H_z, \qquad & E_z &= \eta_4 Z H_x \qquad & (y = 0-) \end{aligned} \tag{15}$$

corresponding to the surface impedances

$$\bar{\bar{\eta}}_+ = \eta_1 \hat{x}\hat{x} + \eta_3 \hat{z}\hat{z} \; , \qquad \bar{\bar{\eta}}_- = \eta_2 \hat{x}\hat{x} + \eta_4 \hat{z}\hat{z} \tag{16}$$

on the upper and lower faces respectively.

If the incident field is independent of z and either E- or H-polarized, the problem for each polarization involves only one pair of impedances and is indistinguishable from that for an isotropic surface. We can now use the boundary conditions for either E_z and H_z or E_y and H_y. Thus, for E polarization the problem is a scalar one for the component E_z satisfying

$$\begin{aligned} &\frac{\partial E_z}{\partial y} + \frac{ik}{\eta_3} E_z = 0 \qquad (y = 0+) \\ &\frac{\partial E_z}{\partial y} - \frac{ik}{\eta_4} E_z = 0 \qquad (y = 0-) \end{aligned} \tag{17}$$

and also for the component H_y satisfying

$$\begin{aligned} &\frac{\partial H_y}{\partial y} + \frac{ik}{\eta_3} H_y = 0 \qquad (y = 0+) \\ &\frac{\partial H_y}{\partial y} - \frac{ik}{\eta_4} H_y = 0 \qquad (y = 0-) \end{aligned} \tag{18}$$

but because the latter were obtained from (15) by a process of tangential differentiation, the solution of the boundary problem (18) may require special care to ensure that the radiation and/or edge conditions are fulfilled. Similarly for H polarization we can use either H_z or E_y satisfying

$$\begin{aligned} &\left(\frac{\partial}{\partial y} + ik\eta_1\right) \begin{matrix} H_z \\ E_y \end{matrix} = 0 \qquad (y = 0+) \\ &\left(\frac{\partial}{\partial y} - ik\eta_2\right) \begin{matrix} H_z \\ E_y \end{matrix} = 0 \qquad (y = 0-) \ . \end{aligned} \tag{19}$$

For a more general incident field including a plane wave at skew incidence, our options are more limited. If the impedances are isotropic, the components E_y and H_y satisfying (19) and (18) with $\eta_3 = \eta_1$, $\eta_4 = \eta_2$ provide a scalarisation of the problem, but for arbitrary anisotropic impedances no scalar boundary conditions have been found. The best that we can then do is to express the four conditions (15) in terms of two field components, for example

$$\begin{aligned}
&\left\{\frac{\partial}{\partial y} + \frac{ik}{\eta_3}\left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)\right\} E_z + \frac{i}{k}\frac{\partial^2}{\partial x \partial z} Z H_z = 0 \\
&\left\{\frac{\partial}{\partial y} + ik\eta_1\left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)\right\} Z H_z - \frac{i}{k}\frac{\partial^2}{\partial x \partial z} E_z = 0
\end{aligned} \qquad (y = 0+)$$

$$\begin{aligned}
&\left\{\frac{\partial}{\partial y} - \frac{ik}{\eta_4}\left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)\right\} E_z + \frac{i}{k}\frac{\partial^2}{\partial x \partial z} Z H_z = 0 \\
&\left\{\frac{\partial}{\partial y} + ik\eta_2\left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)\right\} Z H_z - \frac{i}{k}\frac{\partial^2}{\partial x \partial z} E_z = 0
\end{aligned} \qquad (y = 0-) \qquad (20)$$

or

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} + ik\eta_1\right) E_y - \eta_1 \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y} + \frac{ik}{\eta_1}\right) Z H_y = 0 \\
&\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} + \frac{ik}{\eta_3}\right) Z H_y + \frac{1}{\eta_3}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y} + ik\eta_3\right) E_y = 0
\end{aligned} \qquad (y = 0+)$$

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} - ik\eta_2\right) E_y + \eta_2 \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y} - \frac{ik}{\eta_2}\right) Z H_y = 0 \\
&\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} - \frac{ik}{\eta_4}\right) Z H_y - \frac{1}{\eta_4}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y} - ik\eta_4\right) E_y = 0 \quad .
\end{aligned} \qquad (y = 0-) \qquad (21)$$

Some factors that might motivate one choice or the other have been discussed by Senior (1975c).

Nevertheless, there are some special anisotropic impedance values for which the boundary conditions simplify. If $\eta_1 = \eta_2 = 0$ it follows rather trivially from (15) that

$$
\begin{aligned}
E_x &= 0 , \quad \frac{\partial H_x}{\partial y} + ik\eta_3 \left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right) H_x = 0 \qquad (y = 0+) \\
E_x &= 0 , \quad \frac{\partial H_x}{\partial y} - ik\eta_4 \left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right) H_x = 0 \qquad (y = 0-)
\end{aligned}
\tag{22}
$$

and the components E_x and H_x now provide a scalarisation for any η_3 and η_4. Similarly, if $\eta_3 = \eta_4 = \infty$

$$
\begin{aligned}
H_x &= 0 , \quad \frac{\partial E_x}{\partial y} + \frac{ik}{\eta_1} \left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right) E_x = 0 \qquad (y = 0+) \\
H_x &= 0 , \quad \frac{\partial E_x}{\partial y} - \frac{ik}{\eta_2} \left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial x^2}\right) E_x = 0 \qquad (y = 0-)
\end{aligned}
\tag{23}
$$

for any η_1 and η_2. On the other hand, if $\eta_1 = \eta_2 = \infty$

$$
\begin{aligned}
H_z &= 0 , \quad \frac{\partial E_z}{\partial y} + \frac{ik}{\eta_3} \left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right) E_z = 0 \qquad (y = 0+) \\
H_z &= 0 , \quad \frac{\partial E_z}{\partial y} - \frac{ik}{\eta_4} \left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right) E_z = 0 \qquad (y = 0-)
\end{aligned}
\tag{24}
$$

for any η_3 and η_4, and if $\eta_3 = \eta_4 = 0$

$$
\begin{aligned}
E_z &= 0 , \quad \frac{\partial H_z}{\partial y} + ik\eta_1 \left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right) H_z = 0 \qquad (y = 0+) \\
E_z &= 0 , \quad \frac{\partial H_z}{\partial y} - ik\eta_2 \left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right) H_z = 0 \qquad (y = 0-)
\end{aligned}
\tag{25}
$$

for any η_1 and η_2, and in both these cases the components E_z and H_z provide the desired scalarisation. These same components are also effective if $\eta_3 = 1/\eta_2$ and $\eta_4 = 1/\eta_1$, or if $\eta_4 = -\eta_3$ and $\eta_2 = -\eta_1$, and in the latter case the boundary conditions (15) are identical on the two sides of the half plane.

As evident from (9), an impedance sheet supports both electric and magnetic currents, and in the particular case of a half plane (or, indeed, any infinitesimally thin sheet) confined to the plane $y = 0$, the boundary conditions (15) imply

$$E_x(+) + E_x(-) = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} J_z^* + \frac{2\eta_1\eta_2}{\eta_1 + \eta_2} Z J_x$$

$$E_z(+) + E_z(-) = -\frac{\eta_3 - \eta_4}{\eta_3 + \eta_4} J_x^* + \frac{2\eta_3\eta_4}{\eta_3 + \eta_4} Z J_z \qquad (26)$$

$$H_x(+) + H_x(-) = \frac{\eta_3 - \eta_4}{\eta_3 + \eta_4} J_z + \frac{2}{\eta_3 + \eta_4} Y J_x^*$$

$$H_x(+) + H_z(-) = -\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} J_x + \frac{2}{\eta_1 + \eta_2} Y J_z^*$$

where

$$\underline{J} = \left[H_z\right]_-^+ \hat{x} - \left[H_x\right]_-^+ \hat{z}$$

$$\underline{J}^* = -\left[E_z\right]_-^+ \hat{x} + \left[E_x\right]_-^+ \hat{z} \qquad (27)$$

are the total electric and magnetic currents respectively borne by the sheet. The conditions (26) are entirely equivalent to (15) and show the coupling between the electric and magnetic currents that can occur with an anisotropic sheet.

The acoustically resistive and conductive membranes also have their electromagnetic analogues supporting only an electric or a magnetic current. An electrically resistive sheet is simply an electric current sheet whose strength is proportional to the tangential electric field at its surface, and in recent years the concept of such a sheet has found many useful applications. As first noted by Levi-Cività (see Bateman, 1915; p. 19), its electromagnetic properties are completely specified by its resistivity, and in the general case when this is anisotropic, the conditions at the surface of the sheet are

$$\left[\hat{n} \wedge \underline{E}\right]_{-}^{+} = 0$$

$$Z\,\bar{\bar{R}} \cdot \left[\hat{n} \wedge \underline{H}\right]_{-}^{+} = -\hat{n} \wedge (\hat{n} \wedge \underline{E}) \tag{28}$$

(c.f. the conditions (6)) where $\hat{n}$ is now the outward normal to the upper (positive) side and $Z\,\bar{\bar{R}}$ is the resistivity in ohms per square. If the sheet is confined to (say) the plane $y = 0$, the fact that the tangential components of the scattered magnetic field are asymmetrical across the sheet enables us to replace the second of the conditions (28) by the equivalent boundary condition

$$2\,Z\,\bar{\bar{R}} \cdot \hat{n} \wedge (\underline{H} - \underline{H}^{i}) = -\hat{n} \wedge (\hat{n} \wedge \underline{E}) \; . \tag{29}$$

However, the first condition still remains and even a planar resistive sheet is necessarily a transition (rather than a boundary value) problem unless the incident field is such that $\hat{n} \wedge \underline{E} = 0$.

The dual problem is that of a (magnetically) conductive sheet supporting only a magnetic current, and by analogy with (28) the conditions at its surface are

$$\left[\hat{n} \wedge \underline{H}\right]_{-}^{+} = 0$$

$$Y\,\bar{\bar{R}}^{*} \cdot \left[\hat{n} \wedge \underline{E}\right]_{-}^{+} = \hat{n} \wedge (\hat{n} \wedge \underline{H}) \tag{30}$$

where $Y\bar{\bar{R}}^{*}$ is an anisotropic conductivity in mhos per square. By writing the second of the conditions (28) and (30) in terms of the sum fields on both sides of the sheet we can also define a combined resistive-conductive sheet at which the conditions are

$$2Z\bar{\bar{R}}\cdot\left[\hat{n}\wedge\underline{H}\right]_-^+ = -\hat{n}\wedge\left(\hat{n}\wedge\left\{\underline{E}(+)+\underline{E}(-)\right\}\right)$$

$$2Y\bar{\bar{R}}^*\cdot\left[\hat{n}\wedge\underline{E}\right]_-^+ = \hat{n}\wedge\left(\hat{n}\wedge\left\{\underline{H}(+)+\underline{H}(-)\right\}\right) \tag{31}$$

and for a particular relationship between the resistivity and conductivity tensors the combined sheet is equivalent to an impedance boundary condition one. This is most easily seen in the case of a planar sheet confined to (say) the surface $y = 0$. Manipulation of (31) then gives

$$E_x(\pm) = Z\left(R_1 \pm \frac{1}{4R_3^*}\right) H_z(+) - Z\left(R_1 \mp \frac{1}{4R_3^*}\right) H_z(-)$$

$$E_z(\pm) = -Z\left(R_3 \pm \frac{1}{4R_1^*}\right) H_x(+) + Z\left(R_3 \mp \frac{1}{4R_1^*}\right) H_x(-) \tag{32}$$

where

$$\bar{\bar{R}} = R_1\hat{x}\hat{x} + R_3\hat{z}\hat{z}\ , \quad \bar{\bar{R}}^* = R_1^*\hat{x}\hat{x} + R_3^*\hat{z}\hat{z}\ .$$

Provided

$$2R_1^* = \frac{1}{2R_3}\ , \quad 2R_3^* = \frac{1}{2R_1}\ , \tag{33}$$

(32) are identical to the impedance boundary conditions (15) with

$$\eta_1 = \eta_2 = 2R_1,\ \eta_3 = \eta_4 = 2R_3\ , \tag{34}$$

but only for this resistivity-conductivity relationship is the combination sheet equivalent to an impedance sheet (Senior, 1975b). For other combination sheets and for a resistive or conductive sheet in isolation, the problem is fundamentally a transition one.

3. Methods of Solution

There are two basic methods of solution of our half plane problems and one or other is applicable in all cases for which solutions have been obtained so far. Each has a number of variations which are of interest more for the physical insight that they afford than for any increase in capability, and there are in addition methods which are effective in rather special cases and which may then provide a rather elementary approach. An example of the latter is the method proposed by Tan and Cheng (1970) for a class of half plane problems. The method assumes a solution consisting of a linear combination of Fresnel integrals of the form appearing in the solution for a metallic half plane, and then chooses the coefficients to satisfy the boundary conditions. In most instances the boundary conditions on the two sides of the half plane must be identical, and though Tan and Cheng refer to one of the geometries as an impedance half plane, we remark that the impedances must be such that $\bar{\bar{\eta}}_- = -\bar{\bar{\eta}}_+$.

The two basic methods have very different starting points and, superficially at least, are quite distinct. Nevertheless, they do have many similarities and with the developments that have taken place in recent months their capabilities are now comparable as regards the type of problem considered here. It is convenient to describe the methods in terms of their application to the simple problem of a plane acoustic wave

$$V^i = e^{-ik\rho\cos(\phi-\phi_0)} \tag{35}$$

incident normally on an impedance half plane $y = 0$, $x \geqslant 0$. The boundary conditions are as shown in (3) or (4) with (possibly) $\eta = \eta_3$ on the upper face and $\eta = \eta_4$ on the lower. We shall then examine how the methods apply to the other problems involving impedance half planes in acoustics and electromagnetics, and conclude with a few comments about the transition problems of resistive and conductive half planes.

The first method is that of Maliuzhinets (1959) and is the more general of the two inasmuch as it is applicable to wedge-shaped regions as well. It is equivalent to the approach which Senior (1959b) developed based on Peters' (1952) analysis of a problem in hydrodynamics, but is more elementary and straightforward, and we shall confine attention to it. The method assumes a representation of the total field in cylindrical polar coordinates in the form of a Sommerfeld integral, viz.

$$V = \frac{1}{2\pi i} \int_\gamma e^{ik\rho \cos \alpha} s(\alpha - \phi) \, d\alpha \tag{36}$$

where γ is the Sommerfeld contour shown in Fig. 1 and $s(\alpha-\pi) - (\alpha-2\pi+\phi_0)^{-1}$ is analytic in the strip $0 \leqslant \mathrm{Re.}\ \alpha \leqslant 2\pi$. On applying the boundary conditions (4) and eliminating the derivatives using integration by parts, we find

$$\begin{aligned} &\int_\gamma e^{ik\rho \cos \alpha} (1 - \eta_3 \sin \alpha) \, s(\alpha) \, d\alpha = 0 \\ &\int_\gamma e^{ik\rho \cos \alpha} (1 + \eta_4 \sin \alpha) \, s(\alpha - 2\pi) \, d\alpha = 0 \end{aligned} \tag{37}$$

and for these to be satisfied it is both necessary and sufficient (Maliuzhinets, 1958) that the integrands be even about $\alpha = \pi$. Hence

$$\begin{aligned} &(1 - \eta_3 \sin \alpha) \, s(\alpha) = (1 + \eta_3 \sin \alpha) \, s(2\pi - \alpha) \\ &(1 + \eta_4 \sin \alpha) \, s(\alpha - 2\pi) = (1 - \eta_4 \sin \alpha) \, s(-\alpha) \end{aligned} \tag{38}$$

and if we now write

$$s(\alpha) = \sigma(\alpha) \frac{\Psi(\alpha)}{\Psi(\pi - \phi_0)} \tag{39}$$

where

$$\sigma(\pm\alpha) = \sigma(\pm 2\pi \mp \alpha) \quad , \tag{40}$$

we can choose

$$\sigma(\alpha) = \frac{1}{2}\sin\frac{\phi_o}{2}\left(\sin\frac{\alpha}{2} - \cos\frac{\phi_o}{2}\right)^{-1} \tag{41}$$

provided $\Psi(\alpha)$ is analytic and free of zeros in the strip.

From (38) by a process of elimination,

$$(1-\eta_3\sin\alpha)(1-\eta_4\sin\alpha)\Psi(\alpha+4\pi) = (1+\eta_3\sin\alpha)(1+\eta_4\sin\alpha)\Psi(\alpha) \quad . \tag{42}$$

This is a first order difference equation of interval 4π, but if $\eta_4 = \eta_3$ the equation can be reduced to

$$(1-\eta_3\sin\alpha)\,\Psi(\alpha+2\pi) = (1+\eta_3\sin\alpha)\,\Psi(\alpha) \tag{43}$$

whose interval is 2π. The solution of (42) having the required properties is

$$\Psi(\alpha) = \psi_\pi(\alpha+\pi+\chi_3)\,\psi_\pi(\alpha+\pi-\chi_3)\,\psi_\pi(\alpha-\pi+\chi_4)\,\psi_\pi(\alpha-\pi-\chi_4) \tag{44}$$

where

$$\eta_{3,4} = \sec\chi_{3,4} \tag{45}$$

and $\psi_\pi(\alpha)$ is the special meromorphic function

$$\psi_\pi(\alpha) = \exp\left\{-\frac{1}{8\pi}\int_0^\alpha\left(\pi\sin v - 2\sqrt{2}\,\pi\sin\frac{v}{2} + 2v\right)\frac{dv}{\cos v}\right\}. \tag{46}$$

$\psi_\pi(\alpha)$ has been computed by Bucci (1974) for a range of complex arguments, and many of its properties are listed by Maliuzhinets (1959) and Bowman (1967).

If the contour γ is closed using two steepest descents paths through $\alpha = 0$ and 2π, V can be expressed as the sum of the residues at the included poles. The corresponding terms are the incident and reflected waves of geometrical optics existing in the appropriate regions, together with any surface wave contributions that may occur. The remaining (diffracted) field is then

$$V^d = -\frac{1}{2\pi i}\int_{S(0)} e^{ik\rho\cos\alpha}\left\{s(\alpha-\phi) - s(2\pi+\alpha-\phi)\right\} d\alpha \tag{47}$$

where S(0) is a path from $\alpha = -\frac{\pi}{2} + i\infty$ to $\alpha = \frac{\pi}{2} - i\infty$ passing through the origin, and on inserting the expression for $s(\alpha)$, we have

$$V^d = \frac{1}{2\pi i}\int_{S(0)} e^{ik\rho\cos\alpha}\frac{\sin\frac{\phi_0}{2}}{\cos(\alpha-\phi)+\cos\phi_0}\left\{\left(\sin\frac{\alpha-\phi}{2} + \cos\frac{\phi_0}{2}\right)\frac{\Psi(\alpha-\phi)}{\Psi(\pi-\phi_0)} + \right.$$

$$\left. + \left(\sin\frac{\alpha-\phi}{2} - \cos\frac{\phi_0}{2}\right)\frac{\Psi(2\pi+\alpha-\phi)}{\Psi(\pi-\phi_0)}\right\} d\alpha \quad . \tag{48}$$

From the relations satisfied by the function $\psi_\pi(\alpha)$, it can be shown that

$$\Psi(2\pi+\alpha) = \left\{\frac{\cos\frac{1}{2}\chi_3 - \cos\frac{1}{2}(\alpha+\frac{\pi}{2})}{\cos\frac{1}{2}\chi_3 - \cos\frac{1}{2}(\alpha-\frac{\pi}{2})}\right\}\left\{\frac{\cos\frac{1}{2}\chi_4 + \cos\frac{1}{2}(\alpha+\frac{\pi}{2})}{\cos\frac{1}{2}\chi_4 + \cos\frac{1}{2}(\alpha-\frac{\pi}{2})}\right\}\Psi(\alpha) \quad , \tag{49}$$

and in particular, when $\eta_4 = \eta_3$ ($= \eta$, say)

$$\Psi(2\pi+\alpha) = \frac{1+\eta\sin\alpha}{1-\eta\sin\alpha}\Psi(\alpha) \tag{50}$$

in which case

$$V^d = \frac{1}{\pi i}\int_{S(0)} e^{ik\rho\cos\alpha}\, \frac{\bar{\Psi}(2\pi+\alpha-\phi)}{\bar{\Psi}(\pi-\phi_o)}\, \frac{1}{1+\eta\sin(\alpha-\phi)}\, \frac{\sin\frac{\alpha-\phi}{2}\sin\frac{\phi_o}{2}}{\cos(\alpha-\phi)+\cos\phi_o}\left(1-2\eta\cos\frac{\alpha-\phi}{2}\cos\frac{\phi_o}{2}\right)d\alpha\,. \tag{51}$$

Away from the geometrical optics boundaries and at large distances ($k\rho \gg 1$) V^d has the character of an edge wave. By a steepest descents evaluation of the integral in (48),

$$V^d \sim \sqrt{\frac{2}{\pi k\rho}}\, e^{i(k\rho - \pi/4)}\, P(\phi, \phi_o)$$

with

$$P(\phi, \phi_o) = \frac{i}{2}\, \frac{\sin\frac{\phi_o}{2}}{\cos\phi + \cos\phi_o}\left\{\left(\sin\frac{\phi}{2} - \cos\frac{\phi_o}{2}\right)\frac{\bar{\Psi}(-\phi)}{\bar{\Psi}(\pi-\phi_o)} + \left(\sin\frac{\phi}{2} + \cos\frac{\phi_o}{2}\right)\frac{\bar{\Psi}(2\pi-\phi)}{\bar{\Psi}(\pi-\phi_o)}\right\} \tag{52}$$

(Bowman, 1967). In certain cases the right hand side can be expressed in terms of trigonometric functions alone. Thus, for a hard half plane ($\eta_4 = \eta_3 = \infty$, implying $\chi_4 = \chi_3 = \pi/2$),

$$\bar{\Psi}(\alpha) = \frac{1}{2}\cos\frac{\alpha}{2}\left\{\psi_\pi(\pi/2)\right\}^2$$

giving

$$\frac{\bar{\Psi}(-\phi)}{\bar{\Psi}(\pi-\phi_o)} = -\frac{\bar{\Psi}(2\pi-\phi)}{\bar{\Psi}(\pi-\phi_o)} = \frac{\cos\frac{\phi}{2}}{\sin\frac{\phi_o}{2}}\,.$$

Hence

$$P^{(h)}(\phi, \phi_o) = -i\,\frac{\cos\frac{\phi}{2}\cos\frac{\phi_o}{2}}{\cos\phi + \cos\phi_o} \tag{53}$$

in accordance with the known result (Bowman et al, 1969; p. 322). Similarly, for a soft half plane ($\eta_4 = \eta_3 = 0$, implying $\chi_4 = \chi_3 = i\infty$), $\bar{\Psi}(\alpha)$ is independent of α and therefore

$$P^{(s)}(\phi, \phi_o) = i \frac{\sin\frac{\phi}{2}\sin\frac{\phi_o}{2}}{\cos\phi + \cos\phi_o} \tag{54}$$

(Bowman et al, 1969; p. 315). If the half plane is soft on top ($\chi_3 = i\infty$) but hard on the bottom ($\chi_4 = \pi/2$), then

$$\frac{\Psi(\alpha)}{\Psi(\pi - \phi_o)} = \frac{\cos\frac{1}{4}(\pi-\alpha)}{\cos\frac{\phi_o}{4}}$$

so that

$$\frac{\Psi(-\phi)}{\Psi(\pi-\phi_o)} = \frac{\cos\frac{1}{4}(\pi+\phi)}{\cos\frac{\phi_o}{4}}, \quad \frac{\Psi(2\pi-\phi)}{\Psi(\pi-\phi_o)} = \frac{\cos\frac{1}{4}(\pi-\phi)}{\cos\frac{\phi_o}{4}}$$

giving

$$P^{(s,h)}(\phi,\phi_o) = i\sqrt{2}\,\frac{\sin\frac{\phi}{4}\sin\frac{\phi_o}{4}}{\cos\phi + \cos\phi_o}\left(1 + \cos\frac{\phi}{2} + \cos\frac{\phi_o}{2}\right) \;. \tag{55}$$

Conversely, if the half plane is hard on top ($\chi_3 = \pi/2$) and soft on the bottom ($\chi_4 = i\infty$),

$$P^{(h,s)}(\phi,\phi_o) = i\sqrt{2}\,\frac{\cos\frac{\phi}{4}\cos\frac{\phi_o}{4}}{\cos\phi + \cos\phi_o}\left(1 - \cos\frac{\phi}{2} - \cos\frac{\phi_o}{2}\right) \tag{56}$$

and we observe that

$$P^{(h,s)}(\phi,\phi_o) = P^{(s,h)}(2\pi-\phi,\, 2\pi-\phi_o)$$

as expected. The idea of backing a soft (absorbing) surface with a hard surface to provide rigid support is important in the design of acoustic baffles, and (55) and (56) are useful for assessing their performance. The expression (55) for $P^{(s,h)}(\phi,\phi_o)$ is equivalent to, but a good deal simpler than, the one obtained by Rawlins (1975) using his extension of the standard Wiener-Hopf technique.

In Cartesian coordinates the half plane is a two-part boundary value problem, and if Green's theorem or some other approach is used to represent the scattered field as a surface integral, application of the boundary conditions leads to a system of Wiener-Hopf integral equations for the 'currents' induced in the half plane. In principle at least these equations can be solved using a Fourier or bilateral Laplace transform supplemented by function-theoretic techniques, and we shall refer to this as the Wiener-Hopf method.

To illustrate this second basic method, we again consider the scalar plane wave (35) incident on an impedance half plane but with the same impedance η at both faces. The boundary conditions are now (3), and by applying Green's theorem to the region exterior to a surface surrounding the half plane and then collapsing the surface to the half plane itself, we have

$$V = V^i - \frac{i}{4}\int_0^\infty \left\{ f_1(x') + f_2^*(x')\frac{\partial}{\partial y}\right\} H_0^{(1)}\left(k\sqrt{(x-x')^2+y^2}\right) dx' \tag{57}$$

where

$$f_1(x) = \left[\frac{\partial V}{\partial y}\right]_-^+ , \quad f_2^*(x) = \left[V\right]_-^+ \tag{58}$$

and the notation has been chosen to correspond to that of Senior (1975c). On adding the limits as $y \to \pm 0$ and using the first of the eqs. (3),

$$-\frac{\eta}{k} f_1(x) = 2i\,V^i + \frac{1}{2}\int_0^\infty f_1(x') H_0^{(1)}(k|x-x'|)dx' \tag{59}$$

valid for $x \geqslant 0$, and this is an inhomogeneous Wiener-Hopf integral equation for $f_1(x)$. Similarly, by differentiating (57) with respect to y and then adding the limits,

$$\frac{k}{\eta} f_2^*(x) = 2i\frac{\partial V^i}{\partial y} + \frac{1}{2}\lim_{y\to 0}\int_0^\infty f_2^*(x')\frac{\partial^2}{\partial y^2} H_0^{(1)}\left(k\sqrt{(x-x')^2+y^2}\right) dx' \tag{60}$$

valid for $x \geqslant 0$, and this is a Wiener-Hopf equation for $f_2^*(x)$.

The procedure for solving equations of this type is fully documented in the literature (Bowman et al, 1969; pp 41-49). Taking for example (59), the equation is extended to apply for all x by taking $f_1(x) = 0$, $x < 0$, with $\Omega(x)$ equal to the right hand side of (59) for $x < 0$ and zero otherwise. If a Fourier transform is defined as

$$g(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} g(x)\, dx$$

with inverse

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} g(\xi)\, d\xi ,$$

it then follows that $f_1(\xi)$ is analytic in a lower half plane and $\Omega(\xi)$ in an upper half plane. If, moreover, k is assumed to have a positive imaginary part (later allowed to vanish), there exists a strip of analyticity permitting the transformation of the extended integral equation. The result is

$$\eta\, \Omega(\xi) = 2ik\, V^{i}(\xi) + \left(\eta + \frac{k}{\sqrt{k^2 - \xi^2}}\right) f_1(\xi) \quad . \qquad (61)$$

The key step is now the factorization (or 'splitting') of the coefficient of $f_1(\xi)$ into lower and upper functions having overlapping half planes of analyticity If

$$\eta + \frac{k}{\sqrt{k^2 - \xi^2}} = \frac{K_-(\xi)}{K_+(\xi)} \qquad (62)$$

implying

$$K_-(\xi)\, K_+(-\xi) = 1$$

(Senior, 1952), $K_-(\xi)$ and $K_+(\xi)$ can be determined using Cauchy's theory, and from the known behavior of $f_1(\xi)$ at infinity, Liouville's theorem then enables us to find $f_1(\xi)$ and $\Omega(\xi)$.

The analysis for (60) is very similar and for future reference we remark that if (61) and its analogue are written as a pair of simultaneous equations, the matrix G (ξ) that must be factored is a diagonal one having

$$G_{11}(\xi) = \eta + \frac{k}{\sqrt{k^2 - \xi^2}}, \quad G_{22}(\xi) = \frac{\sqrt{k^2 - \xi^2}}{k} \left(\eta + \frac{k}{\sqrt{k^2 - \xi^2}}\right).$$

The functions $f_1(x)$ and $f_2^*(x)$ can be obtained from $f_1(\xi)$ by Fourier inversion, and when their expressions are inserted into (57) we have

$$V = V^i - \frac{1}{2\pi i} \int_C \frac{K_+(-k \cos \phi_o)}{K_-(k \cos \alpha)} \; \frac{1 - 2\eta \frac{y}{|y|} \cos\frac{\alpha}{2} \cos\frac{\phi_o}{2}}{\cos \alpha + \cos \phi_o} \; e^{ik(x \cos \alpha + |y| \sin \alpha)} \, d\alpha$$

where $\xi = k \cos \alpha$ and the path C in the α plane runs from $i\infty$ to the origin and thence to π and $\pi - i\infty$, passing above the pole at $\alpha = \pi - \phi_o$. On introducing the polar coordinates (ρ, ϕ), the integrand is seen to have a saddle point at $\alpha = \phi$ ($y \geq 0$) or $\alpha = 2\pi - \phi$ ($y < 0$), and when the path is deformed into a steepest descents path through this point, the contribution of the pole at $\alpha = \pi - \phi_o$ combines with V^i to yield the geometrical optics terms. The resulting diffracted field is

$$V^d = -\frac{1}{2\pi i} \int_{S(0)} e^{ik\rho \cos \alpha} \; \frac{K_+(-k \cos \phi_o)}{K_-(k \cos \{\alpha - \phi\})} \; \frac{1 - 2\eta \cos\frac{\alpha - \phi}{2} \cos\frac{\phi_o}{2}}{\cos(\alpha - \phi) + \cos \phi_o} \, d\alpha \tag{63}$$

and when this is compared with (51) it is obvious that the split functions must be closely related to Maliuzhinets' function $\psi(\alpha)$ for $\eta_4 = \eta_3 = \eta$. From the expressions for $K_+(\xi)$ and $K_-(\xi)$ given by Senior (1952), it can be shown that

$$K_+(k \cos \alpha) = \cos\frac{\alpha}{2} \exp(2p/\pi) \left\{(\sqrt{2} + 1)^2 (\eta/2)^{1/2} \Psi(\alpha)\right\}^{-1}$$

$$K_-(k \cos \alpha) = \frac{1 + \eta \sin \alpha}{\sin\frac{\alpha}{2}} \exp(2p/\pi) \left\{(\sqrt{2} + 1)^2 (2\eta)^{1/2} \Psi(\alpha)\right\}^{-1} \tag{64}$$

where $p = 0.9159656...$ is Catalan's constant, and with this identification (51) and (63) are identical. A further consequence of (64) is that $K_-(\xi)$ is also the solution of a difference equation, and by treating the (cut) ξ plane as one sheet of a two-sheeted Riemann surface, we have (Senior, 1959b)

$$K_-(\xi\, e^{2i\pi}) = \frac{k + \eta\sqrt{k^2 - \xi^2}}{k - \eta\sqrt{k^2 - \xi^2}}\, K_-(\xi) \tag{65}$$

c.f. (50), which is a difference equation with (angular) interval 2π.

A Wiener-Hopf problem such as this can be formulated in several different ways but a requirement which is common to them all is the factorisation of a function of the transform variable ξ. An approach due to Jones (1952) and embraced by Noble (1958) is to apply the transform to the partial differential equation before imposing the boundary conditions. The complex variable equation in the transform plane is thereby obtained directly without the necessity of formulating an integral equation for (say) the currents induced in the half plane. In still another approach the rather cumbersome derivation of the integral equation is circumvented by using separation of variables to develop dual integral equations for a quantity related to the far field amplitude. This procedure was expounded by Karp (1950) and exploited by Clemmow (1951, 1966), and in the case of the present problem the representation employed is

$$V - V^i = \int_C \left\{ Q_1(\cos\alpha) \mp \sin\alpha\, Q_2(\cos\alpha) \right\} e^{ik\rho\cos(\alpha \mp \phi)}\, d\alpha$$

for $y \gtrless 0$, where C is the path defined earlier and $Q_1(\xi)$, $Q_2(\xi)$ are angular spectra proportional to the Fourier transforms of $f_1(x)$, $f_2^*(x)$ respectively. If C is now deformed into a steepest descents path through $\alpha = \phi$ ($y \geqslant 0$) or $2\pi - \phi$ ($y<0$), a simple change of variable produces a result strikingly similar to the representation (47) inherent in Maliuzhinets' method.

In this discussion of the Wiener-Hopf method we have assumed that both sides of the half plane have the same impedance. If, instead, the impedances on the upper and lower faces are η_3 and η_4 respectively with $\eta_4 \neq \eta_3$, application of the boundary conditions to the representation (57) gives

$$-\frac{1}{k}\,\frac{2\eta_3\eta_4}{\eta_3+\eta_4}\,f_1(x) + i\,\frac{\eta_3-\eta_4}{\eta_3+\eta_4}\,f_2^*(x) = 2iV^i + \frac{1}{2}\int_0^\infty f_1(x')\,H_0^{(1)}(k\,|x-x'|)\,dx'$$

$$-i\,\frac{\eta_3-\eta_4}{\eta_3+\eta_4}\,f_1(x) + \frac{2k}{\eta_3+\eta_4}\,f_2^*(x) = 2i\,\frac{\partial V^i}{\partial y} + \frac{1}{2}\lim_{y\to 0}\int_0^\infty f_2^*(x')\,\frac{\partial^2}{\partial y^2}\,H_0^{(1)}\left(k\sqrt{(x-x')^2+y^2}\right)dx' \tag{66}$$

which now constitute a pair of <u>coupled</u> Wiener-Hopf integral equations for $f_1(x)$ and $f_2^*(x)$. In the plane of the transform variable ξ, the matrix $G(\xi)$ multiplying the column vector $\{f_1(\xi),\ ik\,f_2^*(\xi)\}$ is

$$G(\xi) = \begin{pmatrix} \frac{2\eta_3\eta_4}{\eta_3+\eta_4} + \frac{k}{\Gamma}, & -\frac{\eta_3-\eta_4}{\eta_3+\eta_4} \\ \frac{\eta_3-\eta_4}{\eta_3+\eta_4}, & \frac{2}{\eta_3+\eta_4} + \frac{\Gamma}{k} \end{pmatrix} \tag{67}$$

where $\Gamma = \sqrt{k^2-\xi^2}$, and though in principle it is possible to factor this into matrices having overlapping half planes of analyticity, no direct method for doing so exists. The matrix can be simplified somewhat by adding γ times the first equation to the second, with $\gamma = 1/\eta_3$ and $-1/\eta_4$, in which case

$$G(\xi) = \begin{pmatrix} 1 + \frac{k}{\eta_3\Gamma}, & \frac{1}{\eta_3} + \frac{\Gamma}{k} \\ -1 - \frac{k}{\eta_4\Gamma}, & \frac{1}{\eta_4} + \frac{\Gamma}{k} \end{pmatrix} \tag{68}$$

but even so it is only recently that a method of factorisation has been found.

A clue as to how we might proceed can be obtained from Maluizhinets' solution for this same diffraction problem. It will be recalled that $s(\alpha)$ and the split functions are closely related if $\eta_4 = \eta_3$ and $s(\alpha)$ then satisfies a difference equation of interval 2π. When $\eta_4 \neq \eta_3$, however, the interval is 4π (see 42), suggesting that we should now invoke a relationship between the values of $G(\xi)$ on adjacent sheets of a Riemann surface. In the particular case of an impedance half plane soft on top ($\eta_3 = 0$) and hard on the bottom ($\eta_4 = \infty$) for which (68) reduces to

$$G(\xi) = \begin{pmatrix} \frac{k}{\Gamma}, & 1 \\ -1, & \frac{\Gamma}{k} \end{pmatrix} ,$$

Rawlins (1975) achieved a factorisation using an intuitive argument based on the change of sign of the radical on crossing the branch cut emanating from $\xi = -k$. More recently still, Hurd (1976) has developed a general theory for matrices $G(\xi)$ whose only singularities are branch points such that

$$G(\xi)\left[G^{(-)}(\xi)\right]^{-1} = \begin{pmatrix} 0 , & \sigma_1 \\ \sigma_2, & 0 \end{pmatrix} \qquad (69)$$

for some σ_1, σ_2, where $G^{(-)}(\xi)$ differs from $G(\xi)$ in having $-\Gamma$ in place of Γ. The conditions are satisfied by (68) (though not by 67), and Hurd has presented the explicit factorisation for any η_3 and η_4. With this extension, the Wiener-Hopf and Maliuzhinets methods now have the same capabilities as regards the acoustic problem of an impedance half plane.

In addition to this problem there are a number of others involving impedance half planes, and it is of interest to see how the basic methods apply to them. If the acoustic plane wave is incident obliquely, with

$$V^i = e^{ik\rho \sin \beta \cos (\phi - \phi_o) + ikz \cos \beta} \qquad (70)$$

in place of (35), the total field must have this same z dependence, and since the boundary conditions (4) can be written as

$$\frac{1}{\rho}\frac{\partial V}{\partial \phi} \pm \frac{ik \sin\beta}{\eta'_{3,4}} V = 0$$

with

$$\eta'_{3,4} = \eta_{3,4} \sin\beta ,$$

both methods work as before. Indeed, the analyses for $V e^{-ikz \sin\beta}$ differ from those for V in the normal incidence case $\beta = \pi/2$ only in having k replaced by $k \sin\beta$ and $\eta_{3,4}$ by $\eta'_{3,4}$.

For an electromagnetic wave at normal incidence on an anisotropic half plane with the impedances (16) it is obviously sufficient to consider E- and H-polarizations separately. The anisotropy then produces no difficulty at all and for each polarization the problem is a scalar one for (say) the z component of the field. Thus, for an E-polarized plane wave with E_z^i given by (35) the total field component E_z satisfies the boundary condition (17) and is identical to the solution V. The analogous result for H-polarization follows on replacing η_3 by $1/\eta_1$ and η_4 by $1/\eta_2$. For oblique incidence however, the problem is not quite so simple and it is desirable to examine the two basic methods separately.

In order to build on the success of Maliuzhinets' method in the above cases, it would seem natural to use the boundary conditions (20) with $\partial/\partial z$ replaced by $ik \cos\beta$, but since these couple E_z and H_z regardless of the anisotropy, we are now forced to represent E_z and H_z simultaneously in the form

$$E_z = \frac{1}{2\pi i}\int_\gamma e^{ik\rho \sin\beta \cos\alpha} s_1(\alpha - \phi)\, d\alpha$$

$$ZH_z = \frac{1}{2\pi i}\int_\gamma e^{ik\rho \sin\beta \cos\alpha} s_2(\alpha - \phi)\, d\alpha . \qquad (41)$$

Application of the boundary conditions then leads to the four functional equations

$$(\sin\beta - \eta_3 \sin\alpha)\, s_1(\alpha) - \cos\alpha\cos\beta\, s_2(\alpha) = (\sin\beta + \eta_3\sin\alpha)\, s_1(2\pi-\alpha) - \cos\alpha\cos\beta\, s_2(2\pi-\alpha)$$
$$(\sin\beta - 1/\eta_1 \sin\alpha)\, s_2(\alpha) + \cos\alpha\cos\beta\, s_1(\alpha) = (\sin\beta + 1/\eta_1\sin\alpha)\, s_2(2\pi-\alpha) + \cos\alpha\cos\beta\, s_1(2\pi-\alpha)$$
$$(\sin\beta + \eta_4 \sin\alpha)\, s_1(\alpha-2\pi) + \cos\alpha\cos\beta\, s_2(\alpha-2\pi) = (\sin\beta - \eta_4\sin\alpha)\, s_1(-\alpha) + \cos\alpha\cos\beta\, s_2(-\alpha)$$
$$(\sin\beta + 1/\eta_2 \sin\alpha)\, s_2(\alpha-2\pi) - \cos\alpha\cos\beta\, s_1(\alpha-2\pi) = (\sin\beta - 1/\eta_2\sin\alpha)\, s_2(-\alpha) - \cos\alpha\cos\beta\, s_1(-\alpha) \tag{72}$$

relating s_1 and s_2, and from these either function can be eliminated. If we again introduce a (common) singular function $\sigma(\alpha)$ as shown in (39), we obtain after much tedious manipulation

$$F(\eta_1, 1/\eta_3)\, F(\eta_2, 1/\eta_4)\, \Psi(\alpha + 4\pi) - \Big\{F(\eta_1, 1/\eta_3)\, F(-\eta_1, -1/\eta_3) + \Lambda + F(\eta_2, 1/\eta_4)\, F(-\eta_2, -1/\eta_4)\Big\}\Psi(\alpha) + F(-\eta_1, -1/\eta_3)\, F(-\eta_2, -1/\eta_4)\Psi(\alpha - 4\pi) = 0 \tag{73}$$

where $\Psi(\alpha) = \Psi_1(\alpha)$ or $\Psi_2(\alpha)$ with

$$F(a, b) = \cos^2\alpha\cos^2\beta + (\sin\alpha + a\sin\beta)(\sin\alpha + b\sin\beta)$$

$$\Lambda = \sin^2\beta\left\{(\eta_1 + \eta_2 - 1/\eta_3 - 1/\eta_4)^2 \sin^2\alpha - \left(\frac{\eta_1}{\eta_3} - \frac{\eta_2}{\eta_4}\right)^2 \sin^2\beta\right\}.$$

This is a second order difference equation of interval 4π, but if $\eta_4 = \eta_3$ and $\eta_2 = \eta_1$ implying $\bar{\bar{\eta}}_+ = \bar{\bar{\eta}}_-$, the equation can be reduced to

$$F(\eta_1, 1/\eta_3)\Psi(\alpha + 2\pi) + \sqrt{2}(\eta_1 - 1/\eta_3)\sin\beta\,\Psi(\alpha) - F(-\eta_1, -1/\eta_3)\Psi(\alpha - 2\pi) = 0 \tag{74}$$

whose interval is only 2π. The decrease in interval is analogous to that achievable in the acoustic problem, but unfortunately the equation is still a second order one for which there is no known method of solution. The only obvious circumstances in which (73) reduces to a first order equation (and is therefore soluble) is when $\Lambda = 0$, requiring

$$\eta_1 = -\eta_2 , \quad \eta_3 = -\eta_4$$

or

$$\eta_1 = 1/\eta_4 , \quad \eta_2 = 1/\eta_3$$

which are the cases noted in Section 2. Nevertheless, if the impedances are isotropic ($\eta_3 = \eta_1$, $\eta_4 = \eta_2$) it is known that the components E_y and H_y provide a simple scalarisation of the problem, implying that the linear combinations

$$\sin\beta\cos\beta\, s_1(\alpha) + \cos\alpha\, s_2(\alpha), \quad \sin\beta\cos\beta\, s_2(\alpha) - \cos\alpha\, s_1(\alpha)$$

satisfy first order difference equations of the form (42), but this fact is hardly evident from the functional equations (71).

In the special case of isotropic impedances the problem is greatly simplified by working with the components E_y and H_y from the outset. These satisfy the (decoupled) scalar boundary conditions (19) and (18) respectively and if

$$E_y = \frac{1}{2\pi i}\int_\gamma e^{ik\rho\sin\beta\cos\alpha}\, s_1(\alpha-\phi)\, d\alpha$$

$$ZH_y = \frac{1}{2\pi i}\int_\gamma e^{ik\rho\sin\beta\cos\alpha}\, s_2(\alpha-\phi)\, d\alpha \qquad (75)$$

it follows immediately that $s_2(\alpha)$ satisfies the difference equation (42) whilst $s_1(\alpha)$ satisfies the same equation with $1/\eta_3$ replaced by η_3 ($= \eta_1$) and $1/\eta_4$ by η_4($= \eta_2$). When the other field components are expressed in terms of $s_1(\alpha)$ and $s_2(\alpha)$, the integrands contain a factor $\cos^2\alpha + \cot^2\beta$ in the denominator, and to ensure that the radiation condition is fulfilled it is now necessary to modify the solution (39) that was appropriate in the acoustic problem. The modification consists of the addition of source-free solutions whose strengths are chosen to suppress the inhomogeneous plane waves contributed by the poles at $\cos\alpha = \pm i \cot\beta$. The details can be found in Bucci and Franceschetti (1976) where the solution is derived for an isotropic impedance half plane with impedances differing on the two sides.

Unfortunately the representations (73) are no more successful than (70) in solving the anisotropic problem, and when the boundary conditions (21) are imposed, the resulting difference equations are again second order ones with coefficients even more complicated than those in (72). It would therefore appear that an intrinsic coupling of the field components via the (vector) boundary conditions manifests itself in second order difference equations, and since there is no procedure for obtaining their solutions, Maliuzhinets' method no longer works.

We now consider the application of the Wiener-Hopf method to this same problem of a plane electromagnetic wave at oblique incidence. If Green's theorem or a Hertz vector representation is used to express the total field in terms of the electric and magnetic currents $\underline{J}$ and $\underline{J}^*$ induced in the half plane, application of the boundary conditions (26) leads to four coupled Wiener-Hopf integral equations for the for (tangential) components of $\underline{J}$ and $\underline{J}^*$. Each equation actually involves only three unknowns but we can increase the symmetry of the system by combining the equations in pairs in a manner similar to that used in deriving (68) from (67). When this is done, the coefficient of the column vector $\left\{J_x(\xi),\ J_z(\xi),\ YJ_x^*(\xi),\ YJ_z^*(\xi)\right\}$ in the ξ plane is the 4 by 4 matrix

$$G(\xi) = \begin{pmatrix} 1+\dfrac{k^2-\xi^2}{\eta_2 k\Gamma}, & \dfrac{\xi\cos\beta}{\eta_2\Gamma}, & \dfrac{\xi\cos\beta}{\Gamma}, & \dfrac{1}{\eta_2}+\dfrac{k\sin^2\beta}{\Gamma} \\ 1+\dfrac{k^2-\xi^2}{\eta_1 k\Gamma}, & \dfrac{\xi\cos\beta}{\eta_1\Gamma}, & \dfrac{-\xi\cos\beta}{\Gamma}, & -\left(\dfrac{1}{\eta_1}+\dfrac{k\sin^2\beta}{\Gamma}\right) \\ -\dfrac{\xi\cos\beta}{\eta_3\Gamma}, & -\left(1+\dfrac{k\sin^2\beta}{\eta_3\Gamma}\right), & \dfrac{1}{\eta_3}+\dfrac{k^2-\xi^2}{k\Gamma}, & \dfrac{\xi\cos\beta}{\Gamma} \\ -\dfrac{\xi\cos\beta}{\eta_4\Gamma}, & -\left(1+\dfrac{k\sin^2\beta}{\eta_4\Gamma}\right), & \dfrac{1}{\eta_4}+\dfrac{k^2-\xi^2}{k\Gamma}, & \dfrac{\xi\cos\beta}{\Gamma} \end{pmatrix} \tag{76}$$

where $\Gamma = \sqrt{k^2\sin\beta - \xi^2}$. For normal incidence ($\beta = \pi/2$), $G(\xi)$ separates into two 2 by 2 matrices of the form shown in (68).

In the special case when the impedances are isotropic ($\eta_1 = \eta_3$, $\eta_2 = \eta_4$) it proves convenient to combine the equations in pairs using multiplying factors ξ and $\pm \kappa \cos \beta$. The procedure is directly analogous to that used by Senior (1959a) and gives

$$G(\xi) = \begin{pmatrix} k\cos\beta(1+\frac{k}{\eta_4\Gamma}) , & \xi(1+\frac{k}{\eta_4\Gamma}) , & -\xi(\frac{1}{\eta_4}+\frac{\Gamma}{k}) , & k\cos\beta(\frac{1}{\eta_4}+\frac{\Gamma}{k}) \\ \xi(1+\frac{\Gamma}{\eta_4 k}) , & -k\cos\beta(1+\frac{\Gamma}{\eta_4 k}) , & k\cos\beta(\frac{1}{\eta_4}+\frac{k}{\Gamma}) , & \xi(\frac{1}{\eta_4}+\frac{k}{\Gamma}) \\ k\cos\beta(1+\frac{k}{\eta_3\Gamma}) , & \xi(1+\frac{k}{\eta_3\Gamma}) , & \xi(\frac{1}{\eta_3}+\frac{\Gamma}{k}) , & -k\cos\beta(\frac{1}{\eta_3}+\frac{\Gamma}{k}) \\ \xi(1+\frac{\Gamma}{\eta_3 k}) , & -k\cos\beta(1+\frac{\Gamma}{\eta_3 k}) , & -k\cos\beta(\frac{1}{\eta_3}+\frac{k}{\Gamma}) , & -\xi(\frac{1}{\eta_3}+\frac{k}{\Gamma}) \end{pmatrix} \tag{77}$$

and if we now introduce new unknowns $g_{1,2}(\xi)$ and $g^*_{1,2}(\xi)$ such that

$$g_1(\xi) = k\cos\beta\ J_x(\xi) + \xi\ J_z(\xi)$$

$$g_2(\xi) = \xi\ J_x(\xi) - k\cos\beta\ J_z(\xi)$$

with corresponding expressions for $g^*_1(\xi)$ and $g^*_2(\xi)$, the matrix coefficient of the column vector $\{g_1(\xi),\ g_2(\xi),\ Y g^*_1(\xi),\ Y g^*_2(\xi)\}$ again separates into two 2 by 2 matrices. We remark that g and g* are the functions that would have occurred naturally if we had expressed the boundary conditions in terms of the field components E_y and H_y (see 19 and 18), and have the same half planes of analyticity as the original current transforms. For impedances which are the same on both sides of the half plane, i.e, $\eta_4 = \eta_3$, addition and subtraction of the equations making up each pair leads to the separated Wiener-Hopf equations obtained by Senior (1959a), whereas for $\eta_4 \neq \eta_3$ the 2 by 2 matrices can be factored using Hurd's (1976) extension of the Wiener-Hopf techniques.

If the impedances are anisotropic but the same on both sides, ie., $\eta_2 = \eta_1$, $\eta_4 = \eta_3$, $\eta_1 \neq \eta_3$, (76) can be simplified by addition and subtraction of the equations in pairs. Once again G (ξ) separates into two 2 by 2 matrices, and the coefficient of the column vector

$$\left\{ J_x(\xi), \quad \frac{\eta_3}{\eta_1} J_z(\xi) \right\}$$

is then

$$G(\xi) = \begin{pmatrix} k \cos\beta \left(1 + \frac{k}{\eta_1 \Gamma}\right), & \xi \left(1 + \frac{k}{\eta_3 \Gamma}\right) \\ \xi \left(1 + \frac{\Gamma}{\eta_1 k}\right), & -k \cos\beta \left(1 + \frac{\Gamma}{\eta_3 k}\right) \end{pmatrix} \tag{78}$$

This cannot be factorised directly, nor does it satisfy the condition (69) required for the application of Hurd's method except in the trivial case of normal incidence ($\beta = \pi/2$). Although it is not impossible that some further simplication could be made such that the resulting matrix would satisfy (69), this does not seem very likely. We are therefore at an impasse and the situation is even more hopeless for a general anisotropic impedance. Like Maliuzhinets' method, the Wiener-Hopf approach is ineffective when the impedance is anisotropic, and the recent extensions of this approach have not overcome the limitation.

Our final remarks concern the transition problem of a resistive or conductive half plane. From a Wiener-Hopf analysis for an impedance half plane we can deduce the corresponding resistive or conductive sheet solution by merely suppressing the contribution of the magnetic or electric current respectively. Since the only solutions which are relevent are those for impedances which are the same on both faces, Hurd's extension is no longer required and if the resistivity (or conductivity) is isotropic, the problem for even a plane wave at oblique incidence is easily obtained in terms of the field components E_y and H_y (Senior, 1975c). In contrast, Maliuzhinets' method is ill-suited to this type of problem, and to illustrate the difficulty that occurs, consider the problem of the acoustic wave (35) incident on a resistive half plane for which the boundary conditions are (6). If we again represent the total field in the form (36), the continuity of V across the sheet requires

$$s(\alpha) - s(\alpha-2\pi) = s(2\pi-\alpha) - s(-\alpha),$$

whilst the jump condition gives

$$\left(\frac{2}{\eta} - \sin\alpha\right) s(\alpha) + \sin\alpha \; s(\alpha-2\pi) = \left(\frac{2}{\eta} + \sin\alpha\right) s(2\pi-\alpha) - \sin\alpha \; s(-\alpha) \quad .$$

Each equation now involves four functions, and though they can still be manipulated to produce a first order difference equation of interval 2π, it is no longer evident how the singular function $\sigma(\alpha)$ (see 39) can be determined. Of course, for anisotropic sheets at oblique incidence, both methods fail for the reasons stated earlier.

Conclusions

These problems of imperfect half planes are of interest to the mathematician as well as to the engineer, and present a variety of challenges to both. To the radar or acoustic engineer, the results are of practical importance in the design and application of absorbers, and are also needed if we are to employ effectively and economically the computer programs which are now in use. To the mathematician, the class of problems created by the various boundary conditions provides an admirable vehicle for exploring the similarities and differences of two distinct methods of analysis, and when two methods are available, each may serve to expand the capability of the other. We have already noted how the work of Hurd (1976) has extended the scope of the Wiener-Hopf method by providing a factorisation for a matrix of the form shown in (69). If a similar method can be developed for the more general matrix (78), this would not only yield the solution for for an anisotropic sheet, but could also result in the solution of the corresponding difference equation (74) in Maliuzhinets' method. We remark that this different equation is not substantially different from that which arises when a wedge with imperfect surfaces is illuminated by a plane electromagnetic wave at oblique incidence.

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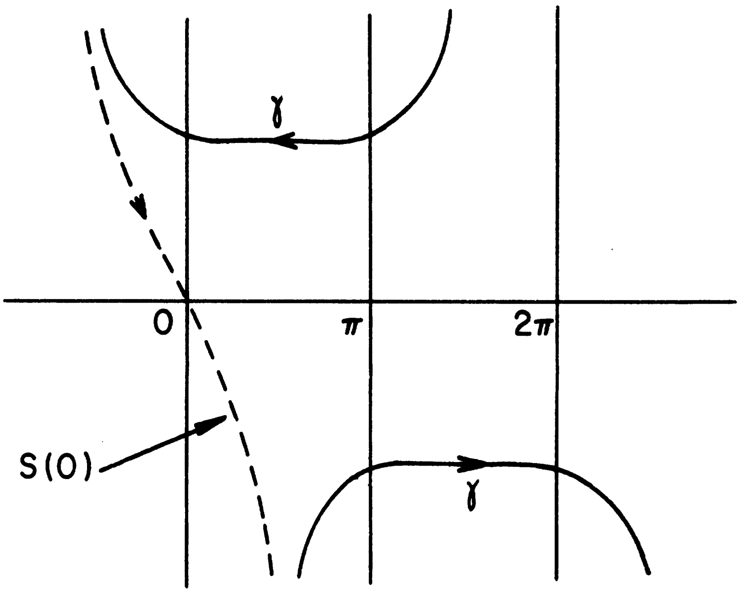

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